

# Time Series Analysis & Forecasting Using R

## 7. Exponential smoothing



# Outline

- 1 Exponential smoothing
- 2 Trend methods
- 3 Lab Session 14
- 4 Seasonal methods
- 5 ETS taxonomy
- 6 Lab Session 15
- 7 Non-Gaussian forecast distributions

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# Pharmaceutical Benefits Scheme



# Pharmaceutical Benefits Scheme

**The Pharmaceutical Benefits Scheme (PBS) is the Australian government drugs subsidy scheme.**

- Many drugs bought from pharmacies are subsidised to allow more equitable access to modern drugs.
- The cost to government is determined by the number and types of drugs purchased. Currently nearly 1% of GDP.
- The total cost is budgeted based on forecasts of drug usage.

# Pharmaceutical Benefits Scheme

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**POLITICS**

**Opp demands drug price  
restriction after PBS budget  
blow-out**

The Federal Opposition has called for tighter controls on drug prices after the Pharmaceutical Benefits Scheme (PBS) budget blew out by almost \$800 million.

The money was spent on two new drugs including the controversial anti-smoking aid Zyban, which dropped in price from \$220 to \$22 after it was listed on the PBS.

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# Pharmaceutical Benefits Scheme

- In 2001: \$4.5 billion budget, under-forecasted by \$800 million.
- Thousands of products. Seasonal demand.
- Subject to covert marketing, volatile products, uncontrollable expenditure.
- Although monthly data available for 10 years, data are aggregated to annual values, and only the first three years are used in estimating the forecasts.
- All forecasts being done with the FORECAST function in MS-Excel!

# Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a “level”, “trend” (slope) and “seasonal” component to describe a time series.
- The rate of change of the components are controlled by “smoothing parameters”:  $\alpha$ ,  $\beta$  and  $\gamma$  respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.



# A model for levels, trends, and seasonalities

We want a model that captures the level ( $\ell_t$ ), trend ( $b_t$ ) and seasonality ( $s_t$ ).

**How do we combine these elements?**

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**Additively?**

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

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**Multiplicatively?**

$$y_t = \ell_{t-1} b_{t-1} s_{t-m} (1 + \varepsilon_t)$$

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**Multiplicatively?**

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**Perhaps a mix of both?**

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

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**How do the level, trend and seasonal components evolve over time?**

# ETS models


General notation

ETS : Exponential Smoothing  
    ↑   ↑   ↑  
Error Trend Season

**Error:** Additive ("A") or multiplicative ("M")

# ETS models

General notation     E T S : Exponen**T**ial **S**oothing



**Error** **Trend** **Season**

**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

# ETS models

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Error Trend Season

**Error:** Additive ("A") or multiplicative ("M")

**Trend:** None ("N"), additive ("A"), multiplicative ("M"), or damped ("Ad" or "Md").

**Seasonality:** None ("N"), additive ("A") or multiplicative ("M")



## ETS(A,N,N): SES with additive errors

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T$$

Measurement equation

$$y_t = \ell_{t-1} + \varepsilon_t$$

State equation

$$\ell_t = \ell_{t-1} + \alpha \varepsilon_t$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

## ETS(A,N,N): SES with additive errors

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where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

- “innovations” or “single source of error” because equations have the same error process,  $\varepsilon_t$ .
- Measurement equation: relationship between observations and states.
- Transition/state equation(s): evolution of state(s) over time.

## ETS(M,N,N): SES with multiplicative errors

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T$$

Measurement equation

$$y_t = \ell_{t-1}(1 + \varepsilon_t)$$

State equation

$$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$$

where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

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where  $\varepsilon_t \sim \text{NID}(0, \sigma^2)$ .

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

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# Holt's linear trend

## Additive errors: ETS(A,A,N)

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T + hb_T$$

Measurement equation

$$y_t = \ell_{t-1} + b_{t-1} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

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## Multiplicative errors: ETS(M,A,N)

Forecast equation

$$\hat{y}_{T+h|T} = \ell_T + hb_T$$

Measurement equation

$$y_t = (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t)$$

State equations

$$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

# Example: Australian population

```
aus_economy <- global_economy ▷  
  filter(Code = "AUS") ▷  
  mutate(Pop = Population / 1e6)  
fit <- aus_economy ▷ model(AAN = ETS(Pop))  
report(fit)
```

```
## Series: Pop  
## Model: ETS(A,A,N)  
##   Smoothing parameters:  
##     alpha = 1  
##     beta  = 0.327  
##  
##   Initial states:  
##   l[0]  b[0]  
## 10.1 0.222  
##  
##   sigma^2: 0.0041  
##  
##   AIC  AICc  BIC
```



