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Time Series Analysis & Forecasting Using R

8. ARIMA models

Outline

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ARIMA models

AR: autoregressive (lagged observations as inputs)

I: integrated (differencing to make series stationary)

MA: moving average (lagged errors as inputs)

ARIMA models

AR: autoregressive (lagged observations as inputs)
I: integrated (differencing to make series stationary)
MA: moving average (lagged errors as inputs)

An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \dots, y_{t+s})$ does not depend on t.

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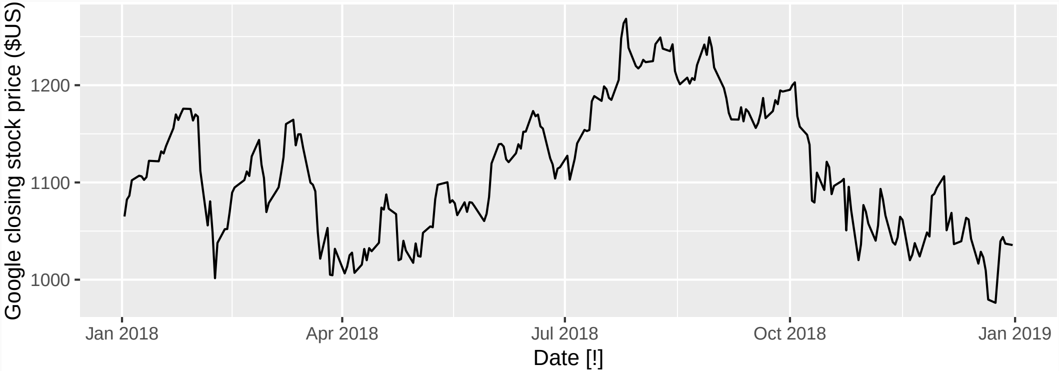
If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \dots, y_{t+s})$ does not depend on t.

A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

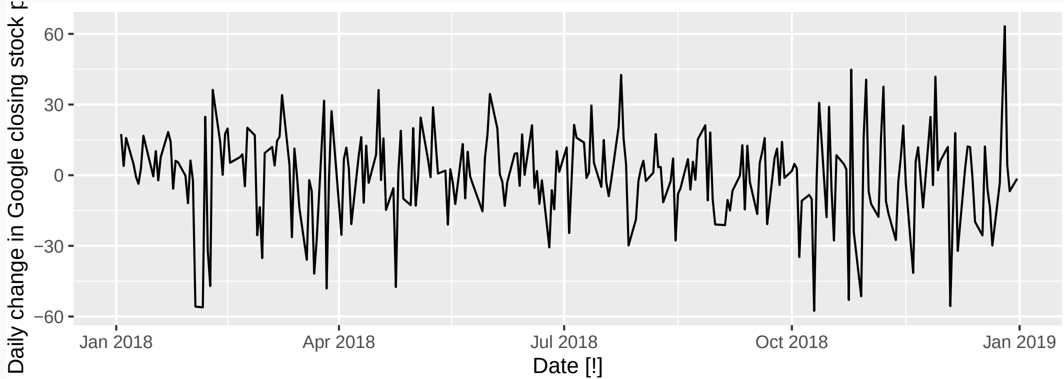
Stationary?

```
gafa_stock |>
  filter(Symbol == "GOOG", year(Date) == 2018) |>
  autoplot(Close) +
  ylab("Google closing stock price ($US)")
```



Stationary?

```
gafa_stock |>
  filter(Symbol == "GOOG", year(Date) == 2018) |>
  autoplot(difference(Close)) +
  ylab("Daily change in Google closing stock price")
```



Differencing

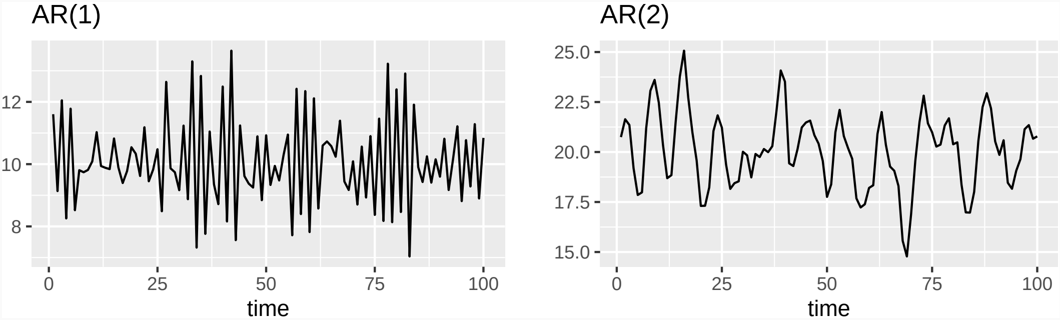
- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series.
- Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time.
- In practice, it is almost never necessary to go beyond second-order differences.

