Problem_1

The following problem is from both the 2023 AMC 10B #1 and 2023 AMC 12B #1, so both problems redirect to this page.

Problem

Mrs. Jones is pouring orange juice into four identical glasses for her four sons. She fills the first three glasses completely but runs out of juice when the fourth glass is only $\frac{1}{3}$ full. What fraction of a glass must Mrs. Jones pour from each of the first three glasses into the fourth glass so that all four glasses will have the same amount of juice?

$\textbf{(A) } \frac{1}{12} \qquad \textbf{(B) } \frac{1}{4} \qquad \textbf{(C) } \frac{1}{6} \qquad \textbf{(D) } \frac{1}{8} \qquad \textbf{(E) } \frac{2}{9}$

Solution 1

The first three glasses each have a full glass. Let's assume that each glass has "1 unit" of juice. It won't matter exactly how much juice everyone has because we're dealing with ratios, and that wouldn't affect our answer. The fourth glass has a glass that is one third. So the total amount of juice will be $1+1+1+\frac{1}{3}=\frac{10}{3}$. If we divide the total amount of juice by 4, we get $\frac{5}{6}$, which should be the amount of juice in each glass. This means that each of the first three glasses will have to contribute $1-\frac{5}{6}=\boxed{\frac{1}{6}}$ to the fourth glass.

~Sir Ian Seo the Great & lprado

Solution 2 (unnecessary numerical values)

Given that the first three glasses are full and the fourth is only $\frac{1}{3}$ full, let's represent their contents with a common denominator, which we'll set as 6. This makes the first three glasses $\frac{6}{6}$ full, and the fourth glass $\frac{2}{6}$ full.

To equalize the amounts, Mrs. Jones needs to pour juice from the first three glasses into the fourth. Pouring $\frac{1}{6}$ from each of the first three glasses will make them all $\frac{5}{6}$ full. Thus, all four glasses will have the same amount of juice. Therefore, the answer is $\boxed{\textbf{(C) } \frac{1}{6}}$.

~Ishaan Garg

Solution 3

We let x denote how much juice we take from each of the first 3 children and give to the 4th child.

We can write the following equation: $1-x=\frac{1}{3}+3x$, since each value represents how much juice each child (equally) has in the end. (Each of the first three children now have $1-x$ juice, and the fourth child has $3x$ more juice on top of their initial $\frac{1}{3}$.)

Solving, we see that $x=\boxed{\textbf{(C) } \frac{1}{6}}$.

~Technodoggo

Problem_2

The following problem is from both the 2023 AMC 10B #2 and 2023 AMC 12B #2, so both problems redirect to this page.

Problem

Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by 20% on every pair of shoes. Carlos also knew that he had to pay a 7.5% sales tax on the discounted price. He had $\$43$ dollars. What is the original (before discount) price of the most expensive shoes he could afford to buy?

$\textbf{(A) }\$46\qquad\textbf{(B) }\$50\qquad\textbf{(C) }\$48\qquad\textbf{(D) }\$47\qquad\textbf{(E) }\$49$

Solution 1 (easy)

We can create the equation:

$$0.8x \cdot 1.075 = 43$$

using the information given. This is because x, the original price, got reduced by 20%, or multiplied by 0.8, and it also got multiplied by 1.075 on the discounted price. Solving that equation, we get

$$\frac{4}{5} \cdot x \cdot \frac{43}{40} = 43$$

$$\frac{4}{5} \cdot x \cdot \frac{1}{40} = 1$$

$$\frac{1}{5} \cdot x \cdot \frac{1}{10} = 1$$

$$x = \boxed{50}$$

~lprado

Solution 2 (One-Step Equation)

The discounted shoe is 20% off the original price. So that means $1-0.2=0.8$. There is also a 7.5% sales tax charge, so $0.8*1.075=0.86$. Now we can set up the equation $0.86x=43$, and solving that we get $x=\boxed{\textbf{(B) }50}$ ~ kabbybear

Solution 3

Let the original price be x dollars. After the discount, the price becomes $80\%x$ dollars. After tax, the price becomes $80\% \times (1+7.5\%) = 86\%x$ dollars. So, $43=86\%x$, $x=\boxed{\textbf{(B) }\$50}$.

~Mintylemon66

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~ Minor tweak:Multpi12
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Solution 4

We can assign a variable c to represent the original cost of the shoes. Next, we set up the equation $80\% \cdot 107.5\% \cdot c = 43$. We can solve this equation for c and get $\boxed{\textbf{(B) }\$50}$.

~vsinghminhas

Solution 5 (Intuition and Guessing)

We know the discount price will be 5/4, and 0.075 is equal to 3/40. So we look at answer choice $\textbf{(B)}$, see that the discoutn price will be 40, and with sales tax applied it will be 43, so the answer choice is $\boxed{\textbf{(B) }\$50}$.

Solution 6 (Not really a solution, DON'T DO THIS ON A REAL TEST)

Open up a coding IDE and use Python to solve this problem. Python code:

```
budget = 43.0
discount_percentage = 20.0
sales_tax_percentage = 7.5
discounted_price = budget / 1.075 / 0.8
print(f"${discounted_price:.2f}")
```

~Ishaan Garg

Problem_3

The following problem is from both the 2023 AMC 10B #3 and 2023 AMC 12B #3, so both problems redirect to this page.

Problem

A $3-4-5$ right triangle is inscribed in circle A, and a $5-12-13$ right triangle is inscribed in circle B. What is the ratio of the area of circle A to the area of circle B?

$\textbf{(A)}\ \frac{9}{25} \qquad \textbf{(B)}\ \frac{1}{9} \qquad \textbf{(C)}\ \frac{1}{5} \qquad \textbf{(D)}\ \frac{25}{169} \qquad \textbf{(E)}\ \frac{4}{25}$

Solution 1

Because the triangle are right triangles, we know the hypotenuses are diameters of circles A and B. Thus, their radii are 2.5 and 6.5 (respectively). Square the two numbers and multiply π to get 6.25π and 42.25π as the areas of the circles. Multiply 4 on both numbers to get 25π and 169π. Cancel out the π, and lastly, divide, to get your answer $= \boxed{\textbf{(D)}\ \frac{25}{169}}$.

~Failure.net

Solution 2

Since the arc angle of the diameter of a circle is 90 degrees, the hypotenuse of each these two triangles is respectively the diameter of circles A and B.

Therefore the ratio of the areas equals the radius of circle A squared : the radius of circle B squared $= 0.5\times$ the diameter of circle A, squared : $0.5\times$ the diameter of circle B, squared $=$ the diameter of circle A, squared: the diameter of circle B, squared $= \boxed{\textbf{(D)}\ \frac{25}{169}}$.

~Mintylemon66

Solution 3

The ratio of areas of circles is the same as the ratios of the diameters squared (since they are similar figures). Since this is a right triangle the hypotenuse of each triangle will be the diameter of the circle. This yields the expression $\frac{5^2}{13^2} = \boxed{\textbf{(D)}\ \frac{25}{169}}$.

~vsinghminhas

Problem_4

The following problem is from both the 2023 AMC 10B #4 and 2023 AMC 12B #4, so both problems redirect to this page.

Problem

Jackson's paintbrush makes a narrow strip with a width of 6.5 millimeters. Jackson has enough paint to make a strip 25 meters long. How many square centimeters of paper could Jackson cover with paint?

$\textbf{(A) } 162,500 \qquad \textbf{(B) } 162.5 \qquad \textbf{(C) } 1,625 \qquad \textbf{(D) } 1,625,000 \qquad \textbf{(E) } 16,250$

Solution 1

6.5 millimeters is equal to 0.65 centimeters. 25 meters is 2500 centimeters. The answer is 0.65×2500, so the answer is $\boxed{\textbf{(C) 1,625}}$.

~Failure.net

Solution 2 (Standard Form)

6.5 millimeters can be represented as 65×10^{-2} centimeters. 25 meters is 25×10^2 centimeters. Multiplying out these results in $(65 \times 10^{-2}) \times (25 \times 10^2)$, which is 65×25 making the answer $\boxed{\textbf{(C) 1,625}}$.

~darrenn.cp

Problem_5

Problem

Maddy and Lara see a list of numbers written on a blackboard. Maddy adds 3 to each number in the list and finds that the sum of her new numbers is 45. Lara multiplies each number in the list by 3 and finds that the sum of her new numbers is also 45. How many numbers are written on the blackboard?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 6 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 9$

Solution

Let there be n numbers in the list of numbers, and let their sum be S. Then we have the following

$$S + 3n = 45$$

$$3S = 45$$

From the second equation, $S = 15$. So, $15 + 3n = 45 \Rightarrow n = \boxed{\textbf{(A)}\ 10}$.

~Mintylemon66 (formatted atictacksh)

Solution 2

Let $x_1, x_2, x_3, \ldots, x_n$ where x_n represents the nth number written on the board. Lara's multiplied each number by 3, so her sum will be $3x_1 + 3x_2 + 3x_3 + \ldots + 3x_n$. This is the same as $3 \cdot (x_1 + x_2 + x_3 + \ldots + x_n)$. We are given this quantity is equal to 45, so the original numbers add to $\frac{45}{3} = 15$. Maddy adds 3 to each of the n terms which yields, $x_1 + 3 + x_2 + 3 + x_3 + 3 + \ldots + x_n + 3$. This is the same as the sum of the original series plus $3 \cdot n$. Setting this equal to 45, $15 + 3n = 45 \Rightarrow n = \boxed{\textbf{(A)}\ 10}$.

~vsinghminhas

Problem_6

Problem

Let $L_1 = 1, L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. How many terms in the sequence $L_1, L_2, L_3, ..., L_{2023}$ are even?

$\textbf{(A) } 673 \qquad \textbf{(B) } 1011 \qquad \textbf{(C) } 675 \qquad \textbf{(D) } 1010 \qquad \textbf{(E) } 674$

Solution 1

We calculate more terms:

$$1, 3, 4, 7, 11, 18, ...$$

We find a pattern: if n is a multiple of 3, then the term is even, or else it is odd. There are $\lfloor \frac{2023}{3} \rfloor = \boxed{\textbf{(E) } 674}$ multiples of 3 from 1 to 2023.

~Mintylemon66

Solution 2

Like in the other solution, we find a pattern, except in a more rigorous way. Since we start with 1 and 3, the next term is 4.

We start with odd, then odd, then (the sum of odd and odd) even, (the sum of odd and even) odd, and so on. Basically the pattern goes: odd, odd, even, odd odd, even, odd, odd even...

When we take $\frac{2023}{3}$ we get 674 with a remainder of one. So we have 674 full cycles, and an extra odd at the end.

Therefore, there are $\boxed{\textbf{(E) } 674}$ evens.

~e_is_2.71828

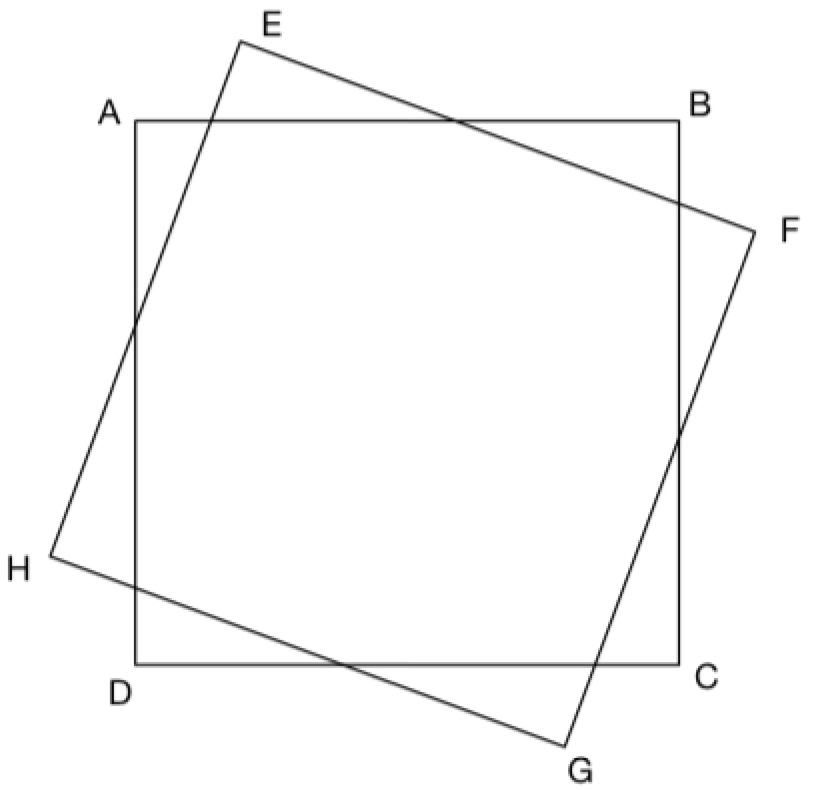
~SpreadTheMathLove

~e_is_2.71828

Problem_7

Problem

Square $ABCD$ is rotated 20° clockwise about its center to obtain square $EFGH$, as shown below.



What is the degree measure of $\angle EAB$?

