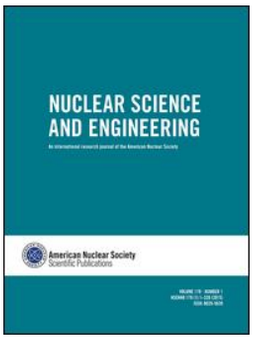

Nuclear Science and Engineering

ISSN: 0029-5639 (Print) 1943-748X (Online) Journal homepage: https://ans.tandfonline.com/loi/unse20

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To cite this article: Masaoki Komata (1969) On the Derivation of Avery's Coupled Reactor Kinetics Equations, Nuclear Science and Engineering, 38:3, 193-204, DOI: 10.13182/NSE69-A21154

To link to this article: https://doi.org/10.13182/NSE69-A21154

Published online: 12 May 2017.

NUCLEAR SCIENCE AND ENGINEERING: 38, 193-204 (1969)

On the Derivation of Avery's Coupled Reactor Kinetics Equations

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Received December 30, 1968

The purpose of this paper is to present the mechanics of the derivation of Avery's coupled reactor kinetics equations, which have been given by his physical consideration. Firstly, the diffusion equation and its adjoint equation are expressed in the matrix form. Then the partial flux and the partial adjoint flux are defined explicitly. The neutron flux, introduced by Henry, is represented as an amplitude $T(t)$ times a shape function $\psi(r,t)$. The adiabatic approximation is adopted in the neutron-flux shape function. Using the commutation law (given in the Appendix) between the diffusion operator and its adjoint operator, Avery's equations are derived from the time-dependent diffusion equations for the partial adjoint flux. The assumptions introduced are; (a) the delayed-neutron fission spectrum is the same as the prompt-neutron fission spectrum, (b) the neutron-flux shape function is approximated by the adiabatic method, (c) the time constant of the amplitude $T(t)$ is much smaller than the minimum time constant of the shape function $\psi(r,t)$ at that instant. As the result of these assumptions, the delay time associated with the transfer of neutron does not appear explicitly in Avery's equations.

I. INTRODUCTION

Although the application of the modal expansion approach to one-dimensional space-kinetics problems has been successful, its extension to actual reactors is not satisfactory because too many modes are required to describe the complex configuration in two or three dimensions. These practical reactor problems may, however, be solved by applying the nodal method first proposed by Avery[1] for a system of coupled reactors. Here, reactors are arbitrarily defined as subregions of the system in which fission neutrons are emitted. The term 'coupled' means that, in each of the reactors, some of the fission neutrons are emitted in fissions induced by neutrons born in other reactors (Fig. 1).

Avery's equations, shown later by Eqs. (66) and (70) in Sec. V, describe the time change of the partial-fission neutron source S_{jk}, given the coupling coefficient k_{jk} and the partial neutron lifetime ℓ_{jk}. If S_j is defined as the total of all fission neutrons in the reactor j, S_{jk} is a part of S_j that results from fission caused by neutrons which originate in reactor k. The coupling coefficient

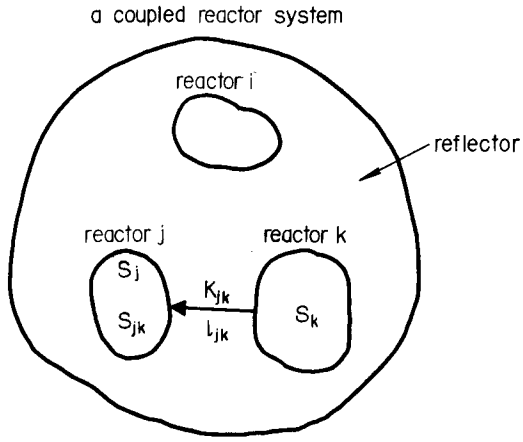

Fig. 1. Illustration of a coupled-reactor system.

[1]R. AVERY, "Theory of Coupled Reactors," *Proc. 2nd Intern. Conf. Peaceful Uses At. Energy*, **12**, 182 (1958).

k_{jk} is given by S_{jk}/S_k and represents the fission neutrons, in reactor j, produced by one neutron which originates in reactor k. The partial lifetime ℓ_{jk} is the lifetime of neutrons which originate in reactor k and are expected to cause fission in reactor j. These parameters k_{jk} and ℓ_{jk} are the integral quantities based on the partial-neutron flux ϕ_k, which is composed of neutrons born in reactor k. The coupling term $k_{jk}\, S_k$ is related to the reactivity effect of reactor k to reactor j. In this sense, Avery's model can be referred to as "the reactivity approach."[2]

Shortly after Avery's proposal, an alternate coupled reactor kinetics model was proposed by Baldwin[3] for a loosely coupled reactor system. The coupling term is introduced here as an additional source to the diffusion equation considered for each reactor and Baldwin's model can be classified as "the effective source approach." The delay time associated with the transfer of neutrons between reactors is introduced explicitly in Baldwin's model.

The mechanics of the derivation of the coupled reactor kinetics equations in those earlier works have not been clarified strictly but rather have been based on the physically intuitive construction.[2] Much work has been done recently to show the mechanics of these derivations from the basic equations such as the diffusion equations and the transport equations. Works of Cockrell[4] and of Hansen[2] may be classified as the reactivity approach and works of Schwalm,[5] Seale,[6] Gage,[7] and Belleni-Morante[8] as the effective source approach. The coupled equations obtained by Cockrell closely resemble and appear to be an extension of Avery's model. But there is a fundamental difference between these two models. While Cockrell's parameters are related only to the integral quantities of one reactor, Avery's parameters are not confined to one reactor. For instance, the partial-fission neutron source S_{jk} in Avery's model relates to the integrations on reactor j and on reactor k and the partial lifetime ℓ_{jk} relates to the integration on an entire coupled system.

The main purpose of this paper is to discuss Avery's model, the foundation of which has been left unclarified, and to show the mechanics of its derivation and the limit of its applications. The following sections, from Sec. II to IV, provide the relations used in Sec. V, in which Avery's equations are derived from the time-dependent adjoint equation for the partial adjoint flux. In Sec. VI, a shape function in the multipoint kinetics model is discussed and the alternate simple forms of Avery's equations are obtained. The illustrations of adjoint operator and commutation law are given in the Appendix.

II. BASIC EQUATIONS

The time-dependent multigroup neutron diffusion equations, in matrix representation, are given by,

$$\{(1-\beta)\chi\cdot\nu P(r,t)-D(r,t)\}\,\boldsymbol{\phi}(r,t) + \sum_i \lambda_i \boldsymbol{R}_i(r,t) = V^{-1}\frac{\partial}{\partial t}\boldsymbol{\phi}(r,t) \qquad (1)$$

$$\beta_i \chi_d \nu P(r,t)\,\boldsymbol{\phi}(r,t) - \lambda_i \boldsymbol{R}_i(r,t) = \frac{\partial}{\partial t}\boldsymbol{R}_i(r,t) \qquad (2)$$

$$(i=1,2,\ldots,6)\ ,$$

where the source term is eliminated and

- $\boldsymbol{\phi}$ is a vector whose elements are the neutron flux in each energy group
- $\boldsymbol{R}_i$ is a vector whose j'th element is the delayed-neutron precursor density of the i'th type, $C_i(r,t)$, times the delayed fission spectrum of the j'th energy group χ_{dj}. In the matrix equations $\boldsymbol{R}_i$ is also designated as the precursor density
- P is a matrix operator to represent the production of fission neutrons
- D is a destruction matrix operator which represents all the neutron destruction terms including absorption, leakage, and the transfer from one energy group to another by the scattering process

[2] F. T. ADLER, S. J. GAGE, and G. C. HOPKINS, "Spatial and Spectral Coupling Effects in Multicore Reactor Systems," *Proc. National Topical Meeting on Coupled Reactor Kinetics*, 521 (1967).