Autoregressive models

Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. A multiple regression with **lagged values** of y_t as predictors.

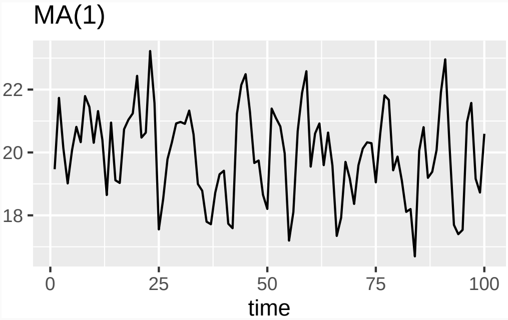


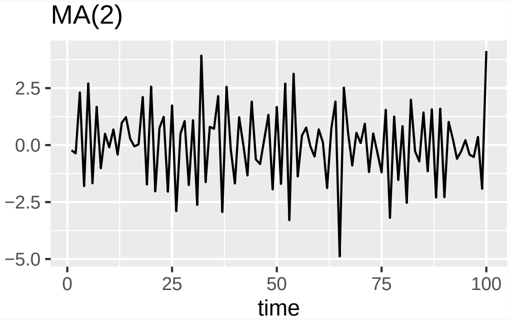
Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q},$$

where ε_t is white noise. A multiple regression with **lagged *errors*** as predictors. *Don't confuse with moving average smoothing!*





ARIMA models

Autoregressive Moving Average models:

$$
\begin{aligned}
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\
+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.
\end{aligned}
$$

ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p}$$
$$+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**

ARIMA models

Autoregressive Moving Average models:

$$
\begin{aligned}
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\
+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.
\end{aligned}
$$

- Predictors include both **lagged values of y_t and lagged errors.**

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing**.
- d-differenced series follows an ARMA model.
- Need to choose p, d, q and whether or not to include c.

ARIMA models

ARIMA(p, d, q) model

$$\begin{aligned} \text{AR:} \quad & p = \text{order of the autoregressive part} \\ \text{I:} \quad & d = \text{degree of first differencing involved} \\ \text{MA:} \quad & q = \text{order of the moving average part.} \end{aligned}$$

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA(p,0,0)
- MA(q): ARIMA(0,0,q)

Example: National populations

```r
fit <- global_economy |>
  model(arima = ARIMA(Population))
fit
```

```
## # A mable: 263 x 2
## # Key:     Country [263]
##    Country                                arima
##    <fct>                                <model>
##  1 Afghanistan                   <ARIMA(4,2,1)>
##  2 Albania                       <ARIMA(0,2,2)>
##  3 Algeria                       <ARIMA(2,2,2)>
##  4 American Samoa                <ARIMA(2,2,2)>
##  5 Andorra              <ARIMA(2,1,2) w/ drift>
##  6 Angola                        <ARIMA(4,2,1)>
##  7 Antigua and Barbuda <ARIMA(2,1,2) w/ drift>
##  8 Arab World                    <ARIMA(0,2,1)>
```

Example: National populations

```
fit |>
  filter(Country == "Australia") |>
  report()
```

```
## Series: Population
## Model: ARIMA(0,2,1)
##
## Coefficients:
##           ma1
##        -0.661
## s.e.    0.107
##
## sigma^2 estimated as 4.063e+09:  log likelihood=-699
## AIC=1401   AICc=1402   BIC=1405
```

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```

$$y_t = 2y_{t-1} - y_{t-2} - 0.7\varepsilon_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim \text{NID}(0, 4 \times 10^9)$$

Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 2$, the long-term forecasts will follow a quadratic trend.

