Swinburne University of Technology

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Polynomials

Polynomials appear in many areas of mathematics and science. They are used in calculus, program analysis, algebraic geometry, and 3D graphics to name a few applications. A basic polynomial is a function with a single variable, f(x), that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0 x^0$$

where $0, \ldots, n$ are integers, $a_0, \ldots, a_{\mathrm{n}}$ are real numbers, and x is the variable of the polynomial. A polynomial can be expressed more concisely by using summation notation:

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

That is, a polynomial can be written as the sum of a finite number of terms $a_i x^i$. Each term consists of the product of a number a_i, called the coefficient, and a variable x raised to integer powers – x^i. The exponent i in x^i is called the degree of the term $a_i x^i$. The degree of a polynomial is the largest exponent of any one term with a non-zero coefficient. For example

- $5x^0$ is a constant polynomial with degree 0,
- $2x^2 + 5x^1 + 3x^0$ is a polynomial of degree 2, that is, a quadratic function.

Polynomials support multiplication. Given two polynomials

$$f(x) = \sum_{i=0}^{n} a_i x^i \qquad \text{and} \qquad g(x) = \sum_{j=0}^{m} b_j x^j$$

the product is defined as

$$f(x)g(x) = \sum_{i=0}^{n} a_i x^i \sum_{j=0}^{m} b_j x^j = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j x^{i+j}$$

In order words, the product of two polynomials yields a new polynomial of degree i+j, where the coefficients $a_i b_j$ are sums over all pairs (i, j), i ≠ j. Consider $f(x) = -0.25x^1 + 4.0x^0$ and the multiplication with itself, f(x)f(x):

$$
\begin{aligned}
& (-0.25x^1 + 4.0x^0) * (-0.25x^1 + 4.0x^0) \\
= \quad & -0.25x^1 * -0.25x^1 + -0.25x^1 * 4.0x^0 + 4.0x^0 * -0.25x^1 + 4.0x^0 * 4.0x^0 \\
= \quad & -0.0625x^2 + -1.0x^1 + -1.0x^1 + 16.0x^0 \\
= \quad & -0.0625x^2 - 2.0x^1 + 16.0x^0
\end{aligned}
$$

Polynomials in Differential Calculus

Let f(x) be a function over a polynomial:

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

As the parameter x varies, so will f(x). Differential calculus deals with the rate of change of such functions. The idea is to start with f(x) at some instant x, and then to change x by a little and see how much f(x) changes. The rate of change is defined as the ratio of the change in f to the change in x. We write Δx to quantify the change in x. The change in f, denoted Δf, is the given by

$$\Delta f = f(x + \Delta x) - f(x)$$

To define the rate of change precisely, we must let shrink Δx to zero. Consequently, when we do that Δf also shrinks to zero, but if we divide Δf by Δx, the ratio will tend to a limit. That limit is the *derivative* of function f(x) with respect to x,

$$\frac{d\ f(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

To illustrate it, consider a computational process whose running time is characterized by a function f(x) = x^2, that is, a simple quadratic function (polynomial of degree 2). To calculate the derivative of f, we begin with

$$f(x + \Delta x) = (x + \Delta x)^2$$

Next, we calculate the right-hand side:

$$f(x + \Delta x) = x^2 + 2x\Delta x + \Delta x^2$$

Now, we subtract f(x):

$$\begin{aligned} f(x + \Delta x) - f(x) &= x^2 + 2x\Delta x + \Delta x^2 - x^2 \\ &= 2x\Delta x + \Delta x^2 \end{aligned}$$

In the next step, we divide by Δx:

$$\frac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$$

Finally, we need to take the limit Δx → 0:

$$\lim_{\Delta x \to 0} 2x + \Delta x = 2x$$

So, the derivative of x^2 is

$$\frac{d\ f(x) = x^2}{dx} = 2x$$

Given a computational process that requires quadratic time to complete (x^2), for each additional input the running time increases by twice the total input size (2x). This is an important piece of information when designing algorithms that require quadratic time to complete.