$\textbf{(A)}\ 24^{\circ} \qquad \textbf{(B)}\ 35^{\circ} \qquad \textbf{(C)}\ 30^{\circ} \qquad \textbf{(D)}\ 32^{\circ} \qquad \textbf{(E)}\ 20^{\circ}$

Solution 1

First, let's call the center of both squares I. Then, $\angle AIE = 20$, and since $\overline{EI} = \overline{AI}$, $\angle AEI = \angle EAI = 80$. Then, we know that AI bisects angle $\angle DAB$, so $\angle BAI = \angle DAI = 45$. Subtracting 45 from 80, we get $\boxed{\text{(B)} 35}$

~jonathanzhou18

Solution 2

First, label the point between A and H point O and the point between A and H point P. We know that $\angle AOP = 20$ and that $\angle A = 90$. Subtracting 20 and 90 from 180, we get that $\angle APO$ is 70. Subtracting 70 from 180, we get that $\angle OPB = 110$. From this, we derive that $\angle APE = 110$. Since triangle APE is an isosceles triangle, we get that $\angle EAP = (180-110)/2 = 35$. Therefore, $\angle EAB = 35$. The answer is $\boxed{\text{(B)} 35}$.

~yourmomisalosinggame (a.k.a. Aaron)

Solution 3

Call the center of both squares point O, and draw circle O such that it circumscribes the squares. $\angle EOF = 90$ and $\angle BOF = 20$, so $\angle EOB = 70$. Since $\angle EAB$ is inscribed in arc $\overset{\frown}{EB}$, $\angle EAB = 70/2 = \boxed{\textbf{(B) } 35}$.

~hpotter2021

Solution 4

Draw EA: we want to find $\angle EAB$. Call P the point at which AB and EH intersect. Reflecting $\triangle APE$ over EA, we have a parallelogram. Since $\angle EPB = 70^\circ$, angle subtraction tells us that two of the angles of the parallelogram are 110°. The other two are equal to $2\angle EAB$ (by properties of reflection).

Since angles on the transversal of a parallelogram sum to 180°, we have $2\angle EAB + 110 = 180$, yielding $\angle EAB = \boxed{\textbf{(B)}\ 35}$

-Benedict T (countmath1)

Problem_8

Problem

What is the units digit of $2022^{2023} + 2023^{2022}$?

$\textbf{(A) }7\qquad\textbf{(B) }1\qquad\textbf{(C) }9\qquad\textbf{(D) }5\qquad\textbf{(E) }3$

Solution

$2022^{2023} + 2023^{2022} \equiv 2^3 + 3^2 \equiv 17 \equiv 7$ (mod 10). $\boxed{\text{A}}$ ~andliu766

Solution 2 (Digit Cycles)

Note that the units digit will be the same regardless of the tens, hundreds, and thousands digits, so we can simplify this problem to finding the last digit of $2^{2023} + 3^{2022}$. We can find the units digit of 2^{2023}, by listing the units digits of the first few powers of two, and trying to find a pattern.

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 6 + 10$

$2^5 = 2 + 30$

$2^6 = 4 + 60$

As we can see the units digits of powers of two repeat after every four iterations. Now we know the units digit of 2^{2020} is 6 and the units digit of $2^{2023} \Rightarrow 2^3 \cdot 2^{2020} \Rightarrow 6 \cdot 8 \Rightarrow 8$. Similarly we can find the last digits of powers of three repeat after every four, so the units digit of 3^{2022} is $1 \cdot 3^2 = 9$. Adding these together, the ones digit is the same as the ones digit of $9 + 8$ which is 7. $\boxed{\text{(A) }7}$

~vsinghminhas

Solution 3

When looking at the units digit patterns of the powers of 2, we see that

$2^1 =$, units digit 2

$2^2 =$, units digit 4

$2^3 =$, units digit 8

$2^4 =$, units digit 6

$2^5 =$, units digit 2

And the pattern repeats. This pattern will apply for the powers of 2022 as well, since the units digit of 2022 is 2. We can find the pattern for the powers of 3 too. The pattern follows with units digits, $3, 9, 7, 1, 3, 9,$ ... Similarly, the units digit of 2023 will follow the same pattern as the powers of 3.

Both of these powers cycle in groups of 4. When diving 2023 by 4, we get 505 remainder 3, meaning 505 complete cycles; or the power being a multiple of 4, 505 times, and 3 extra. So the units digit of 2022^{2023} is 8. 2022 divided by 4 is 505 reminder 2, which means 505 complete cycles, or the power being a multiple of 4, 505 times, and 2 extra. So the units digit of 2023^{2022} is 9.

We only need to find the units digit in the end, so we just add those 2 already found units digits, to get a new units digit of 7. Therefore the answer is $\boxed{\text{(A) }7}$

~mk

Problem_9

Problem

The numbers 16 and 25 are a pair of consecutive postive squares whose difference is 9. How many pairs of consecutive positive perfect squares have a difference of less than or equal to 2023?

$\text{(A) } 674 \qquad \text{(B) } 1011 \qquad \text{(C) } 1010 \qquad \text{(D) } 2019 \qquad \text{(E) } 2017$

Solution 1

Let x be the square root of the smaller of the two perfect squares. Then, $(x+1)^2 - x^2 = x^2 + 2x + 1 - x^2 = 2x + 1 \le 2023$. Thus, $x \le 1011$. So there are $\boxed{\text{(B)} 1011}$ numbers that satisfy the equation.

~andliu766

A very similar solution offered by ~darrenn.cp and ~DarkPheonix has been combined with Solution 1.

Minor corrections by ~milquetoast

Note from ~milquetoast: Alternatively, you can let x be the square root of the larger number, but if you do that, keep in mind that $x = 1$ must be rejected, since $(x-1)$ cannot be 0.

Solution 2

The smallest number that can be expressed as the difference of a pair of consecutive positive squares is 3, which is $2^2 - 1^2$. The largest number that can be expressed as the difference of a pair of consecutive positive squares that is less than or equal to 2023 is 2023, which is $1012^2 - 1011^2$. Since these numbers are in the form $(x+1)^2 - x^2$, which is just $2x+1$. These numbers are just the odd numbers from 3 to 2023, so there are $[(2023-3)/2] + 1 = 1011$ such numbers. The answer is $\boxed{\text{(B)} 1011}$.

~Aopsthedude

Problem_10

The following problem is from both the 2023 AMC 10B #10 and 2023 AMC 12B #5, so both problems redirect to this page.

Problem

You are playing a game. A 2×1 rectangle covers two adjacent squares (oriented either horizontally or vertically) of a 3×3 grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

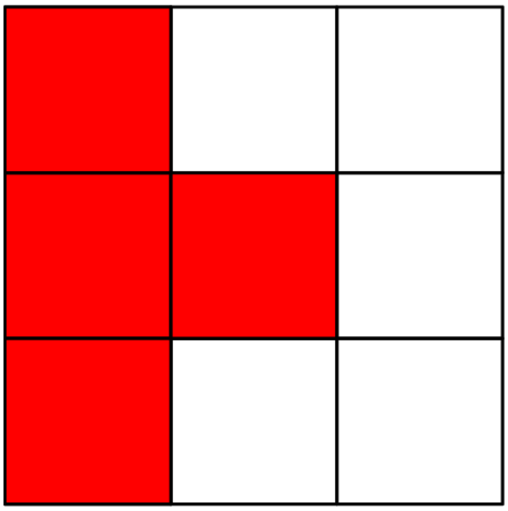
$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 5 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 8 \qquad \textbf{(E)}\ 6$

Solution 1

Notice that the 3×3 square grid has a total of 12 possible 2×1 rectangles.

Suppose you choose the middle square for one of your turns. The middle square is covered by 4 rectangles, and each of the remaining 8 squares is covered by a maximum of 2 uncounted rectangles. This means that the number of turns is at least $1+\frac{12-4}{2}=1+4=5.$

Now suppose you don't choose the middle square. The squares on the middle of the sides are covered by at most 3 uncounted rectangles, and the squares on the corners are covered by at most 2 uncounted rectangles. In this case, we see that the least number of turns needed to account for all 12 rectangles is $12 \div 3 = 4.$ To prove that choosing only side squares indeed does cover all 12 rectangles, we need to show that the 3 rectangles per square that cover each side square do not overlap. Drawing the rectangles that cover one square, we see they form a T shape and they do not cover any other side square. Hence, our answer is $4.$



Solution 2

First, note that since the rectangle covers 2 squares, we only need to guess squares that are not adjacent to any of our other guesses. To minimize the amount of guesses, each of our guessed squares should try to touch another guess on one vertex and one vertex only. There are only two ways to do this: one with 5 guesses, and one with 4. Since the problem is asking for the minimum number, the answer is $\boxed{\text{(C) } 4}$.

~yourmomisalosinggame (a.k.a. Aaron)

Solution 3

Since the hidden rectangle can only hide two adjacent squares, we may think that we eliminate 8 squares and we're done, but think again. This is the AMC 10, so there must be a better solution (also note that every other solution choice is below 8 so we're probably not done) So, we think again, we notice that we haven't used the adjacent condition, and then it clicks. If we eliminate the four squares with only one edge on the boundary of the 9x9 square. We are left with 5 diagonal squares, since our rectangle cant be diagonal, we can ensure that we find it in 4 moves. So our answer is : $\boxed{\text{(C) } 4}$

~arrowskyknight22

Solution 4 (checkerboard)

The 3x3 grid can be colored like a checkerboard with alternating blacks and whites. Let the top left square be white, and the rest of the squares alternate colors.

Each 2×1 rectangle will always cover exactly 1 white square and 1 black square. You can ensure that at least one of your guessed squares is covered by the rectangle by guessing either all the white squares only (5 turns) or all the black squares only (4 turns).

In our case, guessing all the black squares takes $\boxed{4}$ turns, which is less than guessing all the white squares.

~ CherryBerry

Solution 5 (Logic)

We realize that every 2×1 rectangle must contain an edge and no more than one edge. There are a total of four edges so the answer is $\boxed{\text{(C) } 4}$. ~darrenn.cp

See Also

Problem_11

Problem

Suzanne went to the bank and withdrew $\$800$. The teller gave her this ammount using $\$20$ bills, $\$50$ bills, and $\$100$ bills, with at least one of each denomination. How many different collections of bills could Suzanne have received?

$\textbf{(A) } 45 \qquad \textbf{(B) } 21 \qquad \text{(C) } 36 \qquad \text{(D) } 28 \qquad \text{(E) } 32$

Solution 1

We let the number of $\$20$, $\$50$, and $\$100$ bills be $a, b,$ and $c,$ respectively.

We are given that $20a + 50b + 100c = 800.$ Dividing both sides by 10, we see that $2a + 5b + 10c = 80.$

We divide both sides of this equation by 5: $\frac{2}{5}a + b + 2c = 16.$ Since $b + 2c$ and 16 are integers, $\frac{2}{5}a$ must also be an integer, so a must be divisible by 5. Let $a = 5d,$ where d is some positive integer.

We can then write $2 \cdot 5d + 5b + 10c = 80.$ Dividing both sides by 5, we have $2d + b + 2c = 16.$ We divide by 2 here to get $d + \frac{b}{2} + c = 8.$ $d + c$ and 8 are both integers, so $\frac{b}{2}$ is also an integer. b must be divisible by 2, so we let $b = 2e.$

We now have $2d + 2e + 2c = 16 \implies d + e + c = 8.$ Every substitution we made is part of a bijection (i.e. our choices were one-to-one); thus, the problem is now reduced to counting how many ways we can have $d, e,$ and c such that they add to 8.

We still have another constraint left, that each of $d, e,$ and c must be at least 1. For $n \in \{d, e, c\}$, let $n' = n - 1.$ We are now looking for how many ways we can have $d' + e' + c' = 8 - 1 - 1 - 1 = 5.$

We use a classic technique for solving these sorts of problems: stars and bars. We have 5 stars and 3 groups, which implies 2 bars. Thus, the total number of ways is $\binom{5+2}{2} = \binom{7}{2} = 21.$

~Technodoggo ~minor edits by lucaswujc

Solution 2

First, we note that there can only be an even number of 50 dollar bills.

Next, since there is at least one of each bill, we find that the amount of 50 dollar bills is between 2 and 12. Doing some casework, we find that the amount of 100 dollar bills forms an arithmetic sequence: $6+5+4+3+2+1$.

Adding these up, we get 21.

~yourmomisalosinggame (a.k.a. Aaron)

Solution 3

Denote by x, y, z the amount of \$20 bills, \$50 bills and \$100 bills, respectively. Thus, we need to find the number of tuples (x, y, z) with $x, y, z \in \mathbb{N}$ that satisfy

$$20x + 50y + 100z = 800.$$

First, this equation can be simplified as

$$2x + 5y + 10z = 80.$$

Second, we must have $5|x$. Denote $x = 5x'$. The above equation can be converted to

$$2x' + y + 2z = 16.$$

Third, we must have $2|y$. Denote $y = 2y'$. The above equation can be converted to

$$x' + y' + z = 8.$$

Denote $x'' = x' - 1$, $y'' = y' - 1$ and $z'' = z - 1$. Thus, the above equation can be written as

$$x'' + y'' + z'' = 5.$$

Therefore, the number of non-negative integer solutions (x'', y'', z'') is $\binom{5+3-1}{3-1} = \boxed{\textbf{(B) } 21}$.

