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THEORY OF COUPLED REACTORS

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COUPLED SYSTEM PARAMETERS

The general formulation of the reactor equations, as given by the time, space, and energy dependent Boltzmann equation, may be applied, formally at least, to any system however complicated. To say then that a system consists of coupled reactors, an irrelevant concept in the general formulation, is a statement only of how one wishes to consider the system, and implies, of course, that it is believed advantageous to do so.

The individual reactors of the coupled system are arbitrarily defined by any prescription which specifies the reactor in which each fission neutron is emitted. The term "coupled" is taken to mean that in each of the reactors some of the fission neutrons are emitted in fissions induced by neutrons born in other reactors. A fission neutron source may be associated with each of the reactors. The total source is the sum of the individual reactor sources. Further, each of the reactor sources gives rise to a next generation source in other reactors.

The formalism that is developed treats the system in terms of integral parameters which explicitly characterize the individual reactors and the coupling between them. k_{ij} is defined as the expectation value that a fission neutron in reactor j gives rise to a next generation fission neutron in reactor i, and ℓ_{ij} is defined as the average prompt neutron lifetime for the process. k_{ij} for $i \neq j$ is a measure of the cross coupling from reactor j to reactor i, and in general $k_{ij} \neq k_{ji}$.

The quantity $1 - k_{ii}$ occurs frequently and it is therefore convenient to define $\Delta_i \equiv 1 - k_{ii}$. Δ_i is a measure of the subcriticality of reactor i without the contribution of the other reactors.

S_i is defined as the total fission neutron source in reactor i, and S_{ij} is defined as the total fission neutron source in reactor i which results from fissions caused by neutrons which originate in reactor j. We then have

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$$S_i = \sum_{j=1}^{N} S_{ij} \tag{1}$$

where N is the number of reactors in the system.

The definitions of the various integral parameters, k_{ij}, ℓ_{ij}, S_i and S_{ij} are applicable for both the steady state and time dependent situation.

k_{ij}, ℓ_{ij}, S_i, and S_{ij} are all integral or average quantities. Their definition requires some sort of averaging procedure or weighting procedure in integration. A detailed discussion of this is deferred until the section "Correlation with General Formulation."

REACTIVITY CONSIDERATIONS

The critical condition that the k_{ij} must satisfy is considered first for the case of two coupled reactors. From the basic definitions of the k_{ij}, S_{ij}, and S_i, the following steady state ratios hold:

$$\frac{S_{11}}{S_1} = k_{11} \tag{2a}$$

$$\frac{S_{12}}{S_2} = k_{12} \tag{2b}$$

$$\frac{S_{21}}{S_1} = k_{21} \tag{2c}$$

$$\frac{S_{22}}{S_2} = k_{22} \quad . \tag{2d}$$

And also, from Eq. (1),

$$S_{11} + S_{12} = S_1 \tag{3a}$$

$$S_{21} + S_{22} = S_2 \quad . \tag{3b}$$

Substitution of Eqs. (2) into Eqs. (3) gives,

$$k_{11} S_1 + k_{12} S_2 = S_1 \tag{4a}$$

$$k_{21} S_1 + k_{22} S_2 = S_2 \quad . \tag{4b}$$

The condition for criticality is given by the condition that a self-consistent solution exists to Eqs. (4), and further that this solution corresponds to non-negative values for the S_i. This latter condition is necessary to rule out cases where one of the reactors is supercritical on its own. The criticality condition for the two reactor case is thus,

$$\begin{vmatrix} (k_{11}-1) & k_{12} \\ k_{21} & (k_{22}-1) \end{vmatrix} = 0 \quad , \tag{5}$$

which may also be given as,

$$k_{12}\, k_{21} = \Delta_1\, \Delta_2 \quad . \tag{6}$$

The above relationships are all valid even if the cross coupling in one or both directions vanishes. If the coupling to one of the reactors from the other vanishes, then it must be critical on its own. The coupling in the other direction and the reactivity of the other reactor may be any value providing the other reactor is subcritical.

After having satisfied the criticality condition, one can solve for the relative values of the S_i. For the two reactor case this gives

$$\frac{S_1}{S_2} = \frac{k_{12}}{\Delta_1} = \frac{\Delta_2}{k_{21}} \quad . \tag{7}$$

Even if the cross coupling vanishes in one direction, the power ratio given in Eq. (7) is still valid, though one of the two expressions will be indeterminate. If both cross couplings vanish, then both expressions for the power ratios are indeterminate, and of course any power ratio can in fact exist between the two independent critical reactors.

We now consider the reactivity of a non-critical system. We do this in the usual way in terms of the fictitious value of the number of neutrons per fission, ν_c, needed to maintain criticality. At non-criticality $\nu \neq \nu_c$ and the fission neutron source in one generation will reproduce itself in the next generation with a magnitude which differs by a factor k, where $k = \nu/\nu_c$. For small deviations from criticality, $k_{ex} \approx \rho$, where $k_{ex} = k-1$, and the reactivity, $\rho = -\delta\nu/\nu = (\nu - \nu_c)/\nu$.

For the two reactor case,

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = k \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \quad . \tag{8}$$

Using $k = 1 + k_{ex}$ we obtain from Eq. (8) the following expression for k_{ex};

$$\Delta_1 \Delta_2 + k_{ex}\,(\Delta_1 + \Delta_2) + k_{ex}^2 = k_{12} k_{21} \quad . \tag{9}$$

This expression is valid without restriction even if one or both of the reactors are supercritical by themselves. For the case where the reactivity is very small compared to the subcriticality of each of the reactors, we can neglect k_{ex}^2 in Eq. (9) and obtain the approximate expression

$$k_{ex} \approx \rho \approx \frac{k_{12}k_{21} - \Delta_1\Delta_2}{\Delta_1 + \Delta_2} \quad . \tag{10}$$

A more useful approximate expression for the reactivity is obtained by expressing it in terms of the deviations from the critical values,

$$\rho \approx \frac{\Delta_1\Delta_2}{\Delta_1 + \Delta_2}\left(-\frac{\delta\Delta_1}{\Delta_1} - \frac{\delta\Delta_2}{\Delta_2} + \frac{\delta k_{12}}{k_{12}} + \frac{\delta k_{21}}{k_{21}}\right) \quad . \tag{11}$$

A straightforward generalization to N reactors gives

$$\begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ k_{N1} & k_{N2} & & k_{NN} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_N \end{pmatrix} = k \begin{pmatrix} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_N \end{pmatrix} \tag{12}$$

which defines the reactivity of a non-critical system and which gives for the criticality condition

$$\left| k_{ij} - \delta_{ij} \right| = 0, \tag{13}$$

where δ_{ij} is the Kronecker delta function. The power ratios can be determined from Eq. (12) and at criticality the solution must be such that no negative values for the S_i result.

