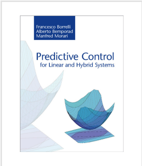

Predictive Control: for Linear and Hybrid Systems
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IV Constrained Optimal Control of Linear Systems
10 Controllability, Reachability and Invariance

10
Controllability, Reachability and Invariance

This chapter is a self-contained introduction to controllability, reachability and invariant set theory. *N*-steps reachable sets are defined for autonomous systems. They represent the set of states which a system can evolve to after *N* steps. *N*-steps controllable sets are defined for systems with inputs. They represent the set of states which a system can be steered to after *N* steps.

Invariant sets are the infinite time versions of *N*-steps controllable and reachable sets. *Invariant sets* are computed for autonomous systems. These types of sets are useful to answer questions such as: “For a given feedback controller $u = k(x)$, find the set of states whose trajectory will never violate the system constraints.” *Control invariant sets* are defined for systems subject to external inputs. These types of sets are useful to answer questions such as: “Find the set of states for which there exists a controller such that the system constraints are never violated.”

This chapter focuses on computational tools for constrained linear systems and constrained linear systems subject to additive and parametric uncertainty. A thorough presentation of the basic notions and algorithms presented in this chapter can be found in the book by Blanchini and Miani [59].

10.1 Controllable and Reachable Sets

In this section we deal with two types of systems, namely, autonomous systems:

$$x(t+1) = g_a(x(t)), \tag{10.1}$$

and systems subject to external inputs:

$$x(t+1) = g(x(t), u(t)). \tag{10.2}$$

Both systems are subject to state and input constraints

$$x(t) \in \mathcal{X},\ \ u(t) \in \mathcal{U},\ \ \forall\, t \geq 0. \tag{10.3}$$

The sets $\mathcal{X}$ and $\mathcal{U}$ are polyhedra.

Definition 10.1 *For the autonomous system (10.1) we denote the* precursor set to the set $\mathcal{S}$ *as*

$$\mathrm{Pre}(\mathcal{S}) = \{x \in \mathbb{R}^n \ : \ g_a(x) \in \mathcal{S}\}. \tag{10.4}$$

$\mathrm{Pre}(\mathcal{S})$ is the set of states which evolve into the target set $\mathcal{S}$ in one time step.

Definition 10.2 *For the system (10.2) we denote the* precursor set to the set $\mathcal{S}$ *as*

$$\mathrm{Pre}(\mathcal{S}) = \{x \in \mathbb{R}^n \ : \ \exists u \in \mathcal{U} \ s.t. \ g(x,u) \in \mathcal{S}\}. \tag{10.5}$$

For a system with inputs, $\mathrm{Pre}(\mathcal{S})$ is the set of states which can be driven into the target set $\mathcal{S}$ in one time step while satisfying input and state constraints.

Definition 10.3 *For the autonomous system (10.1) we denote the* successor set from the set $\mathcal{S}$ *as*

$$\mathrm{Suc}(\mathcal{S}) = \{x \in \mathbb{R}^n \ : \ \exists \ x(0) \in \mathcal{S} \ s.t. \ x = g_a(x(0))\}.$$

Definition 10.4 *For the system (10.2) with inputs we will denote the* successor set from the set $\mathcal{S}$ *as*

$$\mathrm{Suc}(\mathcal{S}) = \{x \in \mathbb{R}^n \ : \ \exists \ x(0) \in \mathcal{S}, \ \exists \ u(0) \in \mathcal{U} \ s.t. \ x = g(x(0), u(0))\}.$$

Therefore, all the states contained in $\mathcal{S}$ are mapped into the set $\mathrm{Suc}(\mathcal{S})$ under the map g_a or under the map g for some input $u \in \mathcal{U}$.

Remark 10.1 The sets $\mathrm{Pre}(\mathcal{S})$ and $\mathrm{Suc}(\mathcal{S})$ are also denoted as 'one-step backward-reachable set' and 'one-step forward-reachable set', respectively, in the literature.

N-step controllable and reachable sets are defined by iterating $\mathrm{Pre}(\cdot)$ and $\mathrm{Suc}(\cdot)$ computations, respectively.

Definition 10.5 (*N*-Step Controllable Set $\mathcal{K}_N(\mathcal{S})$) *For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step controllable set $\mathcal{K}_N(\mathcal{S})$ of the system (10.1) or (10.2) subject to the constraints (10.3) is defined recursively as:*

$$\mathcal{K}_j(\mathcal{S}) = \mathrm{Pre}(\mathcal{K}_{j-1}(\mathcal{S})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{S}) = \mathcal{S}, \qquad j \in \{1, \ldots, N\} \tag{10.6}$$

From Definition 10.5, all states x_0 of the system (10.1) belonging to the *N*-Step Controllable Set $\mathcal{K}_N(\mathcal{S})$ will evolve to the target set $\mathcal{S}$ in N steps, while satisfying state constraints.

Also, all states x_0 of the system (10.2) belonging to the *N*-Step Controllable Set $\mathcal{K}_N(\mathcal{S})$ can be driven, by a suitable control sequence, to the target set $\mathcal{S}$ in N steps, while satisfying input and state constraints.

Definition 10.6 (N-Step Reachable Set $\mathcal{R}_N(\mathcal{X}_0)$) *For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ of the system (10.1) or (10.2) subject to the constraints (10.3) is defined as:*

$$\mathcal{R}_{i+1}(\mathcal{X}_0) = \mathrm{Suc}(\mathcal{R}_i(\mathcal{X}_0)) \cap \mathcal{X}, \quad \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0, \qquad i = 0, \dots, N-1 \tag{10.7}$$

From Definition 10.6, all states x_0 belonging to $\mathcal{X}_0$ will evolve to the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ in N steps.

10.1.1 Computation of Controllable and Reachable Sets

Next, we will show through simple examples the main steps involved in the computation of controllable and reachable sets for constrained linear systems. Later in this section we will provide compact formulas based on polyhedral operations.

Example 10.1 Consider the second order autonomous stable system

$$x(t+1) = Ax(t) = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x(t) \tag{10.8}$$

subject to the state constraints

$$x(t) \in \mathcal{X} = \left\{ x \ : \ \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \ \forall t \ge 0. \tag{10.9}$$

The set Pre($\mathcal{X}$) can be obtained as follows: Since the set $\mathcal{X}$ is a polytope, it can be represented as a $\mathcal{H}$-polytope (Section 4.2)

$$\mathcal{X} = \{x \ : \ Hx \le h\}, \tag{10.10}$$

where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } h = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}.$$

By using this $\mathcal{H}$-presentation and the system equation (10.8), the set Pre($\mathcal{X}$) can be derived:

$$\mathrm{Pre}(\mathcal{X}) = \{x \ : \ Hg_a(x) \le h\} \tag{10.11}$$

$$= \{x \ : \ HAx \le h\}. \tag{10.12}$$

The set (10.12) may contain redundant inequalities which can be removed by using Algorithm

4.1 in Section 4.4.1 to obtain its minimal representation. Note that by using the notation in Section 4.4.11, the set Pre($\mathcal{X}$) in (10.12) is simply $\mathcal{X} \circ A$.

The set Pre($\mathcal{X}$) is

$$\mathrm{Pre}(\mathcal{X}) = \left\{ x \ : \ \begin{bmatrix} 1 & 0 \\ 1 & -0.5 \\ -1 & 0 \\ -1 & -0.5 \end{bmatrix} x \le \begin{bmatrix} 20 \\ 10 \\ 20 \\ 10 \end{bmatrix} \right\}.$$

The one-step controllable set to $\mathcal{X}$, $\mathcal{K}_1(\mathcal{X})$ = Pre($\mathcal{X}$) ∩ $\mathcal{X}$ is

$$\mathrm{Pre}(\mathcal{X}) \cap \mathcal{X} = \left\{ x \ : \ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

and it is depicted in Figure 10.1.

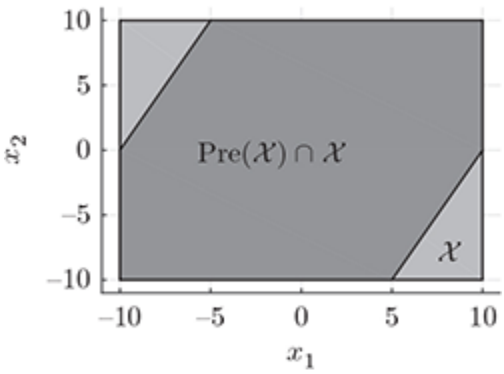

Figure 10.1 Example 10.1. One-step controllable set Pre($\mathcal{X}$) ∩ $\mathcal{X}$ for system (10.8) under constraints (10.9).

The set Suc($\mathcal{X}$) is obtained by applying the map A to the set $\mathcal{X}$. Let us write $\mathcal{X}$ in $\mathcal{V}$-representation (see Section 4.1)

$$\mathcal{X} = \mathrm{conv}(V), \tag{10.13}$$

and let us map the set of vertices V through the transformation A. Because the transformation is linear, the successor set is simply the convex hull of the transformed vertices

$$\mathrm{Suc}(\mathcal{X}) = A \circ \mathcal{X} = \mathrm{conv}(AV). \tag{10.14}$$

We refer the reader to Section 4.4.11 for a detailed discussion on linear transformations of polyhedra.

The set Suc($\mathcal{X}$) in $\mathcal{H}$-representation is

$$\mathrm{Suc}(\mathcal{X}) = \left\{ x \;:\; \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \le \begin{bmatrix} 5 \\ 5 \\ 2.5 \\ 2.5 \end{bmatrix} \right\}$$

and is depicted in Figure 10.2.