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

Solution 4

To start, we simplify things by dividing everything by 10, the resulting equation is $2x + 5y + 10z = 80$, and since the problem states that we have at least one of each, we simplify this to $2x + 5y + 10z = 63$. Note that since the total is odd, we need an odd number of 5 dollar bills. We proceed using casework.

Case 1: One 5 dollar bill

$2x + 10z = 58$, we see that $10z$ can be $10, 20, 30, 40, 50$ or 0. 6 Ways

Case 2: Three 5 dollar bills

$2x + 10z = 48$, like before we see that $10z$ can be $0, 10, 20, 30, 40$, so 5 way.

Now we should start to see a pattern emerges, each case there is 1 less way to sum to 80, so the answer is just $\frac{6(6+1)}{2}$, 21 or (B)

~andyluo

Solution 5

We notice that each \$100 can be split 3 ways: 5 \$20 dollar bills, 2 \$50 dollar bills, or 1 \$100 dollar bill.

There are 8 of these \$100 chunks in total--take away 3 as each split must be used at least once.

Now there are five left--so we use stars and bars.

5 chunks, 3 categories or 2 bars. This gives us $\binom{5+2}{2} = \boxed{\textbf{(B) } 21}$

~not_slay

Solution 6 (kind of bash)

Casework on if there is one 100. There are 6 ways. Casework on if there are two 100s. There are 5 ways. Notice that this continues all the way until there are 6 100s. Our sum is $1 + 2 + 3 + \ldots + 5 + 6 = \frac{6 \cdot 7}{2} = \frac{42}{2} = 21$, or $\boxed{B}$. ~MC413551

Problem_12

The following problem is from both the 2023 AMC 10B #12 and 2023 AMC 12B #6, so both problems redirect to this page.

Problem

When the roots of the polynomial

$$P(x) = (x-1)^1(x-2)^2(x-3)^3 \cdots (x-10)^{10}$$

are removed from the number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is $P(x)$ positive?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 5 \qquad \textbf{(D)}\ 6 \qquad \textbf{(E)}\ 7$

Solution 1

$P(x)$ is a product of $(x - r_n)$ or 10 terms. When $x < 1$, all terms are < 0, but $P(x) > 0$ because there is an even number of terms. The sign keeps alternating $+, -, +, -, ..., +$. There are 11 intervals, so there are $\boxed{\textbf{6}}$ positives and 5 negatives. $\boxed{\textbf{(D) 6}}$

~$\textbf{Techno}\textcolor{red}{doggo}$

(This solution makes no sense, it is correct by luck) ~SpencerD.

(The method is incorrect, but the answer is correct by chance. The actual alternating sign is $-, +, +, -, -, +, +, -, -, +, +$ for all 11 intervals. We count 6 intervals being positive which is our answer, $\boxed{\textbf{(D) 6}}$.) ~Bread10

Solution 2

Denote by I_k the interval $(k-1, k)$ for $k \in \{2, 3, \cdots, 10\}$ and I_1 the interval $(-\infty, 1)$.

Therefore, the number of intervals that $P(x)$ is positive is

$$\begin{aligned} 1 + \sum_{i=1}^{10} \mathbb{I} \left\{ \sum_{j=i}^{10} j \text{ is even} \right\} & = 1 + \sum_{i=1}^{10} \mathbb{I} \left\{ \frac{(i+10)(11-i)}{2} \text{ is even} \right\} \\ & = 1 + \sum_{i=1}^{10} \mathbb{I} \left\{ \frac{-i^2 + i + 110}{2} \text{ is even} \right\} \\ & = 1 + \sum_{i=1}^{10} \mathbb{I} \left\{ \frac{i^2 - i}{2} \text{ is odd} \right\} \\ & = \boxed{\textbf{(D) 6}}. \end{aligned}$$

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

Solution 3

The roots of the factorized polynomial are intervals from numbers 1 to 10. We take each interval as being defined as the number behind it. To make the function positive, we need to have an even number of negative expressions. Real numbers raised to even powers are always positive, so we only focus on $x-1$, $x-3$, $x-5$, $x-7$, and $x-9$. The intervals 1 and 2 leave 4 negative expressions, so they are counted. The same goes for intervals 5, 6, 9, and 10. Intervals 3 and 4 leave 3 negative expressions and intervals 7 and 8 leave 1 negative expression. The solution is the number of intervals which is $\boxed{\textbf{(D) 6}}$.

~darrenn.cp ~DarkPheonix

Solution 4

We can use the turning point behavior at the roots of a polynomial graph to find out the amount of intervals that are positive.

First, we evaluate any value on the interval $(-\infty, 1)$. Since the degree of $P(x)$ is $1+2+...+9+10$ = $\frac{10 \times 11}{2}$ = 55, and every term in P(x) is negative, multiplying 55 negatives gives a negative value. So $(-\infty, 0)$ is a negative interval.

We know that the roots of P(x) are at $1, 2, ..., 10$. When the degree of the term of each root is odd, the graph of P(x) will pass through the graph and change signs, and vice versa. So at $x=1$, the graph will change signs; at $x=2$, the graph will not, and so on.

This tells us that the interval $(1,2)$ is positive, $(2,3)$ is also positive, $(3,4)$ is negative, $(4,5)$ is also negative, and so on, with the pattern being $+,+,-,-,+,+,-,-,...$.

The positive intervals are therefore $(1,2)$, $(2,3)$, $(5,6)$, $(6,7)$, $(9,10)$, and $(10,\infty)$, for a total of $\boxed{\textbf{(D) } 6}$.

~nm1728

Solution 5

The expressions to the power of even powers are always positive, so we don't need to care about those. We only need to care about $(x-1)^1(x-3)^3(x-5)^5(x-7)^7(x-9)^9$. We need 0, 2, or 4 of the expressions to be negative. The 9 through 10 interval and 10 plus interval make all of the expressions positive. The 5 through 6 and 6 through 7 intervals make two of the expressions negative. The 1 through 2 and 2 through 3 intervals make four of the expressions negative. There are $\boxed{\textbf{(D) } 6}$ intervals.

~Aopsthedude

See Also

Problem_13

The following problem is from both the 2023 AMC 10B #13 and 2023 AMC 12B #9, so both problems redirect to this page.

Problem

What is the area of the region in the coordinate plane defined by

$||x|-1|+||y|-1|\leq 1$?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 8 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 15 \qquad \textbf{(E)}\ 12$

Solution 1

First consider, $|x-1|+|y-1| \leq 1.$ We can see that it's a square with radius 1 (diagonal $\sqrt{2}$). The area of the square is $\sqrt{2}^2 = 2.$

Next, we add one more absolute value and get $|x-1|+||y|-1| \leq 1.$ This will double the square reflecting over x-axis.

So now we got 2 squares.

Finally, we add one more absolute value and get $||x|-1|+||y|-1| \leq 1.$ This will double the squares reflecting over y-axis.

In the end, we got 4 squares. The total area is $4\cdot2=\boxed{\text{(B)} 8}$

~Technodoggo ~Minor formatting change: e_is_2.71828

Solution 2 (Graphing)

We first consider the lattice points that satisfy $||x|-1|=0$ and $||y|-1|=1$. The lattice points satisfying these equations are $(1,0),(1,2),(1,-2),(-1,0),(-1,2),$ and $(-1,-2).$ By symmetry, we also have points $(0,1),(2,1),(-2,1),(0,-1),(2,-1),$ and $(-2,-1)$ when $||x|-1|=1$ and $||y|-1|=0$. Graphing and connecting these points, we form 5 squares. However, we can see that any point within the square in the middle does not satisfy the given inequality (take $(0,0)$, for instance). As noted in the above solution, each square has a diagonal 2 for an area of $\frac{2^2}{2}=2$, so the total area is $4\cdot2=\boxed{\text{(B)} 8}$.

~ Brian__Liu

Note

This problem is very similar to a past AIME problem (1997 P13)

~ CherryBerry

Solution 3 (Logic)

The value of $|x|$ and $|y|$ can be a maximum of 1 when the other is 0. Therefore the value of x and y range from -2 to 2. This forms a diamond shape which has area $4 \times \frac{2^2}{2}$ which is $\boxed{\text{(B)} 8}$.

~ darrenn.cp ~ DarkPheonix

See Also

Problem_14

Problem

How many ordered pairs of integers (m, n) satisfy the equation $m^2 + mn + n^2 = m^2n^2$?

$\textbf{(A) }7\qquad\textbf{(B) }1\qquad\textbf{(C) }3\qquad\textbf{(D) }6\qquad\textbf{(E) }5$

Solution 1

Clearly, $m = 0, n = 0$ is 1 solution. However there are definitely more, so we apply to get this:

$$\begin{aligned}
m^2 + mn + n^2 &= m^2n^2 \\
m^2 + mn + n^2 + mn &= m^2n^2 + mn \\
(m+n)^2 &= m^2n^2 + mn \\
(m+n)^2 &= mn(mn+1)
\end{aligned}$$

This basically say that the product of two consecutive numbers $mn, mn+1$ must be a perfect square which is practically impossible except $mn = 0$ or $mn + 1 = 0$. $mn = 0$ gives $(0, 0)$. $mn = -1$ gives $(1, -1), (-1, 1)$.

~Technodoggo ~minor edits by lucaswujc

Solution 2

Case 1: $mn = 0$.

In this case, $m = n = 0$.

Case 2: $mn \neq 0$.

Denote $k = \gcd(m, n)$. Denote $m = ku$ and $n = kv$. Thus, $\gcd(u, v) = 1$.

Thus, the equation given in this problem can be written as

$$u^2 + uv + v^2 = k^2u^2v^2.$$

Modulo u, we have $v^2 \equiv 0 \pmod{u}$. Because $(u, v) = 1$, we must have $|u| = |v| = 1$. Plugging this into the above equation, we get $2 + uv = k^2$. Thus, we must have $uv = -1$ and $k = 1$.

Thus, there are two solutions in this case: $(m, n) = (1, -1)$ and $(m, n) = (-1, 1)$.

Putting all cases together, the total number of solutions is $\boxed{\textbf{(C) }3}$.

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

Solution 3 (Discriminant)

We can move all terms to one side and wrote the equation as a quadratic in terms of n to get

$$(1 - m^2)n^2 + (m)n + (m^2) = 0.$$

The discriminant of this quadratic is

$$\Delta = m^2 - 4(1 - m^2)(m^2) = m^2(4m^2 - 3).$$

For n to be an integer, we must have $m^2(4m^2 - 3)$ be a perfect square. Thus, either $4m^2 - 3$ is a perfect square or $m^2 = 0$ and $m = 0$. The first case gives $m = -1, 1$, which result in the equations $-n + 1 = 0$ and $n - 1 = 0$, for a total of two pairs: $(-1, 1)$ and $(1, -1)$. The second case gives the equation $n^2 = 0$, so it's only pair is $(0, 0)$. In total, the total number of solutions is $\boxed{\textbf{(C) } 3}$.

~A_MatheMagician

Problem_15

Problem

What is the least positive integer m such that $m \cdot 2! \cdot 3! \cdot 4! \cdot 5!...16!$ is a perfect square?

$\textbf{(A) }30\qquad\textbf{(B) }30030\qquad\textbf{(C) }70\qquad\textbf{(D) }1430\qquad\textbf{(E) }1001$

Solution 1

Consider 2 -- there are odd number of 2's in $2! \cdot 3! \cdot 4! \cdot 5!...16!$ (We're not counting 3 2's in 8, 2 3's in 9, etc).

There are even number of 3's in ...etc,

So, we can reduce our original expression to

$$\begin{aligned} m \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 &\equiv m \cdot 2^8 \cdot (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8) \\ &\equiv m \cdot 2 \cdot 3 \cdot (2 \cdot 2) \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot (2 \cdot 2 \cdot 2) \\ &\equiv m \cdot 2 \cdot 5 \cdot 7 \\ m &= 2 \cdot 5 \cdot 7 = 70 \end{aligned}$$

~Technodoggo ~minor edits by lucaswujc

Solution 2

Perfect squares have all of the powers in their prime factorization even. To evaluate $2! \cdot 3! \cdot 4! \cdot 5!...16!$ we get the following:

$$(2^{15}) \times (3^{14}) \times ((2^2)^{13}) \times (5^{12}) \times ((2 \times 3)^{11}) \times (7^{10}) \times ((2^3)^9) \times ((3^2)^8) \times$$
$$((5 \times 2)^7) \times (11^6) \times (((2^2) \times 3)^5) \times (13^4) \times ((7 \times 2)^3) \times ((3 \times 5)^2) \times ((2^4)^1)$$

.

Taking all powers $\mod 2$ we get:

$$(2^1) \times ((2 \times 3)^1) \times (2^1) \times ((5 \times 2)^1) \times (3^1) \times ((7 \times 2)^1)$$

Simplifying again, we finally get:

$$(2^1) \times (5^1) \times (7^1)$$

To make all the powers left even, we need to multiply by $(2 \times 5 \times 7)$ which is $\boxed{\text{(C) }70}$.