We do not have to go through this process all the time. Many rules for finding the derivative of functions have already been discovered, including one for polynomials:

$$\frac{d}{dx}\sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n-1}(i+1)a_{i+1}x^i$$

Clearly, the above rule is written in such a way to make it easy to represent it programmatically. Consider, a polynomial f(x) = $-0.125x^2 + 4.0x^1 + 0x^0$:

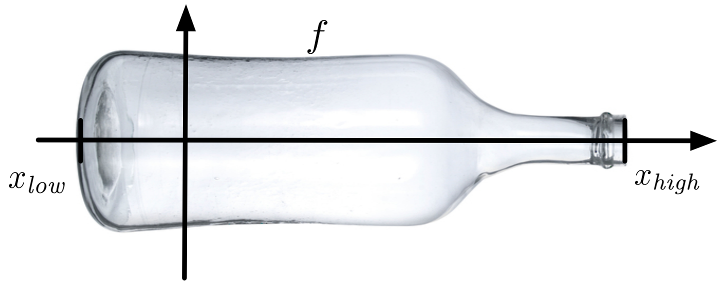
$$\begin{aligned} & \frac{d}{dx} - 0.125x^2 + 4.0x^1 + 0x^0 \\ = \quad & 2 * -0.125x^1 + 1 * 4.0x^0 \\ = \quad & -0.25x^1 + 4.0x^0 \end{aligned}$$

The constant $0x^0$ disappears.

The derivative of function f(x) = $-0.125x^2 + 4.0x^1 + 0x^0$ is $-0.25x^1 + 4.0x^0$, the rate of change at point x in function f(x).

Polynomials in Integral Calculus

Polynomials allow us to describe non-regular shaped objects and to determine their volume:



The volume of this bottle is given by

$$\pi \int_{x_{low}}^{x_{high}} f(x)\ dx$$

This formula can be derived from the formula for the volume of a cylinder: $\pi r^2 h$. For a bottle, whose shape is given by f, we replace the term $r^2 h$ with the definite integral

$$\int_{x_{low}}^{x_{high}} f(x)\ dx$$

that gives us the area under the curve f between the points x_{low} and x_{high}. The volume of the bottle is obtained by rotating this area (infinitely thin slices) 360 degrees around the center and add all of them.

We shall use a polynomial as function f. To compute the volume of an object whose shape is given by a polynomial, we need to compute its *area under the curve*. There exist numerical methods to approximate the area. However, the most precise way of finding the area is to calculate the integral of the polynomial.

The *indefinite integral* of a polynomial is the sum of the integrals of its terms. A general term of a polynomial can be written as ax^k, read the coefficient of the k^{th} degree, and the indefinite integral of that term is

$$\int ax^k\ dx = \frac{a}{k+1}x^{k+1}$$

In other words, we obtain the indefinite integral of the k^{th} degree polynomial term by raising it to the power $k+1$ and dividing it by $k+1$. Using the summation notation, we have

$$\int f(x) = \sum_{i=0}^{n} a_i x^i\ dx = Cx^0 + \sum_{i=1}^{n+1} \frac{a_{i-1}}{i} x^i$$

A key observation here is that the terms in the indefinite integral move one position to the right – the next higher degree. The term at index 0 becomes zero. For example,

$$\int -0.25x^1 + 4.0x^0 \ dx = -0.125x^2 + 4.0x^1 + Cx^0 = -0.125x^2 + 4.0x^1 + 0x^0$$

The *definite integral* of a polynomial is a function with two arguments: x_{low} and x_{high} – the limits of the integral. We can write, without loss of generality,

$$\int_{x_{low}}^{x_{high}} \sum_{i=0}^{n} a_i x^i \ dx = \sum_{i=1}^{n+1} \frac{a_{i-1}}{i} {x_{high}}^i - \sum_{i=1}^{n+1} \frac{a_{i-1}}{i} {x_{low}}^i = \sum_{i=1}^{n+1} \frac{a_{i-1}}{i} ({x_{high}}^i - {x_{low}}^i)$$

That is, to compute the definite integral of a polynomial between x_{low} and x_{high}, we need to construct its indefinite integral first, calculate it for x_{low} and x_{high}, and subtract the result for x_{low} from that obtained for x_{high}. For example,

$$\begin{aligned}
\int_0^{12} -0.25x + 4.0 \ dx &= -0.125x^2 + 4.0x^1 + 0 \Big|_0^{12} \\
&= (-0.125 * 12^2 + 4.0 * 12^1) - (-0.125 * 0^2 + 4.0 * 0^1) \\
&= -18 + 48 \\
&= 30
\end{aligned}$$