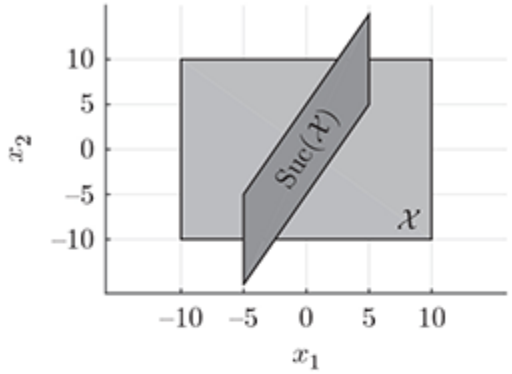

Figure 10.2 Example 10.1. Successor set for system (10.8).

Example 10.2 Consider the second order unstable system

$$x(t+1) = Ax + Bu = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{10.15}$$

subject to the input and state constraints

$$u(t) \in \mathcal{U} = \{u \;:\; -5 \le u \le 5\}, \;\forall t \ge 0 \tag{10.16a}$$

$$x(t) \in \mathcal{X} = \left\{ x \;:\; \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \;\forall t \ge 0. \tag{10.16b}$$

For the nonautonomous system (10.15), the set Pre($\mathcal{X}$) can be computed using the $\mathcal{H}$-representation of $\mathcal{X}$ and $\mathcal{U}$,

$$\mathcal{X} = \{x \;:\; Hx \le h\}, \quad \mathcal{U} = \{u \;:\; H_u u \le h_u\}, \tag{10.17}$$

to obtain

$$\mathrm{Pre}(\mathcal{X}) = \left\{ x \in \mathbb{R}^2 \;:\; \exists u \in \mathcal{U} \text{ s.t. } g(x,u) \in \mathcal{X}, \right\} \tag{10.18}$$

$$= \left\{ x \in \mathbb{R}^2 \;:\; \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}. \tag{10.19}$$

The half-spaces in (10.19) define a polytope in the state-input space, and a projection operation (see Section 4.4.6) is used to derive the half-spaces which define Pre($\mathcal{X}$) in the state space. The one-step controllable set Pre($\mathcal{X}$) ∩ $\mathcal{X}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1.5 \\ -1 & 1.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

is depicted in Figure 10.3.

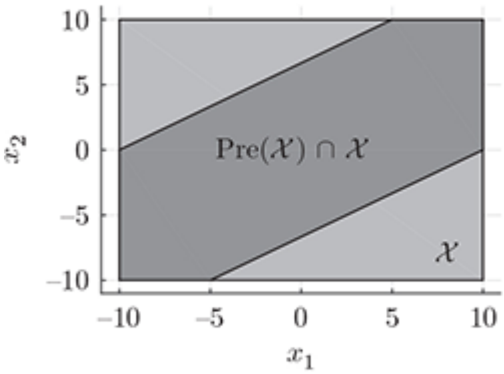

Figure 10.3 Example 10.2. One-step controllable set Pre($\mathcal{X}$) ∩ $\mathcal{X}$ for system (10.15) under constraints (10.16).

Note that by using the definition of the Minkowski sum given in Section 4.4.9 and the affine operation on polyhedra in Section 4.4.11 we can write the operations in (10.19) compactly as follows:

$$\begin{aligned} \mathrm{Pre}(\mathcal{X}) = & \{x \ : \ \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu \in \mathcal{X}\} \\ & \{x \ : \ y = Ax + Bu, \ y \in \mathcal{X}, \ u \in \mathcal{U}\} \\ & \{x \ : \ Ax = y + (-Bu), \ y \in \mathcal{X}, \ u \in \mathcal{U}\} \\ & \{x \ : \ Ax \in \mathcal{C}, \ \mathcal{C} = \mathcal{X} \oplus (-B) \circ \mathcal{U}\} \\ & \{x \ : \ x \in \mathcal{C} \circ A, \ \mathcal{C} = \mathcal{X} \oplus (-B) \circ \mathcal{U}\} \\ & \{x \ : \ x \in (\mathcal{X} \oplus (-B) \circ \mathcal{U}) \circ A\}. \end{aligned} \tag{10.20}$$

The set Suc($\mathcal{X}$) = {$Ax + Bu \in \mathbb{R}^2 : x \in \mathcal{X}, u \in \mathcal{U}$} is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the input $u \in \mathcal{U}$. As shown before,

$$A \circ \mathcal{X} = \mathrm{conv}(AV) \tag{10.21}$$

and therefore

$$\mathrm{Suc}(\mathcal{X}) = \{y + Bu \ : \ y \in A \circ \mathcal{X}, \ u \in \mathcal{U}\}.$$

We can use the definition of the Minkowski sum given in Section 4.4.9 and rewrite the set $\text{Suc}(\mathcal{X})$ as

$$\text{Suc}(\mathcal{X}) = (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}).$$

We can compute the Minkowski sum via projection or vertex enumeration as explained in Section 4.4.9 and obtain the set $\text{Suc}(\mathcal{X})$ in $\mathcal{H}$-representation

$$\text{Suc}(\mathcal{X}) = \left\{ x \ : \ \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & -1.5 \\ -1 & 1.5 \end{bmatrix} x \leq \begin{bmatrix} 20 \\ 20 \\ 25 \\ 25 \\ 27.5 \\ 27.5 \end{bmatrix} \right\},$$

which is depicted in Figure 10.4.

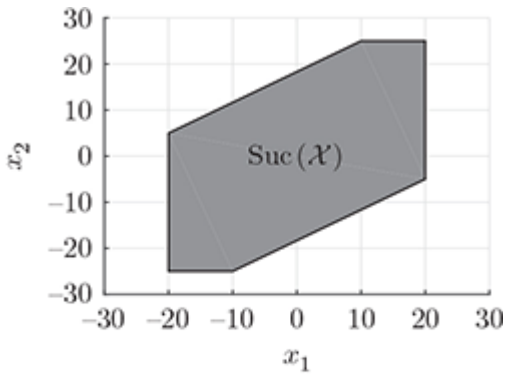

Figure 10.4 Example 10.2. Successor set for system (10.15) under constraints (10.16).

In summary, the sets $\text{Pre}(\mathcal{X})$ and $\text{Suc}(\mathcal{X})$ are the results of linear operations on the polyhedra $\mathcal{X}$ and $\mathcal{U}$ and therefore are polyhedra. By using the definition of the Minkowski sum given in Section 4.4.9 and of affine operation on polyhedra in Section 4.4.11 we can compactly summarize the Pre and Suc operations on linear systems in Table 10.1.

Table 10.1 Pre and Suc operations for linear systems subject to polyhedral state and input constraints $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$

	$x(t+1) = Ax(t)$	$x(t+1) = Ax(t) + Bu(t)$
$\text{Pre}(\mathcal{X})$	$\mathcal{X} \circ A$	$(\mathcal{X} \oplus (-B \circ \mathcal{U})) \circ A$
$\text{Suc}(\mathcal{X})$	$A \circ \mathcal{X}$	$(A \circ \mathcal{X}) \oplus (B \circ \mathcal{U})$

The N-step controllable set $\mathcal{K}_N(\mathcal{S})$ and the N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ can be computed by using their recursive formulas (10.6), (10.7) and computing the Pre and Suc operations as in Table 10.1.

Example 10.3 Consider the second order unstable system (10.15) subject to the input and state constraints (10.16). Consider the target set

$$\mathcal{S} = \left\{ x \ : \ \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le x \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

The N-step controllable set $\mathcal{K}_N(\mathcal{S})$ of the system (10.15) subject to the constraints (10.16) can be computed by using the recursive formula (10.6)

$$\mathcal{K}_j(\mathcal{S}) = \mathrm{Pre}(\mathcal{K}_{j-1}(\mathcal{S})) \cap \mathcal{X}, \quad \mathcal{K}_0(\mathcal{S}) = \mathcal{S}, \qquad j = 1, \dots, N$$

and the steps described in Example 10.2 to compute the Pre(·) set.
The sets $\mathcal{K}_j(\mathcal{S})$ for j = 1, 2, 3, 4 are depicted in Figure 10.5.

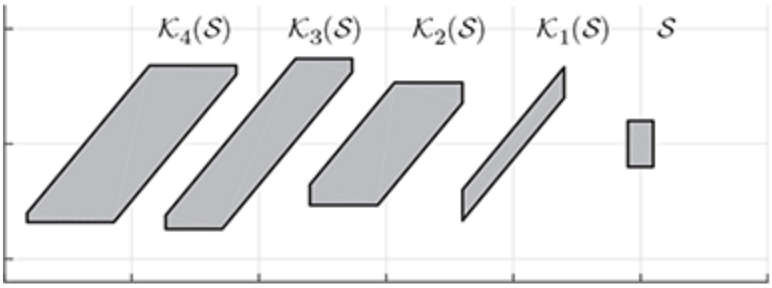

Figure 10.5 Example 10.3. Controllable sets $\mathcal{K}_j(\mathcal{S})$ for system (10.15) under constraints (10.16) for j = 1, 2, 3, 4. Note that the sets are shifted along the x-axis for a clearer visualization.

Example 10.4 Consider the second order unstable system (10.15) subject to the input and state constraints (10.16). Consider the initial set

$$\mathcal{X}_0 = \left\{ x \ : \ \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le x \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

The N-step reachable set $\mathcal{R}_N(\mathcal{X}_0)$ of the system (10.15) subject to the constraints (10.16) can be computed by using the recursively formula (10.7)

$$\mathcal{R}_{j+1}(\mathcal{X}_0) = \mathrm{Suc}(\mathcal{R}_j(\mathcal{X}_0)) \cap \mathcal{X}, \quad \mathcal{R}_0(\mathcal{X}_0) = \mathcal{X}_0, \qquad j = 0, \dots, N-1$$

and the steps described in Example 10.2 to compute the Suc(·) set.

The sets $\mathcal{R}_j(\mathcal{X}_0)$) for j = 1, 2, 3, 4 are depicted in Figure 10.6. The sets are shifted along the x-axis for a clearer visualization.