~darrenn.cp

Solution 3

We can prime factorize the solutions: A = $2 \cdot 3 \cdot 5$, B = $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, C = $2 \cdot 5 \cdot 7$, D = $2 \cdot 5 \cdot 11 \cdot 13$, E = $7 \cdot 11 \cdot 13$,

We can immediately eliminate B, D, and E since 13 only appears in $13!, 14!, 15, 16!$, so $13 \cdot 13 \cdot 13 \cdot 13$ is a perfect square. Next, we can test if 7 is possible (and if it is not we can use process of elimination). 7 appears in $7!$ to $16!$ and 14 appears in $14!$ to $16!$. So, there is an odd amount of 7's since there are 10 7's from $7!$ to $16!$ and 3 7's from $14!$ to $16!$ since 7 appears in 14 once, and $10 + 3 = 13$ which is odd. So we need to multiply by 7 to get a perfect square. Since 30 is not a divisor of 7, our answer is 70 which is $\boxed{\text{C}}$.

~aleyang

Solution 4

First, we note that $3! = 2! \cdot 3, 5! = 4! \cdot 5, ...15! = 14! \cdot 15$. So, $2! \cdot 3! = 2!^2 \cdot 3 \equiv 3, 4! \cdot 5! = 4!^2 \cdot 5 \equiv 5, ...14! \cdot 15! = 14!^2 \cdot 15 \equiv 15$. Simplifying the whole sequence and cancelling out the squares, we get $3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 16!$. Prime factoring $16!$ and cancelling out the squares, the only numbers that remain are $2, 5,$ and 7. Since we need to make this a perfect square, $m = 2 \cdot 5 \cdot 7$. Multiplying this out, we get $\boxed{\text{(C) }70}$.

~yourmomisalosinggame (a.k.a. Aaron) & ~Technodoggo (add more examples)

Solution 5 (Bashy method)

We know that a perfect square must be in the form $2^{2a_1} \cdot 3^{2a_2} \cdot 5^{2a_3} ... p^{2a_n}$ where $a_1, a_2, a_3, ..., a_n$ are nonnegative integers, and p is the largest and nth prime factor of our square number.

Let's assume $r = m \cdot 2! \cdot 3! \cdot 4! \cdot 5!...16!$. We need to prime factorize r and see which prime factors are raised to an odd power. Then, we can multiply one factor each of prime number with an odd number of factors to m. We can do this by finding the number of factors of 2, 3, 5, 7, 11, and 13.

Case 1: Factors of 2

We first count factors of 2^1 in each of the factorials. We know there is one factor of 2^1 each in $2!$ and $3!$, two in $4!$ and $5!$, and so on until we have 8 factors of 2^1 in $16!$. Adding them all up, we have $1 + 1 + 2 + 2 + ...7 + 7 + 8 = 64$.

Now, we count factors of 2^2 in each of the factorials. We know there is one factor of 2^2 each in $4!$, $5!$, $6!$, and $7!$, two in $8!$, $9!$, 10, and $11!$, and so on until we have 4 factors of 2^1 in $16!$. Adding them all up, we have $1 + 1 + 1 + 1 + 2 + 2 + 2 + 2 + 3 + 3 + 3 + 3 + 4 = 28$.

Now we count factors of 2^3 in each of the factorials. Using a similar method as above, we have a sum of $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 = 10$.

Now we count factors of 2^4 in each of the factorials. Using a similar method as above, we have a factor of 2^4 in 16, so there is 1 factor of 2^4.

Adding all the factors of 2, we have 103. Since 103 is odd, m has one factor of 2.

Case 2: Factors of 3

We use a similar method as in case 1. We first count factors of 3^1. We obtain the sum $1 + 1 + 1 + 2 + 2 + 2 + ...4 + 4 + 4 + 5 + 5 = 50$.

We count factors of 3^2. We obtain the sum $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 8$.

Adding all the factors of 3, we have 58. Since 58 is even, m has 0 factors of 3.

Case 3: Factors of 5

We count the factors of 5^1: $1+1+1+1+1+2+2+2+2+2+3+3=21$. Since 21 is odd, m has one factor of 5.

Case 4: Factors of 7

We count the factors of 7^1: $1+1+1+1+1+1+1+2+2+2=13$. Since 13 is odd, m has one factor of 7.

Case 5: Factors of 11

We count the factors of 11^1: $1+1+1+1+1+1=6$. Since 6 is even, m has 0 factors of 11.

Case 6: Factors of 13

We count the factors of 13^1: $1+1+1+1=4$. Since 4 is even, m has 0 factors of 13.

Multiplying out all our factors for m, we obtain $2 \cdot 5 \cdot 7 = \boxed{\text{(C) } 70}$.

~arjken

Problem_16

Problem

Define an $upno$ to be a positive integer of 2 or more digits where the digits are strictly increasing moving left to right. Similarly, define a $downno$ to be a positive integer of 2 or more digits where the digits are strictly decreasing moving left to right. For instance, the number 258 is an upno and 8620 is a downno. Let U equal the total number of $upnos$ and let D equal the total number of $downnos$. What is $|U - D|$?

$\textbf{(A)}\ 512 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 0 \qquad \textbf{(D)}\ 9 \qquad \textbf{(E)}\ 511$

Solution 1

First, we know that D is greater than U, since there are less upnos than downnos. To see why, we examine what determines an upno or downno.

We notice that, given any selection of unique digits (notice that "unique" constrains this to be a finite number), we can construct a unique downno. Similarly, we can also construct an upno, but the selection can not include the digit 0 since that isn't valid.

Thus, there are 2^{10} total downnos and 2^9 total upnos. However, we are told that each upno or downno must be at least 2 digits, so we subtract out the 0-digit and 1-digit cases.

For the downnos, there are 10 1-digit cases, and for the upnos, there are 9 1-digit cases. There is 1 0-digit case for both upnos and downnos.

Thus, the difference is $\left(\left(2^{10}-10-1\right)-\left(2^9-9-1\right)\right) = 2^9 - 1 = \textbf{(E)}\ 511.$

~Technodoggo ~minor edits by lucaswujc

Solution 2

Since Upnos do not allow 0s to be in their first -- and any other -- digit, there will be no zeros in any digits of an Upno. Thus, Upnos only contain digits [1,2,3,4,5,6,7,8,9].

Upnos are 2 digits in minimum and 9 digits maximum (repetition is not allowed). Thus the total number of Upnos will be (9C2)+(9C3)+(9C4)+...+(9C9), since every selection of distinct numbers from the set [1,2,3,4,5,6,7,8,9] can be arranged so that it is an Upno. There will be (9C2) 2 digit Upnos, (9C3) 3 digit Upnos and so on.

Thus, the total number of Upnos will be (9C2)+(9C3)+(9C4)+...+(9C9) = 2^9-(9C0)-(9C1) = 512 - 10 = 502.

Notice that the same combination logic can be done for Downnos, but Downnos DO allow zeros to be in their last digit. Thus, there are 10 possible digits [0,1,2,3,4,5,6,7,8,9] for Downnos.

Therefore, it is visible that the total number of Downnos are (10C2)+(10C3)+(10C4)+...+(10C10) = 2^10-(10C0)-(10C10) = 1024 - 11 = 1013.

Thus abs(#Upno-#Downno) = abs(1013-502) = 511.

~yxyxyxcxcxcx

Solution 3

Note that you can obtain a downo by reversing an upno (like 235 is an upno, and you can obtain 532). So, we need to find the amount of downos that end with 0 since if you 'flip' the numbers, the upno starts with a 0. We can find the cases that end with a 0:

$$\sum_{n=0}^{9} \binom{9}{n} = \binom{9}{0} + \binom{9}{1} + \cdots + \binom{9}{9}$$

to get 512. However, 0 itself is not a valid case (since it has 1 digit) so we subtract 1. Our answer is 511.

-aleyang

-ap246(LaTeX)

Solution 4 (Educated Guess)

First, note that the only $downnos$ that are not contained by the set of $upnos$ is every $downno$ that ends in 0.

Next, listing all the two digits $downnos$, we find that the answer is more than 9, since there are more digits to be tested and there are 9 two digit $downnos$. This leaves us with 512 or 511.

Next, we notice that all the possibilities for 2 through 9 digit $downnos$ ending in 0 pair up with one another, as the possibilities are equal (e.g. possibilities for 2 digits = possibilities for 9 digits, etc.).

This leaves us with one last possibility, the ten digit $downno$ 9876543210.

Since all the previous possibilities form an even number, adding one more possibility will make the total odd. Therefore, we need to choose the odd number from the set $[511, 512]$.

Our answer is $E)$ 511.

~yourmomisalosinggame (a.k.a. Aaron)

Solution 5

We start by calculating the number of upnos. Suppose we are constructing an upno of n digits such that $n \geq 2$. An upno can't start with a "0", so there are 9 digits to choose from. There are $\binom{9}{n}$ ways to choose an upno with n digits. This is because for each combination of digits, only one combination can form an upno. Therefore, for $2 \leq n \leq 9$, the total number of upnos is

$$\binom{9}{2} + \binom{9}{3} + \binom{9}{4} + \cdots + \binom{9}{9} = 2^9 - \binom{9}{1} - \binom{9}{0} = 2^9 - 10.$$

Similarly, the digits of a downo of n digits can be chosen among 10 digits to choose from, since 0 can be a digit of the downo as the last digit. Thus, the number of downos is

$$\binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \cdots + \binom{10}{9} + \binom{10}{10} = 2^{10} - \binom{10}{1} - \binom{10}{0} = 2^{10} - 11.$$

Thus,

$$|U - D| = |(2^9 - 10) - (2^{10} - 11)| = (2^{10} - 11) - (2^9 - 10) = 2^{10} - 2^9 - 1 = \boxed{\textbf{(E) } 511}$$

~rnatog337

Solution 6

Note that since the only way upno and downo can be different is if the downo ends in 0, so that the corresponding upno cannot exist. Therefore we just have to calculate the number to downos that end with 0.This ends up being 2^9 (2 options of whether or not the number exists in the upnoto or not) However we then need to subtract by 1 the $upno/downo$ needs to have 2 digits and 0 would have been a possible $upno/downo$ without this restiction (although we would still need to remove it since 0 would have been both an upno and downo)giving us $2^9 - 1$, or $\boxed{\textbf{(E) } 511}$

Problem_17

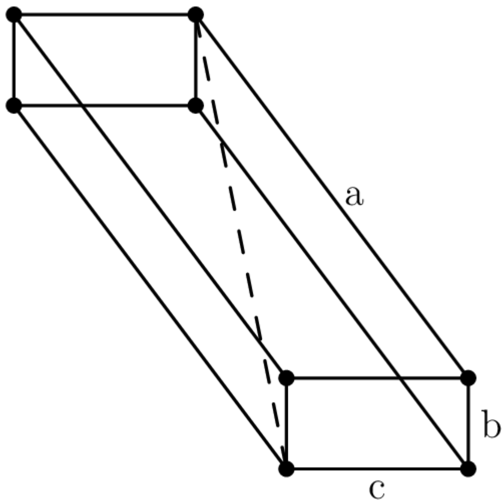
The following problem is from both the 2023 AMC 10B #17 and 2023 AMC 12B #13, so both problems redirect to this page.

Problem

A rectangular box P has distinct edge lengths a, b, and c. The sum of the lengths of all 12 edges of P is 13, the areas of all 6 faces of P is $\frac{11}{2}$, and the volume of P is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of P?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ \frac{3}{8} \qquad \textbf{(C)}\ \frac{9}{8} \qquad \textbf{(D)}\ \frac{9}{4} \qquad \textbf{(E)}\ \frac{3}{2}$

Solution 1 (algebraic manipulation)



We can create three equations using the given information.

$$4a + 4b + 4c = 13$$

$$2ab + 2ac + 2bc = \frac{11}{2}$$

$$abc = \frac{1}{2}$$

We also know that we want $\sqrt{a^2 + b^2 + c^2}$ because that is the length that can be found from using the Pythagorean Theorem. We cleverly notice that $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$. We know that $a + b + c = \frac{13}{4}$ and $2(ab + ac + bc) = \frac{11}{2}$, so $a^2 + b^2 + c^2 = \left(\frac{13}{4}\right)^2 - \frac{11}{2} = \frac{169 - 88}{16} = \frac{81}{16}$. So our answer is $\sqrt{\frac{81}{16}} = \boxed{\frac{9}{4}}$.

Interestingly, we don't use the fact that the volume is $\frac{1}{2}$.