[3] G. C. BALDWIN, *Nucl. Sci. Eng.*, **6**, 320 (1959).

[4] R. G. COCKRELL and R. B. PEREZ, "Kinetic Theory of Spatial and Spectral Coupling of the Reactor Neutron Field," *Proc. Conf. on Neutron Dynamics and Control*, USAEC, CONF-650413, 323 (1966).

[5] D. SCHWALM, "On a Derivation of the Multigroup Kinetic Equations of Coupled Reactors," EUR 2416, e, EURATOM, Ispra, Italy (1965).

[6] R. L. SEALE, "Coupled Core Reactors," LAMS-2967, Los Alamos Scientific Laboratory (1964).

[7] S. J. GAGE, F. T. ADLER, and P. N. POWERS, "Investigations on Nonlinear Stability of Coupled Nuclear Systems," *Proc. Conf. on Neutron Dynamics and Control*, CONF-650413, 45 (1965).

[8] A. BELLENI-MORANTE, "Neutron Transport and Kinetics in a Reactor Composed of Two Loosely Coupled Cores," *Proc. National Topical Meeting on Coupled Reactor Kinetics*, 176 (1967).

χ and χ_d are diagonal matrices whose diagonal elements are the prompt-neutron fission spectrum and the delayed-neutron fission spectrum, respectively

V^{-1} is a diagonal matrix whose diagonal elements are the reciprocals of the mean neutron velocity of each energy group

β_i and λ_i are the fraction and the decay constant of the i'th type delayed-neutron precursor

β is the total delayed-neutron fraction and is equal to the sum of β_i

ν is the average fission-neutron yields.

In a two-group model; matrices and vectors are

$$P = \begin{pmatrix} \Sigma_{f1} & \Sigma_{f2} \\ \Sigma_{f1} & \Sigma_{f2} \end{pmatrix} , \quad D = - \begin{pmatrix} \nabla D_1 \nabla - \Sigma_{T1} & 0 \\ \Sigma_{R1} & \nabla D_2 \nabla - \Sigma_{T2} \end{pmatrix}$$

$$\chi = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} , \quad \chi_d = \begin{pmatrix} \chi_{d1} & 0 \\ 0 & \chi_{d2} \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{v_1} & 0 \\ 0 & \frac{1}{v_2} \end{pmatrix} , \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\boldsymbol{R}_i = \begin{pmatrix} R_{i1} \\ R_{i2} \end{pmatrix} \equiv \begin{pmatrix} \chi_{d1} C_i \\ \chi_{d2} C_i \end{pmatrix} . \tag{3}$$

The source-free adjoint equations corresponding to Eqs. (1) and (2) are given as[9,10]

$$\{(1-\beta)\nu P^* \cdot \chi - D^*\} \boldsymbol{\phi}^* + \sum_i \beta_i \nu P^* \cdot \chi_d \boldsymbol{R}_i^* = -V^{-1} \frac{\partial}{\partial t} \boldsymbol{\phi}^* \tag{4}$$

$$\lambda_i \boldsymbol{\phi}^* - \lambda_i \boldsymbol{R}_i^* = - \frac{\partial}{\partial t} \boldsymbol{R}_i^* \qquad (i = 1, 2, \ldots, 6) . \tag{5}$$

An asterisk denotes an adjoint quantity. Again in a two-group model, adjoint matrices and adjoint vectors are given as

$$P^* = \begin{pmatrix} \Sigma_{f1} & \Sigma_{f1} \\ \Sigma_{f2} & \Sigma_{f2} \end{pmatrix} , \quad D^* = - \begin{pmatrix} \nabla D_1 \nabla - \Sigma_{T1} & \Sigma_{R1} \\ 0 & \nabla D_2 \nabla - \Sigma_{T2} \end{pmatrix}$$

$$\boldsymbol{\phi}^* = \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} , \quad \boldsymbol{R}_i^* = \begin{pmatrix} R_{i1}^* \\ R_{i2}^* \end{pmatrix} . \tag{6}$$

[9] J. LEWINS, *J. Nucl. Energy*, Part A, **13**, 1 (1960).

[10] G. I. MARCHUCK and V. V. ORLOV, "The Theory of Adjoint Functions," Soviet Progress in Neutron Physics, Consultants Bureau, New York, 24 (1963).

III. PARTIAL FLUX AND PARTIAL ADJOINT FLUX

A partial flux $\boldsymbol{\phi}_j(r,t)$ and a partial precursor density $\boldsymbol{R}_{ij}(r,t)$ are defined as the solutions of the following equations:

$$(1-\beta)\chi \cdot [\nu P(r,t)]_j \boldsymbol{\phi}(r,t) - D(r,t)\boldsymbol{\phi}_j(r,t) + \sum_i \lambda_i \boldsymbol{R}_{ij}(r,t) = V^{-1} \frac{\partial}{\partial t} \boldsymbol{\phi}_j(r,t) \tag{7}$$

$$\beta_i \chi_d \cdot [\nu P(r,t)]_j \boldsymbol{\phi}(r,t) - \lambda_i \boldsymbol{R}_{ij}(r,t) = \frac{\partial}{\partial t} \boldsymbol{R}_{ij}(r,t) \qquad (i = 1, 2, \ldots, 6) , \tag{8}$$

where

$\boldsymbol{\phi}(r,t)$ is given by Eqs. (1) and (2)

$[\nu P(r,t)]_j$ is defined as the operator which is equal to $\nu P(r,t)$ in reactor j and vanishes elsewhere

$\boldsymbol{\phi}_j(r,t)$ can be expressed in the following integral form:

$$\boldsymbol{\phi}_j(r,t) = \int_j dr' \times \int_{-\infty}^{t} dt' \boldsymbol{G}(r,t;\, r',t') \left\{ \sum_g \Sigma_{fg}(r',t') \phi_g(r',t') \right\} , \tag{9}$$

where $\int_j dr'$ is an integral operating only on reactor j and $\sum_g$ denotes a summation over all the neutron energy groups. $\boldsymbol{G}(r,t;\, r',t')$ is a Green's function which stands for a neutron flux at (r,t) resulting from a fission that occurred at (r',t') and is given as

$$(1-\beta)\chi \nu \mathbf{1} \delta(r-r', t-t') - D(r,t)\boldsymbol{G}(r,t;\, r',t') + \sum_i \lambda_i \boldsymbol{G}_i(r,t;\, r',t') = V^{-1} \frac{\partial}{\partial t} \boldsymbol{G}(r,t;\, r',t') \tag{10}$$

$$\beta_i \chi_d \nu \mathbf{1} \delta(r-r', t-t') - \lambda_i \boldsymbol{G}_i(r,t;\, r',t') = \frac{\partial}{\partial t} \boldsymbol{G}_i(r,t;\, r',t') , \tag{11}$$

where δ is Dirac's delta function and $\mathbf{1}$ is a vector, all elements of which are unity and the number of the elements is the same as that of the energy group. $\boldsymbol{G}_i(r,t;\, r',t')$ is a Green's function which stands for the i'th-type precursor density at (r,t) resulting from a fission occurring at (r',t'). The following is derived from the fact that the precursor is not transferred in reactor space:

$$\boldsymbol{G}_i(r,t;\, r',t') = \begin{cases} 0 & (t < t') \\ \beta_i \chi_d \nu \mathbf{1} \delta(r-r') \exp[-\lambda_i (t-t')] & (t \geq t') . \end{cases} \tag{12}$$

Between the usual flux $\phi(r,t)$ and the partial flux $\phi_j(r,t)$, there is a relation

$$\phi(r,t) = \sum_j \phi_j(r,t) \quad , \tag{13}$$

since the usual flux is also expressed in the integral form as

$$\phi(r,t) = \int_{sys} dr' \times \int_{-\infty}^{t} dt' \boldsymbol{G}(r,t;\, r',t') \left\{ \sum_g \Sigma_{fg}(r',t') \phi_g(r',t') \right\}, \tag{14}$$

where $\int_{sys} dr'$ is an integral over a whole coupled-reactor space.