We now consider a general hypothetical problem. Assume in the two reactor case that in each of the reactors the value of ν is changed by amount $\delta\nu_1$ in reactor 1 and $\delta\nu_2$ in reactor 2. With the corresponding changes in the k_{ij}, it can be shown from Eq. (11) that

$$\rho \approx \frac{\frac{1}{\Delta_1}}{\frac{1}{\Delta_1} + \frac{1}{\Delta_2}} \frac{\delta\nu_1}{\nu_1} + \frac{\frac{1}{\Delta_2}}{\frac{1}{\Delta_1} + \frac{1}{\Delta_2}} \frac{\delta\nu_2}{\nu_2} \quad . \tag{14}$$

We can generalize the above result to N coupled reactors and find that

$$\rho \approx \sum_{i=1}^{N} \alpha_i \frac{\delta\nu_i}{\nu_i} \quad , \tag{15}$$

where

$$\alpha_i = \frac{\frac{1}{\Delta_i}}{\sum_{j=1}^{N} \frac{1}{\Delta_j}} \quad . \tag{16}$$

The previous considerations lead to the concept of division of reactivity. α_i, the fraction of reactivity in reactor i, is defined as the ratio of the over-all reactivity to $\delta\nu_i/\nu_i$, where the reactivity results from the change of ν_i to $\nu_i + \delta\nu_i$.

KINETICS EQUATIONS

In terms of the previously defined integral parameters the following equations describe the kinetic behavior of the coupled system:

$$\ell_{jk} \frac{dS_{jk}}{dt} = k_{jk} (1 - \beta) \sum_{m=1}^{N} S_{km} - S_{jk} + k_{jk} \sum_{i=1}^{D} \lambda_i C_{ki} \tag{17a}$$

$$\frac{dC_{ki}}{dt} = \beta_i \sum_{m=1}^{N} S_{km} - \lambda_i C_{ki} \quad . \tag{17b}$$

In Eqs. (17), β is the total effective delayed neutron fraction, β_i and λ_i are the effective delayed neutron fraction and decay constant respectively, of the i^{th} delayed neutron precursor, D is the total number of such delayed neutron precursor types, and C_{ki} is a properly weighted measure of the number of delayed neutron emitters of the i^{th} type in reactor k.

For simplicity we have assumed not the most general characteristics for the delayed neutrons, but that the β, β_i, λ_i are all independent of the reactor and also that the k_{jk} and ℓ_{jk} for delayed neutrons are the same as for prompt neutrons. If these assumptions are not made, the only consequence is that there are additional variables and the notation is more involved, but no basic complication results.

Before further discussing the form of the kinetics equations, we introduce the redundant variables N_{jk}, defined by

$$S_{jk} \equiv \frac{N_{jk}}{\ell_{jk}} \quad . \tag{18}$$

We do this only to aid in the exposition, since now the equations will be put in a form that is close in analogy to the usual kinetics equations. Using N_{jk}, instead of S_{jk}, Eqs. (17) become

$$\frac{dN_{jk}}{dt} = k_{jk} (1 - \beta) \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}} - \frac{N_{jk}}{\ell_{jk}} + k_{jk} \sum_{i=1}^{D} \lambda_i C_{ki} \tag{19a}$$

$$\frac{dC_{ki}}{dt} = \beta_i \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}} - \lambda_i C_{ki} \quad . \tag{19b}$$

N_{jk} may be thought of as a quantity similar to a neutron density. It serves as a measure of the number of neutrons, properly weighted, in the system which were born in reactor k and are destined to produce next generation neutrons in reactor j.

The physical arguments leading to Eqs. (19) are the usual ones for kinetics equations. dN_{jk}/dt is given by the difference in production and loss rates for N_{jk} type neutrons. The production rate consists of two terms; production by prompt neutron emission and production by delayed neutron emission. $S_k = \sum_{m=1}^{N} S_{km} = \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}}$ represents the total number of source neutrons in reactor k of which $(1-\beta) S_k$ are prompt. Therefore, $k_{jk} (1-\beta) S_k$, the first term on the right hand side of Eq. (19a) is the production rate of the N_{jk} type neutrons by prompt neutron emission. βS_k represents the total number of delayed source neutrons in reactor k, and of these $\beta_i S_k$ are of the i^{th} precursor type, so that this term represents the production rate term in Eq. (19b) for C_{ki}. The i^{th} precursor decays with a decay constant λ_i, so that $\lambda_i C_{ki}$ represents the loss rate for C_{ki}. The difference between production and loss rates for C_{ki} gives dC_{ki}/dt from which Eq. (19b) follows. The total number of delayed source neutrons in reactor k is $\sum_{i=1}^{D} \lambda_i C_{ki}$, so that $k_{jk} \sum_{i=1}^{D} \lambda_i C_{ki}$, the last term on the right hand side of Eq. (19a), is the production rate of N_{jk} type neutrons by delayed neutron emission. Finally N_{jk}/ℓ_{jk} represents the loss rate of N_{jk} type neutrons from which Eq. (19a) follows.

The reason the coupled equations are as complicated as they are, and the reason it was necessary to introduce the partial sources, S_{jk}, or the related N_{jk}, results from the fact that there is no correlation between the branching ratios, k_{jk}, that a neutron born in reactor k may take in giving rise to fission neutrons in the various final reactors, j = 1,2, . . .N, and the lifetimes, ℓ_{jk}, that it takes to do so. If the various k_{jk} were inversely proportional to the ℓ_{jk}, as is usually the case when there are competing modes of decay, then a more compact description of the neutron kinetic behavior would have been possible.

For the two reactor case, Eqs. (19) become,

$$\frac{dN_{11}}{dt} = k_{11}\,(1-\beta)\left(\frac{N_{11}}{\ell_{11}} + \frac{N_{12}}{\ell_{12}}\right) - \frac{N_{11}}{\ell_{11}} + k_{11}\sum_{i=1}^{D} \lambda_i C_{1i} \tag{20a}$$

$$\frac{dN_{21}}{dt} = k_{21}\,(1-\beta)\left(\frac{N_{11}}{\ell_{11}} + \frac{N_{12}}{\ell_{12}}\right) - \frac{N_{21}}{\ell_{21}} + k_{21}\sum_{i=1}^{D} \lambda_i C_{1i} \tag{20b}$$

$$\frac{dN_{12}}{dt} = k_{12}\,(1-\beta)\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right) - \frac{N_{12}}{\ell_{12}} + k_{12}\sum_{i=1}^{D} \lambda_i C_{2i} \tag{20c}$$

$$\frac{dN_{22}}{dt} = k_{22}\,(1-\beta)\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right) - \frac{N_{22}}{\ell_{22}} + k_{22}\sum_{i=1}^{D} \lambda_i C_{2i} \tag{20d}$$

$$\frac{dC_{1i}}{dt} = \beta_i\left(\frac{N_{11}}{\ell_{11}} + \frac{N_{12}}{\ell_{12}}\right) - \lambda_i C_{1i} \tag{20e}$$

$$\frac{dC_{2i}}{dt} = \beta_i\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right) - \lambda_i C_{2i} \quad . \tag{20f}$$

The usual procedure for solving the coupled kinetics equations is to assume that the k_{jk}, but not the ℓ_{jk}, may be time dependent, and then to solve for the time dependence of the N_{jk} and C_{ki}. We could assume that the ℓ_{jk} are also time dependent, but just as one ordinarily does not assume in the usual kinetics equations that the prompt lifetime, ℓ, is time dependent, so we will usually assume constant ℓ_{jk}. The time dependence of the k_{jk} can be given explicitly and/or as some function or functional of the various N_{jk}. The latter is fully analogous to the dependence of k on n in the usual kinetics equations. We expect that in the coupled case the equations will generally have to be solved numerically, although we are able to obtain some results from analytic considerations.