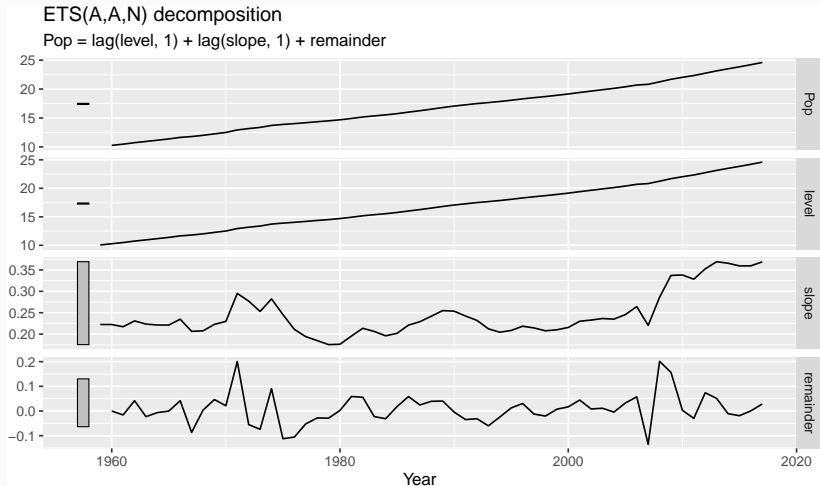
# Example: Australian population

```
components(fit)
```

```
## # A dable: 59 x 7 [1Y]
## # Key:      Country, .model [1]
## # :      Pop = lag(level, 1) + lag(slope, 1) + remainder
##   Country   .model Year   Pop level slope remainder
##   <fct>      <chr>  <dbl> <dbl> <dbl> <dbl>      <dbl>
## 1 Australia AAN     1959  NA    10.1 0.222  NA
## 2 Australia AAN     1960  10.3  10.3 0.222 -0.000145
## 3 Australia AAN     1961  10.5  10.5 0.217 -0.0159
## 4 Australia AAN     1962  10.7  10.7 0.231  0.0418
## 5 Australia AAN     1963  11.0  11.0 0.223 -0.0229
## 6 Australia AAN     1964  11.2  11.2 0.221 -0.00641
## 7 Australia AAN     1965  11.4  11.4 0.221 -0.000314
## 8 Australia AAN     1966  11.7  11.7 0.235  0.0418
## 9 Australia AAN     1967  11.8  11.8 0.206 -0.0869
```

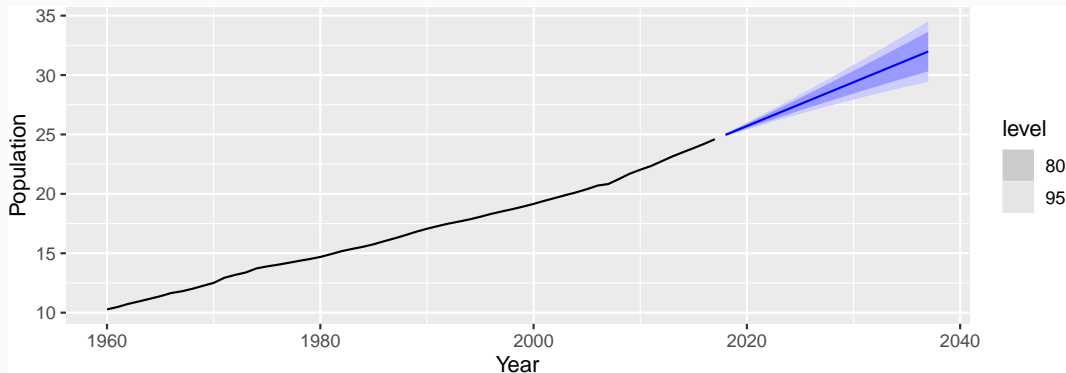
# Example: Australian population

```
components(fit) ▷ autoplot()
```



# Example: Australian population

```
fit ▷  
forecast(h = 20) ▷  
autoplot(aus_economy) +  
labs(y = "Population", x = "Year")
```



