École polytechnique fédérale de Lausanne

Master thesis

MSc Mathematics

Tropical linear systems and the realizability problem

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Contents

1 Introduction

The goal of this thesis is to explore linear systems on metric graphs, which despite being relatively simple objects to understand, have much in common with their counterparts on algebraic curves. There are suitable notions divisors, rational functions and linear systems on metric graphs, which closely mimic how these objects behave on algebraic curves. It turns out that this uncovers many interesting connections between the world of algebraic geometry and combinatorics. Baker and Norine defined in [BN07] the notion of *rank* of a divisor, which behaves in very similar ways on algebraic curves and metric graphs. For example, the Riemann-Roch theorem for algebraic curves may be stated using the rank, and an amazing result shown in [BN07] is that the Riemann-Roch theorem holds also for metric graphs.

Complete linear systems on metric graphs have plenty of interesting combinatorial structure. On one hand, a complete linear system $|D|$ is an abstract polyhedral complex, and on the other hand, the set of rational functions $R(D)$ associated to $|D|$ forms a tropical module (a semi-module equipped with the element-wise maximum and addition operations). The set $|D|$ may be identified with the tropical projectivization $R(D)/\mathbb{R}$, and so one may study the subspaces of $|D|$ that appear as the projectivization of submodules of $R(D)$. We will call such a subspace $\mathfrak{d} \subseteq |D|$ a *tropical linear system* (or a tropical linear *series*). In similar fashion to complete linear systems, it turns out that in many cases tropical linear systems also have an induced abstract polyhedral complex structure.

The theories of linear systems on metric graphs and algebraic curves are far from being just in simple analogy, as it is possible to link them via a process called *tropicalization.* Given an algebraic curve with only ordinary double points as singularities, one may associate to it a graph, called its *dual graph.* When such a curve appears as a closed fiber of a fibered surface, one may furthermore uniquely attribute edge lengths to the dual graph and so give it the structure of a metric graph. There is then a way to transfer divisors from the generic fiber of the surface to the metric graph through a process called *specialization.* Matt Baker has shown in [Bak07] the *specialization lemma*, which states that the rank of a divisor can only go up under specialization. This comparison theorem allows one to derive results about divisors and linear systems on algebraic curves by studying metric graphs.

One interpretation of the specialization lemma is that there are "more" divisors on metric graphs than on algebraic curves. It is then reasonable to ask which divisors on metric graphs come from a divisor on an algebraic curve, if we also require the rank to be preserved. This question is called the realizability problem and there have been only a few specific classes of divisors for which the realizable divisors were fully characterized. For example in [MUW17] the authors give a complete characterization of realizability for canonical divisors and this result was later extended to pluri-canonical divisors in [RS21]. In general, it is an important open problem awaiting to be solved.

One object of study of this thesis is the set of realizable divisors (also called the *realizability locus*) in the canonical linear system. We reinterpret the characterization for realizability from [MUW17] and use the resulting criteria to show that the realizability locus is an abstract polyhedral complex and that it is a tropical linear system.

A natural extension of the realizability question concerns the realizability of linear systems. Tropicalizing a linear system on a curve yields a tropical linear system and one may again ask the realizability question in this context. This question is more complicated, because even if all divisors in a tropical linear system $\mathfrak{d}$ are realizable, it is possible that $\mathfrak{d}$ does not appear as the tropicalization of a linear system on any given curve. The theory of tropical linear systems is presented in [JP22], and further in [FJP23], but it is a very new topic, and remains largely unexplored.

Since the rank of a linear system on an algebraic curve is equal to its dimension as a projective space, it is natural to try to establish such a link for metric graphs. In this thesis we define a suitable notion of local dimension of a tropical linear system and show that the local dimension is bounded from below by the rank. In [JP22], the authors show that when the tropical linear system is finitely generated and satisfies a further combinatorial condition, the dimension may also be bound from above by the rank. Along with the results from [FJP23], this shows that tropicalizations of linear systems on algebraic curves are equi-dimensional abstract polyhedral complexes of dimension equal to the rank, establishing a strong link between the rank and dimension of realizable tropical linear systems and largely limiting what kinds tropical linear systems may be realizable.

Structure of the thesis

Section 2 focuses on the theory of divisors and linear systems on metric graphs. We start by covering the essential definitions concerning metric graphs and divisors in subsections 2.1 through 2.3. In subsection 2.4 we will introduce tropical modules and in subsection 2.5 we describe how complete linear systems admit the structure of an abstract polyhedral complex. We then give a sufficient condition for a subset to also admit the structure of an abstract polyhedral complex, and describe how we can detect its dimension at a point. In subsection 2.6 we show that the local dimension of a complete linear system is bounded from below by the rank (Proposition 2.78). In subsection 2.7, we give some characterizations of the canonical linear system. In particular we show that the lower bound on the dimension is attained in the case of canonical linear systems. In subsection 2.8 we extend Proposition 2.78 to the setting of tropical linear systems (Corollary 2.95.1).

In section 4 we make the links between the worlds of tropical and algebraic geometry. We first go into the details of the tropicalization process in subsection 3.1. In subsection 3.2, we explain the specialization of divisors from algebraic curves to metric graphs and the specialization lemma. In subsection 3.3 we describe the condition for realizability shown in [MUW17] and give a cleaner characterization of inconvenient vertices. We then use this characterization in subsection 3.4 to show that the realizability locus of the canonical linear system is tropically convex and an abstract polyhedral complex. In subsection 3.5 we give a sufficient condition for realizability of canonical divisors (Proposition 3.27) and deduce that the realizable locus always contains a maximal cell of dimension $g - 1$. In subsection 3.6 we explain that specialization preserves linear equivalences and discuss the image of a linear series under the specialization map. We then describe the advances made in [JP22] and [FJP23] on this topic and deduce that tropicalizations of linear series are equi-dimensional.

Finally, in section 4 we describe the theory of linear systems on graphs (without edge lengths) and explain how it relates to the theory on metric graphs. We describe useful results that can be used to work efficiently with these discretizations and allow their implementation with algorithms.

Implementation

I have used the concepts and results from Section 4 to explore linear systems with a computer program. Concretely, I wrote code for working with metric graphs, which can among other things:

- Find all divisors in $|D|$ supported on a fixed model
- Find the extremals of $|D|$
- Check the realizability of a divisor in the canonical linear system
- Test whether a rational function belongs to the span of a generating set
- Find the maximal cells of $|D|$ and calculate their dimensions

This helped me build an intuition, find counter-examples and form hypotheses regarding linear systems on metric graphs.

The code is freely accessible on following GitHub repository: `https://github.com/MattDupraz/Graph-Linear-Systems.git`.

Acknowledgements

I would like to express my deepest gratitude to my advisor, Francesca Carocci, for her unwavering support and guidance, which extended well beyond the scope of the thesis. I have learned much from her not only in terms of mathematics, but also on a personal level.

2 Metric graphs and linear systems

2.1 Metric graphs

We will start by introducing the notion of *metric graph*. Intuitively, a metric graph is just a metric space isomorphic to the geometric realization of a graph with given edge lengths. However, to be more precise, we will define it using the notion of *length space*. This approach is largely inspired by [Mug21]. For the definition of length space, we follow [BBI01]. Throughout this paper, we allow distance functions that admit infinite values.

Let (X, d) be a metric space. A *path* is a continuous map $\gamma : [a, b] \to X$. We will now define the length of a path.

Definition 2.1. [BBI01, Definition 2.3.1.] Let $\gamma : [a, b] \to X$ be a path. A *partition* of $[a, b]$ is a finite collection of points $\{x_0, \dots, x_N\} \subseteq [a, b]$ with

$$a = x_0 < x_1 < \cdots < x_N = b.$$

We define the *length* of γ as

$$L(\gamma) = \sup \sum_{i=1}^{N} d(\gamma(x_{i-1}), \gamma(y_i)),$$

where the supremum is taken over all the partitions of $[a, b]$. A curve is said to be *rectifiable* if its length is finite.

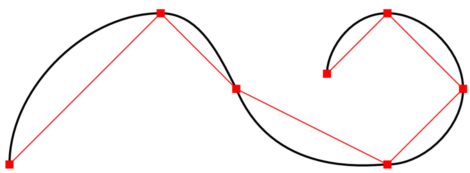

Figure 1: Example of a path and rectification

The notion of path length allows us to define a new distance on X.

Definition 2.2. [BBI01, §2.1.2.] Let (X, d) be a metric space. We define the *induced intrinsic metric* to be

$$d_I(x, y) = \inf L(\gamma),$$

where the infimum is taken over all the paths $\gamma : [a, b] \to X$, with $\gamma(a) = x$ and $\gamma(b) = y$. If there is no path between x and y (when X is disconnected), we let $d_I(x, y) = +\infty$.

A metric space whose distance function is the same as the induced intrinsic metric is called a *length space*.

Remark 2.3. When (X, d) is a metric space, the topology induced by the intrinsic metric d_I is finer than the one induced by X. To see this, notice that for all $x, y \in X$,

$$d_I(x, y) \geq d(x, y).$$

Indeed, when x, y are connected by a path γ, then $L(\gamma) \geq d(x, y)$ by definition of path length. When x, y lie in different path-connected components of X, then $d_I(x, y) = +\infty$ which also directly implies the above inequality. In other words, the identity map $(X, d_I) \to (X, d)$ is continuous.

Definition 2.4. A *metric graph* is a *compact* length space Γ such that each point $x \in \Gamma$ has a neighbourhood U_x that is homeomorphic to $\bigsqcup_{i=1}^{v}[0, \epsilon)/\sim$ for some $\epsilon > 0$, where the equivalence relation $\sim$ identifies the zeroes of the intervals, and such that x corresponds to the 0 via this isomorphism. We call such a neighbourhood a *star-shaped neighbourhood.* We say v is the *valence* of x and denote it by $\mathrm{val}(x)$.

Remark 2.5. Some authors use the terminology *abstract tropical curve* to designate metric graphs.

Remark 2.6. We necessarily have that the set of points $x \in \Gamma$ with $\mathrm{val}(x) \neq 2$ is finite.

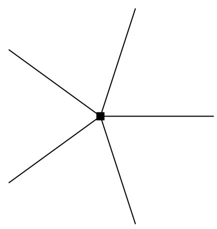

Figure 2: Star-shaped neighbourhood of a point of valence 5

Definition 2.7. Let $V \subseteq \Gamma$ be a finite subset such that $\Gamma \setminus V$ consists of disjoint union of open intervals. Then V determines a *model* $G = (V, E)$ of the metric graph Γ, where E is the set of undirected edges corresponding to the open intervals of $\Gamma \setminus V$. For $e \in E$ an edge, we define $l(e)$ to be the length of the corresponding open interval.

When X is a length space and $\sim$ is an equivalence relation, we would like to equip $X/\sim$ with the structure of a length space. Following [BBI01, Definition 3.1.12.] we may define the following *semi*-metric on $X/\sim$:

$$d_\sim([x], [y]) = \inf \left\{ \sum_{i=1}^{k} d(p_i, q_i) \right\},$$

where the infimum is taken over sequences $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ of points in X, such that $p_1 \sim x$, $q_k \sim y$ and $q_i \sim p_{i+1}$ for $1 \leq i \leq k-1$. This is a semi-metric, because it might happen that $d_\sim([x], [y]) = 0$ even when $[x] \neq [y]$. A prototypical example of this happening is the line with two origins obtaining by gluing two copies of $\mathbb{R}$ along $\mathbb{R} \setminus \{0\}$.

When Γ is a metric graph, we would like to be able to glue some vertices together to obtain a new graph. Let u, v be two points of Γ, we may take the quotient $\Gamma/\{u, v\}$, where the equivalence relation simply identifies these two points. It turns out that the semi-metric $d_\sim$ defined on this quotient is actually a metric.

Proposition 2.8. *The metric $d_\sim$ on $\Gamma/\{u, v\}$ is a well-defined metric.*

Proof. Clearly, $d_\sim$ is symmetric and non-negative and satisfies the triangle inequality. We have to verify that $d_\sim([x], [y]) = 0$ if and only if $[x] = [y]$. Suppose $d_\sim([x], [y]) = 0$. So for all $\epsilon > 0$ there exist sequences (p_i), (q_i) with the properties above, such that

$$\inf \left\{ \sum_{i=1}^{k} d(p_i, q_i) \right\} < \epsilon. \tag{1}$$

We may assume without loss of generality that $q_i \neq p_{i+1}$ for $1 \leq i \leq k-1$, since otherwise we have

$$d(p_i, q_i) + d(p_{i+1}, q_{i+1}) \leq d(p_i, q_{i+1})$$

by the triangle inequality, so we could just remove the terms q_i and p_{i+1} from the sequences. So for $1 \leq i \leq k-1$, we may assume that $q_i, p_{i+1} \in \{u, v\}$.

If $x \notin \{u, v\}$, then $d(x, u), d(x, v) > \delta$ for some δ small enough, so (1) implies that $k = 1$ and $d(x, y) < \epsilon$ for all $\epsilon < \delta$, which in turn implies $d(x, y) = 0$ and so $x = y$. By symmetry we obtain the same result when $y \notin \{u, v\}$, The last case is $x, y \in \{u, v\}$, but then $x \sim y$, so we are done. $\square$

Remark 2.9. The metric space $\Gamma/\{u, v\}$ is in fact a length space, as explained in [BBI01, §3.1], so it is in fact a metric graph, as around the image of $x \sim y$ we will again obtain a star-shaped neighbourhood of valence $\mathrm{val}(x) + \mathrm{val}(y)$.

By induction, for any finite set of vertices $\{v_1, \ldots, v_n\}$, the quotient space $\Gamma/\{v_1, \ldots, v_n\}$ is also a metric graph. More in general, if $A_1, \ldots, A_n$ are finite disjoint sets of vertices, we define $\Gamma/(A_1, \ldots, A_n)$ to be the quotient by the equivalence relation $x \sim y$ if and only if $x = y$ or $\{x, y\} \subseteq A_i$ for some i. This is again a metric graph.

As seen in [BBI01, Exercise 3.1.14.], the topology induced by the metric coincides with the quotient topology, and so in particular the quotient map $\Gamma \to \Gamma/(A_1, \ldots, A_n)$ is continuous.

Definition 2.10. We say $\Gamma/(A_1, \ldots, A_n)$ is a *gluing* of Γ. Equivalently, we say that Γ is a *cut* of $\Gamma/(A_1, \ldots, A_n)$.

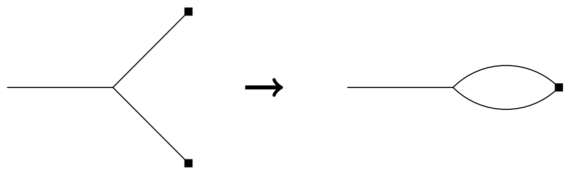

Figure 3: Example of a gluing of two vertices

Definition 2.11. Suppose $G = (V, E)$ is a graph and $l : E \to \mathbb{R}_{>0}$ a map that assigns to each edge a length. We may construct from this a metric graph. Let

$$\mathcal{E} = \bigsqcup_{e \in E} [0, l(e)].$$

The metric on the disjoint union is given by

$$d((x_1, e_1), (x_2, e_2)) = \begin{cases} |x_1 - x_2| & \text{if } e_1 = e_2, \\ 0 & \text{otherwise.} \end{cases}$$

This clearly gives $\mathcal{E}$ the structure of length space and of a metric graph. Fix an ordering $v_1, \ldots, v_n$ on vertices and suppose e is an edge between v_i, v_j with $i \leq j$. We denote $s(e) = v_i$ and $t(e) = v_j$. For each $i \in \{1, \ldots, n\}$ let

$$V_i = \{(e, 0) \in \mathcal{E} : s(e) = v_i\} \cup \{(e, l(e)) \in \mathcal{E}, t(e) = v_i\}$$

Let $\Gamma = \mathcal{E}/(V_1, \ldots, V_n)$. We may identify $V = \{v_1, \ldots, v_n\}$ with the images of $V_1, \ldots, V_n$ under the gluing and this induces a model on Γ that agrees with G. In other words, we have constructed a metric graph that admits G as a model and whose edge lengths agree with the function l. We say Γ is a *realization* of (G, l).

Definition 2.12. We define the *genus* $g(\Gamma)$ of a metric graph Γ to be its first Betti number. In other words it corresponds to the maximal number of independent cycles it contains.

Remark 2.13. If a metric graph Γ with n connected components admits a model $G = (V, E)$, then we have the relation

$$g(\Gamma) = |E| - |V| + n.$$

Indeed, the simplicial homology groups of Γ are calculated from the chain complex

$$\begin{aligned} \cdots \to 0 \to \mathbb{Z}^E &\xrightarrow{\partial_1} \mathbb{Z}^V \to 0 \\ (u, v) &\mapsto u - v, \end{aligned}$$

where we fixed an arbitrary orientation for each edge.

We have that $H^0(\Gamma) = \operatorname{coker}(\partial_1)$. Vertices that are joined by an edge are identified in the cokernel, so we deduce that $\operatorname{rk} H^0(\Gamma) = n$. We have the exact sequence

$$0 \to \ker \partial_1 \to \mathbb{Z}^E \to \mathbb{Z}^V \to \operatorname{coker} \partial_1 \to 0,$$

from where it follows that

$$\operatorname{rk} \ker \partial_1 - |E| + |V| - \operatorname{rk} \operatorname{coker}(\partial_1) = 0.$$

The formula for the genus then follows by remarking that $H^1(\Gamma) = \ker(\partial_1)$.

We will now define tangent vectors on metric graphs in analogy to the definition of tangent vectors on manifolds via tangent curves.

Definition 2.14. For $\epsilon > 0$, let $I_{\epsilon,x}(\Gamma)$ be the set of isometries $\gamma : [0, \epsilon) \to \Gamma$, with $\gamma(0) = x$. For $\epsilon > \epsilon'$, we have a natural map $I_{\epsilon,x}(\Gamma) \to I_{\epsilon',x}(\Gamma)$ given by the restriction, and this defines a direct system over $[0, \infty)$. Let $T_x\Gamma = \varinjlim I_{\epsilon,x}$ be the direct limit of this system. We call this the set of (unit) *tangent vectors* of Γ at x.

Remark 2.15. For ϵ small enough, the open ball $B(x, \epsilon)$ is a star-shaped neighbourhood, so in this case the elements of $I_{\epsilon,x}(\Gamma)$ correspond to the identification of $[0, \epsilon)$ to one of the copies of $[0, \epsilon)$ in

$$B(x, \epsilon) \cong \bigsqcup_{i=1}^{\operatorname{val}(x)} [0, \epsilon)/\sim.$$

In other words, there is a bijective correspondence between the tangent vectors at x and the half-edges of Γ adjacent to x.

Definition 2.16. Let $U \subseteq \Gamma$ be an open subset with a finite number of connected components. We endow U with the induced intrinsic metric and let $\hat{U}$ be the completion of U with respect to this metric. Another way to see $\hat{U}$ is as the space obtained from U by adding a point to each open half-edge of U. From this description it is clear that $\hat{U}$ is also a metric graph.

By remark 2.3, the inclusion $U \hookrightarrow \Gamma$ is continuous and so as Γ is compact, there is unique map $\phi : \hat{U} \to \Gamma$ that extends $U \hookrightarrow \Gamma$. Its image is the closure of U in Γ.

For any model of Γ, the metric graph $\hat{U}$ naturally inherits a model structure, which is the minimal model for the property that it contains $V \cap U \hookrightarrow \hat{U}$ in its set of vertices.

Definition 2.17. If U is an open subset with a finite number of connected components, we define its genus $g(U)$ to be the genus of its completion $\hat{U}$.

We will now prove a useful lemma which may be used to calculate the genus of the graph obtained after cutting the graph in a finite number of points.