ADJOINT FORMULATION

We now consider the equations adjoint to the coupled kinetics equations. For simplicity we assume no delayed neutrons. For the uses that we will make of the adjoint formulation, this assumption will in no way limit us. For the case of all neutrons prompt we obtain for the coupled kinetics equation.

$$\frac{dN_{jk}}{dt} = k_{jk} \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}} - \frac{N_{jk}}{\ell_{jk}} \quad . \tag{21}$$

The equations adjoint to Eq. (21) are

$$\frac{dN_{jk}^{+}}{dt} = \sum_{m=1}^{N} k_{mj} \frac{N_{mj}^{+}}{\ell_{jk}} - \frac{N_{jk}^{+}}{\ell_{jk}} \quad . \tag{22}$$

For the two reactor case Eq. (22) becomes

$$\frac{dN_{11}^{+}}{dt} = k_{11} \frac{N_{11}^{+}}{\ell_{11}} + k_{21} \frac{N_{21}^{+}}{\ell_{11}} - \frac{N_{11}^{+}}{\ell_{11}} \tag{23a}$$

$$\frac{dN_{21}^{+}}{dt} = k_{12} \frac{N_{12}^{+}}{\ell_{21}} + k_{22} \frac{N_{22}^{+}}{\ell_{21}} - \frac{N_{21}^{+}}{\ell_{21}} \tag{23b}$$

$$\frac{dN_{12}^{+}}{dt} = k_{11} \frac{N_{11}^{+}}{\ell_{12}} + k_{21} \frac{N_{21}^{+}}{\ell_{12}} - \frac{N_{12}^{+}}{\ell_{12}} \tag{23c}$$

$$\frac{dN_{22}^{+}}{dt} = k_{12} \frac{N_{12}^{+}}{\ell_{22}} + k_{22} \frac{N_{22}^{+}}{\ell_{22}} - \frac{N_{22}^{+}}{\ell_{22}} \quad . \tag{23d}$$

We can solve for the ratios of the steady state values of the N_{jk}^{+} in this case and obtain

$$N_{11}^{+} = N_{12}^{+} \equiv N_{1}^{+} \tag{24a}$$

$$N_{21}^{+} = N_{22}^{+} \equiv N_{2}^{+} \tag{24b}$$

and in general

$$N_{jk}^{+} = N_{j\ell}^{+} \equiv N_{j}^{+}$$

$$N_1^+ = \frac{k_{21}}{\Delta_1} N_2^+ = \frac{\Delta_2}{k_{12}} N_2^+ \tag{24c}$$

$$\frac{N_{11}^+ S_{11} + N_{12}^+ S_{12}}{N_{21}^+ S_{21} + N_{22}^+ S_{22}} = \frac{N_1^+ S_1}{N_2^+ S_2} = \frac{\Delta_2}{\Delta_1} \quad . \tag{24d}$$

α_i, given by Eq. (16), reduces for the two reactor case to

$$\alpha_i = \frac{\frac{1}{\Delta_i}}{\frac{1}{\Delta_1} + \frac{1}{\Delta_2}} \quad . \tag{25}$$

α_i may then also be expressed by

$$\alpha_i = \frac{N_i^+ S_i}{N_1^+ S_1 + N_2^+ S_2} \quad . \tag{26}$$

From the definition of α_i an alternate expression for its value can be obtained. We consider the usual perturbation formula where only the value of ν is perturbed. The resulting reactivity change is given by

$$\rho \approx \frac{\int \chi(v') \, \phi^+(r,v') \frac{\delta\nu}{\nu} \nu\sigma_f(r,v) \, \phi(r,v) \, dr \, dv \, dv'}{\int \chi(v') \phi^+(r,v') \, \nu\sigma_f(r,v) \, \phi(r,v) \, dr \, dv \, dv'} \quad ,$$

where $\chi(v')$ is the normalized fission spectrum, $\phi(r,v)$ is the neutron flux, $\phi^+(r,v')$ is the adjoint function, and $\sigma_f(r,v)$ is the macroscopic fission cross section.

Assume that $\delta\nu/\nu$ is non-vanishing only over reactor i, and that it is there constant, so that it can be taken outside the integral. Therefore,

$$\rho \approx \frac{\delta\nu_i}{\nu_i} \, \frac{\int_{\text{reactor } i} \chi(v') \, \phi^+(r,v') \, \nu\sigma_f(r,v) \, \phi(r,v) \, dr \, dv \, dv'}{\int_{\text{entire system}} \chi(v') \phi^+(r,v') \, \nu\sigma_f(r,v) \, \phi(r,v) \, dr \, dv \, dv'} \quad .$$

We see from Eqs. (15) and (26) that α_i is given by the coefficient of $\delta\nu_i/\nu_i$ in Eq. (26), and is equal to the fraction of the importance production rate due to all fission source neutrons born in reactor i. By comparison with Eq. (26) the interpretation of N_i^+ as the average importance of a fission neutron born in reactor i follows.

The rate at which importance is removed from or born into the j, k^{th} region, and also the entire system, may be given in terms of the S_j, S_{jk}, and N_k^+.

Rate at which importance removed from j, k^{th} region

$$= N_{jk}^+ S_{jk} = N_j^+ S_{jk} \quad . \tag{27}$$

Rate at which importance removed from entire system

$$= \sum_{j,k} N_{jk}^+ S_{jk} = \sum_j N_j^+ S_j \quad . \tag{28}$$

Rate at which importance born into j, k^{th} region

$$= k_{jk} N_{jk}^+ S_k = k_{jk} N_j^+ S_k \quad . \tag{29}$$

Rate at which importance born into entire system

$$= \sum_{j,k} k_{jk} N_j^+ S_k \quad . \tag{30}$$

Using the steady state ratios at criticality it can easily be shown that values from Eqs. (27) and (28) are identical to those of Eqs. (29) and (30); i.e., at steady state the ratio of the rate of production of importance to the rate of removal of importance is unity.

