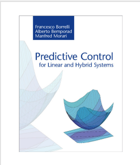

Predictive Control: for Linear and Hybrid Systems
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I Basics of Optimization
2 Linear and Quadratic Optimization

2
Linear and Quadratic Optimization

This chapter focuses on two widely known and used subclasses of convex optimization problems: linear and quadratic programs. They are popular because many important practical problems can be formulated as linear or quadratic programs and because they can be solved efficiently. Also, they are the basic building blocks of many other optimization algorithms. This chapter presents their formulation together with their main properties and some fundamental results.

2.1 Polyhedra and Polytopes

We first introduce a few concepts needed for the geometric interpretation of linear an quadratic optimization. They will be discussed in more detail in Section 4.2.

A *polyhedron* $\mathcal{P}$ in $\mathbb{R}^n$ denotes an intersection of a finite set of closed halfspaces in $\mathbb{R}^n$:

$$\mathcal{P} = \{x \in \mathbb{R}^n : \; Ax \leq b\}, \tag{2.1}$$

where $Ax \leq b$ is the usual shorthand form for a system of inequalities, namely $a_i'x \leq b_i, \;\; i = 1, \ldots, m,$ where $a_1', \ldots, a_m'$ are the rows of A, and $b_1, \ldots, b_m$ are the components of b. A *polytope* is a bounded polyhedron. In Figure 2.1 a two-dimensional polytope is plotted.

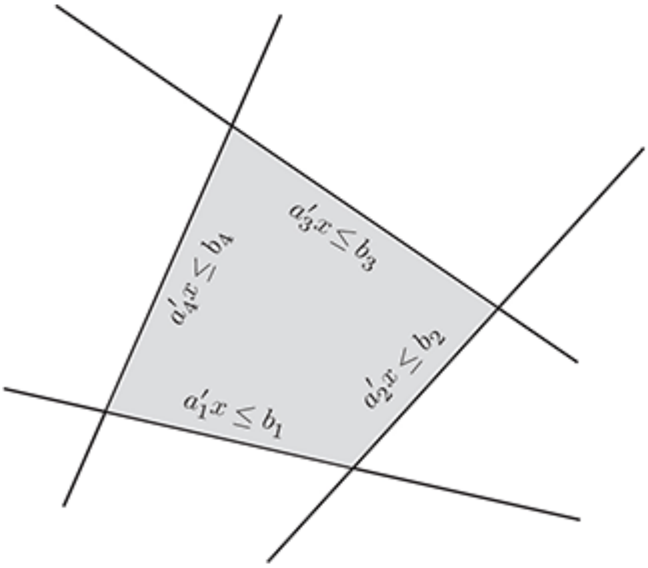


Figure 2.1 Polytope. A polytope is a bounded polyhedron defined by the intersection of closed halfspaces. The planes (here lines) defining the boundary of the halfspaces are $a_i'x - b_i = 0$.

A linear inequality $c'z \leq c_0$ is said to be *valid* for $\mathcal{P}$ if it is satisfied for all points $z \in \mathcal{P}$. A *face* of $\mathcal{P}$ is any nonempty set of the form

$$\mathcal{F} = \mathcal{P} \cap \{z \in \mathbb{R}^s : \; c'z = c_0\}, \tag{2.2}$$

where $c'z \leq c_0$ is a *valid* inequality for $\mathcal{P}$. All faces of $\mathcal{P}$ satisfying $\mathcal{F} \subset \mathcal{P}$ are called proper faces and have dimension less than dim($\mathcal{P}$). The faces of dimension 0,1, dim($\mathcal{P}$)-2 and dim($\mathcal{P}$)-1 are called *vertices*, *edges*, *ridges*, and *facets*, respectively.

2.2 Linear Programming

When the cost and the constraints of the continuous optimization problem (1.4) are affine, then the problem is called a *linear program* (LP). The most general form of a linear program is

$$\begin{array}{ll} \inf_z & c'z \\ \text{subj. to} & Gz \leq w, \end{array} \tag{2.3}$$

where $G \in \mathbb{R}^{m \times s}$, $w \in \mathbb{R}^m$. Linear programs are convex optimization problems.

Two other common forms of linear programs include both equality and inequality constraints:

$$\begin{array}{ll} \inf_z & c'z \\ \text{subj. to} & Gz \leq w \\ & Az = b, \end{array} \tag{2.4}$$

where $A \in \mathbb{R}^{p \times s}$, $b \in \mathbb{R}^p$, or only equality constraints and positive variables:

$$\begin{array}{ll} \inf_z & c'z \\ \text{subj. to} & Az = b \\ & z \geq 0. \end{array} \tag{2.5}$$

By standard simple manipulations [65, p. 146] it is always possible to convert one of the three forms (2.3), (2.4) and (2.5) into the others.