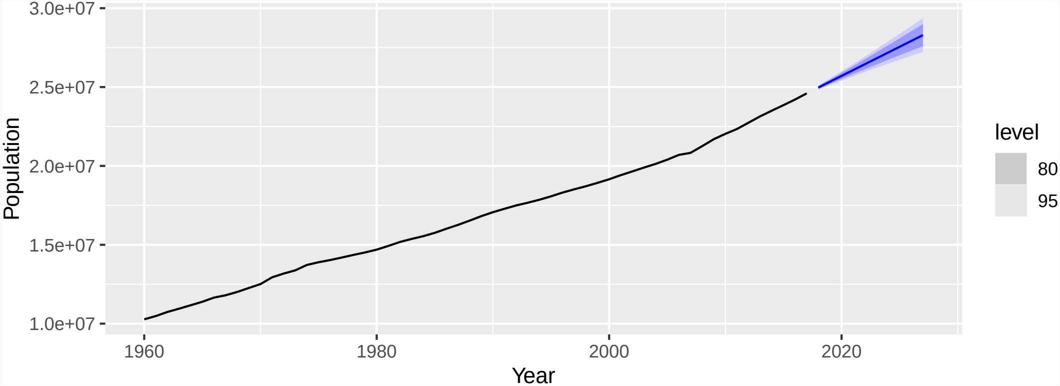
Understanding ARIMA models

Forecast variance and d

- The higher the value of d, the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Example: National populations

```
fit ▷
  forecast(h = 10) ▷
  filter(Country == "Australia") ▷
  autoplot(global_economy)
```



How does ARIMA() work?

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d via KPSS test.
- Select p, q and inclusion of c by minimising AICc.
- Use stepwise search to traverse model space.

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- Select no. differences d via KPSS test.
- Select p, q and inclusion of c by minimising AICc.
- Use stepwise search to traverse model space.

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k=1$ if $c \neq 0$ and $k=0$ otherwise.

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where L is the maximised likelihood fitted to the *differenced* data, $k=1$ if $c\neq 0$ and $k=0$ otherwise.

Note: Can't compare AICc for different values of d.

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

ARIMA$(2, d, 2)$
ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

$\text{ARIMA}(2, d, 2)$

$\text{ARIMA}(0, d, 0)$

$\text{ARIMA}(1, d, 0)$

$\text{ARIMA}(0, d, 1)$

Step 2: Consider variations of current model:

- vary one of $p, q,$ from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

$\text{ARIMA}(2, d, 2)$

$\text{ARIMA}(0, d, 0)$

$\text{ARIMA}(1, d, 0)$

$\text{ARIMA}(0, d, 1)$

Step 2: Consider variations of current model:

- vary one of $p, q,$ from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

Repeat Step 2 until no lower AICc can be found.

Outline

Lab Session 16

For the United States GDP data (from `global_economy`):

- Fit a suitable ARIMA model for the logged data.
- Produce forecasts of your fitted model. Do the forecasts look reasonable?

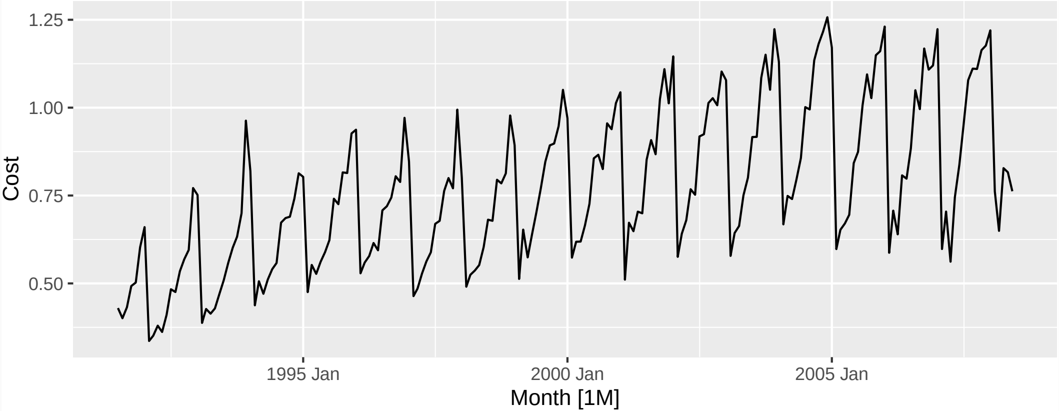
Outline

Cortecosteroid drug sales

```
h02 <- PBS %>%
  filter(ATC2 == "H02") %>%
  summarise(Cost = sum(Cost)/1e6)
```