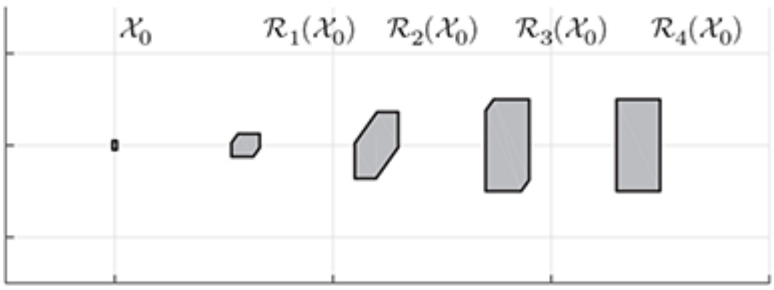

Figure 10.6 Example 10.3. Reachable sets $\mathcal{R}_j(\mathcal{X}_0)$) for system (10.15) under constraints (10.16) for j = 1, 2, 3, 4.

10.2 Invariant Sets

Consider the constrained autonomous system (10.1) and the constrained system subject to external inputs (10.2) defined in Section 10.1.

Two different types of sets are considered in this section: *invariant sets* and *control invariant sets*. We will first discuss invariant sets.

Positive Invariant Sets

Invariant sets are used for characterizing the behavior of autonomous systems. These types of sets are useful to answer questions such as: "For a *given* feedback controller $u = f(x)$, find the set of initial states whose trajectory will never violate the system constraints." The following definitions, derived from [172, 58, 54, 49, 179, 137, 139, 140], introduce the different types of invariant sets.

Definition 10.7 (Positive Invariant Set) *A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a positive invariant set for the autonomous system (10.1) subject to the constraints in (10.3), if*

$$x(0) \in \mathcal{O} \quad \Rightarrow \quad x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}_+.$$

Definition 10.8 (Maximal Positive Invariant Set $\mathcal{O}_\infty$) *The set $\mathcal{O}_\infty \subseteq \mathcal{X}$ is the maximal invariant set of the autonomous system (10.1) subject to the constraints in (10.3) if $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all the invariant sets contained in $\mathcal{X}$.*

Remark 10.2 The maximal invariant sets defined here are often referred to as "maximal admissible sets" or "maximal output admissible sets" in the literature (e.g., [124]), depending on whether the system state or output is constrained.

Remark 10.3 Note that, in general, the nonlinear system (10.1) may have multiple equilibrium points, and thus $\mathcal{O}_\infty$ might be the union of disconnected sets each containing an equilibrium point.

Theorem 10.1 (Geometric condition for invariance [100]) *A set $\mathcal{O} \subseteq \mathcal{X}$ is a positive invariant set for the autonomous system (10.1) subject to the constraints in (10.3), if and only if*

$$\mathcal{O} \subseteq \mathrm{Pre}(\mathcal{O}). \tag{10.22}$$

Proof: We prove both the necessary and sufficient parts by contradiction. (⇐:) If $\mathcal{O} \nsubseteq \mathrm{Pre}(\mathcal{O})$ then $\exists \bar{x} \in \mathcal{O}$ such that $\bar{x} \notin \mathrm{Pre}(\mathcal{O})$. From the definition of $\mathrm{Pre}(\mathcal{O})$, $g_a(\bar{x}) \notin \mathcal{O}$ and thus $\mathcal{O}$ is not positive invariant. (⇒:) If $\mathcal{O}$ is not a positive invariant set then $\exists \bar{x} \in \mathcal{O}$ such that $g_a(\bar{x}) \notin \mathcal{O}$. This implies that $\bar{x} \in \mathcal{O}$ and $\bar{x} \notin \mathrm{Pre}(\mathcal{O})$ and thus $\mathcal{O} \nsubseteq \mathrm{Pre}(\mathcal{O})$ ■

It is immediate to prove that condition (10.22) of Theorem 10.1 is equivalent to the following condition

$$\mathrm{Pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}. \tag{10.23}$$

Based on condition (10.23), the following algorithm provides a procedure for computing the maximal positive invariant subset $\mathcal{O}_\infty$ for system (10.1),(10.3) [10, 49, 172, 124].

Algorithm 10.1 *Computation of $\mathcal{O}_\infty$*

Input: g_a, $\mathcal{X}$

Output: $\mathcal{O}_\infty$

$\Omega_0 \leftarrow \mathcal{X}$, $k \leftarrow -1$

Repeat
  $k \leftarrow k+1$
  $\Omega_{k+1} \leftarrow \mathrm{Pre}(\Omega_k) \cap \Omega_k$
Until $\Omega_{k+1} = \Omega_k$
$\mathcal{O}_\infty \leftarrow \Omega_k$

Algorithm 10.1 generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k$, $\forall k \in \mathbb{N}$ and it terminates when $\Omega_{k+1} = \Omega_k$. If it terminates, then Ω_k is the maximal positive invariant set $\mathcal{O}_\infty$ for the system (10.1)–(10.3). If $\Omega_k = \emptyset$ for some integer k then the simple conclusion is that $\mathcal{O}_\infty = \emptyset$.

In general, Algorithm 10.1 may never terminate. If the algorithm does not terminate in a finite number of iterations, it can be proven that [179]

$$\mathcal{O}_\infty = \lim_{k \to +\infty} \Omega_k.$$

Conditions for finite time termination of Algorithm 10.1 can be found in [124]. A simple sufficient condition for finite time termination of Algorithm 10.1 requires the system $g_a(x)$ to be linear and stable, and the constraint set $\mathcal{X}$ to be bounded and to contain the origin.

Example 10.5 Consider the second order stable system in Example 10.1. The maximal positive invariant set of system (10.8) subject to constraints (10.9)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

is depicted in Figure 10.7.

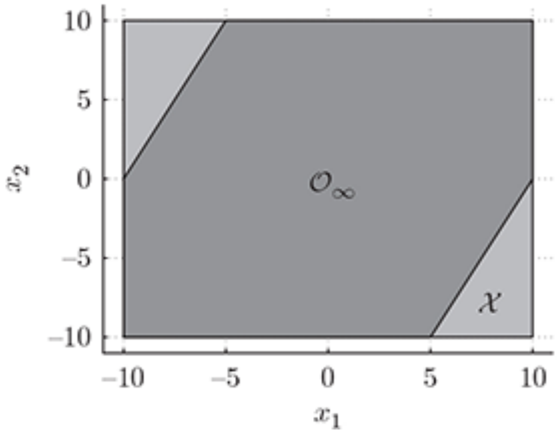

Figure 10.7 Example 10.5. Maximal Positive Invariant Set of system (10.8) under constraints (10.9).

Note from the previous discussion of the example and from Figure 10.1 that here the maximal positive invariant set $\mathcal{O}_\infty$ is obtained after a single step of Algorithm 10.1, i.e.,

$$\mathcal{O}_\infty = \Omega_1 = \text{Pre}(\mathcal{X}) \cap \mathcal{X}.$$

Control Invariant Sets

Control invariant sets are defined for systems subject to external inputs. These types of sets are useful to answer questions such as: "Find the set of initial states for which *there exists* a controller such that the system constraints are never violated." The following definitions, adopted from [172, 58, 54, 49, 179], introduce the different types of control invariant sets.

Definition 10.9 (Control Invariant Set) *A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a control invariant set for the system (10.2) subject to the constraints in (10.3), if*

$$x(t) \in \mathcal{C} \quad \Rightarrow \quad \exists u(t) \in \mathcal{U} \ \textit{such that} \ g(x(t), u(t)) \in \mathcal{C}, \quad \forall t \in \mathbb{N}_+.$$

Definition 10.10 (Maximal Control Invariant Set $\mathcal{C}_\infty$) *The set $\mathcal{C}_\infty \subseteq \mathcal{X}$ is said to be the maximal control invariant set for the system (10.2) subject to the constraints in (10.3), if it is control invariant and contains all control invariant sets contained in $\mathcal{X}$.*

Remark 10.4 The geometric conditions for invariance (10.22), (10.23) hold for control invariant sets.

The following algorithm provides a procedure for computing the maximal control invariant set $\mathcal{C}_\infty$ for system (10.2),(10.3) [10, 49, 172, 124].

Algorithm 10.2 *Computation of $\mathcal{C}_\infty$*

Input: g, $\mathcal{X}$, $\mathcal{U}$

Output: $\mathcal{C}_\infty$

$\Omega_0 \leftarrow \mathcal{X}$, $k \leftarrow -1$

Repeat

$k \leftarrow k+1$
$\Omega_{k+1} \leftarrow \text{Pre}(\Omega_k) \cap \Omega_k$
Until $\Omega_{k+1} = \Omega_k$
$\mathcal{C}_\infty \leftarrow \Omega_{k+1}$

Algorithm 10.2 generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k$, $\forall k \in \mathbb{N}$. Algorithm 10.2 terminates when $\Omega_{k+1} = \Omega_k$. If it terminates, then Ω_k is the maximal control invariant set $\mathcal{C}_\infty$ for the system (10.2)–(10.3). In general, Algorithm 10.2 may never terminate [10, 49, 172, 164]. If the algorithm does not terminate in a finite number of iterations, in general, convergence to the maximal control invariant set is not guaranteed

$$\mathcal{C}_\infty \neq \lim_{k \to +\infty} \Omega_k. \tag{10.24}$$

The work in [50] reports examples of nonlinear systems where (10.24) can be observed. A sufficient condition for the convergence of Ω_k to $\mathcal{C}_\infty$ as $k \to +\infty$ requires the polyhedral sets $\mathcal{X}$ and $\mathcal{U}$ to be bounded and the system $g(x, u)$ to be continuous [50].