2.2.1 Geometric Interpretation and Solution Properties

Let $\mathcal{P}$ be the feasible set (1.6) of problem (2.3). As $Z = \mathbb{R}^s$, this implies that $\mathcal{P}$ is a polyhedron defined by the inequality constraints in (2.3). If $\mathcal{P}$ is empty, then the problem is infeasible. We will assume for the following discussion that $\mathcal{P}$ is not empty. Denote by f^* the optimal value and by Z^* the set of optimizers of problem (2.3)

$$Z^* = \text{argmin}_{z\in\mathcal{P}} \;\; c'z.$$

Three cases can occur.

Case 1. The LP solution is unbounded, i.e., $f_* = -\infty$.

Case 2. The LP solution is bounded, i.e., $f^* > -\infty$ and the optimizer is unique. $z^* = Z^*$ is a singleton.

Case 3. The LP solution is bounded and there are multiple optima. Z^* is an subset of $\mathbb{R}^s$ which can be bounded or unbounded.

The two-dimensional geometric interpretation of the three cases discussed above is depicted in Figure 2.2.

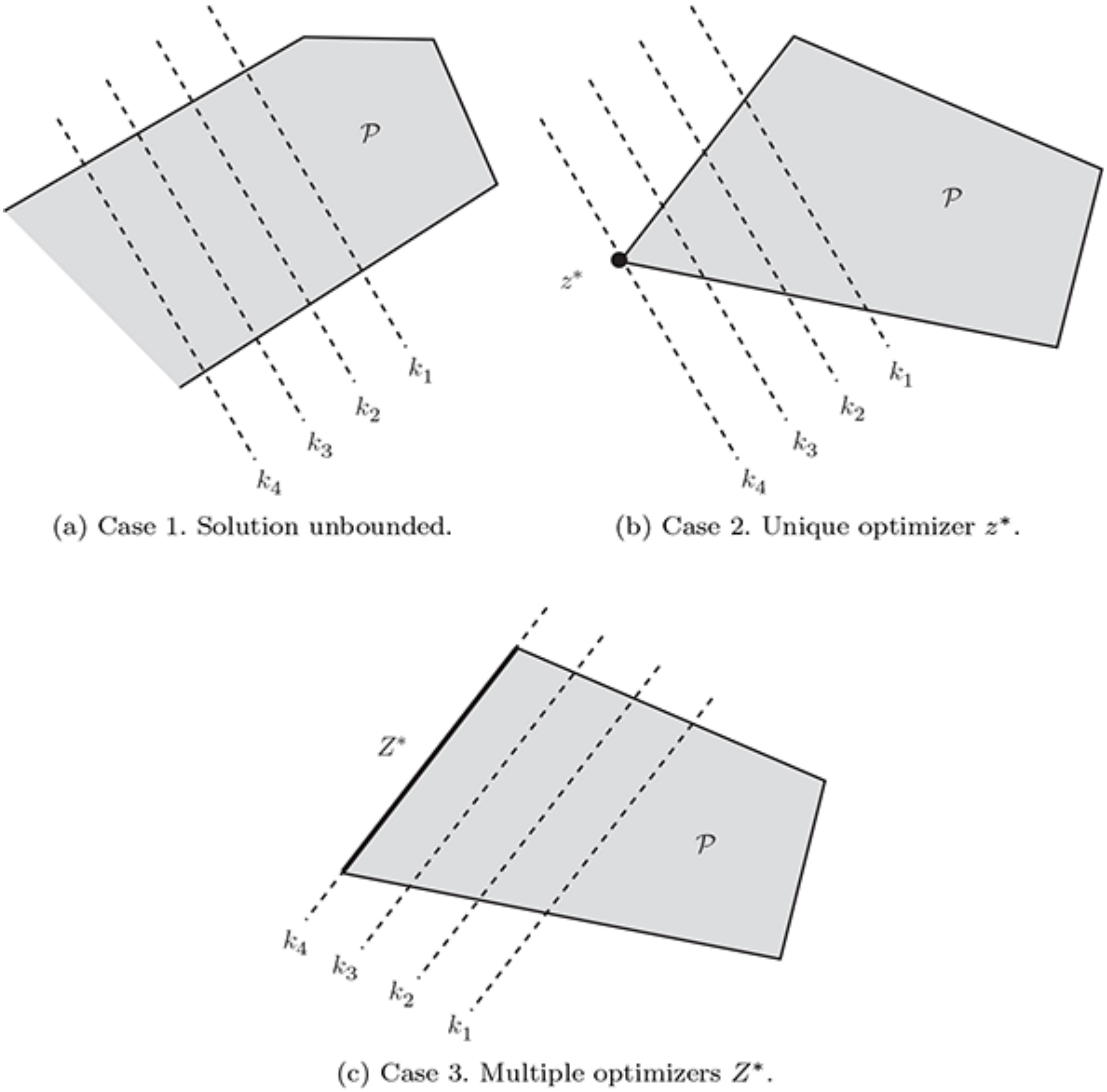


Figure 2.2 Linear program. Geometric interpretation, level curve parameters $k_i < k_{i-1}$.

The level curves of the cost function $c'z$ are represented by the parallel lines. All points z belonging both to the line $c'z = k_i$ and to the polyhedron $\mathcal{P}$ are feasible points with an associated cost k_i, with $k_i < k_{i-1}$. Solving (2.3) amounts to finding a feasible z which belongs to the level curve with the smallest cost k_i. Since the gradient of the cost is c, the direction of steepest descent is $-c/||c||$.

