Progress in Nuclear Energy 105 (2018) 287–293

Contents lists available at ScienceDirect

Progress in Nuclear Energy

journal homepage: www.elsevier.com/locate/pnucene

Mathematical treatment for two-point reactor kinetics model of reflected systems

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ARTICLE INFO

Keywords:
Two-point kinetics model
Delayed neutrons
Reflected reactors
Fundamental matrix
Eigenvalues and eigenvectors

ABSTRACT

An appropriate precise mathematical model of physical systems has to be developed to predict the dynamic behavior of reflected systems, which comprise core and reflector regions. The present paper developed an accurate and economic mathematical method based on the fundamental matrix to describe the spectrum behavior for one- and two-point kinetics model with multi-group of delayed neutrons. The developed method utilizes the eigenvalues and eigenvectors of the coefficient matrix for the homogeneous linear differential equations resulting from the stiff system of coupled partial differential equations in two-point kinetics model. Moreover, the inverse of the fundamental matrix is calculated analytically. It was evident that the fundamental matrix method is proven to be an excellent solution for cases in which the reactivity is represented by a series of steps and improves accuracy for more general cases of time varying reactivity including Newtonian temperature feedback. The stability of the system is analyzed for different types of reactivity. Finally, the numerical results obtained with these algorithms are applied and verified for different applications of reflected reactors.

1. Introduction

The study of the neutron density is a challenging mathematical and physical problem since it constitutes a fundamental topic in the design and safety of nuclear reactors. An extensive knowledge of the spatial power distribution is required for the design and analysis of different types of current-generation reactors, and that requires the development of more sophisticated theoretical methods. Reflected reactor is one of the most important types of nuclear reactors. Since, the experimental results show that the reactor kinetics behavior of some types of reflector systems cannot be adequately characterized using the point kinetics model. In addition, the neutron lifetimes using theories based on the standard point kinetics model have not always agreed well with the neutron lifetimes predicted by the numerical solutions of the multi-group, multi-dimensional diffusion or transport equations. In view of that, a strongly reflected reactor must be analyzed using two-point reactor kinetics model.

The relatively simple two-point kinetics model has been adopted by Cohn (1962); van Dam (1996). Spriggs et al. (1997) have presented a simplified methodology for determining the coupling parameters and the neutron lifetimes based on simple probability relationships, which describe the aggregate migration of neutrons between core and reflectors. When the coupled kinetics systems have constant coefficients, analytical solutions are easily established, Aboanber and El Mhlawy (2008, 2009), but they are elusive when the coefficients vary with time. In light of Spriggs' two-region kinetics model, a two-group point reactor kinetics model is developed for prompt time constants of a reflected reactor, Tao et al. (2006). Recently, Aboanber and El Mhlawy (2008, 2009) have developed the analytical inversion method (AIM) to solve the derived new version of the reflected core inhour equation that distinct in arrangement from the analogous traditional equation. Furthermore, Aboanber (2009) discussed the effect of fuel temperature feedback for two-point kinetics reactor model due to increasing neutron density in the reactor core. The two-point kinetics model for two energy groups of two-point reactor model of reflected reactors has been developed by Aboanber (2010). The non-linear sliding mode observer which has the robust characteristics facing the parameter uncertainties and disturbances is proposed by Ansarifar et al. (2015), based on the two-point nuclear reactor model to estimate the xenon concentration and delayed neutron precursor density of the Pressurized-Water Nuclear Reactor (PWR) using reactor power measurement.

The present work aims to introduce an effective mathematical method to solve the coupled two-point kinetics equations with multi-group of delayed neutrons for reflected reactors. The developed method utilizes the fundamental matrix of homogenous differential equation for the matrix of two-point kinetics model. The two-point kinetics equations of multi-group of delayed neutrons are introduced in section 2. The fundamental matrix of the homogenous differential equations is

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https://doi.org/10.1016/j.pnucene.2018.02.015
Received 21 November 2016; Received in revised form 12 January 2018; Accepted 12 February 2018
Available online 22 February 2018
0149-1970/

applied, in section 3, to solve the matrix formula of two-point kinetics model, where the inverse of the fundamental matrix is calculated analytically. In section 4, the numerical results of the proposed method are discussed and compared to the traditional methods. Finally, the general conclusion is presented.

2. Two-point kinetics equations

The two-point reactor kinetics equations of multi-group delayed neutrons are a system of stiff coupled differential equations. The system includes lumped variables and parameters for the core and reflector regions. The two-point kinetics equations can be stated as (van Dam, 1996; Spriggs et al., 1997; Aboanber and El Mhlawy, 2008, 2009; Aboanber, 2009, 2010):

$$\frac{dN_c(t)}{dt} = [\nu\Sigma_f(1-\beta) - \Sigma_{ac} - \Sigma_{cr}]\mathrm{v}N_c(t) + \Sigma_{rc}\mathrm{v}N_r(t) + \sum_{i=1}^{I}\lambda_i C_i(t) + Q_c \tag{1}$$

$$\frac{dN_r(t)}{dt} = \Sigma_{cr}\mathrm{v}N_c(t) - [\Sigma_{ar} + \Sigma_{lr} + \Sigma_{rc}]\mathrm{v}N_r(t) + Q_r \tag{2}$$