Example 10.6 Consider the second order unstable system in Example 10.2. Algorithm 10.2 is used to compute the maximal control invariant set of system (10.15) subject to constraints (10.16). Algorithm 10.2 terminates after 45 iterations and the maximal control invariant set $\mathcal{C}_\infty$ is:

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 0.55 & -0.83 \\ -0.55 & 0.83 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 4 \\ 4 \\ 2.22 \\ 2.22 \\ 10 \\ 10 \end{bmatrix}.$$

The results of the iterations and $\mathcal{C}_\infty$ are depicted in Figure 10.8.

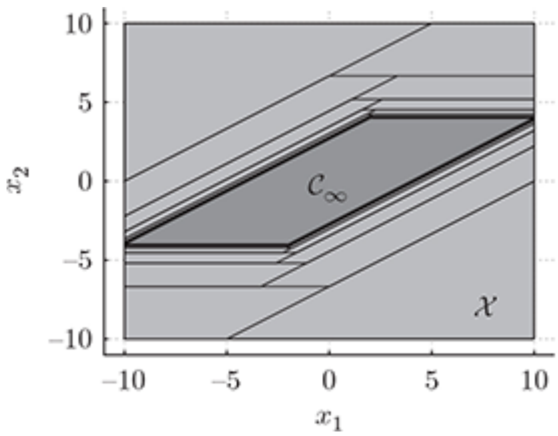

Figure 10.8 Example 10.6. Maximal Control Invariant Set of system (10.15) subject to constraints (10.16).

Definition 10.11 (Finitely determined set) *Consider Algorithm 10.1 (Algorithm 10.2). The set*

$\mathcal{O}_\infty$ ($\mathcal{C}_\infty$) is finitely determined if and only if $\exists\ i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$. The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ is called the determinedness index.

Remark 10.5 From the results in Section 10.1.1, for linear system with linear constraints the sets $\mathcal{O}_\infty$ and $\mathcal{C}_\infty$ are polyhedra if they are finitely determined.

For all states contained in the maximal control invariant set $\mathcal{C}_\infty$ there exists a control law such that the system constraints are never violated. This does not imply that there exists a control law which can drive the state into a user-specified target set. This issue is addressed in the following by introducing the concepts of maximal controllable sets and stabilizable sets.

Definition 10.12 (Maximal Controllable Set $\mathcal{K}_\infty(\mathcal{O})$) *For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the maximal controllable set $\mathcal{K}_\infty(\mathcal{O})$ for system (10.2) subject to the constraints in (10.3) is the union of all N-step controllable sets $\mathcal{K}_N(\mathcal{O})$ contained in $\mathcal{X}$ ($N \in \mathbb{N}$).*

We will often deal with controllable sets $\mathcal{K}_N(\mathcal{O})$ where the target $\mathcal{O}$ is a control invariant set. They are special sets, since in addition to guaranteeing that from $\mathcal{K}_N(\mathcal{O})$ we reach $\mathcal{O}$ in N steps, one can ensure that once it has reached $\mathcal{O}$, the system can stay there at all future time instants.

Definition 10.13 (N-step (Maximal) Stabilizable Set) *For a given control invariant set $\mathcal{O} \subseteq \mathcal{X}$, the N-step (maximal) stabilizable set of the system (10.2) subject to the constraints (10.3) is the N-step (maximal) controllable set $\mathcal{K}_N(\mathcal{O})$ ($\mathcal{K}_\infty(\mathcal{O})$).*

The set $\mathcal{K}_\infty(\mathcal{O})$ contains all states which can be steered into the control invariant set $\mathcal{O}$ and hence $\mathcal{K}_\infty(\mathcal{O}) \subseteq \mathcal{C}_\infty$. The set $\mathcal{K}_\infty(\mathcal{O}) \subseteq \mathcal{C}_\infty$ can be computed as follows [58, 50]:

Algorithm 10.3 *Computation of $\mathcal{K}_\infty(\mathcal{O})$*

Input: g, $\mathcal{X}$, $\mathcal{U}$

Output: $\mathcal{K}_\infty(\mathcal{O})$

$\mathcal{K}_0 \leftarrow \mathcal{O}$, where $\mathcal{O}$ is a control invariant set

$c \leftarrow -1$
Repeat
  $c \leftarrow c+1$
  $\mathcal{K}_{c+1} \leftarrow \text{Pre}(\mathcal{K}_c)\ \mathcal{X}$

Until $\mathcal{K}_{c+1} = \mathcal{K}_c$

$\mathcal{K}_\infty(\mathcal{O}) \leftarrow \mathcal{K}_c$

Since $\mathcal{O}$ is control invariant, it holds $\forall c \in \mathbb{N}$ that $\mathcal{K}_c(\mathcal{O})$ is control invariant and $\mathcal{K}_c \subseteq \mathcal{K}_{c+1}$. Note that Algorithm 10.3 is not guaranteed to terminate in finite time.

Remark 10.6 In general, the maximal stabilizable set $\mathcal{K}_\infty(\mathcal{O})$ is not equal to the maximal control invariant set $\mathcal{C}_\infty$, even for linear systems. $\mathcal{K}_\infty(\mathcal{O}) \subseteq \mathcal{C}_\infty$ for all control invariant sets $\mathcal{O}$. The set $\mathcal{C}_\infty \mathcal{K}_\infty(\mathcal{O})$ includes all initial states from which it is not possible to steer the system to the stabilizable region $\mathcal{K}_\infty(\mathcal{O})$ and hence $\mathcal{O}$.

Example 10.7 Consider the simple constrained one-dimensional system

$$x(t+1) = 2x(t) + u(t) \tag{10.25a}$$

$$|x(t)| \le 1, \text{ and } |u(t)| \le 1 \tag{10.25b}$$

and the state feedback control law

$$u(t) = \begin{cases} 1 & \text{if} \quad x(t) \in \left[-1,\ -\frac{1}{2}\right] \\ -2x(t) & \text{if} \quad x(t) \in \left[-\frac{1}{2},\ \frac{1}{2}\right] \\ -1 & \text{if} \quad x(t) \in \left[\frac{1}{2},\ 1\right]. \end{cases} \tag{10.26}$$

The closed-loop system has three equilibria at –1, 0, and 1 and system (10.25) is always feasible for all initial states in [–1, 1] and therefore $\mathcal{C}_\infty$ = [–1, 1]. The equilibrium point 0 is, however, asymptotically stable only for the open set (–1, 1). In fact, $u(t)$ and any other feasible control law cannot stabilize the system from $x = 1$ and from $x = -1$ and therefore when $\mathcal{O} = \emptyset$ then $\mathcal{K}_\infty(\mathcal{O})$ = (–1, 1) $\subset \mathcal{C}_\infty$. We note in this example that the maximal stabilizable set is open. One can easily argue that, in general, if the maximal stabilizable set is closed then it is equal to the maximal control invariant set.

10.3 Robust Controllable and Reachable Sets

In this section we deal with two types of systems, namely, autonomous systems

$$x(k+1) = g_a(x(k), w(k)), \tag{10.27}$$

and systems subject to external controllable inputs

$$x(k+1) = g(x(k), u(k), w(k)). \tag{10.28}$$

Both systems are subject to the disturbance $w(k)$ and to the constraints

$$x(k) \in \mathcal{X},\ \ u(k) \in \mathcal{U},\ \ w(k) \in \mathcal{W}\ \forall\ k \ge 0. \tag{10.29}$$

The sets $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ are polyhedra.

Definition 10.14 *For the autonomous system (10.27) we will denote the* robust precursor set to the set $\mathcal{S}$ *as*

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \{x \in \mathbb{R}^n \;:\; g_a(x,w) \in \mathcal{S},\ \forall w \in \mathcal{W}\}. \tag{10.30}$$

Pre($\mathcal{S}$, $\mathcal{W}$) defines the set of states of system (10.27) which evolve into the target set $\mathcal{S}$ in one time step for all possible disturbances $w \in \mathcal{W}$.

Definition 10.15 *For the system (10.28) we will denote the* robust precursor set to the set $\mathcal{S}$ *as*

$$\mathrm{Pre}(\mathcal{S},\mathcal{W}) = \{x \in \mathbb{R}^n \;:\; \exists u \in \mathcal{U}\ s.t.\ g(x,u,w) \subseteq \mathcal{S},\ \forall w \in \mathcal{W}\}. \tag{10.31}$$

For a system with inputs, Pre($\mathcal{S}$, $\mathcal{W}$) is the set of states which can be robustly driven into the target set $\mathcal{S}$ in one time step for all admissible disturbances.

Definition 10.16 *For the autonomous system (10.27) we will denote the* robust successor set from the set $\mathcal{S}$ *as*

$$\mathrm{Suc}(\mathcal{S},\mathcal{W}) = \{x \in \mathbb{R}^n \;:\; \exists\ x(0) \in \mathcal{S},\ \exists\ w \in \mathcal{W}\ such\ that\ x = g_a(x(0),w)\}.$$

Definition 10.17 *For the system (10.28) with inputs we will denote the* robust successor set from the set $\mathcal{S}$ *as*

$$\mathrm{Suc}(\mathcal{S},\mathcal{W}) = \{x \in \mathbb{R}^n : \exists\ x(0) \in \mathcal{S},\ \exists\ u \in \mathcal{U},\ \exists\ w \in \mathcal{W},\ such\ that\ x = g(x(0),u,w)\}.$$

Thus, all the states contained in $\mathcal{S}$ are mapped into the set Suc($\mathcal{S}$, $\mathcal{W}$) under the map g_a for all disturbances $w \in \mathcal{W}$, and under the map g for all inputs $u \in \mathcal{U}$ and for all disturbances $w \in \mathcal{W}$.

Remark 10.7 The sets Pre($\mathcal{S}$, $\mathcal{W}$) and Suc($\mathcal{S}$, $\mathcal{W}$) are also denoted as "one-step robust backward-reachable set" and "one-step robust forward-reachable set," respectively, in the literature.

N-step robust controllable and robust reachable sets are defined by iterating Pre(·, ·) and Suc(·, ·) computations, respectively.