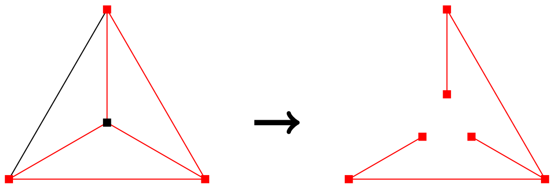

Figure 4: Example of completion of an open subgraph (in red)

Lemma 2.18. *Suppose Γ is connected. Let $A \subset \Gamma$ be a finite set of points. Then*

$$g(\Gamma) = g(\Gamma \setminus A) + \sum_{x \in A} (\mathrm{val}(x) - 1) + 1 - N,$$

where N is the number of connected components of $\Gamma \setminus A$

Proof. We will denote $V(G)$ and $E(G)$ the set of vertices and edges of a graph G respectively. Up to subdividing the model of Γ, we may assume the set A is contained in $V(\Gamma)$.

For C a connected component of $\Gamma \setminus A$, $\hat{C} \to \Gamma$ is a one-to-one mapping, except for points laying above some $x \in A$. For such x, there are exactly $\mathrm{val}_{\overline{C}}(x)$ points in the preimage.

We deduce that

$$\#V(\hat{C}) = \#(V(\Gamma) \cap C) + \sum_{x \in A} \mathrm{val}_{\overline{C}}(x).$$

By summing over the connected components, we get that

$$\begin{aligned}\sum_C \#V(\hat{C}) &= \#(V(\Gamma) \setminus A) + \sum_{x \in A} \mathrm{val}(x) \\ &= \#V(\Gamma) + \sum_{x \in A} (\mathrm{val}(x) - 1)\end{aligned}$$

We get by Remark 2.13 that

$$\begin{aligned}\sum_C g(\hat{C}) &= \sum_C (\#E(\hat{C}) - \#V(\hat{C}) + 1) \\ &= \#E(\Gamma) - \#V(\Gamma) - \sum_{x \in A} (\mathrm{val}(x) - 1) + N \\ &= g(\Gamma) - 1 - \sum_{x \in A} (\mathrm{val}(x) - 1) + N\end{aligned}$$

whence the result follows directly from the fact that $g(C \setminus A) = \sum_C g(\hat{C})$. ☐

Definition 2.19. Let $v \in \Gamma$ a vertex, then for $U = \Gamma \setminus \{v\}$, we have that as a set $\widehat{U} = U \sqcup \{v_1, \ldots, v_n\}$, where $n = \mathrm{val}(v)$. The set $A = \{v_1, \ldots, v_n\}$ corresponds naturally to the sets of tangent vectors $T_v\Gamma$. Let $\zeta \in T_v\Gamma$ be a tangent and let and let $S \subseteq A$, be the subset of points corresponding to the tangents other than ζ. Then we have that the quotient map $\widehat{U} \to \Gamma$ factors as

$$\widehat{U} \to \widehat{U}/S \to \Gamma.$$

We say $\widehat{U}/S$ is the *cut of Γ obtained by cutting along ζ*.

2.2 Divisors and complete linear systems

Definition 2.20. A *divisor* on Γ is an element of the free abelian group generated by the points of Γ, which we denote by $\mathrm{Div}(\Gamma)$. An element of this group is written as

$$D = \sum_{x \in \Gamma} D(x) \cdot x,$$

where $D(x) = 0$ for all but finitely many x. Examples of two divisors are depicted in Fig. 5.

We define the *support* of D to be

$$\mathrm{supp}(D) := \{x \in \Gamma : D(x) \neq 0\}.$$

We say D is *effective*, denoted by $D \geq 0$, when $D(x) \geq 0$ for all $x \in \Gamma$. The *degree* of D is the sum of its coefficients, that is

$$\deg(D) = \sum_{x \in \Gamma} D(x).$$

When Z is a subgraph of Γ, we call

$$D|_Z := \sum_{x \in Z} D(x) \cdot x$$

the *restriction* of D to Z.

Definition 2.21. The *canonical divisor* of Γ is the divisor defined by

$$K = \sum_{x \in \Gamma} (\mathrm{val}(x) - 2) \cdot x$$

Remark 2.22. The canonical divisor $K \in \mathrm{Div}(\Gamma)$ has $\deg K = 2g - 2$.

(a) Divisor of degree 2

(b) Canonical divisor

Figure 5: Two divisors on the same metric graph. The points in the support of the divisors are represented using circles and labeled with their multiplicity.

Definition 2.23. A piece-wise linear (PL) function is a continuous function $f : \Gamma \to \mathbb{R}$ for which there exists a model $G = (V, E)$ such that f is linear when restricted to the edges $e \in E$. We denote the set of piece-wise linear functions on Γ by $\mathrm{PL}(\Gamma)$. A *rational* function is a PL function with integral slopes, and we denote the set of rational functions on Γ by $\mathrm{Rat}(\Gamma)$.

Since Γ is compact, the image of a PL function is compact, and so this lets us define a norm on $\mathrm{PL}(\Gamma)$ by

$$\|f\|_\infty = \max f - \min f.$$

Let f a rational function f on Γ. For any $\zeta \in T_x\Gamma$, represented by an isometric path $\gamma : [0, \epsilon) \to \Gamma$, we define the slope of f along ζ by

$$s_\zeta(f) := \lim_{t \to 0} \frac{f(\gamma(t)) - f(x)}{t}.$$

Since f is piece-wise linear, $f \circ \gamma|_{[0,\delta]}$ is linear for some $\delta > 0$ small enough, so this is well-defined, and clearly this does not depend on the choice of γ.

The *order* of f at x, denoted by $\mathrm{ord}_x(f)$ is the sum of the outgoing slopes of f along each edge emanating from x. In other words,

$$\mathrm{ord}_x(f) = \sum_{\zeta \in T_x\Gamma} s_\zeta(f).$$

The *principal divisor* associated to f is the divisor given by

$$\mathrm{div}(f) := \sum_{x \in \Gamma} \mathrm{ord}_x(f) \cdot x.$$

Note that this is well defined, as for any model (V, E) such that f is linear when restricted to the edges, we have that $\mathrm{ord}_x(f) = 0$ for all $x \in \Gamma \setminus V$ and V is finite.

We define the *bend locus* of f, denoted by $\mathrm{bend}(f)$, to be the support of the associated divisor $\mathrm{div}(f)$.

Proposition 2.24. *For any rational function f on a compact metric graph Γ, we have that* $\deg(\mathrm{div}(f)) = 0$.

Proof. Let $G = (V, E)$ a model of Γ containing $\mathrm{bend}(f)$ in its set of vertices. For e an edge and x one of its vertices, there is a unique tangent vector $\zeta \in T_x\Gamma$ that comes from some $\gamma : [0, \epsilon) \to \Gamma$ whose image lies inside the closure of e (in the future we will simply say that ζ is the tangent of x along e, and $s_\zeta(f)$ is the outgoing slope of f at x along e). Denote $\zeta_{e,1}, \zeta_{e,2}$ the two tangent vectors corresponding to the vertices of e. Since f is linear along each edge, it follows that $s_{\zeta_{e,1}}(f) = -s_{\zeta_{e,2}}(f)$. We the obtain the desired result, as

$$\begin{aligned}\deg(\mathrm{div}(f)) &= \sum_{x \in V} \sum_{\zeta \in T_x\Gamma} s_\zeta(f) \\ &= \sum_{e \in E} (s_{\zeta_{e,1}}(f) + s_{\zeta_{e,2}}(f)) = 0\end{aligned}$$

□

Definition 2.25. A divisor $D \in \mathrm{Div}(\Gamma)$ is called *principal* when there exists a rational function f with $D = \mathrm{div}(f)$.

Two divisors $D, D' \in \mathrm{Div}(\Gamma)$ are said to be linearly equivalent, denoted by $D \sim D'$, when $D - D'$ is principal.

Remark 2.26. By Proposition 2.24, it follows that that any two linearly equivalent divisors have the same degree.

Remark 2.27. The two divisors depicted in Figure 5 are linearly equivalent.

Definition 2.28. Let D be an effective divisor, we define $R(D)$ to be the set of rational functions f such that $D + \mathrm{div}(f) \geq 0$.

Definition 2.29. For D an effective divisor, we define the *complete linear system* associated to D to be the set

$$|D| := \{D' \geq 0 : D' \sim D\}.$$

Remark 2.30. Let $R(D)/\mathbb{R}$ be the quotient of $R(D)$ modulo tropical scaling, that is, we take the quotient by the equivalence relation defined by $f \sim g$ if and only if $f = c + g$ for some $c \in \mathbb{R}$. Then we have a bijective correspondence

$$\begin{aligned} R(D)/\mathbb{R} &\to |D| \\ f &\mapsto D + \operatorname{div}(f) \end{aligned}$$

The set $\operatorname{Div}_d^+(\Gamma)$ of divisors of degree d on Γ may be naturally identified with the symmetric product Γ^d/S_d. The latter is a topological space, and so we may give $\operatorname{Div}_d^+(\Gamma)$ the structure of a topological space induced by this identification. Since $|D|$ is a subset of $\operatorname{Div}_d^+(\Gamma)$, we may equip it with the subspace topology. As seen below, $|D|$ naturally admits the structure of a metric space.

Proposition 2.31. *The norm $\|\cdot\|_\infty$ on $PL(\Gamma)$ induces a metric on $|D|$, which we will also denote by d_∞. Furthermore, the topology induced by this metric agrees with that induced by the inclusion $|D| \hookrightarrow \Gamma^d/S_d$.*

Proof. For any divisors $D_1, D_2 \in |D|$, there is some $f \in \operatorname{Rat}(\Gamma)$ such that $D_1 = D_2 + \operatorname{div}(f)$. We define the metric d_∞ by

$$d_\infty(D_1, D_2) = \|f\|_\infty.$$

We verify that this is well-defined. Suppose $\operatorname{div}(f) = \operatorname{div}(g)$. This implies in particular that $\operatorname{div}(f - g) = 0$ and so $f - g$ is constant. Indeed, if $f - g$ was not constant, then if Z was the subgraph of Γ on which $f - g$ admits its minimum, there would be some $x \in \partial Z$, and for such an x we would necessarily have $\operatorname{div}(f - g)(x) > 0$. Now, it follows directly from the definition of $\|\cdot\|_\infty$ that for any $c \in \mathbb{R}$, $\|f + c\|_\infty = \|f\|_\infty$, and hence taking $c = g - f$, we get that $\|f\|_\infty = \|g\|_\infty$.

The fact that the topology induced by this metric agrees with that induced from Γ^d/S_d is detailed in [Luo18, Proposition B.1]. □

Definition 2.32. The *rank* of a divisor $D \in \operatorname{Div}(\Gamma)$ is the number

$$r(D) := \max\{d \in \mathbb{N} \mid |D - E| \neq \emptyset \text{ for all effective divisor } E \text{ of degree } d\},$$

where if $|D| = \emptyset$ we set $r(D) = -1$.

Theorem 2.33 (Tropical Riemann-Roch)**.** *[BN07, Theorem 1.12] Let $D \in \operatorname{Div}(\Gamma)$ be a divisor, then*

$$r(D) - r(K - D) = \deg(D) - g + 1$$

Definition 2.34. For a closed, not necessarily connected subgraph $Z \subseteq \Gamma$, we define

$$Z_\epsilon := \{x \in \Gamma : \operatorname{dist}(x, Z) < \epsilon\}.$$

A *chip firing move* is the data of a closed subgraph $Z \subseteq \Gamma$ with a finite number of connected components, and a distance $\epsilon > 0$ such that $Z_\epsilon \setminus Z$ is a disjoint union of open intervals.

To such data we can associate the rational function

$$CF(Z, \epsilon)(x) := -\min\{\operatorname{dist}(x, Z), \epsilon\}.$$

This function is identically 0 on Z, it is identically $-\epsilon$ on Z_ϵ^c and it interpolates linearly between these two regions on $Z_\epsilon \setminus Z$.

For $D \in \operatorname{Div}(\Gamma)$ a divisor, we say that we obtain a divisor D' by *firing* Z (by a distance ϵ) when $D' = D + \operatorname{div}(CF(Z, \epsilon))$. We say that $Z \subseteq \Gamma$ *can fire* if there is some $\epsilon > 0$, such that for all $x \in \partial Z$, $D(x) + \operatorname{ord}_x CF(Z, \epsilon) \geq 0$.

Remark 2.35. When $Z \subset \Gamma$ is a closed subgraph and $x \in \partial Z$, we define $\deg_Z^{\mathrm{out}}(x)$ to be the valence of x in the closed subgraph $\Gamma \setminus Z^\circ$. It follows that for $x \in \partial Z$, $\mathrm{ord}_x CF(Z, \epsilon) = -\deg_Z^{\mathrm{out}}(x)$ and so $Z \subseteq \Gamma$ can fire if and only if for all $x \in \partial Z$, $D(x) \geq \deg_Z^{\mathrm{out}}(x)$.

Remark 2.36. It follows directly from the definition that $\|CF(Z, \epsilon)\|_\infty = \epsilon$.

Definition 2.37. A *weighted chip firing move* is a non-constant rational function f, for which there exist two disjoint proper closed subgraphs Z_1 and Z_2, such that $\Gamma \setminus (Z_1 \cup Z_2)$ consists only of open segments such that f is constant on Z_1 and Z_2 and linear on each component of $\Gamma \setminus (Z_1 \cup Z_2)$.

Lemma 2.38. *Every weighted chip firing move f can be written as a sum of chip firing moves (up to a constant)*

$$f = f_1 + \cdots + f_n,$$

where $\|f_i\|_\infty \leq \|f\|_\infty$. Furthermore, if $f \in R(D)$ for some effective divisor D, then $f_k \in R(D + \mathrm{div}(f_1 + \cdots + f_{k-1}))$ for all k, that is $f_1, \ldots, f_n$ is a sequence of legal chip firing moves.

Proof. We will proceed as in [HMY09, Lemma 1]. Let Z_1, Z_2 be as in the definition of weighted chip firing move and let $d = f(Z_2) - f(Z_1)$. Without loss of generality, suppose that $d > 0$. Denote $L_1, \ldots, L_r$ the open segments making up $\Gamma \setminus (Z_1 \cup Z_2)$. Let l_i be the length of L_i. Let also $s_i > 0$ be the slope of f along L_i from Z_1 to Z_2 (so that $s_i = d/l_i$), and let $s = \mathrm{lcm}(s_1, \ldots, s_r)$.

Let $k_i = s/s_i$ and $\delta = d/s$. For $j = 0, \ldots, s-1$, we let Y_j be the subgraphs obtained by attaching the adjacent subsegment of L_i of length $\lfloor j/s_i \rfloor \delta$ to Z_2. We then define $f_j := CF(Y_j, \delta)$. Let $g = f_1 + \cdots + f_{s-1}$. Clearly g is constant on Z_1 and Z_2 and its slope along any given L_i is by definition equal to s_i, hence $f - g$ is a constant.

Suppose $f \in R(D)$ for some effective divisor D. To show that $f_1, \ldots, f_{s-1}$ is a sequence of legal chip firing moves, we will focus on a single L_i. Identify $L_i = (0, l_i)$, where we orient L_i from Z_2 to Z_1. We have that $g_k := f_1 + \cdots + f_k$ is always concave on L_i. Indeed, let ζ_x be the tangent vector at $x \in L_i$, then by the definition of the f_j, we have that

$$s_{\zeta_x}(f_1 + \cdots + f_k) = \begin{cases} -s_i & \text{if } x \in (0, \lfloor k/s_i \rfloor \delta), \\ -s_i\{k/s_i\} & \text{if } x \in [\lfloor j/s_i \rfloor \delta, (\lfloor k/s_i \rfloor + 1)\delta, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\{a\} := a - \lfloor a \rfloor$ denotes the fractional part of a. So g_k is concave on L_i, whence

$$\mathrm{ord}_x g_k \geq 0 \geq -D(x)$$

for any $x \in L_i$. If x is the point corresponding to $0 \in \overline{L_i}$, then the slope of g_k at x along L_i is at least $-s_i$, so we deduce that $\mathrm{ord}_x g_k \geq \mathrm{ord}_x f \geq -D(x)$. Finally, if x is the point corresponding to $l_i \in \overline{L_i}$, then the slope of g_k at x along L_i (in the opposite direction) is non-negative, and hence $\mathrm{ord}_x g_k \geq 0 \geq -D(x)$. We deduce that $g_k \in R(D)$, or equivalently that

$$f_k \in R(D + \deg(f_1 + \cdots + f_{k-1})).$$

□

Lemma 2.39. *Every tropical rational function is a sum of chip firing moves (up to a constant). If we denote the sum by $f = f_1 + \cdots + f_n$, where the f_i are the chip firing moves, then the f_i can be chosen such that $\|f_i\|_\infty \leq \|f\|_\infty$ and furthermore if $f \in R(D)$ for some divisor D, then $f_k \in R(D + \mathrm{div}(f_1 + \cdots + f_{k-1}))$ for all k, that is $f_1, \ldots, f_n$ is a sequence of legal chip firing moves.*

Proof. We will proceed as in [HMY09, Lemma 2]. Let $Y = f(\mathrm{bend}(f) \cup V)$, where V is a set of vertices for any chosen model of Γ. Then Y is finite so denote $y_1 > \cdots > y_r$ its elements. By construction we have that

$$g_i := \max\{\min\{f, y_i\}, y_{i+1}\}$$

is a weighted chip-firing move. Note that

$$\|g_i\|_\infty = y_{i+1} - y_i \leq y_r - y_1 = \|f\|_\infty,$$

and that for all k,

$$g_1 + \cdots + g_k = \max\{f, y_{k+1}\} + c_k,$$

where c_k is some constant. We show this by induction. We have that $g_1 = \max\{f, y_2\}$, so we can set $c_1 = 0$ and

$$\max\{f, y_k\} + c_{k-1} + g_k = \max\{f, y_k\} + \max\{\min\{f, y_k\}, y_{k+1}\} + c_{k-1}.$$

Evaluated in $x \in \Gamma$, we obtain

$$\begin{cases} f(x) + y_k + c_{k-1} & \text{if } f(x) \geq y_k, \\ y_k + f(x) + c_{k-1} & \text{if } y_{k+1} \leq f(x) \leq y_k, \\ y_k + y_{k+1} & \text{if } f(x) \leq y_{k+1}. \end{cases}$$

And hence if we set $c_k = c_{k-1} + y_k$, we get that

$$\max\{f, y_k\} + c_{k-1} + g_k = \max\{f, y_{k+1}\} + c_k.$$

By Lemma 2.38, we have that $g_k = f_1^{(k)} + \cdots + f_{n_k}^{(k)}$ with $\|f_i^{(k)}\|_\infty \leq \|g_k\|_\infty$. So let D an effective divisor such that $f \in R(D)$, then since $g_1 + \cdots + g_k = \max\{f, y_{k+1}\} + c_k$, we have that $g_1 + \cdots + g_k \in R(D)$. This implies that $g_k \in R(D + \mathrm{div}(g_1 + \cdots + g_{k-1}))$ and hence by Lemma 2.38 we have that

$$f_l^{(k)} \in R(D + \mathrm{div}(g_1 + \cdots + g_{k-1} + f_1^{(k)} + \cdots + f_{l-1}^{(k)}))$$

for all l.

Now it is clear that the desired properties hold if we set

$$(f_1, \ldots, f_n) = (f_1^{(1)}, \ldots, f_{n_1}^{(1)}, \ldots, f_1^{(r-1)}, \ldots, f_{n_{r-1}}^{(r-1)}).$$

□

2.3 Reduced divisors

We now describe reduced divisors, which are distinguished divisors in a linear system. The purpose of this subsection is to introduce some theoretical background that will be useful in section 4 and is used in the implementation.

