A Short Proof

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To one loop order, the QCD beta function can be integrated to give

$$r\Lambda^{q\bar{q}}_{(n_f)} = \exp\left(-\frac{1}{2b_0^{(n_f)} g_{q\bar{q}}^2}\right) \tag{1}$$

Converting to the $\overline{MS}$ scheme using equation (16) of the paper gives

$$r\Lambda^{\overline{MS}}_{(n_f)} = \exp\left(\frac{t^{q\bar{q}}_{1(n_f)}}{2b_0^{(n_f)}}\right)\exp\left(-\frac{1}{2b_0^{(n_f)} g_{q\bar{q}}^2}\right) \tag{2}$$

Getting rid of the scale r by taking ratios, we find then that

$$\frac{\Lambda^{\overline{MS}}_{(2)}}{\Lambda^{\overline{MS}}_{(0)}} = \frac{\exp\left(\frac{t^{q\bar{q}}_{1(2)}}{2b_0^{(2)}}\right)\exp\left(-\frac{1}{2b_0^{(2)} g_{q\bar{q}}^2}\right)}{\exp\left(\frac{t^{q\bar{q}}_{1(0)}}{2b_0^{(0)}}\right)\exp\left(-\frac{1}{2b_0^{(0)} g_{q\bar{q}}^2}\right)} \tag{3}$$

As $g_{q\bar{q}} \to \infty$, the second exponentials in both the numerator and denominator quickly vanish, leaving the simple relation

$$\frac{\Lambda^{\overline{MS}}_{(2)}}{\Lambda^{\overline{MS}}_{(0)}} = \exp\left(\frac{t^{q\bar{q}}_{1(2)}}{2b_0^{(2)}} - \frac{t^{q\bar{q}}_{1(0)}}{2b_0^{(0)}}\right) \tag{4}$$

Equation (50) of the paper gives the value

$$t^{q\bar{q}}_{1(n_f)} = \frac{1}{(4\pi)^2}\left[\frac{4}{3}n_f\left(\gamma_E - \frac{1}{6}\right) - 22\left(\gamma_E - \frac{35}{66}\right)\right] \tag{5}$$

so plugging this into equation 4 and using the usual beta function coefficients b_i, we find that the asymptotic value for $\Lambda^{\overline{MS}}_{(2)}/\Lambda^{\overline{MS}}_{(0)}$ is

$$\frac{\Lambda^{\overline{MS}}_{(2)}}{\Lambda^{\overline{MS}}_{(0)}} = 1.0544 \tag{6}$$

While I don't have an easy proof for why this should hold to higher loops, this at least shows that something fishy is going on with Figure 7 of the paper.