Nuclear Engineering and Design 240 (2010) 4083–4090

Contents lists available at ScienceDirect

Nuclear Engineering and Design

journal homepage: www.elsevier.com/locate/nucengdes

Inhomogeneous point kinetics equations and the source contribution

T. Sathiyasheela*

Reactor Physics Division, Indira Gandhi Centre for Atomic Research, 101, B, CDO, Kalpakkam 603102, Kanchipuram District, Tamil Nadu, India

ARTICLE INFO

Article history:
Received 6 July 2010
Received in revised form 2 September 2010
Accepted 8 September 2010

ABSTRACT

Inhomogeneous point reactor kinetics equations with one-group of delayed neutrons are solved analytically for linear reactivity insertion as well as for step reactivity insertion in the presence of external neutron source using the prompt jump approximation. The solution is obtained as an infinite series. The methodology is found to be a promising tool for analyzing nuclear reactor kinetics with positive or negative ramp reactivity insertion on a sub-critical or a zero power delayed critical reactor, where the temperature reactivity feedback is negligibly small. To check the consistency and the accuracy of the analytical solution, the results are compared with the numerical solution for different sub-critical and delayed critical states. The comparison is found to be good for all kinds of positive and negative step and ramp reactivity insertions. The analytical solution is arranged into two terms, one as a function of source contribution the other without that. Using the newly rearranged solution, the importance of the source term and the contribution to the error while neglecting source term to the reactor kinetics analysis, can be realized. Contribution to the error is small (less than 0.1%) when the equilibrium power is more than about one megawatt for a medium sized LMFBR. Similarly, the importance of source contribution to the total reactor period as a function of initial equilibrium power is also realized with the newly rearranged analytical solution. The total reactor period is over predicted (larger period in place of smaller period) which is not conservative, if the source contribution is not considered, for considerably small initial equilibrium power. The percentage of error in not considering the source term in period calculation varies as a function of net reactivity and ramp rate. The percentage of error in period determination without considering the source is comparatively high for small ramp rates.

1. Introduction

During reactor start up, the neutron source gives a considerable contribution to the net reactor power. When a sub-critical reactor is brought to delayed-critical state in the presence of external neutron source, the sub-critical reactor kinetics analysis with source term is important to determine the reactor power as a function of reactivity insertion rate with respect to the initial sub-criticality (Keepin, 1964). There are many numerical methods available in the literature to study the reactor kinetics such as Quintero-Leyva (2009) solved the point kinetics by approximating the neutron density with piecewise polynomial and exponential functions, and Li (2009) efficiently provides the solution of the point kinetics equations using better basis function (BBF). The analytical solutions available in the literature for step reactivity insertion (Aboanber, 2003) and also the solution available for ramp reactivity insertion in a critical reactor without considering the source term (Akcasu et al., 1971; Hetrick, 1993),the prompt jump approximation was used in deriving the analytical expression using one-group delayed neutrons.

There are analytical methods available in the literature to study the sub-critical kinetics, with considering source term. Li et al. (2010) derived analytical solution of point reactor kinetics equations for a step reactivity input with considering the neutron source. Zhang et al. (2008) derived analytical solution of point reactor kinetics equations for linear reactivity input with the prompt jump approximation. The solution (Zhang et al., 2008) is useful when the reactivity ramp is inserted linearly in discontinuous jumps, because here with the prompt jump approximation, constant source approximation is also used. Palma et al. (2009) also used the prompt jump approximation (Sathiyasheela, 2009a) to derive the analytical solution of point reactor kinetics equations for a linear reactivity input without using constant source approximation. Palma et al. (2009) arranged the analytical solution in the form of incomplete gamma functions. So, the solution may be applicable for any positive ramp reactivity insertions on a sub-critical reactor, but it may not be useful for negative ramps such as control rod drop. Though the results of derived analytical solutions are expected to converge into true solutions, accumulation of error due to precision incompatibility might leads the solutions to false

* Tel.: +91 44 27480088; fax: +91 44 27480096.
E-mail address: sheela@igcar.gov.in.

0029-5493/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.nucengdes.2010.09.036

results for some selected ramp reactivity insertions on sub-critical reactor (Sathiyasheela, 2010).

In the present work, point kinetics equations with one-group of delayed neutrons are solved analytically in the presence of a neutron source using the prompt jump approximation. The solution is obtained as a combination of two infinite series [Sathiyasheela and Harish, 2009]. The problem is generalized by combining both the step reactivity insertions and ramp reactivity insertion for a given sub-critical level of the nuclear reactor. The general formulation could be useful in many reactor operating conditions, like the gas bubble passing through the core, stuck rod conditions (step reactivity insertion), uncontrolled withdrawal of a control rod (linear reactivity insertion), discontinuous lifting or lowering of a control rod (combination of step and positive/negative ramp reactivity insertion). Since the formulation is based on the prompt jump approximation, the solution is not valid in the super-prompt critical condition. The methodology is a promising tool to analyze any sub-critical transients with positive and negative external reactivity insertions [Sathiyasheela, 2009b]. The methodology also can be used in the transient analysis of zero power delayed critical reactor, where there are no reactivity feedbacks.

Through analytical solution the transient reactor power at time t can be determined straight away without performing all the previous time step calculation. To check the consistency and the accuracy of the analytical solution, results of the recommended solutions are compared with the numerical code POKIN (Sharada and Singh, 1990), which is developed based on Cohen's method (Cohen, 1958). The comparison is found to be good for all kinds of positive and negative ramp reactivity insertions.

The final solution is arranged into two term one with source term contribution the other without the source term. Arranging the solution, in this form is useful, mainly the importance of source term and the contribution to error while neglecting source term is realized. The contribution to error is found to be small (less than 0.1%) when the initial equilibrium power is small of the order of about one megawatt for a medium sized LMFBR. Similarly, the importance of source contribution to the total reactor period as a function of initial equilibrium power is also realized with the newly rearranged analytical solution. The total reactor period is over predicted (larger period in place of smaller period) which is not conservative, if the source contribution is not considered, for considerably small initial equilibrium power. The percentage of error in not considering the source term in period calculation varies as a function of net reactivity and ramp rate. The percentage of error in period determination without considering the source is comparatively high for small ramp rates.