~lprado

~Technodoggo

~minor edits and add-ons by lucaswujc

~andliu766

Solution 2 (vieta's)

We use the equations from Solution 1 and manipulate it a little:

$$a+b+c=\frac{13}{4}$$

$$ab+ac+bc=\frac{11}{4}$$

$$abc=\frac{1}{2}$$

Notice how these are the equations for the vieta's formulas for a polynomial with roots of a, b, and c. Let's create that polynomial. It would be $x^3-\frac{13}{4}x^2+\frac{11}{4}x-\frac{1}{2}$. Multiplying each term by 4 to get rid of fractions, we get $4x^3-13x^2+11x-2$. Notice how the coefficients add up to 0. Whenever this happens, that means that $(x-1)$ is a factor and that 1 is a root. After using synthetic division to divide $4x^3-13x^2+11x-2$ by $x-1$, we get $4x^2-9x+2$. Factoring that, you get $(x-2)(4x-1)$. This means that this polynomial factors to $(x-1)(x-2)(4x-1)$ and that the roots are 1, 2, and $1/4$. Since we're looking for $\sqrt{a^2+b^2+c^2}$, this is equal to $\sqrt{1^2+2^2+\frac{1}{4}^2}=\sqrt{\frac{81}{16}}=\boxed{\frac{9}{4}}$

~lprado

Solution 3 (Cheese Method)

Incorporating the solution above, we know $a+b+c$ = $14/4 \Rightarrow a+b+c>3$. The side lengths are larger than $1\cdot1\cdot1$ (a unit cube). The side length of the interior of a unit cube is $\sqrt{3}$, and we know that the side lengths are larger than $1\cdot1\cdot1$, so that means the diagonal has to be larger than $\sqrt{3}$, and the only answer choice larger than $\sqrt{3} \Rightarrow \boxed{\textbf{(D) } 9/4}$

~kabbybear

Note that the real number $\sqrt{3}$ is around 1.73. Option A is also greater than $\sqrt{3}$ meaning there are two options greater than $\sqrt{3}$. Option A is an integer so educationally guessing we arrive at answer $D \Rightarrow \boxed{\textbf{(D) } 9/4}$

~atictacksh

See Also

Problem_18

The following problem is from both the 2023 AMC 10B #18 and 2023 AMC 12B #15, so both problems redirect to this page.

Problem

Suppose a, b, and c are positive integers such that

$$\frac{a}{14}+\frac{b}{15}=\frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a,14)=1$ or $\gcd(b,15)=1$ or both, then $\gcd(c,210)=1$.

II. If $\gcd(c,210)=1$, then $\gcd(a,14)=1$ or $\gcd(b,15)=1$ or both.

III. $\gcd(c,210)=1$ if and only if $\gcd(a,14)=\gcd(b,15)=1$.

$\textbf{(A)}$ I, II, and III $\qquad\textbf{(B)}$ I only $\qquad\textbf{(C)}$ I and II only $\qquad\textbf{(D)}$ III only $\qquad\textbf{(E)}$ II and III only

Solution 1 (Guess and check + Contrapositive)

We examine each of the conditions.

The first condition is false. A simple counterexample is $a=3$ and $b=5$. The corresponding value of c is $17\cdot 15=255$. Clearly, $\gcd(3,14)=1$ and $\gcd(5,15)=5$, so condition I would imply that $\gcd(c,210)=1$. However, $\gcd(255,210)$ is clearly not 1 (they share a common factor of 5). Obviously, condition I is false, so we can rule out choices A, B, and C.

We are now deciding between the two answer choices D and E. What differs between them is the validity of condition II, so it suffices to simply check II.

We look at statement II's contrapositive to prove it. The contrapositive states that if $\gcd(a,14)\neq 1$ and $\gcd(b,15)\neq 1$, then $\gcd(c,210)\neq 1$. In other words, if a shares some common factor that is not 1 with 14 and b shares some common factor that is not 1 with 15, then c also shares a common factor with 210. Let's say that $a=a'\cdot n$, where a' is a factor of 14 not equal to 1. (So a' is the common factor.)

We can rewrite the given equation as $15a+14b=c \implies 15(a'n)+14b=c$. We can express 14 as $a'\cdot n'$, for some positive integer n' (this n' can be 1). We can factor a' out to get $a'(15n+14n')=c$.

We know that all values in this equation are integers, so c must be divisible by a'. Since a' is a factor of 14, a' must also be a factor of 210, a multiple of 14. Therefore, we know that c shares a common factor with 210 (which is a'), so $\gcd(c,210)\neq 1$. This is what II states, so therefore II is true.

Thus, our answer is $\boxed{\textbf{(E)}\ \text{II and III only}}$. ~Technodoggo

Solution 2

The equation given in the problem can be written as

$$15a+14b=c. \qquad (1)$$

$\textbf{First, we prove that Statement I is not correct.}$

A counter example is $a=1$ and $b=3$. Thus, $\gcd(c,210)=3\neq 1$.

$\textbf{Second, we prove that Statement III is correct.}$

First, we prove the ``if *part.*

Suppose $\gcd(a,14)=1$ and $\gcd(b,15)=1$. However, $\gcd(c,210)\neq 1$.

Thus, c must be divisible by at least one factor of 210. W.L.O.G, we assume c is divisible by 2.

Modulo 2 on Equation (1), we get that $2|a$. This is a contradiction with the condition that $\gcd(a, 14) = 1$. Therefore, the ``if *part in Statement III is correct.*

Second, we prove the ``only if *part.*

Suppose $\gcd(c, 210) \neq 1$. Because $210 = 14 \cdot 15$, there must be one factor of 14 or 15 that divides c. W.L.O.G, we assume there is a factor $q > 1$ of 14 that divides c. Because $\gcd(14, 15) = 1$, we have $\gcd(q, 15) = 1$. Modulo q on Equation (1), we have $q|a$.

Because $q|14$, we have $\gcd(a, 14) \geq q > 1$.

Analogously, we can prove that $\gcd(b, 15) > 1$.

$\textbf{Third, we prove that Statement II is correct.}$

This is simply a special case of the ``only if *part of Statement III. So we omit the proof.*

All analyses above imply $\boxed{\textbf{(E) II and III only}}$.

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

Solution 3 (Answer Choices)

It can easily be shown that statement I is false (a counterexample would be $a = 1, b = 5, c = 85$), meaning the only viable answer choices are D and E. Since both of these answer choices include statement III, this means III is true. Since III is true, this actually implies that statement II is true, as III is just a stronger version of statement II (or it's contrapositive, to be precise). Therefore the answer is $\boxed{\textbf{(E) II and III only}}$.

~SpencerD. ~edited by A_MatheMagician

See Also

Problem_19

Problem

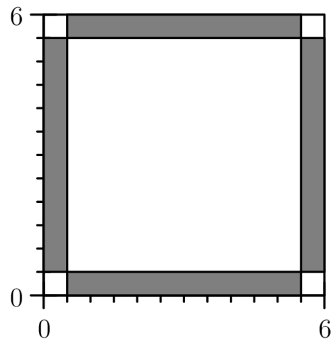
Sonya the frog chooses a point uniformly at random lying within the square $[0, 6] \times [0, 6]$ in the coordinate plane and hops to that point. She then randomly chooses a distance uniformly at random from $[0, 1]$ and a direction uniformly at random from {north, south, east, west}. All he choices are independent. She now hops the distance in the chosen direction. What is the probability that she lands outside the square?

$\textbf{(A) } \frac{1}{6} \qquad \textbf{(B) } \frac{1}{12} \qquad \textbf{(C) } \frac{1}{4} \qquad \textbf{(D) } \frac{1}{10} \qquad \textbf{(E) } \frac{1}{9}$

Solution 1

WLOG, we assume Sonya jumps 0.5 units every time, since that is her expected value.

If Sonya is within 0.5 blocks of an edge, she can jump off the board. Let us examine the region that is at most 0.5 blocks from exactly one edge.



If Sonya starts in this region, she has a $\frac{1}{4}$ chance of landing outside (there's exactly one direction she can hop to get out). The total area of this region is $4 \cdot 0.5 \cdot 5 = 10.$ For this region, Sonya has a $\frac{1}{4}$ chance, so we multiply 10 by $\frac{1}{4}$ to get $2.5.$

If Sonya is in one of the corner squares, she can go two directions to get out, so she has a $\frac{2}{4} = \frac{1}{2}$ chance to get out. The total area is $0.5 \cdot 0.5 \cdot 4 = 1$, so this region yields $\frac{1}{2} \cdot 1 = \frac{1}{2}$.

Adding the two, we get 3. This is out of 36 square units of area, so our answer is thus $\frac{1}{12}$.

~Technodoggo

Solution 2

Since all the actions are independent, we can switch the orders. Let Sonya choose the direction first. And the problem is symmetric, so we consider just one direction. WLOG, let's say she choose $south$. When she first pick the location, she'll have to be within 1 unit of the x axis to have a chance to jump out of the boundary southward. That's $\frac{1}{6}$. With in that region, the expected y coordinate would be 0.5 which is 0.5 unit from the boundary (x-axis). Now, the jumping distance required to jump out of the boundary on average has to be greater than 0.5. That's another $\frac{1}{2}$. So the final probability is $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$.

~Technodoggo

Solution 3

We denote by (x, y) the frog's initial coordinates. We denote by $k \in \{n, e, s, w\}$ the direction to hop. We denote by z the hopping distance. In this analysis, we say that the frog wins if landing outside the square.

We have

$$ (Error compiling LaTeX. Unknown error_msg)\begin{align*} P \left(win \right) & = \sum_{k \in \left\{ n, e, s, w \right\}} P\left(win | k \right) P \left(k \right) \\ & = P \left(win | k = w \right) \sum_{k \in \left\{ n, e, s, w \right\}} P \left(k \right) \\ & = P \left(win | k = w \right) \\ & = \int_{y=0}^1 P \left(win | k = w, y \right) dy \\ & = P \left(win | k = w, y = 0 \right) \\ & = P \left(win | k = w, y = 0, x \in [0,1] \right) P \left(x \in [0,1] \right) + P \left(win | k = w, y = 0, x \in (1,6] \right) P \left(x \in (1,6] \right) \\ & = P \left(win | k = w, y = 0, x \in [0,1] \right) \cdot \frac{1}{6} + 0 \cdot \frac{5}{6} \\ & = \frac{1}{6} P \left(win | k = w, y = 0, x \in [0,1] \right) \\ & = \frac{1}{6} \int_{x=0}^1 \int_{z=x}^1 dz dx \\ & = \frac{1}{6} \cdot \frac{1}{2} \\ & = \boxed{\textbf{(B) $\frac{1}{12}$}}. \end{align*}$$ (Error compiling LaTeX. Unknown error_msg) The second equality above follows from symmetry that $P \left(win | k \right)$ are all the same for all $k \in \left\{ n, e, s, w \right\}$. The fifth equality above follows from symmetry that $P \left(win | k = w, y \right)$ are all the same for all $y \in [0, 1]$.

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

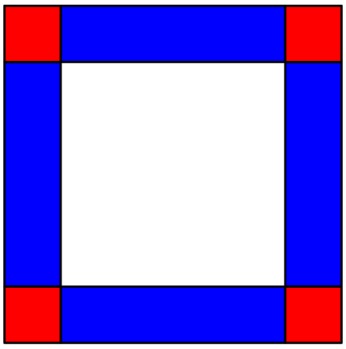
Solution 4

We can notice that Sonya can only jump out if she first picks a point that is at most 1 unit from the border. Let's separate this region into 4 different strips each with overlapping corners. Since each strip is exactly the same, let's first consider the probability of Sonya leaving the square given that she first lands on the top strip. Using expected value, one can get that the probability that Sonya gets the distance needed to leave the square is $\frac{1}{2}$. Now the probability she gets the direction needed (north) is $\frac{1}{4}$. Now with the probability of landing in the strip being $\frac{1}{6}$, we get the probability that she lands on the strip and leaves the square to be $\frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{48}$. Since there are four strips, we add the probabilities giving a final answer of $\boxed{\textbf{(B) } \frac{1}{12}}$.

~Daniel C (Minor edits by Ishaan Garg)

Solution 5

We know that Sonya can only jump out if she is within 1 unit of the border. We can calculate the probability that Sonya can jump out.



The total area of the colored regions is 20, so the probability that Sonya lands in a colored region is $\frac{20}{36} = \frac{5}{9}$. We can calculate the probability that Sonya gets out of each type of region.

Case 1: Sonya chooses a blue region.

The probability that Sonya chooses a blue region is $\frac{16}{20} = \frac{4}{5}$. One direction can let her out, so the probability that she chooses the right one is $\frac{1}{4}$. Finally, the probability that Sonya chooses a distance to get her out is $\frac{1}{2}$. So, the probability that she chooses a blue region and gets out is $\frac{5}{9} \cdot \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{18}$.

Case 2: Sonya choose a red region.

The probability that Sonya chooses a red region is $\frac{4}{20} = \frac{1}{5}$. Two directions can let her out, so the probability that she chooses one of them is $\frac{1}{2}$. Finally, the probability that Sonya chooses a distance to get her out is $\frac{1}{2}$. So, the probability that she chooses a red region and gets out is $\frac{5}{9} \cdot \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{36}$.

So, the probability that Sonya gets out is $\frac{1}{18} + \frac{1}{36} = \frac{3}{36} = \boxed{\textbf{(B)}\ \frac{1}{12}}$.