$$\frac{dC_i(t)}{dt} = \beta_i\nu\Sigma_f\mathrm{v}N_c(t) - \lambda_i C_i(t), \quad i = 1,2,3,\ \cdots, I \tag{3}$$

where, $N_c(t)$ and $N_r(t)$ are the neutron density in core and reflector regions, $C_i(t)$ is the i-group delayed neutron precursors concentration, Q_c and Q_r are the external neutron source in the core and reflector regions, v is the neutron velocity, ν is the average number of neutrons produced per fission, Σ_f is the macroscopic fission cross section, Σ_{ac} and Σ_{ar} is the macroscopic absorption cross section in core and reflector regions, Σ_{cr} is the macroscopic cross section for neutron transfer from core to reflector, Σ_{rc} is the macroscopic cross section for neutron transfer from reflector to core, Σ_{lr} is the macroscopic leakage cross section for the reflector, λ_i is the decay constant of i-group of delayed neutrons, β_i is the fraction of i-group delayed neutrons and I is the total number of delayed neutron groups.

The interpretation of the individual terms in the ordinary differential equations (1)–(3) are as follows: $\frac{dN_c(t)}{dt}$ and $\frac{dN_r(t)}{dt}$ represent the time rate of change of the neutron density in the core and reflector regions, $\frac{dC_i(t)}{dt}$ is the time rate of change of the i-group delayed neutron precursors concentration, $\nu\Sigma_f(1-\beta)\mathrm{v}N_c(t)$ is the fraction of the prompt neutron density to the fission neutron density after fission process in core region, $\Sigma_{ac}\mathrm{v}N_c(t)$ is the absorption neutron density in the core, $\Sigma_{cr}\mathrm{v}N_c(t)$ is the neutron current from core to reflector, $\Sigma_{rc}\mathrm{v}N_r(t)$ is the neutron current from reflector to core, $\sum_{i=1}^{I}\lambda_i C_i(t)$ represents the delayed neutrons which appear in the core region due to decay of the precursor concentrations, $\Sigma_{ar}\mathrm{v}N_r(t)$ is the absorption neutron density in the reflector region, $\Sigma_{lr}\mathrm{v}N_r(t)$ is the leakage neutron current from the reflector region to outside and $\beta_i\nu\Sigma_f\mathrm{v}N_c(t)$ is the i-group of the delayed neutron precursors in the core region, Fig. 1.

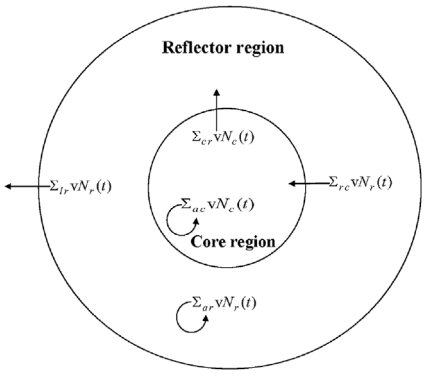

Fig. 1. Relation between the core and reflector regions.

Equations (1)–(3) are simplified as:

$$\frac{dN_c(t)}{dt} = \frac{\rho - \beta - \alpha_{cr}\alpha_{rc}}{\Lambda_c}N_c(t) + \frac{\alpha_{rc}}{\Lambda_r}N_r(t) + \sum_{i=1}^{I}\lambda_i C_i(t) + Q_c \tag{4}$$

$$\frac{dN_r(t)}{dt} = \frac{\alpha_{cr}}{\Lambda_c}N_c(t) - \frac{1}{\Lambda_r}N_r(t) + Q_r \tag{5}$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\Lambda_c}N_c(t) - \lambda_i C_i(t), \quad i = 1,2,3,\ \cdots, I \tag{6}$$

where, $\rho = 1 - \frac{\Sigma_{ac}}{\nu\Sigma_f} - \frac{\Sigma_{cr}(\Sigma_{ar}+\Sigma_{lr})}{\nu\Sigma_f(\Sigma_{ar}+\Sigma_{lr}+\Sigma_{rc})}$ is the reactivity in core, $\Lambda_c = \frac{1}{\mathrm{v}\nu\Sigma_f}$ is the neutron generation time in core, $\Lambda_r = \frac{1}{\mathrm{v}(\Sigma_{ar}+\Sigma_{lr}+\Sigma_{rc})}$ is the neutron lifetime in the reflector, $\alpha_{cr} = \frac{\Sigma_{cr}}{\nu\Sigma_f}$ is the fraction of fission neutrons produced that escapes to the reflector, $\alpha_{rc} = \frac{\Sigma_{rc}}{\Sigma_{ar}+\Sigma_{lr}+\Sigma_{rc}}$ is the fraction of reflector neutrons flowing to the core (the reflection coefficient or albedo) and the initial conditions are as follows: $N_c(0) = 1.0(neutron/cm^3)$, $N_r(0) = \frac{\alpha_{cr}\Lambda_r}{\Lambda_c}N_c(0) + \Lambda_r Q_r$, and $C_i(0) = \frac{\lambda_i\beta_i}{\Lambda_c}N_c(0)$.

Equations (4)–(6) are called *two-point kinetics equations with multi-group delayed neutrons*. The solution of this system of ordinary differential equations is our target in the following section using the fundamental matrix method.