The amount of importance in the j, k^{th} region; i.e., the total amount of importance carried by all the neutrons in the system at any time which were born in reactor k and which are going to cause fissions in reactor j, is obtained from the product of the rate at which importance is removed from (or born into) the j, k^{th} reactor and the lifetime for neutrons in the j, k^{th} region. We then have

Amount of importance in j, k^{th} region

$$= N_j^+ S_{jk} \ell_{jk} = N_j^+ N_{jk} \quad . \tag{31}$$

Amount of importance in entire system

$$= \sum_{j,k} N_j^+ N_{jk} \quad . \tag{32}$$

We now develop a perturbation formula which relates the change in reactivity with a change from the critical values of the various k_{ij}. We have already obtained this result from previous considerations. The result is re-derived as a means of illustration of the significance of the adjoint formulation. We assume that the k_{ij} are perturbed by an amount δk_{ij} and that the steady state is maintained by means of a fictitious change in ν of amount $\delta\nu$, where $\delta\nu/\nu = -\rho$. As a consequence of these perturbations the N_{jk} are themselves perturbed by an amount δN_{jk}. The resulting steady state equations are,

$$(k_{jk} + \delta k_{jk})\,(1-\rho) \sum_{m=1}^{N} \frac{(N_{km} + \delta N_{km})}{\ell_{km}} - \frac{(N_{jk} + \delta N_{jk})}{\ell_{jk}} = 0. \tag{33}$$

The unperturbed adjoint steady state equations are

$$\sum_{m=1}^{N} k_{mj} \frac{N^{+}_{mj}}{\ell_{jk}} - \frac{N^{+}_{jk}}{\ell_{jk}} = 0 \quad . \tag{34}$$

We now apply the usual techniques of perturbation theory. We multiply the j, k^th real perturbed equation by N^{+}_{jk} and sum over all j, k. From this we subtract the sum over all j, k of the product of the j, k^th unperturbed adjoint equation and $(N_{jk} + \delta N_{jk})$, yielding

$$\sum_{j,k} N^{+}_{jk} \left\{ (k_{jk} + \delta k_{jk})\,(1-\rho) \sum_{m=1}^{N} \frac{(N_{km} + \delta N_{km})}{\ell_{km}} - \frac{(N_{jk} + \delta N_{jk})}{\ell_{jk}} \right\}$$

$$- \sum_{j,k} (N_{jk} + \delta N_{jk}) \left\{ \sum_{m=1}^{N} k_{mj} \frac{N^{+}_{mj}}{\ell_{jk}} - \frac{N^{+}_{jk}}{\ell_{jk}} \right\} = 0 \quad .$$

The zero^th order terms; i.e., terms not involving any δ terms, cancel out because of the steady state condition of the unperturbed system. We ignore all terms higher than the first order. For the first order terms all terms involving δN_{jk} cancel out identically. We then obtain

$$\rho = \frac{\sum_{j,k} \delta k_{jk} N^{+}_{jk} \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}}}{\sum_{j,k} k_{jk} N^{+}_{jk} \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}}} = \frac{\sum_{j,k} \delta k_{jk} N^{+}_{j} S_{k}}{\sum_{j,k} k_{jk} N^{+}_{j} S_{k}} \quad . \tag{35}$$

For the two reactor case this becomes

$$\rho = \frac{\delta k_{11} N_{11}^{+}\left(\frac{N_{11}}{\ell_{11}} + \frac{N_{12}}{\ell_{12}}\right) + \delta k_{21} N_{21}^{+}\left(\frac{N_{11}}{\ell_{11}} + \frac{N_{12}}{\ell_{12}}\right) + \delta k_{12} N_{12}^{+}\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right) + \delta k_{22} N_{22}^{+}\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right)}{k_{11} N_{11}^{+}\left(\frac{N_{11}}{\ell_{11}} + \frac{N_{12}}{\ell_{12}}\right) + k_{21} N_{21}^{+}\left(\frac{N_{11}}{\ell_{11}} + \frac{N_{12}}{\ell_{12}}\right) + k_{12} N_{12}^{+}\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right) + k_{22} N_{22}^{+}\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right)}, \quad (36)$$

which on substitution of the steady state ratios reduces to the result obtained previously in Eq. (11).

NEUTRON LIFETIME

The neutron lifetime, ℓ, serves as a measure of the average time between successive fission events caused by prompt neutrons. It is defined[1] as the ratio of the total neutron importance in the system to the rate at which importance is removed and in terms of the coupling parameters is given by

$$\ell = \frac{\sum_{j,k} N_{jk}^{+} N_{jk}}{\sum_{j,k} N_{jk}^{+} S_{jk}} = \frac{\sum_{j,k} N_{jk}^{+} S_{jk} \ell_{jk}}{\sum_{j,k} N_{jk}^{+} S_{jk}} . \quad (37)$$

Since the relative steady state values of the various N_{jk}^{+} and also the relative steady state values of the S_{jk} do not involve the various ℓ_{jk}, Eq. (37) gives the neutron lifetime as a linear combination of the partial lifetimes.

For the two reactor case the lifetime, ℓ, as given by Eq. (37) and the previously obtained steady state ratios, becomes

$$\ell = \frac{\Delta_2 k_{11}}{\Delta_1 + \Delta_2} \ell_{11} + \frac{\Delta_1 k_{22}}{\Delta_1 + \Delta_2} \ell_{22} + \frac{\Delta_1 \Delta_2}{\Delta_1 + \Delta_2} (\ell_{12} + \ell_{21}) . \quad (38)$$

It is clear from the definition of the lifetime that it is not the average time between fissions, since there is a non-constant weighting factor attached to intervals between fissions which depends on the importance and thus on location of the fission events. We can explicitly demonstrate this by giving the value for the average time between fissions for the two reactor case. For this average the weighting factor for ℓ_{jk} is

$$\frac{S_{jk}}{\sum_{j,k} S_{jk}}$$

which then gives for the average time between fissions, $\bar{\ell}$,

$$\bar{\ell} = \frac{\Delta_2 k_{11}}{\Delta_2 + k_{21}} \ell_{11} + \frac{\Delta_1 \Delta_2}{\Delta_1 + k_{12}} \ell_{21} + \frac{\Delta_1 \Delta_2}{\Delta_2 + k_{21}} \ell_{12} + \frac{\Delta_1 k_{22}}{\Delta_1 + k_{12}} \ell_{22} \quad . \tag{39}$$

In the expression for the lifetime in the two reactor case, Eq. (38), it may be seen that the contributions of the two cross lifetimes, ℓ_{12} and ℓ_{21} enter with the same weighting factor even though there are in general a different number of fissions associated with each lifetime, as may be seen from Eq. (39). This is a consequence of the fact that the relative importance for each type event is such as to maintain an equal total importance production for neutrons going from reactor 1 to 2 as for neutrons going from reactor 2 to 1.

An alternative definition of the neutron lifetime, but which is equivalent to the ratio of total importance to importance rate removal, considers an infinitely small perturbation in the value of ν of amount $\delta\nu$. Assuming all neutrons are prompt, the system will asymptotically approach an exponential time behavior, $e^{\omega t}$. The neutron lifetime is then defined by

$$\ell = \left| \frac{\frac{\delta \nu}{\nu}}{\omega} \right|_{\lim \delta\nu,\ \omega \to 0} \tag{40}$$

The definition of Eq. (40) can be used as the basis for another derivation for the expression for the neutron lifetime in the coupled formalism.