2. Analytical solution of point kinetics equations

Point kinetics equations with the source term for one-group of delayed neutrons are,

$$\frac{dP(t)}{dt} = \frac{\rho'(t)-\beta}{\Lambda}P(t) + \lambda C(t) + S_P(t) \qquad (1)$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda}P(t) - \lambda C(t) \qquad (2)$$

where P is the reactor power, C is the delayed neutron precursor concentration, ρ' is the net reactivity (expressed in absolute units), $S_P(t)$ is the source term (source rated power measured in W/s). Though source strength is the varying function of time, it is assumed to be constant in the few hours of problem times like reactor start up. So, it is assumed S_P is the time independent source term. β is the delayed neutron fraction, λ is the delayed neutron decay constant and Λ is prompt neutron generation time. The source term S_P is related to neutron source strength S (neutrons/s) by the relation (Singh, 1987),

$$S_P = \frac{S\mu}{\nu\Lambda}$$

where μ is the energy released per fission (Joules per fission) and ν is the number of neutrons emitted per fission. When a reactor is at a sub-critical level ρ'_s, the reactor power at sub-critical equilibrium with respect to the source term is P_0 then,

$$S_P = \frac{|\rho'_s|}{\Lambda}P_0 \quad \text{or} \quad P_0 = \frac{\Lambda S_P}{|\rho'_s|}$$

If $|\rho'_s|$ is expressed in dollars, $|\rho_s| = |\rho'_s/\beta$

$$P_0 = \frac{\Lambda S_P}{\beta|\rho_s|} \qquad (3a)$$

Suppose if the reactor is brought from the initial sub-critical equilibrium to another equilibrium, with the step reactivity addition ρ'_e. Then, the net reactivity is the sum of the initial sub-critical reactivity and the added external step reactivity. *i.e.* $\rho'_0 = \rho'_s + \rho'_e$ if ρ'_0 expressed in dollars then, $\rho_0 = \rho'_0/\beta$. That is at time $t \to 0$, the reactor is brought from the initial sub-critical equilibrium to a new equilibrium, it is assumed $dP(t)/dt = dC(t)/dt = 0$. The instantaneous power corresponding to the new equilibrium changes from P_0 to $P(0)$.

$$P(0) = \frac{P_0 + (S_P\Lambda/\beta)}{1-\rho_0} \qquad (3b)$$

In (1), if the rate of change of power in a mean generation time is small so that, $(\Lambda(dP(t)/dt))/P \ll (\beta-\rho)$ then the term $\Lambda(dP(t)/dt)$ in point kinetics is neglected in comparison with $(\beta-\rho)P$, which is the prompt jump approximation (Bell and Glasstone, 1970). Eq. (1) with the prompt jump approximation,

$$[\rho'(t)-\beta]P(t) + \Lambda\lambda C(t) + \Lambda S_P = 0$$

Differentiating the above equation with respect to t,

$$[\rho'(t)-\beta]\frac{dP(t)}{dt} + \frac{d\rho'(t)}{dt}P(t) + \Lambda\lambda\frac{dC(t)}{dt} + \Lambda\frac{dS_P}{dt} = 0 \qquad (4)$$

Substituting (1) with the prompt jump approximation and (2) into (4),

$$[\beta-\rho'(t)]\frac{dP(t)}{dt} = P(t)\left[\frac{d\rho'(t)}{dt} + \lambda\rho'(t)\right] + \Lambda\lambda S_P \qquad (5)$$

Eq. (5) is general for any kind of reactivity insertion, with the condition that the maximum reactivity be less than one dollar. Eq. (5) is also derived by Hetrick (1993), based on the expansion of neutron density (reactor power) in powers of the small parameter Λ, the prompt neutron generation time.

2.1. The prompt jump approximation and its validity

Basic criterion for validity of the prompt jump approximation is,

$$\frac{\Lambda(dP(t)/dt)}{P(t)} \ll (\beta-\rho) \quad i.e. \quad \left|\frac{dP(t)}{dt}\right| \ll \frac{\beta-\rho}{\Lambda}P(t)$$

Other than the above criterion it is required to find out the maximum permissible reactivity within which the prompt jump approximation is valid, and which is how closer to the prompt critical (Hetrick, 1993). By combining the above condition with (5) gives,

$$[\beta-\rho'(t)]^2\frac{P(t)}{\Lambda} \gg P(t)\left[\frac{d\rho'(t)}{dt} + \lambda\rho'(t)\right] + \Lambda\lambda S_P$$

$$[\beta-\rho'(t)] \gg \sqrt{\Lambda\left|\frac{d\rho'(t)}{dt}+\lambda\rho'(t)\right|+\frac{\Lambda^2\lambda S_P}{P}}$$

The values of β, λ and Λ of the considered medium sized LMFBR are β = 337.16 pcm, λ = 0.0867 s^{-1}, prompt neutron life time l = 0.39412 µs and the prompt neutron generation time $\Lambda = l/k$. The source rated power S_P is 1,420,887 W/s. Suppose the reactor is at zero power delayed criticality with $\rho'(t)$ = 100 pcm, $d\rho'(t)/dt$ = 20 pcm/s, P = 300 W, the above inequality corresponding to the given data is, $[\beta-\rho'(t)] \gg 2.0\times10^{-5}$.

From the above discussions, the prompt jump approximation is valid in the complete sub-critical region and about 98% closer to prompt critical region. Though in the nominal power delayed critical reactor, there are reactivity feedbacks to be considered, in which case analytical solution may not be good enough, and one should go for numerical analysis. But for a sub-critical and zero power delayed critical reactor (temperatures are small enough to ignore the feedback reactivity), analytical solution with a combination of the prompt jump approximation is useful in determining power as a varying function of time.