3. Fundamental matrix

Accurate values for the model parameters should be obtained from detailed transport calculations on the system. To suit our case the set of the two-point kinetics model with multi-group of delayed neutrons may be written in matrix notation as:

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{A}\mathbf{X}(t) + \mathbf{B} \tag{7}$$

where,

$$\mathbf{X}(t) = \begin{bmatrix} N_c(t) \\ N_r(t) \\ C_1(t) \\ C_2(t) \\ \vdots \\ C_I(t) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} Q_c \\ Q_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \alpha_1 & \alpha_2 & \lambda_1 & \lambda_2 & \cdots & \lambda_I \\ \xi_1 & -\xi_2 & 0 & 0 & \cdots & 0 \\ \mu_1 & 0 & -\lambda_1 & 0 & \cdots & 0 \\ \mu_2 & 0 & 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_I & 0 & 0 & 0 & \cdots & -\lambda_I \end{bmatrix},$$

and $\alpha_1 = \frac{\rho-\beta-\alpha_{cr}\alpha_{rc}}{\Lambda_c}$, $\alpha_2 = \frac{\alpha_{rc}}{\Lambda_r}$, $\xi_1 = \frac{\alpha_{cr}}{\Lambda_c}$, $\xi_2 = \frac{1}{\Lambda_r}$, and $\mu_i = \frac{\beta_i}{\Lambda_c}$.

The solution of equation (7) depends on the coefficient matrix **A** for the following two cases:

Case I: The matrix **A** is time independent, analytical solution e.g. (Aboanber and Nahla, 2002; Aboanber, 2003a, b; Jordan and Smith, 2007; Spriggs et al., 1997), of equation (7) is easily obtained:

$$\mathbf{X}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{X}(t_0) + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\mathbf{B}ds \tag{8}$$

where, $\Phi(t)$ is a fundamental matrix defined as follows

$$\Phi(t) = \left[\mathbf{U}_0 e^{\omega_0 t} \quad \mathbf{U}_1 e^{\omega_1 t} \quad \mathbf{U}_2 e^{\omega_2 t} \quad \cdots \quad \mathbf{U}_{I+1} e^{\omega_{I+1} t}\right] \tag{9}$$

ω_k and $\mathbf{U}_k$ are the eigenvalues and eigenvectors of the matrix **A** respectively, $k = 0,1,2,\ \cdots, I+1$.

Case II: The matrix **A** is time dependent (Aboanber, 2009), the solution of equation (7) over time interval $[t_m,\ t_{m+1}]$ can be obtained in the following procedure:

$$\mathbf{X}(t_{m+1}) = \Phi(t_{m+1})\Phi^{-1}(t_m)\mathbf{X}(t_m) + \Phi(t_{m+1})\int_{t_m}^{t_{m+1}}\Phi^{-1}(s)\mathbf{B}ds \tag{10}$$

Here $\Phi(t_{m+1})$ is the fundamental matrix at t_{m+1} which takes the mode:

Table 1
Parameters data for four reflected reactors.

Reactor	Cases	System	ρ_∞	Λ_c	Λ_r	α_{rc}	α_{cr}
PROTEUS	Case I	1-Point	0.4	2.0 (m s)	0.0 (m s)	0.0	0.0
		2-Point	0.4	0.4 (m s)	4.0 (m s)	0.5	0.8
AGN 201	Case II	1-Point	0.27142	26.0 (μ s)	0.0 (μ s)	0.0	0.0
		2-Point	0.27142	26.0 (μ s)	127.7 (μ s)	0.4429	0.4872
	Case III	1-Point	0.2567	96.3 (μ s)	0.0 (μ s)	0.0	0.0
		2-Point	0.2567	26.0 (μ s)	238.6 (μ s)	0.4737	0.4877
	Case IV	1-Point	0.25	169.3 (μ s)	0.0 (μ s)	0.0	0.0
		2-Point	0.25	26.0 (μ s)	472.3 (μ s)	0.4877	0.488
Decay constant ($1/s$)	λ_i	0.012444	0.030535	0.111438	0.301368	1.136307	3.013683
Parameters (PROTEUS)	β_i	2.371E-4	1.583E-3	1.417E-3	2.856E-3	8.314E-4	3.037E-4
Parameters (AGN 201)	β_i	2.460E-4	1.642E-3	1.470E-3	2.962E-3	8.625E-4	3.150E-4

Table 2
Core and reflector neutron density for different types of reflected reactors at step reactivity.