# ETS(A,Ad,N): Damped trend method

## Additive errors

Forecast equation  $\hat{y}_{T+h|T} = \ell_T + (\phi + \dots + \phi^{h-1})b_T$

Measurement equation  $y_t = (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t$

State equations  $\ell_t = (\ell_{t-1} + \phi b_{t-1}) + \alpha \varepsilon_t$

$$b_t = \phi b_{t-1} + \beta \varepsilon_t$$

# ETS(A,Ad,N): Damped trend method

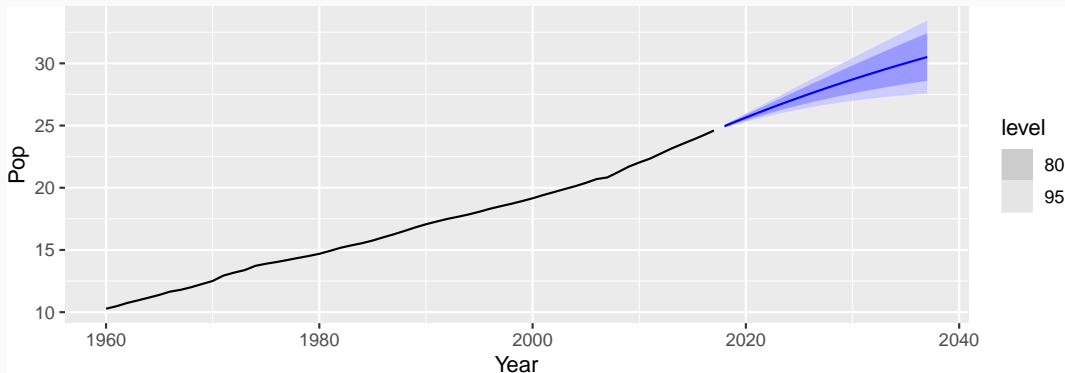
## Additive errors

Forecast equation	$\hat{y}_{T+h T} = \ell_T + (\phi + \dots + \phi^{h-1})b_T$
Measurement equation	$y_t = (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t$
State equations	$\ell_t = (\ell_{t-1} + \phi b_{t-1}) + \alpha \varepsilon_t$ $b_t = \phi b_{t-1} + \beta \varepsilon_t$

- Damping parameter  $0 < \phi < 1$ .
- If  $\phi = 1$ , identical to Holt's linear trend.
- As  $h \rightarrow \infty$ ,  $\hat{y}_{T+h|T} \rightarrow \ell_T + \phi b_T / (1 - \phi)$ .
- Short-run forecasts trended, long-run forecasts constant.

# Example: Australian population

```
aus_economy ▷  
  model(holt = ETS(Pop ~ trend("Ad"))) ▷  
  forecast(h = 20) ▷  
  autoplot(aus_economy)
```



# Example: National populations

```
fit <- global_economy ▷  
  mutate(Pop = Population / 1e6) ▷  
  model(ets = ETS(Pop))  
fit
```

```
## # A mable: 263 x 2  
## # Key:      Country [263]  
##   Country          ets  
##   <fct>           <model>  
## 1 Afghanistan    <ETS(A,A,N)>  
## 2 Albania         <ETS(M,A,N)>  
## 3 Algeria         <ETS(M,A,N)>  
## 4 American Samoa <ETS(M,A,N)>  
## 5 Andorra         <ETS(M,A,N)>  
## 6 Angola          <ETS(M,A,N)>  
## 7 Antigua and Barbuda <ETS(M,A,N)>  
## 8 Arab World      <ETS(M,A,N)>  
## 9 Argentina       <ETS(A,A,N)>  
## 10 Armenia        <ETS(M,A,N)>
```

# Example: National populations

```
fit ▷  
  forecast(h = 5)
```

```
## # A tibble: 1,315 x 5 [1Y]  
## # Key:      Country, .model [263]  
##   Country      .model Year      Pop .mean  
##   <fct>        <chr> <dbl>    <dist> <dbl>  
## 1 Afghanistan ets    2018    N(36, 0.012) 36.4  
## 2 Afghanistan ets    2019    N(37, 0.059) 37.3  
## 3 Afghanistan ets    2020    N(38, 0.16) 38.2  
## 4 Afghanistan ets    2021    N(39, 0.35) 39.0  
## 5 Afghanistan ets    2022    N(40, 0.64) 39.9  
## 6 Albania     ets    2018    N(2.9, 0.00012) 2.87  
## 7 Albania     ets    2019    N(2.9, 6e-04) 2.87  
## 8 Albania     ets    2020    N(2.9, 0.0017) 2.87
```



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## Lab Session 14

Try forecasting the Chinese GDP from the `global_economy` data set using an ETS model.