Case 1 is depicted in Figure 2.2(a). The feasible set $\mathcal{P}$ is unbounded. One can move in the direction of steepest descent $-c$ and be always feasible, thus decreasing the cost to $-\infty$. Case 2 is depicted in Figure 2.2(b). The optimizer is unique and it coincides with one of the vertices of the feasible polyhedron. Case 3 is depicted in Figure 2.2(c). The whole facet of the feasible polyhedron $\mathcal{P}$ is optimal, i.e., the cost for any point z belonging to the facet equals the optimal value f^*. In general, the optimal facet will be a facet of the polyhedron $\mathcal{P}$ parallel to the hyperplane $c'z = 0$.

From the analysis above we can conclude that the optimizers of any bounded LP always lie on the boundary of the feasible polyhedron $\mathcal{P}$.

2.2.2 Dual of LP

Consider the LP (2.3)

$$\begin{array}{ll} \inf_z & c'z \\ \text{subj. to} & Gz \leq w, \end{array} \tag{2.6}$$

with $z \in \mathbb{R}^s$ and $G \in \mathbb{R}^{m \times s}$.

The Lagrange function as defined in (1.14) is

$$L(z,u) = c'z + u'(Gz - w).$$

The dual cost is

$$d(u) = \inf_z L(z,u) = \inf_z (c' + u'G)z - u'w = \begin{cases} -u'w & \text{if} -G'u = c \\ -\infty & \text{if} -G'u \neq c. \end{cases}$$

Since we are interested only in cases where d is finite, from the relation above we conclude that the dual problem is

$$\begin{array}{ll} \sup_u & -u'w \\ \text{subj. to} & -G'u = c \\ & u \geq 0, \end{array} \tag{2.7}$$

which can be rewritten as

$$\begin{array}{ll} \inf_u & w'u \\ \text{subj. to} & G'u = -c \\ & u \geq 0. \end{array} \tag{2.8}$$

Note that for LPs, feasibility implies strong duality (Remark 1.3).

2.2.3 KKT condition for LP

The KKT conditions (1.29a)–(1.29e) for the LP (2.3) become

$$G'u + c = 0, \tag{2.9a}$$

$$u_i(G_i z - w_i) = 0, \; i = 1, \dots, m \tag{2.9b}$$

$$u \geq 0, \tag{2.9c}$$

$$Gz - w \leq 0. \tag{2.9d}$$

They are: stationarity condition (2.9a), complementary slackness conditions (2.9b), dual feasibility (2.9c) and primal feasibility (2.9d). Often dual feasibility in linear programs refers to both (2.9a) and (2.9c).

2.2.4 Active Constraints and Degeneracies

Consider the LP (2.3). Let $I = \{1,..., m\}$ be the set of constraint indices. For any $A \subseteq I$, let G_A and w_A be the submatrices of G and w, respectively, comprising the rows indexed by A and denote with G_j and w_j the j-th row of G and w, respectively. Let z be a feasible point and consider the set of active and inactive constraints at z:

$$\begin{aligned} A(z) &= \{i \in I : G_i z = w_i\} \\ NA(z) &= \{i \in I : G_i z < w_i\}. \end{aligned} \tag{2.10}$$

From (2.10) we have

$$\begin{array}{l} G_{A(z^*)} z^* = w_{A(z^*)} \\ G_{NA(z^*)} z^* < w_{NA(z^*)}. \end{array} \tag{2.11}$$

Definition 2.1 (Linear Independence Constraint Qualification (LICQ)) *We say that LICQ holds at z^* if the matrix $G_{A(z^*)}$ has full row rank.*

Lemma 2.1 *Assume that the feasible set $\mathcal{P}$ of problem (2.3) is bounded. If the LICQ is violated at $z_1^* \in Z^*$ then there exists $z_2^* \in Z^*$ such that $|A(z_2^*)| > s$.*

Consider $z_1^* \in Z^*$ on an optimal facet. Lemma 2.1 states the simple fact that if LICQ is violated at z_1^*, then there is a vertex $z_2^* \in Z^*$ on the same facet where $|A(z_2^*)| > s$. Thus, violation of LICQ is equivalent to having more than s constraints active at an optimal vertex.

Definition 2.2 *The LP (2.3) is said to be primal degenerate if there exists a $z^* \in Z^*$ such that the LICQ does not hold at z^*.*

Figure 2.3 depicts a case of primal degeneracy with four constraints active at the optimal vertex, i.e., more than the minimum number two.