We use a method analogous to that ordinarily used to evaluate the perturbation theory expression for the lifetime in the usual kinetics formulation. Two separate eigenvalue problems are considered. The first, already considered, has as the eigenvalue the number of neutrons emitted per fission. One perturbs the system, i.e., the values of the k_{ij}, and determines the fictitious change in ν, $\delta\nu$, necessary to maintain criticality. The reactivity, $\rho = -\delta\nu/\nu$, corresponding to the perturbation is thus determined. The result of this analysis is given by Eq. (35). The other eigenvalue problem which we now consider deals with a fixed value of ν and considers the time constant, ω, corresponding to an exponential time behavior, $e^{\omega t}$, as the eigenvalue. The system is perturbed again by changing the values of the k_{ij}, and the change in ω is determined. The expression for ω contains quantities that can be recognized as the reactivity from the first eigenvalue problem. The lifetime, ℓ, is then determined by the relation, $\ell = \rho/\omega$. Proceeding, the perturbed kinetics equations with the time constant as the eigenvalue are

$$(k_{jk} + \delta k_{jk}) \sum_{m=1}^{N} \frac{(N_{km} + \delta N_{km})}{\ell_{km}} - \frac{(N_{jk} + \delta N_{jk})}{\ell_{jk}} = \omega (N_{jk} + \delta N_{jk}). \tag{41}$$

We consider along with Eq. (41) the unperturbed adjoint steady state equations, Eq. (34), and then form the usual combination of

$$\sum_{j,k} N_{jk}^{+} \left\{ (k_{jk} + \delta k_{jk}) \sum_{m=1}^{N} \frac{(N_{km} + \delta N_{km})}{\ell_{km}} - \frac{(N_{jk} + \delta N_{jk})}{\ell_{jk}} - \omega (N_{jk} + \delta N_{jk}) \right\}$$

$$- \sum_{j,k} (N_{jk} + \delta N_{jk}) \left\{ \sum_{m=1}^{N} \frac{k_{mj} N_{mj}^{+}}{\ell_{jk}} - \frac{N_{jk}^{+}}{\ell_{jk}} \right\} = 0 \quad .$$

The zero order terms cancel out and we ignore higher order terms. The first order terms involving the δN_{jk} cancel out. We obtain

$$\omega = \frac{\sum_{j,k} \delta k_{jk} N_{jk}^{+} \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}}}{\sum_{j,k} N_{jk}^{+} N_{jk}} \quad ,$$

which along with Eq. (35) gives

$$\ell = \frac{\rho}{\omega} = \frac{\sum_{j,k} N_{jk}^{+} N_{jk}}{\sum_{j,k} k_{jk} N_{jk}^{+} \sum_{m=1}^{N} \frac{N_{km}}{\ell_{km}}} = \frac{\sum_{j,k} N_{j}^{+} S_{jk} \ \ell_{jk}}{\sum_{j,k} k_{jk} N_{j}^{+} S_{k}} \quad . \tag{42}$$

The expression for the lifetime in Eq. (42) is consistent with that of Eq. (37), since at criticality the rate of production of importance, the denominator in Eq. (42), and the rate of removal of importance, the denominator in Eq. (37), are equal.

INHOUR EQUATION

We consider the problem of obtaining for the two reactor case the equivalent of the inhour equation, i.e., the equation relating the possible time constants with the parameters characterizing the system. We assume exponential solutions

$$N_{jk}(t) = N_{jk}^{0} e^{\omega t}; \ C_{ji} = C_{ji}^{0} e^{\omega t}$$

and substitute into Eqs. (20). The condition that the determinant of the matrix of coefficients vanish yields after some algebra

$$\left(\Delta_1 + \omega \ell_{11} + \omega k_{11} \sum_{i=1}^{D} \frac{\beta_i}{\omega + \lambda_i} \right) \left(\Delta_2 + \omega \ell_{22} + \omega k_{22} \sum_{i=1}^{D} \frac{\beta_i}{\omega + \lambda_i} \right)$$

$$(1 + \omega \ell_{12})\,(1 + \omega \ell_{21}) = k_{12}\, k_{21}\, (1 + \omega \ell_{11})\,(1 + \omega \ell_{22}) \left(1 - \omega \sum_{i=1}^{D} \frac{\beta_i}{\omega + \lambda_i} \right)^2 , \tag{43}$$

which is the desired equation.

If we assume that $\omega \ell << \Delta_1, \Delta_2$, i.e., take the limit for very long periods, so that terms of the order of $\omega \ell$ or higher can be neglected, the results yield the correct value of the neutron lifetime. In this approximation Eq. (43) reduces to

$$\omega \left[\frac{\Delta_2 k_{11}}{\Delta_1 + \Delta_2} \ell_{11} + \frac{\Delta_1 k_{22}}{\Delta_1 + \Delta_2} \ell_{22} + \frac{\Delta_1 \Delta_2}{\Delta_1 + \Delta_2} (\ell_{12} + \ell_{21}) + \sum_{i=1}^{D} \frac{\beta_i}{\lambda_i} + 2\rho \sum_{i=1}^{D} \frac{\beta_i}{\lambda_i} \right]$$

$$= \frac{k_{12} k_{21} - \Delta_1 \Delta_2}{\Delta_1 + \Delta_2} . \tag{44}$$

When the assumptions of this derivation are satisfied

$$\frac{k_{12}\, k_{21} - \Delta_1 \Delta_2}{\Delta_1 + \Delta_2} \approx \rho .$$

Neglecting the higher order last term in the brackets on the left hand side of Eq. (44) we obtain

$$\omega \left(\ell + \sum_{i=1}^{D} \frac{\beta_i}{\lambda_i} \right) \approx \rho , \tag{45}$$

where ℓ has the same form as in Eq. (38). The relation of Eq. (45) is, of course, the one expected for very small reactivities.

PROMPT JUMP

The prompt jumps in neutron densities under step changes in the coupling parameters, k_{ij}, can be studied analytically with the aid of the coupled kinetics equations. We consider the situation when the step change in reactivity does not exceed the amount required to go prompt critical and determine the source strength in each region a short time after the change is made. By a short time one means a time that is short compared to the delayed neutron periods, but long compared to the lifetime of a prompt neutron chain. If we define S_1^0 and S_2^0 as the initial values of S_1 and S_2, and S_1^F and S_2^F as the values after the prompt jump, then

$$S_1^F = \frac{k_{11}\,\beta\, S_1^0 + k_{12}\,[(1-\beta)\, S_2^F + \beta\, S_2^0]}{1 - k_{11}\,(1-\beta)} \tag{46a}$$

$$S_2^F = \frac{k_{21}\,[(1-\beta)\, S_1^F + \beta\, S_1^0] + k_{22}\,\beta S_2^0}{1 - k_{22}\,(1-\beta)}\quad , \tag{46b}$$

where all the k_{ij} are the values after the step change and where,

$$\beta S_1^0 = \sum_{i=1}^{D} \lambda_i\, C_{1i}^0$$

$$\beta\, S_2^0 = \sum_{i=1}^{D} \lambda_i\, C_{2i}^0$$

can be evaluated from the initial values of the k_{ij}.

