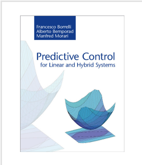

Predictive Control: for Linear and Hybrid Systems
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IV Constrained Optimal Control of Linear Systems
11 Constrained Optimal Control

11
Constrained Optimal Control

In this chapter we study the finite time and infinite time optimal control problem for linear systems with linear constraints on inputs and state variables. We establish the structure of the optimal control law and derive algorithms for its computation. For finite time problems with linear and quadratic objective functions we show that the time varying feedback law is piecewise affine and continuous. The value function is a convex piecewise linear for linear objective functions and convex piecewise quadratic for quadratic objective functions.

We describe how the optimal control action for a given initial state can be computed by means of linear or quadratic programming. We also describe how the optimal control law can be computed by means of multiparametric linear or quadratic programming. Finally, we show how to compute the infinite time optimal controller for linear and quadratic objective functions and prove that, when it exists, the infinite time controller inherits all the structural properties of the finite time optimal controller.

11.1 Problem Formulation

Consider the linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \tag{11.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and input vectors, respectively, subject to the constraints

$$x(t) \in \mathcal{X}, \;\; u(t) \in \mathcal{U}, \;\; \forall t \geq 0. \tag{11.2}$$

The sets $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are polyhedra.

Remark 11.1 The results of this chapter also hold for more general forms of linear constraints such as mixed input and state constraints

$$[x(t)', u(t)'] \in \mathcal{P}_{x,u}, \tag{11.3}$$

where $\mathcal{P}_{x,u}$ is a polyhedron in $\mathbb{R}^{n+m}$ of mixed input and state constraints over a finite time, or, even more general, constraints of the type:

$$[x(0)', \dots, x(N-1)', u(0)', \dots, u(N-1)'] \in \mathcal{P}_{x,u,N}, \tag{11.4}$$

where $\mathcal{P}_{x,u,N}$ is a polyhedron in $\mathbb{R}^{N(n+m)}$. Note that constraints of the type (11.4) can arise, for example, from constraints on the input rate $\Delta u(t) = u(t) - u(t-1)$. In this chapter, for the sake of simplicity, we will use the less general form (11.2).

Define the cost function

$$J_0(x(0), U_0) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k), \tag{11.5}$$

where x_k denotes the state vector at time k obtained by starting from the state $x_0 = x(0)$ and applying to the system model

$$x_{k+1} = Ax_k + Bu_k \tag{11.6}$$

the input sequence $u_0, \dots, u_{k-1}$.

If the 1-norm or ∞-norm is used in the cost function (11.5), then we set $p(x_N) = ||Px_N||_p$ and $q(x_k, u_k) = ||Qx_k||_p + ||Ru_k||_p$ with $p = 1$ or $p = \infty$ and P, Q, R full column rank matrices. Cost (11.5) is rewritten as

$$J_0(x(0), U_0) = \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p. \tag{11.7}$$

If the squared Euclidian norm is used in the cost function (11.5), then we set $p(x_N) = x_N' P x_N$ and $q(x_k, u_k) = x_k' Q x_k + u_k' R u_k$ with $P \succeq 0$, $Q \succeq 0$ and $R \succ 0$. Cost (11.5) is rewritten as

$$J_0(x(0), U_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k. \tag{11.8}$$

Consider the constrained finite time optimal control problem (CFTOC)

$$\begin{array}{lll} J_0^*(x(0)) = \min_{U_0} & J_0(x(0), U_0) \\ \text{subj. to} & x_{k+1} = Ax_k + Bu_k, \ k = 0, \dots, N-1 \\ & x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \ k = 0, \dots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(0), \end{array} \tag{11.9}$$

where N is the time horizon and $\mathcal{X}_f \subseteq \mathbb{R}^n$ is a terminal polyhedral region. In (11.5)–(11.9) $U_0 = [u_0', \dots, u_{N-1}']' \in \mathbb{R}^s$, $s = mN$ is the optimization vector. We denote with $\mathcal{X}_0 \subseteq \mathcal{X}$ the set of initial states $x(0)$ for which the optimal control problem (11.5)–(11.9) is feasible, i.e.,

$$\begin{aligned}\mathcal{X}_0 = \{x_0 \in \mathbb{R}^n :\ &\exists (u_0, \dots, u_{N-1}) \text{ such that } x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \dots, N-1,\\ &x_N \in \mathcal{X}_f \text{ where } x_{k+1} = Ax_k + Bu_k,\ k = 0, \dots, N-1\}.\end{aligned} \tag{11.10}$$

Remark 11.2 Note that we distinguish between the *current* state $x(k)$ of system (11.1) at time k and the variable x_k in the optimization problem (11.9), that is the *predicted* state of system (11.1) at time k obtained by starting from the state $x_0 = x(0)$ and applying to system (11.6) the input sequence $u_0, \dots, u_{k-1}$. Analogously, $u(k)$ is the input applied to system (11.1) at time k while u_k is the k-th optimization variable of the optimization problem (11.9).

If we use cost (11.8) with the squared Euclidian norm and set

$$\{(x, u) \in \mathbb{R}^{n+m} \ :\ x \in \mathcal{X},\ u \in \mathcal{U}\} = \mathbb{R}^{n+m},\ \mathcal{X}_f = \mathbb{R}^n, \tag{11.11}$$

problem (11.9) becomes the standard unconstrained finite time optimal control problem (Chapter 8) whose solution (under standard assumptions on A, B, P, Q and R) can be expressed through the time varying state feedback control law (8.28)

$$u^*(k) = F_k x(k) \quad k = 0, \dots, N-1. \tag{11.12}$$

From (8.25) the optimal cost is given by

$$J_0^*(x(0)) = x(0)' P_0 x(0). \tag{11.13}$$

If we let $N \to \infty$ as discussed in Section 8.5, then problem (11.8), (11.9), (11.11) becomes the standard infinite horizon linear quadratic regulator (LQR) problem whose solution (under standard assumptions on A, B, P, Q and R) can be expressed as the state feedback control law (see (8.33))

$$u^*(k) = F_\infty x(k), \quad k = 0, 1, \dots \tag{11.14}$$

In the following chapters we will show that the solution to problem (11.9) can again be expressed in feedback form where now $u^*(k)$ is a continuous piecewise affine function on polyhedra of the state $x(k)$, i.e., $u^*(k) = f_k(x(k))$ where

$$f_k(x) = F_k^j x + g_k^j \quad \text{if} \quad H_k^j x \le K_k^j,\ j = 1, \dots, N_k^r. \tag{11.15}$$

Matrices H_k^j and K_k^j in equation (11.15) describe the j-th polyhedron $CR_k^j = \{x \in \mathbb{R}^n : H_k^j x \le K_k^j\}$ inside which the feedback optimal control law $u^*(k)$ at time k has the affine form $F_k^j x + g_k^j$. The set of polyhedra $CR_k^j,\ j = 1, \dots, N_k^r$ is a *polyhedral partition* of the set of feasible states $\mathcal{X}_k$ of problem (11.9) at time k. The sets $\mathcal{X}_k$ are discussed in detail in the next section. Since the functions $f_k(x(k))$ are continuous, the use of polyhedral partitions rather than strict polyhedral partitions (Definition 4.5) will not cause any problem, indeed it will simplify the exposition.

In the rest of this chapter we will characterize the structure of the value function and describe how the optimal control law can be efficiently computed by means of multiparametric linear and quadratic programming. We will distinguish the cases 1- or ∞-norm and squared 2-norm.

11.2 Feasible Solutions

We denote with $\mathcal{X}_i$ the set of states x_i at time i for which (11.9) is feasible, for $i = 0, \ldots, N$. The sets $\mathcal{X}_i$ for $i = 0, \ldots, N$ play an important role in the solution of (11.9). They are independent of the cost function (as long as it guarantees the existence of a minimum) and of the algorithm used to compute the solution to problem (11.9). There are two ways to rigorously define and compute the sets $\mathcal{X}_i$: the *batch approach* and the *recursive approach*. In the batch approach

$$\begin{aligned}\mathcal{X}_i = \{x_i \in \mathcal{X}:\ \exists(u_i, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = i, \ldots, N-1,\\ x_N \in \mathcal{X}_f \text{ where } x_{k+1} = Ax_k + Bu_k,\ k = i, \ldots, N-1\}.\end{aligned} \tag{11.16}$$

The definition of $\mathcal{X}_i$ in (11.16) requires that for any initial state $x_i \in \mathcal{X}_i$ there exists a feasible sequence of inputs $U_i = [u_i', \ldots, u_{N-1}']$ which keeps the state evolution in the feasible set $\mathcal{X}$ at future time instants $k = i+1, \ldots, N-1$ and forces x_N into $\mathcal{X}_f$ at time N. Clearly $\mathcal{X}_N = \mathcal{X}_f$. Next we show how to compute $\mathcal{X}_i$ for $i = 0, \ldots, N-1$. Let the state and input constraint sets $\mathcal{X}$, $\mathcal{X}_f$ and $\mathcal{U}$ be the $\mathcal{H}$-polyhedra $A_x x \le b_x$, $A_f x_N \le b_f$, $A_u u \le b_u$, respectively. Define the polyhedron $\mathcal{P}_i$ for $i = 0, \ldots, N-1$ as follows

$$\mathcal{P}_i = \{(U_i, x_i) \in \mathbb{R}^{m(N-i)+n}:\ G_i U_i - E_i x_i \le w_i\}, \tag{11.17}$$

where G_i, E_i and w_i are defined as follows

$$G_i = \begin{bmatrix} A_u & 0 & \ldots & 0 \\ 0 & A_u & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & A_u \\ 0 & 0 & \ldots & 0 \\ A_x B & 0 & \ldots & 0 \\ A_x AB & A_x B & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_f A^{N-i-1}B & A_f A^{N-i-2}B & \ldots & A_f B \end{bmatrix},\ E_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^{N-i} \end{bmatrix},\ w_i = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}. \tag{11.18}$$

The set $\mathcal{X}_i$ is a polyhedron as it is the projection of the polyhedron $\mathcal{P}_i$ in (11.17)– (11.18) on the x_i space.

In the *recursive approach*,

$$\begin{aligned} \mathcal{X}_i &= \{x \in \mathcal{X} : \ \exists u \in \mathcal{U} \text{ such that } Ax + Bu \in \mathcal{X}_{i+1}\}, \\ &\quad i = 0, \dots, N-1 \\ \mathcal{X}_N &= \mathcal{X}_f. \end{aligned} \tag{11.19}$$

The definition of $\mathcal{X}_i$ in (11.19) is recursive and requires that for any feasible initial state $x_i \in \mathcal{X}_i$ there exists a feasible input u_i which keeps the next state $Ax_i + Bu_i$ in the feasible set $\mathcal{X}_{i+1}$. It can be compactly written as

$$\mathcal{X}_i = \text{Pre}(\mathcal{X}_{i+1}) \cap \mathcal{X}. \tag{11.20}$$

Initializing $\mathcal{X}_N$ to $\mathcal{X}_f$ and solving (11.19) backward in time yields *the same* sets $\mathcal{X}_i$ as the batch approach. This recursive formulation, however, leads to an alternative approach for computing the sets $\mathcal{X}_i$. Let $\mathcal{X}_i$ be the $\mathcal{H}$-polyhedra $A_{\mathcal{X}_i}x \le b_{\mathcal{X}_i}$. Then the set $\mathcal{X}_{i-1}$ is the projection of the following polyhedron

$$\begin{bmatrix} A_u \\ 0 \\ A_{\mathcal{X}_i}B \end{bmatrix} u_i + \begin{bmatrix} 0 \\ A_x \\ A_{\mathcal{X}_i}A \end{bmatrix} x_i \le \begin{bmatrix} b_u \\ b_x \\ b_{\mathcal{X}_i} \end{bmatrix} \tag{11.21}$$

on the x_i space.

Consider problem (11.9). The set $\mathcal{X}_0$ is the set of all initial states x_0 for which (11.9) is feasible. The sets $\mathcal{X}_i$ with $i = 1, \dots, N-1$ are hidden. A given $\bar{U}_0 = [\bar{u}_0, \dots, \bar{u}_{N-1}]$ is feasible for problem (11.9) if and only if at all time instants i, the state x_i obtained by applying $\bar{u}_0, \dots, \bar{u}_{i-1}$ to the system model $x_{k+1} = Ax_k + Bu_k$ with initial state $x_0 \in \mathcal{X}_0$ belongs to $\mathcal{X}_i$. Also, $\mathcal{X}_i$ is the set of feasible initial states for problem

$$\begin{aligned} J_i^*(x(0)) = \min_{U_i} \quad & p(x_N) + \sum_{k=i}^{N-1} q(x_k, u_k) \\ \text{subj. to} \quad & x_{k+1} = Ax_k + Bu_k, \ k = i, \dots, N-1 \\ & x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \ k = i, \dots, N-1 \\ & x_N \in \mathcal{X}_f. \end{aligned} \tag{11.22}$$

Next, we provide more insights into the set $\mathcal{X}_i$ by using the invariant set theory of Section 10.2. We consider two cases: (1) $\mathcal{X}_f = \mathcal{X}$ which corresponds to effectively “removing” the terminal constraint set and (2) $\mathcal{X}_f$ chosen to be a control invariant set.