Definition 2.40. Let $v \in \Gamma$ be a point. We say a divisor $D \in \mathrm{Div}(\Gamma)$ is *effective away from* v if $D(x) \geq 0$ for all $x \neq v$.

A v-*reduced* divisor is a divisor $D \in \mathrm{Div}(\Gamma)$ that is effective away from v and such that for all subgraphs $Z \subset V$ with $v \notin Z$, Z cannot fire with respect to D.

Proposition 2.41. *[Ami12, Theorem 2] Let $D \in \mathrm{Div}(\Gamma)$. There exists a unique v-reduced divisor linearly equivalent to D.*

Definition 2.42. Let f a piece-wise linear function on Γ. We say a closed connected subset $C \subseteq \Gamma$ is a *local maximum* of f if f is constant on C and there exists some open neighbourhood U of C, with $f(U \setminus C) < f(C)$ (in the sense that for all $x \in U \setminus C$, $y \in C$, we have $f(x) < f(y)$).

Proposition 2.43. *Let $D \in \mathrm{Div}(\Gamma)$ be a v-reduced divisor. Then for all $f \in R(D)$, if $Z \subseteq \Gamma$ is a local maximum for f, then $v \in Z$.*

Proof. This will follow if we show that Z can fire with respect to D. Since Z is a local maximum, f has strictly negative integral slope along all outgoing tangents on ∂Z. It follows that for any $x \in \partial Z$,

$$\mathrm{ord}_x(f) \leq -\deg_Z^{\mathrm{out}}(x),$$

and so since $(D + \mathrm{div}(f))(x) \geq 0$, this implies $D(x) - \deg_Z^{\mathrm{out}}(x) \geq 0$ and so Z can fire. □

Corollary 2.43.1. *Let $D \in \mathrm{Div}(\Gamma)$ be a v-reduced divisor and $f \in R(D)$ a rational function. Then f admits its maximum in v.*

Proof. Let $Z \subseteq \Gamma$ be the set on which f admits its maximum, then Z is a local maximum of f and so by Proposition 2.43, Z contains v. □

Corollary 2.43.2. *Let $D \in \mathrm{Div}(\Gamma)$ be a v-reduced divisor and $f \in R(D)$ a rational function. For all $a \leq f(v)$, the subgraph $f^{-1}([a, \infty))$ is connected.*

Proof. Suppose there was some a with $Y = f^{-1}([a, \infty))$ disconnected. Then let C be a connected component of Y such that $v \notin C$, then if we denote $Z \subseteq \Gamma$ the set on which $f|_C$ admits its maximum, we have that Z is a local maximum of f and hence $v \in Z$, a contradiction. □

Proposition 2.44. *Suppose $D \in \mathrm{Div}(\Gamma)$ is a v-reduced divisor. Then D is linearly equivalent to an effective divisor if and only if D is effective.*

Proof. If D is effective then the statement is clear. Suppose $D + \mathrm{div}(f) \geq 0$ for some f. Then f admits its maximum in v by Corollary 2.43.1. In particular, the outgoing slopes of f at v are all negative (or zero). This implies that $\mathrm{ord}_v(f) \leq 0$ and hence

$$D(v) \geq D(v) + \mathrm{ord}_v(f) = (D + \mathrm{div}(f))(v) \geq 0,$$

so D is effective. □

2.4 Tropical modules

We will now discuss tropical modules, which are a natural structure that appear in the context of linear systems. We build on the discussion in [HMY09, Section 3].

Definition 2.45. The *tropical semifield* $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is the set of real numbers $\mathbb{R} \cup \{-\infty\}$ with infinity with the two tropical operations defined by

$$a \oplus b = \max(a, b),$$
$$a \odot b = a + b.$$

A *tropical module* $(M, \oplus, \odot, -\infty)$ is a semi-module over the tropical semi-ring.

For any set E, the space $\mathbb{R}^E \cup \{-\infty\}$ is naturally a tropical module. Clearly, $\mathrm{PL}(\Gamma)$ and $\mathrm{Rat}(\Gamma)$ are stable under tropical addition and scaling, so we have an inclusion of tropical modules $\mathbb{R}^\Gamma \cup \{-\infty\} \supset \mathrm{PL}(\Gamma) \supset \mathrm{Rat}(\Gamma)$, where by abuse of notation we implicitly consider $-\infty$ to be part of these tropical modules.

Proposition 2.46. *Let D be an effective divisor. Then the space $R(D)$ with the point-wise tropical operations is a tropical module.*

Proof. For any $c \in \mathbb{R}$ and $f \in R(D)$, we have that $\operatorname{div}(c \odot f) = \operatorname{div}(f)$ as adding a constant does not change the slopes of f, so clearly $R(D)$ is stable under tropical scaling.

Let $f, g \in R(D)$ and $x \in \Gamma$. If $f(x) \neq g(x)$, w.l.o.g. $f(x) > g(x)$, then for all tangent vectors $\zeta \in T_x\Gamma$, $s_\zeta(f \oplus g) = s_\zeta(f)$. It follows that $\operatorname{ord}_x(f \oplus g) = \operatorname{ord}_x(f)$ and so

$$(D + \operatorname{div}(f \oplus g))(x) = (D + \operatorname{div}(f))(x) \geq 0$$

If instead $f(x) = g(x)$, then for all tangent vectors $\zeta \in T_x\Gamma$,

$$s_\zeta(f \oplus g) = \max(s_\zeta(f), s_\zeta(g)) \geq s_\zeta(f),$$

and so in particular $\operatorname{ord}_x(f \oplus g) \geq \operatorname{ord}_x(f)$. We deduce that

$$(D + \operatorname{div}(f \oplus g))(x) \geq (D + \operatorname{div}(f))(x) \geq 0,$$

and so we conclude that $f \oplus g \in R(D)$.

Hence $R(D)$ is stable under tropical addition and scaling, so it is a tropical submodule of $\operatorname{Rat}(\Gamma)$. $\square$

As seen before, we have that $|D| \cong R(D)/\mathbb{R}$ and this gives $|D|$ additional structure we can work with.

Definition 2.47. Let M a tropical module, we define the *tropical projectivization* of M to be

$$\mathbb{T}(M) = (M \setminus \{-\infty\})/\mathbb{R},$$

which is the quotient of $M \setminus \{-\infty\}$ modulo tropical scaling. We call such a space a *tropical projective space.*

Definition 2.48. Let $X = \mathbb{T}(M)$ be a tropical projective space. We say a subset $Y \subset X$ is *tropically convex*, if $Y = \mathbb{T}(N)$ where N is a tropical submodule of M.

Remark 2.49. When $D \sim D'$, then $D' = D + \operatorname{div}(f)$, so the tropical modules $R(D)$ and $R(D')$ are isomorphic via the mapping

$$\begin{aligned} \phi : R(D) &\to R(D') \\ g &\mapsto g - f. \end{aligned}$$

This is clearly a morphism as for $g, h \in R(D)$,

$$\begin{aligned} \phi(g \oplus h) &= \phi(\max(g, h)) \\ &= \max(g, h) - f \\ &= \max(g - f, h - f) \\ &= \phi(g) \oplus \phi(h) \end{aligned}$$

and for $c \in \mathbb{R} \cup \{-\infty\}$,

$$\phi(c \odot g) = c + g - f = c \odot \phi(g).$$

We deduce that the tropical projective space structure on $|D|$ does not depend on the chosen divisor D.

We will now define a binary operator useful for studying tropical modules. This operator has been used in the implementation to check whether a rational function belongs to the submodule spanned by a chosen set of generators.

Definition 2.50. Let $M \subseteq \mathbb{R}^E \cup \{-\infty\}$ be a tropical submodule. We define the binary operator $\langle \cdot, \cdot \rangle$ by

$$\langle f, g \rangle = \inf_{x \in E} \{f(x) - g(x)\}$$

for all $f, g \in M$, and $g \neq -\infty$.

For $f, g \in M$ we say $\langle f, g \rangle \odot g$ is the *projection* of g on f.

Remark 2.51. When M is a submodule of $\mathrm{PL}(\Gamma) \subseteq \mathbb{R}^\Gamma \cup \{-\infty\}$, or when E is finite, the infimum is attained.

Remark 2.52. We have that

$$(\langle f, g \rangle \odot g)(x) = \inf_{y \in E} \{f(y) - g(y)\} + g(x) \leq f(x)$$

for all $x \in E$, and so $\langle f, g \rangle \odot g \leq f$.

Proposition 2.53. *Let $M \subseteq \mathbb{R}^E \cup \{-\infty\}$ be a finitely generated submodule and $G \subseteq M$ a finite generating set. Then for all $f \in M$,*

$$f = \bigoplus_{g \in G} \langle f, g \rangle \odot g.$$

Proof. By Remark 2.52, we already know that

$$f \geq \bigoplus_{g \in G} \langle f, g \rangle \odot g.$$

Since G is a generating set for M, there exist some $a_g \in \mathbb{R} \cup \{-\infty\}$ for all $g \in G$ such that

$$f = \bigoplus_{g \in G} a_g \odot g.$$

Now, for all g, we have that $f \geq a_g \odot g$. It follows that $f - g \geq a_g$ and so $\langle f, g \rangle \geq a_g$. We deduce that

$$\bigoplus_{g \in G} \langle f, g \rangle \odot g \geq \bigoplus_{g \in G} a_g \odot g = f,$$

which shows the other inequality. □

Corollary 2.53.1. *Let $M \subseteq \mathbb{R}^E \cup \{-\infty\}$ a tropical submodule and G a finite subset. Then $f \in M$ belongs to the submodule spanned by G if and only if*

$$f = \bigoplus_{g \in G} \langle f, g \rangle \odot g.$$

These properties are quite useful for studying tropical modules when we know their set of generators. Unfortunately, it is in general not easy to find a set of generators of a tropical submodule. Luckily, when M is a finitely generated submodule of $\mathbb{R}^E \cup \{-\infty\}$, we can give an explicit characterization of the elements belonging to a minimal generating set and in some cases it is even possible to find these exhaustively.

Definition 2.54. An element $f \in M$ is called *extremal* if for any $g, h \in M$ such that $f = g \oplus h$, it holds that either $f = g$ or $f = h$.

Remark 2.55. An element $f \in M$ is extremal if and only if for all $c \in \mathbb{R}$, $c \odot f$ is extremal as well.

Proposition 2.56. *[HMY09, Proposition 8] Let $M \subseteq \mathbb{R}^E \cup \{-\infty\}$ a finitely generated tropical module. The set of extremals of M is finite (up to tropical scalar multiplication), and is a minimal generating set of of M.*

Proposition 2.57. *[HMY09, Theorem 6] The tropical semi-module $R(D)$ finitely generated.*

The description of the generating set of a tropical module in terms of its extremals is especially useful when working with $R(D)$, and in this case we may even find a generating set explicitly.

Proposition 2.58. *[HMY09, Lemma 5] A rational function $f \in R(D)$ is an extremal of $R(D)$ if and only if there are no proper subgraphs Γ_1, Γ_2 covering Γ (in the sense that $\Gamma_1 \cup \Gamma_2 = \Gamma$), such that each can fire on $D + \operatorname{div}(f)$.*

Definition 2.59. We say a finite subset $A \subset \Gamma$ is a *cut set* if $\Gamma \setminus A$ is not connected and w e denote by $\mathcal{S}$ the set of rational functions $f \in R(D)$, such that $\operatorname{supp}(D + \operatorname{div}(f))_E$ is not a cut set.

Proposition 2.60. *[HMY09, Theorem 6(a)] The set $\mathcal{S}$ is finite (up to tropical scalar multiplication) and contains the set of extremals of $R(D)$.*

Remark 2.61. Since Proposition 2.58 gives us a way to check whether a function is extremal in a finite number of steps, this yields an algorithm that finds a minimal generating set of $R(D)$ in finite time.

Remark 2.62. One could also check that a function $f \in \mathcal{S}$ is extremal directly from the definition using Corollary 2.53.1, since $\mathcal{S}$ is a finite generating set. Indeed, f is extremal if and only if it is not in the span of $\mathcal{S} \setminus \{f\}$.

2.5 Abstract polyhedral complex structure

Let X be a Hausdorff topological space X. An *abstract polyhedral complex* structure on X is a finite set $\mathcal{K}$ of subspaces $\sigma \subseteq X$, with partial order given by inclusion, such that

- Each $\sigma \in \mathcal{K}$ is homeomorphic to a convex rational polyhedron $|\sigma|$.
- The homeomorphism $\sigma \cong |\sigma|$ induces an isomorphism between the posets

  $$\mathcal{K}_{\leq \sigma} := \{\tau \in \mathcal{K} \mid \tau \subseteq \sigma\}$$

  and the poset of faces of $|\sigma|$, denoted by F_σ.
- When $\tau \subseteq \sigma$ is a face corresponding to $\eta \in F_\sigma$, $|\tau|$ and η are isometric.
- For $\sigma, \tau \in \mathcal{K}$, we have that $\sigma \cap \tau \in \mathcal{K}$.

As described in [Ami12], a complete linear system $|D|$ admits a natural structure of an abstract polyhedral complex. We will briefly summarize how it is characterized, but we will avoid going into the technical details.

A model $G = (V, E)$ of Γ induces an abstract polyhedral complex structure on it. Indeed, if we start with V with the discrete topology, Γ is obtained by gluing for each edge $e \in E$ a closed interval of length $l(e)$ to the corresponding vertices. There is then a naturally induced abstract polyhedral complex structure on the product Γ^d. When G has no self-loops, we also get an induced abstract polyhedral complex structure on the d^{th} symmetric product Γ^d / S_d, where S_d is the symmetric group acting on Γ^d. Fix for each edge $e \in E$ a direction, making it a directed edge. This allows us to identify an edge e with the interval $[0, l(e)]$. The relative interior of a face σ of Γ^d / S_d is described by maps

- $m_V : V \to \mathbb{N}$,
- $m_E : E \to \bigcup_{k=0}^{\infty} \mathbb{N}_{>0}^k$,

A point $x = (x_1, \dots, x_d)$ belongs to $\operatorname{relint}(\sigma)$ if and only if the following conditions hold:

- For each vertex $v \in V$, the number of x_i such that $x_i = v$ is $m_V(v)$.
- For each edge $e \in E$ let $y_1, \dots, y_r$ be the points of e appearing in $(x_1, \dots, x_d)$, ordered according to the fixed orientation of e. Then the number of x_i such that $x_i = y_j$ is $m_E(e)_j$.

Fix an ordering $e_1, \dots, e_s$ of edges. Let k_i be the length of the sequence $m_E(e_i)$. Let $n = \sum_{e \in E} k_e$, then $\operatorname{relint}(\sigma)$ may be identified with the following subset of $\mathbb{R}^n$.

$$\begin{aligned}\operatorname{relint}(\sigma) \cong \{x \in \mathbb{R}^n \mid\ & 0 < x_1 < \cdots < x_{k_1} < l(e_1), \\ & 0 < x_{k_1+1} < \cdots < x_{k_1+k_2} < l(e_2), \dots, \\ & 0 < x_{k_1+\cdots+k_{s-1}+1} < \cdots < x_n < l(e_s)\}\end{aligned}$$

When D is a divisor of degree d, $|D|$ embeds into the abstract polyhedral complex complex Γ^d/S_d. Denote $\mathcal{K}$ the poset of faces (also called *cells*) of Γ^d/S_d. The intersection of $|D|$ with the relative interior of a face $\tau \in \mathcal{K}$ consist of a finite number of connected components. It turns out that the set $\mathcal{K}_D$ consisting of the closures of all such connected components gives $|D|$ the structure of an abstract polyhedral complex. A face $\tau \in \mathcal{K}_D$ is by definition contained in some face $\sigma \in \mathcal{K}$. The homeomorphism $\tau \cong |\tau|$ is then given by co-restriction of the embedding

$$\tau \hookrightarrow \sigma \cong |\sigma|.$$

Similarly as before, we may describe the relative interior of a face $\tau \in \mathcal{K}_D$ with a triple (m_V, m_E, s) given by maps

- $m_V : V \to \mathbb{N}$,
- $m_E : E \to \bigcup_{k=0}^{\infty} \mathbb{N}_{>0}^k$,
- $s : E \to \mathbb{Z}$.

A divisor $D' = D + \operatorname{div}(f)$ then belongs to $\operatorname{relint}(\tau)$ if and only if the following conditions are satisfied:

- For each vertex $v \in V$, $D'(v) = m_V(v)$.
- For each edge $e \in E$, $D'|_e = \sum_i m_E(e)_i x_i$, where $0 < x_1 < \cdots < x_k < l(e)$.
- For each edge $e \in E$ with corresponding tangent ζ based at the origin of the edge, $s_\zeta(f) = s(e)$.

Remark 2.63. Not all triples (m_V, m_E, s) will correspond to a non-empty face of $|D|$.

We will now define the notion of *definable* subset, following [JP22].

Definition 2.64. A subset of $\mathbb{R}^n$ is called *definable* if it can be written as a finite expression involving intersections, unions, and complements of closed half-spaces

$$H_i = \{u \in \mathbb{R}^n \mid \langle u, v_i \rangle \geq a_i\}.$$

A subset $X \subseteq |D|$ is *definable* if the image of $X \cap \sigma \hookrightarrow |\sigma|$ is definable for all $\sigma \in \mathcal{K}_D$.

The reason why this notion is nice is that it allows us to define a reasonable notion of dimension around a point for a large class of subspaces of $|D|$.

Proposition 2.65. *When $Y \subseteq \mathbb{R}^n$ is a closed and definable subset, then Y is a finite union of (possibly unbounded) polyhedra.*

Proof. By assumption, $Y \subseteq \mathbb{R}^n$ is written as a finite expression involving intersections, unions, and complements of closed half-spaces H_i. We can rewrite this expression to write it under the form

$$Y = \bigcup_{i=1}^{k} \bigcap_{j=1}^{l} H_{i,j},$$

where the $H_{i,j}$ are either closed, or open half-spaces. We call an expression of this form a *disjunctive normal form* (it is not unique). We may assume that for all i

$$\bigcap_{j=1}^{l} H_{i,j}$$

is non-empty, as otherwise we can just remove this term from the expression.

However as we assumed that Y is closed, we have that

$$Y = \overline{Y} = \bigcup_{i=1}^{k} \overline{\bigcap_{j=1}^{l} H_{i,j}} = \bigcup_{i=1}^{k} \bigcap_{j=1}^{l} \overline{H_{i,j}}.$$

To see the last equality, note that

$$\overline{\bigcap_{j=1}^{l} H_{i,j}} \subseteq \bigcap_{j=1}^{l} \overline{H_{i,j}}.$$

Take any $x \in \bigcap_{j=1}^{l} \overline{H_{i,j}}$ and $y \in \bigcap_{j=1}^{l} H_{i,j}$ (such an y exists as we assumed the intersection is non-empty). Then define

$$x_n = \frac{1}{n} y + \left(1 - \frac{1}{n}\right) x.$$

We will show that this sequence is contained in $\bigcap_{j=1}^{l} H_{i,j}$. Suppose for the sake of contradiction that there are some i, j, n such that $x_n \notin H_{i,j}$. If $H_{i,j}$ was a closed half-space, then $x \in H_{i,j}$, but this would imply that $[x, y] \subseteq H_{i,j}$ as $H_{i,j}$ is convex. So $H_{i,j}$ has to be an open half-space. Write

$$H_{i,j} = \{u \in \mathbb{R}^n \mid \langle u, v \rangle < a\},$$

then since $x \in \overline{H_{i,j}}$, it follows that $\langle x, v \rangle \leq a$. Furthermore $\langle y, v \rangle < a$. But by bilinearity we get that

$$\langle x_n, v \rangle = \frac{1}{n} \langle y, v \rangle + \left(1 - \frac{1}{n}\right) \langle x, v \rangle < a.$$

This shows that $x_n \in H_{i,j}$, a contradiction.