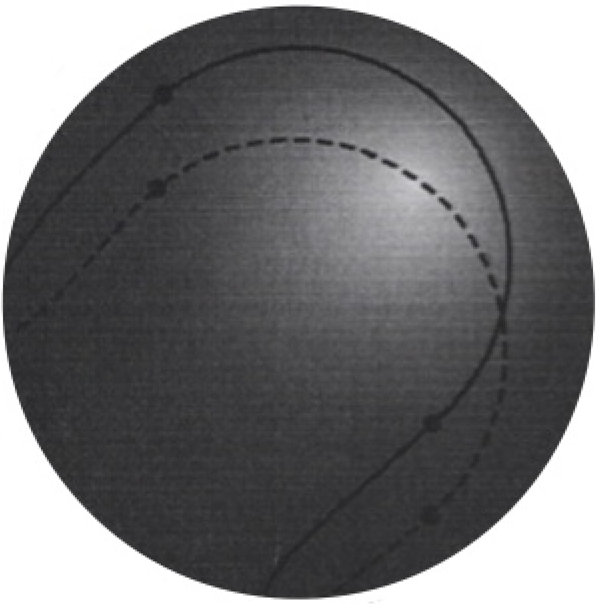
~ MathCactus0_0

Problem_20

Problem

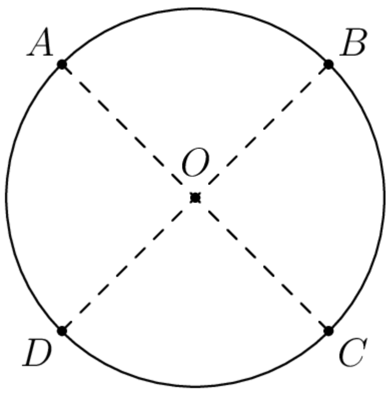
Four congruent semicircles are drawn on the surface of a sphere with radius 2, as shown, creating a close curve that divides the surface into two congruent regions. The length of the curve is $\pi\sqrt{n}$. What is n?

$\textbf{(A) } 32 \qquad \textbf{(B) } 12 \qquad \textbf{(C) } 48 \qquad \textbf{(D) } 36 \qquad \textbf{(E) } 27$



Solution 1

There are four marked points on the diagram; let us examine the top two points and call them A and B. Similarly, let the bottom two dots be C and D, as shown:



This is a cross-section of the sphere seen from the side. We know that $\overline{AO} = \overline{BO} = \overline{CO} = \overline{DO} = 2$, and by Pythagorean therorem, $\overline{AB} = 2\sqrt{2}.$

Each of the four congruent semicircles has the length AB as a diameter (since AB is congruent to $BC, CD,$ and DA), so its radius is $\dfrac{2\sqrt{2}}{2} = \sqrt{2}.$ Each one's arc length is thus $\pi \cdot \sqrt{2} = \sqrt{2}\pi.$

We have 4 of these, so the total length is $4\sqrt{2}\pi = \sqrt{32}\pi$, so thus our answer is $\boxed{\textbf{(A) } 32.}$

~Technodoggo

Solution 2

Assume A, B, C, and D are the four points connecting the semicircles. By law of symmetry, we can pretty confidently assume that $ABCD$ is a square. Then, $\overline{AB}=2\sqrt{2}.$, and the rest is the same as the second half of solution 1.

~jonathanzhou18

Solution 3

We put the sphere to a coordinate space by putting the center at the origin. The four connecting points of the curve have the following coordinates: $A=(0,0,2)$, $B=(2,0,0)$, $C=(0,0,-2)$, $D=(-2,0,0)$.

Now, we compute the radius of each semicircle. Denote by M the midpoint of A and B. Thus, M is the center of the semicircle that ends at A and B. We have $M=(1,0,1)$. Thus, $OM=\sqrt{1^2+0^2+1^2}=\sqrt{2}$.

In the right triangle $\triangle OAM$, we have $MA=\sqrt{OA^2-OM^2}=\sqrt{2}$.

Therefore, the length of the curve is

$$4\cdot\frac{1}{2}2\pi\cdot MA=\pi\sqrt{32}.$$

Therefore, the answer is $\boxed{\textbf{(A) 32}}$.

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

Solution 4

Note that each of the diameters are the chord of the sphere of a quarter arc. Thus, the semicircles diameter's length is $2\sqrt{2}$. Thus, the entire curve is $2\sqrt{2}\cdot\pi\cdot\frac{1}{2}\cdot 4=4\sqrt{2}\pi=\sqrt{32}\pi$. Therefore, the answer is $\boxed{\textbf{(A) 32}}$. ~andliu766

Problem_21

The following problem is from both the 2023 AMC 10B #21 and 2023 AMC 12B #19, so both problems redirect to this page.

Problem

Each of 2023 balls is randomly placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

$\textbf{(A) } \frac{2}{3} \qquad \textbf{(B) } \frac{3}{10} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } \frac{1}{3} \qquad \textbf{(E) } \frac{1}{4}$

Important Clarification

Stars and Bars does not provide an exact probability. However, it does provide a good estimate for the approximate answer as on average, the number of arrangements will be almost the same when each container has an odd # of balls or when each container has an even # of balls. (similar binomial distributions)

Solution 1

Because each bin will have an odd number, they will have at least one ball. So we can put one ball in each bin prematurely. We then can add groups of 2 balls into each bin, meaning we now just have to spread 1010 pairs over 3 bins. This will force every bin to have an odd number of balls. Using stars and bars, we find that this is equal to $\binom{1012}{2}$. This is equal to $\frac{1012 \cdot 1011}{2}$. The total amount of ways would also be found using stars and bars. That would be $\binom{2023+3-1}{3-1} = \binom{2025}{2}$. Dividing our two quantities, we get $\frac{1012 \cdot 1011 \cdot 2}{2 \cdot 2025 \cdot 2024}$. We can roughly cancel $\frac{1012 \cdot 1011}{2025 \cdot 2024}$ to get $\frac{1}{4}$. The 2 in the numerator and denominator also cancels out, so we're left with $\boxed{\frac{1}{4}}$.

~lprado

~AtharvNaphade ~eevee9406 ~Teddybear0629

Solution 2 (Solution 1 with more steps)

Suppose the numbers are a_1, a_2, and a_3. First, we try to calculate the amount of ways for all three balls to be placed in a bin so the number of balls in each bin is odd. $a_1 + a_2 + a_3 = 2023$ and each bin has at least one ball because they are positive odd numbers. Changing the equation, we see that $\frac{[(a_1)+1]}{2} + \frac{[(a_2)+1]}{2} + \frac{[(a_3)+1]}{2} = \frac{(2023+3)}{2}$. Let $a_1 + 1 = 2b_1$, $a_2 + 1 = 2b_2$, and $a_3 + 1 = 2b_3$. Thus $b_1 + b_2 + b_3 = 1013$. We can also see that b_1, b_2, and b_3 are all positive. Using the positive version of stars and bars, we get $\binom{1013-1}{3-1}$ = $\binom{1012}{2}$ choices.

Now, we want to find the total amount of cases. Using the non-negative version of stars and bars, we find that the total is $\binom{2023+3-1}{3-1}$ = $\binom{2025}{2}$.

Now we need to calculate $\binom{1012}{2}$ / $\binom{2025}{2}$, which is just $\frac{1012 \cdot 1011 \cdot 2}{2 \cdot 2025 \cdot 2024}$. Cancelling the twos, we get $\frac{1012 \cdot 1011}{2025 \cdot 2024}$. This is roughly equal to $\frac{1}{4}$. The answer is $\boxed{\textbf{(E)} \frac{1}{4}}$.

~Aopsthedude

Solution 3

Having 2 bins with an odd number of balls means the 3rd bin also has an odd number. The probability of the first bin having an odd number of balls is $\frac{1}{2}$, since even and odd have roughly the same probability. The probability of the second bin having an odd number of balls is also $\frac{1}{2}$ for the same reason. If both of these bins have an odd number of balls, the number of balls remaining for the third bin is also odd. Therefore the probability is $\frac{1}{2} \cdot \frac{1}{2} = \boxed{\textbf{(E)} \frac{1}{4}}$.

~Yash C

Solution 4

We first examine the possible arrangements for parity of number of balls in each box for 2022 balls.

If a E denotes an even number and a O denotes an odd number, then the distribution of balls for 2022 balls could be $EEE, EOO, OEO,$ or OOE. With the insanely overpowered magic of cheese, we assume that each case is about equally likely.

From EEE, it is not possible to get to all odd by adding one ball; we could either get $OEE, EOE,$ or EEO. For the other 3 cases, though, if we add a ball to the exact right place, then it'll work.

For each of the working cases, we have 1 possible slot the ball can go into (for OEO, for example, the new ball must go in the center slot to make OOO) out of the 3 slots, so there's a $\frac{1}{3}$ chance. We have a $\frac{3}{4}$ chance of getting one of these working cases, so our answer is $\frac{3}{4} \cdot \frac{1}{3} = \boxed{\textbf{(E)}\ \frac{1}{4}.}$

~pengf ~Technodoggo

Solution 5

2023 is an arbitrary large number. So, we proceed assuming that an arbitrarily large number of balls have been placed.

For an odd-numbered amount of balls case, the 3 bins can only be one of these 2 combinations:

OEE (OEE,EOE,EEO)

OOO (OOO)

Let the probability of achieving the OOO case to be $P(OOO) = p$ and any of the OEE permutations to be $P(OEE) = 1 - p$.

Because the amount of balls is arbitrarily large, $P(OOO) = p$ even after another two balls are be placed.

There are two cases for which placing another two balls results in OOO:

OOO: The two balls are placed in the same bin ($OOO \rightarrow OOE \rightarrow OOO$)

OEE: The two balls are placed in the two even bins ($OEE \rightarrow OOE \rightarrow OOO$)

So,

$$P(OOO) = P(OOO) * \frac{1}{3} + P(OEE) * \frac{2}{3} * \frac{1}{3}$$

$$p = p * \frac{1}{3} + (1 - p) * \frac{2}{3} * \frac{1}{3}$$

$$\frac{8}{9}p = \frac{2}{9}$$

$p = \boxed{\textbf{(E)}\frac{1}{4}}$

~Dissmo

Solution 6

We use the generating functions approach to solve this problem. Define $\Delta = \{(a,b,c) \in \mathbb{Z}_+ : a+b+c = 2023\}$.

We have

$$(x+y+z)^{2023} = \sum_{(a,b,c)\in\Delta} \binom{2023}{a,b,c} x^a y^b z^c.$$

First, we set $x \leftarrow 1, y \leftarrow 1, z \leftarrow 1$. We get

$$3^{2023} = \sum_{(a,b,c)\in\Delta} \binom{2023}{a,b,c} 1. \qquad (1)$$

Second, we set $x \leftarrow 1, y \leftarrow -1, z \leftarrow 1$. We get

$$1 = \sum_{(a,b,c)\in\Delta} \binom{2023}{a,b,c} (-1)^b. \qquad (2)$$

Third, we set $x \leftarrow 1, y \leftarrow 1, z \leftarrow -1$. We get

$$1 = \sum_{(a,b,c)\in\Delta} \binom{2023}{a,b,c} (-1)^c. \qquad (3)$$

Fourth, we set $x \leftarrow 1, y \leftarrow -1, z \leftarrow -1$. We get

$$-1 = \sum_{(a,b,c)\in\Delta} \binom{2023}{a,b,c} (-1)^{b+c}. \qquad (4)$$

Taking $\dfrac{(1)-(2)-(3)+(4)}{4}$, we get

$$\begin{aligned}
\frac{3^{2023} - 1 - 1 + (-1)}{4} &= \frac{1}{4} \sum_{(a,b,c)\in\Delta} \binom{2023}{a,b,c} \left(1 - (-1)^b - (-1)^c + (-1)^{b+c}\right) \\
&= \frac{1}{4} \sum_{(a,b,c)\in\Delta} \binom{2023}{a,b,c} \left(1 - (-1)^b\right)\left(1 - (-1)^c\right) \\
&= \sum_{\substack{(a,b,c)\in\Delta \\ a,b,c \text{ are odds}}} \binom{2023}{a,b,c}.
\end{aligned}$$

The last expression above is the number of ways to get all three bins with odd numbers of balls. Therefore, this happens with probability

$$\frac{\frac{3^{2023}-1-1+(-1)}{4}}{3^{2023}} \approx \boxed{\textbf{(E) } \frac{1}{4}}.$$

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

Solution 7

Four even-odd splittings divides 2023 in to three, namely (O, O, O), (E, E, O), (E, O, E), and (O, E, E). Here if we define a "move" as relocated one ball, then we will notice in each case, that a random "move" will be evenly likely to be one of the other three splittings. Hence by Group theory (or by intuition), we will find the structure of this splitting is V_4 group, and it's symmetric for all four elements in this Group.

Thus, no matter what is the initial starting point, four cases will be evenly likely to appear when repeated many times. The answer is $\boxed{\textbf{(E)} \frac{1}{4}}$.

~Prof. Joker

Solution 8

Really simple way to solve it. To have 3 numbers that are all odd, you need to get odd for the first two bins, and the last one will always be odd. There are 2023 ball, so the chance of having a odd number in the first bin is 1012/2023 and the chance of having another odd is 1/2. 1012/2023 * 1/2 is closest to 1/4.

~Jack Bai(only 9 years old)

See Also

Problem_22

Problem

How many distinct values of x satisfy $\lfloor x \rfloor^2 - 3x + 2 = 0$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x?