We can define a partial adjoint flux $\phi_j^*(r,t)$ and a partial adjoint precursor density $\boldsymbol{R}_{ij}^*(r,t)$ as follows using the results of Eqs. (4) and (5):

$$(1-\beta)[\nu P^*]_j \cdot \chi \phi^*(r,t) - D^* \phi_j^*(r,t) + \sum_i \beta_i [\nu P^*]_j \cdot \chi_d \boldsymbol{R}_i^*(r,t) = -V^{-1} \frac{\partial}{\partial t} \phi_j^*(r,t) \tag{15}$$

$$\lambda_i \phi_j^*(r,t) - \lambda_i \boldsymbol{R}_{ij}^*(r,t) = -\frac{\partial}{\partial t} \boldsymbol{R}_{ij}^*(r,t) \qquad (i = 1, 2, \ldots, 6) \quad . \tag{16}$$

By following the same procedure from Eq. (7) to Eq. (14), we obtain

$$\phi^*(r,t) = \sum_j \phi_j^*(r,t) \quad . \tag{17}$$

Figure 2 shows an example of a thermal-neutron-group partial flux and a thermal-neutron-group partial adjoint flux in the three energy group, one-dimensional calculations in slab geometry.

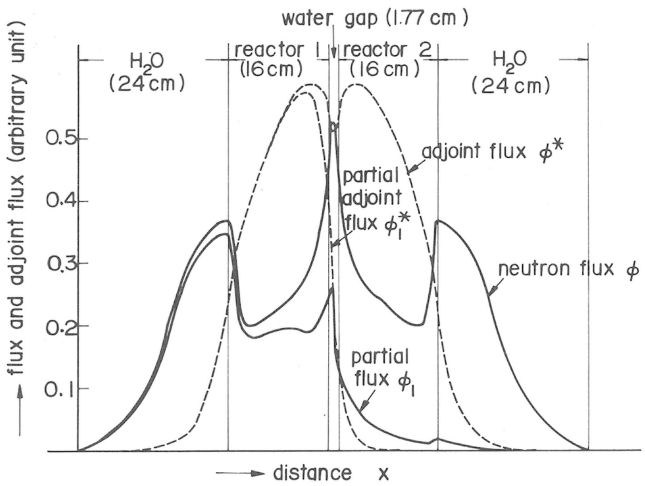

Fig. 2. An example of a thermal-neutron-group partial flux ϕ_j and a thermal-neutron-group partial adjoint flux ϕ_j^* in the three energy-group, one-dimensional calculations in slab geometry.

It is worth noting that ϕ_j^* is not an adjoint quantity of ϕ_j because a matrix operator, synthesized from Eqs. (15) and (16), is not an adjoint operator of that synthesized from Eqs. (7) and (8). This is recognized easily in the simple case of a one-energy group and a two-reactor system in which the delayed neutrons are omitted. Synthesized equations for the column vectors

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{ and } \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}$$

are

$$\begin{pmatrix} [\nu P]_1 - D & [\nu P]_1 \\ [\nu P]_2 & [\nu P]_2 - D \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{18}$$

and

$$\begin{pmatrix} [\nu P]_1 - D & [\nu P]_1 \\ [\nu P]_2 & [\nu P]_2 - D \end{pmatrix} \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix} = - \begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \phi_1^* \\ \phi_2^* \end{pmatrix}, \tag{19}$$

where the matrix operator in the left-hand side of Eq. (19) is the same as and not transposed from that in the left-hand side of Eq. (18).

As shown in Sec. II, we are only concerned with the source-free diffusion equations and their adjoint equations. This implies that a neutron source is postulated outside of an interval in which we are concerned and its effect is taken in the equations through an initial condition. It is also implied in the adjoint quantities that a neutron detector is postulated at some final time t_f outside of our concerned interval and its effect is introduced in the adjoint equations through a final condition. The physical meaning of the adjoint flux $\phi^*(r,t)$ is the effect of a neutron at (r,t), through its progeny, on the final neutron detector.[11] The meaning of the partial adjoint flux $\phi_j^*(r,t)$ is also the effect of a neutron at (r,t) on the final detector, with the additional condition that the neutron causes a fission in reactor j if it reacts on nuclei of reactor material at the end of its lifetime. The meaning of the partial flux $\phi_j(r,t)$ is clear from Eq. (7); that is, it is a neutron flux, at (r,t), composed of neutrons born in reactor j (see Fig. 3).

IV. ADIABATIC APPROXIMATION

The neutron flux $\phi(r,t)$ can be expressed as an amplitude $T(t)$ times a shape function $\psi(r,t)$,[12]

$$\phi(r,t) = T(t)\, \psi(r,t) \quad , \tag{20}$$

[11] J. LEWINS, *Nucl. Sci. Eng.*, **7**, 268 (1960).
[12] A. F. HENRY, *Nucl. Sci. Eng.*, **3**, 52 (1958).

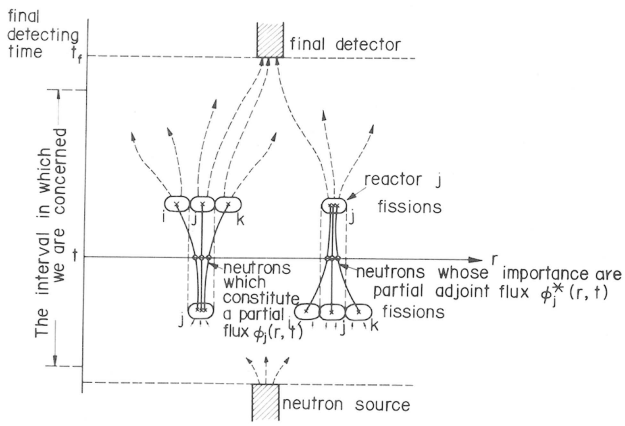

Fig. 3. Illustrative diagram of the physical meaning of a partial flux ϕ_j and a partial adjoint flux ϕ_j^*.

where $T(t)$ and $\psi(r,t)$ are defined as

$$T(t) = \langle \boldsymbol{W} , \phi(r,t) \rangle \tag{21}$$

$$\langle \boldsymbol{W} , \psi(r,t) \rangle = 1 , \tag{22}$$

and $\boldsymbol{W}$ is an arbitrary weighting vector which can be chosen for a particular purpose. An amplitude $T(t)$ is a main part of $\phi(r,t)$ which changes with time and $\psi(r,t)$ represents the shape of the neutron flux at that instant and is relatively insensitive to a time change of $\phi(r,t)$.