Cases	System	Methods	1.0 (s)		2.0 (s)		3.0 (s)	
			Core	Reflector	Core	Reflector	Core	Reflector
PROTEUS	1-Point	AEM	2.359369		3.293477		4.300094	
Case I		FMM	2.359369		3.293477		4.300093	
	2-Point	AEM	2.360462	18.852188	3.294816	26.328411	4.302018	34.381290
		AIM Padé01	2.360458	18.852160	3.294808	26.328348	4.302004	34.381181
		AIM Padé02	2.360463	18.852194	3.294817	26.328424	4.302021	34.381313
		AIM Padé03	2.360464	18.852201	3.294820	26.328440	4.302024	34.381341
		AIM Padé11	2.359930	18.847977	3.293640	26.319062	4.299989	34.365135
		AIM Padé12	2.360109	18.849366	3.294034	26.322148	4.300666	34.370468
		FMM	2.360463	18.852197	3.294818	26.328428	4.302021	34.381319
AGN 201	1-Point	AEM	8.283412		16.506964		31.962611	
Case II		FMM	8.283283		16.506540		31.961468	
	2-Point	AEM	8.038840	18.481360	15.892310	36.536780	30.506500	70.135260
		AIM Padé01	8.037142	18.477463	15.887071	36.524756	30.492807	70.103826
		AIM Padé02	8.037720	18.479336	15.888860	36.529945	30.497487	70.116652
		AIM Padé03	8.036096	18.475054	15.883928	36.517526	30.484702	70.085181
		AIM Padé11	7.951514	18.279497	15.631902	35.935944	29.838258	68.594851
		AIM Padé12	7.982254	18.351603	15.721555	36.144880	30.065525	69.122735
		FMM	8.038863	18.481408	15.892371	36.536923	30.506648	70.135610

$$\Phi(t_{m+1}) = \left[\mathbf{U}_0 e^{\omega_0(t_m+h)} \quad \mathbf{U}_1 e^{\omega_1(t_m+h)} \quad \mathbf{U}_2 e^{\omega_2(t_m+h)} \quad \cdots \quad \mathbf{U}_{I+1} e^{\omega_{I+1}(t_m+h)}\right], \tag{11}$$

and $\Phi(t_m)$ is giving by:

$$\Phi(t_m) = \left[\mathbf{U}_0 e^{\omega_0 t_m} \quad \mathbf{U}_1 e^{\omega_1 t_m} \quad \mathbf{U}_2 e^{\omega_2 t_m} \quad \cdots \quad \mathbf{U}_{I+1} e^{\omega_{I+1} t_m}\right], \tag{12}$$

where $t_{m+1} = t_m + h$ and h is the step time interval.

3.1. Eigenvalues and eigenvectors of coefficient matrix **A**

The system of differential equation (7) may be solved for the case of an initially critical reactor in which the properties are changed at t = 0 in such a way. The eigenvalues of the coefficient matrix **A** are given by the roots of the famous inhour equation for a two-point model, the basis of reactor kinetics:

$$(\omega - \alpha_1) - \frac{\xi_1 \alpha_2}{(\omega + \xi_2)} - \sum_{i=1}^{I} \frac{\lambda_i \mu_i}{(\omega + \lambda_i)} = 0 \tag{13}$$

Consequently, we get the average value of reactivity ρ over the small time interval $[t_m, t_{m+1}]$.

$$\rho = \omega\Lambda_c + \frac{\omega\alpha_{cr}\alpha_{rc}\Lambda_r}{(1 + \omega\Lambda_r)} + \sum_{i=1}^{I} \frac{\omega\beta_i}{(\omega + \lambda_i)} \tag{14}$$

This equation is represented by $(I + 2)^{th}$ degree of algebraic equation which has $(I + 2)$ real roots. To suit our case and to obtain the roots of this equation, let us simplify equation (14) to the following (Aboanber, 2009; Aboanber and El Mhlawy, 2008, 2009; van Dam, 1996)

$$\begin{aligned}\rho(1 + \omega\Lambda_r)\prod_{i=1}^{I}(\omega + \lambda_i) &= \omega\Lambda_c(1 + \omega\Lambda_r)\prod_{i=1}^{I}(\omega + \lambda_i) \\ &+ \omega\alpha_{cr}\alpha_{rc}\Lambda_r\prod_{i=1}^{I}(\omega + \lambda_i) \\ &+ \omega(1 + \omega\Lambda_r)\sum_{i=1}^{I}\beta_i\prod_{\substack{l=1\\ l\neq i}}^{I}(\omega + \lambda_l)\end{aligned}$$

and consequently, we get

$$\sum_{k=0}^{I+2}\left\{\Lambda_c\Lambda_r D_{k,0} + (\alpha_{cr}\alpha_{rc}\Lambda_r + \Lambda_c - \rho\Lambda_r)D_{k-1,0} - \rho D_{k-2,0} + \Lambda_r\sum_{l=1}^{I}\beta_l D_{k-1,l} + \sum_{l=1}^{I}\beta_l D_{k-2,l}\right\}\omega^{I+2-k} = 0 \tag{16}$$

where,

$$D_{k,l} = \begin{cases} 0, & \text{for } k < 0 \text{ or} k > I; \\ 1, & \text{for } k = 0; \\ \sum_{\substack{l_1=1\\ l_1\neq l}}^{I}\sum_{\substack{l_2=l_1+1\\ l_1\neq l}}^{I}\cdots\sum_{\substack{l_k=l_{k-1}+1\\ l_k\neq l}}^{I}\lambda_{l_1}\lambda_{l_2}\cdots\lambda_{l_k}, & \text{for } 0 < k \le I \end{cases}$$

and $\sum_{\substack{l_k=l_{k-1}+1\\ l_k\neq l}}^{I}\lambda_{l_1}\lambda_{l_2}\cdots\lambda_{l_k} = 0 \; for \; l_k > I.$

The benefit of equation (3.1) is that it presents a simple form to get the required coefficients, which make it easy to algorithm the inhour equation using Visual FORTRAN and/or Matlab program. This program

Table 3
Core and reflector neutron density for different types of reflected reactors at ramp reactivity (0.1$/s).