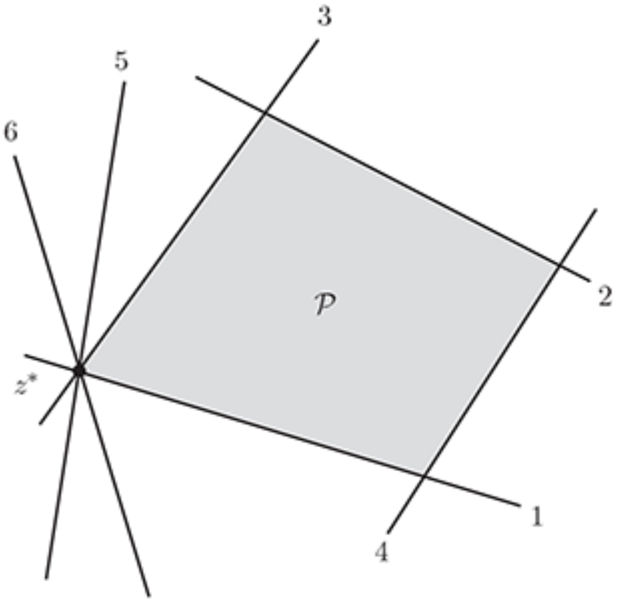


Figure 2.3 Primal degeneracy in a linear program.

Definition 2.3 *The LP (2.3) is said to be dual degenerate if its dual problem is primal degenerate.*

The LICQ condition is invoked so that the optimization problem is well behaved in a way we will explain next. Let us look at the equality constraints of the dual problem (2.8) at the optimum $G'u^* = -c$ or equivalently $G'_A u^*_A = -c$ since $u^*_{NA} = 0$. If LICQ is satisfied, then the equality constraints will allow only a unique optimizer u^*. If LICQ is not satisfied, then the dual *may* have multiple optimizers. Thus we have the following Lemma.

Lemma 2.2 *If the primal problem (2.3) is not degenerate then the dual problem has a unique optimizer. If the dual problem (2.8) is not degenerate, then the primal problem has a unique optimizer.*

Multiple dual optimizers imply primal degeneracy and multiple primal optimizers imply dual degeneracy but the reverse is not true as we will illustrate next. In other words, LICQ is only a sufficient condition for uniqueness of the dual.

Example 2.1 Primal and dual degeneracies

Consider the following pair of primal and dual LPs

$$\begin{array}{l} \text{Primal} \\ \inf\ [-1\ \ -1]x \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \tag{2.12}$$

$$\begin{aligned}&\text{Dual}\\&\inf\ [1\ 1\ 0\ 0]u\\&\begin{bmatrix}0 & 1 & -1 & 0\\ 1 & 1 & 0 & -1\end{bmatrix}u=\begin{bmatrix}1\\1\end{bmatrix}\\&u\geq 0\end{aligned} \tag{2.13}$$

Substituting for u_3 and u_4 from the equality constraints in (2.13) we can rewrite the dual as

$$\begin{aligned}&\text{Dual}\\&\inf\ [1\ \ 1]\begin{bmatrix}u_1\\u_2\end{bmatrix}\\&\begin{bmatrix}1 & 0\\0 & 1\\1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}\geq\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\end{aligned} \tag{2.14}$$

The situation is portrayed in Figure 2.4.