Shape functions can be approximated by solving diffusion equations in which a time derivative term is eliminated by changing a real fission neutron yield ν to some fictitious value ν'. The basic idea of this approximation is to replace the true shape function at any instant by the fundamental lambda mode, corresponding to the reactor condition at that instant. The value ν' varies with the change in the system. This method is usually called the adiabatic method.[13] After changing ν to ν' in Eqs. (1) and (2), the following equations are obtained:

$$\{(1-\beta)\chi \cdot \nu'(t) P(r,t) - D(r,t)\}\psi_a(r,t) + \sum_i \lambda_i \boldsymbol{\Gamma}_{ai}(r,t) = 0 \tag{23}$$

$$\beta_i \chi_d \cdot \nu'(t) P(r,t) \psi_a(r,t) - \lambda_i \boldsymbol{\Gamma}_{ai}(r,t) = 0 , \tag{24}$$

where the subscript 'a' denotes the quantities in the adiabatic approximation and ψ_a satisfies the following constraint:

$$\langle \boldsymbol{W} , \psi_a(r,t) \rangle = 1 . \tag{25}$$

The function $\boldsymbol{\Gamma}_{ai}$ represents a shape of an i'th type delayed-neutron precursor density distribution in the adiabatic approximation, its magnitude being defined from Eq. (24). The quantities $\phi_a(r,t)$ and $\boldsymbol{R}_{ai}(r,t)$ are defined as

$$\phi_a(r,t) = T(t)\, \psi_a(r,t) \tag{26}$$

$$\boldsymbol{R}_{ai}(r,t) = T(t)\, \boldsymbol{\Gamma}_{ai}(r,t) . \tag{27}$$

In order to derive Avery's equations, $\chi = \chi_d$ is assumed and the following relations are obtained:

$$\chi \cdot \nu'(t) P(r,t) \phi_a(r,t) = D(r,t) \phi_a(r,t) \tag{28}$$

$$\beta_i \chi \cdot \nu'(t) P(r,t) \phi_a(r,t) = \lambda_i \boldsymbol{R}_{ai}(r,t) , \quad (i = 1, 2, \ldots, 6) . \tag{29}$$

The adjoint neutron flux and the adjoint precursor density are expressed approximately as follows by using the adiabatic method:

$$\phi_a^*(r,t) = T^*(t) \psi_a^*(r,t) \tag{30}$$

$$\boldsymbol{R}_{ai}^*(r,t) = T^*(t)\, \boldsymbol{\Gamma}_{ai}^*(r,t) , \tag{31}$$

where $T^*(t)$ is an amplitude and $\psi_a^*(r,t)$ and $\boldsymbol{\Gamma}_{ai}^*(r,t)$ are adiabatic shape functions of the adjoint neutron flux and of the adjoint precursor density, respectively, and are defined, as in case of T, ψ_a and $\boldsymbol{\Gamma}_{ai}$, by using an arbitrary weighting vector. The adiabatic shape functions ψ_a^* and $\boldsymbol{\Gamma}_{ai}^*$ are obtained from Eqs. (4) and (5) by changing ν to $\nu'(t)$. Assuming again that $\chi = \chi_d$, we have

$$\nu'(t) P^*(r,t) \cdot \chi \phi_a^*(r,t) = D^*(r,t) \phi_a^*(r,t) \tag{32}$$

$$\phi_a^*(r,t) = \boldsymbol{R}_{ai}^*(r,t), \quad (i = 1, 2, \ldots, 6) . \tag{33}$$

After tracing the same procedure as the above, the following relations are obtained for the partial flux from Eqs. (7) and (8):

$$\chi \cdot [\nu'(t) P(r,t)]_j \phi_a(r,t) = D(r,t) \phi_{aj}(r,t) \tag{34}$$

$$\beta_i \chi \cdot [\nu'(t) P(r,t)]_j \phi_a(r,t) = \lambda_i \boldsymbol{R}_{aij}(r,t) \quad (i = 1, 2, \ldots, 6) , \tag{35}$$

and for the partial adjoint flux from Eqs. (15) and (16)

$$[\nu'(t) P^*(r,t)]_j \cdot \chi\, \phi_a^*(r,t) = D^*(r,t) \phi_{aj}^*(r,t) \tag{36}$$

$$\phi_{aj}^*(r,t) = \boldsymbol{R}_{aij}^*(r,t) , \quad (i = 1, 2, \ldots, 6) . \tag{37}$$

The partial flux ϕ_{aj} and the partial precursor density $\boldsymbol{R}_{aij}$ are expressed as an amplitude $T(t)$ times adiabatic shape functions $\psi_{aj}(r,t)$ and $\boldsymbol{\Gamma}_{aij}(r,t)$, respectively. The partial adjoint flux ϕ_{aj}^* and the partial adjoint precursor density $\boldsymbol{R}_{aij}^*$ are also expressed as an amplitude $T^*(t)$ times shape functions $\psi_{aj}^*(r,t)$ and $\boldsymbol{\Gamma}_{aij}^*(r,t)$, respectively.

[13] S. KAPLAN, A. F. HENRY, S. G. MARGOLIS, and J. J. TAYLOR, "Space-Time Reactor Dynamics," *Proc. 3rd Intern. Conf. Peaceful Uses At. Energy*, **4**, 41 (1964).

V. DERIVATION OF AVERY'S COUPLED REACTOR KINETICS EQUATIONS

In order to derive Avery's coupled reactor kinetics equations, Eq. (15) for a partial adjoint flux must be chosen as a starting point.

$$(1-\beta)[\nu P^*(r,t)]_j \cdot \chi \phi^*(r,t) - D^*(r,t)\phi_j^*(r,t) + \sum_i \beta_i [\nu P^*(r,t)]_j \cdot \chi_d \boldsymbol{R}_i^*(r,t) = -V^{-1}\frac{\partial}{\partial t}\phi_j^*(r,t) \quad . \tag{38}$$

Multiplying both sides of Eq. (38) by a partial flux $\phi_k(r,t)$, from the right, and integrating over an entire reactor space and over an interval (t_i,t), we obtain

$$\int_{t_i}^{t} \langle -V^{-1}\frac{\partial}{\partial t}\phi_{aj}^*, \phi_{ak}\rangle dt = \int_{t_i}^{t} \Big\{ \langle (1-\beta)[\nu P^*]_j \cdot \chi \phi_a^*, \phi_{ak}\rangle - \langle D^*\phi_{aj}^*, \phi_{ak}\rangle + \sum_i \langle \beta_i [\nu P^*]_j \cdot \chi_d \boldsymbol{R}_{ai}^*, \phi_{ak}\rangle \Big\} dt \quad , \tag{39}$$

where the shape functions are approximated by using the adiabatic method. The second term, in parentheses, of the right-hand side of Eq. (39) can be rewritten as follows by using Eq. (36):

$$\langle D^*\phi_{aj}^*, \phi_{ak}\rangle = \langle [\nu'(t)P^*(r,t)]_j \cdot \chi \phi_a^*, \phi_{ak}\rangle \quad , \tag{40}$$

where $\chi = \chi_d$ is assumed in order to obtain Avery's equations. Because $[\nu P^*]_j$ is defined as the same operator as νP^* in reactor j and vanishes elsewhere, the right-hand side of Eq. (40) relates the integral only to reactor j and it is rewritten as

$$\langle [\nu'(t)P^*]_j \cdot \chi \phi_a^*, \phi_{ak}\rangle = \langle \nu'(t)P^* \cdot \chi \phi_a^*, \phi_{ak}\rangle_j \quad . \tag{41}$$

Other terms, in parentheses, of the right-hand side of Eq. (39) also relate the integrals only to reactor j.