This shows that the sequence of x_n is contained in $\bigcap_{j=1}^{l} H_{i,j}$, and hence since x is the limit of this sequence, it lies in $\overline{\bigcap_{j=1}^{l} H_{i,j}}$. This shows the other inclusion.

As a result, we may assume that all the $H_{i,j}$ are actually closed half-spaces and so this concludes the proof. □

Corollary 2.65.1. *Let $Y \subseteq \mathbb{R}^n$ a closed and definable subset. Then Y admits the structure of a polyhedral complex.*

Proof. Since Y is a finite union of polyhedra, up to subdividing the polyhedra we may assume that the intersection of any two given polyhedra is a face of each respective polyhedron. It follows that Y admits a natural structure of polyhedral complex. $\square$

Corollary 2.65.2. *If $X \subseteq |D|$ is a closed and definable subset, then X admits the structure of an abstract polyhedral complex.*

Proof. For all $\sigma \in \mathcal{K}_D$, $X \cap \sigma$ admits the structure of a polyhedral complex $\mathcal{K}_\sigma$, and up to refining the polyhedral complex structure, we may assume that for any $\tau \subseteq \sigma$, $\mathcal{K}_\sigma$ restricted to faces contained in τ agrees with $\mathcal{K}_\tau$. The abstract polyhedral complex structure $\mathcal{K}_X$ on X is then given by the union of the $\mathcal{K}_\sigma$, where for $\tau \subseteq \sigma$ we identify $\mathcal{K}_\tau$ with the corresponding subset of $\mathcal{K}_\sigma$. $\square$

Definition 2.66. Let X be an abstract polyhedral complex, and $x \in X$. We define the dimension of X at x to be

$$\dim_x X := \max\{\dim(\sigma) \mid \sigma \in \mathcal{K}_X, x \in \sigma\}.$$

Here $\dim(\sigma)$ denotes the dimension of the smallest affine subspace containing $|\sigma|$.

Definition 2.67. Let X be an abstract polyhedral complex. We define the relative interior of X to be

$$\operatorname{relint}(X) := \bigcup_{\substack{\sigma \in \mathcal{K}_X \\ \sigma \text{ maximal}}} \operatorname{relint}(\sigma).$$

Remark 2.68. Let X be a closed definable subset of $|D|$, equipped with the induced abstract polyhedral complex structure. Let $D' \in X$ be a divisor, which belongs to the relative interior of a unique face σ of Γ^d/S_d and also of a unique face τ of X. Then $\tau \subseteq \sigma$ and in addition τ embeds as a polytope in σ, so we get naturally an embedding of manifolds $\operatorname{relint}(\tau) \hookrightarrow \operatorname{relint}(\sigma)$. If $D' \in \operatorname{relint}(X)$, then τ is a maximal face of X, and so τ has dimension $\dim_x X$. In particular, this implies that we may detect the dimension of X at D' from the dimension of the tangent space of $\operatorname{relint}(\tau)$ at D', which we may naturally identify as a subspace of the tangent space of $\operatorname{relint}(\sigma)$ at D'.

Recall that $\operatorname{relint}(\sigma)$ can be naturally identified as a subspace of $\mathbb{R}^n$ for some n, with coordinates corresponding to the positions of the chips of D' along the edge they are supported on. A tangent vector at D' may be therefore coordinatized by the rate at which it moves each of the chips of D' in a given direction along each edge. In other words, it corresponds to an infinitesimal transformation of D' and may be represented by a continuously differentiable path $[0, \epsilon) \to \operatorname{relint}(\sigma)$

The above discussion motivates a definition of tangent space of an arbitrary subspace of Γ^d/S_d at a given point.

Definition 2.69. Let $X \subseteq \Gamma^d/S_d$ be a subspace. We define the tangent space of X at x, denoted by $T_x X$ as the linear subspace of $T_x\Gamma^d/S_d$ generated by the tangent vectors that may be represented as a continuously differentiable path $[0, \epsilon) \to \Gamma^d/S_d$ whose image is contained in X.

Remark 2.70. When X is an abstract polyhedral complex and $x \in \operatorname{relint}(X)$, then

$$\dim_x X = \dim T_x X.$$

2.6 Structure of complete linear systems

In what follows we fix a model $G = (V, E)$ of Γ without self-loops. As discussed previously, a complete linear system $|D_0|$ inherits the structure of an abstract polyhedral complex which depends on this model.

Definition 2.71. Let $D \in |D_0|$ be a divisor. We say D is *generic* if $D \in \operatorname{relint}(|D_0|)$.

Remark 2.72. For any divisor $D \in |D_0|$, there is a unique cell Δ_D of $|D_0|$ such that D belongs to the relative interior of Δ_D. It follows that D is generic if and only if Δ_D is an inclusion-wise maximal cell.

Remark 2.73. The set of generic divisors of $|D_0|$ is dense in $|D_0|$.

For D a divisor, we will split it as a sum of the divisors $D_V := D|_V$ and $D_E := D|_{\Gamma \setminus V}$.

Proposition 2.74. *[HMY09, Prop. 13] Let $D \in |D_0|$ be a divisor, then*

$$\dim \Delta_D = \#\{\textit{connected components of } \Gamma \setminus \operatorname{supp} D_E\} - 1$$

Definition 2.75. Let A be a finite subset of Γ. We say a divisor $D \in \operatorname{Div}(\Gamma)$ is *A-unsaturated* whenever there is no closed subgraph $C \subseteq \Gamma$ with $\partial C \cap A \neq \emptyset$ that can fire.

Remark 2.76. We are mostly interested in the case where the set A is the set of vertices V of our model G.

Proposition 2.77. *Let D be a divisor. Then D is generic if and only if the following conditions are satisfied:*

1) For all $x \in \Gamma$, we have that $D(x) < \operatorname{val}(x)$.

2) D is V-unsaturated

Proof. Suppose D is generic, then it follows immediately from Proposition 2.74 that there is no subgraph C that can fire, that satisfies

$$\# \operatorname{supp}(\operatorname{div}(D + CF(Z, \epsilon))_E) > \# \operatorname{supp} D_E$$

for all $\epsilon > 0$ small enough.

If for some $x \in \Gamma$ we had that $D(x) \geq \operatorname{val}(x)$, then taking $C = \{x\}$, we would get that the subgraph C can fire, and by firing it we would get a divisor with more points supported on the edges. So condition 1) is satisfied.

If there was a closed subgraph $C \subseteq \Gamma$ with $\partial C \cap V \neq \emptyset$, then firing C would move at least one chip from a vertex to the interior of an edge. Again, this would yield a divisor with more points supported on the edges. Hence D is V-unsaturated and so condition 2) is satisfied as well.

Now, suppose D satisfies the two conditions. For the sake of contradiction, suppose D is not generic, then for all $\epsilon > 0$, there exists some $D' = D + \operatorname{div}(f) \in |D|$, such that $\|f\|_\infty \leq \epsilon$ and D' belongs to some higher dimensional cell. By Lemma 2.39, we may write $f = f_1 + \cdots + f_n$, where the $f_i = CF(Z_i, \epsilon_i)$ are chip firing moves with $\epsilon_i = \|f_i\|_\infty \leq \|f\|_\infty$. Furthermore, the Lemma also ensures that $f_1 \in R(D)$. By condition 1), Z_1 contains no isolated point, and since D is V-unsaturated, ∂Z_1 does not contain any vertices. We deduce that for ϵ small enough, $D + \operatorname{div}(f_1)$ has the same combinatorial type as D. In addition, f_1 is constant in a neighbourhood of each vertex, so we deduce that $D + \operatorname{div}(f_1)$ belongs to the relative interior of the same cell of $|D|$ as D. We can repeat the argument to get by induction that D' belongs to the relative interior of the same cell as D, a contradiction. □

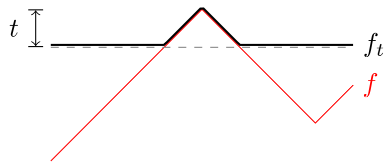

Figure 6: Example of f and $f_t = f \oplus (\sup(f) - t)$

Proposition 2.78. *For any generic divisor $D \in |D_0|$, we have that*

$$\dim \Delta_D \geq r(D).$$

Proof. We will show by induction on r that if $r(D) \geq r$, then $\dim \Delta_D \geq r$, which implies the result. The base case $r = 0$ is trivially verified, so suppose $r \geq 1$.

Since $r(D) \geq r \geq 1$, there exists some non-constant rational function $f \in R(D)$, then define

$$f_t = f \oplus (\sup(f) - t).$$

See Figure 6 for a depiction of f_t on an interval. The map $t \in [0, \epsilon) \mapsto D + \operatorname{div}(f_t)$ determines a tangent vector ζ in $|D|$.

Let $S := V \cup \operatorname{bend}(f)$, then $f(S)$ is a finite set and so for $\epsilon > 0$ small enough, the set

$$f(S) \cap [\sup(f) - \epsilon, \infty)$$

will consist only of a single point, which implies that f_ϵ will be a weighted chip-firing move (and so will be all the f_t for $0 < t < \epsilon$). Let Z be the subgraph on which f_ϵ attains its maximum, then Z can fire as

$$D(x) \geq -\operatorname{div}(f_\epsilon)(x) \geq \deg_Z^{\text{out}}(x).$$

Since D was assumed to be generic, by Proposition 2.77 we know that Z contains no isolated points (as otherwise for such an x we would get $\deg_Z^{\text{out}}(x) = \operatorname{val}(x)$ and so $D(x) \geq \operatorname{val}(x)$), and $\partial Z \cap V = \emptyset$, as D is V-unsaturated. In particular, $\partial Z \cap \operatorname{supp} D$ consists only of points on the interior of edges, and for such x, we know that $D(x) < \operatorname{val}(x) = 2$. We deduce that f_ϵ can only be an ordinary chip firing move $CF(Z, \epsilon)$. Now, take any $x \in \partial Z \cap \operatorname{supp} D$. Clearly,

$$r(D - x) \geq r(D) - 1 \geq r - 1.$$

Furthermore, $D - x$ is generic as it clearly satisfies the conditions of Proposition 2.77. By the induction hypothesis, we obtain that $\Delta_{D-x} \geq r - 1$ and so let $\zeta_1, \ldots, \zeta_{n-1}$ be linearly independent tangent vectors at $D - x$. Since Δ_{D-x} naturally embeds into Δ_D via the map $D' \mapsto D' + x$, we may see the ζ_i as vectors in the tangent space at D.

We claim that the vectors $\zeta, \zeta_1, \ldots, \zeta_{n-1}$ are linearly independent, which will imply that the tangent space at D has dimension at least n and so $\dim \Delta_D \geq n$. To see this, notice that the ζ_i correspond to infinitesimal transformations of D that fix the chip at x. This follows from the fact that $D(x) = 1$ and so $D - x$ has no chip at x. However, x was chosen in the boundary of Z, so ζ has a non-zero component along the coordinate corresponding to the chip at x, so ζ is independent from the ζ_i, and this completes the proof. □

Corollary 2.78.1. *For any divisor $D \in |D_0|$, we have that*

$$\dim_D |D_0| \geq r(D_0).$$

Proof. For any maximal cell $\sigma \in \mathcal{K}_{D_0}$, its relative interior is non-empty and so there exists some $D' \in \mathrm{relint}(\sigma)$. Note that this implies $\Delta_{D'} = \sigma$. In particular,

$$\dim \Delta_{D'} \geq r(D') = r(D_0)$$

Since the dimension of $|D_0|$ at D is at least the dimension of a maximal cell containing D, the statement follows. $\square$

(a) Rank 0 (b) Rank 1

Figure 7: Divisors whose complete linear system is locally of dimension strictly greater than the rank.

Remark 2.79. One may wonder whether the rank is fully encoded in the dimension of the complete linear system. In general this is not true, as it might happen that $|D|$ only has cells of dimension strictly higher than $r(D)$. The dumbbell graph provides simple counter-examples, as illustrated in the Figure 7. The complete linear system corresponding to the divisor in (a) is just an interval, so equi-dimensional of dimension 1. The complete linear system corresponding to the divisor in (b) has maximal cells of dimensions at least 2. Indeed, the maximal cells are of three sorts:

- All three chips on the bridge (dimension 3)
- Two chips on one circle and the third on the bridge (dimension 2)
- All three chips on the same circle (dimension 2)

2.7 Structure of the canonical linear system

We will now study the case of the canonical linear system. We recall that the canonical divisor is defined by

$$K = \sum_{x \in \Gamma} (\mathrm{val}(x) - 2) \cdot x.$$

The canonical linear system depends tightly on the cycles present in the metric graph as we will soon see. We will start with a few lemmas.

Lemma 2.80. *Let D be a V-unsaturated divisor. Then for any connected component C of $\Gamma \setminus \mathrm{supp}\, D_E$, we have that $g(C) \geq \deg D|_C$*

Proof. We proceed by induction on $\deg D|_C$. The fact is clearly true when $\deg D|_C = 0$, as $g(C)$ is always non-negative. So suppose the Lemma holds when $\deg D|_C < n$ for some fixed $n \in \mathbb{N}$, we will now show that it also holds when $\deg D|_C = n$.

We may see $D|_C$ as a divisor in $\hat{C}$. Fix some $x \in C \setminus \mathrm{supp}\, D|_C$ and let A be the connected component of x in $\hat{C} \setminus \mathrm{supp}\, D|_C$ and let B be the complement of A. We claim that there is some $y \in \mathrm{supp}\, D|_C$ such that $\deg_B^{\mathrm{out}}(y) > D(y)$. For the sake of contradiction, suppose that for all $y \in \mathrm{supp}\, D|_C$, $\deg_B^{\mathrm{out}}(y) \leq D(y)$, then we can fire B in $\hat{C}$. Let B' be the image of B in Γ, then $\partial B' \setminus C$ consists of points in $\mathrm{supp}\, D_E$. Let y be such a point, then $\mathrm{val}(y) = 2$, which implies that $\deg_{B'}^{\mathrm{out}}(y) = 1$ as y is on the boundary of B'. Furthermore, $D(y) \geq 1$ since $y \in \mathrm{supp}\, D_E$. We deduce that B' can fire in Γ. Since we assumed D is V-unsaturated, this implies $\partial B' \cap V = \emptyset$ and so $\partial B' \subseteq \operatorname{supp} D_E$. We deduce that B' is all of $\overline{C}$, but this would imply that B is all of $\hat{C}$ (since only a finite number of points are identified in the gluing $\hat{C} \to \overline{C}$ and B is closed), which is absurd as A is non-empty and open.

So let $y \in \operatorname{supp} D|_C$ such that $\deg_B^{\text{out}}(y) > D(y)$. Then choose $\zeta \in T_y\Gamma$ along some edge in A. Cut Γ along ζ to obtain a new metric graph Γ', and let C' be the set of points of Γ' lying above C. We claim that C' is connected. Indeed, since $y \in \operatorname{supp} D|_C$, it follows that $D(y) \geq 1$ and so $\deg_B^{\text{out}}(y) \geq 2$. In particular, this means there are at least two independent paths between x and y in A, which go along a different tangent at y. So after the cut, one of the two paths has to stay intact, which implies the connectedness of C'. In Γ' there are two points lying above y, say y_0, y_1 and suppose y_1 is the leaf that corresponds to the cut along ζ. Lift the model of Γ to Γ' by letting

$$V' = (V \setminus \{y\}) \cup \{y_0, y_1\}.$$

Define a divisor D' on Γ' by lifting D to Γ' on $\Gamma \setminus \{y\}$ and let $D'(y_0) = D(y) - 1$ and $D(y_1) = 0$. We claim that D' is a V'-unsaturated divisor of Γ'. Suppose there is a closed subgraph $Z' \subseteq \Gamma'$ with $\partial Z' \cap V' \neq \emptyset$ that can fire. Then for all $z \in \partial Z'$, we have that $\deg_{Z'}^{\text{out}}(z) \leq D'(z)$. Let Z be the image of Z' under the gluing $\Gamma' \to \Gamma$. It follows that for $z \in \partial Z \setminus \{y\}$,

$$\deg_Z^{\text{out}}(z) = \deg_{Z'}^{\text{out}}(z) \leq D'(z) = D(z).$$

If $y \in \partial Z$, then we have by construction that

$$\deg_Z^{\text{out}}(y) \leq \deg_{Z'}^{\text{out}}(y_0) + 1 \leq D'(y_0) + 1 = D(y).$$

We conclude that Z can fire and so $\partial Z \cap V = \emptyset$ since D is V-unsaturated. In particular, ∂Z is contained in the image of $\partial Z'$, so we deduce that $\partial Z' \cap V' = \emptyset$ by choice of model V'.

So we may apply the induction hypothesis to Γ', D', C' to get that $g(C') \geq \deg D'|_{C'}$. Now, notice that $\hat{C}'$ has the same number of edges as $\hat{C}$ but one extra vertex ($\hat{C}'$ is the cut of $\hat{C}$ along ζ), so by Remark 2.13, we have that $g(C') = g(C) - 1$. Furthermore, we have by construction of D' that $\deg D'|_{C'} = \deg D|_C - 1$, and so the conclusion follows. □

Lemma 2.81. *Suppose Γ is connected. Let D be a generic divisor. Let $C_1, \ldots, C_n$ be the connected components of $\Gamma \setminus \operatorname{supp} D_E$. Then*

$$\dim \Delta_D = \deg D - g(\Gamma) + \sum_{i=1}^{n} (g(C_i) - \deg D|_{C_i})$$

Proof. The points of $\operatorname{supp} D_E$ are all of valence 2, furthermore, since D is generic, by Proposition 2.77, $D(x) = 1$ for all $x \in \operatorname{supp} D_E$. So we deduce from Lemma 2.18 that

$$g(\Gamma) = g(\Gamma \setminus \operatorname{supp} D_E) + \deg D_E + 1 - N,$$

where N is the number of connected components of $\Gamma \setminus D_E$. By Proposition 2.74, we have that

$$\begin{aligned}\dim \Delta_D &= N - 1 \\ &= \deg D_E - g(\Gamma) + g(\Gamma \setminus \operatorname{supp} D_E) \\ &= \deg D - g(\Gamma) + g(\Gamma \setminus \operatorname{supp} D_E) - \deg D_V.\end{aligned}$$

Now the result follows from the fact that

$$g(\Gamma \setminus \operatorname{supp} D_E) = \sum_{i=1}^{n} g(C_i), \qquad \text{and} \qquad \deg D_V = \sum_{i=1}^{n} \deg D|_{C_i}.$$

□

By the Riemann-Roch theorem (Theorem 2.33) we have that

$$r(K) = \deg(K) - g(\Gamma) + 1.$$

The degree of K is $2g-2$, which may be computed using Remark 2.13, and hence the rank of K is $g-1$. By Proposition 2.78, we know that maximal cells have dimension at least $g-1$. As we will now see, there is a general class of graphs for which the canonical linear system always has cells of higher dimension.

Definition 2.82. We say two cycles are *disjoint* if they don't intersect.

Proposition 2.83. *If Γ has at least two disjoint cycles, then there is a cell σ of $|K|$ of dimension at least g.*

Proof. The union of the two cycles $Z = Z_1 \cup Z_2$ is a subgraph where each vertex v has $\mathrm{val}(v) - 2$ outgoing edges. Since $K(v) = \mathrm{val}(v) - 2$, we deduce Z can fire and firing Z will remove all the chips from it. So fire Z by a small amount $\epsilon > 0$ to obtain a divisor $F \in |K|$. Since generic divisors are dense in $|K|$, we can find a generic divisor D close enough to F, so that $\operatorname{supp} D \cap Z = \emptyset$. It follows that $\Gamma \setminus D_E$ has at least two connected components containing a cycle and no point of D_V (the connected components containing Z_1 and Z_2 respectively). By Proposition 2.77 and Lemma 2.80, it follows that

$$\sum_C (g(C) - \deg D|_C) \geq 2,$$

where the sum is over the connected components of $\Gamma \setminus \operatorname{supp} D_E$. Since $\deg D = \deg K = 2g-2$, the result follows from Lemma 2.81. ☐

Remark 2.84. The converse statement is not true, for example the bipartite graph on 6 vertices does not contain two disjoint cycles, however $|K|$ has a cell of dimension $5 \geq g(\Gamma) = 4$. This divisor is represented in Figure 8.