Experiment with the various options in the `ETS( )` function to see how much the forecasts change with damped trend, or with a Box-Cox transformation. Try to develop an intuition of what each is doing to the forecasts.

[Hint: use `h=20` when forecasting, so you can clearly see the differences between the various options when plotting the forecasts.]

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## ETS(A,A,A): Holt-Winters additive method

Forecast equation

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

Observation equation

$$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$$

State equations

$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$$

$$b_t = b_{t-1} + \beta\varepsilon_t$$

$$s_t = s_{t-m} + \gamma\varepsilon_t$$

- $k = \text{integer part of } (h - 1)/m$ .
- $\sum_i s_i \approx 0$ .
- Parameters:  $0 \leq \alpha \leq 1$ ,  $0 \leq \beta^* \leq 1$ ,  $0 \leq \gamma \leq 1 - \alpha$  and  $m = \text{period of seasonality (e.g. } m = 4 \text{ for quarterly data)}$ .

## ETS(M,A,M): Holt-Winters multiplicative method

Forecast equation	$\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t+h-m(k+1)}$
Observation equation	$y_t = (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t)$
State equations	$\ell_t = (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t)$ $b_t = b_{t-1}(1 + \beta\varepsilon_t)$ $s_t = s_{t-m}(1 + \gamma\varepsilon_t)$

- $k$  is integer part of  $(h - 1)/m$ .
- $\sum_i s_i \approx m$ .
- Parameters:  $0 \leq \alpha \leq 1$ ,  $0 \leq \beta^* \leq 1$ ,  $0 \leq \gamma \leq 1 - \alpha$  and  $m = \text{period of seasonality (e.g. } m = 4 \text{ for quarterly data)}$ .

# Example: Australian holiday tourism

```
holidays <- tourism ▷  
  filter(Purpose = "Holiday")  
fit <- holidays ▷ model(ets = ETS(Trips))  
fit
```

```
## # A mable: 76 x 4
```

```
## # Key:      Region, State, Purpose [76]
```

##	Region	State	Purpose	ets
##	<chr>	<chr>	<chr>	<model>
## 1	Adelaide	South Australia	Holiday	<ETS(A,N,A)>
## 2	Adelaide Hills	South Australia	Holiday	<ETS(A,A,N)>
## 3	Alice Springs	Northern Territory	Holiday	<ETS(M,N,A)>
## 4	Australia's Coral Coast	Western Australia	Holiday	<ETS(M,N,A)>
## 5	Australia's Golden Outback	Western Australia	Holiday	<ETS(M,N,M)>
## 6	Australia's North West	Western Australia	Holiday	<ETS(A,N,A)>
## 7	Australia's South West	Western Australia	Holiday	<ETS(M,N,M)>
## 8	Ballarat	Victoria	Holiday	<ETS(M,N,A)>
## 9	Barkly	Northern Territory	Holiday	<ETS(A,N,A)>

# Example: Australian holiday tourism

```
fit ▷  
  filter(Region = "Snowy Mountains") ▷  
  report()
```

```
## Series: Trips  
## Model: ETS(M,N,A)  
## Smoothing parameters:  
##   alpha = 0.157  
##   gamma = 1e-04  
##  
## Initial states:  
## l[0] s[0] s[-1] s[-2] s[-3]  
## 142 -61 131 -42.2 -27.7  
##  
## sigma^2: 0.0388  
##  
## AIC AICc BIC  
## 852 854 869
```