2.2. Step reactivity insertion

For step reactivity addition ρ'_e the reactor is brought from the initial sub-criticality ρ'_s to the new sub-criticality $\rho'_0=\rho'_s+\rho'_e$, Eq. (5) corresponds to the new sub-criticality ρ'_0 is,

$$\frac{dP(t)}{dt}-\frac{\lambda\rho'_0}{\beta-\rho'_0}P(t)=\frac{\Lambda\,\lambda S_P}{\beta-\rho'_0} \tag{6a}$$

if ρ'_0 is expressed in dollars, $\rho_0=\rho'_0/\beta$

$$\frac{dP(t)}{dt}-\frac{\lambda\rho_0}{1-\rho_0}P(t)=\frac{1}{1-\rho_0}\frac{\Lambda\,\lambda S_P}{\beta} \tag{6b}$$

Integrating the above equation,

$$P(t)\exp\left(-\frac{\lambda\rho_0}{1-\rho_0}t\right)=-\frac{\Lambda S_P}{\beta\rho_0}\exp\left(-\frac{\lambda\rho_0}{1-\rho_0}t\right)+P(0)+\frac{\Lambda S_P}{\beta\rho_0} \tag{7}$$

Rearranging Eq. (7) leads to,

$$P(t)=\frac{\Lambda S_P}{\beta\rho_0}\left(\exp\left(\frac{\lambda\rho_0}{1-\rho_0}t\right)-1\right)+P(0)\exp\left(\frac{\lambda\rho_0}{1-\rho_0}t\right) \tag{8}$$

Above equation also can be derived from the generalized equation presented by Zhang et al. (2008). With external step reactivity when the reactor is brought from the initial sub-criticality ρ'_s to the new sub-criticality ρ'_0, the sub-critical equilibrium power P_0 is assumed to change instantaneously to the new power $P(0)$, at $t\to0$. After $t>0$, the course of the transient is determined by the new sub-criticality, and the other relevant neutronic parameters. Based on the added reactivity, the reactor may continue to be in sub-critical with different sub-critical level, or may go to delayed critical. Eq. (8) is useful in analyzing transients of any sub-critical or zero power delayed critical reactor for a given positive/negative step reactivity insertions.

2.3. Linear reactivity insertion

Bringing the reactor from sub critical to delayed critical and then to super critical could be done by discontinuous linear reactivity insertion. But in case if there is an uncontrolled withdrawal of a control rod due to some unexpected anomalous reactor condition, then the reactivity insertion is continuous. So, to analyze the start up incidents and accidents, it is necessary to have a general solution for any linear reactivity insertion, with and without the combination of step reactivity for any sub-critical reactor. If γ' is the ramp reactivity insertion rate, then the linear reactivity addition is $\rho'(t)=\rho'_0+\gamma' t$, ρ_0 is step reactivity addition measured in dollars, γ is the ramp reactivity insertion rate measured in dollar/s, $\rho_0=\rho'_0/\beta$, $\gamma=\gamma'/\beta$. Eq. (5) for linear reactivity addition is,

$$\frac{dP(t)}{dt}-\frac{\gamma(1+\lambda t)+\lambda\rho_0}{1-(\rho_0+\gamma t)}P(t)=\frac{\Lambda\lambda S_P}{\beta[1-(\rho_0+\gamma t)]} \tag{9}$$

Whose solution is,

$$P(t)[1-(\rho_0+\gamma t)]^{1+(\lambda/\gamma)}e^{\lambda t}=\frac{\Lambda\lambda S_P}{\beta}\int[1-(\rho_0+\gamma t)]^{\lambda/\gamma}e^{\lambda t}\,dt \tag{10}$$

The integral on the RHS of (10) upon integrating by parts is straight forwardly obtained as,

$$\int[1-(\rho_0+\gamma t)]^{\lambda/t}e^{\lambda/t}\,dt$$
$$=-\frac{e^{\lambda/t}}{\gamma}\sum_{n=1}^{\infty}\left(\frac{\lambda}{\gamma}\right)^{n-1}\frac{[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}((\lambda/\gamma)+k)}+C_2 \tag{11}$$

The complete solution of (10) is given by,

$$P(t)[[1-(\rho_0+\gamma t)]^{1+(\lambda/\gamma)}e^{\lambda t}]$$
$$=-\frac{\Lambda\lambda S_P}{\beta}\frac{e^{\lambda t}}{\gamma}\sum_{n=1}^{\infty}\left(\frac{\lambda}{\gamma}\right)^{n-1}\frac{[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}((\lambda/\gamma)+k)}+C_2 \tag{12}$$

Convergence of Eq. (12) is obtained consistently for all reactivity addition $-\alpha<(\rho_0+\gamma t)\leq1$. The Eq. (12) can be further simplified as,

$$P(t)[[1-(\rho_0+\gamma t)]^{1+(\lambda/\gamma)}e^{\lambda t}]$$
$$=-\frac{\Lambda S_P}{\beta}e^{\lambda t}\sum_{n=1}^{\infty}\frac{[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}(1+k(\gamma/\lambda))}+C_2$$

Let

$$Q_t=[[1-(\rho_0+\gamma t)]^{1+(\lambda/\gamma)}e^{\lambda t}],\qquad Q_s=\frac{\Lambda S_P}{\beta}$$

where C_2 is the integration constant given in terms of the initial conditions as,

$$C_2=\left\{P(0)(1-\rho_0)^{1+(\lambda/\gamma)}+\frac{\Lambda S_P}{\beta}\sum_{n=1}^{\infty}\frac{[1-\rho_0]^{n+(\lambda/\gamma)}}{\prod_{k=1}^{n}(1+k(\gamma/\lambda))}\right\} \tag{13}$$

$$P(t)=\frac{Q_s}{Q_t}\sum_{n=1}^{\infty}\frac{-e^{\lambda t}[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}+[1-\rho_0]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}(1+k(\gamma/\lambda))}$$
$$+\frac{P(0)}{Q_t}(1-\rho_0)^{1+(\lambda/\gamma)} \tag{14}$$

Truncation of the series can be chosen based on the required accuracy. Consistency check can be made on Eq. (14), *i.e.* when $t\to0$, $P(t)=P(0)$, which is the instantaneous power corresponding to the new sub-criticality ρ_0. Suppose there is no external step reactivity addition, then the sub-critical reactivity is simply equal to the initial sub-criticality $|\rho_s|$. Then at $t=0$, $P(t)=P(0)=P_0$ from Eq. (3b),

$$P(0)=\frac{P_0+(S_P\Lambda/\beta)}{1-\rho_0}=\frac{(S_P\,\Lambda/|\rho_s|\beta)(1+|\rho_s|)}{1+\rho_s}=\frac{S_P\Lambda}{|\rho_s|\beta}=P_0 \tag{15}$$

This is the sub-critical equilibrium power corresponds to the initial sub-criticality $|\rho_s|$ as mentioned earlier.