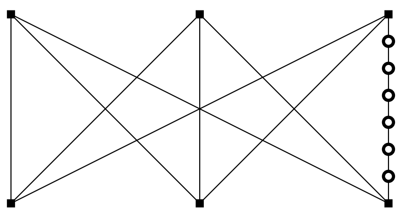

Figure 8: Divisor D on bipartite graph on six vertices with $\dim \Delta_D = 5$. The points in the support of D are all of multilpicity 1.

Unlike in the case of arbitrary complete linear systems, the lower bound for dimension is always attained in the case of canonical linear systems.

Proposition 2.85. *For any metric graph Γ, there always exists a maximal cell σ of $|K|$ of dimension $g-1$. Furthermore, for any $D \in \mathrm{relint}(\sigma)$, we have that $\operatorname{supp} D \cap V = \emptyset$.*

Proof. By the tropical Riemann-Roch theorem (Theorem 2.33), we get that $r(K) = g-1$. So choose $P_1, \ldots, P_{g-1}$ in $\Gamma \setminus V$, such that $\Gamma \setminus \{P_1, \ldots, P_{g-1}\}$ has genus 1. By the definition of rank, there exists some divisor $R \in |K - P_1 - \cdots - P_{g-1}|$, so let $S = R + P_1 + \cdots + P_{g-1}$.

Let $D \in |K|$ be a generic divisor sufficiently close enough to S, so that $B(P_i, \epsilon) \cap \operatorname{supp} D$ is non-empty for some small ϵ (we may always find such a divisor D, since generic divisors are dense in the complete linear system). For such a D we necessarily have $g(\Gamma \setminus \operatorname{supp} D_E) \leq 1$.

Now we may apply Lemma 2.18 to get

$$\begin{aligned}\#\{\text{conn. comp. of } \Gamma \setminus D_E\} &= 1 + \deg D_E - (g(\Gamma) - g(\Gamma \setminus D_E)) \\ &\leq 1 + \deg D - (g-1) = g.\end{aligned}$$

By Proposition 2.74, we get that $\dim \Delta_D \leq g - 1$. We also have that $\dim \Delta_D \geq g - 1$, by Proposition 2.78, so we must get $\dim \Delta_D = g - 1$. In particular, this forces $\deg D_E = \deg D$ and so $D_E = D$. $\square$

2.8 Tropical linear systems

We would now like to define the notion of linear systems on metric graphs. In algebraic geometry linear systems are projective subspaces of the complete linear system, so in analogy we will call tropical linear system the projectivizations of tropical submodules of $R(D_0)$.

Definition 2.86. Let $\mathfrak{d}$ be a tropically convex subset of the complete linear system $|D_0|$, we say $\mathfrak{d}$ is a *tropical linear system* (or tropical linear *series*).

For any $D \in |D_0|$, we will denote by $R(\mathfrak{d}, D)$ the cone over $\mathfrak{d}$ in $R(D)$. In other words,

$$R(\mathfrak{d}, D) := \{f \in R(D) \mid D + \operatorname{div}(f) \in \mathfrak{d}\}$$

is the tropical submodule of $R(D)$ whose tropical projectivization is $\mathfrak{d}$.

Remark 2.87. Note that $R(\mathfrak{d}, D)$ contains the constant functions if and only if $D \in \mathfrak{d}$.

Remark 2.88. Some authors choose to start with a tropical submodule $\Sigma \subseteq R(D_0)$ and denote the associated tropical linear system by

$$|\Sigma| := \mathbb{T}(\Sigma) \subseteq |D_0|.$$

This is the convention used in [JP22] and [FJP23]. Note that in these two papers the authors use the term "tropical linear series" for Σ rather than $|\Sigma|$.

Definition 2.89. Let E be an effective divisor, we define

$$\mathfrak{d}(-E) = \{D \in \mathfrak{d} \mid D - E \geq 0\}.$$

Note that $\mathfrak{d}(-E)$ is also a tropical linear series as for any choice of $D \in |D_0|$,

$$R(\mathfrak{d}(-E), D) = R(\mathfrak{d}, D) \cap R(D - E) \subseteq R(D),$$

is a tropical submodule.

Definition 2.90. We define the *rank* of $\mathfrak{d}$ to be the integer

$$r(\mathfrak{d}) = \max\{d \in \mathbb{N} \mid \mathfrak{d}(-E) \neq \emptyset \text{ for all effective divisor } E \text{ of degree } d\},$$

where if $\mathfrak{d}(-E) = \emptyset$, we set $r(\mathfrak{d}) = -1$.

Definition 2.91. We will say that a tropical linear system $\mathfrak{d} \subseteq |D_0|$ is finitely generated if $R(\mathfrak{d}, D_0)$ is finitely generated as a tropical module.

Proposition 2.92. *[JP22, Lemma 2.8] If $\mathfrak{d} \subseteq |D_0|$ is a finitely generated tropical linear series, then $\mathfrak{d}$ is a closed, definable subset of $|D_0|$.*

Corollary 2.92.1. *Any finitely generated tropical linear series $\mathfrak{d}$ admits the structure of an abstract polyhedral complex.*

Proof. This follows from Corollary 2.65.2. □

We would now like to generalize Proposition 2.78 to tropical linear systems. The problem we face is that the notion of *generic* divisor we defined in the previous section is not as tractable in this setting, as the abstract polyhedral complex structure is not so easy to characterize. However, we can show the result for a dense class of divisors, which will then imply the dimension bound for all maximal cells. Fix $\mathfrak{d} \subseteq |D_0|$ a complete linear system.

Definition 2.93. Let $D \in \mathfrak{d}$ be a divisor. Let $f \in R(\mathfrak{d}, D)$ be a non-constant function and let Z be the set on which f attains its maximum. Then we say that f *splits* D (at x) if $D(x) + \mathrm{ord}_x(f) > 0$ for some $x \in \partial Z$.

We will say D *does not split* (in $\mathfrak{d}$) if there is no $f \in R(\mathfrak{d}, D)$ that splits D.

Proposition 2.94. *The set of divisors that do not split in* $\mathfrak{d}$ *is dense in* $\mathfrak{d}$.

Proof. Let $D \in \mathfrak{d}$. We will show that for all $\epsilon > 0$ there exists a $D' \in \mathfrak{d}$ that does not split and $d_\infty(D, D') < \epsilon$. We proceed by induction on $n = \deg D - \#\operatorname{supp} D$. If $n = 0$, then D has multiplicity 1 at all points in its support, so there cannot be any function that splits D.

Now, suppose $n > 0$ and suppose there exists an $f \in R(\mathfrak{d}, D)$ that splits D, then let Z be the set on which f attains its maximum and let

$$f_\epsilon = f \oplus (\sup(f) - \epsilon).$$

Then f_ϵ is a weighted chip-firing move. For $\epsilon > 0$ small enough, $Z_\epsilon \cap \operatorname{supp} D = Z \cap \operatorname{supp} D$, and $Z_\epsilon^\circ \setminus Z$ consists only of open intervals, where we let

$$Z_\epsilon = \{x \in \Gamma \mid \mathrm{dist}(x, Z) \leq \epsilon\}.$$

Since f splits D, there exists some $x \in \partial Z$ such that $D(x) + \mathrm{ord}_x(f) > 0$, in particular $D(x) + \mathrm{ord}_x(f_{\epsilon/2}) > 0$. Let $D' = D + \mathrm{div}(f_{\epsilon/2})$. If D' does not split, we're done since $d_\infty(D, D') = \epsilon/2 < \epsilon$. Otherwise, since $\#\operatorname{supp} D' \cap Z_\epsilon > \#\operatorname{supp} D \cap Z_\epsilon$ and $D'|_{Z_\epsilon^c} = D|_{Z_\epsilon^c}$, we see that we can apply the induction hypothesis to get divisor D'' that does not split with $d_\infty(D', D'') < \epsilon/2$. The result the follows by the triangle inequality. □

Proposition 2.95. *If* $D \in \mathfrak{d}$ *does not split, then the dimension of the tangent space of* $\mathfrak{d}$ *at* D *is at least* $r(\mathfrak{d})$.

Proof. We will show by induction on r that if $r(\mathfrak{d}) \geq r$, then $\dim T_D\mathfrak{d} \geq r(\mathfrak{d})$, which will imply the result. When $r = 0$ there is nothing to show, so suppose $r > 0$.

Since $r(\mathfrak{d}) \geq 1$, there exists some non-constant $f \in R(\mathfrak{d}, D)$. Define

$$f_t = f \oplus (\sup(f) - t).$$

Then $t \in [0, \epsilon) \mapsto D + \mathrm{div}(f_t)$ determines a tangent vector in $\mathfrak{d}$.

Let Z be the set on which f attains its maximum and choose some $x \in \partial Z$. Clearly, $D(x) \geq 1$, so $D \in \mathfrak{d}(-x)$. We claim that $R(\mathfrak{d}(-x), D)$ contains no function g that attains its maximum at x and $\mathrm{ord}_x(g) < 0$. Indeed, if g was such a function and we denote by Y the set on which g attains its maximum, then $x \in Y$ and $D(x) + \mathrm{ord}_x(g) \geq 1$ by definition of $\mathfrak{d}(-x)$, which contradicts the fact that D does not split.

Now, it follows from the definition of rank that

$$r(\mathfrak{d}(-x)) \geq r(\mathfrak{d}) - 1 \geq r - 1,$$

and hence $\dim_D \mathfrak{d}(-x) \geq r-1$ by the induction hypothesis, so let $\zeta_1, \ldots, \zeta_{r-1}$ be independent tangents in the tangent space of $\mathfrak{d}(-x)$ at D. Since $\mathfrak{d}(-x) \subseteq \mathfrak{d}$, we may see ζ_i as tangents in the tangent space of $\mathfrak{d}$ at D. Since there is no function $g \in R(\mathfrak{d}(-x), D)$ such that $\mathrm{ord}_x(g) \neq 0$ attains its maximum at x, we deduce that the ζ_i all correspond to infinitesimal transformations of D that fix the chips at x. However, since x was chosen in the boundary of Z, ζ has a non-zero component among the coordinates corresponding to the chips of D at x, so ζ is independent from the ζ_i, which completes the proof. $\square$

Corollary 2.95.1. *If $\mathfrak{d}$ admits the structure of an abstract polyhedral complex, then all its maximal faces have dimension at least $r(\mathfrak{d})$.*

Proof. For any maximal face σ of $\mathfrak{d}$, there exists some $D \in \mathrm{relint}(\sigma)$ which does not split (since $\mathrm{relint}(\sigma)$ is open in $\mathfrak{d}$) and so

$$\dim(\mathfrak{d}) = \dim T_D(\mathfrak{d}) = \dim T_D\mathfrak{d} \geq r(\mathfrak{d}).$$

$\square$

Corollary 2.95.2. *If $\mathfrak{d}$ is finitely generated, then $\mathfrak{d}$ is an abstract polyhedral complex whose maximal faces are all of dimension at least $r(\mathfrak{d})$.*

3 The realizability problem

3.1 Tropicalization

We will now describe the tropicalization process, which attributes to a smooth projective curve a metric graph. We will first recall a few definitions on algebraic curves and models.

Definition 3.1. An algebraic curve C over k is called *pre-stable* if it is reduced and has only ordinary double points as singularities. It is called *stable* if in addition

1. C is connected and projective, of arithmetic genus $p_a(C) \geq 2$.
2. If Y is an irreducible component of C, which is isomorphic to $\mathbb{P}^1_k$, then Y meets the other components of C in at least three points.

If in the above we replace the requirement for three intersection points with only two intersection points, we obtain the definition of a *semi-stable* curve. The curve is called *totally degenerate* if all of its irreducible components are isomorphic to the projective line over k and all singularities of C are k-rational.

We may naturally associate to a pre-stable curve a weighted graph called its *dual graph*, the vertices of which correspond to the irreducible components and the edges to the nodes. The weights of the vertices are given by the genus of the respective components. Figure 9 depicts an example of a stable curve (in the center) and its corresponding dual graph (on the right).

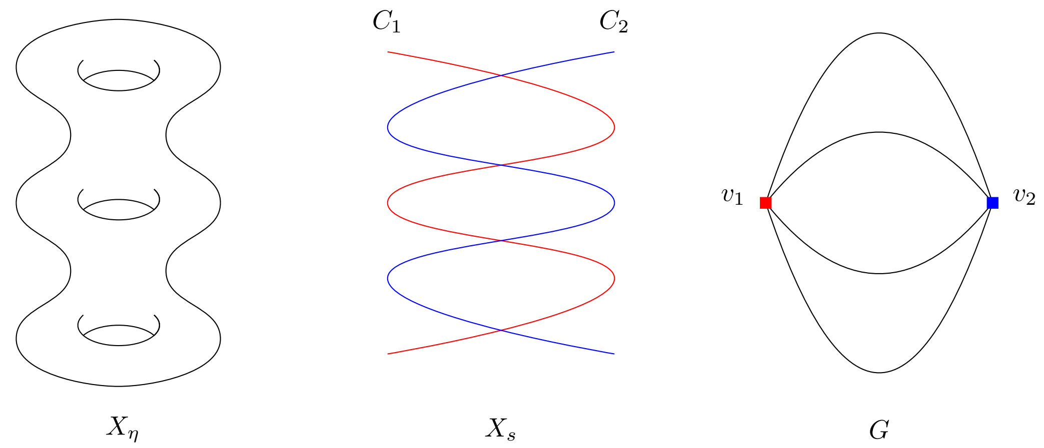

Figure 9: Example of a smooth curve of genus 3 (left) degenerating to a stable curve (center) whose dual graph is a genus 3 banana graph (right). The four edges of the dual graph correspond to the four intersection points between the two irreducible components of the stable curve.

We may give this graph further structure if we consider the pre-stable curve inside a fixed smoothing. Let K be a valued field with ring of integers $R = \mathcal{O}_K$, and corresponding maximal ideal $\mathfrak{m}_{\mathcal{O}_K} = \mathfrak{m}$. Denote $k = R/\mathfrak{m}$ the residue field. For simplicity, we will assume the residue field k is algebraically closed and that R is complete.

Definition 3.2. A *fibered surface* over $S = \operatorname{Spec} R$ is an integral, projective, flat scheme $\pi : X \to S$ of dimension 1 over S. Let η be the generic point of S and s its only closed point. The fiber X_η is called the *generic fiber* and X_s the *special fiber*.

We say X is a *regular (resp. normal) fibered surface*, whenever X is a regular (resp. normal) scheme. We also call a regular fibered surface an *arithmetic surface*. We will also say that X is (pre/semi-)stable and/or totally degenerate whenever these properties hold for the special fiber X_s.

When X is pre-stable, we may equip the dual graph of X_s with the structure of a metric graph.

Proposition 3.3. *[Liu02, Corollary 10.3.22] Let X be a pre-stable fibered surface over S such that X_η is smooth. Let $x \in X_s$ be a singular point of X_s. Then we have an isomorphism*

$$\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_K[[u,v]]/(uv - c)$$

for some $c \in \mathfrak{m}_{\mathcal{O}_K}$.

Definition 3.4. In the setting of Proposition 3.3, let $w_x \geq 1$ be the valuation of c. We call w_x the *thickness* (or *width*) of x in X.

Definition 3.5. Let X be a smooth, geometrically connected, projective curve over K. A normal fibered surface $\mathfrak{X} \to S$ such that $\mathfrak{X}_\eta \cong X$ is a called *model of X over S*.

Remark 3.6. There may be many different models of any given curve X, but they might not be pre-stable. As we will soon see, we will still be able to uniquely attribute a metric graph to X, which will be what we call the *tropicalization* of X.

Theorem 3.7. *[DM69, Corollary 2.7] When X is a smooth, projective, geometrically connected curve over K, with $g(X) \geq 2$, there exists a finite algebraic extension L of K, such that $X_L = X \times_K L$ has a unique stable model $\mathfrak{X}_L$ over $\mathcal{O}_L$ with generic fiber isomorphic to X_L. Moreover, L can be taken separable over K.*

Remark 3.8. The theorem stated in this form can be found in [Liu02, Theorem 10.4.3].

Definition 3.9 (Tropicalization)**.** Let X be a smooth, projective, geometrically connected curve over K. Let L be a finite algebraic extension of K such that X_L has a unique stable model $\mathfrak{X}_L$ as in Theorem 3.7.

We equip the dual graph G of $\mathfrak{X}_s$ with the structure of a metric graph by defining the length function

$$\begin{aligned} l : E(G) &\to \mathbb{Q} \\ e &\mapsto \frac{w_{n_e}}{[L:K]}, \end{aligned}$$

where n_e denotes the node of $\mathfrak{X}_s$ corresponding to the edge e. The resulting metric graph Γ is called the *tropicalization* of X.

Remark 3.10. The fact that the above definition does not depend on choice of the field extension $L \mid K$ is verified in [Viv13, Lemma 2.2.4].

3.2 Specialization of divisors

A first natural object we might want to transfer from the algebraic curve to the tropicalization are divisors. We will now describe the specialization process as introduced in [Bak07].

Let X be a smooth, geometrically connected, projective curve of genus ≥ 2. Up to performing base change to a finite extension of X, suppose X admits a stable model $\mathfrak{X}$. So let Γ be the tropicalization of X. By taking the minimal desingularization of $\mathfrak{X}$ we obtain the *minimal regular model* of X, denoted $\mathfrak{X}_{\min}$. [Liu02, Corollary 10.3.25] tells us that this model is semi-stable and that the dual graph corresponds to the subdivision of the edges of Γ, so that each edge has length 1.

Denote by C the special fiber of $\mathfrak{X}_{\min}$ and let $C_1, \ldots, C_n$ be its irreducible components, corresponding to the vertices $v_1, \ldots, v_n$ of the dual graph. For a K-rational point P of X, we may take its Zariski closure in $\mathfrak{X}_{\min}$ to obtain a Weil divisor on $\mathfrak{X}_{\min}$ which we denote by $\overline{P}$. We have that $\overline{P}$ intersects the special fiber C in its smooth locus in a single point by [Liu02, Proposition 9.1.32] and so $\overline{P}$ intersects a unique irreducible component of C. Let $v(P)$ be the corresponding vertex in Γ.

This allows us to define a map $\rho : \mathrm{Div}(X(K)) \to \mathrm{Div}(\Gamma)$ by setting

$$\rho(D) = \sum_{P \in X(K)} D(P) \cdot v(P).$$

We would like to extend the definition of ρ to all of $X(\overline{K})$. In order to do so, we need to check that ρ is compatible with base change. Let L be a finite extension of K, and denote $\rho_K : \mathrm{Div}(X(K)) \to \mathrm{Div}(\Gamma)$ and $\rho_L : \mathrm{Div}(X(L)) \to \mathrm{Div}(\Gamma)$ the corresponding maps. We want to show that $\rho_L|_{\mathrm{Div}(X(K))} = \rho_K$. Taking the base change $\mathfrak{X}_{\min,K} \times_{\mathcal{O}_K} \mathcal{O}_L$ yields a semi-stable model of X_L. By [Liu02, Corollary 10.3.22(c)], this has the effect of multiplying the thickness of the nodes of the special fiber by $[L : K]$. This introduces new singularities to the model, and the minimal regular model of X_L is given by repeatedly blowing up those singularities. The effect of this on the dual graph is to subdivide the edges into $[L : K]$ segments of equal length. Note moreover that if P is a K-rational point, it will specialize to the same connected component after performing base-change and desingularizing. So in fact the two specialization maps agree on divisors that are supported on K-rational points. See [CR93, Section 2] for more details on the compatibility with base change.