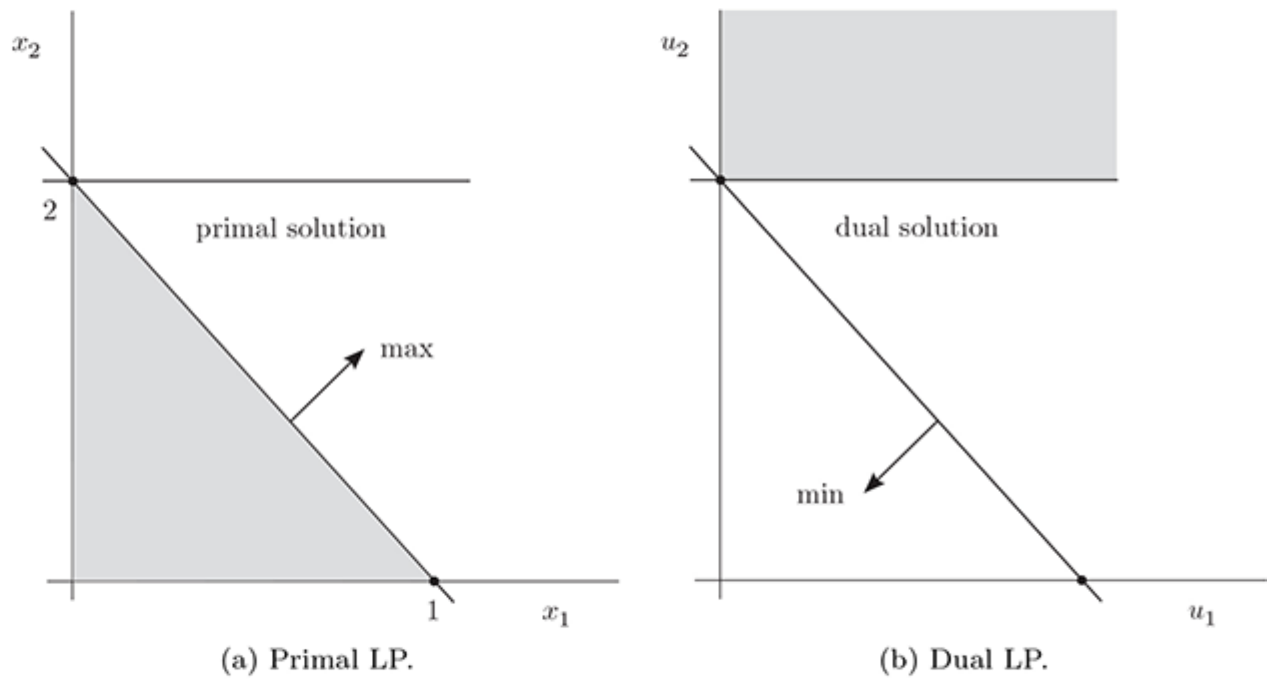


Figure 2.4 Example 2.1. LP with primal and dual degeneracy. The vectors max and min point in the direction for which the objective improves. The feasible sets are shaded.

Consider the two solutions for the primal LP denoted with 1 and 2 in Figure 2.4(a) and referred to as “basic” solutions. Basic solution 1 is primal nondegenerate, since it is defined by exactly as many active constraints as there are variables. Basic solution 2 is primal degenerate, since it is defined by three active constraints, i.e., more than two. Any convex combination of optimal solutions 1 and 2 is also optimal. This continuum of optimal solutions in the primal problem corresponds to a degenerate solution in the dual space, that is, the dual problem is primal-degenerate. Hence the primal problem is dual-degenerate. In conclusion, Figures 2.4(a) and 2.4(b) show an example of a primal problem with multiple optima and the corresponding dual problem being primal degenerate.

Next we want to show that the statement "if the dual problem is primal degenerate then the primal problem has multiple optima" is, in general, not true. Switch dual and primal problems, i.e., call the "dual problem" primal problem and the "primal problem" dual problem (this can be done since the dual of the dual problem is the primal problem). Then, we have a dual problem which is primal degenerate in solution 2 while the primal problem does not present multiple optimizers.

2.2.5 Convex Piecewise Affine Optimization

Consider a continuous and convex piecewise affine function $f : \mathcal{R} \subseteq \mathbb{R}^s \rightarrow \mathbb{R}$:

$$f(z) = c_i'z + d_i \text{ if } z \in \mathcal{R}_i,\ \ i = 1, \ldots, p, \tag{2.15}$$

where $\{\mathcal{R}_i\}_{i=1}^p$ are polyhedral sets with disjoint interiors, $\mathcal{R} = \bigcup_{i=1}^p \mathcal{R}_i$ is a polyhedron and $c_i \in \mathbb{R}^s$, $d_i \in \mathbb{R}$. Then, $f(z)$ can be rewritten as [257]

$$f(z) = \max_{i=1,\ldots,k} \{c_i'z + d_i\},\ \ z \in \mathcal{R}, \tag{2.16}$$

or [65]:

$$\begin{array}{rl} f(z) = \min_\varepsilon & \varepsilon \\ & c_i'z + d_i \leq \varepsilon,\ \ i = 1, \ldots, k \\ & z \in \mathcal{R}. \end{array} \tag{2.17}$$

See Figure 2.5 for an illustration of the idea.

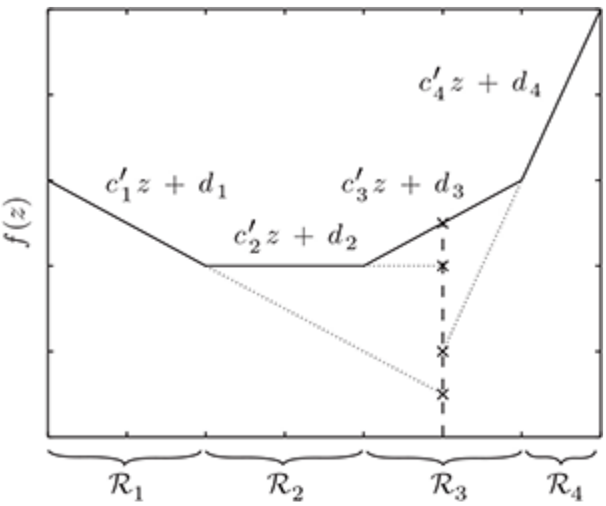


Figure 2.5 Convex piecewise affine (PWA) function described as the max of affine functions.

Consider the following optimization problem

$$\begin{array}{lll} f^* = \min_z & f(z) \\ \text{subj. to} & Gz \leq w \\ & z \in \mathcal{R}, \end{array} \tag{2.18}$$

where the cost function has the form (2.15). Substituting (2.17) this becomes

$$\begin{array}{lll} f^* = \min_{z,\varepsilon} & \varepsilon \\ \text{subj. to} & Gz \leq w \\ & c_i' z + d_i \leq \varepsilon, \quad i = 1, \dots, k \\ & z \in \mathcal{R}. \end{array} \tag{2.19}$$

The previous result can be extended to the sum of continuous and convex piecewise affine functions. Let $f : \mathcal{R} \subseteq \mathbb{R}^s \to \mathbb{R}$ be defined as:

$$f(z) = \sum_{j=1}^{r} f^j(z), \tag{2.20}$$

with

$$f^j(z) = \max_{i=1,\dots,k^j} \{c_i^{j\prime} z + d_i^j\}, \;\; z \in \mathcal{R}. \tag{2.21}$$