Definition 10.18 (*N*-Step Robust Controllable Set $\mathcal{K}_N$ ($\mathcal{S}$, $\mathcal{W}$)) *For a given target set $\mathcal{S} \subseteq \mathcal{X}$, the N-step robust controllable set $\mathcal{K}_N$ ($\mathcal{S}$, $\mathcal{W}$) of the system (10.27) or (10.28) subject to the constraints (10.29) is defined recursively as:*

$$\mathcal{K}_j(\mathcal{S},\mathcal{W}) = \mathrm{Pre}(\mathcal{K}_{j-1}(\mathcal{S},\mathcal{W}),\mathcal{W}) \cap \mathcal{X},\quad \mathcal{K}_0(\mathcal{S},\mathcal{W}) = \mathcal{S},\quad j \in \{1,\ldots,N\}. \tag{10.32}$$

From Definition 10.18, all states x_0 belonging to the *N*-Step Robust Controllable Set $\mathcal{K}_N$ ($\mathcal{S}$, $\mathcal{W}$)

can be robustly driven, through a time-varying control law, to the target set $\mathcal{S}$ in N steps, while satisfying input and state constraints for all possible disturbances.

N-step robust reachable sets are defined analogously to N-step robust controllable set.

Definition 10.19 (N-Step Robust Reachable Set $\mathcal{R}_N(\mathcal{X}_0, \mathcal{W})$) *For a given initial set $\mathcal{X}_0 \subseteq \mathcal{X}$, the N-step robust reachable set $\mathcal{R}_N(\mathcal{X}_0, \mathcal{W})$ of the system (10.27) or (10.28) subject to the constraints (10.29) is defined recursively as:*

$$\mathcal{R}_{i+1}(\mathcal{X}_0, \mathcal{W}) = \text{Suc}(\mathcal{R}_i(\mathcal{X}_0, \mathcal{W}), \mathcal{W}) \cap \mathcal{X}, \quad \mathcal{R}_0(\mathcal{X}_0, \mathcal{W}) = \mathcal{X}_0, \qquad i = 0, \ldots, N-1. \tag{10.33}$$

From Definition 10.19, all states x_0 belonging to $\mathcal{X}_0$ will evolve to the N-step robust reachable set $\mathcal{R}_N(\mathcal{X}_0, \mathcal{W})$ in N steps.

Next, we will show through simple examples the main steps involved in the computation of robust controllable and robust reachable sets for certain classes of uncertain constrained linear systems.

10.3.1 Linear Systems with Additive Uncertainty and without Inputs

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Example 10.8 Consider the second order autonomous system

$$x(t+1) = Ax(t) + w(t) = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x(t) + w(t) \tag{10.34}$$

subject to the state constraints

$$x(t) \in \mathcal{X} = \left\{ x : \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \ \forall t \ge 0, \tag{10.35}$$

and where the additive disturbance belongs to the set

$$w(t) \in \mathcal{W} = \left\{ w : \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \ \forall t \ge 0. \tag{10.36}$$

The set $\text{Pre}(\mathcal{X}, \mathcal{W})$ can be obtained as described next. Since the set $\mathcal{X}$ is a polytope, it can be represented as an $\mathcal{H}$-polytope (Section 4.2)

$$\mathcal{X} = \{x : \ Hx \le h\}, \tag{10.37}$$

where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } h = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}.$$

By using this $\mathcal{H}$-presentation and the system equation (10.34), the set Pre($\mathcal{X}$, $\mathcal{W}$) can be rewritten as

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) = \{x \;:\; Hg_a(x, w) \leq h, \;\forall w \in \mathcal{W}\} \tag{10.38a}$$

$$= \{x \;:\; HAx \leq h - Hw, \;\forall w \in \mathcal{W}\}. \tag{10.38b}$$

The set (10.38) can be represented as a the following polyhedron

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) = \{x \in \mathbb{R}^n \;:\; HAx \leq \tilde{h}\} \tag{10.39}$$

with

$$\tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w). \tag{10.40}$$

In general, a linear program is required to solve problems (10.40). In this example H_i and $\mathcal{W}$ have simple expressions and we get $\tilde{h} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$. The set (10.39) might contain redundant inequalities which can be removed by using Algorithm 4.1 in Section 4.4.1 to obtain its minimal representation.

The set Pre($\mathcal{X}$, $\mathcal{W}$) is

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x \;:\; \begin{bmatrix} 1 & 0 \\ 1 & -0.5 \\ -1 & 0 \\ -1 & 0.5 \end{bmatrix} x \leq \begin{bmatrix} 18 \\ 9 \\ 18 \\ 9 \end{bmatrix} \right\}.$$

The one-step robust controllable set $\mathcal{K}_1$($\mathcal{X}$, $\mathcal{W}$) = Pre($\mathcal{X}$, $\mathcal{W}$) ∩ $\mathcal{X}$ is

$$\mathrm{Pre}(\mathcal{X}, \mathcal{W}) \cap \mathcal{X} = \left\{ x \;:\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 9 \\ 9 \end{bmatrix} \right\}$$

and is depicted in Figure 10.9.

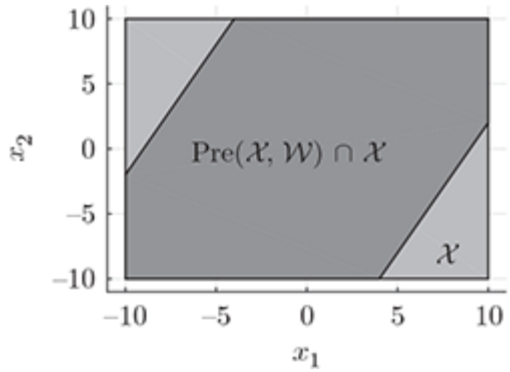

Figure 10.9 Example 10.8. One-step robust controllable set $\mathrm{Pre}(\mathcal{X}, \mathcal{W}) \cap \mathcal{X}$ for system (10.34) under constraints (10.35)–(10.36).

Note that by using the definition of the Pontryagin difference given in Section 4.4.8 and affine operations on polyhedra in Section 4.4.11 we can compactly summarize the operations in (10.38) and write the set Pre in (10.30) as

$$\begin{aligned}\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \{x \in \mathbb{R}^n \;:\; Ax + w \in \mathcal{X},\ \forall w \in \mathcal{W}\} = \{x \in \mathbb{R}^n \;:\; Ax \in \mathcal{X} \ominus \mathcal{W}\} = \\ = (\mathcal{X} \ominus \mathcal{W}) \circ A.\end{aligned}$$

The set

$$\mathrm{Suc}(\mathcal{X},\mathcal{W}) = \{y \;:\; \exists x \in \mathcal{X},\ \exists w \in \mathcal{W} \text{ such that } y = Ax + w\} \tag{10.41}$$

is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the disturbance $w \in \mathcal{W}$. Let us write $\mathcal{X}$ in $\mathcal{V}$-representation (see Section 4.1)

$$\mathcal{X} = \mathrm{conv}(V), \tag{10.42}$$

and let us map the set of vertices V through the transformation A. Because the transformation is linear, the composition of the map A with the set $\mathcal{X}$, denoted as $A \circ \mathcal{X}$, is simply the convex hull of the transformed vertices

$$A \circ \mathcal{X} = \mathrm{conv}(AV). \tag{10.43}$$

We refer the reader to Section 4.4.11 for a detailed discussion on linear transformations of polyhedra. Rewrite (10.41) as

$$\mathrm{Suc}(\mathcal{X},\mathcal{W}) = \{y \in \mathbb{R}^n \;:\; \exists\, z \in A \circ \mathcal{X},\ \exists w \in \mathcal{W} \text{ such that } y = z + w\}.$$

We can use the definition of the Minkowski sum given in Section 4.4.9 and rewrite the Suc set as

$$\mathrm{Suc}(\mathcal{X},\mathcal{W}) = (A \circ \mathcal{X}) \oplus \mathcal{W}.$$

We can compute the Minkowski sum via projection or vertex enumeration as explained in Section 4.4.9. The set $\mathrm{Suc}(\mathcal{X}, \mathcal{W})$ in $\mathcal{H}$-representation is

$$\mathrm{Suc}(\mathcal{X}, \mathcal{W}) = \left\{ x \;:\; \begin{bmatrix} 1 & -0.5 \\ 0 & -1 \\ -1 & 0 \\ -1 & 0.5 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 4 \\ 16 \\ 6 \\ 4 \\ 16 \\ 6 \end{bmatrix} \right\},$$

and is depicted in Figure 10.10.

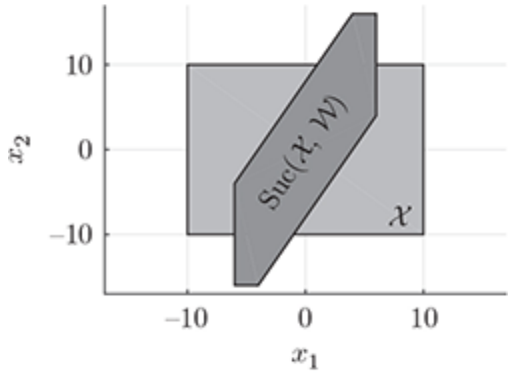

Figure 10.10 Example 10.8. Robust successor set for system (10.34) under constraints (10.36).