The arguments leading to these expressions are similar to those that can be used in connection with the prompt jump in the normal kinetics equations; i.e., that the fission neutron level is determined by the product of the number of delayed neutrons coming into the system and the prompt multiplication. In the coupled case the argument is generalized in the following manner. The fission neutron level in each reactor is given by the product of the number of neutrons coming into the reactor, either as delays originating in the reactor or as neutrons originating in the other reactor, and the prompt multiplication of the reactor. The assumption in all cases, i.e., the normal or the coupled case, is that the rate of delayed neutrons emitted is unchanged immediately after the step change. We can solve for the unknowns S_1^F, S_2^F from Eqs. (46). S_1^F and S_2^F can be divided into the partial sources S_{jk}^F,

$$S_{11}^F = \frac{k_{11}\,\beta S_1^0 + k_{11}\,(1-\beta)\, k_{12}\,[(1-\beta)\, S_2^F + \beta S_2^0]}{1 - k_{11}\,(1-\beta)} \tag{47a}$$

$$S_{21}^F = k_{21}\,[(1-\beta)\, S_1^F + \beta\, S_1^0] \tag{47b}$$

$$S_{12}^F = k_{12}\, 1(1-\beta)\, S_2^F + \beta\, S_2^0] \tag{47c}$$

$$S^F_{22} = \frac{k_{22}\,(1-\beta)\,k_{21}\,[(1-\beta)\,S^F_1 + \beta\,S^0_1] + k_{22}\beta\,S^0_2}{1 - k_{22}\,(1-\beta)} \quad . \tag{47d}$$

CORRELATION WITH GENERAL FORMULATION

We now consider the correlation of the coupled formalism that has been developed with a general formulation of the reactor equations. In so doing, we determine the exact definitions of the various coupling parameters and how to explicitly evaluate them.

We first consider the steady state. In the general formulation we make use of the following quantities:

$\phi(r,v)$, the neutron flux as a function of position and velocity

$\phi^+(r,v)$, the function adjoint to the neutron flux (the adjoint flux)

$\nu\sigma_f(r,v)$, the product of the average number of neutrons emitted per fission and the macroscopic fission cross section (the product may be a function of position and of incoming neutron velocity)

$[\nu\sigma_f(r,v)]_j$, the part of $\nu\sigma_f(r,v)$ which prescribes the reactor j; and

$$\sum_j [\nu\sigma_f(r,v)]_j = \nu\sigma_f(r,v)$$

$\chi(v)$, the fission spectrum; $\int\chi(v)\,dv = 1$

$\phi_j(r,v)$, for this quantity we make use of concepts related to the source iterative technique(2) of solution of the reactor equations. To determine the critical flux distribution one uses as the fission neutron source

$$\chi(v)\int\nu\sigma_f(r,v')\,\phi(r,v')\,dv'.$$

One then determines the resulting neutron flux, treating fission events as removal events. After the flux has been completely determined one can then evaluate a new fission neutron source, which after the method has converged and the correct flux obtained, agrees with the initial fission neutron source. $\phi_j(r,v)$ is the resultant single iteration flux from a source

$$\chi(v)\int[\nu\sigma_f(r,v')]_j\,\phi(r,v')\,dv'.$$

It is the part of the steady state flux that results from neutrons born in reactor j; and

$$\sum_j \phi_j(r,v) = \phi(r,v).$$

$\phi_j^+(r,v)$, a similar iterative technique, can be used to solve the adjoint problem. In this case the "source" is $\nu\sigma_f(r,v) \int \chi(v') \; \phi^+(r,v')\,dv'$.

$\phi_j^+(r,v)$ is the resultant adjoint flux from a "source" $[\nu\sigma_f(r,v)]_j \int \chi(v') \; \phi^+(r,v')\,dv'$;

and

$\sum_j \phi_j^+(r,v) = \phi^+(r,v)$.

As is well known $\phi^+(r,v)$ can be interpreted as the importance function; i.e., $\phi^+(r,v)$ serves as a measure of the extent to which a neutron at r with velocity v ultimately contributes to the maintenance of the chain reaction. $\phi_j^+(r,v)$ can be interpreted as the part of the importance function which is contributed by neutrons which will cause their next fission in reactor j.

In correlating the coupling parameters with the general quantities just defined, we begin with the definition that S_j is equal to the total fission neutron source in reactor j,

$$S_j = \int [\nu\sigma_f(r,v)]_j \; \phi(r,v)\;dr\;dv. \tag{48}$$

Analogously we have for the delayed neutron emitters,

$$C_{ji} = \frac{\beta_i}{\lambda_i} \int [\nu\sigma_f(r,v)]_j \; \phi(r,v)\;dr\;dv. \tag{49}$$

In dividing the total source in reactor j into the partial sources, S_{jk}, arising from neutrons originating in the various other reactors, we do so by taking the fraction of the total importance arising from fission neutrons in the reactor j which results from neutrons from the various other reactors. Thus,

$$S_{jk} = \frac{\int [\nu\sigma_f(r,v)]_j \; \phi(r,v)\;dr\;dv \int \chi(v') \; \phi^+(r,v') [\nu\sigma_f(r,v)]_j \; \phi_k(r,v)\;dr\;dv\;dv'}{\int \chi(v') \; \phi^+(r,v') \, [\nu\sigma_f(r,v)]_j \, \phi(r,v)\;dr\;dv\;dv'} \tag{50}$$

Because of the definition used in Eq. (50) we are able to assign the same average importance to all the partial sources in reactor j; i.e., $N_{jk}^+ = N_{j\ell}^+ \equiv N_j^+$.

At the steady state

$$k_{jk} = \frac{S_{jk}}{S_k}$$

so that

$$k_{jk} = \frac{\int [\nu\sigma_f(r,v)]_j \; \phi(r,v) \; dr \; dv \int \chi(v') \; \phi^+(r,v')[\nu\sigma_f(r,v)]_j \phi_k(r,v) \, dr \; dv \; dv'}{\int [\nu\sigma_f(r,v)]_k \, \phi(r,v) \; dr \; dv \int \chi(v') \; \phi^+(r,v')[\nu\sigma_f(r,v)]_j \phi(r,v) \; dr \; dv \; dv'} \quad . \tag{51}$$

$N_j^+ \; S_{jk}$ is to be interpreted as the total importance of the fission neutrons in reactor j which result from neutrons born in reactor k. Therefore,

$$N_j^+ \; S_{jk} = \int \chi(v') \; \phi^+(r,v') \; [\nu\sigma_f(r,v)]_j \; \phi_k(r,v) \; dr \; dv \; dv'$$

and thus

$$N_j^+ = \frac{\int \chi(v') \; \phi^+(r,v') \, [\nu\sigma_f(r,v)]_j \; \phi(r,v) \; dr \; dv \; dv'}{\int [\nu\sigma_f(r,v)]_j \; \phi(r,v) \; dr \; dv \; dv'} \quad . \tag{52}$$

We see that N_j^+ just represent the average importance of all fission neutrons born in reactor j.