Cases	System	Methods	0.1 (s)		0.5 (s)		1.0 (s)		3.0 (s)	
			Core	Reflector	Core	Reflector	Core	Reflector	Core	Reflector
PROTEUS	1-Point	Exact	1.00163		1.02805		1.08306		1.47946	
Case I		AIM Padé02	1.00163		1.02805		1.08306		1.47972	
		FMM	1.00163		1.02805		1.08306		1.47972	
	2-Point	Exact	1.00177	8.01315	1.02923	8.23085	1.08644	8.68733	1.50416	12.0234
		AIM Padé02	1.00177	8.01316	1.02923	8.23089	1.05643	8.68740	1.50414	12.0236
		FMM	1.00171	8.01271	1.02817	8.22249	1.08316	8.66127	1.47994	11.8302
AGN 201	1-Point	Exact	1.00930		1.05610		1.12878		1.65270	
Case II		AIM Padé02	1.00948		1.05629		1.12876		1.65337	
		FMM	1.00948		1.05629		1.12876		1.65337	
	2-Point	Exact	1.00948	2.32050	1.05630	2.42800	1.12878	2.59470	1.65310	3.79970
		AIM Padé02	1.00916	2.31995	1.05591	2.42750	1.12843	2.59403	1.65239	3.79906
		FMM	1.00958	2.32122	1.05644	2.42893	1.12897	2.59570	1.65423	3.80331
AGN 201	1-Point	Exact	1.00885		1.05542		1.12753		1.64878	
Case III		AIM Padé02	1.00895		1.05554		1.12767		1.64904	
		FMM	1.00895		1.05554		1.12767		1.64904	
	2-Point	Exact	1.00990	4.52000	1.05680	4.72970	1.12931	5.05424	1.65350	7.40009
		AIM Padé02	1.01256	4.51523	1.05986	4.72348	1.13292	5.04607	1.66117	7.37747
		FMM	1.00917	4.51652	1.05585	4.72540	1.12812	5.04883	1.65083	7.38800
AGN 201	1-Point	Exact	1.00795		1.05407		1.12555		1.64035	
Case IV		AIM Padé02	1.00795		1.05406		1.12554		1.64060	
		FMM	1.00795		1.05406		1.12554		1.64060	
	2-Point	Exact	1.00821	8.93703	1.05452	9.34740	1.12626	9.98322	1.64336	14.5662
		AIM Padé02	1.01241	8.92024	1.05936	9.32588	1.13199	9.95537	1.65544	14.4928
		FMM	1.00836	8.93842	1.05468	9.34892	1.12643	9.98485	1.64415	14.5732

calculates all roots of an algebraic equation with real coefficients using numerical Laguerre's method.

The eigenvector $\mathbf{U}_k$ of the coefficient matrix $\mathbf{A}$ corresponding to ω_k takes the following form:

$$\mathbf{U}_k = \left[1 \quad \frac{\xi_1}{(\omega_k+\xi_2)} \quad \frac{\mu_1}{(\omega_k+\lambda_1)} \quad \frac{\mu_2}{(\omega_k+\lambda_2)} \quad \cdots \quad \frac{\mu_I}{(\omega_k+\lambda_I)}\right]^T \tag{17}$$

where $k = 0,1,2, \cdots, I+1$.

3.2. Inverse of fundamental matrix

The inverse of the fundamental matrix $\Phi(t_m)$, equation (12), is also calculated analytically, according to Aboanber et al. (2014) as follows:

$$\Phi^{-1}(t_m) = \begin{bmatrix} \phi_0 e^{-\omega_0 t_m} & \frac{\varphi_0\phi_0 e^{-\omega_0 t_m}}{(\omega_0+\xi_2)} & \frac{\varphi_1\phi_0 e^{-\omega_0 t_m}}{(\omega_0+\lambda_1)} & \frac{\varphi_2\phi_0 e^{-\omega_0 t_m}}{(\omega_0+\lambda_2)} & \cdots & \frac{\varphi_I\phi_0 e^{-\omega_0 t_m}}{(\omega_0+\lambda_I)} \\ \phi_1 e^{-\omega_1 t_m} & \frac{\varphi_0\phi_1 e^{-\omega_1 t_m}}{(\omega_1+\xi_2)} & \frac{\varphi_1\phi_1 e^{-\omega_1 t_m}}{(\omega_1+\lambda_1)} & \frac{\varphi_2\phi_1 e^{-\omega_1 t_m}}{(\omega_1+\lambda_2)} & \cdots & \frac{\varphi_I\phi_1 e^{-\omega_1 t_m}}{(\omega_1+\lambda_I)} \\ \phi_2 e^{-\omega_2 t_m} & \frac{\varphi_0\phi_2 e^{-\omega_2 t_m}}{(\omega_2+\xi_2)} & \frac{\varphi_1\phi_2 e^{-\omega_2 t_m}}{(\omega_2+\lambda_1)} & \frac{\varphi_2\phi_2 e^{-\omega_2 t_m}}{(\omega_2+\lambda_2)} & \cdots & \frac{\varphi_I\phi_2 e^{-\omega_2 t_m}}{(\omega_2+\lambda_I)} \\ \phi_3 e^{-\omega_3 t_m} & \frac{\varphi_0\phi_3 e^{-\omega_3 t_m}}{(\omega_3+\xi_2)} & \frac{\varphi_1\phi_3 e^{-\omega_3 t_m}}{(\omega_3+\lambda_1)} & \frac{\varphi_2\phi_3 e^{-\omega_3 t_m}}{(\omega_3+\lambda_2)} & \cdots & \frac{\varphi_I\phi_3 e^{-\omega_3 t_m}}{(\omega_3+\lambda_I)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{I+1} e^{-\omega_{I+1} t_m} & \frac{\varphi_0\phi_{I+1} e^{-\omega_I+1 t_m}}{(\omega_{I+1}+\xi_2)} & \frac{\varphi_1\phi_{I+1} e^{-\omega_I+1 t_m}}{(\omega_{I+1}+\lambda_1)} & \frac{\varphi_2\phi_{I+1} e^{-\omega_I+1 t_m}}{(\omega_{I+1}+\lambda_2)} & \cdots & \frac{\varphi_I\phi_{I+1} e^{-\omega_I+1 t_m}}{(\omega_{I+1}+\lambda_I)} \end{bmatrix} \tag{18}$$