Theorem 11.1 *[172, Theorem 5.3]. Let the terminal constraint set $\mathcal{X}_f$ be equal to $\mathcal{X}$. Then,*

1. *The feasible set $\mathcal{X}_i$, $i = 0, \ldots, N-1$ is equal to the $(N-i)$-step controllable set:*

$$\mathcal{X}_i = \mathcal{K}_{N-i}(\mathcal{X}).$$

2. *The feasible set $\mathcal{X}_i$, $i = 0, \ldots, N-1$ contains the maximal control invariant set:*

$$\mathcal{C}_\infty \subseteq \mathcal{X}_i.$$

3. *The feasible set $\mathcal{X}_i$ is control invariant if and only if the maximal control invariant set is finitely determined and $N-i$ is equal to or greater than its determinedness index $\bar{N}$, i.e.,*

$$\mathcal{X}_i \subseteq Pre(\mathcal{X}_i) \Leftrightarrow \mathcal{C}_\infty = \mathcal{K}_{N-i}(\mathcal{X}) \quad for\ all\ i \leq N - \bar{N}.$$

4. $\mathcal{X}_i \subseteq \mathcal{X}_j$ *if $i < j$ for $i = 0, \ldots, N-1$. The size of the feasible set $\mathcal{X}_i$ stops decreasing (with decreasing i) if and only if the maximal control invariant set is finitely determined and $N-i$ is larger than its determinedness index, i.e.,*

$$\mathcal{X}_i \subset \mathcal{X}_j \ \ if\ N - \bar{N} < i < j < N.$$

Furthermore,

$$\mathcal{X}_i = \mathcal{C}_\infty \ \ if\ i \leq N - \bar{N}.$$

Theorem 11.2 *[172, Theorem 5.4]. Let the terminal constraint set $\mathcal{X}_f$ be a control invariant subset of $\mathcal{X}$. Then,*

1. *The feasible set $\mathcal{X}_i$, $i = 0, \ldots, N-1$ is equal to the $(N-i)$-step stabilizable set:*

$$\mathcal{X}_i = \mathcal{K}_{N-i}(\mathcal{X}_f).$$

2. *The feasible set $\mathcal{X}_i$, $i = 0, \ldots, N-1$ is control invariant and contained within the maximal control invariant set:*

$$\mathcal{X}_i \subseteq \mathcal{C}_\infty.$$

3. $\mathcal{X}_i \supseteq \mathcal{X}_j$ *if $i < j$, $i = 0, \ldots, N-1$. The size of the feasible $\mathcal{X}_i$ set stops increasing (with decreasing i) if and only if the maximal stabilizable set is finitely determined and $N-i$ is larger than its determinedness index, i.e.,*

$$\mathcal{X}_i \supset \mathcal{X}_j \ \ if\ N - \bar{N} < i < j < N.$$

Furthermore,

$$\mathcal{X}_i = \mathcal{K}_\infty(\mathcal{X}_f) \ \ if\ i \leq N - \bar{N}.$$

Remark 11.3 Theorems 11.1 and 11.2 help us understand how the feasible sets $\mathcal{X}_i$ propagate backward in time as a function of the terminal set $\mathcal{X}_f$. In particular, when $\mathcal{X}_f = \mathcal{X}$ the set $\mathcal{X}_i$ shrinks as *i* becomes smaller and stops shrinking when it becomes the maximal control invariant set. Also, depending on *i*, either it is not a control invariant set or it is the maximal control invariant set. We have the opposite if a control invariant set is chosen as terminal constraint $\mathcal{X}_f$. The set $\mathcal{X}_i$ grows as i becomes smaller and stops growing when it becomes the maximal stabilizable set. Both cases are shown in the Example 11.1 below.

Remark 11.4 In this section we investigated the behavior of $\mathcal{X}_i$ as i varies for a fixed horizon N. Equivalently, we could study the behavior of $\mathcal{X}_0$ as the horizon N varies. Specifically, the sets $\mathcal{X}_{0\to N_1}$ and $\mathcal{X}_{0\to N_2}$ with $N_2 > N_1$ are equal to the sets $\mathcal{X}_{N_2-N_1\to N}$ and $\mathcal{X}_{0\to N}$, respectively, with $N = N_2$.

Example 11.1 Consider the double integrator

$$\begin{cases} x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases} \tag{11.23}$$

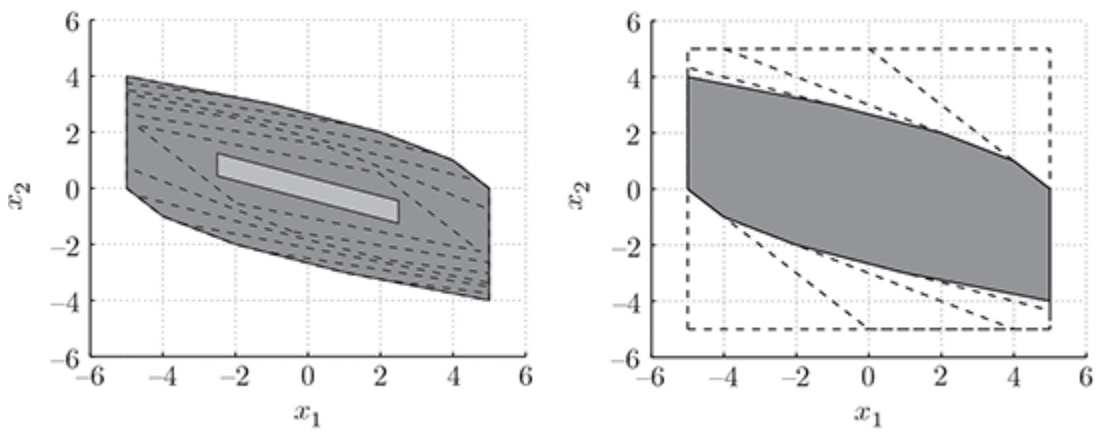
subject to the input constraints

$$-1 \le u(k) \le 1 \text{ for all } k \ge 0 \tag{11.24}$$

and the state constraints

$$\begin{bmatrix} -5 \\ -5 \end{bmatrix} \le x(k) \le \begin{bmatrix} 5 \\ 5 \end{bmatrix} \text{ for all } k \ge 0. \tag{11.25}$$

We compute the feasible sets $\mathcal{X}_i$ and plot them in Figure 11.1 in two cases.



(a) Case 1: $\mathcal{X}_f$ = control invariant set in (11.26).

(b) Case 2: $\mathcal{X}_f = \mathcal{X}$.

Figure 11.1 Example 11.1. Propagation of the feasible sets $\mathcal{X}_i$ to arrive at $\mathcal{C}_\infty$ (shaded dark).

Case 1. $\mathcal{X}_f$ is the control invariant set

$$\begin{bmatrix} -0.32132 & -0.94697 \\ 0.32132 & 0.94697 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 0.3806 \\ 0.3806 \\ 2.5 \\ 2.5 \end{bmatrix}. \tag{11.26}$$

After six iterations the sets $\mathcal{X}_i$ converge to the following $\mathcal{K}_\infty(\mathcal{X}_f)$

$$\begin{bmatrix} -0.44721 & -0.89443 \\ -0.24254 & -0.97014 \\ -0.31623 & -0.94868 \\ 0.24254 & 0.97014 \\ 0.31623 & 0.94868 \\ 0.44721 & 0.89443 \\ 1 & 0 \\ -1 & 0 \\ 0.70711 & 0.70711 \\ -0.70711 & -0.70711 \end{bmatrix} x \leq \begin{bmatrix} 2.6833 \\ 2.6679 \\ 2.5298 \\ 2.6679 \\ 2.5298 \\ 2.6833 \\ 5 \\ 5 \\ 3.5355 \\ 3.5355 \end{bmatrix}. \tag{11.27}$$

Note that in this case $\mathcal{C}_\infty = \mathcal{K}_\infty(\mathcal{X}_f)$ and the determinedness index is six.

Case 2. $\mathcal{X}_f = \mathcal{X}$. After six iterations the sets $\mathcal{X}_i$ converge to $\mathcal{K}_\infty(\mathcal{X}_f)$ in (11.27).

11.3 2-Norm Case Solution

Consider problem (11.9) with $J_0(\cdot)$ defined by (11.8). In this chapter we always assume that $Q = Q' \succeq 0,\ R = R' \succ 0,\ P = P' \succeq 0.$

$$\begin{array}{rl} J_0^*(x(0)) = \min_{U_0} & J_0(x(0), U_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \\ \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(0). \end{array} \tag{11.28}$$

11.3.1 Solution via QP

As shown in Section 8.2, problem (11.28) can be rewritten as

$$\begin{array}{rl} J_0^*(x(0)) = \min\limits_{U_0} & J_0(x(0), U_0) = U_0' H U_0 + 2x'(0) F U_0 + x'(0) Y x(0) \\ = \min\limits_{U_0} & J_0(x(0), U_0) = [U_0' \ x'(0)] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0' \ x(0)']' \\ \text{subj. to} & G_0 U_0 \leq w_0 + E_0 x(0), \end{array} \tag{11.29}$$

with G_0, w_0 and E_0 defined in (11.18) for $i = 0$ and H, F, Y defined in (8.8). As $J_0(x(0), U_0) \geq 0$ by definition it follows that $\left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] \succeq 0$.

For a given vector $x(0)$ the optimal input sequence U_0^* solving problem (11.29) can be computed by using a Quadratic Program (QP) solver (see Section 2.3 for QP definition and properties and Chapter 3 for fast numerical methods for solving QPs).

To obtain the problem (11.29) we have eliminated the state variables and equality constraints $x_{k+1} = Ax_k+Bu_k$ by successive substitution so that we are left with $u_0, \ldots, u_{N-1}$ as the only decision variables and $x(0)$ as a parameter vector. In general, it might be more efficient to solve a QP problem with equality and inequality constraints so that sparsity can be exploited. To this aim we can define the variable $\tilde{z}$ as

$$\tilde{z} = \left[\begin{array}{cccccc} x_1' & \ldots & x_N' & u_0' & \ldots & u_{N-1}' \end{array}\right]'$$

and rewrite problem (11.28) as

$$\begin{array}{rll} J_0^*(x(0)) = & \min\limits_{\tilde{z}} & [\tilde{z}'\ x(0)'] \left[\begin{smallmatrix} \bar{H} & 0 \\ 0 & Q \end{smallmatrix}\right] [\tilde{z}'\ x(0)']' \\ & \text{subj. to} & G_{0,\text{eq}}\tilde{z} = E_{0,\text{eq}}x(0) \\ & & G_{0,\text{in}}\tilde{z} \le w_{0,\text{in}} + E_{0,\text{in}}x(0). \end{array} \tag{11.30}$$

For a given vector $x(0)$ the optimal input sequence U_0^* solving problem (11.30) can be computed by using a Quadratic Program (QP) solver.

To obtain problem (11.30) we have rewritten the equalities from system dynamics $x_{k+1} = Ax_k + Bu_k$ as $G_{0,\text{eq}}\tilde{z} = E_{0,\text{eq}}x(0)$ where

$$G_{0,\text{eq}} = \left[\begin{array}{ccccc|cccc} I & & & & & -B & & & \\ -A & I & & & & & -B & & \\ & -A & I & & & & & -B & \\ & & \ddots & \ddots & & & & & \ddots \\ & & & -A & I & & & & -B \end{array}\right],\quad E_{0,\text{eq}} = \left[\begin{array}{c} A \\ 0 \\ \vdots \\ 0 \end{array}\right],$$

and rewritten state and input constraints as $G_{0,\text{in}}\tilde{z} \le w_{0,\text{in}} + E_{0,\text{in}}x(0)$ where

$$G_{0,\text{in}} = \left[\begin{array}{ccccc|ccccc} 0 & & & & & 0 & & & & \\ A_x & 0 & & & & 0 & & & & \\ & A_x & & & & & 0 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & A_x & & & & & 0 & \\ & & & & A_f & & & & & 0 \\ \hline 0 & & & & & A_u & & & & \\ & 0 & & & & & A_u & & & \\ & & \ddots & & & & & \ddots & & \\ & & & 0 & & & & & A_u & \\ & & & & 0 & & & & & A_u \end{array}\right],\quad w_{0,\text{in}} = \left[\begin{array}{c} b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ \hline b_u \\ b_u \\ \vdots \\ b_u \\ b_u \end{array}\right],$$

$$E_{0,\text{in}} = \left[\begin{array}{cccc} -A_x' & 0 & \cdots & 0 \end{array}\right]',$$

and constructed the cost matrix $\overline{H}$ as

$$\bar{H} = \left[\begin{array}{cccc|ccc} Q & & & & & & \\ & \ddots & & & & & \\ & & Q & & & & \\ & & & P & & & \\ \hline & & & & R & & \\ & & & & & \ddots & \\ & & & & & & R \end{array}\right].$$

11.3.2 State Feedback Solution via Batch Approach

As shown in Section 8.2, problem (11.28) can be rewritten as

$$\begin{aligned} J_0^*(x(0)) = \min_{U_0} \quad & J_0(x(0), U_0) = [U_0' \; x'(0)] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0{}' \; x(0)']' \\ \text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0), \end{aligned} \tag{11.31}$$

with G_0, w_0 and E_0 defined in (11.18) for $i = 0$ and H, F, Y defined in (8.8). As $J_0(x(0), U_0) \geq 0$ by definition it follows that $\left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] \succeq 0$. Note that the optimizer U_0^* is independent of the term involving Y in (11.31).

We view $x(0)$ as a vector of parameters and our goal is to solve (11.31) for all values of $x(0) \in \mathcal{X}_0$ and to make this dependence *explicit*. The computation of the set $\mathcal{X}_0$ of initial states for which problem (11.31) is feasible was discussed in Section 11.2.