10.3.2 Linear Systems with Additive Uncertainty and Inputs

Example 10.9 Consider the second order unstable system

$$\left\{ \; x(t+1) = Ax + Bu = \begin{bmatrix} 1.5 & 0 \\ 1 & -1.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + w(t) \right. \tag{10.44}$$

subject to the input and state constraints

$$u(t) \in \mathcal{U} = \{u \;:\; -5 \leq u \leq 5\}, \; \forall t \geq 0 \tag{10.45a}$$

$$x(t) \in \mathcal{X} = \left\{ x \;:\; \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \; \forall t \geq 0, \tag{10.45b}$$

where

$$w(t) \in \mathcal{W} = \{w \;:\; -1 \leq w \leq 1\}, \; \forall t \geq 0. \tag{10.46}$$

For the nonautonomous system (10.44), the set Pre($\mathcal{X}$, $\mathcal{W}$) can be computed using the $\mathcal{H}$-presentation of $\mathcal{X}$ and $\mathcal{U}$,

$$\mathcal{X} = \{x \;:\; Hx \leq h\}, \quad \mathcal{U} = \{u \;:\; H_u u \leq h_u\}, \tag{10.47}$$

to obtain

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \{x \in \mathbb{R}^2 \ : \ \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + w \in \mathcal{X}, \ \forall\, w \in \mathcal{W}\} \tag{10.48a}$$

$$= \left\{x \in \mathbb{R}^2 \ : \ \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} h - Hw \\ h_u \end{bmatrix}, \ \forall\, w \in \mathcal{W}\right\}. \tag{10.48b}$$

As in Example 10.8, the set $\mathrm{Pre}(\mathcal{X}, \mathcal{W})$ can be compactly written as

$$\mathrm{Pre}(\mathcal{X},\mathcal{W}) = \left\{x \in \mathbb{R}^2 \ : \ \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{bmatrix} \tilde{h} \\ h_u \end{bmatrix}\right\}, \tag{10.49}$$

where

$$\tilde{h}_i = \min_{w \in \mathcal{W}} (h_i - H_i w). \tag{10.50}$$

In general, a linear program is required to solve problems (10.50). In this example H_i and $\mathcal{W}$ have simple expressions and we get $\tilde{h} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$.

The halfspaces in (10.49) define a polytope in the state-input space, and a projection operation (see Section 4.4.6) is used to derive the halfspaces which define $\mathrm{Pre}(\mathcal{X}, \mathcal{W})$ in the state space. The set $\mathrm{Pre}(\mathcal{X}, \mathcal{W}) \cap \mathcal{X}$ is depicted in Figure 10.11 and reported below:

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & -1.5 \\ -1 & 1.5 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 9.3 \\ 9.3 \\ 9 \\ 9 \\ 10 \\ 10 \end{bmatrix}.$$

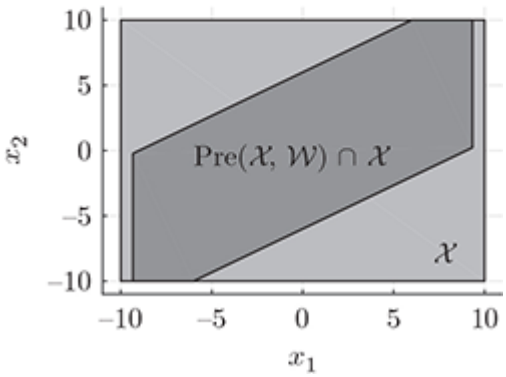

Figure 10.11 Example 10.9. One-step robust controllable set $\mathrm{Pre}(\mathcal{X}, \mathcal{W}) \cap \mathcal{X}$ for system (10.44) under constraints (10.45)–(10.46).

Note that by using the definition of a Minkowski sum given in Section 4.4.9 and the affine operation on polyhedra in Section 4.4.11 we can compactly write the operations in (10.48) as follows:

$$\begin{aligned}\mathrm{Pre}(\mathcal{X},\mathcal{W}) &= \{x \ : \ \exists u \in \mathcal{U} \text{ s.t. } Ax+Bu+w \in \mathcal{X},\ \forall\, w \in \mathcal{W}\}\\ &= \{x \ : \ \exists y \in \mathcal{X},\ \exists u \in \mathcal{U} \text{ s.t. } y = Ax+Bu+w,\ \forall\, w \in \mathcal{W}\}\\ &= \{x \ : \ \exists y \in \mathcal{X},\ \exists u \in \mathcal{U} \text{ s.t. } Ax = y+(-Bu)-w,\ \forall\, w \in \mathcal{W}\}\\ &= \{x \ : \ Ax \in \mathcal{C} \text{ and } \mathcal{C} = \mathcal{X} \oplus (-\mathrm{B}) \circ \mathcal{U} \ominus \mathcal{W}\}\\ &= \{x \ : \ x \in \mathcal{C} \circ A,\ \mathcal{C} = \mathcal{X} \oplus (-B) \circ \mathcal{U} \ominus \mathcal{W}\}\\ &= \{x \ : \ x \in ((\mathcal{X} \ominus \mathcal{W}) \oplus (-B \circ \mathcal{U})) \circ A\}\,.\end{aligned} \tag{10.51}$$

Remark 10.8 Note that in (10.51) we have used the fact that if a set $\mathcal{S}$ is described as $\mathcal{S} = \{v : \exists z \in \mathcal{Z}, \text{ s.t. } v = z - w, \forall\, w \in \mathcal{W}\}$, then $\mathcal{S} = \{v : \exists z \in \mathcal{Z}, \text{ s.t. } z = v + w, \forall\, w \in \mathcal{W}\}$ or $\mathcal{S} = \{v : v + w \in \mathcal{Z}, \forall\, w \in \mathcal{W}\} = \mathcal{Z} \ominus \mathcal{W}$. Also, to derive the last equation of (10.51) we have used the associative property of the Pontryagin difference.

The set $\mathrm{Suc}(\mathcal{X}, \mathcal{W}) = \{y : \exists x \in \mathcal{X}, \exists u \in \mathcal{U}, \exists w \in \mathcal{W} \text{ s.t. } y = Ax + Bu + w\}$ is obtained by applying the map A to the set $\mathcal{X}$ and then considering the effect of the input $u \in \mathcal{U}$ and of the disturbance $w \in \mathcal{W}$. We can use the definition of Minkowski sum given in Section 4.4.9 and rewrite $\mathrm{Suc}(\mathcal{X}, \mathcal{W})$ as

$$\mathrm{Suc}(\mathcal{X},\mathcal{W}) = (A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}.$$

The set $\mathrm{Suc}(\mathcal{X}, \mathcal{W})$ is depicted in Figure 10.12.

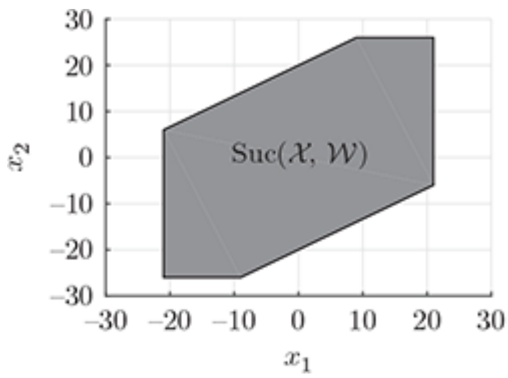

Figure 10.12 Example 10.9. Robust successor set for system (10.44) under constraints (10.45)–(10.46).

In summary, for linear systems with additive disturbances the sets $\mathrm{Pre}(\mathcal{X}, \mathcal{W})$ and $\mathrm{Suc}(\mathcal{X}, \mathcal{W})$ are the results of linear operations on the polytopes $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ and therefore are polytopes. By using the definition of Minkowski sum given in Section 4.4.9, Pontryagin difference given in Section 4.4.8 and affine operation on polyhedra in Section 4.4.11 we can compactly summarize the operations in Table 10.2. Note that the summary in Table 10.2 applies also to the class of systems $x(k+1) = Ax(t) + Bu(t) + E\tilde{d}(t)$ where $\tilde{d} \in \tilde{\mathcal{W}}$. This can be transformed into $x(k+1) = Ax(t) + Bu(t) + w(t)$ where $w \in \mathcal{W} = E \circ \tilde{\mathcal{W}}$.

Table 10.2 Pre and Suc operations for uncertain linear systems subject to polyhedral input and state constraints $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$ with additive polyhedral disturbances $w(t) \in \mathcal{W}$.

	$x(t+1) = Ax(t) + w(t)$	$x(t+1) = Ax(t) + Bu(t) + w(t)$
$\text{Pre}(\mathcal{X}, \mathcal{W})$	$(\mathcal{X} \ominus \mathcal{W}) \circ A$	$((\mathcal{X} \ominus \mathcal{W}) \oplus (-B \circ \mathcal{U})) \circ A$
$\text{Suc}(\mathcal{X}, \mathcal{W})$	$(A \circ \mathcal{X}) \oplus \mathcal{W}$	$(A \circ \mathcal{X}) \oplus (B \circ \mathcal{U}) \oplus \mathcal{W}$

The N-step robust controllable set $\mathcal{K}_N(\mathcal{S}, \mathcal{W})$ and the N-step robust reachable set $\mathcal{R}_N(\mathcal{X}_0, \mathcal{W})$ can be computed by using their recursive formulas (10.32), (10.33) and computing the Pre and Suc operations as described in Table 10.2.

10.3.3 Linear Systems with Parametric Uncertainty

The next Lemma 10.1 will help us computing Pre and Suc sets for linear systems with parametric uncertainty.

Lemma 10.1 *Let* $g : \mathbb{R}^{n_z} \times \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_g}$ *be a function of* (z, x, w) *convex in* w *for each* (z, x)*. Assume that the variable* w *belongs to the polytope* $\mathcal{W}$ *with vertices* $\{\bar{w}_i\}_{i=1}^{n_\mathcal{W}}$*. Then, the constraint*

$$g(z, x, w) \le 0 \ \forall w \in \mathcal{W} \tag{10.52}$$

is satisfied if and only if

$$g(z, x, \bar{w}_i) \le 0, \ i = 1, \dots, n_\mathcal{W}. \tag{10.53}$$

Proof: Easily follows from the fact that the maximum of a convex function over a compact convex set is attained at an extreme point of the set. ■

Lemma 10.2 shows how to reduce the number of constraints in (10.53) for a specific class of constraint functions.