# Example: Australian holiday tourism

```
fit >
  filter(Region = "Snowy Mountains") >
  components(fit)
```

```
## # A dable: 84 x 9 [1Q]
## # Key:      Region, State, Purpose, .model [1]
## # :        Trips = (lag(level, 1) + lag(season, 4)) * (1 + remainder)
##   Region      State Purpose .model Quarter Trips level season remainder
##   <chr>        <chr> <chr>  <chr>    <qtr>  <dbl>  <dbl>  <dbl>      <dbl>
## 1 Snowy Mountai~ New ~ Holiday ets    1997 Q1  NA      NA    -27.7    NA
## 2 Snowy Mountai~ New ~ Holiday ets    1997 Q2  NA      NA    -42.2    NA
## 3 Snowy Mountai~ New ~ Holiday ets    1997 Q3  NA      NA    131.     NA
## 4 Snowy Mountai~ New ~ Holiday ets    1997 Q4  NA     142.   -61.0    NA
## 5 Snowy Mountai~ New ~ Holiday ets    1998 Q1 101.    140.   -27.7   -0.113
## 6 Snowy Mountai~ New ~ Holiday ets    1998 Q2 112.    142.   -42.2    0.154
## 7 Snowy Mountai~ New ~ Holiday ets    1998 Q3 310.    148.   131.     0.137
## 8 Snowy Mountai~ New ~ Holiday ets    1998 Q4  89.8   148.   -61.0    0.0335
## 9 Snowy Mountai~ New ~ Holiday ets    1999 Q1 112.    147.   -27.7   -0.0687
```

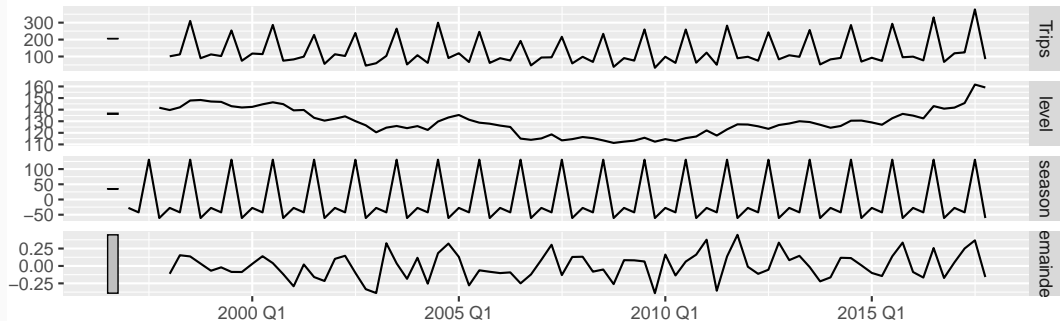


# Example: Australian holiday tourism

```
fit ▷  
  filter(Region = "Snowy Mountains") ▷  
  components(fit) ▷  
  autoplot()
```

ETS(M,N,A) decomposition

$\text{Trips} = (\text{lag}(\text{level}, 1) + \text{lag}(\text{season}, 4)) * (1 + \text{remainder})$



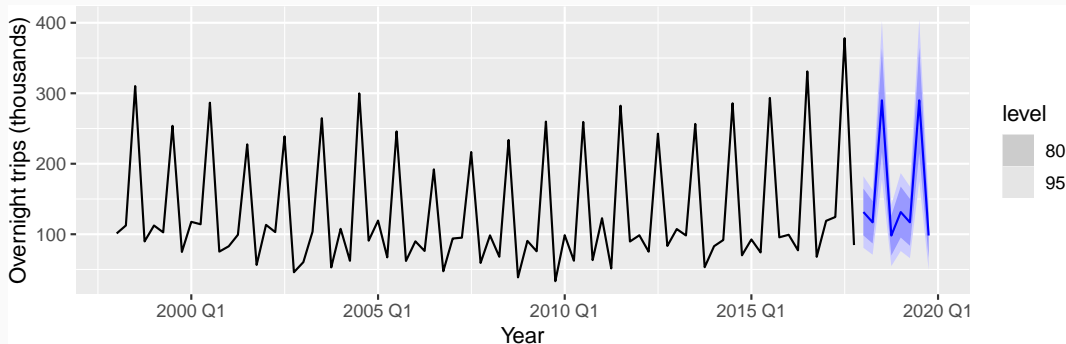
# Example: Australian holiday tourism

```
fit ► forecast()
```

```
## # A fable: 608 x 7 [1Q]
## # Key:      Region, State, Purpose, .model [76]
##   Region      State      Purpose .model Quarter      Trips .mean
##   <chr>        <chr>        <chr>  <chr>    <qtr>      <dist> <dbl>
## 1 Adelaide     South Australia Holiday ets    2018 Q1 N(210, 457) 210.
## 2 Adelaide     South Australia Holiday ets    2018 Q2 N(173, 473) 173.
## 3 Adelaide     South Australia Holiday ets    2018 Q3 N(169, 489) 169.
## 4 Adelaide     South Australia Holiday ets    2018 Q4 N(186, 505) 186.
## 5 Adelaide     South Australia Holiday ets    2019 Q1 N(210, 521) 210.
## 6 Adelaide     South Australia Holiday ets    2019 Q2 N(173, 537) 173.
## 7 Adelaide     South Australia Holiday ets    2019 Q3 N(169, 553) 169.
## 8 Adelaide     South Australia Holiday ets    2019 Q4 N(186, 569) 186.
## 9 Adelaide Hills South Australia Holiday ets    2018 Q1  N(19, 36)  19.4
## 10 Adelaide Hills South Australia Holiday ets    2018 Q2  N(20, 36)  19.6
## # ... with 598 more rows
```