Before proceeding further, it is convenient to define

$$z = U_0 + H^{-1}F'x(0), \tag{11.32}$$

$z \in \mathbb{R}^s$, remove $x(0)'Y x(0)$ and to transform (11.31) to obtain the equivalent problem

$$\begin{aligned} \hat{J}^*(x(0)) = \min_{z} \quad & z'Hz \\ \text{subj. to} \quad & G_0 z \leq w_0 + S_0 x(0), \end{aligned} \tag{11.33}$$

where $S_0 = E_0 + G_0 H^{-1} F'$, and $\hat{J}^*(x(0)) = J_0^*(x(0)) - x(0)'(Y - FH^{-1}F')x(0)$. In the transformed problem the parameter vector $x(0)$ appears only on the right-hand side of the constraints.

Problem (11.33) is a multiparametric quadratic program that can be solved by using the algorithm described in Section 6.3.1. Once the multiparametric problem (11.33) has been solved, the solution $U_0^* = U_0^*(x(0))$ of CFTOC (11.28) and therefore $u^*(0) = u^*(x(0))$ is available explicitly as a function of the initial state $x(0)$ for all $x(0) \in \mathcal{X}_0$.

Theorem 6.7 states that the solution $z^*(x(0))$ of the mp-QP problem (11.33) is a continuous and piecewise affine function on polyhedra of $x(0)$. Clearly the same properties are inherited by the controller. The following corollaries of Theorem 6.7 establish the analytical properties of the optimal control law and of the value function.

Corollary 11.1 *The control law $u^*(0) = f_0(x(0))$, $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, obtained as a solution of the CFTOC (11.28) is continuous and piecewise affine on polyhedra*

$$f_0(x) = F_0^j x + g_0^j \quad \textit{if} \quad x \in CR_0^j, \quad j = 1, \ldots, N_0^r, \tag{11.34}$$

where the polyhedral sets $CR_0^j = \{x \in \mathbb{R}^n \ : \ H_0^j x \le K_0^j\}$, $j = 1, \ldots, N_0^r$ are a partition of the feasible polyhedron $\mathcal{X}_0$.

Proof: From (11.32) $U_0^*(x(0)) = z^*(x(0)) - H^{-1}F'x(0)$. From Theorem 6.7 we know that $z^*(x(0))$, solution of (11.33), is PPWA and continuous. As $U_0^*(x(0))$ is a linear combination of a linear function and a PPWA function, it is PPWA. As $U_0^*(x(0))$ is a linear combination of two continuous functions it is continuous. In particular, these properties hold for the first component $u^*(0)$ of U_0^*. ■

Remark 11.5 Note that, as discussed in Remark 6.8, the critical regions defined in (6.4) are in general sets that are neither closed nor open. In Corollary 11.1 the polyhedron CR_0^i describes the closure of a critical region. The function $f_0(x)$ is continuous and therefore it is simpler to use a polyhedral partition rather than a strict polyhedral partition.

Corollary 11.2 *The value function $J_0^*(x(0))$ obtained as solution of the CFTOC (11.28) is convex and piecewise quadratic on polyhedra. Moreover, if the mp-QP problem (11.33) is not degenerate, then the value function $J_0^*(x(0))$ is $C^{(1)}$.*

Proof: By Theorem 6.7 $\hat{J}^*(x(0))$ is a convex function of $x(0)$. As $\left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] \succeq 0$, its Schur complement[1] $Y - FH^{-1}F' \succeq 0$, and therefore $J_0^*(x(0)) = \hat{J}^*(x(0)) + x(0)'(Y - FH^{-1}F')x(0)$ is a convex function, because it is the sum of convex functions. If the mp-QP problem (11.33) is not degenerate, then Theorem 6.9 implies that $\hat{J}^*(x(0))$ is a $C^{(1)}$ function of $x(0)$ and therefore $J_0^*(x(0))$ is a $C^{(1)}$ function of $x(0)$. The results of Corollary 11.1 imply that $J_0^*(x(0))$ is piecewise quadratic. ■

Remark 11.6 The relation between the design parameters of the optimal control problem (11.28) and the degeneracy of the mp-QP problem (11.33) is complex, in general.

The solution of the multiparametric problem (11.33) provides the state feedback solution $u^*(k) = f_k(x(k))$ of CFTOC (11.28) for $k = 0$ and it also provides the open-loop optimal control $u^*(k)$ as function of the initial state, i.e., $u^*(k) = u^*(k, x(0))$. The state feedback PPWA optimal controllers $u^*(k) = f_k(x(k))$ with $f_k : \mathcal{X}_k \mapsto \mathcal{U}$ for $k = 1, \ldots, N$ are computed in the following way. Consider the same CFTOC (11.28) over the shortened time-horizon $[i, N]$

$$\begin{array}{ll} \min_{U_i} & x_N' P x_N + \sum_{k=i}^{N-1} x_k' Q x_k + u_k' R u_k \\ \text{subj. to} & x_{k+1} = A x_k + B u_k,\ k = i, \dots, N-1 \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = i, \dots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_i = x(i), \end{array} \tag{11.35}$$

where $U_i = [u_i', \dots, u_{N-1}']$. As defined in (11.16) and discussed in Section 11.2, $\mathcal{X}_i \subseteq \mathbb{R}^n$ is the set of initial states $x(i)$ for which the optimal control problem (11.35) is feasible. We denote by U_i^* the optimizer of the optimal control problem (11.35).

Problem (11.35) can be translated into the mp-QP

$$\begin{array}{ll} \min & U_i{}' H_i U_i + 2x'(i) F_i U_i + x'(i) Y_i x(i) \\ \text{subj. to} & G_i U_i \le w_i + E_i x(i), \end{array} \tag{11.36}$$

where $H_i = H_i' \succ 0$, F_i, Y_i are appropriately defined for each i and G_i, w_i, E_i are defined in (11.18). The first component of the multiparametric solution of (11.36) has the form

$$u_i^*(x(i)) = f_i(x(i)),\ \forall x(i) \in \mathcal{X}_i, \tag{11.37}$$

where the control law $f_i : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PPWA

$$f_i(x) = F_i^j x + g_i^j \quad \text{if} \quad x \in CR_i^j,\ j = 1, \dots, N_i^r, \tag{11.38}$$

and where the polyhedral sets $CR_i^j = \{x \in \mathbb{R}^n \ :\ H_i^j x \le K_i^j\}$, $j = 1, \dots, N_i^r$ are a partition of the feasible polyhedron $\mathcal{X}_i$. Therefore the feedback solution $u^*(k) = f_k(x(k))$, $k = 0, \dots, N-1$ of the CFTOC (11.28) is obtained by solving N mp-QP problems of decreasing size. The following corollary summarizes the final result.

Corollary 11.3 *The state feedback control law $u^*(k) = f_k(x(k))$, $f_k : \mathcal{X}_k \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq \mathbb{R}^m$, obtained as a solution of the CFTOC (11.28) and $k = 0, \dots, N-1$ is time-varying, continuous and piecewise affine on polyhedra*

$$f_k(x) = F_k^j x + g_k^j \quad \textit{if} \quad x \in CR_k^j, \quad j = 1, \dots, N_k^r, \tag{11.39}$$

where the polyhedral sets $CR_k^j = \{x \in \mathbb{R}^n \ :\ H_k^j x \le K_k^j\}$, $j = 1, \dots, N_k^r$ are a partition of the feasible polyhedron $\mathcal{X}_k$.

11.3.3 State Feedback Solution via Recursive Approach

Consider the dynamic programming formulation of the CFTOC (11.28)

$$
\begin{aligned}
J_j^*(x_j) = \min_{u_j} \quad & x_j' Q x_j + u_j' R u_j + J_{j+1}^*(Ax_j + Bu_j) \\
\text{subj. to} \quad & x_j \in \mathcal{X},\ u_j \in \mathcal{U}, \\
& Ax_j + Bu_j \in \mathcal{X}_{j+1}
\end{aligned}
\tag{11.40}
$$

for $j = 0, \dots, N-1$, with boundary conditions

$$J_N^*(x_N) = x_N' P x_N \tag{11.41}$$

$$\mathcal{X}_N = \mathcal{X}_f, \tag{11.42}$$

where $\mathcal{X}_j$ denotes the set of states x for which the CFTOC (11.28) is feasible at time j (as defined in (11.16)). Note that according to Corollary 11.2, $J_{j+1}^*(Ax_j + Bu_j)$ is piecewise quadratic for $j < N-1$. Therefore (11.40) is not simply an mp-QP and, contrary to the unconstrained case (Section 8.2), the computational advantage of the iterative over the batch approach is not obvious. Nevertheless an algorithm was developed and can be found in Section 17.6.

11.3.4 Infinite Horizon Problem

Assume $Q \succ 0$, $R \succ 0$ and that the constraint sets $\mathcal{X}$ and $\mathcal{U}$ contain the origin in their interior.[2] Consider the following infinite-horizon linear quadratic regulation problem with constraints (CLQR)

$$
\begin{aligned}
J_\infty^*(x(0)) = \min_{u_0, u_1, \dots} \quad & \sum_{k=0}^{\infty} x_k' Q x_k + u_k' R u_k \\
\text{subj. to} \quad & x_{k+1} = Ax_k + Bu_k,\ k = 0, \dots, \infty \\
& x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \dots, \infty \\
& x_0 = x(0)
\end{aligned}
\tag{11.43}
$$

and the set (see Remark 7.1 for notation)

$$\mathcal{X}_\infty = \{x(0) \in \mathbb{R}^n \ : \ \text{Problem (11.43) is feasible and } J_\infty^*(x(0)) < +\infty\}. \tag{11.44}$$

Because $Q \succ 0$, $R \succ 0$ *any* optimizer u_k^* of problem (11.43) *must* converge to the origin $(u_k^* \to 0)$ and so must the state trajectory resulting from the application of u_k^* $(x_k^* \to 0)$. Thus the origin $x = 0$, $u = 0$ must lie in the interior of the constraint set $(\mathcal{X}, \mathcal{U})$ (if the origin were not contained in the constraint set then $J_\infty^*(x(0))$ would be infinite). For this reason, the set $\mathcal{X}_\infty$ in (11.44) is the maximal stabilizable set $\mathcal{K}_\infty(\mathcal{O})$ of system (11.1) subject to the constraints (11.2) with $\mathcal{O}$ being the origin (Definition 10.13).

If the initial state $x_0 = x(0)$ is sufficiently close to the origin, then the constraints will never become active and the solution of problem (11.43) will yield the same control input as the *unconstrained* LQR (8.33). More formally we can define a corresponding invariant set around the origin.

Definition 11.1 (Maximal LQR Invariant Set $\mathcal{O}_\infty^{\mathrm{LQR}}$) *Consider the system* $x(k+1) = Ax(k) + Bu(k)$. $\mathcal{O}_\infty^{LQR} \subseteq \mathbb{R}^n$ *denotes the maximal positively invariant set for the autonomous constrained linear system:*

$$x(k+1) = (A + BF_\infty)x(k),\ x(k) \in \mathcal{X},\ u(k) \in \mathcal{U},\ \forall\ k \geq 0,$$

where $u(k) = F_\infty x(k)$ *is the unconstrained LQR control law (8.33) obtained from the solution of the ARE (8.32).*

Therefore, from the previous discussion, there is some finite time $\bar{N}(x_0)$, depending on the initial state x_0, at which the state enters $\mathcal{O}_\infty^{\mathrm{LQR}}$ and after which the system evolves in an unconstrained manner $(x_k^* \in \mathcal{X},\ u_k^* \in \mathcal{U},\ \forall k > \bar{N})$. This consideration allows us to split problem (11.43) into two parts by using the dynamic programming principle, one up to time $k = \bar{N}$ where the constraints may be active and one for longer times $k > \bar{N}$ where there are no constraints.

$$\begin{array}{lll} J_\infty^*(x(0)) = \min_{u_0,u_1,\ldots} & \sum_{k=0}^{\bar{N}-1} x_k' Q x_k + u_k' R u_k + J_{\bar{N}\to\infty}^*(x_{\bar{N}}) & \\ \text{subj. to} & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0,\ldots,\bar{N}-1 & \\ & x_{k+1} = Ax_k + Bu_k,\ k \geq 0 & \\ & x_0 = x(0), & \end{array} \tag{11.45}$$

where

$$\begin{array}{ll} J_{\bar{N}\to\infty}^*(x_{\bar{N}}) = \min_{u_{\bar{N}},u_{\bar{N}+1},\ldots} & \sum_{k=\bar{N}}^{\infty} x_k' Q x_k + u_k' R u_k \\ \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k \geq \bar{N} \\ = x_{\bar{N}}' P_\infty x_{\bar{N}}. & \end{array} \tag{11.46}$$

This key insight due to Sznaier and Damborg [272] is formulated precisely in the following.

Theorem 11.3 (Equality of Finite and Infinite Optimal Control, [260]) *For any given initial state* $x(0)$, *the solution to (11.45, 11.46) is equal to the infinite time solution of (11.43), if the terminal state* $x_{\bar{N}}$ *of (11.45) lies in the positive invariant set* $\mathcal{O}_\infty^{LQR}$ *and no terminal set constraint is applied in (11.45), i.e., the state 'voluntarily' enters the set* $\mathcal{O}_\infty^{LQR}$ *after* $\bar{N}$ *steps.*

Theorem 11.3 suggests that we can obtain the infinite horizon constrained linear quadratic regulator CLQR by solving the finite horizon problem for a horizon of $\bar{N}$ with a terminal weight of $P = P_\infty$ and *no* terminal constraint. The critical question of how to determine $\bar{N}(x_0)$ or at least an upper bound was studied by several researchers. Chmielewski and Manousiouthakis [85] presented an approach that provides a conservative estimate N_{est} of the finite horizon $\bar{N}(x_0)$ for all x_0 belonging to a compact set of initial conditions $\mathcal{S} \subseteq \mathcal{X}_\infty = \mathcal{K}_\infty(\mathbf{0})$ $(N_{est} \geq \bar{N}_{\mathcal{S}}(x_0),\ \forall x_0 \in \mathcal{S})$.