Because νP^* has no derivative, we have the following commutation laws even though ϕ_{ak}, ϕ_a^*, and $\boldsymbol{R}_{ai}^*$ do not disappear on the boundary of reactor j (see Appendix):

$$\langle (1-\beta)\nu P^* \cdot \chi \phi_a^*, \phi_{ak}\rangle_j = \langle \phi_a^*, (1-\beta)\chi \cdot \nu P \phi_{ak}\rangle_j \tag{42}$$

$$\langle \nu'(t)P^* \cdot \chi \phi_a^*, \phi_{ak}\rangle_j = \langle \phi_a^*, \chi \cdot \nu'(t) P \phi_{ak}\rangle_j \tag{43}$$

$$\langle \beta_i \nu P^* \cdot \chi_d \boldsymbol{R}_{ai}^*, \phi_{ak}\rangle_j = \langle \boldsymbol{R}_{ai}^*, \beta_i \chi_d \cdot \nu P \phi_{ak}\rangle_j . \tag{44}$$

If a particular open interval (t_i,t) is chosen, the following commutation law is obtained for the left-hand side of Eq. (39):

$$\int_{t_i}^{t} \langle -V^{-1}\frac{\partial}{\partial t}\phi_{aj}^*, \phi_{ak}\rangle dt = \int_{t_i}^{t} \langle \phi_{aj}^*, V^{-1}\frac{\partial}{\partial t}\phi_{ak}\rangle dt \quad . \tag{45}$$

Then, after differentiating the resulting equation, with respect to time and using the relation $\phi_a^* = \boldsymbol{R}_{ai}^*$ [see Eq. (33)], we have

$$\langle \phi_{aj}^*, V^{-1}\frac{\partial}{\partial t}\phi_{ak}\rangle = \langle \phi_a^*, (1-\beta)\chi \cdot [\nu P]_j \phi_{ak}\rangle - \langle \phi_a^*, \chi \cdot [\nu'(t)P]_j \phi_{ak}\rangle + \sum_i \langle \phi_a^*, \beta_i \chi_d \cdot [\nu P]_j \phi_{ak}\rangle \quad . \tag{46}$$

Dividing both sides of Eq. (46) by $T^*(t)$, we have

$$\langle \psi_{aj}^*, V^{-1}\frac{\partial}{\partial t}\phi_{ak}\rangle = \langle \psi_a^*, (1-\beta)\chi \cdot [\nu P]_j \phi_{ak}\rangle - \langle \psi_a^*, \chi \cdot [\nu'(t)P]_j \phi_{ak}\rangle + \sum_i \langle \psi_a^*, \beta_i \chi_d \cdot [\nu P]_j \phi_{ak}\rangle \quad . \tag{47}$$

The left-hand side of Eq. (47) can be written as follows:

$$\langle \psi_{aj}^*, V^{-1}\frac{\partial}{\partial t}\phi_{ak}\rangle = \langle \psi_{aj}^*, V^{-1}\psi_{ak}\rangle \frac{d}{dt}T(t) + \langle \psi_{aj}^*, V^{-1}T(t)\frac{\partial}{\partial t}\psi_{ak}\rangle \quad . \tag{48}$$

If we put $\tau(T)$ and $\tau(\psi)$ as time constants of T and ψ_{ak} near the instant t, respectively, Eq. (48) is expressed as

$$\langle \psi_{aj}^*, V^{-1}\frac{\partial}{\partial t}\phi_{ak}\rangle = \langle \psi_{aj}^*, V^{-1}\psi_{ak}\rangle \tau^{-1}(T) \cdot T(t) + \langle \psi_{aj}^*, V^{-1}\psi_{ak}\tau^{-1}(\psi)\rangle T(t) \quad . \tag{49}$$

Except in the case when strong local perturbations occur, we can neglect the second term of the right-hand side of Eq. (49) because ψ_{ak} is insensitive to time, i.e., $\tau(T) \ll \min \tau(\psi)$. Therefore, this leads to

$$\langle \psi_{aj}^*, V^{-1}\psi_{ak}\rangle \frac{d}{dt}T(t) = (1-\beta)\langle \psi_a^*, \chi \cdot [\nu P]_j \psi_{ak}\rangle T(t) - \langle \psi_a^*, \chi \cdot [\nu'(t)P]_j \psi_{ak}\rangle T(t) + \sum_i \beta_i \langle \psi_a^*, \chi_d [\nu P]_j \psi_{ak}\rangle T(t) \quad . \tag{50}$$

Here, intending to derive Avery's coupled reactor kinetics equations, we multiply each term of Eq. (50) by complex factors whose overall values are equivalent to unity. The left-hand side of Eq. (50) becomes

$$\langle \psi_{aj}^*,\ V^{-1}\psi_{ak}\rangle \frac{d}{dt}T(t) = \frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1},\chi\cdot[\nu'(t)P]_j\psi_a\rangle}\ \frac{\langle \psi_{aj}^*,\ V^{-1}\psi_{ak}\rangle}{\langle \psi_a^*,\chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle}\ \langle \mathbf{1},\chi\cdot[\nu'(t)P]_j\psi_a\rangle$$

$$\times \frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle}{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}\ \frac{d}{dt}T(t)\ , \tag{51}$$

where $\mathbf{1}$ is a vector, all elements of which are unity and the number of its elements is equal to that of ϕ. The first term of the right-hand side of Eq. (50) becomes

$$(1-\beta)\langle \psi_a^*,\chi\cdot[\nu P]_j\psi_{ak}\rangle T(t) = (1-\beta)\frac{\langle \psi_a^*,\chi\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1},\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}\left(\frac{\nu}{\nu'(t)}\right)\frac{\langle \mathbf{1},\chi\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1},\chi\cdot[\nu'(t)P]_k\psi_a\rangle}$$

$$\times \frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle}{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}\ \sum_m\left\{\langle \mathbf{1},\ \chi\cdot[\nu'(t)P]_k\psi_a\rangle\cdot\frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_k\psi_{am}\rangle}{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_k\psi_a\rangle}\right\}\cdot T(t)\ , \tag{52}$$

where the following relation, derived from the adiabatic representation of Eq. (13) is used:

$$\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_k\psi_a\rangle = \sum_m \langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_k\psi_{am}\rangle\ . \tag{53}$$

The second term of the right-hand side of Eq. (50) becomes

$$\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle T(t) = \frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1},\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}\cdot\langle \mathbf{1},\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle\frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle}{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}T(t)\ . \tag{54}$$