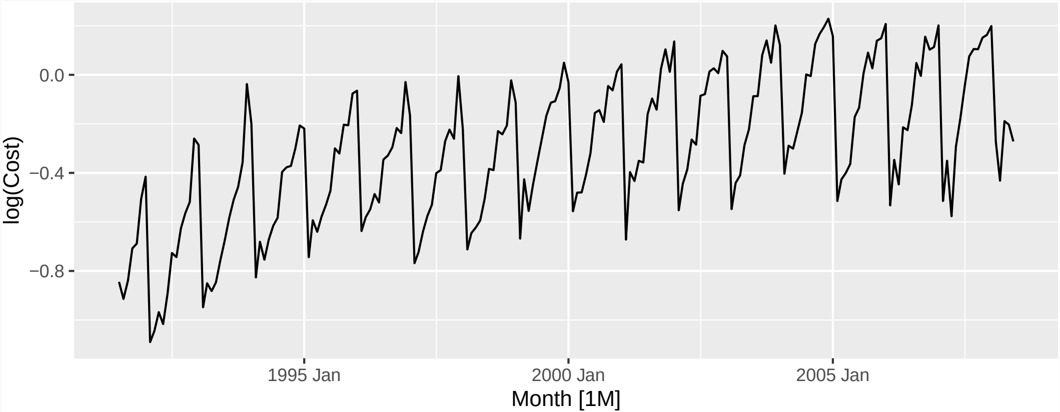
Cortecosteroid drug sales

```
h02 %>% autoplot(
  Cost
)
```



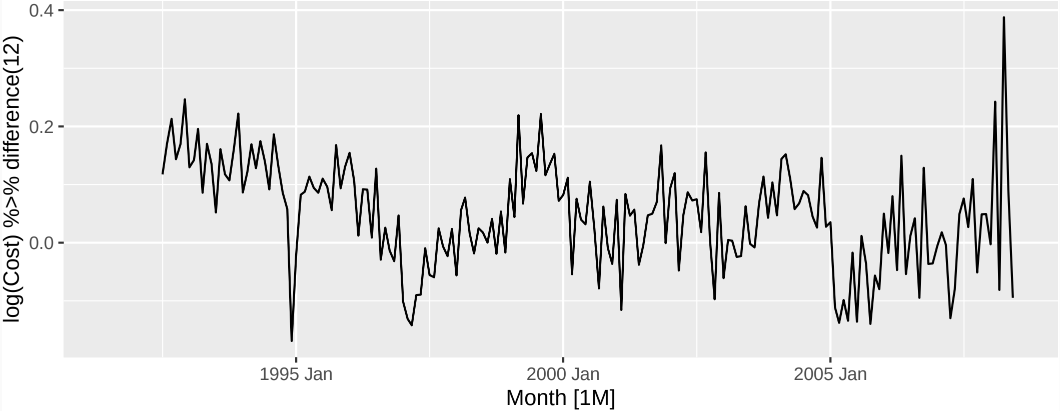
Cortecosteroid drug sales

```
h02 %>% autoplot(
  log(Cost)
)
```



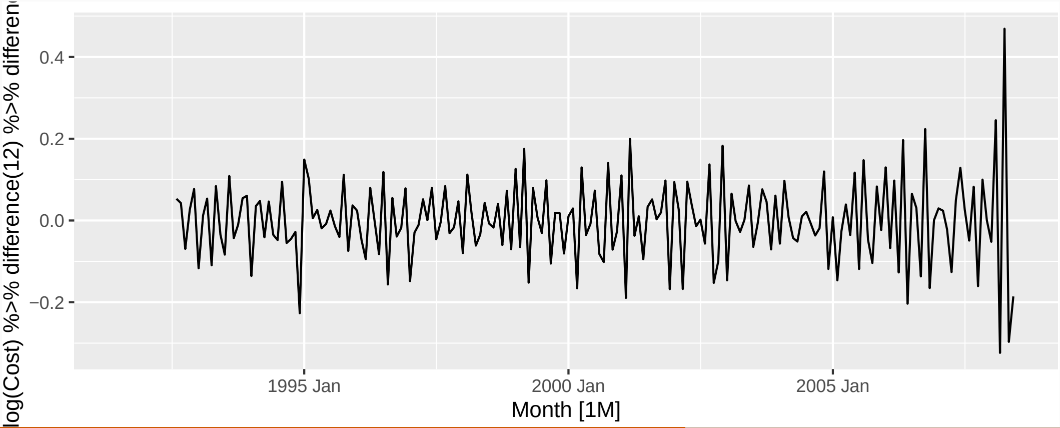
Cortecosteroid drug sales

```
h02 %>% autoplot(
  log(Cost) %>% difference(12)
)
```



