Integrating the QCD beta function: A letter to the QCDSF Collaboration

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An inconsistency

In QCD, the beta function is of central importance to understanding how the theory's gauge coupling parameter g depends on an energy scale μ. The beta function is defined as

$$\frac{\partial g(\mu)}{\partial \ln \mu} = -g^3 \sum_{k=0}^{\infty} b_k g^k(\mu) \equiv \beta(g(\mu)), \tag{1}$$

with the first four of these coefficients known in the $\overline{MS}$ scheme [1, 2]. According to the QCDSF Collaboration [3, 4, 5, 6, 7] and a recent lattice review [8], this differential equation can be solved by the equation

$$\frac{\mu}{\Lambda} = \exp\left(\frac{1}{2b_0 g^2}\right)\left(\kappa g^2\right)^{\frac{b_1}{2b_0^2}} \exp\left[\int_\tau^g dg' \left(\frac{1}{\beta(g')} + \frac{1}{b_0 g'^3} - \frac{b_1}{b_0^2 g'}\right)\right]. \tag{2}$$

Both QCDSF and the lattice review use $\kappa = b_0$ and $\tau = 0$, though these parameters are arbitrary if the sole requirement is for eq. (2) to satisfy eq. (1). That is to say, κ and τ are determined by internal consistency relations and by the boundary conditions imposed upon the solution. This arbitrariness may easily be seen by first taking the natural logarithm of eq. (2) then differentiating both sides with respect to g. Cancellations occur entirely independently of the parameters κ and τ, and the reciprocal of eq. (1) is recovered.

The goal of this letter is to establish a concrete method of determining these parameters κ and τ. In regards to τ, this goal is best achieved by first taking the natural logarithm of (2)

$$\ln\frac{\mu}{\Lambda} = \frac{1}{2b_0 g^2} + \frac{b_1}{2b_0^2}\ln \kappa g^2 + \int_\tau^g dg' \left(\frac{1}{\beta(g')} + \frac{1}{b_0 g'^3} - \frac{b_1}{b_0^2 g'}\right), \tag{3}$$

then differentiating with respect to τ and rearranging,

$$\frac{1}{\beta(\tau)} = -\frac{1}{b_0\tau^3} + \frac{b_1}{b_0^2\tau}. \tag{4}$$

Expanding the reciprocal of the beta function about $\tau = 0$ correctly reproduces the right hand side of eq. (4) up to $\mathcal{O}(\tau)$ corrections. For consistency, the $\mathcal{O}(\tau)$ corrections must vanish, which implies that $\tau \to 0$, as given by QCDSF.

To determine κ, the strategy is to solve for $\ln(\mu/\Lambda)$ using two different methods. The first method involves directly evaluating the integral in eq. (2), while the second method involves integrating eq. (1) using the boundary condition $g(\Lambda) \to \infty$. For both methods, the beta function will be truncated to its two loop form, $\beta(g) = -b_1 g^3(\frac{b_0}{b_1} + g^2)$. This is for the sake of clarity – the method will work for higher orders in the perturbative expansion, as should be made clear in the following section.

Both methods require the reciprocal of the beta function to be integrated, which can be achieved using the partial fraction decomposition,

$$\frac{1}{-b_1 g^3\left(\frac{b_0}{b_1} + g^2\right)} = \frac{A}{g} + \frac{B}{g^2} + \frac{C}{g^3} + \frac{gD}{g^2 + \frac{b_0}{b_1}}, \tag{5}$$

where

$$A = \frac{b_1}{b_0^2} \quad ; \quad B = 0 \quad ; \quad C = -\frac{1}{b_0} \quad ; \quad D = -\frac{b_1}{b_0^2} \tag{6}$$

Inserting this decomposition now into eq. (3) to solve for $\ln(\mu/\Lambda)$, the A/g and C/g^3 terms exactly cancel the singular terms, ensuring the integral is finite:

$$\ln\frac{\mu}{\Lambda} = \frac{1}{2b_0 g^2} + \frac{b_1}{2b_0^2}\ln\kappa g^2 + \int_0^g dg' \left(-\frac{b_1}{b_0^2}\frac{g'}{g'^2 + \frac{b_0}{b_1}}\right). \tag{7}$$

Once integrated, this gives the final solution for $\ln(\mu/\Lambda)$

$$\ln\frac{\mu}{\Lambda} = \frac{1}{2b_0 g^2} + \ln\left(\frac{b_1}{\kappa b_0} + \frac{1}{\kappa g^2}\right)^{-\frac{b_1}{2b_0^2}}. \tag{8}$$

The second method begins by integrating the definition of the beta function with the boundary conditions $g(\Lambda) \to \infty$:

$$\int_\infty^g \frac{dg'}{\beta(g')} = \int_\Lambda^\mu d\ln\mu' = \ln\frac{\mu}{\Lambda}. \tag{9}$$