The third term of the right-hand side of Eq. (50), representing delayed-neutron precursor density, becomes

$$\sum_i \beta_i\langle \psi_a^*,\ \chi_d\cdot[\nu P]_j\psi_{ak}\rangle T(t) = \sum_i \lambda_i\frac{\langle \psi_a^*,\ \chi_d\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1},\ \chi_d\cdot[\nu'(t)P]_j\psi_a\rangle}\left(\frac{\nu}{\nu'(t)}\right)\frac{\langle \mathbf{1},\ \chi_d\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1},\ \chi_d\cdot[\nu'(t)P]_k\psi_a\rangle}$$

$$\times \frac{\langle \psi_a^*,\ \chi_d\cdot[\nu'(t)P]_j\psi_{ak}\rangle}{\langle \psi_a^*,\ \chi_d\cdot[\nu'(t)P]_j\psi_a\rangle}\langle \mathbf{1},\ \Gamma_{aik}\rangle T(t) \tag{55}$$

after including the relation

$$\beta_i\langle \mathbf{1},\chi_d\cdot[\nu'(t)P]_k\psi_a\rangle = \lambda_i\langle \mathbf{1},\ \Gamma_{aik}\rangle \tag{56}$$

derived from Eq. (35). Assuming that $\chi = \chi_d$, we obtain

$$N_j^{0*}(t)\ell_{jk}^0(t)S_{jk}^0(t)\frac{d}{dt}T(t) = (1-\beta)N_j^{0*}(t)\left(\frac{\nu}{\nu'(t)}\right)k_{jk}^0(t)\sum_m S_{km}^0(t)T(t) - N_j^{0*}(t)S_{jk}^0(t)T(t)$$

$$+\sum_i \lambda_i N_j^{0*}(t)\left(\frac{\nu}{\nu'(t)}\right)k_{jk}^0(t)C_{ik}^0(t)T(t)\ , \tag{57}$$

where S_{jk}^0, C_{ik}^0, k_{jk}^0, ℓ_{jk}^0 and N_j^{0*} are defined as

$$S_{jk}^0(t) = \langle \mathbf{1},\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle\cdot\frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle}{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle} \tag{58}$$

$$C_{jk}^0(t) = \langle \mathbf{1},\ \Gamma_{aik}\rangle \quad (i = 1, 2, \ldots, 6) \tag{59}$$

$$k_{jk}^0(t) = \frac{\langle \mathbf{1},\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1},\ \chi\cdot[\nu'(t)P]_k\psi_a\rangle}\cdot\frac{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle}{\langle \psi_a^*,\ \chi\cdot[\nu'(t)P]_j\psi_a\rangle} \tag{60}$$

$$\ell^0_{jk}(t) = \frac{\langle \psi_{aj}^*,\ V^{-1}\psi_{ak}\rangle}{\langle \psi_a^*, \chi\cdot[\nu'(t)P]_j\psi_{ak}\rangle} \tag{61}$$

$$N_j^{0*}(t) = \frac{\langle \psi_a^*, \chi\cdot[\nu'(t)P]_j\psi_a\rangle}{\langle \mathbf{1}, \chi\cdot[\nu'(t)P]_j\psi_a\rangle} \quad . \tag{62}$$

The definitions above are those given by Avery. The superscript '0' denotes that these quantities are related only to the shape functions and are relatively insensitive to a time change of the total power.

If we drop N_j^{0*} from both sides of Eq. (57) and further define

$$S_{jk}(t) = T(t)S^0_{jk}(t) \tag{63}$$

$$C_{ik}(t) = T(t)C^0_{ik}(t) \tag{64}$$

$$k_{jk}(t) = \frac{\nu}{\nu'(t)}\, k^0_{jk}(t) \quad , \tag{65}$$

Eq. (57) results in Avery's coupled reactor kinetics equations

$$\ell^0_{jk}(t)\frac{d}{dt}S_{jk}(t) = (1-\beta)k_{jk}(t)\sum_m S_{km}(t) - S_{jk}(t) + k_{jk}(t)\sum_i \lambda_i C_{ik}(t) \quad , \tag{66}$$

where the following approximation is introduced:

$$S^0_{jk}(t)\frac{d}{dt}T(t) \simeq \frac{d}{dt}S_{jk}(t) \tag{67}$$

which is equivalent to the assumption made on deriving Eq. (50) from Eq. (46).

In order to derive the equations for the delayed-neutron precursor densities in Avery's model, Eq. (8),

$$\beta_i\chi_d\cdot[\nu P(r,t)]_j\phi(r,t) - \lambda_i \boldsymbol{R}_{ij}(r,t) = \frac{\partial}{\partial t}\boldsymbol{R}_{ij}(r,t) \quad , \tag{68}$$

must be chosen as a starting point. Approximating shape functions are again obtained by the adiabatic method and after multiplying both sides of Eq. (68) by $\mathbf{1}$ from the left and integrating over an entire reactor space, we have

$$\beta_i\langle \mathbf{1},\ \chi_d\cdot[\nu P]_j\psi_a\rangle T(t) - \lambda_i\langle \mathbf{1},\ \boldsymbol{\Gamma}_{aij}\rangle T(t) = \langle \mathbf{1},\ \boldsymbol{\Gamma}_{aij}\rangle \frac{d}{dt}T(t) + \langle \mathbf{1},\ \frac{\partial}{\partial t}\boldsymbol{\Gamma}_{aij}\rangle T(t) \quad . \tag{69}$$

Assuming that $\chi = \chi_d$ and applying the definitions for S_{jk} and C_{ij}, Eq. (69) results directly in the following precursor equation in Avery's model:

$$\beta_i\sum_k S_{jk}(t) - \lambda_i C_{ij} = \frac{d}{dt}C_{ij} \quad . \tag{70}$$

VI. DISCUSSIONS

We may call Eqs. (66) and (70) the results of a "multipoint kinetics model" as opposed to a "one-point kinetics model." The $k_{jk}(t)$ and $\ell^0_{jk}(t)$ constitute the multipoint kinetic operators and $T(t)$ and $S^0_{jk}(t)$ are an amplitude and a shape function in the multipoint kinetics model, respectively. This is easy to understand if we define $S_j(t)$ and $S^0_j(t)$ as

$$S_j(t) = \sum_k S_{jk}(t) \quad , \quad S^0_j(t) = \sum_k S^0_{jk}(t) \tag{71}$$

which gives

$$S_j(t) = T(t)\ S^0_j(t) \quad . \tag{72}$$

Comparing Eq. (72) with Eq. (20), it is recognized that $S^0_j(t)$ is a shape function in the multipoint kinetics model where the suffix j corresponds to the space variable r in $\psi(r,t)$.

The parameter $k^0_{jk}(t)$ is a coupling coefficient defined only by shape functions in the adiabatic approximation and satisfies the critical conditions

$$\sum_k k^0_{jk}(t)\ S^0_k(t) = S^0_j(t) \quad , \tag{73}$$

which can be verified by the definitions of $k^0_{jk}(t)$ and $S^0_j(t)$. The conditions established by Eq. (73), when it has a non-trivial solution, are expressed as

$$\det.\ \left|a(jk)\right| = 0 \quad , \tag{74}$$

where

$$a(jk) = \begin{cases} k^0_{jk}(t) - 1 & (j = k) \\ k^0_{jk}(t) & (j \neq k) \end{cases} .$$

In a two-reactor system, Eq. (74) becomes

$$(k^0_{11} - 1)(k^0_{22} - 1) = k^0_{12}\, k^0_{21} \quad . \tag{75}$$

For the time-dependent multipoint kinetics model, the following simplified form can be obtained by summing both sides of Eq. (66) with respect to k and replacing $\sum_k S_{jk}$ by S_j:

$$\ell^0_j(t)\frac{d}{dt}S_j(t) = (1-\beta)\sum_k k_{jk}(t)S_k(t) - S_j(t) + \sum_k k_{jk}(t)\sum_i {}_iC_{ik}(t) \quad , \tag{76}$$

where $\ell^0_j(t)$ is defined by

$$\ell^0_j(t) = \frac{\langle \psi_{aj}^*,\ V^{-1}\psi_a\rangle}{\langle \psi_a^*, \chi\cdot[\nu'(t)P]_j\psi_a\rangle} \tag{77}$$

and is identified as the mean lifetime, in a fictitious equilibrium reactor system, of neutrons which are destined to cause fissions in reactor j. In deriving Eq. (76), it is again assumed that the

time constant of the amplitude $\tau(T)$ is much smaller than that of the shape function $\tau(\psi)$.