Cortecosteroid drug sales

```
h02 %>% autoplot(
  log(Cost) %>% difference(12) %>% difference(1)
)
```



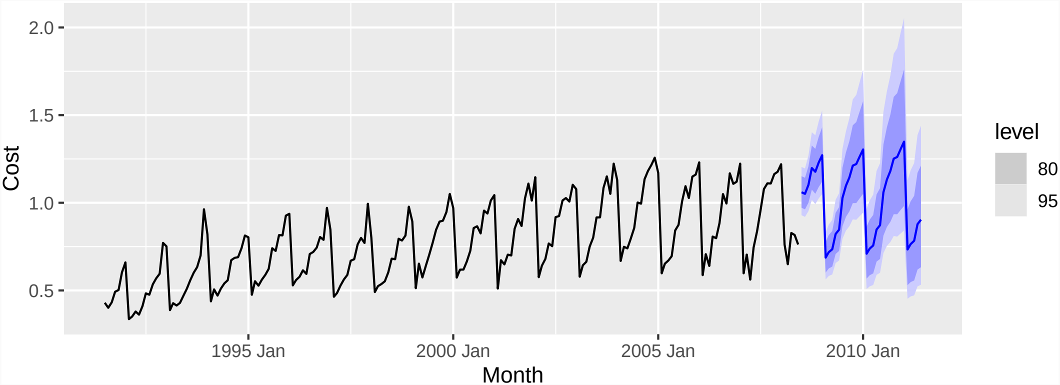
Example: US electricity production

```
h02 ▷
  model(arima = ARIMA(log(Cost))) ▷
  report()
```

```
## Series: Cost
## Model: ARIMA(2,1,0)(0,1,1)[12]
## Transformation: log(Cost)
##
## Coefficients:
##           ar1      ar2     sma1
##       -0.8491  -0.4207  -0.6401
## s.e.   0.0712   0.0714   0.0694
##
## sigma^2 estimated as 0.004387:  log likelihood=245
## AIC=-483   AICc=-483   BIC=-470
```

Example: US electricity production

```
h02 |>
  model(arima = ARIMA(log(Cost))) |>
  forecast(h = "3 years") |>
  autoplot(h02)
```



Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$ Non-seasonal part of the model	$\uparrow$ Seasonal part of of the model

- $m =$ number of observations per year.
- d first differences, D seasonal differences
- p AR lags, q MA lags
- P seasonal AR lags, Q seasonal MA lags

Seasonal and non-seasonal terms combine multiplicatively

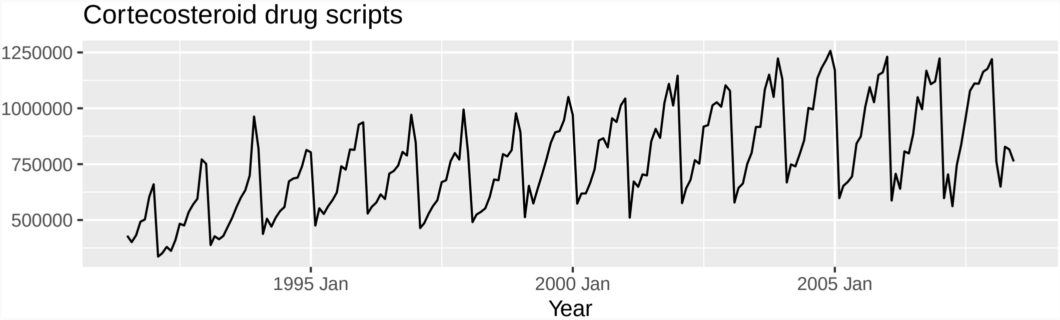
Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA(0,1,1)(0,1,1)$_m$	with log transformation
ARIMA(0,1,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,0)(0,1,1)$_m$	with log transformation
ARIMA(0,2,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,2)(0,1,1)$_m$	with no transformation