# Example: Australian holiday tourism

```
fit ▷  
  forecast() ▷  
  filter(Region = "Snowy Mountains") ▷  
  autoplot(holidays) +  
  labs(x = "Year", y = "Overnight trips (thousands)")
```



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# Exponential smoothing models

## Additive Error

### Trend Component

## Seasonal Component

### N (None)      A (Additive)      M (Multiplicative)

N	(None)	A,N,N	A,N,A	<del>A,N,M</del>
A	(Additive)	A,A,N	A,A,A	<del>A,A,M</del>
A <sub>d</sub>	(Additive damped)	A,A <sub>d</sub> ,N	A,A <sub>d</sub> ,A	<del>A,A<sub>d</sub>,M</del>

## Multiplicative Error

### Trend Component

## Seasonal Component

### N (None)      A (Additive)      M (Multiplicative)

N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A <sub>d</sub>	(Additive damped)	M,A <sub>d</sub> ,N	M,A <sub>d</sub> ,A	M,A <sub>d</sub> ,M

# Estimating ETS models

- Smoothing parameters  $\alpha, \beta, \gamma$  and  $\phi$ , and the initial states  $\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1}$  are estimated by maximising the “likelihood” = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

# Model selection

## Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters & initial states estimated in the model.

# Model selection

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## Corrected AIC

$$\text{AIC}_c = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).



# Model selection

## Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2k$$

where  $L$  is the likelihood and  $k$  is the number of parameters & initial states estimated in the model.

## Corrected AIC

$$\text{AIC}_c = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

## Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k(\log(T) - 2).$$

## AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

# Automatic forecasting

## From Hyndman et al. (IJF, 2002):

- 1 Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE.
  - 2 Select best method using AICc.
  - 3 Produce forecasts using best method.
  - 4 Obtain forecast intervals using underlying state space model.
- Method performed very well in M3 competition.
  - Used as a benchmark in the M4 competition.

# Outline

- 1 Exponential smoothing
- 2 Trend methods
- 3 Lab Session 14
- 4 Seasonal methods
- 5 ETS taxonomy
- 6 Lab Session 15
- 7 Non-Gaussian forecast distributions

## Lab Session 15

Find an ETS model for the Gas data from `aus_production`.

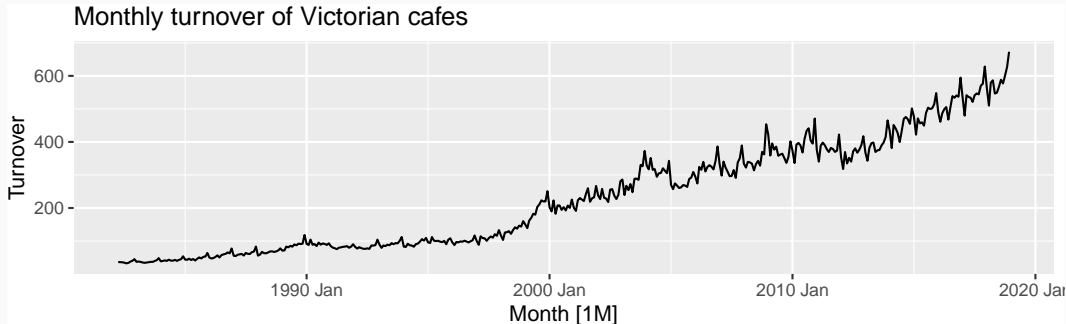
- Why is multiplicative seasonality necessary here?
- Experiment with making the trend damped. Does it improve the forecasts?