The over-all neutron lifetime, ℓ, is given by the well known expression

$$\ell = \frac{\int \frac{\phi^+(r,v) \, \phi(r,v) \; dr \; dv}{v}}{\int \chi(v') \; \phi^+(r,v') \; \nu\sigma_f(r,v) \, \phi(r,v) \; dr \; dv \; dv'} \quad ,$$

which may also be written in the form

$$\ell = \frac{\sum_{j,k} \int \frac{\phi_j^+(r,v) \phi_k(r,v) \; dr \; dv}{v}}{\int \chi(v') \; \phi^+(r,v) \; \nu\sigma_f(r,v) \, \phi(r,v) \; dr \; dv \; dv'} \quad . \tag{53}$$

In terms of the coupled formalism, ℓ is given by Eq. (37). The denominators of Eqs. (37) and (53) are equivalent. Recalling the interpretations of $\phi_j^+(r,v)$ and $\phi_k(r,v)$, we associate each of the j,k terms in the numerator of Eq. (53) with the corresponding j,k term of the numerator of Eq. (37) and obtain

$$\ell_{jk} = \frac{\int \frac{\phi_j^+(r,v) \, \phi_k(r,v) \; dr \; dv}{v}}{\int \chi(v') \; \phi^+(r,v') \; [\nu\sigma_f(r,v)]_j \, \phi_k(r,v) \; dr \; dv \; dv'} \tag{54}$$

from which it also follows that

$$N_{jk} = S_{jk}\, \ell_{jk} = \frac{\int [\nu\sigma_f(r,v)]_j \phi(r,v)\, dr\, dv \int \frac{\phi_j^+(r,v)\, \phi_k(r,v)\, dr\, dv}{v}}{\int \chi(v')\, \phi^+(r,v')\, [\nu\sigma_f(r,v)]_j\, \phi(r,v)\, dr\, dv\, dv'} \quad . \tag{55}$$

In order to determine the k_{jk} for a non-critical configuration, we first consider the steady state problem with a fictitious value of number of neutrons emitted per fission, ν_c. We use the fluxes as they result from the problem, evaluate the k_{jk} in the usual manner and obtain the desired results by multiplying the values by ν/ν_c.

APPLICATION TO FAST-THERMAL SYSTEM

The preceding formalism was developed in connection with a study of coupled fast-thermal systems, particularly as they may be applied to nuclear power breeders.(3,4) We use this system to give some illustrations of the application of the formalism.

Useful qualitative considerations may be made from the power ratio given by Eq. (7). In the fast-thermal system one wants the ratio of fast to slow power, S_F/S_S, large for breeding gain purposes and one also wants the subcriticality of the fast part, Δ_F, substantial for safety purposes. It is clear from consideration of

$$\frac{S_F}{S_S} \doteq \frac{k_{FS}}{\Delta_F}$$

that it is important to make k_{FS} as large as possible in order to satisfy both objectives. In practice, obtaining a large value of k_{FS} will have to be achieved subject to a competing requirement of maintaining a barrier between the fast and thermal parts which keeps the neutron spectrum high in the fast part.

As an illustration of the use of Eq. (11) we consider a change in the thermal utilization of the thermal reactor and we wish to determine the resultant change of reactivity. In the change of thermal utilization only Δ_S and k_{SF} are changed to first order,

$$\frac{\delta k_{SS}}{k_{SS}} = \frac{\delta k_{SF}}{k_{SF}} = \frac{\delta \eta_f}{\eta_f} \quad .$$

Therefore,

$$\rho \approx \frac{\Delta_F \Delta_S}{\Delta_F + \Delta_S} \left(- \frac{\delta \Delta_S}{\Delta_S} + \frac{\delta k_{SF}}{k_{SF}} \right)$$

or

$$\rho \approx \frac{\frac{1}{\Delta_S}}{\frac{1}{\Delta_S} + \frac{1}{\Delta_F}} \frac{\delta\eta_f}{\eta_f} \quad . \tag{56}$$

If we apply Eq. (38) to the fast-thermal system we obtain

$$\ell = \frac{\Delta_S k_{FF}}{\Delta_F + \Delta_S} \ell_{FF} + \frac{\Delta_F k_{SS}}{\Delta_F + \Delta_S} \ell_{SS} + \frac{\Delta_F \Delta_S}{\Delta_F + \Delta_S} (\ell_{SF} + \ell_{FS}) \quad . \tag{57}$$

In this system ℓ_{FF} and ℓ_{FS} are very short since they are associated with neutrons that cause fissions while still fast. ℓ_{SS} and ℓ_{SF} are orders of magnitude longer since they are associated with neutrons that slow down and diffuse about before causing thermal fissions. As a first approximation for determining the neutron lifetime in a fast-thermal system, we may neglect ℓ_{FF} and ℓ_{FS} with respect to ℓ_{SS} and ℓ_{SF} and set $\ell_{SS} = \ell_{SF} = \ell_S$. We then obtain

$$\ell \approx \frac{\frac{1}{\Delta_S}}{\frac{1}{\Delta_S} + \frac{1}{\Delta_F}} \ell_S ; \tag{58}$$

i.e., the neutron lifetime is approximately equal to the product of the fraction of reactivity in the thermal reactor and the slow neutron lifetime.

The expression for the neutron lifetime given previously refers to the value at the steady state. However, the more important quantity is the neutron lifetime during a transient. In the usual system it is invariably assumed, and usually with much validity, that the neutron lifetime is a constant of the system and therefore does not change during a transient. That this might not be the case in a fast-thermal system can be seen from the following example. Assume that reactivity is added linearly to the fast part (k_{FF} is increased linearly in time) so that the system finally becomes critical on fast neutrons alone. Then during the transient the neutron lifetime changes drastically. We can see from this type of consideration that there is a possibility for much different kinetic behavior in, for example, two systems which have the same neutron lifetime, but in one case $k_{SS} = k_{FF} = .995$ and in the other $k_{SS} = k_{FF} = .8$. In both cases the over-all lifetime is half the thermal lifetime, but in the first case it might be very easy for the system to outrun the thermal neutrons and take off on a period characteristic of fast systems. Thus, in some cases we have to study the kinetic behavior of the system on the basis of the coupled kinetics equations and not be able to deduce the kinetic behavior from the steady state neutron lifetime and the solutions of the ordinary kinetics equations.

The study on the basis of the coupled formalism yields a method of studying the kinetic behavior of coupled systems in a realistic manner which circumvents the difficulties encountered in the above example and which, while it is somewhat more complicated than the usual kinetics formulation, is far less involved than a general space and time dependent formulation.