They solve a single, finite dimensional, convex program to obtain N_{est}. Their estimate can be used to compute the PWA solution of (11.45) for a particular set $\mathcal{S}$.

Alternatively, the quadratic program with horizon N_{est} can be solved to determine $u_0^*,\ u_1^*,\ldots,u_{N_{\text{est}}}^*$ for a particular $x(0) \in \mathcal{S}$. For a given initial state $x(0)$, rather then a set $\mathcal{S}$, Scokaert and Rawlings [260] presented an algorithm that attempts to identify $\overline{N}(x(0))$ iteratively. In summary, we can state the following Theorem.

Theorem 11.4 (Explicit solution of CLQR) *Assume that* (A, B) *is a stabilizable pair and* $(Q^{1/2}, A)$ *is an observable pair,* $R \succ 0$*. The state feedback solution to the CLQR problem (11.43) in a compact set of the initial conditions* $\mathcal{S} \subseteq \mathcal{X}_\infty = \mathcal{K}_\infty(\mathbf{0})$ *is time-invariant, continuous and piecewise affine on polyhedra*

$$u^*(k) = f_\infty(x(k)),\quad f_\infty(x) = F^j x + g^j \quad \textit{if} \quad x \in CR_\infty^j,\quad j = 1,\ldots,N_\infty^r, \tag{11.47}$$

where the polyhedral sets $CR_\infty^j = \{x \in \mathbb{R}^n\ :\ H^j x \le K^j\}$, $j = 1,\ldots,N_\infty^r$ *are a finite partition of the feasible compact polyhedron* $\mathcal{S} \subseteq \mathcal{X}_\infty$.

As argued previously, the complexity of the solution manifested by the number of polyhedral regions depends on the chosen horizon. As the various discussed techniques yield an N_{est} that may be too large by orders of magnitude this is not a viable proposition. An efficient algorithm for computing the PPWA solution to the CLQR problem is presented next.

11.3.5 CLQR Algorithm

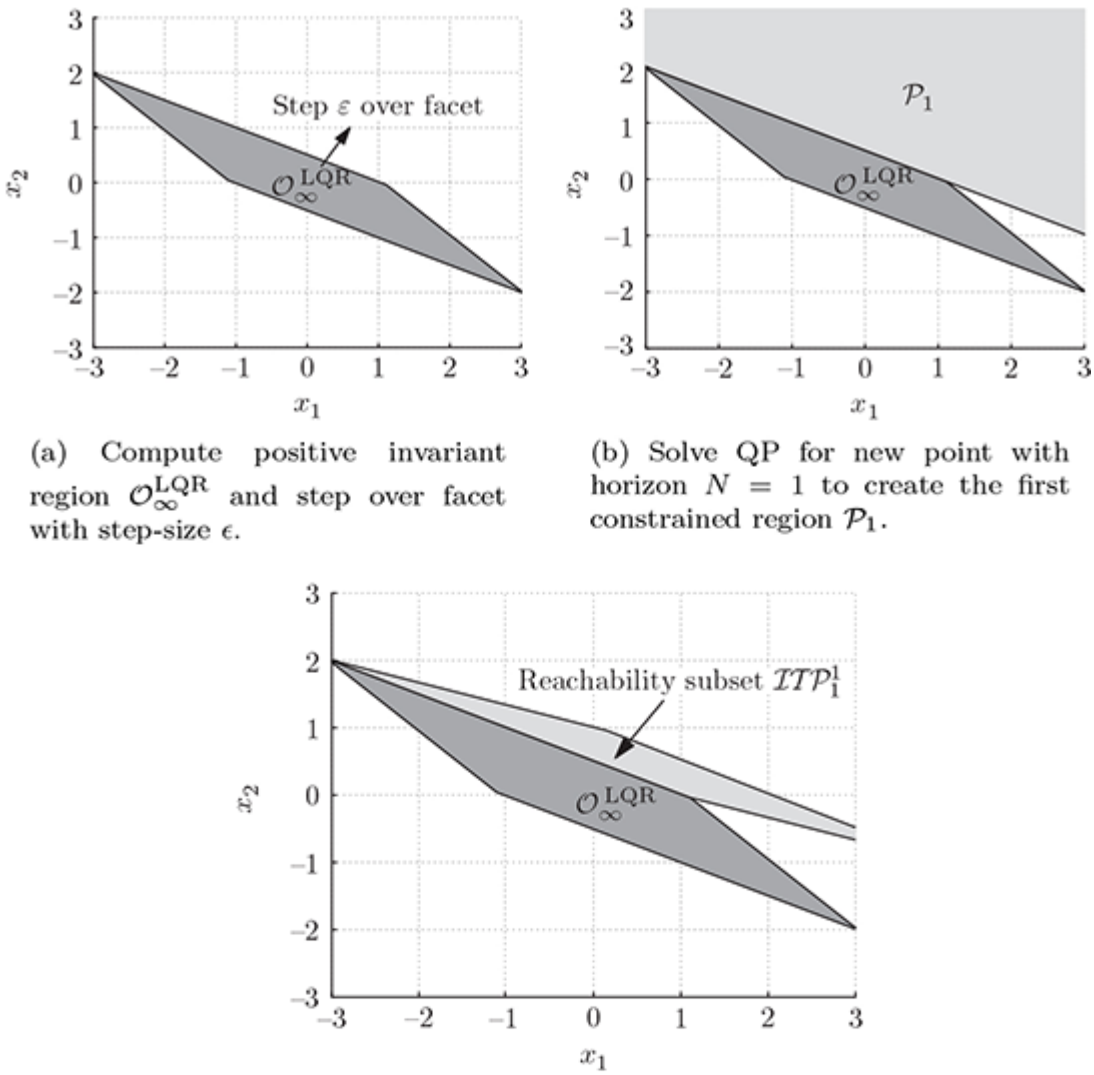
In this section we will sketch an efficient algorithm to compute the PWA solution to the CLQR problem in (11.43) for a given set $\mathcal{S}$ of initial conditions. Details are available in [132, 133]. As a side product, the algorithm also computes $\overline{N}_{\mathcal{S}}$, the shortest horizon $\overline{N}$ for which the problem (11.45), (11.46) is equivalent to the infinite horizon problem (11.43).

The idea is as follows. For the CFTOC problem (11.28) with a horizon N with no terminal constraint ($\mathcal{X}_f = \mathbb{R}^n$) and terminal cost $P = P_\infty$, where P_∞ is the solution to the ARE (8.32), we solve an mp-QP and obtain the PWA control law. From Theorem 11.3 we can conclude that for all states which enter the invariant set $\mathcal{O}_\infty^{\text{LQR}}$ introduced in Definition 11.1 with the computed control law in N steps, the infinite-horizon problem has been solved. For these states, which we can identify via a reachability analysis, the computed feedback law is infinite-horizon optimal.

In more detail, we start the procedure by computing the Maximal LQR Invariant Set $\mathcal{O}_\infty^{\text{LQR}}$ introduced in Definition 11.1, the polyhedron $\mathcal{P}_0 = \mathcal{O}_\infty^{\text{LQR}} = \{x \in \mathbb{R}^n\ :\ H_0 x \le K_0\}$. Figure 11.2(a) depicts $\mathcal{O}_\infty^{\text{LQR}}$. Then, the algorithm finds a point $\bar{x}$ by stepping over a facet of $\mathcal{O}_\infty^{\text{LQR}}$ with a small step ϵ, as described in [19]. If (11.28) is feasible for horizon $N = 1$ (terminal set constraint $\mathcal{X}_f = \mathbb{R}^n$, terminal cost $P = P_\infty$ and $x(0) = \bar{x}$), the active constraints will define the neighboring polyhedron $\mathcal{P}_1 = \{x \in \mathbb{R}^n : H_1 x \le K_1\}$ ($\bar{x} \in \mathcal{P}_1$, see Figure 11.2(b)) [44]. By Theorem 11.3, the finite time optimal solution computed above equals the infinite time optimal solution if $x_1 \in \mathcal{O}_\infty^{\text{LQR}}$. Therefore we extract from $\mathcal{P}_1$ the set of points that will enter $\mathcal{O}_\infty^{\text{LQR}}$ in $N = 1$ time-steps, provided that the optimal control law associated with $\mathcal{P}_1$ (i.e., $U_1^* = F_1 x(0) + g_1$) is applied. The Infinite Time Polyhedron $(\mathcal{ITP}_1^1)$ is therefore defined by the intersection of the following two polyhedra:

$$x_1 \in \mathcal{O}_\infty^{\mathrm{LQR}}, \; x_1 = Ax_0 + BU_1^*, \tag{11.48a}$$

$$x_0 \in \mathcal{P}_1. \tag{11.48b}$$



(a) Compute positive invariant region $\mathcal{O}_\infty^{\mathrm{LQR}}$ and step over facet with step-size ϵ.

(b) Solve QP for new point with horizon $N = 1$ to create the first constrained region $\mathcal{P}_1$.

(c) Compute reachability subset of $\mathcal{P}_1$ to obtain $\mathcal{ITP}_1^1$.

Figure 11.2 CLQR Algorithm. Region Exploration.

Equation (11.48a) is the reachability constraint and (11.48b) defines the set of states for which the computed feedback law is feasible and optimal over $N = 1$ steps (see [44] for details). The intersection is the set of points for which the control law is infinite time optimal.

A general step r of the algorithm involves stepping over a facet to a new point $\bar{x}$ and determining the polyhedron $\mathcal{P}_r$ and the associated control law $(U_N^* = F_r x(0) + g_r)$ from (11.28) with horizon N. Then we extract from $\mathcal{P}_r$ the set of points that will enter $\mathcal{O}_\infty^{\mathrm{LQR}}$ in N time-steps, provided that the optimal control law associated with $\mathcal{P}_r$ is applied. The Infinite Time Polyhedron $(\mathcal{ITP}_r^N)$ is therefore defined by the intersection of the following two polyhedra:

$$x_N \in \mathcal{O}_\infty^{\mathrm{LQR}} \tag{11.49a}$$

$$x_0 \in \mathcal{P}_r. \tag{11.49b}$$

This intersection is the set of points for which the control law is infinite time optimal. Note that, as for x_1 in the one-step case, x_N in (11.49a) can be described as a linear function of x_0 by substituting the feedback sequence $U_N^* = F_r x_0 + g_r$ into the LTI system dynamics (11.1).

We continue exploring the facets increasing N when necessary. The algorithm terminates when we have covered $\mathcal{S}$ or when we can no longer find a new feasible polyhedron $\mathcal{P}_r$. The following theorem shows that the algorithm also provides the horizon $\bar{N}_\mathcal{S}$ for compact sets. Exact knowledge of $\bar{N}_\mathcal{S}$ can serve to improve the performance of a wide array of algorithms presented in the literature.

Theorem 11.5 (Exact Computation of $\bar{N}_\mathcal{S}$, [132, 133]) *If we explore any given compact set $\mathcal{S}$ with the proposed algorithm, the largest resulting horizon is equal to $\bar{N}_\mathcal{S}$, i.e.,*

$$\bar{N}_\mathcal{S} = \max_{\mathcal{ITP}_r^N \quad r=0,\ldots,R} N.$$

Often the proposed algorithm is more efficient than standard multiparametric solvers, even if finite horizon optimal controllers are sought. The initial polyhedral representation $\mathcal{P}_r$ contains redundant constraints which need to be removed in order to obtain a minimal representation of the controller region. The intersection with the reachability constraint, as proposed here, can simplify this constraint removal.