$\textbf{(A) } \text{an infinite number} \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 3 \qquad \textbf{(E) } 0$

Solution 1(three cases)

First, let's take care of the integer case--clearly, only $x = 1, 2$ work. Then, we know that $3x$ must be an integer. Set $x = \frac{a}{3}$.

Now, there are two cases for the value of $\lfloor x \rfloor$. Case 1: $\lfloor x \rfloor = \frac{a-1}{3}$

$$\frac{a^2 - 2a + 1}{9} = a - 2 \to a^2 - 2a + 1 = 9a - 18 \to a^2 - 11a + 19 = 0.$$

There are no solutions in this case. Case 2: $\lfloor x \rfloor = \frac{a-2}{3}$

$$\frac{a^2 - 4a + 4}{9} = a - 2 \to a^2 - 4a + 4 = 9a - 18 \to a^2 - 13a + 22 = 0.$$

This case provides the two solutions $\frac{2}{3}$ and $\frac{11}{3}$ as two more solutions. Our final answer is thus $\boxed{4}$.

~wuwang2002

Solution 2

First, $x = 2, 1$ are trivial solutions

We assume from the shape of a parabola and the nature of the floor function that any additional roots will be near 2 and 1

We can now test values for $\lfloor x \rfloor$:

$\lfloor x \rfloor = 0$

We have $0 - 3x + 2 = 0$. Solving, we have $x = \frac{2}{3}$. We see that $\lfloor \frac{2}{3} \rfloor = 0$, so this solution is valid

$\lfloor x \rfloor = -1$

We have $1 - 3x + 2 = 0$. Solving, we have $x = 1$. $\lfloor 1 \rfloor \neq -1$, so this is not valid. We assume there are no more solutions in the negative direction and move on to $\lfloor x \rfloor = 3$

$\lfloor x \rfloor = 3$

We have $9 - 3x + 2 = 0$. Solving, we have $x = \frac{11}{3}$. We see that $\lfloor \frac{11}{3} \rfloor = 3$, so this solution is valid

$\lfloor x \rfloor = 4$

We have $16 - 3x + 2 = 0$. Solving, we have $x = 6$. $\lfloor 6 \rfloor \neq 4$, so this is not valid. We assume there are no more solutions.

Our final answer is $\boxed{\textbf{(B) } 4}$

~kjljixx

Solution 3

Denote $a = \lfloor x \rfloor$. Denote $b = x - \lfloor x \rfloor$. Thus, $b \in [0, 1)$.

The equation given in this problem can be written as

$$a^2 - 3(a + b) + 2 = 0.$$

Thus,

$$3b = a^2 - 3a + 2.$$

Because $b \in [0, 1)$, we have $3b \in [0, 3)$. Thus,

$$a^2 - 3a + 2 = 0, 1, \text{ or } 2.$$

If $a^2 - 3a + 2 = 0$, $(a-2)(a-1) = 0$ so a can be $1, 2$.

If $a^2 - 3a + 2 = 1$, $a^2 - 3a + 1 = 0$ which we find has no integer solutions after finding the discriminant.

If $a^2 - 3a + 2 = 2$, $a^2 - 3a = 0$ -> $a(a-3) = 0$ so a can also be $0, 3$.

Therefore, $a = 1$, 2, 0, 3. Therefore, the number of solutions is $\boxed{\textbf{(B) } 4}$.

~Steven Chen (Professor Chen Education Palace, www.professorchenedu.com)

Solution 4(Quick)

A quadratic equation can have up to 2 real solutions. With the $\lfloor x \rfloor$ it could also help generate another pair. We have to verify that the solutions are real and distinct.

First, we get the trivial solution by ignoring the floor. $(x-2)(x-1) = 0$, we get $(2, 1)$ as our first pair of solutions.

Up to this point, we can rule out A,E.

Next, we see that $\lfloor x \rfloor^2 - 3x = 0$. This implies that $-3x$ must be an integer. We can guess and check x as $\frac{k}{3}$ which yields $\left(\frac{2}{3}, \frac{11}{3}\right)$.

So we got 4 in total $\left(\frac{2}{3}, 1, 2, \frac{11}{3}\right)$.

~Technodoggo

Problem_23

Problem

An arithmetic sequence of positive integers has $\text{n} \geq 3$ terms, initial term a, and common difference $d > 1$. Carl wrote down all the terms in this sequence correctly except for one term, which was off by 1. The sum of the terms he wrote was 222. What is $a + d + n$?

$\textbf{(A) } 24 \qquad \textbf{(B) } 20 \qquad \textbf{(C) } 22 \qquad \textbf{(D) } 28 \qquad \textbf{(E) } 26$

Solution 1

Since one of the terms was either 1 more or 1 less than it should have been, the sum should have been $222 - 1 = 221$ or $222 + 1 = 223.$

The formula for an arithmetic series is $an + d\left(\frac{(n-1)n}{2}\right) = \frac{n}{2}\left(a + d(n-1)\right).$ This can quickly be rederived by noticing that the sequence goes $a, a+d, a+2d, a+3d, \dots, a+(n-1)d$, and grouping terms.

We know that $\frac{n}{2}(2a + d(n-1)) = 221$ or 223. Let us now show that 223 is not possible.

If $\frac{n}{2}(2a + d(n-1)) = 223$, we can simplify this to be $n(a + d(n-1)) = 223 \cdot 2.$ Since every expression in here should be an integer, we know that either $n = 2$ and $a + d(n-1) = 223$ or $n = 223$ and $a + d(n-1) = 2.$ The latter is not possible, since $n \geq 3, d > 1,$ and $a > 0.$ The former is also impossible, as $n \geq 3.$ Thus, $\frac{n}{2}(2a + d(n-1)) \neq 223 \implies \frac{n}{2}(2a + d(n-1)) = 221.$

We can factor 221 as $13 \cdot 17$. Using similar reasoning, we see that $221 \cdot 2$ can not be paired as 2 and 221, but rather must be paired as 13 and 17 with a factor of 2 somewhere.

Let us first try $n = 13.$ Our equation simplifies to $2a + 12d = 34 \implies a + 6d = 17.$ We know that $d > 1,$ so we try the smallest possible value: $d = 2.$ This would give us $a = 17 - 2 \cdot 6 = 17 - 12 = 5.$ (Indeed, this is the only possible d.)

There is nothing wrong with the values we have achieved, so it is reasonable to assume that this is the only valid solution (or all solutions sum to the same thing), so we answer $a + d + n = 5 + 2 + 13 = \boxed{\textbf{(B) } 20.}$

For the sake of completeness, we can explore $n = 17.$ It turns out that we reach a contradiction in this case, so we are done.

~Technodoggo

Solution 2

There are n terms, the xth term is $a + (x-1)d$, summation is $an + dn(n-1)/2 = n(a + \frac{d(n-1)}{2})$.

The summation of the set is $222 \pm 1 = 221, 223$. First, 221: its only possible factors are $1, 13, 17, 221$, and as said by the problem, $n \geq 3$, so n must be $13, 17,$ or 221. Let's start with $n = 13$. Then, $a + 6d = 17$, and this means $a = 5$, $d = 2$. Summing gives $13 + 5 + 2 = 20$. We don't need to test any more cases, since the problem writes that all $a + d + n$ are the same, so we don't need to test other cases.

-HIA2020

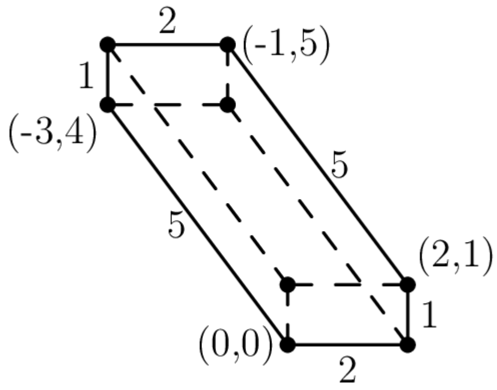
Problem_24

Problem

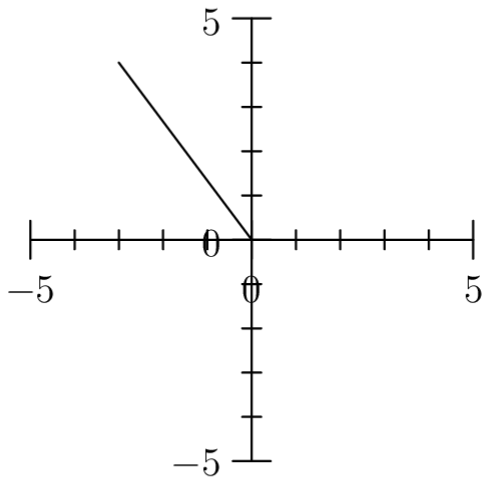
What is the perimeter of the boundary of the region consisting of all points which can be expressed as $(2u-3w, v+4w)$ with $0\le u\le1, 0\le v\le1,$ and $0\le w\le1$?

$\textbf{(A)}\ 10\sqrt{3} \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 12 \qquad \textbf{(D)}\ 18 \qquad \textbf{(E)}\ 16$

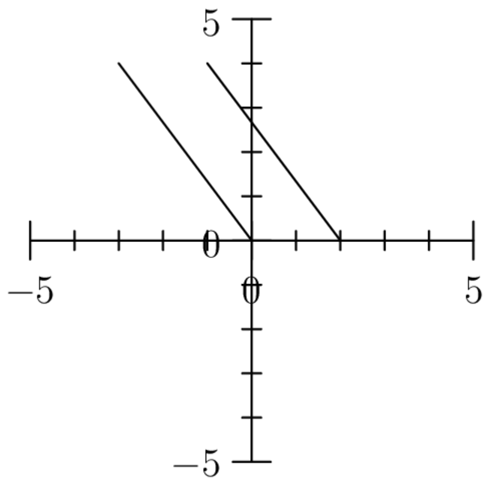
Solution



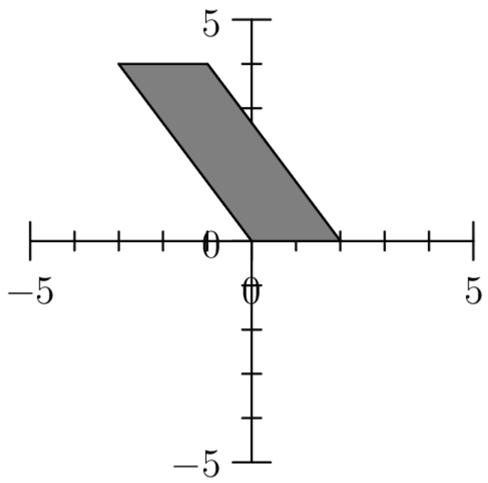
Notice that this we are given a parametric form of the region, and w is used in both x and y. We first fix u and v to 0, and graph $(-3w, 4w)$ from $0\le w\le1$:



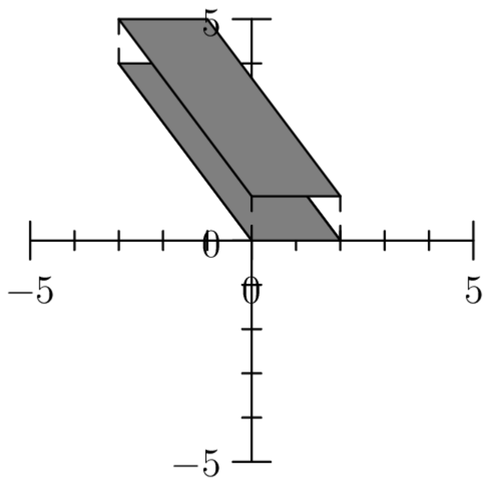
Now, when we vary u from 0 to 2, this line is translated to the right 2 units:

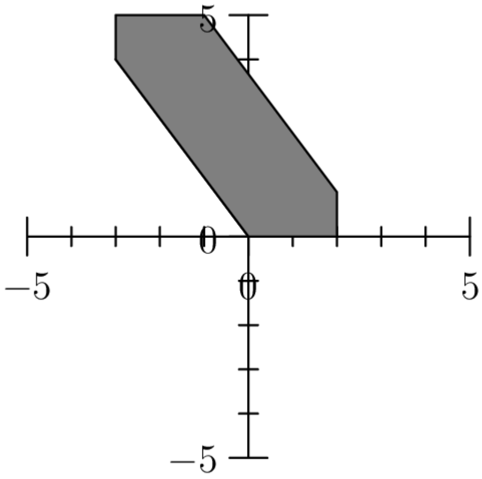


We know that any points in the region between the line (or rather segment) and its translation satisfy w and u, so we shade in the region:



We can also shift this quadrilateral one unit up, because of v. Thus, this is our figure:





The length of the boundary is simply $1+2+5+1+2+5$ (5 can be obtained by Pythagorean theorem, since we have side lengths 3 and 4.). This equals $\boxed{\textbf{(E) } 16.}$