One might be wary of integrating the beta function over large values of g, since the beta function becomes non-perturbative in this region. However, from a purely mathematical point of view, the value of the integral is well-defined, so for the physical results to be internally consistent, eq. (9) is necessarily a true statement. Once again using the partial fraction decomposition to evaluate the integral in eq. (9),

$$\begin{aligned}
\int_\infty^g \frac{dg'}{\beta(g')} &= \int_\infty^g dg' \left(\frac{b_1}{b_0^2 g'} - \frac{1}{b_0 g'^3} - \frac{b_1}{b_0^2}\frac{g'}{g'^2 + \frac{b_0}{b_1}}\right) \\
&= \left[\frac{1}{2b_0 g'^2} + \ln\left(1 + \frac{b_0}{b_1 g'^2}\right)^{-\frac{b_1}{2b_0^2}}\right]_\infty^g \\
&= \frac{1}{2b_0 g^2} + \ln\left(1 + \frac{b_0}{b_1 g^2}\right)^{-\frac{b_1}{2b_0^2}}
\end{aligned} \tag{10}$$

Equation (9) establishes an equality between equations (8) and (10), allowing κ to be determined. Comparing the two equations, the only valid solution is $\kappa = b_1/b_0$, in tension with the QCDSF value of $\kappa = b_0$.

Pushing forward

When written in terms of the roots of the beta function, the integral in eq. (2) can be performed analytically to any finite order in perturbation theory. To one loop order, only b_0 is non-zero, and the expression for μ/Λ is

$$\frac{\mu}{\Lambda} = \exp\left(\frac{1}{2b_0 g^2}\right). \tag{11}$$

To higher orders, integration depends on finding the roots of the beta function. If N denotes the index of the last non-vanishing coefficient b_k, then the beta function can be rewritten in terms of its non-zero roots r_k:

$$\beta(g) = -b_N g^3 \prod_{k=1}^{N}(g^2 - r_k) \tag{12}$$

The partial fraction decomposition for the reciprocal of eq. (12) is

$$\frac{1}{-b_N g^3}\prod_{k=1}^{N}\frac{1}{g^2-r_k}=\frac{A}{g}+\frac{B}{g^2}+\frac{C}{g^3}+\sum_{k=1}^{N}\frac{gP_k}{g^2-r_k} \tag{13}$$

with

$$\begin{aligned}
A &= \frac{(-1)^{N+1}}{b_N}\left[\prod_{k=1}^{N}\frac{1}{r_k}\right]\left[\sum_{k=1}^{N}\frac{1}{r_k}\right]\\
B &= 0\\
C &= \frac{(-1)^{N+1}}{b_N}\prod_{k=1}^{N}\frac{1}{r_k}\\
P_k &= -\frac{1}{b_N r_k^2}\prod_{\substack{j=1\\ j\neq k}}^{N}\frac{1}{r_k-r_j}
\end{aligned} \tag{14}$$

By expanding eq. (12) and collecting powers of g^2, it can be shown that the expressions for A and C simplify to $A=b_1/b_0^2$ and $C=-1/b_0$. These values for A and C are what allow the singular terms of eq. (2) to be exactly cancelled, regardless of expansion order. Using the values $\kappa=b_1/b_0$ and $\tau=0$ determined in the previous section, the closed form expression for μ/Λ is then

$$\frac{\mu}{\Lambda}=\exp\left(\frac{1}{2b_0g^2}\right)\left(\frac{b_1}{b_0}g^2\right)^{\frac{b_1}{2b_0^2}}\prod_{k=1}^{N}\left(1-\frac{g^2}{r_k}\right)^{\frac{P_k}{2}}. \tag{15}$$

Equation (15) holds provided none of the roots r_k are repeated. The nonzero roots of the beta function are, in general, complex, but the imaginary component of eq. (15) vanishes exactly, as it must for a purely real integral. One should be aware, however, that computational rounding errors can introduce a small imaginary component when evaluating the expression numerically. Taking the real part of the result solves any problems that may be connected to this issue.

No closed form expression exists for the roots of the 6 loop beta function, since this involves solving the quintic. Conversely, however, eq. (15) can be fully written down in terms of radicals for any perturbative expansion to 5 loops or less, owing to well known general solutions of quartic functions.

Appendix A contains plots comparing analytic solutions to μ/Λ with $\kappa=b_0$ and $\kappa=b_1/b_0$, as well as the numerically inverted solutions for $\alpha(\mu)\equiv g^2(\mu)/4\pi$ up to four loops.