11.3.6 Examples

Example 11.2 Consider the double integrator (11.23). We want to compute the state feedback optimal controller that solves problem (11.28) with $N = 6$, $Q = \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$, $R = 0.1$, P is equal to the solution of the Riccati equation (8.32), $\mathcal{X}_f = \mathbb{R}^2$. The input constraints are

$$-1 \leq u(k) \leq 1, \ \ k = 0, \ldots, 5 \tag{11.50}$$

and the state constraints

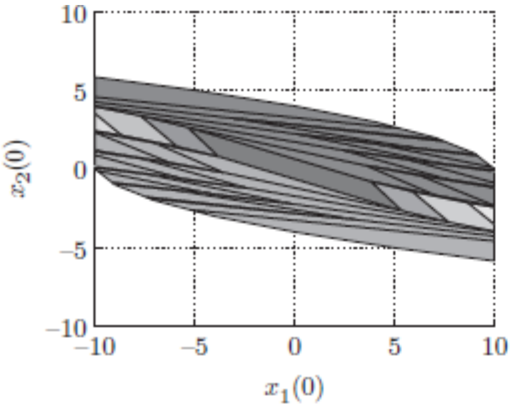
$$\begin{bmatrix}-10\\-10\end{bmatrix} \leq x(k) \leq \begin{bmatrix}10\\10\end{bmatrix}, \ \ k = 0, \ldots, 5. \tag{11.51}$$

This task is addressed as shown in Section (11.3.2). The feedback optimal solution $u^*(0), \ldots, u^*(5)$ is computed by solving six mp-QP problems and the corresponding polyhedral partitions of the state space are depicted in Figure 11.3. Only the last two optimal control moves are reported below:

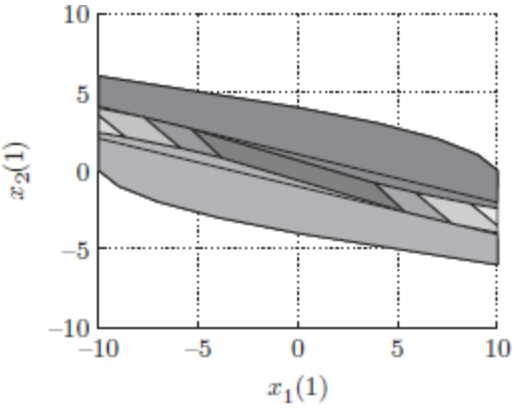
$$
u^*(5) = \begin{cases}
\begin{bmatrix} -0.58 & -1.55 \end{bmatrix} x(5) & \text{if } \begin{bmatrix} -0.35 & -0.94 \\ 0.35 & 0.94 \\ 1.00 & 0.00 \\ -1.00 & 0.00 \end{bmatrix} x(5) \le \begin{bmatrix} 0.61 \\ 0.61 \\ 10.00 \\ 10.00 \end{bmatrix} \quad \text{(Region \#1)} \\
1.00 & \text{if } \begin{bmatrix} 0.35 & 0.94 \\ 0.00 & -1.00 \\ -0.71 & -0.71 \\ 1.00 & 0.00 \\ -1.00 & 0.00 \end{bmatrix} x(5) \le \begin{bmatrix} -0.61 \\ 10.00 \\ 7.07 \\ 10.00 \\ 10.00 \end{bmatrix} \quad \text{(Region \#2)} \\
-1.00 & \text{if } \begin{bmatrix} -0.35 & -0.94 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.71 & 0.71 \end{bmatrix} x(5) \le \begin{bmatrix} -0.61 \\ 10.00 \\ 10.00 \\ 10.00 \\ 7.07 \end{bmatrix} \quad \text{(Region \#3)}
\end{cases}
$$

$$
u^*(4) = \begin{cases}
\begin{bmatrix} -0.58 & -1.55 \end{bmatrix} x(4) & \text{if } \begin{bmatrix} -0.35 & -0.94 \\ 0.35 & 0.94 \\ -0.77 & -0.64 \\ 0.77 & 0.64 \end{bmatrix} x(4) \le \begin{bmatrix} 0.61 \\ 0.61 \\ 2.43 \\ 2.43 \end{bmatrix} \quad \text{(Region \#1)} \\
1.00 & \text{if } \begin{bmatrix} 0.29 & 0.96 \\ 0.00 & -1.00 \\ -0.71 & -0.71 \\ -0.45 & -0.89 \\ 1.00 & 0.00 \\ -1.00 & 0.00 \end{bmatrix} x(4) \le \begin{bmatrix} -0.98 \\ 10.00 \\ 7.07 \\ 4.92 \\ 10.00 \\ 10.00 \end{bmatrix} \quad \text{(Region \#2)} \\
1.00 & \text{if } \begin{bmatrix} 0.29 & 0.96 \\ 0.35 & 0.94 \\ 0.00 & -1.00 \\ -0.71 & -0.71 \\ -0.45 & -0.89 \\ 1.00 & 0.00 \\ -1.00 & 0.00 \end{bmatrix} x(4) \le \begin{bmatrix} -0.37 \\ -0.61 \\ 10.00 \\ 7.07 \\ 4.92 \\ 10.00 \\ 10.00 \end{bmatrix} \quad \text{(Region \#3)} \\
-1.00 & \text{if } \begin{bmatrix} -0.29 & -0.96 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.71 & 0.71 \\ 0.45 & 0.89 \end{bmatrix} x(4) \le \begin{bmatrix} -0.98 \\ 10.00 \\ 10.00 \\ 10.00 \\ 7.07 \\ 4.92 \end{bmatrix} \quad \text{(Region \#4)} \\
-1.00 & \text{if } \begin{bmatrix} -0.29 & -0.96 \\ -0.35 & -0.94 \\ 1.00 & 0.00 \\ 0.00 & 1.00 \\ -1.00 & 0.00 \\ 0.71 & 0.71 \\ 0.45 & 0.89 \end{bmatrix} x(4) \le \begin{bmatrix} -0.37 \\ -0.61 \\ 10.00 \\ 10.00 \\ 10.00 \\ 7.07 \\ 4.92 \end{bmatrix} \quad \text{(Region \#5)} \\
\begin{bmatrix} -0.44 & -1.43 \end{bmatrix} x(4) - 0.46 & \text{if } \begin{bmatrix} -0.29 & -0.96 \\ 0.29 & 0.96 \\ -0.77 & -0.64 \\ 1.00 & 0.00 \end{bmatrix} x(4) \le \begin{bmatrix} 0.98 \\ 0.37 \\ -2.43 \\ 10.00 \end{bmatrix} \quad \text{(Region \#6)} \\
\begin{bmatrix} -0.44 & -1.43 \end{bmatrix} x(4) + 0.46 & \text{if } \begin{bmatrix} -0.29 & -0.96 \\ 0.29 & 0.96 \\ 0.77 & 0.64 \\ -1.00 & 0.00 \end{bmatrix} x(4) \le \begin{bmatrix} 0.37 \\ 0.98 \\ -2.43 \\ 10.00 \end{bmatrix} \quad \text{(Region \#7)}
\end{cases}
$$

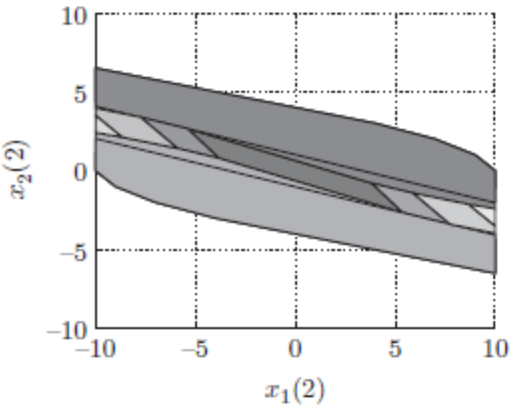
Note that by increasing the horizon N, the control law changes only far away from the origin. This must be expected from the results of Section 11.3.5. The control law does not change anymore with increasing N in the set where the CFTOC law becomes equal to the constrained infinite-horizon linear quadratic regulator (CLQR) problem. This set gets larger as N increases [85, 260].



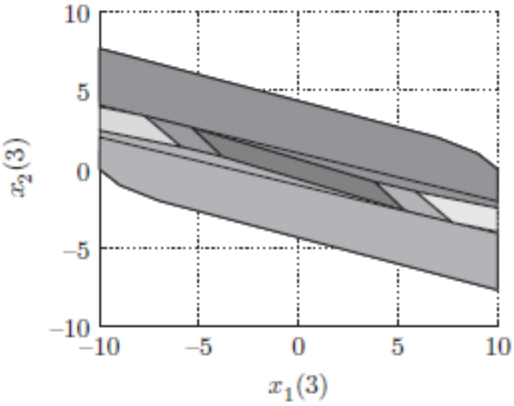
(a) Partition of the state space for the affine control law $u^*(0)$ ($N_0^r = 13$).



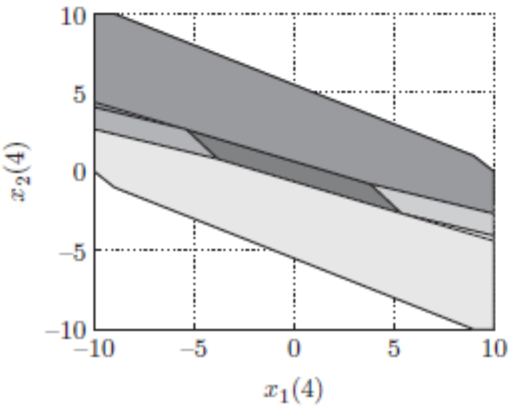
(b) Partition of the state space for the affine control law $u^*(1)$ ($N_1^r = 13$).



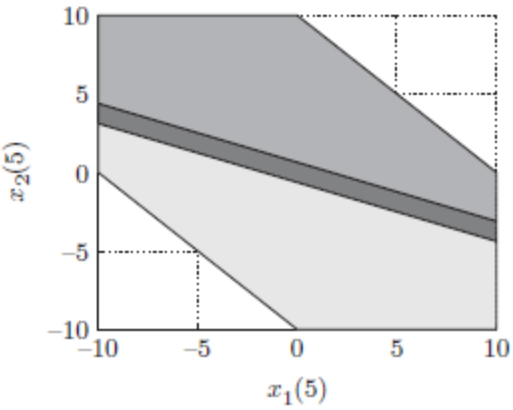
(c) Partition of the state space for the affine control law $u^*(2)$ ($N_2^r = 13$).



(d) Partition of the state space for the affine control law $u^*(3)$ ($N_3^r = 11$).



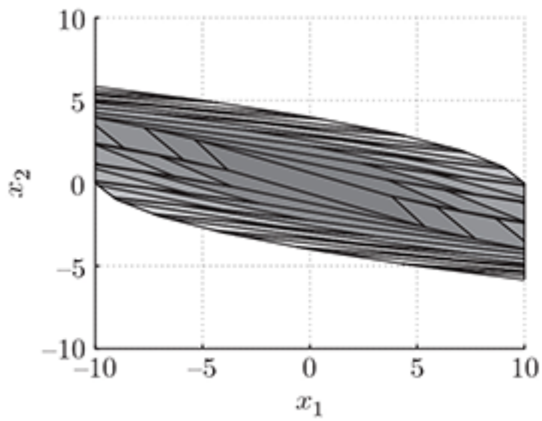
(e) Partition of the state space for the affine control law $u^*(4)$ ($N_4^r = 7$).



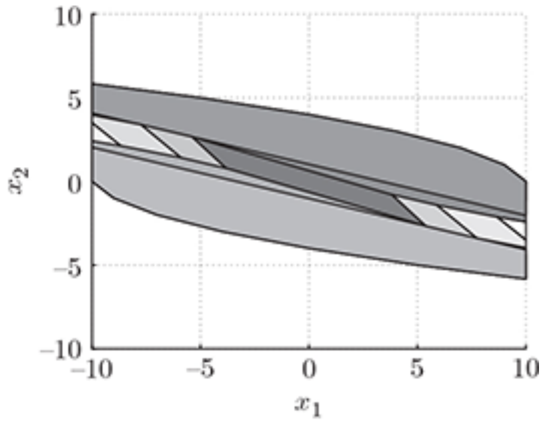
(f) Partition of the state space for the affine control law $u^*(5)$ ($N_5^r = 3$).

Figure 11.3 Example 11.2. Double Integrator, 2-norm objective function, horizon N = 6. Partition of the state space for the time-varying optimal control law. Polyhedra with the same control law were merged.

Example 11.3 The infinite time CLQR (11.43) was determined for Example 11.2 by using the approach presented in Section 11.3.4. The resulting $\overline{N}_S$ is 12. The state space is divided into 117 polyhedral regions and is depicted in Figure 11.4(a). In Figure 11.4(b) the same control law is represented where polyhedra with the same affine control law were merged.



(a) Partition before merging (N^r_∞ = 117).



(b) Partition after merging (N^r_∞ = 13).

Figure 11.4 Example 11.3. Double Integrator, 2-norm objective function, horizon $N = \infty$. Partition of the state space for the time invariant optimal control law.

11.4 1-Norm and ∞-Norm Case Solution

Next, we consider problem (11.9) with $J_0(\cdot)$ defined by (11.7) with $p = 1$ or $p = \infty$. In the following section we will concentrate on the ∞-norm, the results can be extended easily to cost functions with 1-norm or mixed 1/∞ norms.

$$\begin{array}{rl} J_0^*(x(0)) = \min_{U_0} & J_0(x(0), U_0) = \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p \\ \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_0 = x(0). \end{array} \tag{11.52}$$

11.4.1 Solution via LP

The optimal control problem (11.52) with $p = \infty$ can be rewritten as a linear program by using the approach presented in Section 9.2. Therefore, problem (11.52) can be reformulated as the following LP problem

$$\min_{z_0} \quad \varepsilon_0^x + \cdots + \varepsilon_N^x + \varepsilon_0^u + \cdots + \varepsilon_{N-1}^u \tag{11.53a}$$

$$\text{subj. to} \quad -1_n \varepsilon_k^x \leq \pm Q \left[A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \right], \tag{11.53b}$$

$$-\mathbf{1}_r\varepsilon_N^x \leq \pm P\left[A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j}\right], \tag{11.53c}$$

$$-\mathbf{1}_m\varepsilon_k^u \leq \pm R u_k, \tag{11.53d}$$

$$A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \in \mathcal{X},\ u_k \in \mathcal{U}, \tag{11.53e}$$

$$A^N x_0 + \sum_{j=0}^{N-1} A^j B u_{N-1-j} \in \mathcal{X}_f, \tag{11.53f}$$

$$\begin{aligned} &k = 0, \dots, N-1 \\ &x_0 = x(0), \end{aligned} \tag{11.53g}$$

where constraints (11.53b)–(11.53f) are componentwise, and ± means that the constraint appears once with each sign.