An amplitude $T(t)$ can be obtained alternately by solving the usual one-point kinetics equation if the reactivity is given as

$$\rho(t) = \frac{\nu - \nu'(t)}{\nu'(t)} \quad . \tag{78}$$

Accordingly we can treat the space-time kinetics problems by combining the one-point kinetics model with Eq. (73) which is the adiabatic approximation in the multipoint kinetics model, if the reactivity $\rho(t)$, therefore $\nu'(t)$, and $k^0_{jk}(t)$ could be obtained from the multipoint kinetics model. Unfortunately, this can not be realized strictly in the present multipoint kinetics model, not in Avery's model, nor in any other models. Some other ideas will be required to make the multipoint model more useful.

VII. CONCLUSION

The preceding discussion reveals the premise assumed by Avery in his coupled-reactor kinetics equations and the limits of their application. The following assumptions are involved:

(A) The delayed-neutron fission spectrum is the same as the prompt-neutron fission spectrum.

(B) The adiabatic approximation is introduced into the neutron-flux shape function $\psi(r,t)$ Here, the neutron flux $\phi(r,t)$ is expressed as

$$\phi(r,t) = T(t)\ \psi(r,t) \quad ,$$

where $T(t)$ is an amplitude which represents an overall change with time and $\psi(r,t)$ is a shape function which is relatively insensitive to time.

(C) The time constant of the amplitude $T(t)$ is much smaller than the minimum time constant of the shape function $\psi(r,t)$ at that instant.

On account of assumptions (B) and (C), it is doubtful that we can apply Avery's model to the analysis of a phenomenon shortly after an abrupt local perturbation is introduced, that is, a phenomenon occurring during several milliseconds after an abrupt insertion of a reactivity. Also, as a result of these assumptions, the delay time associated with the transfer of neutrons does not appear explicitly in Avery's model. Conversely, Avery's model is simplified by introducing the above assumptions and it is applicable to the actual analysis without including the complicated formulas of Cockrell's model, though its use is not justified in case of a large local perturbation.

Most of the existing multipoint kinetics models, including Avery's, lack consistency when applied to problems which include feedback phenomena. The difficulty comes from the process for solving the diffusion model in order to obtain the coupling parameters $k^0_{jk}(t)$ and $\ell^0_{jk}(t)$ in the presence of feedback. One method by which this difficulty is overcome will soon be discussed in a separate publication.

NOMENCLATURE

A. *Quantities in the Diffusion Equations*

Operators [See Eqs. (3) and (6)]

P a matrix operator representing the production of fission neutrons

D a destructive matrix operator which represents all the neutron destruction terms including absorption, leakage, and the transfer from one energy group to another by the scattering process

χ and χ_d diagonal matrices whose diagonal elements are the prompt-neutron fission spectrum and the delayed-neutron fission spectrum, respectively

V^{-1} a diagonal matrix whose diagonal elements are the reciprocals of the mean neutron velocity of each energy group

β_i and λ_i the fraction and the decay constant of the i'th-type delayed-neutron precursor

β the total delayed neutron fraction

ν the average fission-neutron yield

$\nu'(t)$ a fictitious fission-neutron yield at time t.

Neutron Fluxes and Precursor Densities

$\boldsymbol{\phi}$ a vector whose elements are the neutron flux in each energy group

$\boldsymbol{R}_i$ a vector whose j'th element is the i'th-type delayed neutron precursor density $C_i(r,t)$ multiplied by the delayed-fission spectrum of the j'th energy group χ_{dj}

$\boldsymbol{\phi}^*$ and $\boldsymbol{R}_i^*$ adjoint vectors of $\boldsymbol{\phi}$ and $\boldsymbol{R}_i$, respectively

$\boldsymbol{\phi}_j$ a vector whose elements are the partial neutron flux of each energy group, given by Eq. (7). The partial neutron flux is a flux of neutrons born in a reactor, such as reactor j

$\boldsymbol{R}_{ij}$ a vector whose k'th element is the partial delayed-neutron precursor density of the i'th-type $C_{ij}(r,t)$ times χ_{dk}, given by Eq. (8). Here,

$$C_{ij}(r,t) = \begin{cases} C_i(r,t) & , \quad \text{at } r \text{ in reactor } j \\ 0 & , \quad \text{elsewhere} \end{cases}$$

$\boldsymbol{\phi}_j^*$ a vector whose elements are the partial adjoint-neutron flux in each energy group, given by Eq. (15). It is not an adjoint vector of $\boldsymbol{\phi}_j$. The partial adjoint neutron flux is physically explained as the importance of a neutron which is expected to cause a fission in reactor j

$\boldsymbol{R}_{ij}^*$ a vector whose elements are the partial adjoint precursor density of the i'th-type in each energy group, given by Eq. (16). The partial adjoint precursor density is explained as the importance of a fraction of precursor, the delayed neutrons of which is expected to cause a fission in reactor j

$\boldsymbol{G}(r,t;\, r',t')$ a vector whose elements are Green's functions which stand for neutron flux of each energy group at (r,t) resulting from a fission occurred at (r',t') given by Eq. (10)

$\boldsymbol{G}_i(r,t;\, r',t')$ a vector whose j'th element is a Green's function which stands for the i'th-type precursor density at (r,t), resulting from a fission that occurred at (r',t'), multiplied by the delayed-neutron fission spectrum of the j'th energy group χ_{dj}. This vector is given by Eq. (11) and by Eq. (12).

Amplitudes and Shape Functions

T a common time function, namely an amplitude defined by Henry, of $\boldsymbol{\phi}$, $\boldsymbol{\phi}_j$, $\boldsymbol{R}_i$, and $\boldsymbol{R}_{ij}$.

T^* a common time function of $\boldsymbol{\phi}^*$, $\boldsymbol{\phi}_j^*$, $\boldsymbol{R}_i^*$, and $\boldsymbol{R}_{ij}^*$

$\boldsymbol{\psi}$, $\boldsymbol{\psi}_j$, $\boldsymbol{\Gamma}_i$, $\boldsymbol{\Gamma}_{ij}$, $\boldsymbol{\psi}^*$, $\boldsymbol{\psi}_j^*$, $\boldsymbol{\Gamma}_i^*$, and $\boldsymbol{\Gamma}_{ij}^*$ shape functions of $\boldsymbol{\phi}$, $\boldsymbol{\phi}_j$, $\boldsymbol{R}_i$, $\boldsymbol{R}_{ij}$, $\boldsymbol{\phi}^*$, $\boldsymbol{\phi}_j^*$, $\boldsymbol{R}_i^*$, and $\boldsymbol{R}_{ij}^*$, respectively

Others

$\mathbf{1}$ a vector, all elements of which are unity and the number of its elements is equal to that of energy groups

δ Dirac's delta function

The subscript 'a' denotes quantities, shape functions of which are approximated by the adiabatic method

The notation $\langle \boldsymbol{a}, \boldsymbol{b} \rangle_G$ means to take the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ over the domain G in a reactor space; that is,

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle_G \equiv \sum_j \int_G a_j b_j dr \ ,$$

where a_j and b_j are the j'th elements of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively

The notation $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ means to take the inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ over an entire reactor space.