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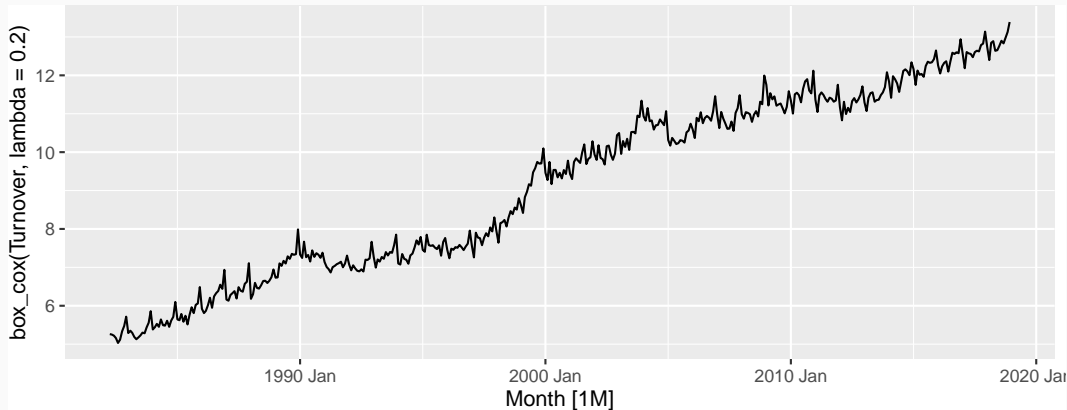
# Non-Gaussian forecast distributions

```
vic_cafe <- tsibbledata::aus_retail ▷  
  filter(State = "Victoria",  
         Industry = "Cafes, restaurants and catering services") ▷  
  select(Month, Turnover)  
vic_cafe ▷  
  autoplot(Turnover) + labs(title = "Monthly turnover of Victorian cafes")
```



# Forecasting with transformations

```
vic_cafe > autoplot(box_cox(Turnover, lambda = 0.2))
```





# Forecasting with transformations

```
fit <- vic_cafe ▷  
  model(ets = ETS(box_cox(Turnover, 0.2)))  
fit
```

```
## # A mable: 1 x 1  
##           ets  
##      <model>  
## 1 <ETS(A,A,A)>
```

```
(fc <- fit ▷ forecast(h = "3 years"))
```

```
## # A fable: 36 x 4 [1M]  
## # Key:      .model [1]  
##   .model      Month      Turnover .mean  
##   <chr>       <mth>      <dist> <dbl>  
## 1 ets       2019 Jan   t(N(13, 0.02)) 608.  
## 2 ets       2019 Feb   t(N(13, 0.028)) 563.  
## 3 ets       2019 Mar   t(N(13, 0.036)) 629.  
## 4 ets       2019 Apr   t(N(13, 0.044)) 615.
```

# Forecasting with transformations

```
fit <- vic_cafe ▷  
  model(ets = ETS(box_cox(Turnover, 0.2)))  
fit
```

```
## # A mable: 1 x 1  
##           ets  
##       <model>  
## 1 <ETS(A,A,A)>
```

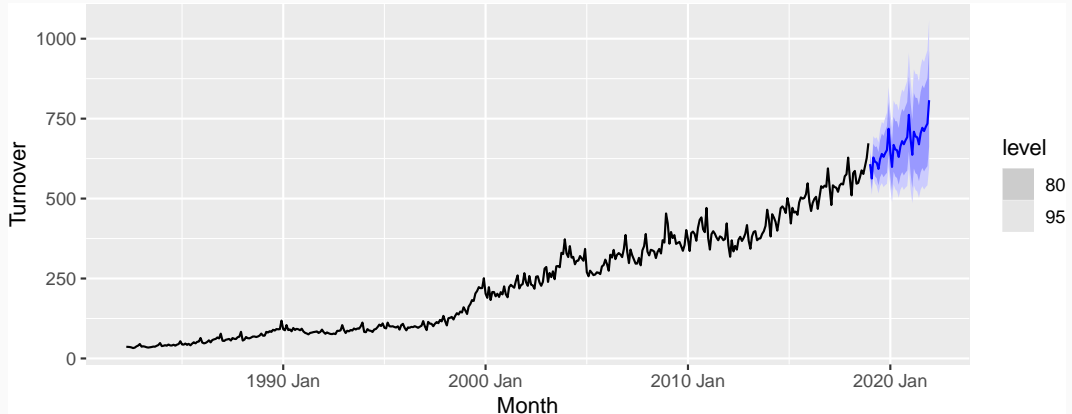
```
(fc <- fit ▷ forecast(h = "3 years"))
```

```
## # A fable: 36 x 4 [1M]  
## # Key:       .model [1]  
##   .model      Month      Turnover .mean  
##   <chr>       <mth>      <dist> <dbl>  
## 1 ets        2019 Jan   t(N(13, 0.02)) 608.  
## 2 ets        2019 Feb   t(N(13, 0.028)) 563.  
## 3 ets        2019 Mar   t(N(13, 0.036)) 629.  
## 4 ets        2019 Apr   t(N(13, 0.044)) 615.
```