Since the specialization map is compatible with base change, it induces a map

$$\rho : \mathrm{Div}(X(\overline{K})) \to \mathrm{Div}(\Gamma)$$

that has image in $\mathrm{Div}_{\mathbb{Q}}(\Gamma)$, which is the set of divisors supported on the points in Γ which have rational distance from any given vertex.

We now define the rank of a divisor on an algebraic curve in analogy to Definition 2.32.

Definition 3.11. If D is a divisor on X, we define

$$r(D) := \max\{d \in \mathbb{N} \mid |D - E| \neq \emptyset \text{ for all effective divisor } E \text{ of degree } d\},$$

to be the *rank* of D.

Remark 3.12. By [Bak07, Lemma 2.4] the rank of a divisor D is equal to $\dim \mathcal{L}(D) - 1$. This explains the formula of the Riemann-Roch theorem (Theorem 2.33) in terms of the rank.

Matt Baker has famously shown that during the specialization process, the rank of a divisor can only increase.

Lemma 3.13 (Specialization lemma)**.** *[Bak07, Lemma 2.8] For all divisors $D \in \mathrm{Div}(X(K))$,*

$$r(\rho(D)) \geq r(D).$$

Example 3.14. Consider the situation in Figure 9. Let $D = v_1 + v_2$. Then D is a divisor of rank 1. We may lift D to a divisor D' of X_η, since by [Bak07, Remark 2.3] the specialization map is surjective. When X_η is not hyperelliptic, the divisor D' must be of rank 0, which gives an example of a case where the inequality in Lemma 3.13 is strict.

We will now construct a model as in Figure 9. Let K be a valued field with ring of integers R, maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let q_1, q_2 be two homogeneous polynomials over R of degree 2 such that $X_s = V(\overline{q_1 q_2}) \subseteq \mathbb{P}^2_k$ is a union of two smooth quadrics in general position. By Bézout's theorem, these quadrics will intersect in four points, hence the dual graph of X_s is G as in Figure 9. Let $c \in \mathfrak{m}$ and let p be a homogeneous degree 4 polynomial such that $X_\eta = V(q_1 q_2 + cp) \subseteq \mathbb{P}^2_K$ is a smooth quartic. It is then clear that $X = V(q_1 q_2 + cp) \subseteq \mathbb{P}^2_R$ is a fibered surface with generic fiber X_η and special fiber X_s. By the degree-genus forumla X_η is of genus 3. To conclude, note that by [Har77, Example IV.5.2.1] any quartic plane curve is non-hyperelliptic.

When Γ is a metric graph, we call a divisor $D \in \mathrm{Div}(\Gamma)$ *realizable*, whenever there exists a curve X and D' an effective divisor on X of the same rank as D such that (Γ, D) is the tropicalization of (X, D'). It is an important open problem to characterize realizable divisors.

3.3 Realizability of canonical divisors

In [MUW17], the authors give a complete characterization for the realizability of divisors in the canonical tropical linear system. We will reinterpret this result in a simpler context and give some sufficient conditions for realizability.

The condition as presented in [MUW17] works with more structure on the graph Γ. First of all, the vertices are decorated with a function $h : V \to \mathbb{N}$. During the tropicalization process, h records the genus of the corresponding irreducible component. With this decoration, the canonical divisor on Γ is defined as

$$K = \sum_{x \in V} (2h(x) - 2 + \mathrm{val}(x)) \cdot x$$

and the genus is

$$g = b_1(\Gamma) + \sum_{x \in V} h(x),$$

where $b_1(\Gamma)$ is the first betti number of Γ. Note that when the semi-stable reduction is totally degenerate, these definitions agree with the previous ones.

Definition 3.15. A *graph with legs* is a length space obtained from a metric graph by attaching to it a finite set of half-rays, which we call *legs*. The notions from Section 2 extend naturally to graphs with legs.

Let Γ be a graph with legs. We call any function $l : V \to \mathbb{Z}_{\leq 0}$ such that $l^{-1}(0) \neq \emptyset$ a *level function* on Γ. Such a level function induces a full order on the vertices of Γ. We call Γ with the data of a level function a *level graph*, and denote it by $\overline{\Gamma}$. For any edge e between two vertices x, y, we say e is *horizontal* whenever $l(x) = l(y)$, otherwise we say e is vertical.

We write $\Lambda = \bigsqcup_{x \in V} T_x\Gamma$ for the set of tangent vectors of Γ based at the vertices. This is naturally identified with the set of half-edges and legs.

An *enhanced level graph* Γ^+ is a level graph $\overline{\Gamma}$ together with a function $k : \Lambda \to \mathbb{Z}$ such that

1. For any edge e with corresponding tangents ζ^+, ζ^- along e, we have that $k(\zeta^+) + k(\zeta^-) = -2$. An edge is horizontal iff $k(\zeta^\pm) = -1$ and when e is vertical with ζ^+ being the tangent at the higher vertex, then $k(\zeta^+) > k(\zeta^-)$.

2. For each vertex v,
$$\sum_{\zeta \in T_v\Gamma} k(\zeta) = 2h(v) - 2.$$

When Γ^+ is an enhanced level graph, we define the *type* $\mu(v)$ of a vertex to be the ordered tuple (in decreasing order) of the $k(\zeta)$, for $\zeta \in T_v(\Gamma)$.

Definition 3.16. Let Γ^+ be an enhanced level graph. A vertex $v \in \Gamma^+$ is called *inconvenient* if $h(v) = 0$ and its type $\mu(v) = (k_1, \dots, k_n)$ has the following properties:

- $k_i \neq -1$ for all i.
- There exists an index i such that

$$k_i > \left(\sum_{k_j < 0} -k_j \right) - \#\{k_j < 0\} - 1$$

We can naturally attribute to a metric graph Γ along with a given canonical divisor $D \in |K|$ an enhanced level graph.

Definition 3.17. Let Γ a metric graph and let $D = K + \mathrm{div}(f) \in |K|$ an effective canonical divisor on Γ. Up to subdividing the model of Γ, we may assume that D is supported on the vertices of Γ.

- For each vertex x, we attach $D(x)$ legs to Γ and call the resulting graph with legs Γ'.
- Extend f to a rational function on Γ' so that f is linear on the legs with $s_\zeta(f) = -2$ for ζ a tangent at a vertex along a leg.
- Give Γ' the structure of level map induced by the function f.
- Define also $k(\zeta) = -s_\zeta(f) - 1$.

This equips Γ' the structure of an enhanced level graph, which we denote by $\Gamma^+(f)$.

Definition 3.18. We will say a vertex v of Γ is inconvenient if it is an inconvenient vertex of the enhanced level graph $\Gamma^+(f)$.

We may now state [MUW17, Theorem 6.3], which gives us a necessary and sufficient condition for the realizability of a divisor on a tropical curve. A *cycle* is a non-trivial path from a vertex from itself. A cycle is *simple* if it's not self-intersecting (other than at the endpoints).

Theorem 3.19. *Let Γ be a metric graph and let $D = K + \mathrm{div}(f)$ be an effective canonical divisor on Γ. We consider the model of Γ subdivided so that* $\mathrm{bend}\, f$ *is supported on the vertices. Then D is realizable if and only if the following conditions are satisfied:*

- *Every inconvenient vertex is contained in a simple cycle that lies above it (in the sense that $f(Z) \geq f(v)$, where Z is the given cycle).*
- *Every horizontal edge (meaning that f is constant on that edge) is contained in a simple cycle that lies above it.*

We will now describe more explicitly what it means for a vertex to be inconvenient.

Proposition 3.20. *Let Γ be a metric graph and $D = K + \mathrm{div}(f)$ an effective canonical divisor in $|K|$. Let v a vertex and denote $s_1, \dots, s_r$ the outgoing slopes of f along the edges adjacent to v. Then v is inconvenient iff $h(v) = 0$, $s_j \neq 0$ for all j, and there is an i such that $s_i < 0$ and*

$$-s_i > \sum_{j, s_j > 0} s_j.$$

Proof. Let v be a vertex, since $k(\zeta) = -s_\zeta(f) - 1$ for tangents $\zeta \in \Lambda$, the first condition in Definition 3.16 translates to $s(\zeta) \neq 0$ for all tangents at v.

Let ζ_j be the tangents corresponding to the k_j appearing in the type of the vertex v. Denote also $s_j = s_{\zeta_j}(f)$. The second condition of Definition 3.16 is equivalent to the existence of an index i such that

$$-s_i > \sum_{s_j > 0} s_j.$$

From Definition 3.17 it follows that for all j such that $s_j > 0$, ζ_j is a tangent along an edge. Now, if ζ_i is tangent along a leg, the left hand side is just 2. In this case we would get $2 > \sum_{s_j>0} s_j$. For this to be satisfied there should be at most one tangent with positive outgoing slope at v and that slope has to be equal to 1. But then there are at least $\mathrm{val}_\Gamma(x) - 1$ other edges with strictly negative outgoing slopes, which would imply that $\mathrm{div}(f)(x) \leq -\mathrm{val}_\Gamma(x) + 2$ (here the divisor of f is taken inside Γ). This in turn means that $(K + \mathrm{div}(f))(x) \leq 0$, which forces $(K + \mathrm{div}(f))(x) = 0$, as $D = K + \mathrm{div}(f)$ is by assumption effective. But then by definition of the enhanced level graph $\Gamma^+(f)$, we would not have attached any legs, so this situation cannot happen. We deduce that all the terms appearing in the inequality can only come from tangents along edges (so tangents of the original metric graph Γ), which finishes the proof. □

Thanks to this, when talking about realizability we will not need to refer to the structure of enhanced level graphs, and so we will restrict our discussion to metric graphs as defined in Section 2.

Denote by $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ the the moduli space parametrizing isomorphism classes of metric graphs (with vertex weights) of genus g with the choice of a canonical divisor. It carries the structure of a generalized cone complex by [LU17, Theorem 4.3]. Let $\mathbb{P}\mathcal{R} \subseteq \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$, the subset of pairs $([\Gamma], D) \in \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ that are realizable. By [MUW17, Theorem 6.6], $\mathbb{P}\mathcal{R}$ is an abstract cone complex whose maximal cones have dimension $4g - 4$. Furthermore, by [MUW17, Proposition 6.9(i)] the graphs appearing in the maximal cones have $h \equiv 0$. Hence the set of pairs $([\Gamma], D) \in \mathbb{P}\mathcal{R}$, which have $h \equiv 0$ is dense in $\mathbb{P}\mathcal{R}$. For this reason in our following discussion we will only focus on the case where $h \equiv 0$ and so we will be looking only at graphs that appear as the dual of a totally degenerate semi-stable curve.

3.4 Realizability locus of the canonical linear system

We will now give some characterizations of the realizability locus of the canonical complete linear system $|K|$, which is the subset of divisors which are realizable.

Proposition 3.21. *Let Γ be a metric graph and let $\mathrm{Real}(|K|) \subseteq |K|$ be the set of realizable canonical divisors. Then $\mathrm{Real}(|K|)$ is tropically convex.*

Proof. Let M be the set of $f \in R(K)$ that correspond to realizable divisors $K + \mathrm{div}(f)$, plus the element $-\infty$. We claim that M is a submodule of $R(K)$.

Let $f \in M$. Since for all $c \in \mathbb{R}$, we have that $\mathrm{div}(c \odot f) = \mathrm{div}(f)$, we deduce that $K + \mathrm{div}(c \odot f)$ and so M is stable under tropical scalar multiplication.

When $f, g \in M$, we will show that $f \oplus g = \max(f, g) \in M$. Consider the model G of Γ, such that both f and g are linear when restricted to any given edge. It follows that when restricted to any fixed edge, $f \oplus g$ is equal to either f or g.

If e is a horizontal edge of Γ with respect to $f \oplus g$, then we must have that e is a horizontal edge with respect to one of f or g. W.l.o.g. f is constant on e, but then as $K + \mathrm{div}(f)$ is realizable, there is a simple cycle $\gamma \subseteq \Gamma$ containing e and which lies above it with respect to f. But $f \oplus g \geq f$ and hence γ also lies above e with respect to $f \oplus g$. We deduce that the condition on the edges in Theorem 3.19 is satisfied.

Now, suppose v is an inconvenient vertex with respect to $f \oplus g$. If $f(v) \neq g(v)$, then w.l.o.g. $f(v) > g(v)$ and so $f \oplus g$ agrees with f in a neighbourhood of v. This implies that v is inconvenient with respect to f as well. Hence there is a simple cycle $\gamma \subseteq \Gamma$ containing v and which lies above it with respect to f, and as before this also implies it lies above v with respect to $f \oplus g$. Now, suppose $f(v) = g(v)$. In this case, for any tangent $\zeta \in T_v\Gamma$, we have that $s_\zeta(f \oplus g) = \max(s_\zeta(f), s_\zeta(g))$. Fix any ordering $\zeta_1, \dots, \zeta_r$ on $T_v\Gamma$ and let $s_i = s_{\zeta_i}(f \oplus g)$ and $s'_i = s_{\zeta_i}(f)$. It follows that $s_i \geq s'_i$ for all i. Since v is inconvenient with respect to $f \oplus g$, we know that there exists some i such that $s_i < 0$ and

$$-s_i > \sum_{j, s_j > 0} s_j$$

Then we have that

$$-s'_i \geq -s_i > \sum_{j, s_j > 0} s_j \geq \sum_{j, s'_j > 0} s_j \geq \sum_{j, s'_j > 0} s'_j.$$

Hence it follows that v is also an inconvenient vertex for f, and as before this yields a simple cycle that lies above v with respect to both f and $f \oplus g$. We deduce that the condition on inconvenient vertices in Theorem 3.19 is satisfied and so $K + f \oplus g$ is realizable.

We conclude that M is a submodule of $R(K)$ and hence $\mathrm{Real}(|K|)$ being the image of this submodule is tropically convex. $\square$

Proposition 3.22. *The realizability locus* $\mathrm{Real}(|K|)$ *is a definable subset of* $|K|$.

Proof. We know that $R(K)$ is finitely generated and so let $\{\phi_1, \dots, \phi_r\}$ be a generating set. Consider a model $G = (V, E)$ of Γ, such that the ϕ_i are all linear on each edge. This is possible since the set

$$\bigcup_{i=1}^{r} \mathrm{bend}(\phi_i)$$

is finite, so we may choose V to contain this set. The set of a_i such that $\max(a_i + \phi_i) \geq 0$ on a fixed edge is clearly a definable subset of $\mathbb{R}^r$. Denote this subset by C_e. Let also γ be a simple path in Γ, then the set of a_i such that $\max(a_i + \phi_i) \geq 0$ an all of γ is just the intersection of all the C_e for each $e \subseteq \gamma$. Hence this set is also definable and we will denote it by C_γ.

Now, choose a vertex v. Let S be any subset of $\{1, \dots, r\}$ such that the rational function

$$\phi_S := \bigoplus_{i \in S} (\phi_i - \phi_i(v))$$

makes v an inconvenient vertex. Then for any choice of $a_1, \dots, a_r$ such that $a_i = -\phi_i(v)$ for all $i \in S$ and $a_i < -\phi_i(v)$, we have that

$$\phi := \bigoplus_{i=1}^{r} a_i \odot \phi_i$$

also makes v inconvenient. Indeed, we have that ϕ and ϕ_S coincide in a neighbourhood of v, and hence the outgoing slopes of ϕ at v are all equal to the outgoing slopes of ϕ_S. Denote the set of such $(a_1, \dots, a_r) \in \mathbb{R}^r$ by $I_{v,S}$. Again, $I_{v,S}$ is clearly a definable set. Define now

$$I_v := \bigcup_{S \subseteq \{1, \dots, r\}} I_{v,S},$$

where the union ranges over the subsets S such that v is inconvenient for ϕ_S. This set corresponds to all the rational functions $f \in R(K)$ such that $f(v) = 0$ and for which v is inconvenient. Indeed, if f is such a function, then

$$f = \bigoplus_{i=1}^{r} a_i \odot \phi_i$$

for some choice of a_i and since $f(v) = 0$, we necessarily have that $a_i \leq -\phi_i(v)$ for all i and $a_i = -\phi_i(v)$ at least for one i. We may then set $S = \{i \mid a_i = -\phi_i(v)\}$ for which $f \in I_{v,S}$.

Now, the set

$$Z_v := I_v \cap \bigcup_{\gamma} C_\gamma,$$

where the union is over the simple cycles of Γ containing v corresponds to all the rational functions $f \in R(K)$ such that $f(v) = 0$, v is inconvenient and contained in some simple cycle that lies above it.

The image of Z_v via the piece-wise affine map

$$\begin{aligned} \eta : \mathbb{R}^r &\to |K| \\ (a_1, \dots, a_r) &\mapsto K + \operatorname{div}\left(\bigoplus_{i=1}^{r} a_i \odot \phi_i\right) \end{aligned}$$

is precisely the subset of divisors of $|K|$ such that v is inconvenient, but contained in a simple cycle that lies above it. Similarly, we have that $\eta(I_v)$ is the set of divisors of $|K|$ such that v is inconvenient (with no further conditions). Note also that since η is piece-wise affine, the image of any definable set is again definable.

Let U be a subset of vertices. We let

$$A_U := \bigcap_{v \in U} \eta(Z_v) \cap \bigcap_{v \notin U} \eta(I_v)^c.$$

This is the set of divisors in $|K|$ for which the set of inconvenient vertices is precisely U and every inconvenient vertex is contained in a simple cycle that lies above it.

It follows that $A := \bigcup_{U \subseteq V} A_U$ is the set of divisors in $|K|$ for which each inconvenient vertex is contained in a simple cycle that lies above it. It is clear from the construction of A that this set is definable.

We will now show using an analogous argument that the set of divisors in $|K|$ for which each horizontal edge is contained in a simple cycle lying above it is also definable, which will finish the proof.

Let e be an edge and let ϕ_j be a rational function that is constant on e. Let $H_{e,j}$ be the set of $(a_1, \dots, a_r) \in \mathbb{R}^r$ with $a_j = -\phi_j(e)$ and

$$\min_{x \in e} \max_{i \neq j} (a_i + \phi_i) < a_j + \phi_j(e).$$

Then $H_{e,j}$ is clearly definable and corresponds to the set of rational functions that have a horizontal segment along e, on which they are equal to $a_j + \phi_j(e) = 0$. Define also $H_e := \bigcup_{j=1}^{r} H_{e,j}$, where we just let $H_{e,j} = \emptyset$ when ϕ_j is not constant on e. Then $\eta(H_e)$ is the set of divisors in $|K|$ that have a horizontal segment on the edge e. Let

$$Y_e := H_e \cap \bigcup_{\gamma} C_\gamma,$$

where the union is over the simple cycles of Γ containing e. Then clearly $\eta(Y_e)$ is the set of divisors in $|K|$ such that e is has a horizontal segment, which is contained in a cycle that lives above it. Finally, if we let

$$B = \bigcup_{F \subseteq E} \bigcap_{e \in F} \eta(Y_e) \cap \bigcap_{e \notin F} \eta(H_e)^c,$$

we deduce that B is the set of divisors in $|K|$ for which each horizontal edge is contained in a simple cycle that lies above it.