where, $\phi_k = \dfrac{(\omega_k+\xi_2)\prod_{i=1}^{I}(\omega_k+\lambda_i)}{\prod_{\substack{j=0 \\ j\neq k}}^{I}(\omega_k-\omega_j)}$, $k = 0,1,2, \cdots, I+1$; $\varphi_0 = -\dfrac{\prod_{k=0}^{I+1}(\omega_k+\xi_2)}{\xi_1\prod_{i=1}^{I}(\xi_2-\lambda_i)}$

and $\varphi_j = -\dfrac{\prod_{k=0}^{I+1}(\omega_k+\lambda_j)}{\mu_j(\lambda_j-\xi_2)\prod_{\substack{i=1 \\ i\neq j}}^{I}(\lambda_j-\lambda_i)}$, $j = 1,2, \cdots, I$.

According to equations (11) and (18), we note that the result of product $\Phi(t_{m+1})$ by $\Phi^{-1}(t_m)$ into equation (10) is independent on t_m explicitly.

4. Stability

Stability was most likely the main question in establishing dynamical frameworks which was managed carefully. At the point when a framework is unstable, the yield of the framework might be vast despite the fact that the contribution to the framework was limited. Furthermore, there is no single concept of stability, and a wide range of definitions are conceivable see e.g. (Jordan and Smith, 2007; Gil, 2005). A necessary and sufficient condition for the stability of the equation,

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}(t)\mathbf{X}(t), \quad for \quad t \geq 0 \tag{19}$$

is the uniform boundedness of its Cauchy operator:

$$\sup_{t\geq 0} \|\Phi(t)\| < \infty \tag{20}$$

Rephrasing the above conditions, we will say that a necessary and sufficient condition for the stability of equation (19) is the existence of a constant $M > 0$ such that any solution $\mathbf{X}(t)$ of equation (19) satisfies the estimate:

$$\|\mathbf{X}(t)\| \leq M\|\mathbf{X}(0)\|, \quad for \quad t \geq 0 \tag{21}$$

The least value M_0 of this constant is clearly given by the equality:

$$M_0 = \sup_{t\geq 0} \|\Phi(t)\| \tag{22}$$

Linear equation (19) is uniformly stable if there exists a constant $M > 0$ such that any solution $\mathbf{X}(t)$ of this equation satisfies the following estimate for all $t \geq s \geq 0$:

$$\|\mathbf{X}(t)\| \leq M\|\mathbf{X}(s)\|, \tag{23}$$

Since $\mathbf{X}(t) = \Phi(t, s)\mathbf{X}(s)$, it can be shown that the uniform stability of the equation is equivalent to the condition

$$\sup_{t\geq s\geq 0} \|\Phi(t, s)\| < \infty \tag{24}$$

Note that if $\mathbf{A}(t) \equiv \mathbf{A}$ is a constant matrix, then one gets the stability condition:

$$\sup_{t\geq 0} \|\exp(t\mathbf{A})\| < \infty \tag{25}$$

The equilibrium point for the linear differential equation (7) without external source (i.e. $\mathbf{B} = 0$) is given by:

$$\mathbf{X}^*(t) = \mathbf{A}^{-1}\mathbf{B} \tag{26}$$

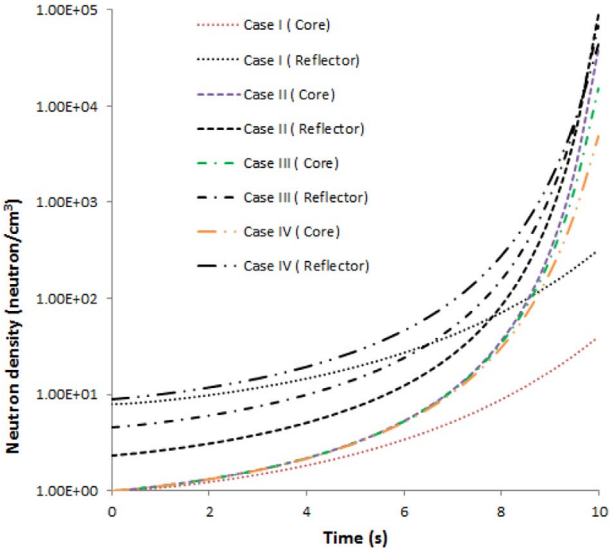

Fig. 2. The core and reflector neutron density for the positive ramp reactivity.