Lemma 10.2 *Assume* $g(z, x, w) = g^1(z, x) + g^2(w)$*. Then the constraint (10.52) can be replaced by* $g^1(z, x) \le -\bar{g}$*, where* $\bar{g} = [\bar{g}_1, \dots, \bar{g}_{n_g}]'$ *is a vector whose i-th component is*

$$\bar{g}_i = \max_{w \in \mathcal{W}} g_i^2(w), \tag{10.54}$$

and $g_i^2(w)$ *denotes the i-th component of* $g^2(w)$*.*

Example 10.10 Consider the second order autonomous system

$$x(t+1) = A(w^p(t))x(t) + w^a(t) = \begin{bmatrix} 0.5 + w^p(t) & 0 \\ 1 & -0.5 \end{bmatrix} x(t) + w^a(t) \qquad (10.55)$$

subject to the constraints

$$\begin{aligned} x(t) \in \mathcal{X} &= \left\{ x \ : \ \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \ \forall t \ge 0 \\ w^a(t) \in \mathcal{W}^a &= \left\{ w^a \ : \ \begin{bmatrix} -1 \\ -1 \end{bmatrix} \le w^a \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \ \forall t \ge 0 \\ w^p(t) \in \mathcal{W}^p &= \{ w^p \ : \ 0 \le w^p \le 0.5 \}, \ \forall t \ge 0. \end{aligned} \qquad (10.56)$$

Let $w = [w^a; w^p]$ and $\mathcal{W} = \mathcal{W}^a \times \mathcal{W}^p$. The set $\text{Pre}(\mathcal{X}, \mathcal{W})$ can be obtained as follows. The set $\mathcal{X}$ is a polytope and it can be represented as an $\mathcal{H}$-polytope (Section 4.2)

$$\mathcal{X} = \{x : \ Hx \le h\}, \qquad (10.57)$$

where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } h = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}.$$

By using this $\mathcal{H}$-presentation and the system equation (10.55), the set $\text{Pre}(\mathcal{X}, \mathcal{W})$ can be rewritten as

$$\begin{aligned} \text{Pre}(\mathcal{X}, \mathcal{W}) &= \{ x \ : \ Hg_a(x, w) \le h, \ \forall w \in \mathcal{W} \} \qquad (10.58) \\ &= \{ x \ : \ HA(w^p)x \le h - Hw^a, \ \forall w^a \in \mathcal{W}^a, \ w^p \in \mathcal{W}^p \}. \qquad (10.59) \end{aligned}$$

By using Lemmas 10.1 and 10.2, the set (10.59) can be rewritten as a the polytope

$$x \in \text{Pre}(\mathcal{X}, \mathcal{W}) = \left\{ x \in \mathbb{R}^n \ : \ \begin{bmatrix} HA(0) \\ HA(0.5) \end{bmatrix} x \le \begin{bmatrix} \tilde{h} \\ \tilde{h} \end{bmatrix} \right\} \qquad (10.60)$$

with

$$\tilde{h}_i = \min_{w^a \in \mathcal{W}^a} (h_i - H_i w^a), \ i = 1, \ldots, 4. \qquad (10.61)$$

The set $\text{Pre}(\mathcal{X}, \mathcal{W}) \cap \mathcal{X}$ is depicted in Figure 10.13 and reported below:

$$\begin{bmatrix} 1 & 0 \\ 0.89 & -0.44 \\ 1 & 0 \\ -0.89 & 0.44 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 9 \\ 8.049 \\ 9 \\ 8.049 \\ 10 \\ 10 \end{bmatrix}.$$

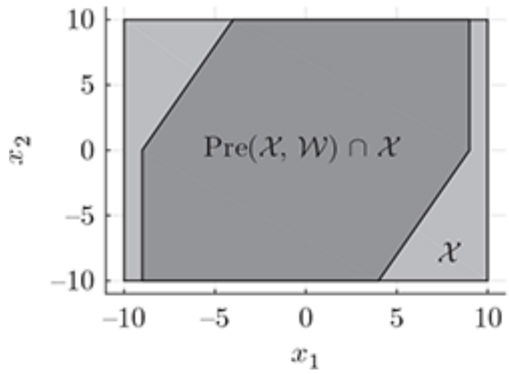

Figure 10.13 Example 10.10. One-step robust controllable set Pre($\mathcal{X}$, $\mathcal{W}$)∩$\mathcal{X}$ for system (10.55) under constraints (10.56).

Example 10.11 Consider the second order autonomous system

$$x(t+1) = A(w^p(t))x(t) = \begin{bmatrix} 0.5 + w^p(t) & 0 \\ 1 & -0.5 \end{bmatrix} x(t) \tag{10.62}$$

subject to the constraints

$$\begin{aligned} & x(t) \in \mathcal{X} = \left\{ x \ : \ \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \ \forall t \ge 0 \\ & w^p(t) \in \mathcal{W}^p = \{ w^p \ : \ 0 \le w^p \le 0.5 \}, \ \forall t \ge 0. \end{aligned} \tag{10.63}$$

Let $w = [w^p]$ and $\mathcal{W} = \mathcal{W}^p$. The set Suc($\mathcal{X}$, $\mathcal{W}$) can be written as infinite union of reachable sets

$$\mathrm{Suc}(\mathcal{X}, \mathcal{W}) = \bigcup_{\bar{w} \in \mathcal{W}} \mathrm{Suc}(\mathcal{X}, \bar{w}) \tag{10.64}$$

where Suc($\mathcal{X}$, $\overline{w}$) is computed as described in Example 10.1 for the system $A(\overline{w})$. In general, the union in equation (10.64) generates a nonconvex set, as can be seen in Figure 10.14. Nonconvexity of successor sets for parametric linear systems is also discussed in [59][Section 6.1.2].

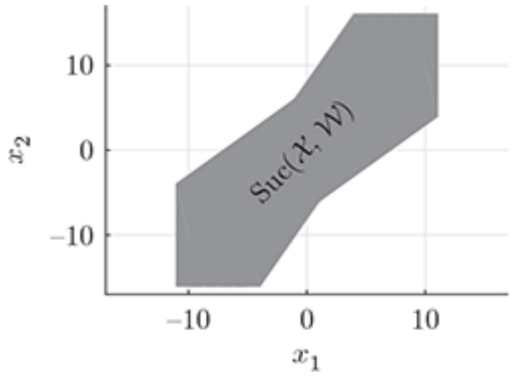

Figure 10.14 Example 10.11. Robust successor set Suc($\mathcal{X}$, $\mathcal{W}$) for system (10.62) under constraints (10.63).

10.4 Robust Invariant Sets

Two different types of sets are considered in this chapter: *robust invariant sets* and *robust control invariant sets*. We will first discuss robust invariant sets.

Robust Positive Invariant Sets

Robust invariant sets are computed for autonomous systems. These types of sets are useful to answer questions such as: "For a *given* feedback controller $u = f(x)$, find the set of states whose trajectory will never violate the system constraints for all possible disturbances." The following definitions introduce the different types of robust invariant sets.

Definition 10.20 (Robust Positive Invariant Set) *A set $\mathcal{O} \subseteq \mathcal{X}$ is said to be a robust positive invariant set for the autonomous system (10.27) subject to the constraints (10.29), if*

$$x(0) \in \mathcal{O} \quad \Rightarrow \quad x(t) \in \mathcal{O}, \quad \forall w(t) \in \mathcal{W}, \ t \in \mathbb{N}_+.$$

Definition 10.21 (Maximal Robust Positive Invariant Set $\mathcal{O}_\infty$) *The set $\mathcal{O}_\infty \subseteq \mathcal{X}$ is the maximal robust invariant set of the autonomous system (10.27) subject to the constraints (10.29) if $\mathcal{O}_\infty$ is a robust invariant set and $\mathcal{O}_\infty$ contains all the robust positive invariant sets contained in $\mathcal{X}$.*

Theorem 10.2 (Geometric condition for invariance) *A set $\mathcal{O} \subseteq \mathcal{X}$ is a robust positive invariant set for the autonomous system (10.27) subject to the constraints (10.29), if and only if*

$$\mathcal{O} \subseteq \mathrm{Pre}(\mathcal{O}, \mathcal{W}). \tag{10.65}$$

The proof of Theorem 10.2 follows the same lines of the proof of Theorem 10.1. ■

It is immediate to prove that condition (10.65) of Theorem 10.2 is equivalent to the following condition

$$\mathrm{Pre}(\mathcal{O}, \mathcal{W}) \cap \mathcal{O} = \mathcal{O}. \tag{10.66}$$

Based on condition (10.66), the following algorithm provides a procedure for computing the maximal robust positive invariant subset $\mathcal{O}_\infty$ for system (10.27)– (10.29) (for reference to proofs and literature see Section 10.1).

Algorithm 10.4 *Computation of $\mathcal{O}_\infty$*

Input: g_a, $\mathcal{X}$, $\mathcal{W}$

Output: $\mathcal{O}_\infty$

$\Omega_0 \leftarrow \mathcal{X}$, $k \leftarrow -1$

Repeat
$k \leftarrow k+1$
$\Omega_{k+1} \leftarrow \mathrm{Pre}(\Omega_k, \mathcal{W}) \cap \Omega_k$

Until $\Omega_{k+1} = \Omega_k$
$\mathcal{O}_\infty \leftarrow \Omega_k$

Algorithm 10.4 generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k$, $\forall k \in \mathbb{N}$ and it terminates when $\Omega_{k+1} = \Omega_k$. If it terminates, then Ω_k is the maximal robust positive invariant set $\mathcal{O}_\infty$ for system (10.27)–(10.29). If $\Omega_k = \emptyset$ for some integer k then the simple conclusion is that $\mathcal{O}_\infty = \emptyset$.