By construction B is definable, and since $\mathrm{Real}(|K|) = A \cap B$, the realizability locus is definable as well. □

Proposition 3.23. *The realizability locus* $\mathrm{Real}(|K|)$ *is closed.*

Proof. Let $D_n = K + \mathrm{div}(f_n)$ be a sequence of realizable divisors in $\mathrm{Real}(|K|)$ converging in $|K|$ to $D + \mathrm{div}(f)$. Suppose v is an inconvenient vertex for f. Let ζ be a tangent vector at v. Let s be the slope of f along ζ and s_n the slope of f_n along ζ. The sequence s_n takes values in the finite set $\{-\deg K, \ldots, \deg K\}$, so up to switching to a subsequence, we may assume that the s_n are all equal to some s'. Let e be the edge corresponding to ζ, then since $K + \mathrm{div}(f_n)$ is effective, all of the f_n are convex along the edge e. This forces $s \geq s'$. Hence up to switching to a subsequence, we may assume that for all tangent vectors ζ at v, we have that

$$s_\zeta(f) \geq s_\zeta(f_n).$$

We deduce the same way as in the proof of Proposition 3.21 that v is also an inconvenient vertex for all of the f_n. Since all the D_n are realizable, we deduce that for all n there exists some simple cycle γ_n that lie above v (with respect to f_n). There are only finitely many simple cycles, so up to switching to a subsequence, we may assume that all of the γ_n are equal to some fixed cycle γ. We have for all n that $f_n(\gamma) \geq f_n(v)$ and so by taking the limit, we also obtain that $f(\gamma) \geq f(v)$. Hence we conclude that for every inconvenient vertex for f, there exists a simple cycle that lies above it.

Now, let e be a horizontal edge for f. We have that

$$\lim_{n \to \infty} \|f - f_n\|_\infty = 0,$$

and the f_n are convex on e and have only integral slopes. For n such that $\|f - f_n\| \leq l(e) < 2$ this implies f_n has to have a horizontal section along e. We deduce that there exists a simple cycle γ_n containing e such that $f_n(\gamma_n) \geq f_n(e)$. Like before, we deduce that there exists a simple cycle γ containing e, such that $f(\gamma) \geq f(e)$.

The two conditions for realizability from Theorem 3.19 are satisfied, and so we conclude that D is realizable, and so $\mathrm{Real}(|K|)$ is closed. □

Corollary 3.23.1. *The realizability locus* $\mathrm{Real}(|K|)$ *admits the structure of an abstract polyhedral complex.*

Proof. By Propositions 3.22 and 3.23, $\mathrm{Real}(|K|)$ is closed and definable, so the statement follows from Corollary 2.65.2. □

Although we have shown that $\mathrm{Real}(|K|)$ is an abstract polyhedral complex, this does not show anything about whether or not it is finitely generated. For example, consider the triangle spanned by $[1:0:0]$, $[0:1:0]$ and $[1:1:0]$ in $\mathbb{T}(\mathbb{R}^3)$. It is impossible to express any point of the form $[a:b:1]$ with $a+b=1$ as a convex combination of other points in the triangle, so the triangle is not finitely generated, despite being a tropically convex subspace.

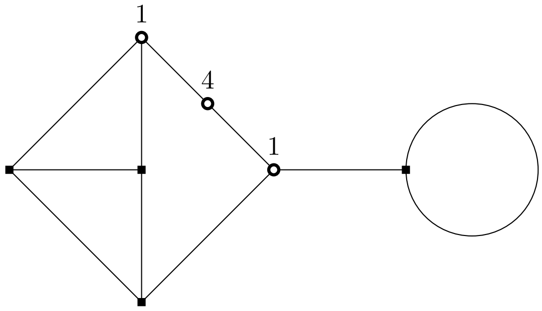

Figure 10: Realizable divisor in the canonical linear system that is not in the span of the realizable extremals of $|K|$.

If one considers the tropically convex subspace of $|K|$ spanned by the realizable extremals of $|K|$, it would appear that this set agrees with $\mathrm{Real}(|K|)$ when one consider genus 3 graphs. Unfortunately, this already fails for genus 4 graphs. For example, the divisor shown in Figure 10 is realizable, but does not belong to the span of realizable extremals of $|K|$ (this is not obvious a priori, but has been checked using a computer program).

Question 3.24. Is the realizability locus in the canonical linear system finitely generated?

An affirmative answer to this question would be very useful, as it would allow to characterize the realizability locus using its extremals.

3.5 Cycles and realizability

In this subsection we will give a sufficient characterization for realizability of divisors in the canonical linear system by studying the cycles that can appear in the metric graph. We will be able to deduce that $\mathrm{Real}(|K|)$ always contains some maximal cell of $|K|$ of dimension $g-1$.

Proposition 3.25. *Let $D = K + \mathrm{div}(f)$ an effective canonical divisor. If C is a local maximum of f, then C does not have vertices of valence 1 in C. In particular, C contains a cycle.*

Proof. For any vertex $v \in C$, since f may not have positive slope along any tangent at x (as C is a local maximum), we deduce that

$$D(v) \leq K(v) - (\mathrm{val}_\Gamma(v) - \mathrm{val}_C(v)) = \mathrm{val}_C(v) - 2.$$

In particular, $\mathrm{val}_C(v) \geq 2 + D(v) \geq 2$ and so v is not a leaf in C.

The second statement follows from the fact that if G is a connected graph with edges E, vertices V, each of valence at least 2, then $\#E = \frac{1}{2}\sum_{v\in V} \mathrm{val}_G(v) \geq \#V$ and so $g(G) = \#E - \#V + 1 \geq 1$. □

Lemma 3.26. *If $v \in V$ is an inconvenient vertex, there are at least two edges with strictly positive outgoing slope.*

Proof. Suppose v is of valence r and denote the outgoing slopes by $s_1, \ldots, s_r$, in increasing order. If s_r is the only strictly positive number, we have by definition of inconvenient vertex (Proposition 3.20) that $-s_1 > s_r$ and $s_i < 0$ for all $i < r$. But then

$$\mathrm{div}(f)(v) = \sum_{i=1}^{r} s_i = (s_1 + s_r) + \sum_{i=2}^{r-1} s_i < -(r-2) = -K(v).$$

This is absurd, as this would imply $(K + \mathrm{div}(f))(v) < 0$. $\square$

Proposition 3.27. *Let $D = K + \mathrm{div}(f)$ be an effective canonical divisor. If there are no two disjoint horizontal cycles (possibly of different heights), then D is realizable.*

Proof. We need to verify the conditions of Theorem 3.19 are satisfied.

If we have an inconvenient vertex v, by Lemma 3.26 there are (at least) two edges e_1, e_2 emanating from v such that the outgoing slope of f along these edges is strictly positive. Select two simple (= not self-intersecting) paths γ_1, γ_2 (seen as functions $[0,1] \to \Gamma$ with $\gamma_i(0) = v$) along e_1, e_2, that are maximal for the property that $f \circ \gamma_i$ is non-decreasing and $f \circ \gamma_i$ is strictly increasing on $(1-\epsilon, 1)$ for some $\epsilon > 0$. Let $x_i := \gamma_i(1)$. Let C_i be the connected component of $f^{-1}(f(x_i))$ containing x_i. By construction, C_i is a local maximum, so by Proposition 3.25, the C_i each contain a horizontal cycle. Since by assumption these cycles have to intersect, we obtain $C_1 = C_2$.

Now, if γ_1, γ_2 intersect, we can choose (t_1, t_2) a pair such that $\gamma_1(t_1) = \gamma_2(t_2)$, minimal for the partial order $(t_1, t_2) \leq (s_1, s_2) \iff t_1 \leq s_1$ and $t_2 \leq s_2$. Then $\gamma_1|_{[0,t_1]} \oplus \overleftarrow{\gamma_2|_{[0,t_2]}}$ is a simple cycle that lies above v. Here by $\overleftarrow{\gamma}$ we mean the reversed path $\overleftarrow{\gamma}(t) = \gamma(1-t)$. By $\oplus$ we mean the concatenation of paths, that is.

$$\alpha \oplus \beta(t) := \begin{cases} \alpha(2t) & \text{if } t \leq 1/2, \\ \beta(2t-1) & \text{if } t \geq 1/2. \end{cases}$$

Lastly, when we write the restriction $\gamma|_{[a,b]}$, it is understood that this new path is reparametrized as to have again domain $[0,1]$, that is

$$\gamma|_{[a,b]}(t) = \gamma(a + t(b-a))$$

If γ_1, γ_2 don't intersect, since $C_1 = C_2$ is connected, there is a simple path τ from x_1 to x_2. Then the path $\gamma_1 \oplus \tau \oplus \overleftarrow{\gamma_2}$ is a simple cycle that lies above v. We conclude that every inconvenient vertex is contained in a simple cycle that lies above it.

Now let e be a horizontal edge between two vertices v_1 and v_2. Let C be the connected component of $f^{-1}(f(e))$ containing e. If $C \setminus e$ is connected, then there exists a simple path γ in $C \setminus e$ from v_1 to v_2. Then going along γ from v_1 to v_2 and then from v_2 to v_1 along e determines a horizontal simple cycle containing e. Hence e is not a problematic horizontal edge.

So suppose $C \setminus e$ is disconnected and let C_1, C_2 be the two components containing the vertices v_1, resp. v_2. We will show that there exist simple paths γ_i in $\Gamma \setminus e$ from v_i to the same horizontal cycle and like before, this would prove that e is contained in in a simple cycle that lies above it.

If C_i has any point x that is a leaf of C, then x has an adjacent edge on which f has strictly positive outgoing slope. Indeed, this follows because all of the other $\mathrm{val}_\Gamma(x) - 1$ edges have non-zero slopes, and if they were all negative, then $\mathrm{div}(f)(x) \leq -(\mathrm{val}_\Gamma(x) - 1)$, but this would imply that $(K + \mathrm{div}(f))(x) \leq -1$, which is absurd as we assumed $D = K + \mathrm{div}(f)$ is effective. So x neighbours an edge with strictly positive outgoing slope, and like before, we could take a path γ starting at x along this edge, which is maximal for the property that $f \circ \gamma$ is non-decreasing and $f \circ \gamma$ is strictly increasing on $(1-\epsilon, 1)$ for some ϵ. Then $\gamma(1)$ would lie on a local maximum which contains a distinguished cycle.

So suppose C_i contains no leaf of C, then the only leaf of C_i is possibly v_i. If we denote V_i the vertices of C_i and E_i the edges of C_i, we know that $\sum_{v \in V_i} \mathrm{val}_{C_i}(v) = 2 \cdot \#E_i$. It follows that the sum is even and so if v_i were of valence 1 in C_i, there would also need to be another vertex of odd valence, and so this one would have to be of valence at least 3. In any case, we get that $\#E_i \geq \#V_i$ and so $g(C_i) \geq 1$. In other words, C_i contains a simple cycle. The two cycles contained in C_1, C_2 are both horizontal, so they have to intersect, but this contradicts the fact that C_1, C_2 are disjoint.

So we conclude that e is contained in a simple cycle that lies above it, and so having verified the conditions of Theorem 3.19, we conclude D is realizable. $\square$

Corollary 3.27.1. *Suppose Γ does not contain disjoint cycles, then every $D \in |K|$ is realizable.*

Corollary 3.27.2. *If $|K|$ is of dimension at most $g-1$, then every $D \in |K|$ is realizable.*

Proof. If Γ contained disjoint cycles, then $|K|$ would contain a cell of dimension at least g by Proposition 2.83. The corollary then follows from the contrapositive of this statement. $\square$

Remark 3.28. The converse of Proposition 3.27 is not true. For example, the canonical divisor is realizable for the graph obtained by joining two cycles by a pair of edges (see Figure 11). On the other hand, the converse of Corollary 3.27.1 is true as the following proposition shows.

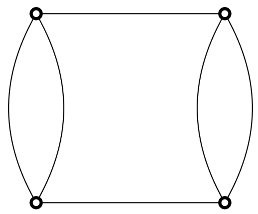

Figure 11: Graph with realizable canonical divisor.

Proposition 3.29. *If Γ contains disjoint cycles, then there exist non-realizable divisors $D \in |K|$.*

Proof. Let $Z_1, Z_2 \subseteq \Gamma$ be the two disjoint cycles. Let also $\gamma \subseteq \Gamma$ be a simple path such that $\gamma(0) \in Z_1$, $\gamma(1) \in Z_2$ and $\gamma((0,1)) \cap (Z_1 \cup Z_2) = \emptyset$.

The union $Z = Z_1 \cup Z_2 \cup \gamma$ is a closed subgraph where each vertex v has at most $\mathrm{val}(v) - 2$ outgoing edges, this means Z can fire (if $Z = \Gamma$, firing doesn't have any effect). So fire Z by a small amount to obtain a divisor $D = K + \mathrm{div}(f) \in |K|$. Now, since Z_1, Z_2 are disjoint, γ is non-trivial and so passes through at least one edge e. By construction, e is a horizontal edge. If $C \subseteq \Gamma$ is any simple cycle passing through e, C must intersect the complement of Z. Indeed, removing e from Z would disconnect it, as $e \subseteq \gamma$, which is a simple path connecting the two disjoint cycles. But because D was obtained by firing Z, it is clear that $f(\Gamma \setminus Z) < f(Z)$, and so C is not a cycle that lies above e. We conclude that D is not realizable. $\square$

Corollary 3.29.1. *There exists a maximal cell σ of $|K|$ of dimension $g-1$ such that $\sigma \subseteq \mathrm{Real}(|K|)$.*

Proof. Let σ be the cell of dimension $g-1$ given by Proposition 2.85. Let $D = K + \mathrm{div}(f) \in \mathrm{relint}(\sigma)$, so that $\mathrm{supp}\, D \cap V = \emptyset$. It follows that that

$$g - 1 = \dim \sigma = \deg D - (g - g(\Gamma \setminus D))$$

and so $g(\Gamma \setminus D) = 1$. We have that $\mathrm{supp}\, D \subseteq \mathrm{bend}\, f$. In particular,

$$g(\Gamma \setminus \mathrm{bend}\, f) \leq g(\Gamma \setminus \mathrm{supp}\, D) = 1,$$

and so there are no two disjoint horizontal cycles. This implies by Proposition 3.27 that D is realizable. Since D was arbitrary, it follows that $\mathrm{relint}(\sigma) \subseteq \mathrm{Real}(|K|)$, and since $\mathrm{Real}(|K|)$ is closed by Proposition 3.23, we deduce that $\sigma \subseteq \mathrm{Real}(|K|)$. $\square$

3.6 Specialization of linear series

We are also interested in the specialization of linear series. Recall that given a smooth projective curve X over a valued field K, which is geometrically connected, and of genus ≥ 2, we have defined the specialization map

$$\rho : \mathrm{Div}(X_{\overline{K}}) \to \mathrm{Div}_{\mathbb{Q}}(\Gamma).$$

First, note that the specialization map ρ by definition preserves the property of being effective. We claim that it also preserves the property of being principal. Let D be a principal divisor on $X_{\overline{K}}$, then since D is a Weil divisor (Weil and Cartier divisors are identified on regular schemes), it may be written as a finite sum

$$\sum_{i=1}^{r} n_i P_i$$

where the $P_i \in X_{\overline{K}}$ are $\overline{K}$-rational points. But then there exists an algebraic extension L of K such that P_i are all L-rational. In particular D may be seen as a principal divisor on X_L, and hence it determines a rational function f on X_L. Let $\mathfrak{X}$ be the minimal regular model of X_L, with $C_1, \ldots, C_s$ corresponding to a set of vertices $v_1, \ldots, v_s$ on the metric graph Γ. Since X_L is open in $\mathfrak{X}$, f determines a rational function on all of $\mathfrak{X}$, and so a principal divisor D' on $\mathfrak{X}$ (again we identify Weil and Cartier divisors, since $\mathfrak{X}$ is regular). Now, f could only acquire new zeroes and poles on the complement of X_L in $\mathfrak{X}$, so only on the special fiber. It follows that the difference

$$D' - \sum_{i=1}^{r} n_i \overline{P_i}$$

is a vertical divisor, which we will write as

$$D_v = \sum_{i=1}^{s} m_i C_i.$$

Now, note that we may rewrite the restriction $\rho : X_L \to \mathrm{Div}(\Gamma)$ as

$$\rho(P) = \sum_{i=1}^{s} (\overline{P} \cdot C_i) v_i,$$

where $\overline{P} \cdot C_i$ denotes the intersection number $\deg(\mathcal{O}_{\mathfrak{X}}(\overline{P})|_{C_i})$. Let

$$\overline{D} = \sum_{i=1}^{r} n_i \overline{P_i},$$

then it follows by bilinearity of the intersection number that

$$\rho(D) = \sum_{i=1}^{s} (\overline{D} \cdot C_i) v_i.$$

Since D' is principal, we have that $D' \cdot C_i = 0$ and hence

$$\overline{D} \cdot C_i = -D_v \cdot C_i.$$

Now, note that for any component C_j, the divisor

$$\sum_{i=1}^{s} (C_j \cdot C_i) v_i$$

corresponds to the divisor obtained by firing the vertex v_j by the unit distance (distance between any adjacent vertices of Γ), in particular it is principal. We conclude that $\rho(D)$ is a sum of principal divisors, and so also principal.

Now, let $|D|$ be a complete linear series on X, then for any divisor $D' \in |D|$, we have that D' is effective and $D - D'$ is principal. By what we just showed, this implies that $\rho(D')$ is effective and $\rho(D - D')$ is principal. As a result, $\rho(D') \in |\rho(D)|$ and so $\rho(|D|) \subseteq |\rho(D)|$ (this is actually the main ingredient for the proof of the Specialization Lemma). We are now interested in studying the specialization of linear series $\mathfrak{d} \subseteq |D|$, which we will define as the topological closure of the subspace $\rho(\mathfrak{d}) \subseteq |\rho(D)|$ and we will denote it by $\mathrm{trop}(\mathfrak{d})$.

To distinguish when we talk about linear systems on the algebraic curve or on the tropical curve, we will denote the objects on the algebraic curve with a subscript. So in what follows, fix a divisor $D_X \in \mathrm{Div}(X_{\overline{K}})$, specializing to a divisor $D \in \mathrm{Div}(\Gamma)$.

Proposition 3.30. *Let $\mathfrak{d}_X \subseteq |D_X|$ be a linear series of rank r. Then $\mathrm{trop}(\mathfrak{d}_X)$ is a finitely generated tropical linear series of rank at least r.*

Proof. This is shown in Lemmas 6.1, 6.2 and Proposition 6.4 of [FJP23]. ☐

Remark 3.31. Note that in my thesis the term "tropical linear series" does not designate the same thing as the same term in [FJP23] and [JP22]. Here it means a tropically convex subset of a complete linear series (in analogy to the nomenclature from algebraic geometry).

Corollary 3.31.1. *For $\mathfrak{d}_X \subseteq |D_X|$ a linear series, $\mathrm{trop}(\mathfrak{d}_X)$ admits the structure of an abstract polyhedral complex.*

Proof. By Proposition 3.30, $\mathrm{trop}(\mathfrak{d}_X)$ is finitely generated and so this follows from Proposition 2.92 and Corollary 2.65.2 ☐

A linear system $\mathfrak{d}_X$ is a projective subspace of $|D_X|$, in particular it has a well-defined notion of independence – a set of vectors of $\mathfrak{d}_X$ of size s is independent if and only if it is not contained in a projective subspace of $\mathfrak{d}_X$ of dimension $s-1$. This gives $\mathfrak{d}_X$ the structure of a matroid and it turns out that this notion translates well through tropicalization.

Definition 3.32. Let $S = \{D + \mathrm{div}(\phi_1), \dots, D + \mathrm{div}(\phi_n)\}$ be a subset of $|D|$. We say that S is *tropically dependent* if there are real numbers a_i such that for every point $v \in \Gamma$, the minimum in $\min_i\{\phi_i(v) + a_i\}$ is achieved at least twice.