Problem (11.53) can be rewritten in the more compact form

$$\begin{aligned} \min_{z_0} \quad & c_0' z_0 \\ \text{subj. to} \quad & \bar{G}_0 z_0 \leq \bar{w}_0 + \bar{S}_0 x(0), \end{aligned} \tag{11.54}$$

where $z_0 = \{\varepsilon_0^x, \dots, \varepsilon_N^x, \varepsilon_0^u, \dots, \varepsilon_{N-1}^u, u_0', \dots, u_{N-1}'\} \in \mathbb{R}^s,\ s = (m+1)N + N + 1$ and

$$\bar{G}_0 = \begin{bmatrix} G_\varepsilon^x & G_\varepsilon^u & G_\varepsilon^c \\ 0 & 0 & G_0 \end{bmatrix},\ \bar{S}_0 = \begin{bmatrix} S_\varepsilon \\ E_0 \end{bmatrix},\ \bar{w}_0 = \begin{bmatrix} w_\varepsilon \\ w_0 \end{bmatrix}, \tag{11.55}$$

where $[G_\varepsilon^x,\ G_\varepsilon^u,\ G_\varepsilon^c]$ is the block partition of the matrix G_ε into the three parts corresponding to the variables ε_i^x, ε_i^u and u_i, respectively. The vector c_0 and the submatrices G_ϵ, w_ϵ, S_ϵ associated with the constraints (11.53b)–(11.53d) are defined in (9.10). The matrices G_0, w_0 and E_0 are defined in (11.18) for $i = 0$.

For a given vector $x(0)$ the optimal input sequence U_0^* solving problem (11.54) can be computed by using a Linear Program (LP) solver (see Section 2.2 for LP definition and properties and Chapter 3 for fast numerical methods for solving LPs).

To obtain the problem (11.54) we have eliminated the state variables and equality constraints $x_{k+1} = Ax_k + Bu_k$ by successive substitution so that we are left with $u_0, \dots, u_{N-1}$ and the slack variables ϵ as the only decision variables, and $x(0)$ as a parameter vector. As in the 2-norm case, it might be more efficient to solve an LP problem with equality and inequality constraints so that sparsity can be exploited. We omit the details and refer the reader to the construction of the QP problem without substitution in Section 11.3.1.

11.4.2 State Feedback Solution via Batch Approach

As shown in the previous section, problem (11.52) can be rewritten in the compact form

$$\begin{array}{ll} \min_{z_0} & c_0' z_0 \\ \text{subj. to} & \bar{G}_0 z_0 \le \bar{w}_0 + \bar{S}_0 x(0), \end{array} \tag{11.56}$$

where $z_0 = \{\varepsilon_0^x, \dots, \varepsilon_N^x, \varepsilon_0^u, \dots, \varepsilon_{N-1}^u, u_0', \dots, u_{N-1}'\} \in \mathbb{R}^s$, $s = (m+1)N + N + 1$. As in the 2-norm case, by treating $x(0)$ as a vector of parameters, problem (11.56) becomes a *multiparametric linear program* (mp-LP) that can be solved as described in Section 6.2. Once the multiparametric problem (11.56) has been solved, the explicit solution $z_0^*(x(0))$ of (11.56) is available as a piecewise affine function of $x(0)$, and the optimal control law $u^*(0)$ is also available explicitly, as the optimal input $u^*(0)$ consists simply of m components of $z_0^*(x(0))$

$$u^*(0) = [0 \ \dots 0 \ I_m \ 0 \ \dots \ 0] z_0^*(x(0)). \tag{11.57}$$

Theorem 6.5 states that there always exists a continuous PPWA solution $z_0^*(x)$ of the mp-LP problem (11.56). Clearly the same properties are inherited by the controller. The following corollaries of Theorem 6.5 summarize the analytical properties of the optimal control law and of the value function.

Corollary 11.4 *There exists a control law $u^*(0) = f_0(x(0))$, $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, obtained as a solution of the CFTOC (11.52) with $p = 1$ or $p = \infty$, which is continuous and PPWA*

$$f_0(x) = F_0^j x + g_0^j \quad \textit{if} \quad x \in CR_0^j, \ j = 1, \dots, N_0^r, \tag{11.58}$$

where the polyhedral sets $CR_0^j = \{H_0^j x \le k_0^j\}$, $j = 1, \dots, N_0^r$, are a partition of the feasible set $\mathcal{X}_0$.

Corollary 11.5 *The value function $J^*(x)$ obtained as a solution of the CFTOC (11.52) is convex and PPWA.*

Remark 11.7 Note that if the optimizer of problem (11.52) is unique for all $x(0) \in \mathcal{X}_0$, then Corollary 11.4 reads: "The control law $u^*(0) = f_0(x(0))$, $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, obtained as a solution of the CFTOC (11.52) with $p = 1$ or $p = \infty$, **is** continuous and PPWA,. . .." From the results of Section 6.2 we know that in case of multiple optimizers for some $x(0) \in \mathcal{X}_0$, a continuous control law of the form (11.58) can always be computed.

The multiparametric solution of (11.56) provides the open-loop optimal sequence $u^*(0), \dots, u^*(N-1)$ as an affine function of the initial state $x(0)$. The state feedback PPWA optimal controllers $u^*(k) = f_k(x(k))$ with $f_k : \mathcal{X}_k \mapsto \mathcal{U}$ for $k = 1, \dots, N$ are computed in the following way. Consider the same CFTOC (11.52) over the shortened time horizon $[i, N]$

$$\begin{array}{ll} \min_{U_i} & \|Px_N\|_p + \sum_{k=i}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p \\ \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = i, \ldots, N-1 \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = i, \ldots, N-1 \\ & x_N \in \mathcal{X}_f \\ & x_i = x(i), \end{array} \tag{11.59}$$

where $U_i = [u'_i, \ldots, u'_{N-1}]$ and $p = 1$ or $p = \infty$. As defined in (11.16) and discussed in Section 11.2, $\mathcal{X}_i \subseteq \mathbb{R}$ is the set of initial states $x(i)$ for which the optimal control problem (11.59) is feasible. We denote by U_i^* one of the optimizers of the optimal control problem (11.59).

Problem (11.59) can be translated into the mp-LP

$$\begin{array}{ll} \min\limits_{z_i} & c'_i z_i \\ \text{subj. to} & \bar{G}_i z_i \leq \bar{w}_i + \bar{S}_i x(i), \end{array} \tag{11.60}$$

where $z_i = \{\varepsilon_i^x, \ldots, \varepsilon_N^x, \varepsilon_i^u, \ldots, \varepsilon_{N-1}^u, u'_i, \ldots, u'_{N-1}\}$ and c_i, $\bar{G}_i$, $\overline{S}_i$, $\overline{w}_i$, are appropriately defined for each i. The component u_i^* of the multiparametric solution of (11.60) has the form

$$u_i^*(x(i)) = f_i(x(i)),\ \forall x(i) \in \mathcal{X}_i, \tag{11.61}$$

where the control law $f_i : \mathbb{R}^n \to \mathbb{R}^m$, is continuous and PPWA

$$f_i(x) = F_i^j x + g_i^j \quad \text{if} \quad x \in CR_i^j,\ j = 1, \ldots, N_i^r \tag{11.62}$$

and where the polyhedral sets $CR_i^j = \{x \in \mathbb{R}^n \ :\ H_i^j x \leq K_i^j\}$, $j = 1, \ldots, N_i^r$ are a partition of the feasible polyhedron $\mathcal{X}_i$. Therefore the feedback solution $u^*(k) = f_k(x(k))$, $k = 0, \ldots, N-1$ of the CFTOC (11.52) with $p = 1$ or $p = \infty$ is obtained by solving N mp-LP problems of decreasing size. The following corollary summarizes the final result.

Corollary 11.6 *There exists a state feedback control law $u^*(k) = f_k(x(k))$, $f_k : \mathcal{X}_k \subseteq \mathbb{R}^n \to \mathcal{U} \subseteq \mathbb{R}^m$, solution of the CFTOC (11.52) for $p = 1$ or $p = \infty$ and $k = 0, \ldots, N-1$ which is time-varying, continuous and piecewise affine on polyhedra*

$$f_k(x) = F_k^j x + g_k^j \quad \textit{if} \quad x \in CR_k^j, \quad j = 1, \ldots, N_k^r, \tag{11.63}$$

where the polyhedral sets $CR_k^j = \{x \in \mathbb{R}^n \ :\ H_k^j x \leq K_k^j\}$, $j = 1, \ldots, N_k^r$ are a partition of the feasible polyhedron $\mathcal{X}_k$.

11.4.3 State Feedback Solution via Recursive Approach

Consider the dynamic programming formulation of (11.52) with $J_0(\cdot)$ defined by (11.7) with $p = 1$

or $p = \infty$

$$J_j^*(x_j) = \min_{u_j} \ \|Qx_j\|_p + \|Ru_j\|_p + J_{j+1}^*(Ax_j + Bu_j)$$
$$\text{subj. to } x_j \in \mathcal{X},\ u_j \in \mathcal{U}$$
$$Ax_j + Bu_j \in \mathcal{X}_{j+1}, \tag{11.64}$$

for $j = 0, \ldots, N-1$, with boundary conditions

$$J_N^*(x_N) = \|Px_N\|_p \tag{11.65}$$

$$\mathcal{X}_N = \mathcal{X}_f. \tag{11.66}$$

Unlike for the 2-norm case the dynamic program (11.64)–(11.66) can be solved as explained in the next theorem.

Theorem 11.6 *The state feedback piecewise affine solution (11.63) of the CFTOC (11.52) for p* = 1 *or p* = ∞ *is obtained by solving the optimization problem (11.64)–(11.66) via N mp-LPs.*

Proof: Consider the first step $j = N-1$ of dynamic programming (11.64)–(11.66)

$$J_{N-1}^*(x_{N-1}) = \min_{u_{N-1}} \ \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + J_N^*(Ax_{N-1} + Bu_{N-1})$$
$$\text{subj. to } \ x_{N-1} \in \mathcal{X},\ u_{N-1} \in \mathcal{U}$$
$$Ax_{N-1} + Bu_{N-1} \in \mathcal{X}_f. \tag{11.67}$$

$J_{N-1}^*(x_{N-1}),\ u_{N-1}^*(x_{N-1})$ and $\mathcal{X}_{N-1}$ are computable via the mp-LP:

$$J_{N-1}^*(x_{N-1}) = \min_{\mu, u_{N-1}} \ \mu$$
$$\text{subj. to } \ \mu \geq \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + \|P(Ax_{N-1} + Bu_{N-1})\|_p$$
$$x_{N-1} \in \mathcal{X},\ u_{N-1} \in \mathcal{U}$$
$$Ax_{N-1} + Bu_{N-1} \in \mathcal{X}_f. \tag{11.68}$$

The constraint $\mu \geq \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + \|P(Ax_{N-1} + Bu_{N-1})\|_p$ in (11.68) is converted into a set of linear constraints as discussed in Remark 2.1 of Section 2.2.5. For instance, if $p = \infty$ we follow the approach of Section 9.3 and rewrite the constraint as

$$\begin{aligned} \mu &\geq \varepsilon_{N-1}^x + \varepsilon_{N-1}^u + \varepsilon_N^x \\ -\mathbf{1}_n \varepsilon_{N-1}^x &\leq \pm Q x_{N-1} \\ -\mathbf{1}_m \varepsilon_{N-1}^u &\leq \pm R u_{N-1} \\ -\mathbf{1}_{r_N} \varepsilon_N^x &\leq \pm P_N \left[A x_{N-1} + B u_{N-1}\right] \end{aligned} \tag{11.69}$$

By Theorem 6.5, J_{N-1}^* is a convex and piecewise affine function of x_{N-1}, the corresponding optimizer u_{N-1}^* is piecewise affine and continuous, and the feasible set $\mathcal{X}_{N-1}$ is a polyhedron. Without any loss of generality we assume J_{N-1}^* to be described as follows: $J_{N-1}^*(x_{N-1}) = \max_{i=1,\ldots,n_{N-1}}\{c_i' x_{N-1} + d_i\}$ (see Section 2.2.5 for convex PPWA functions representation) where n_{N-1} is the number of affine components comprising the value function J_{N-1}^*. At step $j = N-2$ of dynamic programming (11.64)–(11.66) we have

$$\begin{array}{rll} J^*_{N-2}(x_{N-2}) = \min_{u_{N-2}} & \|Qx_{N-2}\|_p + \|Ru_{N-2}\|_p + J^*_{N-1}(Ax_{N-2} + Bu_{N-2}) \\ \text{subj. to} & x_{N-2} \in \mathcal{X},\ u_{N-2} \in \mathcal{U} \\ & Ax_{N-2} + Bu_{N-2} \in \mathcal{X}_{N-1}. \end{array} \tag{11.70}$$

Since $J^*_{N-1}(x)$ is a convex and piecewise affine function of x, the problem (11.70) can be recast as the following mp-LP (see Section 2.2.5 for details)

$$\begin{array}{rll} J^*_{N-2}(x_{N-2}) = \min_{\mu, u_{N-2}} & \mu \\ \text{subj. to} & \mu \geq \|Qx_{N-2}\|_p + \|Ru_{N-2}\|_p + c_i(Ax_{N-2} + Bu_{N-2}) + d_i \\ & i = 1, \ldots, n_{N-1} \\ & x_{N-2} \in \mathcal{X},\ u_{N-2} \in \mathcal{U} \\ & Ax_{N-2} + Bu_{N-2} \in \mathcal{X}_{N-1}. \end{array} \tag{11.71}$$

$J^*_{N-2}(x_{N-2})$, $u^*_{N-2}(x_{N-2})$ and $\mathcal{X}_{N-2}$ are computed by solving the mp-LP (11.71). By Theorem 6.5, J^*_{N-2} is a convex and piecewise affine function of x_{N-2}, the corresponding optimizer u^*_{N-2} is piecewise affine and continuous, and the feasible set $\mathcal{X}_{N-2}$ is a convex polyhedron.