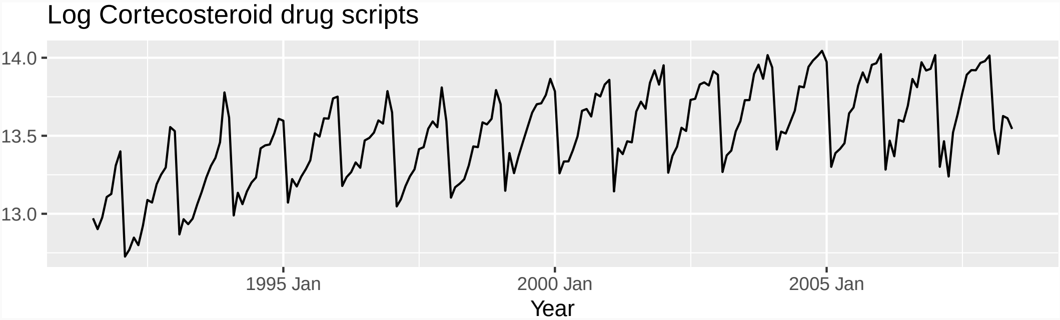
Cortecosteroid drug sales

```
h02 <- PBS |>
  filter(ATC2 == "H02") |>
  summarise(Cost = sum(Cost))
h02 |> autoplot(Cost) +
  xlab("Year") + ylab("") +
  ggtitle("Cortecosteroid drug scripts")
```



Cortecosteroid drug sales

```
h02 <- PBS |>
  filter(ATC2 == "H02") |>
  summarise(Cost = sum(Cost))
h02 |> autoplot(log(Cost)) +
  xlab("Year") + ylab("") +
  ggtitle("Log Cortecosteroid drug scripts")
```



Cortecosteroid drug sales

```r
fit <- h02 |>
  model(auto = ARIMA(log(Cost)))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(2,1,0)(0,1,1)[12]
## Transformation: log(Cost)
##
## Coefficients:
##           ar1      ar2     sma1
##       -0.8491  -0.4207  -0.6401
## s.e.   0.0712   0.0714   0.0694
##
## sigma^2 estimated as 0.004399:  log likelihood=245
## AIC=-483   AICc=-483   BIC=-470
```

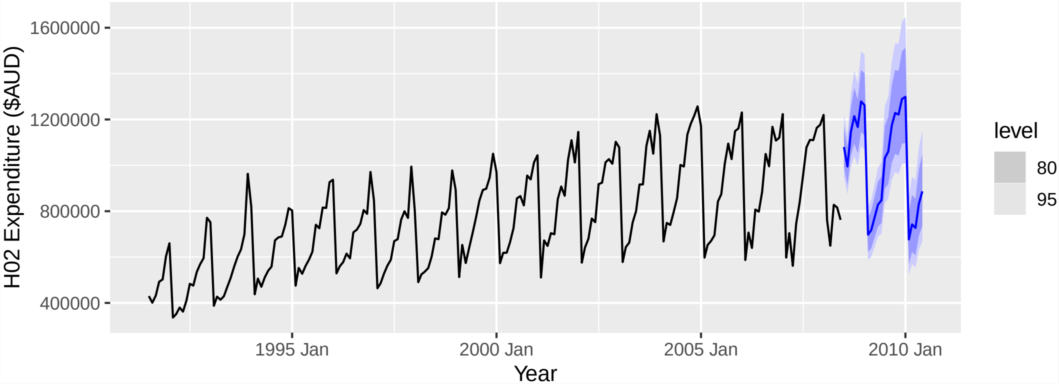
Cortecosteroid drug sales

```
fit <- h02 |>
  model(best = ARIMA(log(Cost),
    stepwise = FALSE,
    approximation = FALSE,
    order_constraint = p + q + P + Q <= 9
  ))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(4,1,1)(2,1,2)[12]
## Transformation: log(Cost)
##
## Coefficients:
##           ar1    ar2    ar3     ar4     ma1   sar1     sar2     sma1   sma2
##       -0.0426  0.210  0.202  -0.227  -0.742  0.621  -0.383  -1.202  0.496
## s.e.   0.2167  0.181  0.114   0.081   0.207  0.242   0.118   0.249  0.214
##
## sigma^2 estimated as 0.004061:  log likelihood=254
## AIC=-489   AICc=-487   BIC=-456
```