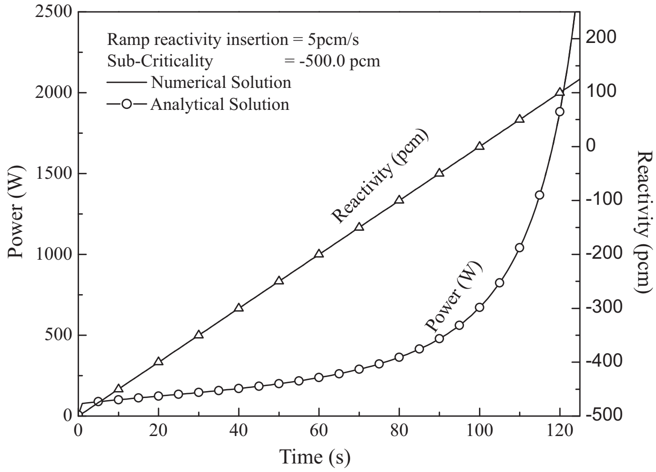

Fig. 1. Comparison of numerical and analytical solution of 5 pcm/s ramp reactivity insertion at −500 pcm sub-critical reactor.

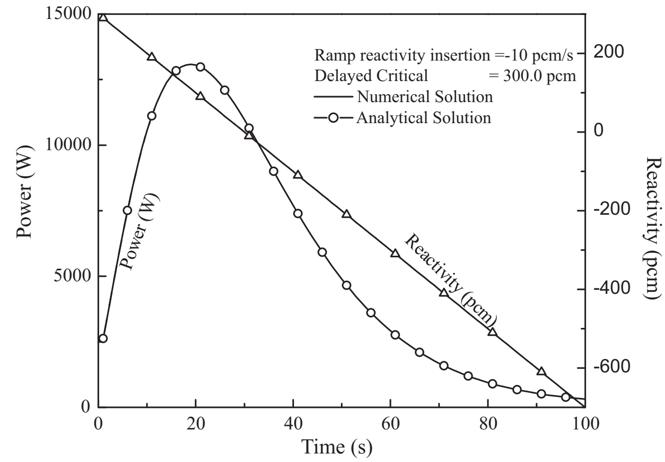

Fig. 3. Comparison of numerical and analytical solution of −10 pcm/s ramp reactivity insertion at 300 pcm delayed critical reactor.

3. Comparisons with numerical results

Analyses are carried out on a typical medium sized LMFBR. The values of β, λ and Λ of the considered reactor are β = 337.16 pcm, λ = 0.0867 s^{-1}, prompt neutron life time l = 0.39412 μs and the prompt neutron generation time $\Lambda = l/k$. The source rated power S_P is 1,420,887 W/s. The sub-critical equilibrium power is 7 W when the reactor is at −8697 pcm sub-criticality. Analyses are carried out for different sub-criticality and ramp reactivity insertion combination. The initial power for different sub-criticality ρ_0 is calculated from the sub-critical equilibrium power P_0 using (3b). Analytical solution of ramp reactivity insertion of 5 pcm/s with sub-criticality of −500 pcm is shown in Fig. 1. Similarly ramp reactivity insertions of 50 pcm/s with a sub-criticality of −5000 pcm is shown in Fig. 2. Analytical solutions are compared against the results of the computer code POKIN which is developed based on Cohen's method. The analytical solutions are found to be matching very well with the numerical results.

From Figs. 1 and 2, it can be seen that the accuracy of analytical solutions is found to be good for any positive ramp reactivity insertions on a sub-critical reactor irrespective of its sub-criticality and the control rod withdrawal speed. In Fig. 1 results are compared for normal control rod withdrawal speed, and in Fig. 2 comparisons are made for control speed which is about ten times more than the typical control rod speed. Since the analytical solution is valid for both positive and negative ramp reactivity insertions, negative ramp analyses are carried out on a zero power reactor, where there are no reactivity feedbacks. Fig. 3 shows the results of negative ramp reactivity insertion of −10 pcm/s with a delayed criticality of 300 pcm. The results are found to be in excellent agreement with the numerical solutions. Fig. 4 shows the results of delayed critical level 300 pcm and ramp reactivity insertion of −10 pcm/s for 20 s. The results are shown up to 50 s. Fig. 4 is simulated to study the stuck rod condition. There is a negative reactivity insertion on a delayed critical reactor for 20 s, after that the control rod is assumed to get struck, so the negative reactivity insertion rate is assumed to be zero after 20 s. The agreement of results as in the previous cases is excellent. From the above study the analytical solution can be useful in analyzing both positive and negative reactivity insertions, on a sub-critical and delayed critical reactor. In all the cases above, the relative error calculated with respect to the results of the code POKIN is less than 0.1%.

4. Discussion

Eq. (14) is convergent for all $-\alpha < (\rho_0 + \gamma t) \leq 1$, although (14) is the correct solution, it fails to give correct results when calculated with limited precision on a computer. The nth term, R_n of the

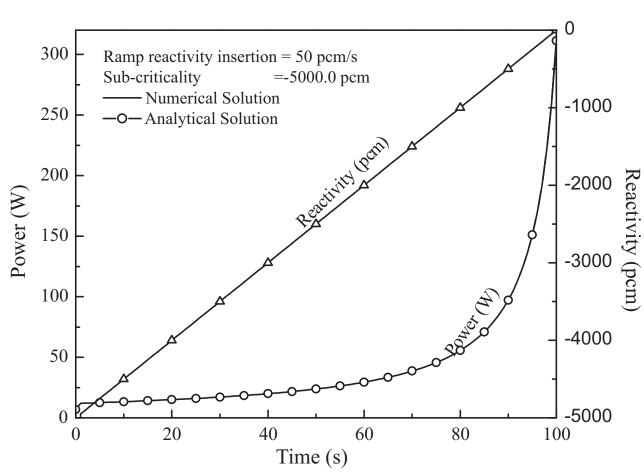

Fig. 2. Comparison of numerical and analytical solution of 50 pcm/s ramp reactivity insertions at −5000 pcm sub-criticality.