- $t(N)$  denotes a transformed normal distribution.
- back-transformation and bias adjustment is done automatically.

# Forecasting with transformations

```
fc > autoplot(vic_cafe)
```



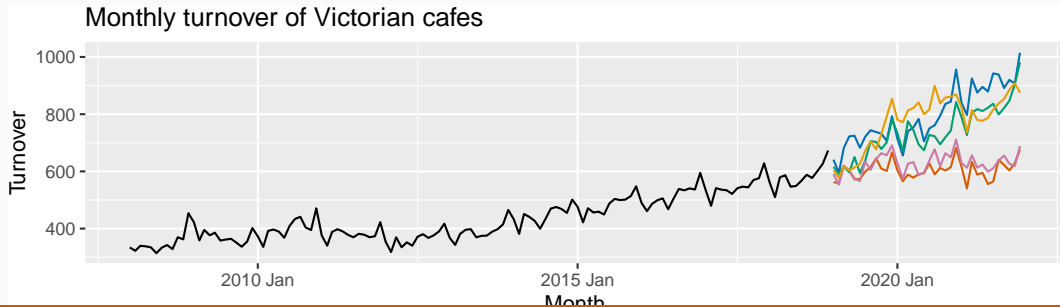
# Bootstrapped forecast distributions

```
sim <- fit ▷ generate(h = "3 years", times = 5, bootstrap = TRUE)
sim
```

```
## # A tsibble: 180 x 4 [1M]
## # Key:           .model, .rep [5]
##   .model      Month .rep   .sim
##   <chr>      <mth> <chr> <dbl>
## 1 ets       2019 Jan 1     562.
## 2 ets       2019 Feb 1     560.
## 3 ets       2019 Mar 1     620.
## 4 ets       2019 Apr 1     602.
## 5 ets       2019 May 1     572.
## 6 ets       2019 Jun 1     573.
## 7 ets       2019 Jul 1     599.
## 8 ets       2019 Aug 1     615.
## 9 ets       2019 Sep 1     646.
## 10 ets      2019 Oct 1     610.
## # ... with 170 more rows
```

# Bootstrapped forecast distributions

```
vic_cafe >
  filter(year(Month) ≥ 2008) >
  ggplot(aes(x = Month)) +
  geom_line(aes(y = Turnover)) +
  geom_line(aes(y = .sim, colour = as.factor(.rep)), data = sim) +
  labs(title = "Monthly turnover of Victorian cafes") +
  guides(col = FALSE)
```



# Bootstrapped forecast distributions

```
fc <- fit %>% forecast(h = "3 years", bootstrap = TRUE)
fc
```

```
## # A tibble: 36 x 4 [1M]
## # Key:   .model [1]
##   .model      Month      Turnover .mean
##   <chr>      <mth>      <dist> <dbl>
## 1 ets       2019 Jan t(sample[5000]) 608.
## 2 ets       2019 Feb t(sample[5000]) 563.
## 3 ets       2019 Mar t(sample[5000]) 628.
## 4 ets       2019 Apr t(sample[5000]) 614.
## 5 ets       2019 May t(sample[5000]) 612.
## 6 ets       2019 Jun t(sample[5000]) 592.
## 7 ets       2019 Jul t(sample[5000]) 623.
## 8 ets       2019 Aug t(sample[5000]) 640.
## 9 ets       2019 Sep t(sample[5000]) 630.
## 10 ets      2019 Oct t(sample[5000]) 642.
## # ... with 26 more rows
```

# Bootstrapped forecast distributions

```
fc > autoplot(vic_cafe) +  
  labs(title = "Monthly turnover of Victorian cafes")
```

