AN INTRODUCTION TO SUM-PRODUCT NETWORKS

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Abstract. Sum-Product Networks (SPNs) are deep probabilistic graphical models (PGMs) that compactly represent tractable probability distributions. Exact inference in SPNs is computed in time linear in the number of edges, an attractive feature that distinguishes SPNs from other PGMs. However, learning SPNs is a tough task. There have been many advances in learning both the structure and parameters of SPNs in the past few years. One interesting feature is the fact that we can make use of SPN's deep architecture and perform deep learning on these models. Since the number of hidden layers not only does not negatively impact the tractability of inference of SPNs but also augments the representability of this model, it is very much desirable to continue research on deep learning of SPNs. In this article we seek to produce a tutorial on Sum-Product Networks in a simpler, clearer way then how it is currently written in literature. We will introduce SPNs and explain how knowledge is represented in this model, how to perform exact inference and describe and analyse in detail a simple structural learning algorithm.

1. Introduction

Conventional probabilistic graphical models (PGMs) can compactly represent complex probability distributions and perform sub-exponential time inference through approximate methods. They are able to learn from data accurately and have very expressive semantics. However, exact inference in the general case is intractable. The alternative to exact inference is through the use of approximation algorithms. Unfortunately, approximate inference is at times unpredictable and analysis of these algorithms is very difficult.

Sum-Product Networks (SPNs) are deep PGMs that are able to compactly represent tractable probability distributions. Inference in SPNs is computed in time linear in the number of edges of the graph, where the number of edges is at most polynomial in the number of variables of the distribution. SPNs are represented by a DAG where internal nodes are either sum or product nodes. Leaf nodes are univariate distributions, though recent work on SPNs have shown that multivariate distributions are also allowed as leaves [RL14]. Learning of SPNs can be achieved through subsequent clusterings of both variables and instanciations, where sum

nodes can be seen as mixtures of distributions and product nodes as variable independencies. An interesting feature of SPNs is its deep architecture. As shown in Delalleau and Bengio's work [DB11], deep SPNs have more representative power than shallow SPNs. Poon and Domingos, on the innaugural SPN article [PD11], were able to learn accurate deep SPNs with 36 layers, as opposed to the few, less than ten layers that is typically learned in other deep models.

In this article we will provide a comprehensive description of Sum-Product Networks, from its graph representation and how to perform exact polynomial time inference, to describing and analysing our implementation of the structural learning algorithm introduced in [GD13], a learning algorithm that is able to learn SPNs of potentially tens of layers.

2. Sum-Product Networks

A Sum-Product Network is a DAG where nodes can either be sums, products or leaves. In this section we will present two definitions of SPNs. The first one can be considered "low level", whilst the second carries heavier semantics due to properties we shall enumerate in this section. These classifications of "low level" vs "high level" will become clearer as we progress through this section.

2.1. *Network Polynomials*

Before we define SPNs more formally, we need to understand network polynomials. First introduced by Darwiche [Dar03], a network polynomial is a multilinear function over the variables of an unnormalized distribution. Computing the marginals of the distribution through its network polynomial is possible by setting all indicator variables to values consistent with evidence. The partial derivatives of the network polynomial can be seen as conditioning the network polynomial to a given event.

Network polynomials are not that attractive by themselves, as the number of terms in the polynomial is exponential in the general case. Arithmetic circuits (ACs) are a way of representing network polynomials with a polynomial number of terms. However, ACs are just another way of compiling a Bayesian network into a polynomial-sized model for tractable distributions.

Definition 2.1 (Network polynomial)**.** *Let $\mathcal{N}$ be a Bayesian network over set of variables $\mathbf{X}$. Let $Pa(X)$ be the set of parents of variable X. The network polynomial of $\mathcal{N}$ is given by*

$$f = \sum_{\mathbf{x}} \prod_{X, Pa(X) \sim \mathbf{x}} \lambda_X \theta_{X|Pa(X)}$$

where $\mathbf{x}$ is the set of possible instantiations of $\mathbf{X}$ and $X \sim x$ means that X is consistent with instantiation x, that is, $\lambda_X = 1$ if $X = x$ and $\lambda_X = 0$ if $X \neq x$. If the set of evidence does not contain x, then all λ_X are 1.

References

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