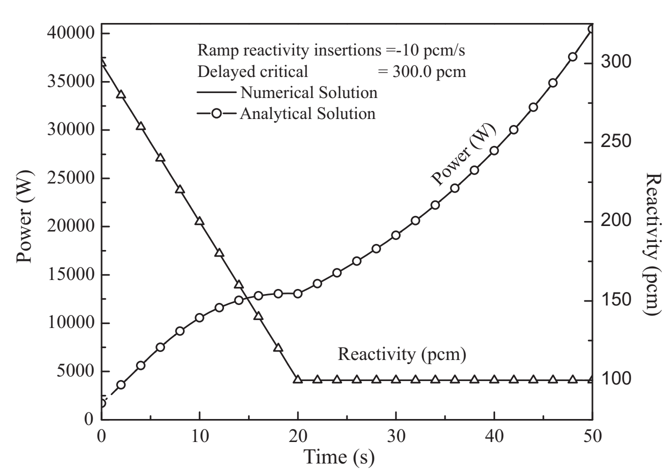

Fig. 4. Comparison of numerical and analytical solution of −10 pcm/s ramp reactivity insertion for 20 s at 300 pcm delayed critical reactor.

infinite series in (14) is,

$$R_n = -\frac{Q_s}{Q_t}\frac{e^{\lambda t}[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}(1+k(\gamma/\lambda))}$$

i.e.,

$$R_n = -\frac{\Lambda\lambda S_P}{\beta}\frac{[1-(\rho_0+\gamma t)]^{n-1}}{\lambda\prod_{k=1}^{n}(1+(k\gamma/\lambda))}$$

The second ratio in R_n can be written as,

$$f = f_n f_d$$

where

$$f_n = [1-(\rho_0+\gamma t)]^{n-1}$$

and

$$f_d = \left[\lambda\prod_{k=1}^{n}\left(1+\frac{k\gamma}{\lambda}\right)\right]^{-1}$$

During control rod withdrawal from a zero power delayed critical reactor, both ρ_0 and the ramp rate γ are positive. In such a case both f_n and f_d decreases with n, so, the convergence of the solution is very fast. About ten terms of the Eq. (14) is good enough in achieving better accuracy, such that addition of more number of terms may not introduce big difference in results. Similarly control rod withdrawal on a sub-critical reactor, ρ_0 is negative while the ramp rate γ is positive. In such case f_n increases with n up to the time $t < |\rho_0|/\gamma$ and f_d decreases with n, so the function f increases with n, but the decreasing function of $|f_d|$ tries to bring down the function f and the solution converges. Beyond time $t < |\rho_0|/\gamma$, both f_n and f_d decreases with n, convergence of the solution is assured. For a sub-critical reactor ρ_0 with a negative ramp rate γ, the function f_n increases with n and f_d also increases with n up to $|n\gamma/\lambda| \sim 1$, after which the sign of the term alternates with n. The maxima of $|f_d|$ occur for an n which satisfies the condition $n\gamma/\lambda \approx 2$. For $n > 2\lambda/\gamma$, $|f_d|$ decreases with n. Though the magnitude of f_n increases with n, the decreasing function of $|f_d|$ after $n > 2\lambda/\gamma$ tries to bring down the function f and the solution converges. In order to get the accuracy up to two decimal places the chosen number of terms n should be such that, the nth term $R_n = \left|(\Lambda S_P/\beta)\left([1-(\rho_0+\gamma t)]^{n-1}/\prod_{k=1}^{n}(1+k\gamma/\lambda)\right)\right| < 0.01$. Since the truncation of (14) is done, at a particular n for which the value of $R_n < 0.01$ is satisfied, the maximum truncated error could be less than 1%.

When the numerical value of f goes beyond the computers precision, error introduced by the summation of very large numbers in infinite series become quite significant leading to erroneous solution of (14) [for example, the sub-criticality $\rho_0 = -3000$ pcm, and $\gamma = 5$ pcm/s]. For some selected combinations of sub-criticality and ramp reactivity insertions, double and quadruple computer precisions are found to be adequate. However, it is imperative to look for techniques which will avoid the precision problem such as the one addressed in the earlier paper [Sathiyasheela, 2010].

5. Contribution of source term: power

Eq. (14) is the analytical solution of Eqs. (1) and (2). Eq. (14) with (3b) is,

$$P(t) = \frac{Q_s}{Q_t}\sum_{n=1}^{\infty}\frac{-e^{\lambda t}[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}+[1-\rho_0]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}(1-k(\gamma/\lambda))} + \frac{S_P\Lambda/\beta}{1-\rho_0}\frac{1}{Q_t}(1-\rho_0)^{1+(\lambda/\gamma)} + \frac{P_0}{1-\rho_0}\frac{1}{Q_t}(1-\rho_0)^{1+(\lambda/\gamma)}$$

$$P(t) = \frac{Q_s}{Q_t}\left(\sum_{n=1}^{\infty}\frac{-e^{\lambda t}[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}+[1-\rho_0]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}(1+k(\gamma/\lambda))} + (1-\rho_0)^{\lambda/\gamma}\right) + \frac{P_0}{Q_t}(1-\rho_0)^{\lambda/\gamma} \quad (16)$$