In general, Algorithm 10.4 may never terminate. If the algorithm does not terminate in a finite number of iterations, it can be proven that [179]

$$\mathcal{O}_\infty = \lim_{k \to +\infty} \Omega_k.$$

Conditions for finite time termination of Algorithm 10.4 can be found in [124]. A simple sufficient condition for finite time termination of Algorithm 10.1 requires the system $g_a(x, w)$ to be linear and stable, and the constraint set $\mathcal{X}$ and disturbance set $\mathcal{W}$ to be bounded and to contain the origin.

Example 10.12 Consider the second order stable system in Example 10.8

$$x(t+1) = Ax(t) + w(t) = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} x(t) + w(t) \tag{10.67}$$

subject to the constraints

$$\begin{aligned} x(t) \in \mathcal{X} &= \left\{ x \ : \ \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \ \forall t \geq 0 \\ w(t) \in \mathcal{W} &= \left\{ w \ : \ \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq w \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \ \forall t \geq 0. \end{aligned} \tag{10.68}$$

The maximal robust positive invariant set of system (10.67) subject to constraints (10.68) is depicted in Figure 10.15 and reported below:

$$\begin{bmatrix} 0.89 & -0.44 \\ -0.89 & 0.44 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x \leq \begin{bmatrix} 8.04 \\ 8.04 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}.$$

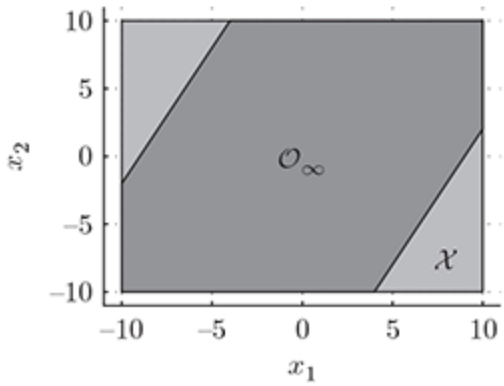

Figure 10.15 Example 10.12. Maximal Robust Positive Invariant Set of system (10.67) subject to constraints (10.68).

Robust Control Invariant Sets

Robust control invariant sets are defined for systems subject to controllable inputs. These types of sets are useful to answer questions such as: "Find the set of states for which *there exists* a controller such that the system constraints are never violated for all possible disturbances." The following definitions introduce the different types of robust control invariant sets.

Definition 10.22 (Robust Control Invariant Set) *A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a robust control invariant set for the system (10.28) subject to the constraints (10.29), if*

$$x(t) \in \mathcal{C} \quad \Rightarrow \quad \exists u(t) \in \mathcal{U} \text{ such that } g(x(t), u(t), w(t)) \in \mathcal{C},\ \forall\, w(t) \in \mathcal{W},\ \forall\, t \in \mathbb{N}_+.$$

Definition 10.23 (Maximal Robust Control Invariant Set $\mathcal{C}_\infty$) *The set $\mathcal{C}_\infty \subseteq \mathcal{X}$ is said to be the maximal robust control invariant set for the system (10.28) subject to the constraints (10.29), if it is robust control invariant and contains all robust control invariant sets contained in $\mathcal{X}$.*

Remark 10.9 The geometric conditions for invariance (10.65)–(10.66) hold for control invariant sets.

The following algorithm provides a procedure for computing the maximal robust control invariant set $\mathcal{C}_\infty$ for system (10.28)–(10.29).

Algorithm 10.5 *Computation of $\mathcal{C}_\infty$*

Input: g, $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$

Output: $\mathcal{C}_\infty$

$\Omega_0 \leftarrow \mathcal{X}$, $k \leftarrow -1$

Repeat

$k \leftarrow k+1$

$\Omega_{k+1} \leftarrow \text{Pre}(\Omega_k, \mathcal{W}) \cap \Omega_k$

Until $\Omega_{k+1} = \Omega_k$

$\mathcal{C}_\infty \leftarrow \Omega_{k+1}$

Algorithm 10.5 generates the set sequence $\{\Omega_k\}$ satisfying $\Omega_{k+1} \subseteq \Omega_k$, $\forall k \in \mathbb{N}$. Algorithm 10.5 terminates when $\Omega_{k+1} = \Omega_k$. If it terminates, then Ω_k is the maximal robust control invariant set $\mathcal{C}_\infty$ for the system (10.28)–(10.29).

In general, Algorithm 10.5 may never terminate [10, 49, 172, 164]. If the algorithm does not terminate in a finite number of iterations, in general, convergence to the maximal robust control invariant set is not guaranteed

$$\mathcal{C}_\infty \neq \lim_{k\to+\infty} \Omega_k. \tag{10.69}$$

The work in [50] reports examples of nonlinear systems where (10.69) can be observed. A sufficient condition for the convergence of Ω_k to $\mathcal{C}_\infty$ as $k \to +\infty$ requires the polyhedral sets $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ to be bounded and the system $g(x, u, w)$ to be continuous [50].

Example 10.13 Consider the second order unstable system in Example 10.9. The maximal robust control invariant set of system (10.44) subject to constraints (10.45)–(10.46) is an empty set. If the uncertain set (10.46) is replaced with

$$w(t) \in \mathcal{W} = \{w \ : \ -0.1 \le w \le 0.1\}, \ \forall t \ge 0$$

the maximal robust control invariant set is

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 0.55 & -0.83 \\ -0.55 & 0.83 \end{bmatrix} x \le \begin{bmatrix} 3.72 \\ 3.72 \\ 2.0 \\ 2.0 \end{bmatrix}$$

which is depicted in Figure 10.16.

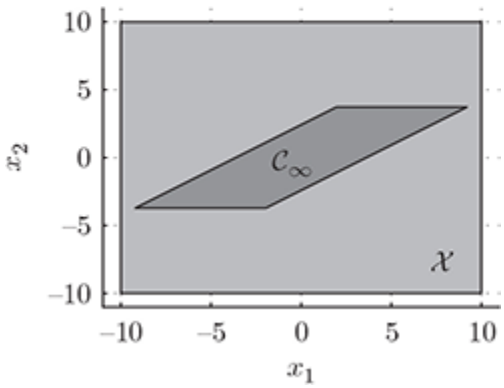

Figure 10.16 Example 10.13. Maximal Robust Control Invariant Set of system (10.44) subject to constraints (10.45).

Definition 10.24 (Finitely determined set) *Consider Algorithm 10.4 (10.5). The set $\mathcal{O}_\infty$ ($\mathcal{C}_\infty$) is finitely determined if and only if $\exists\ i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$. The smallest element $i \in \mathbb{N}$ such that $\Omega_{i+1} = \Omega_i$ is called the determinedness index.*

For all states contained in the maximal robust control invariant set $\mathcal{C}_\infty$ there exists a control law, such that the system constraints are never violated for all feasible disturbances. This does not imply that there exists a control law which can drive the state into a user-specified target set. This issue is addressed in the following by introducing the concept of robust controllable and stabilizable sets.

Definition 10.25 (Maximal Robust Controllable Set $\mathcal{K}_\infty(\mathcal{O}, \mathcal{W})$) *For a given target set $\mathcal{O} \subseteq \mathcal{X}$, the maximal robust controllable set $\mathcal{K}_\infty(\mathcal{O}, \mathcal{W})$ for the system (10.28) subject to the constraints (10.29) is the union of all N-step robust controllable sets contained in $\mathcal{X}$ for $N \in \mathbb{N}$.*

Robust controllable sets $\mathcal{K}_N(\mathcal{O}, \mathcal{W})$ where the target $\mathcal{O}$ is a robust control invariant set are special sets, since in addition to guaranteeing that from $\mathcal{K}_N(\mathcal{O}, \mathcal{W})$ we robustly reach $\mathcal{O}$ in N steps, one can ensure that once reached $\mathcal{O}$, the system can stay there at all future time instants and for all possible disturbance realizations.

Definition 10.26 (N-step (Maximal) Robust Stabilizable Set) *For a given robust control invariant set $\mathcal{O} \subseteq \mathcal{X}$, the N-step (maximal) robust stabilizable set of the system (10.28) subject to the constraints (10.29) is the N-step (maximal) robust controllable set $\mathcal{K}_N(\mathcal{O}, \mathcal{W})$ ($\mathcal{K}_\infty(\mathcal{O}, \mathcal{W})$).*

The set $\mathcal{K}_\infty(\mathcal{O}, \mathcal{W})$ contains all states which can be robustly steered into the robust control invariant set $\mathcal{O}$ and hence $\mathcal{K}_\infty(\mathcal{O}, \mathcal{W}) \subseteq \mathcal{C}_\infty$. The set $\mathcal{K}_\infty(\mathcal{O}, \mathcal{W}) \subseteq \mathcal{C}_\infty$ can be computed as follows:

Algorithm 10.6 *Computation of* $\mathcal{K}_\infty(\mathcal{O}, \mathcal{W})$

Input: g_a, $\mathcal{X}$, $\mathcal{W}$

Output: $\mathcal{K}_\infty(\mathcal{O}, \mathcal{W})$

$\mathcal{K}_0 \leftarrow \mathcal{O}$, where $\mathcal{O}$ is a robust control invariant set

$c \leftarrow -1$

Repeat

$c \leftarrow c+1$

$\mathcal{K}_{c+1} \leftarrow \mathrm{Pre}(\mathcal{K}_c, \mathcal{W})\ \mathcal{X}$

Until $\mathcal{K}_{c+1} = \mathcal{K}_c$

$\mathcal{K}_\infty(\mathcal{O}, \mathcal{W}) = \mathcal{K}_c$

Since $\mathcal{O}$ is robust control invariant, it holds $\forall c \in \mathbb{N}$ that $\mathcal{K}_c$ is robust control invariant and $\mathcal{K}_c \subseteq \mathcal{K}_{c+1}$. Note that Algorithm 10.6 is not guaranteed to terminate in finite time.