Then the following optimization problem

$$\begin{array}{lll} f^* = \min_z & f(z) \\ \text{subj. to} & Gz \leq w \\ & z \in \mathcal{R}, \end{array} \tag{2.22}$$

where the cost function has the form (2.20), can be solved by the following linear program:

$$\begin{array}{lll} f^* = \min_{z,\varepsilon^1,\dots,\varepsilon^r} & \varepsilon^1 + \cdots + \varepsilon^r \\ \text{subj. to} & Gz \leq w \\ & c_i^{1\prime} z + d_i^1 \leq \varepsilon^1, \quad i = 1, \dots, k^1 \\ & c_i^{2\prime} z + d_i^2 \leq \varepsilon^2, \quad i = 1, \dots, k^2 \\ & \vdots \\ & c_i^{r\prime} z + d_i^r \leq \varepsilon^r, \quad i = 1, \dots, k^r \\ & z \in \mathcal{R}. \end{array} \tag{2.23}$$

Remark 2.1 Note that the results of this section can be immediately applied to the minimization of one or infinity norms. For any $y \in \mathbb{R}$, $|y| = \max\{y, -y\}$. Therefore for any $Q \in \mathbb{R}^{k \times s}$ and $p \in \mathbb{R}^k$:

$$\|Qz - p\|_\infty = \max\{Q_1{}'z + p_1, -Q_1{}'z - p_1, \dots, Q_k{}'z + p_k, -Q_k{}'z - p_k\},$$

and

$$\|Qz - p\|_1 = \sum_{i=1}^{k} |Q_i{}'z + p_i| = \sum_{i=1}^{k} \max\{Q_i{}'z + p_i, -Q_i{}'z - p_i\}.$$

2.3 Quadratic Programming

The continuous optimization problem (1.4) is called a *quadratic program* (QP) if the constraint functions are affine and the cost function is a convex quadratic function. In this book we will use the form:

$$\begin{array}{ll} \min_z & \frac{1}{2}z'Hz + q'z + r \\ \text{subj. to} & Gz \leq w, \end{array} \tag{2.24}$$

where $z \in \mathbb{R}^s$, $H = H' \succ 0 \in \mathbb{R}^{s \times s}$, $q \in \mathbb{R}^s$, $G \in \mathbb{R}^{m \times s}$. In (2.24) the constant term can be omitted if one is only interested in the optimizer.

Other QP forms often include equality and inequality constraints:

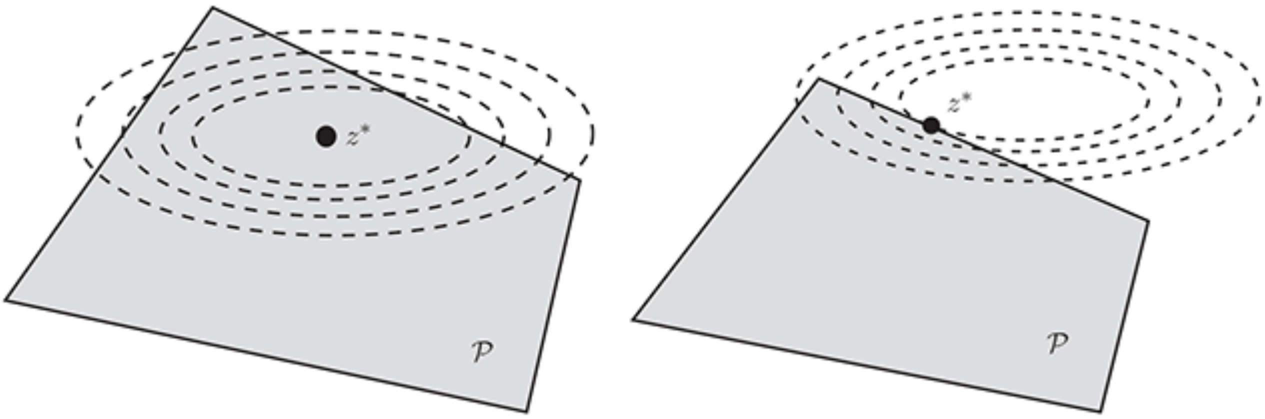
$$\begin{array}{ll} \min_z & \frac{1}{2}z'Hz + q'z + r \\ \text{subj. to} & Gz \leq w \\ & Az = b. \end{array} \tag{2.25}$$

2.3.1 Geometric Interpretation and Solution Properties

Let $\mathcal{P}$ be the feasible set (1.6) of problem (2.24). As $Z = \mathbb{R}^s$, this implies that $\mathcal{P}$ is a polyhedron defined by the inequality constraints in (2.24). The two dimensional geometric interpretation is depicted in Figure 2.6. The level curves of the cost function $\frac{1}{2}z'Hz+q'z+r$ are represented by the ellipsoids. All the points z belonging both to the ellipsoid $\frac{1}{2}z'Hz + q'z + r = k_i$ and to the polyhedron $\mathcal{P}$ are feasible points with an associated cost k_i. The smaller the ellipsoid, the smaller is its cost k_i. Solving (2.24) amounts to finding a feasible z which belongs to the level curve with the smallest cost k_i. Since H is strictly positive definite, the QP (2.24) cannot have multiple optima nor unbounded solutions. If $\mathcal{P}$ is not empty the optimizer is unique. Two cases can occur if $\mathcal{P}$ is not empty:

Case 1. The optimizer lies strictly inside the feasible polyhedron (Figure 2.6(a)).