The above equation is a combination of two terms. The first one is a function of Q_s, which is again a function of source strength S_P, and another one without the function of source strength. Arranging the solution in this form made it is possible to recognize the contribution of source term to the solution and the one without that. The solution is valid for both sub-critical and critical reactor analysis. In the early part of start up, the contribution of source term can not be neglected. But, in the critical reactor analysis, if the initial equilibrium power is high enough, the source term can be neglected and the inhomogeneous differential equations can be converted into simple homogeneous point kinetics equations *vice versa*. But the question is, what is the initial equilibrium power boundary beyond which the source term can be neglected, and what could be the error in case if the source term is neglected. Since the analytical solution is arranged as a function of source term, it is easy to address the above issues. From Eq. (16),

$$Q_s\left(\sum_{n=1}^{\infty}\frac{-e^{\lambda t}[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}+[1-\rho_0]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}(1+k(\gamma/\lambda))} + (1-\rho_0)^{\lambda/\gamma}\right) \prec\prec P_0(1-\rho_0)^{\lambda/\gamma}$$

if the above condition is satisfied, then the source term can be comfortably neglected. The first term is a function of source strength, the reactivity inputs such as ρ_0, γ with time t. It is a function of ρ_0, because the amount of fuel available to interact with the source neutron varies with ρ_0, and it is a function of γ, with time not

Table 1
Contribution of the source term to the total reactor power and the percentage of error, for a ramp reactivity input of 20 pcm/s at 10 s as a function of initial equilibrium power.

S. no.	Power at criticality (W)	Transient power (W)	Power without considering source term (W)	Source contribution[a] (W)	% of error[b]
1	250.0	0.21962E+04	963.21	0.12330E ± 04	56.14
2	1250.0	0.60491E+04	4816.06	0.12330E ± 04	20.38
3	6250.0	0.25313E+05	24080.30	0.12330E ± 04	4.87
4	31250.0	0.12163E+06	120401.50	0.12330E ± 04	1.01
5	156250.0	0.60324E+06	602007.50	0.12330E ± 04	0.20
6	781250.0	0.30113E+07	3010037.50	0.12330E ± 04	0.04
7	3906250.0	0.15051E+08	15050188.00	0.12330E ± 04	0.01
8	19531250.0	0.75252E+08	75250936.00	0.12330E ± 04	0.00
9	97656250.0	0.37626E+09	376254688.00	0.12330E ± 04	0.00
10	488281250.0	0.18813E+10	1881273470.00	0.12330E ± 04	0.00

[a] Source contribution to the total reactor power is a function of reactivity and independent of the initial reactor power.
[b] % of error if the source term is not considered.

Table 2
Contribution of the source term to the total reactor power and the percentage of error, for a ramp reactivity input of 30 pcm/s on a critical reactor with initial equilibrium power 250 W.

Time (s)	Net reactivity (pcm)	Transient power (W)	Power without considering source term (W)	Source contribution[a] (W)	% of error[b]
1	30.0	0.49184E ± 03	275.55	0.21629E ± 03	43.98
2	60.0	0.57175E ± 03	309.53	0.26222E ± 03	45.86
3	90.0	0.67913E ± 03	355.89	0.32323E ± 03	47.60
4	120.0	0.82905E ± 03	421.40	0.40765E ± 03	49.17
5	150.0	0.10490E ± 04	518.35	0.53065E ± 03	50.59
6	180.0	0.13938E ± 04	671.30	0.72249E ± 03	51.84
7	210.0	0.19871E ± 04	935.63	0.10514E ± 04	52.91
8	240.0	0.31617E ± 04	1460.49	0.17012E ± 04	53.81
9	270.0	0.61091E ± 04	2779.57	0.33295E ± 04	54.50
10	300.0	0.18248E ± 05	8216.48	0.10032E ± 05	54.97

[a] Source contribution to the total reactor power is function of reactivity.
[b] % of error if the source term is not considered.

Table 3
Contribution of the source term to the total reactor power and the percentage of error for a ramp reactivity input of 30 pcm/s on a critical reactor with initial equilibrium power 1000 W.

Time (s)	Net reactivity (pcm)	Transient power (W)	Power without considering source term (W)	Source contribution[a] (W)	% of error[b]
1	30.0	0.13185E ± 04	1102.20	0.21629E ± 03	16.4041
2	60.0	0.15003E ± 04	1238.10	0.26222E ± 03	17.4776
3	90.0	0.17468E ± 04	1423.57	0.32323E ± 03	18.5043
4	120.0	0.20932E ± 04	1685.59	0.40765E ± 03	19.4747
5	150.0	0.26041E ± 04	2073.40	0.53065E ± 03	20.3779
6	180.0	0.34077E ± 04	2685.21	0.72249E ± 03	21.2018
7	210.0	0.47939E ± 04	3742.53	0.10514E ± 04	21.9323
8	240.0	0.75432E ± 04	5841.95	0.17012E ± 04	22.5531
9	270.0	0.14448E ± 05	11118.29	0.33295E ± 04	23.0452
10	300.0	0.42898E ± 05	32865.92	0.10032E ± 05	23.3851

[a] Source contribution to the total reactor power is function of reactivity.
[b] % of error if the source term is not considered.

only the amount of fuel available to interact with the source neutron varies, the amount of time available for the source neutron interaction also varies. Though the importance of source neutron may differ from fission neutron, it is assumed to be the same in the present study. In Table 1 it is shown that, the source contributions vary as a function of input reactivity, and it is independent of the initial rector equilibrium power at criticality as it is expected. The percentage of error in the solution, if the source is not considered is a varying function of the initial equilibrium power and ramp insertion rates. From Tables 2 and 3 it is shown, that the error contribution for the same ramp reactivity insertion rate with a different initial equilibrium power varies as a function of power. From the table it is understood that, the percentage of error decreases as the initial equilibrium reactor power increases.

Comparison of same reactivity input of 300 pcm, with different withdrawal rate of 30 pcm/s and 5 pcm/s respectively is shown in Tables 3 and 4. As the withdrawal speed decreases, the percentage of source neutron contribution increases, and the percentage of error for not considering the source term also increases. This may be due to the amount of time available for the source neutron interaction increases as the ramp reactivity rate decreases. In the present study, the percentage of error decreases less than 0.1% as the initial equilibrium power go more than about 1 MW. So in low power transient analyses the source term cannot be ignored, it needs to be retained till the initial equilibrium reactor power reaches the order of about a megawatt for a medium sized LMFBR.