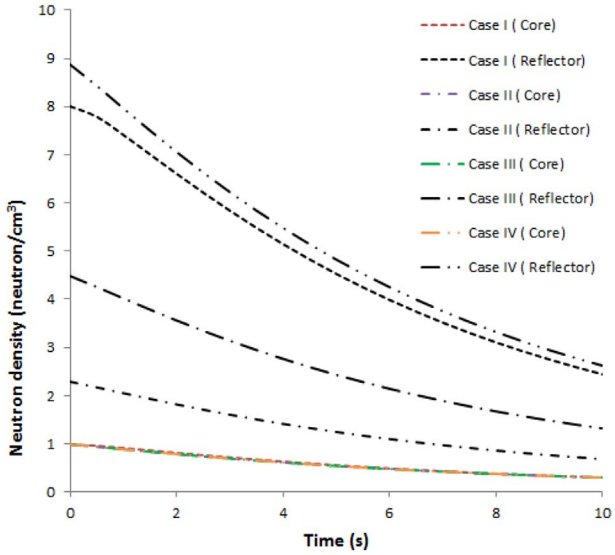

Fig. 3. The core and reflector neutron density for the negative ramp reactivity.

Consequently, the equilibrium point is $\mathbf{X}^*(t) = 0$, so the matrix model of the linear differential equation (7) is uniformly asymptotically (exponentially) stable if the zero solution of equation (7) is uniformly asymptotically (exponentially) stable.

5. Results and discussions

To check the accuracy and efficiency of the proposed method, (FMM), the method is applied for the two-point kinetics equations with six-group of delayed neutrons. Several different computational benchmark examples are presented in this section. The neutron density in the core and reflector as well as the precursor concentrations has been obtained using the algorithm coded by visual FORTRAN. This code is applied to the step, ramp and temperature feedback reactivities. The reactor kinetic parameters for the investigated four different cases of the reflected reactor and the delayed neutron data are summarized in Table 1. The first case represents one- and two-point parameters for the experimental zero power PROTEUS at the Paul Scherrer Institute (Villigen-Switzerland) with a beta-effective of $\beta = 0.00723$ (van Dam, 1996; Jatuff et al., 2003). It consists of a relatively small core, one cubic meter, surrounded by a thick graphite reflector. The other three cases represent one- and two-point parameters of the core plus graphite reflector; the core plus graphite reflector and lead shield; the core plus graphite reflector and the lead and water shields for AGN-201 reactor (Spriggs et al., 1997). The beta effective of AGN-201 is $\beta = 0.0075$. The stability was considered for negative, zero and positive reactivity of the proposed method.

5.1. Step reactivity

The first computational example simulates a step reactivity insertion to check the accuracy and efficiency of the fundamental matrix method. This method is applied to the one- and two-point kinetics equations with constant reactivity $\rho = (1 - \rho_\infty)\beta$, where ρ_∞ is the fraction of core neutrons that return to the core region after having leaked into the reflector. The core and reflector neutron density for four reflected reactors are presented in Table 2. The results of comparison for the fundamental matrix method (FMM) with the analytical exponential method (AEM) (Aboanber and El Mhlawy, 2008, 2009), AIM Padé01, AIM Padé02, AIM Padé03, AIM Padé11 and AIM Padé12 approximations (Aboanber and El Mhlawy, 2009) are presented in Table 2. All comparisons are taken under the same conditions with time step $h = 0.001(s)$.

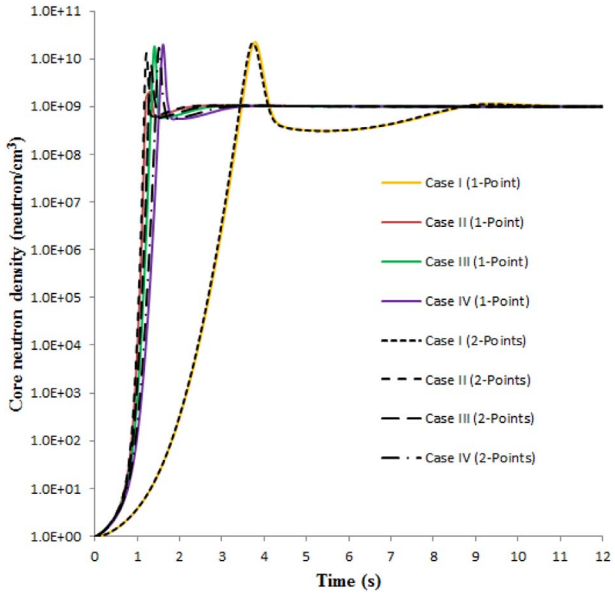

Fig. 4. Core neutron density for nonlinear one-point and two-point kinetics equations.