Case 2. The optimizer lies on the boundary of the feasible polyhedron (Figure 2.6(b)).



(a) Case 1. Optimizer z^* in interior of $\mathcal{P}$. (b) Case 2. Optimizer z^* on boundary of $\mathcal{P}$.

Figure 2.6 Geometric interpretation of the quadratic program solution.

In Case 1 the QP (2.24) is unconstrained and we can find the minimizer by setting the gradient equal to zero

$$Hz^* + q = 0. \tag{2.26}$$

Since $H \succ 0$ we obtain $z^* = -H^{-1}q$.

2.3.2 Dual of QP

Consider the QP (2.24)

$$\begin{array}{ll} \min_z & \frac{1}{2}z'Hz + q'z \\ \text{subj. to} & Gz \leq w. \end{array}$$

The Lagrange function as defined in (1.14) is

$$L(z,u) = \frac{1}{2}z'Hz + q'z + u'(Gz - w).$$

The dual cost is

$$d(u) = \min_z \ \frac{1}{2}z'Hz + q'z + u'(Gz - w) \tag{2.27}$$

and the dual problem is

$$\max_{u \geq 0} \min_z \frac{1}{2}z'Hz + q'z + u'(Gz - w). \tag{2.28}$$

For a given u the Lagrange function $\frac{1}{2}z'Hz + q'z + u'(Gz - w)$ is convex. Therefore it is necessary and sufficient for optimality that the gradient is zero

$$Hz + q + G'u = 0.$$

From the equation above we can derive $z = -H^{-1}(q + G'u)$ and, substituting this in Equation (2.27), we obtain:

$$d(u) = -\frac{1}{2}u'(GH^{-1}G')u - u'(w + GH^{-1}q) - \frac{1}{2}q'H^{-1}q. \tag{2.29}$$

By using (2.29) the dual problem (2.28) can be rewritten as:

$$\begin{array}{ll} \min_u & \frac{1}{2}u'(GH^{-1}G')u + u'(w + GH^{-1}q) + \frac{1}{2}q'H^{-1}q \\ \text{subj. to} & u \geq 0. \end{array} \tag{2.30}$$

Note that for convex QPs feasibility implies strong duality (Remark 1.3).

2.3.3 KKT conditions for QP

Consider the QP (2.24). Then, $\nabla f(z) = Hz + q$, $g_i(z) = G_i z - w_i$ (where G_i is the i-th row of G), $\nabla g_i(z) = G_i'$. The KKT conditions become

$$Hz + q + G'u = 0 \tag{2.31a}$$

$$u_i(G_i z - w_i) = 0,\ i = 1, \ldots, m \tag{2.31b}$$

$$u \geq 0 \tag{2.31c}$$

$$Gz - w \leq 0. \tag{2.31d}$$

2.3.4 Active Constraints and Degeneracies

Consider the definition of active set $A(z)$ in (2.10). Note that $A(z^*)$ may be empty in the case of a QP. We define primal and dual degeneracy as in the LP case.

Definition 2.4 *The QP (2.24) is said to be primal degenerate if there exists a $z^* \in Z^*$ such that the LICQ does not hold at z^*.*

Note that if the QP (2.24) is not primal degenerate, then the dual QP (2.30) has a unique solution since $u^*_{NA} = 0$ and $(G_A H^{-1} G'_A)$ is invertible.

Definition 2.5 *The QP (2.24) is said to be dual degenerate if its dual problem is primal degenerate.*

We note from (2.30) that all the constraints are independent. Therefore LICQ always holds for dual QPs and dual degeneracy can never occur for QPs with $H \succ 0$.

2.3.5 Constrained Least-Squares Problems

The problem of minimizing the convex quadratic function

$$\|Az - b\|_2^2 = z'A'Az - 2b'Az + b'b \tag{2.32}$$

is an (unconstrained) QP. It arises in many fields and has many names, e.g., linear regression or least-squares approximation. From (2.26) we find the minimizer

$$z^* = (A'A)^{-1}A'b = A^\dagger b,$$

where $A^\dagger$ is the *generalized inverse* of A. When linear inequality constraints are added, the problem is called constrained linear regression or *constrained least-squares*, and there is no longer a simple analytical solution. As an example we can consider regression with lower and upper bounds on the variables, i.e.,

$$\begin{array}{ll} \min_z & \|Az - b\|_2^2 \\ \text{subj. to} & l_i \le z_i \le u_i, \quad i = 1, \ldots, n, \end{array} \tag{2.33}$$

which is a QP. In Chapter 6.3.1 we will show how to compute an analytical solution to the constrained least-squares problem. In particular we will show how to compute the solution z^* as a function of b, u_i and l_i.