It turns out that linearly dependent subsets of $|D_X|$ specialize to tropically dependent subsets of $|D|$. For $\mathfrak{d}_X \subseteq |D_X|$ a linear series, its rank is equal to its dimension as a projective subspace of $|D_X|$ and so any subset of $r+2$ points of $\mathfrak{d}_X$ is linearly dependent. Hence we expect the same property to hold after tropicalizing.

Proposition 3.33. *[FJP23, Lemma 6.2] Any subset of $\mathrm{trop}(\mathfrak{d}_X)$ of size at least $r+2$ is tropically dependent.*

The same way the notion of rank provides a lower bound on the dimension of a tropical linear series, the notion of tropical independence yields an upper bound.

Proposition 3.34. *[JP22, Corollary 4.7] Let $\mathfrak{d} \subseteq |D|$ be a finitely generated submodule such that any set of $r+2$ functions of $\mathfrak{d}$ is tropically dependent, then $\dim \mathfrak{d} \leq r$.*

Corollary 3.34.1. *If $\mathfrak{d} \subseteq |D|$ is a finitely generated tropical linear series of rank $\geq r$, such that any set of $r+2$ points of $\mathfrak{d}$ is tropically dependent, then $\mathfrak{d}$ is of rank r. Furthermore, $\mathfrak{d}$ is equi-dimensional of dimension r (in the sense that all the maximal cells have dimension r).*

Proof. Corollary 2.95.2 implies that all the maximal faces of $\mathfrak{d}$ have dimension at least $r(\mathfrak{d})$, and so $\dim \mathfrak{d} \geq r(\mathfrak{d}) \geq r$. But by Proposition 3.34 this forces $\dim \mathfrak{d} = r$ and so in particular $r(\mathfrak{d}) = r$, and the maximal dimensional faces of $\mathfrak{d}$ need to have dimension exactly r. $\square$

Corollary 3.34.2. *If $\mathfrak{d}_X \subseteq |D_X|$ is a linear series of rank r, then* $\mathrm{trop}(\mathfrak{d}_X)$ *is of rank r, and equi-dimensional of dimension r.*

4 Discrete representations

In this section, we are going to briefly discuss possible approaches to working with tropical curves using computer techniques.

One approach, which is the closest to the original definitions is to represent a metric graph as a set of vertices V and edges E with a fixed orientation and length. Divisors and rational functions can be entirely determined by specifying their value on a finite set of points of the graph Γ. Furthermore, a point of Γ is either a vertex, in which case it corresponds to an element of V, or a point along an edge, in which case it may be specified by an edge $e \in E$, and a distance along that edge. This approach is well-suited for working with tropical submodules, as the tropical operations are fairly easy to implement.

Another approach is to restrict our attention to rational functions and divisors on a given model of Γ, for which each edge has identical length. The advantage of this approach is that operations on such a curve may be represented by matrix operations, which can make many things significantly faster and easier to implement. It is much easier to represent and work with chip-firing moves, find v-reduced divisors and go back and forth between divisors and rational functions. An important aspect of using this representation is that in a linear system there may only be a finite number of divisors which are supported on a given model of Γ. This allows us to develop an algorithm for finding the set of these divisors exhaustively.

We are now going to describe in detail how to work with linear systems on graphs and in particular justify how these finite models carry information about linear systems on the whole metric graphs.

4.1 Graphs

Let Γ be a metric graph. We are mainly interested in metric graphs with sides of integer length as these are those that appear as the tropicalization of an algebraic curve. We can choose the model $G = (V, E)$ of Γ that has all edge lengths of size 1. If we restrict rational maps and divisors to be supported on V, we have a nice description.

Definition 4.1. Let $G = (V, E)$ be a graph. A *level map* is a function

$$f : V \to \mathbb{Z}.$$

A level map f uniquely determines a rational function on Γ if we interpolate linearly between the vertices. Indeed, as we assumed all edge lengths are of size 1, this implies the resulting function has integral slopes.

Reciprocally, up to adding a constant, we may assume a rational function $f : \Gamma \to \mathbb{R}$ supported on V admits integral values on V. So by restricting f to V this determines a level map up to a constant.

Definition 4.2. Let $G = (V, E)$ be a graph. A *divisor* on G is a divisor D on Γ, which is supported on the set of vertices V. We denote the set of divisors on G by $\mathrm{Div}(G)$.

When we scale a metric graph by an integer, we do not affect the combinatorial structure of the metric graph, nor the structure of linear systems. Scaling the metric graph m times has the effect of uniformly subdividing the associated model where all edges have length 1 by splitting each edge into m edges. Let $D \in \mathrm{Div}(\Gamma)$ be a $\mathbb{Q}$-rational divisor. Then that all the points in the support of D are a rational distance away from any given vertex. If we take the common denominator of these numbers, say d, we know that D will be supported on the d^{th} subdivision of the graph G. So we deduce that by subdividing the graph G we may obtain a more and more faithful representation of the whole set of divisors $\mathrm{Div}(\Gamma)$.

Definition 4.3. Let $G = (V, E)$ be a graph. We define the *incidence map* to be the map

$$\phi : E \to \{\{x, y\} : x, y \in V\},$$

which maps each edge to its respective vertices.

For $x \in V$, denote $E_x = \{e \in E : x \in \phi(E)\}$ the set of edges adjacent to x and for $e \in E_x$, with $\phi(e) = \{x, y\}$, we denote $\nu_x(e) = y$ the vertex adjacent to x along the edge e.

Definition 4.4. Let f be a level map on G. Then the *order* of f at x is defined by

$$\mathrm{ord}_x(f) = \sum_{e \in E_x} (f(\nu_x(e)) - f(x)).$$

The divisor $\mathrm{div}(f) \in \mathrm{Div}(G)$ associated to f is defined by

$$\mathrm{div}(f) = \sum_{x \in V} \mathrm{ord}_x(f) \cdot x$$

Remark 4.5. This definition is compatible with the analogous definitions on rational maps.

Definition 4.6. Let $A \subseteq V$ be any subset. The level map associated to a chip firing move is

$$CF(A)(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Remark 4.7. The rational map associated to this chip firing move on a metric graph is $\sum_{x \in A} CF(\{x\}, 1)$, or equivalently if $Z \subseteq \Gamma$ is the subgraph obtained by adding all the edges between the vertices in A, then the chip firing move is the same as $CF(Z, 1)$. We say Z is the subgraph *spanned* by the set of vertices A.

Remark 4.8. Any level map is a sum of chip-firing moves (up to a constant).

The advantage of working with such a discretization is that we can express level maps and divisors as vectors, and chip-firing moves as matrix operations. To this end, fix an ordering $\{v_1, \ldots, v_n\}$ on the vertices V.

Definition 4.9. Let f be a level map, the *vector associated to* f, denoted by $[f]$ is the column-vector

$$[f] = \begin{bmatrix} f(v_1) \\ \vdots \\ f(v_n) \end{bmatrix}.$$

Similarly, for D a divisor, the *vector associated to* D is

$$[D] = \begin{bmatrix} D(v_1) \\ \vdots \\ D(v_n) \end{bmatrix}$$

Definition 4.10. We define the *adjacency matrix* of G to be the symmetric matrix $\mathrm{Adj}(G) \in \mathbb{Z}_{\geq 0}^{n \times n}$ defined by

$$\mathrm{Adj}(G)_{i,j} = \#\{e \in E : \phi(e) = \{v_i, v_j\}\}$$

We define also the *firing matrix* of G to be the matrix

$$F = F(G) = \mathrm{Adj}(G) - \mathrm{diag}(\mathrm{val}(v_1), \ldots, \mathrm{val}(v_n))$$

Proposition 4.11. *For f a level map, we have that*

$$[\mathrm{div}(f)] = F \cdot [f]$$

Proof. This follows from the observation that

$$\mathrm{ord}_x(f) = \sum_{e \in E_x} f(\nu_x(e)) - \mathrm{val}(x) f(x)$$

□

Remark 4.12. Let D be a divisor. To say that D is principal is the same as saying that there exists some $\vec{u} \in \mathbb{Z}^{n \times n}$ such that

$$[D] = F \cdot \vec{u}.$$

In other words, D is principal if and only if $[D] \in \mathrm{im}(F)$.

Let F^+ be the pseudo-inverse of F. By the properties of the pseudo-inverse, FF^+ is the projection on the image of F. It follows that when D is principal,

$$FF^+[D] = [D].$$

Let g be the level map given by the vector $F^+[D]$, we deduce that $D = \mathrm{div}(g)$. So when D is principal, this gives us a way to find a level map whose associated divisor is D.

Definition 4.13. A *path* γ in G is a sequence of vertices $x_0, x_1, \ldots, x_n$ such that for each $i \in \{1, \ldots, n\}$, there is an edge in G between the vertices x_{i-1} and x_i. The *length* of the path γ is $L(\gamma) = n$.

We define the distance between two vertices x, y on the graph to be

$$d(x, y) := \inf L(\gamma).$$

Remark 4.14. The length of a path and distance between points clearly agrees with the notions defined for metric graphs.

Remark 4.15. For a given vertex x, an efficient way to find the distance of x from each other vertex is via a single pass of breadth-first search (BFS).

Definition 4.16. Let $A \subseteq V$ be a set of vertices. We say A is *connected* when for all $x, y \in A$, there exists a path from x to y contained in A.

4.2 Reduced divisors on graphs

Unless stated otherwise, we suppose G is a connected graph.

Proposition 4.17. *A divisor $D \in \mathrm{Div}(G)$ is v-reduced if and only if it is effective away from v and for all $A \subset V$ with $v \notin A$, we have that A cannot fire with respect to D.*

Proof. Suppose that D satisfies these assumptions. If $Z \subseteq \Gamma$ is a subgraph that can fire with respect to, then $\partial Z \subseteq \mathrm{supp}\, D$. But then the set of vertices $V \cap Z$ can fire on G, which implies that $v \notin V \cap Z$, and so in particular $v \notin Z$, which shows that D is v-reduced.

Reciprocally, suppose D is v-reduced. Let A be a subset of V that can fire on G. If we let Z be the subgraph spanned by A, then Z can fire. So by assumption $v \notin Z$ and so in particular $v \notin A$. □

We will now describe an algorithm that can be used to find v-reduced divisors.

Lemma 4.18. *Let $D \in \mathrm{Div}(G)$. There exists a divisor on G effective away from v that is linearly equivalent to D.*

Proof. Let $n = \max_{x \in V} d(x, v)$ and for $i \in \{0, n\}$ let

$$A_i := \{x \in V : d(v, x) \leq i\}.$$

We will proceed by induction to define D_i such that D_i is effective away from A_{n-i} and D_i is linearly equivalent to D. In particular D_n will be effective away from $A_0 = \{v\}$ and this would show the lemma.

We start with $D_0 = D$. Suppose we have found D_{i-1} for some i, then let

$$m = \min\{D_{i-1}(x) : x \in A_{n-i+1} \setminus A_{n-i}\}.$$

We set $D_i = D_{i-1} + \mathrm{div}(m \cdot CF(A_{n-i}))$, then D_i is effective away from A_{n-i}. Indeed for all $x \in A_{n-i+1} \setminus A_{n-i}$, there is at least one edge connecting x to a vertex in A_{n-i} and so by firing A_{n-i}, m chips are moved to x along this edge. In particular $D_i(x) \geq 0$. Furthermore, it is clear that the vertices in $V \setminus A_{n-i+1}$ are not affected, hence D_i is indeed effective away from A_{n-i}. □

Proposition 4.19. *Let $D \in \mathrm{Div}(G)$. There exists a unique v-reduced divisor on G linearly equivalent to D.*

Proof. By Lemma 4.18, we may assume that D is effective away from v.

Suppose D is not v-reduced. Let A be the maximal subset of $\Gamma \setminus \{v\}$ that can fire. There is a unique such subset, since when A_1, A_2 are two subsets that can fire, then firing $A_1 \cup A_2$ corresponds to the chip-firing move $CF(A_1) \oplus CF(A_2)$. Since $R(D)$ is a tropical semi-module, we deduce that $A_1 \cup A_2$ can fire. Let m be maximal for the property that $D + \mathrm{div}(m \cdot CF(A))$ is effective away from v and set $D' = D + \mathrm{div}(m \cdot CF(A))$. If D' is not v-reduced, we may repeat this procedure until we obtain a v-reduced divisor.

It remains to check that this algorithm terminates. Define as earlier

$$B_i := \{x \in V : d(v, x) \leq i\}.$$

We define the partial order $\prec$ on the set of divisors on G given by $D_1 \prec D_2$ if and only if there exists some $n \in \mathbb{N}$ such that $\deg D_1|_{B_i} = \deg D_2|_{B_i}$ for $i < n$ and $\deg D_1|_{B_n} < \deg D_2|_{B_n}$.

Let D and D' as before, we claim that $D \prec D'$. Indeed, let A be the subset from before and let n maximal for the property that $A \cap B_n = \emptyset$. It follows that there is some $x \in B_{n+1} \cap A$ and so by the definition of distance there is some $y \in B_n$ that is adjacent to x. Firing A will move at least one chip from x to y and since no chip is moved away from B_n when firing A, we deduce that $\deg D|_{B_n} < \deg D|_{B_n}$. Clearly, firing A does not affect the vertices in B_i for $i < n$, so we deduce that $D \prec D'$. Since $D \prec D'$ and there is only a finite number of divisors of a given degree, we deduce that the algorithm has to terminate.

Unicity then follows from the unicity of v-reduced divisors on metric graphs. □

Remark 4.20. We can find the set A from the above proof using *Dhar's burning algorithm.*

Start by distributing chips on the vertices, according to the divisor D, and light a fire at v. Then repeat the following steps:

1. If there is an unburned edge adjacent to a burned vertex, burn it. Otherwise terminate the algorithm.
2. If there is at least one chip on the vertex adjacent to the corresponding edge, remove one chip. Otherwise burn the vertex.

3. Go back to step 1.

The set A of unburned vertices is our desired set.

Remark 4.21. It is clear from the proof of the proposition that we may also easily find the corresponding level map such that $D + \mathrm{div}(f)$ is v-reduced. When $D \sim D'$, and we know that $D + \mathrm{div}(f)$ and $D' + \mathrm{div}(g)$ are v-reduced, then in fact $D + \mathrm{div}(f) = D' + \mathrm{div}(g)$ by unicity. It follows that $D - D' = \mathrm{div}(g - f)$ and so this gives us another way to find level maps that relate two linearly equivalent divisors.

Corollary 4.21.1. *Suppose Γ is a graph and G is a model with equal edge lengths. If $D \in \mathrm{Div}(\Gamma)$ is a divisor supported on the vertices of G, and v is a vertex, then the v-reduced divisor linearly equivalent to D is also supported on the vertices.*

Proof. This follows from Proposition 4.19 and the unicity of v-reduced divisors on metric graphs. □

Proposition 4.22. *Let D be a v-reduced divisor on G and let f be a level map such that $D + \mathrm{div}(f) \geq 0$. Let $v \in A \subset V$ and*

$$m = \max\{f(x) : x \in A, \deg_A^{out}(x) > 0\},$$

then $f(V \setminus A) \leq m$.

Proof. Then let B the subset of $V \setminus A$ on which $f|_{V\setminus A}$ attains its maximum and suppose for the sake of contradiction that $f(B) > m$. Let $Z \subseteq \Gamma$ be the subgraph spanned by B. Then Z is a local maximum for f. Indeed, for any vertex $x \notin B$ adjacent to a point in B, we have that either $x \in V \setminus A$, in which case $f(x) < f(B)$ by assumption, or $x \in A$. In this case, since we assumed x is adjacent to a point in B, we have that $\deg_A^{\mathrm{out}}(x) > 0$ and hence by definition of m, $f(x) \leq m < f(B)$. So Z is indeed a local maximum but this implies by Proposition 2.43 that $v \in Z$, a contradiction as $v \in A$. □

Remark 4.23. This is the same as saying that the subgraph spanned by the vertices in A is connected.

Proposition 4.24. *Let D be a v-reduced divisor on G and f a level map such that $D + \mathrm{div}(f) \geq 0$. For any $x \in V$, there exists a non-decreasing path from x to v.*

Proof. We will proceed by induction on $f(v) - f(x)$. If $f(v) = f(x)$, by Corollary 2.43.2, $f^{-1}(f(v))$ is connected and so there exists a path from x to v.

If $f(v) > f(x)$, then we know that the set $f^{-1}([f(x), \infty))$ is connected by Corollary 2.43.2. Let y be the closest vertex to x such that $f(y) > f(x)$ and let $x = x_0, x_1, \ldots, x_n = y$ be a minimal path from x to y. This path is clearly non-decreasing as $f(x_i) = f(x)$ for all $i < n$ by assumption. By induction, there exists a non-decreasing path from y to v. By concatenating these paths we obtain a non-decreasing path from x to v. □

References

[Ami12] Omid Amini. *Reduced Divisors and Embeddings of Tropical Curves*. 2012. arXiv: 1007.5364 [math.CO].

[Bak07] Matthew Baker. *Specialization of linear systems from curves to graphs*. 2007. arXiv: math/0701075 [math.NT].

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A Course in Metric Geometry*. American Mathematical Society, 2001.

[BN07] Matthew Baker and Serguei Norine. *Riemann-Roch and Abel-Jacobi theory on a finite graph*. 2007. arXiv: math/0608360 [math.CO].

[CR93] Ted CHINBURG and Robert Rumely. In: *Journal für die reine und angewandte Mathematik* 1993.434 (1993), pp. 1–44. DOI: doi:10.1515/crll.1993.434.1. URL: https://doi.org/10.1515/crll.1993.434.1.

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". In: *Publications mathématiques de l'IHÉS* 36.1 (Jan. 1969), pp. 75–109. ISSN: 1618-1913. DOI: 10.1007/bf02684599. URL: http://dx.doi.org/10.1007/BF02684599.

[FJP23] Gavril Farkas, David Jensen, and Sam Payne. *The Kodaira dimensions of* $\overline{\mathcal{M}}_{22}$ *and* $\overline{\mathcal{M}}_{23}$. 2023. arXiv: 2005.00622 [math.AG].

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer New York, 1977. ISBN: 9781475738490. DOI: 10.1007/978-1-4757-3849-0. URL: http://dx.doi.org/10.1007/978-1-4757-3849-0.

[HMY09] Christian Haase, Gregg Musiker, and Josephine Yu. *Linear Systems on Tropical Curves*. 2009. arXiv: 0909.3685 [math.AG].

[JP22] David Jensen and Sam Payne. *Tropical Linear Series and Tropical Independence*. 2022. arXiv: 2209.15478 [math.AG].

[Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, 2002.

[LU17] Bo Lin and Martin Ulirsch. "Towards a Tropical Hodge Bundle". In: *Combinatorial Algebraic Geometry*. Springer New York, 2017, pp. 353–368. ISBN: 9781493974863. DOI: 10.1007/978-1-4939-7486-3_16. URL: http://dx.doi.org/10.1007/978-1-4939-7486-3_16.

[Luo18] Ye Luo. *Idempotent Analysis, Tropical Convexity and Reduced Divisors*. 2018. arXiv: 1808.01987 [math.CO].

[Mug21] Delio Mugnolo. *What is actually a metric graph?* 2021. arXiv: 1912.07549 [math.CO].

[MUW17] Martin Moeller, Martin Ulirsch, and Annette Werner. *Realizability of tropical canonical divisors*. 2017. arXiv: 1710.06401 [math.AG].

[RS21] Felix Röhrle and Johannes Schwab. *Realizability of tropical pluri-canonical divisors*. 2021. arXiv: 2109.03579 [math.AG].

[Viv13] Filippo Viviani. *Tropicalizing vs Compactifying the Torelli morphism*. 2013. arXiv: 1204.3875 [math.AG].