~Technodoggo

Problem_25

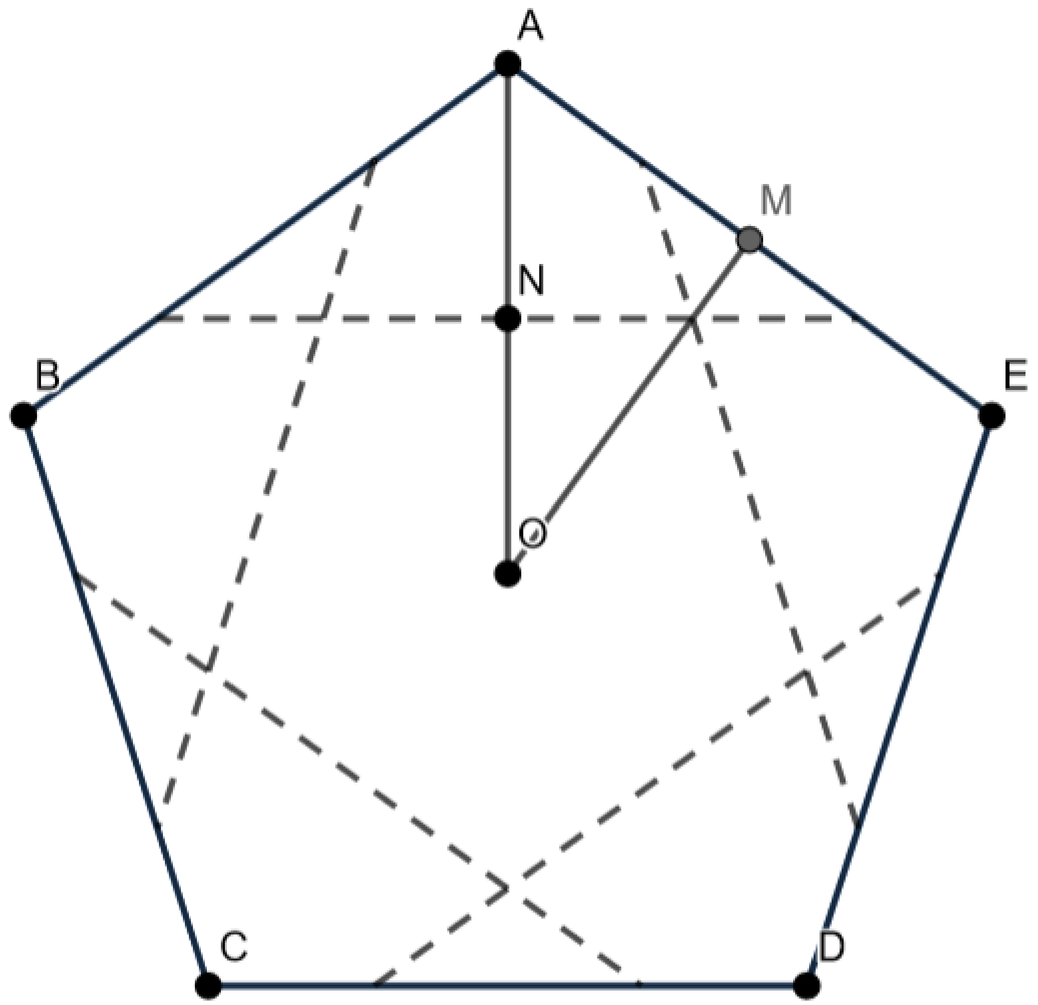
The following problem is from both the 2023 AMC 10B #25 and 2023 AMC 12B #25, so both problems redirect to this page.

Problem

A regular pentagon with area $\sqrt{5}+1$ is printed on paper and cut out. The five vertices of the pentagon are folded into the center of the pentagon, creating a smaller pentagon. What is the area of the new pentagon?

$\textbf{(A)}\ 4-\sqrt{5} \qquad \textbf{(B)}\ \sqrt{5}-1 \qquad \textbf{(C)}\ 8-3\sqrt{5} \qquad \textbf{(D)}\ \frac{\sqrt{5}+1}{2} \qquad \textbf{(E)}\ \frac{2+\sqrt{5}}{3}$

Solution 1



Let the original pentagon be $ABCDE$ centered at O. The dashed lines represent the fold lines. WLOG, let's focus on vertex A. Since A is folded onto O, $AN = NO$ where N is the intersection of AO and the creaseline between A and O. Note that the inner pentagon is regular, and therefore similar to the original pentagon, due to symmetry.

Because of their similarity, the ratio of the inner pentagon's area to that of the outer pentagon can be represented by

$$(\frac{ON}{OM})^2 = (\frac{\frac{OA}{2}}{OA\sin(\angle OAE)})^2 = \frac{1}{4\sin^2 54}$$

Option 1: Knowledge

Remember that $\sin 54 = \dfrac{1+\sqrt{5}}{4}$.

Option 2: Angle Identities

$$\sin 54 = \cos 36$$

$$4\cos^3 18 - 3\cos 18 = 2\sin 18 \cos 18$$

$$4(1-\sin^2 18) - 3 - 2\sin 18 = 0$$

$$4\sin^2 18 + 2\sin 18 - 1 = 0$$

$$\sin 18 = \frac{-1+\sqrt{5}}{4}$$

$$\sin 54 = \cos 36 = 1 - 2\sin^2 18 = \frac{1+\sqrt{5}}{4}$$

$$\sin^2 54 = \frac{3+\sqrt{5}}{8}$$

Let the inner pentagon be Z.

$$[Z] = \frac{1}{4\sin^2 54}[ABCDE]$$

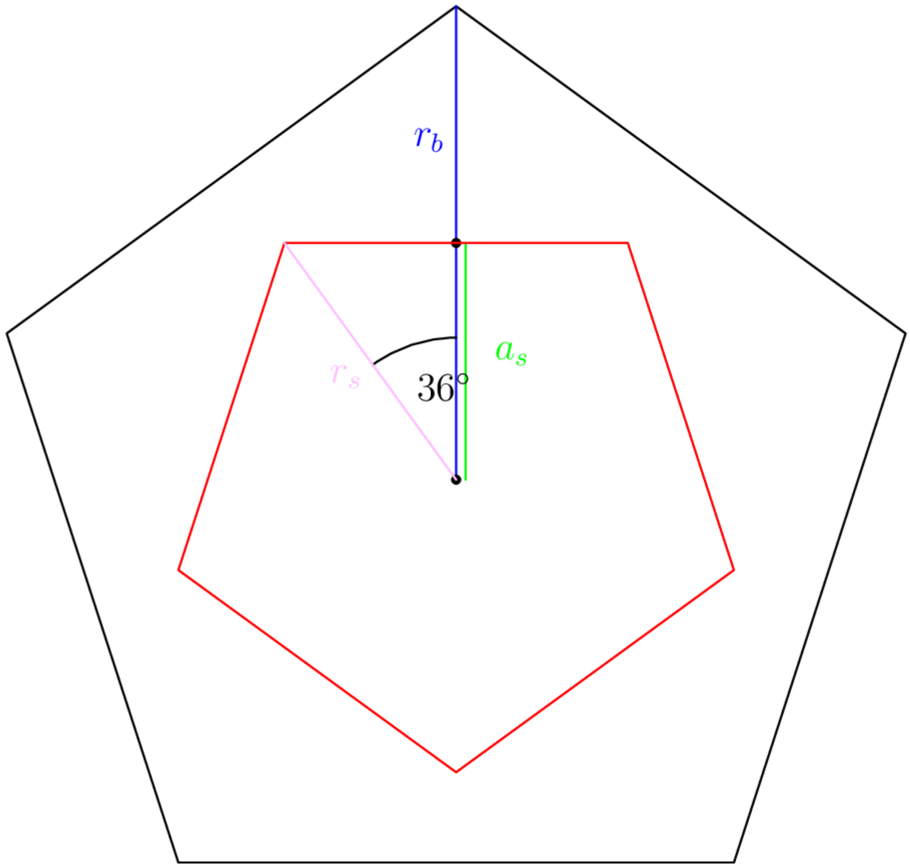
$$= \frac{2(1+\sqrt{5})}{3+\sqrt{5}}$$

$$= \sqrt{5} - 1$$

$$\boxed{B}$$

-Dissmo

Solution 2



Let r_b and r_s be the circumradius of the big and small pentagon, respectively. Let a_s be the apothem of the smaller pentagon and A_s and A_b be the areas of the smaller and larger pentagon, respectively.

From the diagram:

$$\begin{aligned}
\cos 36^\circ &= \frac{a_s}{r_s} = \frac{\phi}{2} = \frac{\sqrt{5}+1}{4} \\
a_s &= \frac{r_b}{2} \\
A_s &= \left(\frac{r_s}{r_b}\right)^2 A_b \\
&= \left(\frac{a_s}{\cos 36^\circ r_b}\right)^2 \left(1+\sqrt{5}\right) \\
&= \left(\frac{r_b}{\frac{\phi}{2} r_b}\right)^2 \left(1+\sqrt{5}\right) \\
&= \left(\frac{1}{2\frac{\phi}{2}}\right)^2 \left(1+\sqrt{5}\right) \\
&= \left(\frac{2}{\sqrt{5}+1}\right)^2 \left(1+\sqrt{5}\right) \\
&= \frac{4}{\sqrt{5}+1} \\
&= \frac{4\left(\sqrt{5}-1\right)}{\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)} \\
&= \sqrt{5}-1
\end{aligned}$$

$$\boxed{\textbf{(B) } \sqrt{5}-1}$$

~Technodoggo

Solution 3

Interestingly, we find that the pentagon we need is the one that is represented by the intersection of the diagonals. Through similar triangles and the golden ratio, we find that the side length ratio of the two pentagons is $\frac{\sqrt{5}-1}{2}$ Thus, the answer is $\sqrt{5}+1 \cdot (\frac{\sqrt{5}-1}{2})^2 = \sqrt{5}-1. \boxed{\text{B}}$ ~andliu766

Solution 4 (answer choices)

After drawing a decent diagram, we can see that the area of the inner pentagon is quite a bit smaller than half the area of the larger pentagon.

Then, we can estimate the values of the answers and choose one that seems the closest to the smallest answer.

We know that $\sqrt{5} \approx 2.236$, so we'll use $\sqrt{5} = 2.2$ for our estimations. The area of the original pentagon is $\sqrt{5}+1 \approx 3.2$, so half of it is roughly 1.6.

A: $4-\sqrt{5} \approx 1.8$ clearly, this is wrong because it is greater than half the area of the pentagon.

B: $\sqrt{5}-1 \approx 1.2$ This answer could be right.

C: $8-3\sqrt{5}\approx 1.4$ This too.

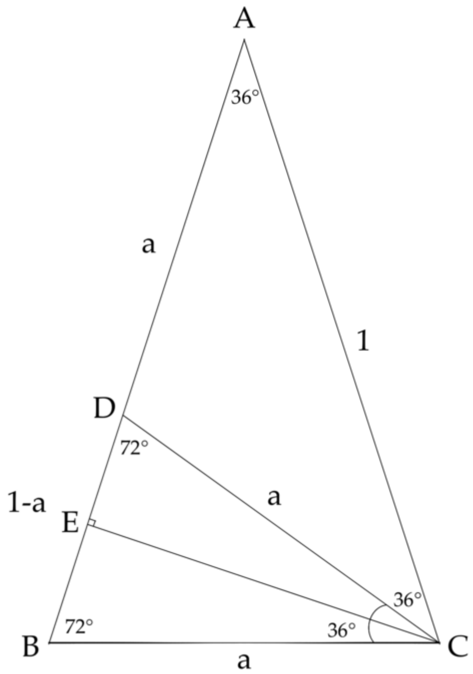
D: $\frac{\sqrt{5}+1}{2}$ This answer is wrong, as it assumes that the area of the inner pentagon is exactly half the area of the larger one.

E: $\frac{2+\sqrt{5}}{3}\approx 1.4$ This answer could be right.

But, from our diagram, assume that the area of the pentagon is significantly less than the area half of the larger pentagon, so we choose the smallest answer choice, giving us $\boxed{\textbf{(B) }\sqrt{5}-1}$. ~erics118

Supplement (Calculating sin54/cos36 from Scratch)

Method 1:



Construct golden ratio triangle $\triangle ABC$ with $\angle A=36^\circ, \angle B=\angle C=72^\circ$ and $\triangle BCD$ with $\angle C=36^\circ$, $\angle DBC=\angle BDC=72^\circ$. WLOG, let $AB=AC=1, BC=CD=AD=a, BD=1-a$. $\triangle ABC\sim\triangle BCD$

$$\frac{AC}{BC}=\frac{BC}{BD},\quad \frac{1}{a}=\frac{a}{1-a},\quad 1-a=a^2,\quad a^2+a-1=0$$

$$a=\frac{-1+\sqrt{1^2-4(-1)}}{2}=\frac{\sqrt{5}-1}{2}$$

$$\cos 36^\circ=\cos\angle A=\frac{AE}{AC}=\frac{\frac{1-a}{2}+1}{}=\frac{a+1}{2}=\frac{\frac{\sqrt{5}-1}{2}+1}{2}=\frac{\sqrt{5}+1}{4}$$

$$\sin 54^\circ=\cos 36^\circ=\frac{\sqrt{5}+1}{4}$$

Method 2: (Writing in progress...)

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