2.4 Mixed-Integer Optimization

As discussed in Section 1.1.2, if the decision set Z in the optimization problem (1.2) is the Cartesian product of a binary set and a real Euclidian space, i.e., $Z \subseteq \{[z_c, z_b] : z_c \in \mathbb{R}^{s_c}, z_b \in \{0, 1\}^{s_b}\}$, then the optimization problem is said to be *mixed-integer*. In this section Mixed Integer Linear Programming (MILP) and Mixed Integer Quadratic Programming (MIQP) are introduced.

When the cost of the optimization problem (1.10) is quadratic and the constraints are affine, then the problem is called a *mixed integer quadratic program* (MIQP). The most general form of an MIQP is

$$\begin{array}{ll} \inf_{[z_c, z_b]} & \frac{1}{2}z'Hz + q'z + r \\ \text{subj. to} & G_c z_c + G_b z_b \le w \\ & A_c z_c + A_b z_b = b \\ & z_c \in \mathbb{R}^{s_c}, \;\; z_b \in \{0, 1\}^{s_b} \\ & z = [z_c, z_b], \end{array} \tag{2.34}$$

where $H \succcurlyeq 0 \in \mathbb{R}^{s \times s}$, $G_c \in \mathbb{R}^{m \times s_c}$, $G_b \in \mathbb{R}^{m \times s_b}$, $w \in \mathbb{R}^m$, $A_c \in \mathbb{R}^{p \times s_c}$, $A_b \in \mathbb{R}^{p \times s_b}$, $b \in \mathbb{R}^p$ and $s = s_c + s_d$. Mixed integer quadratic programs are nonconvex optimization problems, in general. When $H = 0$ the problem is called a *mixed integer linear program* (MILP). Often the term r is omitted from the cost since it does not affect the optimizer, but r has to be considered when computing the optimal value. In this book, we will often use the form of MIQP with inequality constraints only

$$\begin{array}{ll} \inf_{[z_c,z_b]} & \frac{1}{2}z'Hz + q'z + r \\ \text{subj. to} & G_c z_c + G_b z_b \le w \\ & z_c \in \mathbb{R}^{s_c},\ \ z_b \in \{0,1\}^{s_b} \\ & z = [z_c, z_b]. \end{array} \tag{2.35}$$

The general form can always be translated into the form with inequality constraints only by standard simple manipulations.

For a fixed integer value $\bar{z}_b$ of z_b, the MIQP (2.34) becomes a quadratic program:

$$\begin{array}{ll} \inf_{z_c} & \frac{1}{2}z_c' H_c z_c + q_c(z_b)' z_c + k(z_b) \\ \text{subj. to} & G_c z_c \le w - G_b \bar{z}_b \\ & A_c z_c = b_{eq} - A_b \bar{z}_b \\ & z_c \in \mathbb{R}^{s_c}. \end{array} \tag{2.36}$$

Therefore the most obvious way to interpret and solve an MIQP is to enumerate all the 2^{s_b} integer values of the variable z_b and solve the corresponding QPs. By comparing the 2^{s_b} optimal costs one can derive the optimizer and the optimal cost of the MIQP (2.34). Although this approach is not used in practice, it gives a simple way for proving what is stated next. Let $\mathcal{P}_{\bar{z}_b}$ be the feasible set (1.6) of problem (2.35) for a fixed $z_b = \bar{z}_b$. The cost is a quadratic function defined over $\mathbb{R}^{s_c}$ and $\mathcal{P}_{\bar{z}_b}$ is a polyhedron defined by the inequality constraints

$$G_c z_c \le w - G_b \bar{z}_b. \tag{2.37}$$

Denote by f^* the optimal value and by Z^* the set of optimizers of problem (2.34) If $\mathcal{P}_{\bar{z}_b}$ is empty for all $\bar{z}_b$, then the problem (2.35) is infeasible. Five cases can occur if $\mathcal{P}_{\bar{z}_b}$ is not empty for at last one $\bar{z}_b \in \{0, 1\}^{s_b}$:

Case 1. The MIQP solution is unbounded, i.e., $f^* = -\infty$. This cannot happen if $H_c \succ 0$.

Case 2. The MIQP solution is bounded, i.e., $f^* > -\infty$ and the optimizer is unique. Z^* is a singleton.

Case 3. The MIQP solution is bounded and there are infinitely many optimizers corresponding to the same integer value. Z^* is the Cartesian product of an infinite dimensional subset of $\mathbb{R}^s$ and an integer number z_b^*. This cannot happen if $H_c \succ 0$.

Case 4. The MIQP solution is bounded and there are finitely many optimizers corresponding to different integer values. Z^* is a finite set of optimizers $\{(z_{1,c}^*, z_{1,b}^*), \ldots, (z_{N,c}^*, z_{N,b}^*)\}$.

Case 5. The union of Case 3 and Case 4.