Neutron Diffusion Equation Discretization

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The steady state neutron diffusion equation for eigenvalue problems is

$$-\nabla \cdot D_g(\mathbf{r})\nabla\phi(\mathbf{r}) + \Sigma_{t,g}\phi_g(\mathbf{r}) = \frac{\chi_g}{k}\sum_{g'}\nu\Sigma_{f,g'}\phi_{g'}(\mathbf{r}) + \sum_{g'}\Sigma_{s,g'g}\phi(\mathbf{r}), \tag{1}$$

where D_g is the diffusion operator, which for a one-dimensional sphere can be simplified via

$$\nabla \cdot D_g(\mathbf{r})\nabla = \frac{1}{\mathbf{r}^2}\frac{d}{d\mathbf{r}}\mathbf{r}^2 D(\mathbf{r})\frac{d}{d\mathbf{r}} \tag{2}$$

$$\phi_i = \frac{1}{V_i}\int_{r-1/2}^{r+1/2}\phi(\mathbf{r})dV \qquad dV = 4\pi\mathbf{r}^2 d\mathbf{r} \tag{3}$$

$$V_i = \frac{4}{3}\pi(r_{i+1/2}^3 - r_{i-1/2}^3) \qquad S_{i\pm1/2} = 4\pi r_{i\pm1/2}^2 \tag{4}$$

Discretizing the spatial variable and using the volume and surface area in Eq. 4, Eq. 1 is converted to

$$-\frac{1}{V_i}\left[D_{i+1/2}S_{i+1/2}\frac{\phi_{i+1}-\phi_i}{\Delta r} - D_{i-1/2}S_{i-1/2}\frac{\phi_i-\phi_{i-1}}{\Delta r}\right] + \Sigma_{t,g,i}\phi_i = \frac{\chi_g}{k}\sum_{g'}\nu\Sigma_{f,g',i}\phi_i + \sum_{g'}\Sigma_{s,g'g,i}\phi_i. \tag{5}$$

For my code, I combined the self-scattering and total cross sections for a removal term ($\Sigma_{r,g} = \Sigma_{t,g} - \Sigma_{s,g,g}$) and setting the self-scattering terms on the right hand side to zero. To convert this to the matrix form of $A\phi = (1/k_{\text{eff}})B\phi$, block matrices for each energy group are used so A and B are of size $G(I+1)\times G(I+1)$. To construct each diagonal block matrix M for matrix A,

$$(\mathbf{M}_{gg})_{i,i} = \begin{cases} \frac{2}{V_i\Delta r}\left[D_{g,i+1/2}S_{i+1/2} - D_{g,i-1/2}S_{i-1/2}\right] + \Sigma_{r,g,i} & i = 0 \ldots I-1 \\ \left(\frac{\mathcal{A}_g}{2} + \frac{\mathcal{B}_g}{\Delta r}\right) & i = I \end{cases} \tag{6}$$

$$(\mathbf{M}_{gg})_{i,i-1} = \begin{cases} -\frac{D_{g,i+1/2}S_{i+1/2}}{V_i\Delta r} & i = 0 \ldots I-1 \\ i = 0 \ldots I-1 \\ \left(\frac{\mathcal{A}_g}{2} - \frac{\mathcal{B}_g}{\Delta r}\right) & i = I \end{cases} \tag{7}$$

$$(\mathbf{M}_{gg})_{i,i+1} = \begin{cases} -\frac{D_{g,i+1/2}S_{i+1/2}}{V_i\Delta r} & i = 0 \ldots I-1 \end{cases} \tag{8}$$

In the downscattering and upscattering block matrices,

$$(\mathbf{M}_{g,g'\neq g})_{i,i} = \begin{cases} -\Sigma_{s,g\to g',i} & i = 0 \ldots I-1 \\ 0 & i = I \end{cases} \tag{9}$$

For the block matrices P for matrix B,

$$(\mathbf{P}_{g,g'})_{i,i} = \chi_{g,i}\nu\Sigma_{f,g',i}. \tag{10}$$