6. Contribution of source term: period

From Keepin (1964), the period is defined as $P(dp/dt)^{-1}$. So, period is determined from the known solution of P and $(dp/dt)^{-1}$, through Eqs. (9) and (14). Eq. (9) is a general equation to determine $dP(t)/dt$ with considering the neutron source. $dP(t)/dt$ without considering the neutron source is,

$$\frac{dP(t)}{dt} = \frac{\gamma(1+\lambda t)+\lambda\rho_0}{1-(\rho_0+\gamma t)}P(t) \tag{17}$$

Table 4
Contribution of the source term to the total reactor power and the percentage of error, for a ramp reactivity input of 5 pcm/s on a critical reactor with initial equilibrium power 1000 W.

Time (s)	Net reactivity (pcm)	Transient power (W)	Power without considering source term (W)	Source contribution[a] (W)	% of error[b]
6.0	30.0	0.14335E ± 04	1125.14	0.30837E ± 03	21.51
12.0	60.0	0.18428E ± 04	1352.10	0.49069E ± 03	26.63
18.0	90.0	0.25410E ± 04	1761.89	0.77909E ± 03	30.66
24.0	120.0	0.38331E ± 04	2542.34	0.12908E ± 04	33.67
30.0	150.0	0.65196E ± 04	4187.77	0.23319E ± 04	35.77
36.0	180.0	0.13109E ± 05	8248.90	0.48601E ± 04	37.07
42.0	210.0	0.33697E ± 05	20968.25	0.12729E ± 05	37.77
48.0	240.0	0.12755E ± 06	78995.85	0.48555E ± 05	38.07
54.0	270.0	0.95777E ± 06	592381.12	0.36539E ± 06	38.15
60.0	300.0	0.33145E ± 08	20496186.00	0.12649E ± 08	38.16

[a] Source contribution to the total reactor power is function of reactivity, independent of equilibrium reactor power.
[b] % of error if the source term is not considered.

Table 5
Contribution of source term to the period and the percentage of error, for a ramp reactivity input of 25 pcm/s on a critical reactor with initial equilibrium power 250 W.

Time (s)	Net reactivity (pcm)	Transient power (W)	Period with considering source term (s)	Period without considering source term (s)	% of error[a]
1.0	25.0	483.6	8.2	11.5	40.3
2.0	50.0	550.1	7.4	9.8	32.8
3.0	75.0	635.1	6.6	8.3	26.5
4.0	100.0	746.5	5.8	7.0	21.1
5.0	125.0	897.4	5.1	5.9	16.5
6.0	150.0	1109.5	4.4	4.9	12.5
7.0	175.0	1423.1	3.7	4.0	9.3
8.0	200.0	1918.2	3.0	3.2	6.5
9.0	225.0	2774.3	2.4	2.5	4.3
10.0	250.0	4468.1	1.8	1.9	2.5

[a] % of error if the source term is not considered.

Table 6
Contribution of source term to the period and the percentage of error, for a ramp reactivity input of 5 pcm/s on a critical reactor with initial equilibrium power 250 W.

Time (s)	Net reactivity (pcm)	Transient power (W)	Period with considering source term (s)	Period without considering source term (s)	% of error[a]
5.0	25.0	558.8	18.7	43.5	132.1
10.0	50.0	736.8	17.4	30.7	76.9
15.0	75.0	998.2	15.6	22.7	46.0
20.0	100.0	1407.2	13.6	17.3	27.5
25.0	125.0	2097.8	11.5	13.4	15.9
30.0	150.0	3378.8	9.5	10.4	8.7
35.0	175.0	6058.7	7.7	8.0	4.3
40.0	200.0	12646.7	6.0	6.1	1.9
45.0	225.0	32966.6	4.5	4.6	0.7
50.0	250.0	121189.8	3.3	3.3	0.2

[a] % of error if the source term is not considered.

The period without considering the neutron source is,

$$\text{Period}\quad T=\frac{P(t)}{dP(t)/dt}=\frac{1-(\rho_0+\gamma t)}{\gamma(1+\lambda t)+\lambda\rho_0} \tag{18}$$

Without external source, the period varies as a function of only the reactivity level and the ramp rate. So, the period is a function of ramp and net reactivity. But in the presence of neutron source the period is also a function of source strength and the initial power. With considering neutron source, the reactor power $P(t)$ from Eq. (14) is,

$$P(t)=\frac{Q_s}{Q_t}\left(\sum_{n=1}^{\infty}\frac{-e^{\lambda t}[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}+[1-\rho_0]^{(\lambda/\gamma)+n}}{\prod_{k=1}^{n}(1+k(\gamma/\lambda))}+(1-\rho_0)^{\lambda/\gamma}\right)+\frac{P_0}{Q_t}(1-\rho_0)^{\lambda/\gamma}$$

Then the period T is,

where $dP(t)/dt$ from Eq. (9) is,

$$\frac{dP(t)}{dt}-\frac{\gamma(1+\lambda t)+\lambda\rho_0}{1-(\rho_0+\gamma t)}P(t)=\frac{\Lambda\lambda S_P}{\beta[1-(\rho_0+\gamma t)]}$$

So, it is important to consider the source contribution in the period calculations especially when the initial equilibrium power is considerably small. Otherwise the results are over predicted (larger period in place of smaller period) which is not conservative. Since the neutron source gives considerable contribution to multiplying neutron in the beginning of reactor start up, the total reactor period is smaller than the system which does not consider the neutron source. Since the period is a varying function of net reactivity, the percentage of error in not considering the source term also varies as a function of reactivity, as it is understood from Eq. (19). From Table 5, the period with source is 40% less than the one without source. The period calculation without source may give a misconception that the period is high, instead of a low period, where SCRAM action may required on low period during reactor start up. As the net reactivity increases for a ramp reactivity insertion, the

$$\text{Period}\quad T=\frac{(Q_s/Q_t)\left(\sum_{n=1}^{\infty}\left((-e^{\lambda t}[1-(\rho_0+\gamma t)]^{(\lambda/\gamma)+n}+[1-\rho_0]^{(\lambda/\gamma)+n})/\left(\prod_{k=1}^{n}1+(k\gamma/\lambda)\right)\right)+(1-\rho_0)^{\lambda/\gamma}\right)}{dP(t)/dt}+\frac{(P_0/Q_t)(1-\rho_0)^{\lambda/\gamma}}{dP(t)/dt} \tag{19}$$

Table 7
Contribution of source term to the period and the percentage of error, for a ramp reactivity input of 5 pcm/s for 10 s, as a function of initial equilibrium power.