Cortecosteroid drug sales

```
fit ▷
  forecast() ▷
  autoplot(h02) +
  ylab("H02 Expenditure ($AUD)") + xlab("Year")
```



Outline

Lab Session 17

For the Australian tourism data (from `tourism`):

- Fit a suitable ARIMA model for all data.
- Produce forecasts of your fitted models.
- Check the forecasts for the “Snowy Mountains” and “Melbourne” regions. Do they look reasonable?

Outline

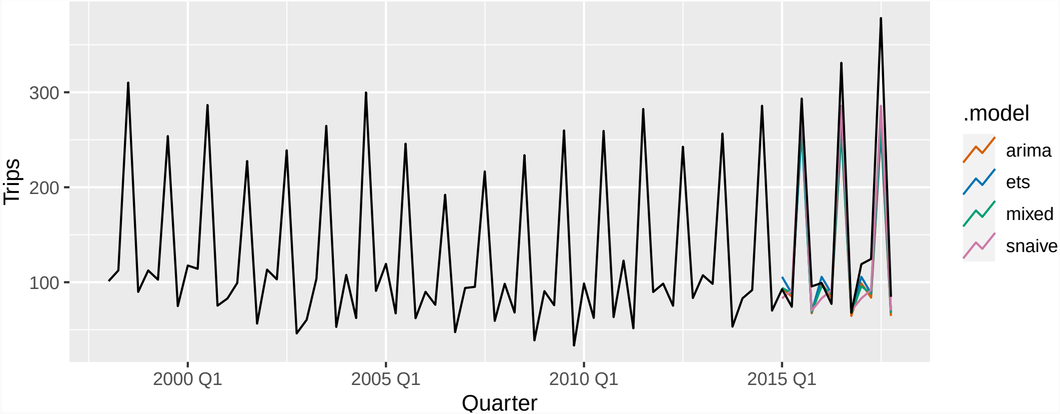
Forecast ensembles

```
train <- tourism |>
  filter(year(Quarter) <= 2014)
fit <- train |>
  model(
    ets = ETS(Trips),
    arima = ARIMA(Trips),
    snaive = SNAIVE(Trips)
  ) |>
  mutate(mixed = (ets + arima + snaive) / 3)
```

- Ensemble forecast `mixed` is a simple average of the three fitted models.
- `forecast()` will produce distributional forecasts taking into account the correlations between the forecast errors of the component models.

Forecast ensembles

```
fc <- fit |> forecast(h = "3 years")
fc |> filter(Region == "Snowy Mountains", Purpose == "Holiday") |>
  autoplot(tourism, level = NULL)
```



Forecast ensembles

```
accuracy(fc, tourism) |>
  group_by(.model) |>
  summarise(
    RMSE = mean(RMSE),
    MAE = mean(MAE),
    MASE = mean(MASE)
  ) |>
  arrange(RMSE)
```

```
## # A tibble: 4 x 4
##   .model  RMSE   MAE  MASE
##   <chr>  <dbl> <dbl> <dbl>
## 1 mixed   19.8  16.0 0.997
## 2 ets     20.2  16.4 1.00
## 3 snaive  21.5  17.3 1.17
## 4 arima   21.9  17.8 1.06
```

Forecast ensembles

Can we do better than equal weights?

Forecast ensembles

Can we do better than equal weights?

- Hard to find weights that improve forecast accuracy.
- Known as the “forecast combination puzzle”.
- Solution: FFORMA

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FFORMA (Feature-based FORecast Model Averaging)

- Vector of time series features used to predict best weights.
- A modification of xgboost is used.
- Method came 2nd in the 2018 M4 international forecasting competition.
- Main author: Pablo Montero-Manso (Monash U)
- Not (yet) available for fable.