A Appendix A

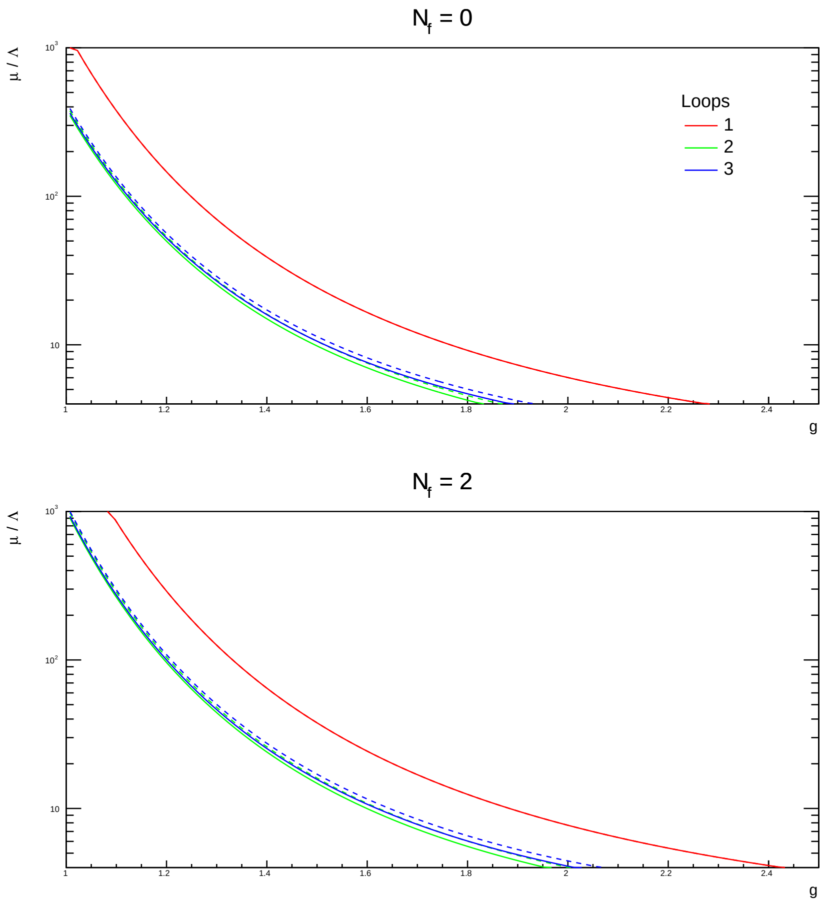

Figure 1: μ/Λ vs g in the $\overline{MS}$ renormalization scheme for the quenched approximation (top) and for $n_f = 2$ (bottom), using increasingly higher orders in the perturbative expansion. $\kappa = b_0$ appears as a dashed line while $\kappa = b_1/b_0$ appears as a solid line.

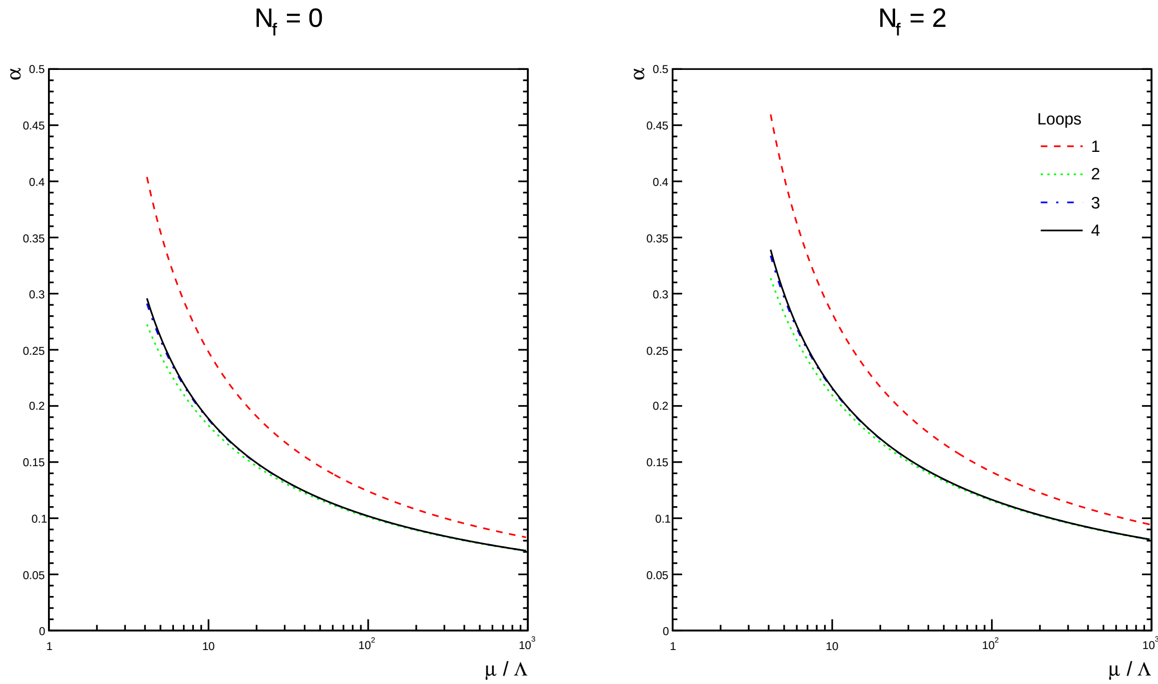

Figure 2: α vs μ/Λ in the $\overline{MS}$ renormalization scheme for the quenched approximation (left) and for $n_f = 2$ (right), using increasingly higher orders in the perturbative expansion, with $\kappa = b_1/b_0$.

References

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