In the fast-thermal system, under some conditions we can make approximations which simplify the coupled kinetics equations considerably. We first assume ℓ_{FF}, $\ell_{FS} << \ell_{SF}$, ℓ_{SS}. We further restrict ourselves to those transients for which the system is never critical on fast neutrons alone. Under these assumptions we can, with considerable validity, set

$$\frac{dN_{FF}}{dt} = 0, \text{ equivalent to setting } \ell_{FF} = 0$$

and

$$\frac{dN_{FS}}{dt} = 0, \text{ equivalent to setting } \ell_{FS} = 0.$$

We shall refer to the resulting equations as the reduced set of coupled kinetic equations.

The assumptions are very similar to that often used in the ordinary kinetics equations when one sets $dn/dt = 0$ in the range below prompt critical. This is equivalent to setting the prompt neutron lifetime equal to zero. In the ordinary kinetics equations this corresponds to saying that when a delayed neutron enters the system, the resulting prompt burst of fissions occurs instantaneously. In the fast-thermal case, the approximation corresponds to saying that when a neutron is emitted either from thermal fission or from a delayed emitter any resulting fast fissions (i.e., fissions occurring in the chain until a thermal fission breaks the branch of the chain) occur instantaneously. Clearly, in the stated ranges, the approximations are quite valid for both the coupled or ordinary kinetic equations.

With the notation of $1 \leftrightarrow F$ and $2 \leftrightarrow S$, the reduced coupled kinetics equations take the form

$$[1 - k_{11}(1 - \beta)]\frac{N_{11}}{\ell_{11}} = k_{11}(1 - \beta)\frac{N_{12}}{\ell_{12}} + k_{11}\sum_{i=1}^{D} \lambda_i C_{1i} \qquad (59)$$

which results from $\frac{dN_{11}}{dt} = 0$, and

$$\frac{N_{12}}{\ell_{12}} = k_{12}(1 - \beta)\left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}}\right) + k_{12}\sum_{i=1}^{D} \lambda_i C_{2i} \qquad (60)$$

from $\frac{dN_{12}}{dt} = 0$.

Substituting Eq. (60) into Eq. (59) we obtain

$$\frac{N_{11}}{\ell_{11}} = \frac{k_{11}(1-\beta)\left[k_{12}(1-\beta)\left(\frac{N_{21}}{\ell_{21}}+\frac{N_{22}}{\ell_{22}}\right)+k_{12}\sum_{i=1}^{D}\lambda_i C_{2i}\right]+k_{11}\sum_{i=1}^{D}\lambda_i C_{1i}}{1-k_{11}(1-\beta)} \tag{61}$$

and summing Eqs. (60) and (61), we obtain

$$\frac{N_{11}}{\ell_{11}}+\frac{N_{12}}{\ell_{12}} = \frac{k_{12}(1-\beta)\left(\frac{N_{21}}{\ell_{21}}+\frac{N_{22}}{\ell_{22}}\right)+k_{11}\sum_{i=1}^{D}\lambda_i C_{1i}+k_{12}\sum_{i=1}^{D}\lambda_i C_{2i}}{1-k_{11}(1-\beta)} \quad . \tag{62}$$

Using Eq. (62), we obtain

$$\frac{dN_{21}}{dt} = \frac{k_{21}(1-\beta)\,k_{12}(1-\beta)}{1-k_{11}(1-\beta)}\left(\frac{N_{21}}{\ell_{21}}+\frac{N_{22}}{\ell_{22}}\right)-\frac{N_{21}}{\ell_{21}}$$

$$+\frac{k_{21}}{1-k_{11}(1-\beta)}\sum_{i=1}^{D}\lambda_i C_{1i}+\frac{k_{12}(1-\beta)\,k_{21}}{1-k_{11}(1-\beta)}\sum_{i=1}^{D}\lambda_i C_{2i} \quad . \tag{63}$$

We also have

$$\frac{dN_{22}}{dt} = k_{22}(1-\beta)\left(\frac{N_{21}}{\ell_{21}}+\frac{N_{22}}{\ell_{22}}\right)-\frac{N_{22}}{\ell_{22}}+k_{22}\sum_{i=1}^{D}\lambda_i C_{2i} \quad . \tag{64}$$

Using Eq. (62) we obtain

$$\frac{dC_{1i}}{dt} = \frac{\beta_i\,k_{12}(1-\beta)}{1-k_{11}(1-\beta)}\left(\frac{N_{21}}{\ell_{21}}+\frac{N_{22}}{\ell_{22}}\right)+\frac{\beta_i\,k_{11}}{1-k_{11}(1-\beta)}\sum_{i=1}^{D}\lambda_i C_{1i}$$

$$+\frac{\beta_i\,k_{12}}{1-k_{11}(1-\beta)}\sum_{i=1}^{D}\lambda_i C_{2i}-\lambda_i C_{1i} \quad . \tag{65}$$

We also have

$$\frac{dC_{2i}}{dt} = \beta_i \left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}} \right) - \lambda_i C_{2i} \quad . \tag{66}$$

Eqs. (63) through (66) form our reduced kinetics equations, along with the equations for N_{11}/ℓ_{11} and N_{12}/ℓ_{12} in Eqs. (60) and (61).

If we write the reduced set of equations ignoring delayed neutrons we obtain

$$\frac{dN_{21}}{dt} = \left(\frac{k_{12}\, k_{21}}{\Delta_1} - 1 \right) \frac{N_{21}}{\ell_{21}} + \frac{k_{12}\, k_{21}}{\Delta_1} \frac{N_{22}}{\ell_{22}} \tag{67}$$

$$\frac{dN_{22}}{dt} = k_{22} \frac{N_{21}}{\ell_{21}} - \Delta_2 \frac{N_{22}}{\ell_{22}} \tag{68}$$

and where N_{11} and N_{12} are given by

$$\frac{N_{11}}{\ell_{11}} = \frac{k_{11}\, k_{12}}{\Delta_1} \left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}} \right) \tag{69}$$

$$\frac{N_{12}}{\ell_{12}} = k_{12} \left(\frac{N_{21}}{\ell_{21}} + \frac{N_{22}}{\ell_{22}} \right) \quad . \tag{70}$$

We now obtain the inhour equation for the reduced set of equations without delayed neutrons. We assume exponential solutions of the form $N_{jk}^0\, e^{\omega t}$, substitute into Eqs. (67) and (68) and obtain from the condition that the determinant of the matrix of coefficients must vanish

$$\Delta_1\, (\Delta_2 + \omega \ell_{22})\, (1 + \omega \ell_{12}) = k_{12}\, k_{21}\, (1 + \omega \ell_{22}) \quad . \tag{71}$$

If we further assume that $\ell_{22} = \ell_{21} = \ell_S$, Eq. (71) reduces to

$$\omega \ell_S = \frac{1}{\Delta_1} \left(k_{12}\, k_{21} - \Delta_1\, \Delta_2 \right)$$

which along with Eq. (10) and the general relation $\omega \approx \rho/\ell$ where ℓ is the effective prompt lifetime during the excursion yields

$$\ell \approx \ell_S \frac{\Delta_1}{\Delta_1 + \Delta_2} \quad .$$

Therefore, the same relation that was obtained for the steady state is also approximately valid during an excursion as long as the system is below prompt fast critical.

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