The convexity and piecewise linearity of J^*_j and the polyhedra representation of $\mathcal{X}_j$ still hold for $j = N - 3, \ldots, 0$ and the procedure can be iterated backwards in time, proving the theorem. ∎

Consider the state feedback piecewise affine solution (11.63) of the CFTOC (11.52) for $p = 1$ or $p = \infty$ and assume we are interested only in the optimal controller at time 0. In this case, by using duality arguments we can solve the equations (11.64)–(11.66) by using vertex enumerations and one mp-LP. This is proven in the next theorem.

Theorem 11.7 *The state feedback piecewise affine solution (11.63) at time $k = 0$ of the CFTOC (11.52) for $p = 1$ or $p = \infty$ is obtained by solving the optimization problem (11.64)–(11.66) via one mp-LP.*

Proof: Consider the first step $j = N - 1$ of dynamic programming (11.64)– (11.66)

$$\begin{array}{rll} J^*_{N-1}(x_{N-1}) = \min_{u_{N-1}} & \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + J^*_N(Ax_{N-1} + Bu_{N-1}) \\ \text{subj. to} & x_{N-1} \in \mathcal{X},\ u_{N-1} \in \mathcal{U} \\ & Ax_{N-1} + Bu_{N-1} \in \mathcal{X}_f \end{array} \tag{11.72}$$

and the corresponding mp-LP:

$$\begin{array}{rll} J^*_{N-1}(x_{N-1}) = \min_{\mu, u_{N-1}} & \mu \\ \text{subj. to} & \mu \geq \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + \|P(Ax_{N-1} + Bu_{N-1})\|_p \\ & x_{N-1} \in \mathcal{X},\ u_{N-1} \in \mathcal{U} \\ & Ax_{N-1} + Bu_{N-1} \in \mathcal{X}_f. \end{array} \tag{11.73}$$

By Theorem 6.5, J^*_{N-1} is a convex and piecewise affine function of x_{N-1}, and the feasible set $\mathcal{X}_{N-1}$ is a polyhedron. J^*_{N-1} and $\mathcal{X}_{N-1}$ are computed without explicitly solving the mp-LP (11.73).

Rewrite problem (11.73) in the more compact form

$$\begin{array}{ll} \min_{z_{N-1}} & c'_{N-1} z_{N-1} \\ \text{subj. to} & \bar{G}_{N-1} z_{N-1} \leq \bar{w}_{N-1} + \bar{S}_{N-1} x_{N-1}, \end{array} \tag{11.74}$$

where z_{N-1} collects the optimization variables μ, u_{N-1} and the auxiliary variables need to transform the constraint $\mu \geq \|Qx_{N-1}\|_p + \|Ru_{N-1}\|_p + \|P(Ax_{N-1} + Bu_{N-1})\|_p$ into a set of linear constraints. Consider the LP dual of (11.74)

$$\begin{array}{ll} \max_v & -(\bar{w}_{N-1} + \bar{S}_{N-1} x_{N-1})' v \\ \text{subj. to} & \bar{G}'_{N-1} v = -c_{N-1} \\ & v \geq 0. \end{array} \tag{11.75}$$

Consider the dual feasibility polyheron $\mathcal{P}_d = \{v \geq 0 \ : \ \bar{G}'_{N-1} v = -c_{N-1}\}$. Let $\{V_1, \ldots, V_k\}$ be the vertices of $\mathcal{P}_d$ and $\{y_1, \ldots, y_e\}$ be the rays of $\mathcal{P}_d$. Since we have a zero duality gap, we have that

$$J^*_{N-1}(x_{N-1}) = \max_{i=1,\ldots,k} \{-(\bar{w}_{N-1} + \bar{S}_{N-1} x_{N-1})' V_i\}$$

i.e.,

$$J^*_{N-1}(x_{N-1}) = \max_{i=1,\ldots,k} \{-(V_i' \bar{S}_{N-1}) x_{N-1} - \bar{w}'_{N-1} V_i\}.$$

Recall that if the dual of a mp-LP is unbounded, then the primal is infeasible (Theorem 6.3). For this reason the feasible set $\mathcal{X}_{N-1}$ is obtained by requiring that the cost of (11.75) does not increase in the direction of the rays:

$$\mathcal{X}_{N-1} = \{x_{N-1} \ : \ -(\bar{w}_{N-1} + \bar{S}_{N-1} x_{N-1})' y_i \leq 0, \ \forall\, i = 1, \ldots, e\}$$

with $J^*_{N-1}(x_{N-1})$ and $\mathcal{X}_{N-1}$ available, we can iterate the procedure backwards in time for steps $N-2, N-3, \ldots, 1$. At step 0 one mp-LP will be required in order to compute $u_0^*(x(0))$. ■

11.4.4 Example

Example 11.4 Consider the double integrator system (11.23). We want to compute the state feedback optimal controller that solves (11.52) with $p = \infty$, $N = 6$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 0.8$, subject to the input constraints

$$\mathcal{U} = \{u \in \mathbb{R} \ : \ -1 \leq u \leq 1\} \tag{11.76}$$

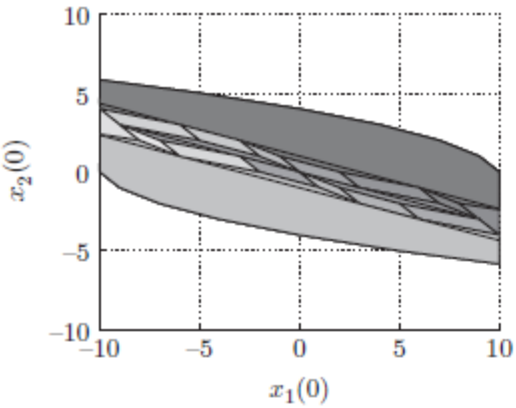
and the state constraints

$$\mathcal{X} = \left\{x \in \mathbb{R}^2 \ : \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix}\right\} \tag{11.77}$$

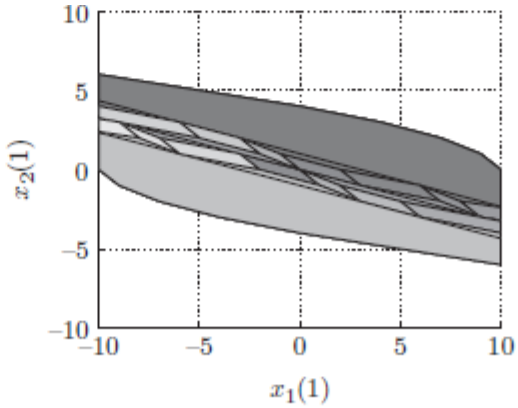
and $\mathcal{X}_f = \mathcal{X}$. The optimal feedback solution $u^*(0), \ldots, u^*(5)$ was computed by solving six mp-LP problems and the corresponding polyhedral partitions of the state space are depicted in Figure 11.5, where polyhedra with the same control law were merged. Only the last optimal control move is reported below:

$$u^*(5) = \begin{cases} 0 & \text{if } \begin{bmatrix} 0.45 & 0.89 \\ 1.00 & 0.00 \\ -0.71 & -0.71 \\ -1.00 & -0.00 \end{bmatrix} x(5) \le \begin{bmatrix} 0.00 \\ 0.00 \\ 7.07 \\ 10.00 \end{bmatrix} & \text{(Region \#1)} \\ 0 & \text{if } \begin{bmatrix} -0.45 & -0.89 \\ -1.00 & 0.00 \\ 0.71 & 0.71 \\ 1.00 & -0.00 \end{bmatrix} x(5) \le \begin{bmatrix} 0.00 \\ 0.00 \\ 7.07 \\ 10.00 \end{bmatrix} & \text{(Region \#2)} \\ [-1.00\ -2.00]\, x(5) & \text{if } \begin{bmatrix} -0.45 & -0.89 \\ 0.45 & 0.89 \\ 0.71 & 0.71 \\ -1.00 & -0.00 \end{bmatrix} x(5) \le \begin{bmatrix} 0.00 \\ 0.45 \\ 0.00 \\ 10.00 \end{bmatrix} & \text{(Region \#3)} \\ [-1.00\ -2.00]\, x(5) & \text{if } \begin{bmatrix} -0.45 & -0.89 \\ 0.45 & 0.89 \\ -0.71 & -0.71 \\ 1.00 & -0.00 \end{bmatrix} x(5) \le \begin{bmatrix} 0.45 \\ 0.00 \\ 0.00 \\ 10.00 \end{bmatrix} & \text{(Region \#4)} \\ [1.00\ 0.00]\, x(5) & \text{if } \begin{bmatrix} -0.71 & -0.71 \\ -1.00 & 0.00 \\ 1.00 & 0.00 \\ -0.00 & 1.00 \end{bmatrix} x(5) \le \begin{bmatrix} 0.00 \\ 1.00 \\ 0.00 \\ 10.00 \end{bmatrix} & \text{(Region \#5)} \\ [1.00\ 0.00]\, x(5) & \text{if } \begin{bmatrix} 0.71 & 0.71 \\ -1.00 & 0.00 \\ 1.00 & 0.00 \\ -0.00 & -1.00 \end{bmatrix} x(5) \le \begin{bmatrix} 0.00 \\ 0.00 \\ 1.00 \\ 10.00 \end{bmatrix} & \text{(Region \#6)} \\ 1.00 & \text{if } \begin{bmatrix} 0.45 & 0.89 \\ -1.00 & 0.00 \\ -0.00 & -1.00 \\ 1.00 & -0.00 \end{bmatrix} x(5) \le \begin{bmatrix} -0.45 \\ -1.00 \\ 10.00 \\ 10.00 \end{bmatrix} & \text{(Region \#7)} \\ -1.00 & \text{if } \begin{bmatrix} -0.45 & -0.89 \\ 1.00 & 0.00 \\ -1.00 & -0.00 \\ -0.00 & 1.00 \end{bmatrix} x(5) \le \begin{bmatrix} -0.45 \\ -1.00 \\ 10.00 \\ 10.00 \end{bmatrix} & \text{(Region \#8)} \end{cases} \tag{11.78}$$

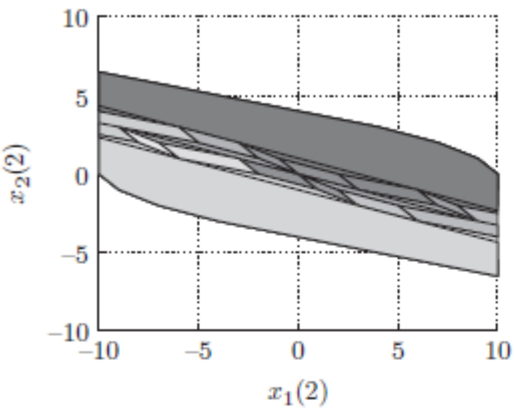
Note that the controller (11.78) is piecewise linear around the origin. In fact, the origin belongs to multiple regions (1 to 6). Note that the number N_i^r of regions is not always increasing with decreasing i ($N_5^r = 8$, $N_4^r = 12$, $N_3^r = 12$, $N_2^r = 26$, $N_1^r = 28$, $N_0^r = 26$). This is due to the merging procedure, before merging we have $N_5^r = 12$, $N_4^r = 22$, $N_3^r = 40$, $N_2^r = 72$, $N_1^r = 108$, $N_0^r = 152$.



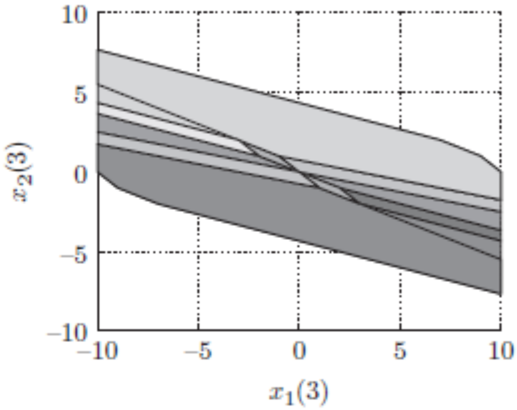
(a) Partition of the state space for the affine control law $u^*(0)(N_0^r = 26)$.



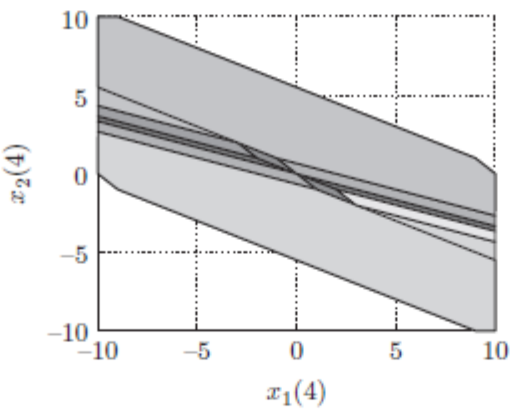
(b) Partition of the state space for the affine control law $u^*(1)$ $(N_1^r = 28)$.



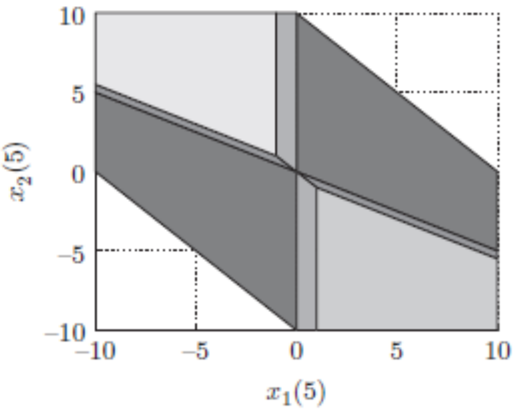
(c) Partition of the state space for the affine control law $u^*(2)$ $(N_2^r = 26)$.