B. Quantities in the Nodal Representation

S_j total fission neutrons in reactor j, given by Eq. (71)

S_{jk} a part of S_j that results from fissions caused by neutrons which originate in reactor k, given by Eq. (63)

C_{ik} total delayed-neutron precursors of the i'th-type in reactor k, given by Eq. (64)

N_j^* an average importance of fission neutrons in reactor j

k_{jk} a coupling coefficient between reactor k and reactor j, given by Eq. (65)

S_j^0 a shape function of S_j, defined by Eq. (71)

S_{jk}^0 a shape function of S_{jk}, defined by Eq. (58)

C_{ik}^0 a shape function of C_{ik}, defined by Eq. (59)

N_j^{0*} a shape function of N_j^* defined by Eq. (62)

k_{jk}^0 a coupling coefficient between reactor k and reactor j in a fictitious equilibrium state of a reactor system, defined by Eq. (60)

ℓ_{jk}^0 a partial-neutron lifetime in a fictitious equilibrium reactor system, defined by Eq. (61); that is, the lifetime of neutrons which originate in reactor k and are expected to cause fissions in reactor j

ℓ_j^0 a mean lifetime, in a fictitious equilibrium reactor system, of neutrons which are expected to cause fissions in reactor j, defined by Eq. (77)

The superscript '0' denotes quantities which are related only to shape functions and not to amplitudes.

APPENDIX

Adjoint Operator and Commutation Law

Let A be any differential operator of order n defined over a domain G of the m-dimensional Cartesian space E^m, written in the form of[14]

$$A = \sum_{|\rho|,|\sigma|=0}^{n} D^{(\rho)}\, a^{\rho;\sigma}(x)\, D^{(\sigma)} \quad , \tag{A.1}$$

where

$$D^{(\rho)} = \partial^{\rho_1+\ldots+\rho_m}/\partial x_1^{\rho_1} \ldots \partial x_m^{\rho_m}, \quad |\rho| = \sum_{j=1}^{m} \rho_j \tag{A.2}$$

$$a^{\rho;\sigma}(x) = a^{\rho_1\rho_2\ldots\rho_m;\sigma_1\sigma_2\ldots\sigma_m}(x_1, \ldots, x_m), \tag{A.3}$$

where $a^{\rho;\sigma}(x)$ is an infinitely differentiable real function defined over G and satisfies a symmetric relation

$$a^{\rho;\sigma}(x) = a^{\sigma;\rho}(x) (\text{when } |\rho| = |\sigma| = n) \ . \tag{A.4}$$

Then an adjoint differential operator A^* is uniquely defined as

$$A^* = \sum_{|\rho|,|\sigma|=0}^{n} (-1)^{|\rho|+|\sigma|}\, D^{(\sigma)} a^{\rho;\sigma}(x)\, D^{(\rho)} \quad , \tag{A.5}$$

and the following commutation relation is obtained:

$$<v \ , Au>_G = <A^* v \ , u>_G \quad , \tag{A.6}$$

where u and v may be arbitrary functions which are $2n$-times continuously differentiable over G and vanish on the boundary of G. Equation (A.6) is verified as follows. Let

$$\omega(x) = \frac{\partial^{\rho_1^{-1}+\rho_2+\ldots\ldots+\rho_m}}{\partial x_1^{\rho_1^{-1}}\, \partial x_2^{\rho_2} \ldots\ldots \partial x_m^{\rho_m}} a^{\rho;\sigma}(x) D^{(\sigma)} u(x) \tag{A.7}$$

and after integrating the left-hand side of Eq. (A.6) by parts, we obtain

$$<v, Au>_G = <v, \frac{\partial \omega}{\partial x_1}>_G = [v\cdot\omega] - <\frac{\partial v}{\partial x_1}, \omega>_G \quad , \tag{A.8}$$

where the term $[v\cdot\omega]$ vanishes on the boundary of G. After repeating this process, Eq. (A.6) is obtained. When A does not contain any derivative operation, that is, A means only to multiply a function $a(x)$, Eq. (A.6) is satisfied without the limitation that u and v vanish on the boundary of G.

There is following an easily established law for the product of two operators:

$$(A\cdot B)^* = B^*\cdot A^* \ . \tag{A.9}$$

When an operator B is given in matrix notation, say

$$B = (b_{ij}) \ , \tag{A.10}$$

the adjoint operator B^* is defined by the transposed matrix of B whose elements are adjoint quantities

$$B^* = (b_{ji}^*) \ . \tag{A.11}$$

Again the adjoint operator is linked with the bilinear form and satisfies the following commutation law,

$$<\boldsymbol{v}, B\boldsymbol{u}>_G = <B^*\boldsymbol{v}, \boldsymbol{u}>_G \ , \tag{A.12}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors whose elements may be arbitrary functions which are $2n$-times continuously differentiable over G and vanish on the boundary of G.

The matrix operators νP and D given in the text are differential operators of order 0 and of order 2, respectively, defined over an entire reactor space. The cross sections appearing in these operators are usually given as discontinuous function, which correspond to the function $a^{\rho;\sigma}(x)$ in Eq. (A.1), at the interface of sub-regions in a reactor system. The commutation laws (A.12) between νP, D and those adjoint operators νP^*, D^* are still obtained over an entire reactor space, given the condition that the neutron flux and the neutron current are continuous at the interface of the subregions. That is,

$$<\phi^*, \nu P\phi> = <\nu P^*\phi^*, \phi> \tag{A.13}$$

$$<\phi^*, D\phi> = <D^*\phi^*, \phi> \ , \tag{A.14}$$

where ϕ and ϕ^* vanish at the outer boundary of the coupled reactor space. If we integrate Eq. (A.13) on a subregion, such as reactor j, the following commutation relation is still obtained:

$$<\phi^*, \nu P\phi>_j = <\nu P^*\phi^*, \phi>_j \ , \tag{A.15}$$

since neither νP nor νP^* have any derivatives. But the same result can not be derived for D and D^*, because the boundary terms do not disappear when the integrations by parts are carried out on the calculation of $<\phi^*, D\phi>_j$. If we define the adjoint operator D^*, conversely, by the commu-

[14] K. YOSHIDA, *Iso Kaiseki (Topological Analysis)*, Modern Applied Mathematics Series, A.4.I., Iwanami Book Co. (1957).

tation law over the subregion j, D^* can not be written as a usual analytic form.

The same situation occurs in a time derivative operator $\partial/\partial t$.[15] When its adjoint $(\partial/\partial t)^*$ is defined under the assumption that the commutation law is formed between $\partial/\partial t$ and $(\partial/\partial t)^*$ in a finite closed time interval $[t_i, t_f]$, that is

$$\int_{t_i}^{t_f} \langle \phi^*, \frac{\partial}{\partial t} \phi \rangle \, dt = \int_{t_i}^{t_f} \langle \left(\frac{\partial}{\partial t}\right)^* \phi^*, \phi \rangle \, dt, \tag{A.16}$$

where ϕ and ϕ^* do not vanish in general at $t = t_i$ and at $t = t_f$, then $(\partial/\partial t)^*$ is given as

$$\left(\frac{\partial}{\partial t}\right)^* = -\frac{\partial}{\partial t} + \delta(t - t_f) - \delta(t - t_i), \tag{A.17}$$

where δ denotes the Dirac's delta-function. In an open interval, $t_i < t < t_f$,

$$\left(\frac{\partial}{\partial t}\right)^* = -\frac{\partial}{\partial t} \quad . \tag{A.18}$$

So if we put $t_i \to -\infty$ and $t_f \to +\infty$ we may use Eq. (A.18) in almost the whole range of interval with which we are usually concerned.

[15] D. E. DOUGHERTY and C. N. SHEN, *Nucl. Sci. Eng.*, **13**, 141 (1962).

ACKNOWLEDGMENT

The author would like to acknowledge the guidance and encouragement of Professor J. Wakabayashi.