S. no.	Power at criticality (W)	Transient power (W)	Period with considering source term (s)	Period without considering source term (s)	% of error[a]
1	250.0	736.8	17.4	30.7	76.9
2	1250.0	1052.2	20.0	30.7	53.8
3	6250.0	1682.9	23.0	30.7	33.7
4	31250.0	2944.4	25.7	30.7	19.2
5	156250.0	5467.3	27.8	30.7	10.4
6	781250.0	10513.0	29.1	30.7	5.4
7	3906250.0	20604.6	29.9	30.7	2.7
8	19531250.0	40787.7	30.3	30.7	1.4
9	97656250.0	81153.9	30.5	30.7	0.7
10	488281250.0	161886.2	30.6	30.7	0.3

[a] % of error if the source term is not considered.

difference in period with and without considering the source term comes down with time, so the percentage of error also comes down with time.

The amount of time available for the source neutron interaction increases and gives a considerable contribution to the net power as the ramp reactivity rate decreases, so, the percentage of error in period determination without considering the source is very high. This can be understood from Tables 5 and 6, from the difference in percentage of error for the same net reactivity, with a different control rod withdrawal speed. From Table 7, as the initial equilibrium power increases, the period also increases and almost approach the period without considering the source strength. The period without considering source over predict (higher period in place of lower period) the solution. The percentage of error for not considering the source decreases as the initial equilibrium power increases.

7. Conclusion

Point reactor kinetics equations with one-group of delayed neutrons are solved analytically for linear reactivity insertion as well as for step reactivity insertion in the presence of an external neutron source using the prompt jump approximation. The solution is obtained as an infinite series. The methodology is found to be a promising tool for analyzing nuclear reactor kinetics for any positive or negative ramp reactivity insertion on a sub-critical or a zero power delayed critical reactor. The general formulation could be useful in analyzing many reactor operating conditions, like air bubble passing through the core, stuck rod conditions, uncontrolled withdrawal of a control rod, discontinuous raising and lowering of a control rod, etc. To check the consistency and the accuracy of the analytical solution, the results are compared with the numerical solution for different sub-critical and delayed critical state. The comparison is found to be good for all kinds of positive and negative step and ramp reactivity insertions. The final solution is arranged into two term one with source term contribution the other without the source term. Arranging the solution in this form is useful, the importance of source term and the contribution to the error while neglecting source term can be viewed. The error contribution is small (less than 0.1%) when the initial equilibrium power is small of the order of about one megawatt for a medium sized LMFBR. Similarly, the importance of source contribution to the total reactor period as a function of initial equilibrium power is also realized with the newly rearranged analytical solution. The total reactor period is over predicted (larger period in place of smaller period) which is not conservative, if the source contribution is not considered, for considerably small initial equilibrium power. The percentage of error in not considering the source term in period calculation varies as a function of net reactivity and ramp rate.

Acknowledgements

The author would like to thank Dr P. Mohanakrishnan Associate Director, RPhG/IGCAR for his support and encouragement. The author also would like to thank Dr R. Harish, Shri A. Riyas Smt M.M. Shanthi and Smt E. Radha for the fruitful discussion.

References

Aboanber, A.E., 2003. Analytical solution of the point kinetics equations by exponential mode analysis. Progress in Nuclear Energy 42 (2), 179–197.

Akcasu, Z., et al., 1971. Mathematical methods in nuclear reactor dynamics. Nuclear Science and Technology, vol. 7. Academic Press.

Bell, G, Glasstone, S., 1970. Nuclear Reactor Theory. Van Nostrand Reinhold Company, New York.

Cohen, R., 1958. Proceedings of the Second United Nations International Conference on the Peaceful use of Atomic Energy, vol. 11, Geneva, p. 302.

Hetrick, D.L., 1993. Dynamics of Nuclear Reactor. American Nuclear Society, Jbc., Illinois, USA.

Keepin, G.R., 1964. Physics of Nuclear Kinetics. Addison-Wesley Publishing Company, London.

Li, H., et al., 2009. A new integral method for solving the point reactor neutron kinetics equation. Annals of Nuclear Energy 36 (4), 427–432.

Li, H., et al., 2010. A new formula of neutron multiplication during startup of PWR. Progress in Nuclear Energy 52, 321–326.

Quintero-Leyva, B., 2009. On the numerical solution of the integro-differential equation of the point kinetics of nuclear reactors. Annals of Nuclear Energy 36, 1280–1284.

Sathiyasheela, T., 2009a. Letter to the editor: Comments to the paper, Analytical solution of point kinetics equations for linear reactivity variation during the start-up of a nuclear reactor (Palma et al., Annals of Nuclear Energy 36 (2009) 1469–1471). Annals of Nuclear Energy.

Sathiyasheela, T., 2009b. Semi analytical solution of point kinetic equation with source from cold start up to delayed critical. In: Paper Accepted for Oral Presentation at the XVI ENFIR, INAC, Rio de Janeiro, Brazil.

Sathiyasheela, T., Harish, R., 2009. Analytical solution of inhomogeneous point kinetics equations as an infinite series. In: Paper Accepted for NSRP-18, Udaipur, November 2009.

Sathiyasheela, T., 2010. Sub-critical reactor kinetics analysis using incomplete gamma functions and binomial expansions. Annals of Nuclear Energy 37, 1248–1253.

Sharada, B., Singh, O.P., 1990. Validation of the computer code POKIN against SEFOR super prompt critical transient and exact analytical results. IGCAR Internal Report, RPD/SNAS-32.

Singh, O.P., 1987. Lecture Notes on Reactor Kinetics. IAEA-PPTN Project, INS/04/018/03, Ban Dung.

Zhang, F., et al., 2008. Analytic method study of point-reactor kinetic equation when cold start-up. Annals of Nuclear Energy 35 (4), 746–749.