(d) Partition of the state space for the affine control law $u^*(3)$ $(N_3^r = 12)$



(e) Partition of the state space for the affine control law $u^*(4)$ $(N_4^r = 12)$.



(f) Partition of the state space for the affine control law $u^*(5)$ $(N_5^r = 8)$.

Figure 11.5 Example 11.4. Double Integrator, ∞-norm objective function, horizon $N = 6$. Partition of the state space for the time-varying optimal control law. Polyhedra with the same control law were merged.

11.4.5 Infinite-Time Solution

Assume that Q and R have full column rank and that the constraint sets $\mathcal{X}$ and $\mathcal{U}$ contain the origin in their interior. Consider the following infinite-horizon problem with constraints

$$\begin{array}{rl} J_\infty^*(x(0)) = \min_{u_0,u_1,\ldots} & \sum_{k=0}^{\infty} \|Qx_k\|_p + \|Ru_k\|_p \\ \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0,\ldots,\infty \\ & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0,\ldots,\infty \\ & x_0 = x(0) \end{array} \tag{11.79}$$

and the set

$$\mathcal{X}_\infty = \{x(0) \in \mathbb{R}^n :\ \text{Problem (11.79) is feasible and } J_\infty^*(x(0)) < +\infty\}. \tag{11.80}$$

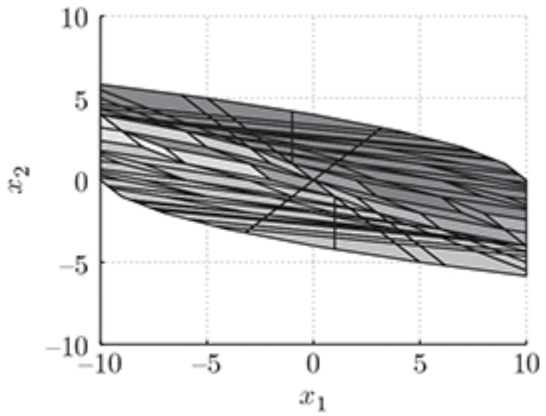
Because Q and R have full column rank, *any* optimizer u_k^* of problem (11.43) *must* converge to the origin $(u_k^* \to 0)$ and so must the state trajectory resulting from the application of u_k^* $(x_k^* \to 0)$. Thus the origin $x = 0$, $u = 0$ must lie in the interior of the constraint set $(\mathcal{X}, \mathcal{U})$. (If the origin were not contained in the constraint set then $J_\infty^*(x(0))$ would be infinite.) Furthermore, if the initial state $x_0 = x(0)$ is sufficiently close to the origin, then the state and input constraints will never become active and the solution of problem (11.43) will yield the *unconstrained* optimal controller (9.31).

The discussion for the solution of the infinite horizon constrained linear quadratic regulator (Section 11.3.4) by means of the batch approach can be repeated here with one precaution. Since the unconstrained optimal controller (if it exists) is PPWA the computation of the Maximal Invariant Set for the autonomous constrained piecewise linear system is more involved and requires algorithms which will be presented later in Chapter 17.

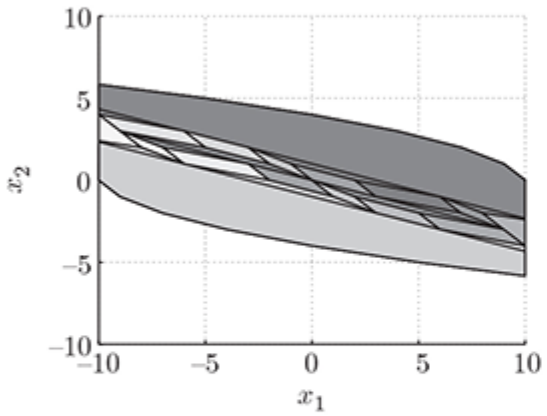
Differently from the 2-norm case, here the use of dynamic programming for computing the infinite horizon solution is a viable alternative to the batch approach. Convergence conditions for the dynamic programming strategy and convergence guarantees for the resulting possibly discontinuous closed-loop system are given in [87]. A computationally efficient algorithm to obtain the infinite time optimal solution, based on a dynamic programming exploration strategy with an mp-LP solver and basic polyhedral manipulations, is also presented in [87].

Example 11.5 We consider the double integrator system (11.23) from Example 11.4 with $N = \infty$.

The partition of the state space for the time invariant optimal control law is shown in Figure 11.6(a) and consists of 202 polyhedral regions. In Figure 11.6(b) the same control law is represented where polyhedra with the same affine control law were merged.



(a) Partition before merging (N^r_∞ = 202).



(b) Partition after merging (N^r_∞ = 26).

Figure 11.6 Example 11.5. Double Integrator, ∞-norm objective function, horizon $N = \infty$. Partition of the state space for the time invariant optimal control law.

11.5 State Feedback Solution, Minimum-Time Control

In this section we consider the solution of minimum-time optimal control problems

$$\begin{array}{lll} J_0^*(x(0)) = & \min\limits_{U_0, N} & N \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0), \end{array} \tag{11.81}$$

where $\mathcal{X}_f \subset \mathbb{R}^n$ is a terminal target set to be reached in minimum time.

We can find the controller that brings the states into $\mathcal{X}_f$ in one time step by solving the following multiparametric program

$$\begin{array}{ll} \min\limits_{u_0} & c(x_0, u_0) \\ \text{subj. to} & x_1 = Ax_0 + Bu_0 \\ & x_0 \in \mathcal{X},\ u_0 \in \mathcal{U} \\ & x_1 \in \mathcal{X}_f, \end{array} \tag{11.82}$$

where $c(x_0, u_0)$ is any convex quadratic function. Let us assume that the solution of the multiparametric program generates R^1 regions $\{\mathcal{P}_r^1\}_{r=1}^{R^1}$ with the affine control law $u_0 = F_r^1 x + g_r^1$ in each region r. By construction we have

$$\mathcal{X}_0 = \mathcal{K}_1(\mathcal{X}_f).$$

Continuing to set up simple multiparametric programs bring the states into $\mathcal{X}_f$ in 2, 3, . . . steps, we have for step j

$$
\begin{aligned}
\min_{u_0} \quad & c(x_0, u_0) \\
\text{subj. to} \quad & x_1 = Ax_0 + Bu_0 \\
& x_0 \in \mathcal{X},\ u_0 \in \mathcal{U} \\
& x_1 \in \mathcal{K}_{j-1}(\mathcal{X}_f),
\end{aligned}
\tag{11.83}
$$

which yields R^j regions $\{\mathcal{P}_r^j\}_{r=1}^{R^j}$ with the affine control law $u_0 = F_r^j x + g_r^j$ in each region r. By construction we have

$$\mathcal{X}_0 = \mathcal{K}_j(\mathcal{X}_f).$$

Thus to obtain $\mathcal{K}_1(\mathcal{X}_f)$, . . . , $\mathcal{K}_N(\mathcal{X}_f)$ we need to solve N multiparametric programs with a prediction horizon of 1. Since the overall complexity of a multiparametric program is exponential in N, this scheme can be exploited to yield controllers of lower complexity than the optimal control schemes introduced in the previous sections.

Since N multiparametric programs have been solved, the controller regions overlap in general. In order to achieve minimum time behavior, the feedback law associated with the region computed for the smallest number of steps c, is selected for any given state x.

Algorithm 11.1 *On-line computation of minimum-time control input*

Input: State measurement x, N controller partitions solution to (11.83) for $j = 1, \ldots, N$
Output: Minimum time control action $u(x)$

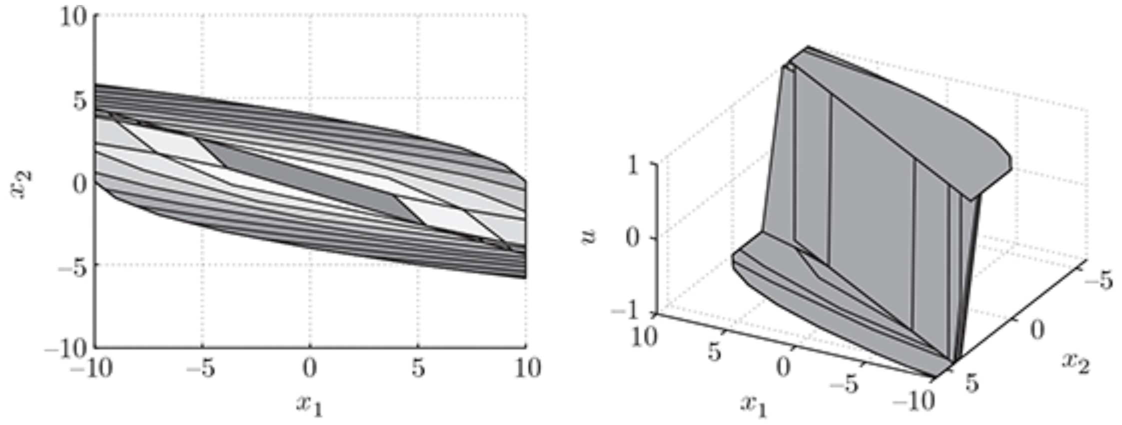
Find controller partition $c_{\min} = \min_{c \in \{0, \ldots, N\}} c$, s.t. $x \in \mathcal{K}_c(\mathcal{X}_f)$

Find controller region r, such that $x \in \mathcal{P}_r^{c_{\min}}$ and compute $u = F_r^{c_{\min}} x + g_r^{c_{\min}}$
Return u

Note that the region identification for this type of controller partition is much more efficient than simply checking all the regions. The two steps of “finding a controller partition” and “finding a controller region” in Algorithm 11.1 correspond to two levels of a search tree, where the search is first performed over the feasible sets $\mathcal{K}_c(\mathcal{X}_f)$ and then over the controller partition $\{\mathcal{P}_r^c\}_{r=1}^{R^c}$.

Furthermore, one may discard all regions $\mathcal{P}_r^i$ which are completely covered by previously computed controllers (i.e., $\mathcal{P}_r^i \subseteq \bigcup_{j \in \{1,\ldots,i-1\}} \mathcal{K}_j(\mathcal{X}_f)$) since they are not time optimal.

Example 11.6 Consider again the double integrator from Example 11.2. The Minimum-Time Controller is computed that steers the system to the Maximal LQR Invariant Set $\mathcal{O}_\infty^{\text{LQR}}$ in the minimum number of time steps N. The Algorithm terminated after 11 iterations, covering the Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{O}_\infty^{\text{LQR}})$. The resulting controller is defined over 33 regions. The regions are depicted in Figure 11.7(a). The Maximal LQR Invariant Set $\mathcal{O}_\infty^{\text{LQR}}$ is the central shaded region.



(a) Minimum-Time State Space Partition (N^r=33).

(b) Minimum-Time Control Law (discontinuous).

Figure 11.7 Example 11.6. Double integrator, minimum-time objective function. Partition of the state space and control law. Maximal LQR Invariant Set $\mathcal{O}_\infty^{\mathrm{LQR}}$ is the central shaded region in (a).

The control law on this partition is depicted in Figure 11.7(b). Note that, in general, the minimum-time control law is not continuous as can be seen in Figure 11.7(b).

11.6 Comparison of the Design Approaches and Controllers

For the design, the storage and the on-line execution time of the control law we are interested in the *controller complexity*. All the controllers we discussed in this chapter are PPWA. Their storage and execution times are closely related to the number of polyhedral regions, which is related to the number of constraints in the mp-LP or mp-QP.

When the control objective is expressed in terms of the 1- or ∞- norm, it is translated into a set of constraints in the mp-LP. When the control objective is expressed in terms of the 2-norm such a translation is not necessary. Thus controllers minimizing the 1- or ∞- norm do generally involve more regions than those minimizing the 2-norm and therefore tend to be more complex.

In mp-LPs degeneracies are common and have to be taken care of in the respective algorithms. Strictly convex mp-QPs for which the Linear Independent Constraint Qualification holds are comparatively well behaved with unique solutions and full-dimensional polyhedral regions. Only for the 1-norm and ∞-norm objective, however, we can use for the controller design an efficient Dynamic Programming scheme involving a sequence of mp-LPs.

As we let the horizon N go to infinity, the number of regions stays finite for the 2-norm objective, for the 1- and ∞-norm nothing can be said in general. In the examples, we have usually observed a finite number, but cases can be constructed where the number can be proven to be infinite.

As we will argue in the following chapter, infinite horizon controllers based on the 2-norm render the closed-loop system exponentially stable, while controllers based on the 1- or ∞-norm render the closed-loop system stable, only.

Among the controllers proposed in this chapter, the minimum-time controller is usually the least complex involving the smallest number of regions. Minimum time controllers are often considered to be too aggressive for practical use. The controller here is different, however. It is minimum time only until it reaches the terminal set $\mathcal{X}_f$. Inside the terminal set a different unconstrained control law can be used. Overall the operation of the minimum-time controller is observed to be very similar to that of the other infinite time controllers in this chapter.

1 Let $X = \begin{bmatrix} A & B' \\ B & C \end{bmatrix}$ and $A \succ 0$. Then $X \succeq 0$ if and only if the Schur complement $S = C - BA^{-1}B' \succeq 0$.

2 As in the unconstrained case, the assumption $Q \succ 0$ can be relaxed by requiring that $(Q^{1/2}, A)$ is observable (Section 8.5).