5.2. Ramp reactivity

To verify the accuracy and efficiency of the proposed method for ramp reactivity, the fundamental matrix method is applied to the one- and two-point kinetics equations with linear ramp reactivity for different four types of reflected reactors. The linear ramp reactivity takes the form $\rho(t) = 0.1\beta t$. The neutron density of the core and reflector for reflected reactors is presented in Table 3. The results of the fundamental matrix method (FMM) are accurate compared with the results of the analytical (Aboanber, 2009) and AIM Padé02 (Aboanber and El Mhlawy, 2009; Aboanber, 2009) methods.

The neutron density for the core and reflector using the fundamental matrix method for the positive ramp reactivity $0.1(\$/s)$ is shown in Fig. 2. In addition, for the negative ramp reactivity $-0.1(\$/s)$ neutron density of the core and reflector is shown in Fig. 3.

The first two cases, negative and zero reactivity, satisfies the stability condition, that is, all eigenvalue of the coefficient matrix has a negative real part.

Table 4
Time variations of neutron density for core and reflector at different peaks for representative reflected reactors at temperature feedback reactivity $\rho(t) = at - b\int_0^t N_c(t')dt'$.

	System	PROTEUS Case I		AGN 201 Case II		AGN 201 Case III		AGN 201 Case IV	
		Core	Reflector	Core	Reflector	Core	Reflector	Core	Reflector
$a = 0.1(1/s)$	1-Point	2.569E11		2.415E11		2.434E11		2.469E11	
$b = 10^{-11}(cm^3/s)$		(1.114 s)		(0.249 s)		(0.298 s)		(0.372 s)	
	2-Point	2.331E11	1.847E12	2.383E11	5.462E11	2.373E11	1.057E12	2.350E11	2.072E12
		(1.058 s)	(1.062 s)	(0.237 s)	(0.237 s)	(0.276 s)	(0.276 s)	(0.336 s)	(0.337 s)
$a = 0.1(1/s)$	1-Point	3.047E13		2.895E13		2.918E13		2.948E13	
$b = 10^{-13}(cm^3/s)$		(1.201 s)		(0.265 s)		(0.317 s)		(0.398 s)	
	2-Point	2.743E13	2.170E14	2.839E13	6.515E13	2.829E13	1.263E14	2.793E13	2.466E14
		(1.136 s)	(1.140 s)	(0.251 s)	(0.252 s)	(0.294 s)	(0.294 s)	(0.359 s)	(0.359 s)
$a = 0.01(1/s)$	1-Point	2.229E10		1.997E10		2.029E10		2.067E10	
$b = 10^{-11}(cm^3/s)$		(3.782 s)		(1.251 s)		(1.391 s)		(1.607 s)	
	2-Point	2.161E10	1.727E11	1.981E10	4.554E10	2.003E10	8.960E10	2.026E10	1.795E11
		(3.731 s)	(3.735 s)	(1.217 s)	(1.218 s)	(1.334 s)	(1.334 s)	(1.517 s)	(1.518 s)
$a = 0.01(1/s)$	1-Point	2.706E12		2.475E12		2.507E12		2.545E12	
$b = 10^{-13}(cm^3/s)$		(4.078 s)		(1.304 s)		(1.459 s)		(1.697 s)	
	2-Point	2.616E12	2.091E13	2.458E12	5.650E12	2.476E12	1.107E13	2.494E12	2.209E13
		(4.018 s)	(4.022 s)	(1.268 s)	(1.268 s)	(1.396 s)	(1.396 s)	(1.597 s)	(1.597 s)

5.3. Temperature feedback reactivity

In this case, the fundamental matrix method is applied to the nonlinear one- and two-point kinetics equations with temperature feedback reactivity. The feedback reactivity is taken as a function of time and core neutron density (Aboanber, 2009; Nahla and Zayed, 2010).

$$\rho(t) = at - b\int_0^t N_c(t')dt'$$

The first term at represents the impressed reactivity variation and b is the shutdown coefficient of the reflected reactor system. As presently provided in our calculations the parameter a takes the value $0.1(s^{-1})$ and $0.01(s^{-1})$ while the parameter b in general ranging from $b \backsim 10^{-13}(cm^3/s)$ for slow systems to $b \backsim 10^{-7}(cm^3/s)$ for fast metal systems. The neutron density in the core region for one- and two-point kinetics equations are shown in Fig. 4. The core and reflector neutron density variations at peaks as a function of time are introduced in Table 4. The fundamental matrix method represents an accurate computational method for the one- and two-point reactor kinetics equations.

6. Conclusions

Two-point kinetics equations for a compact core and reflector system were developed. The numerical results indicate that the two-point model improves the accuracy close to the space dependent approach. The fundamental matrix method was applied to solve the one- and two-point reactor kinetics equations for six-groups of delayed neutrons for step, ramp, and Newtonian temperature feedback reactivities. This method is based on the eigenvalues and eigenvectors of the coefficient matrix for the homogenous system of stiff ordinary differential equations. The fundamental matrix method is applied to four cases of reflected reactors. The first case is the experimental zero power PROTEUS and the other three cases are the core plus graphite reflector; the core plus graphite reflector and lead shield; the core plus graphite reflector and the lead and water shields for AGN-201 reactor. The results of the fundamental matrix method are compared with the results of the analytical exponential model (AEM), several types of AIM Padé approximations and exact analytical solution. These comparisons substantiate the accuracy of the fundamental matrix method for solving linear/nonlinear one- and two-point kinetics equations. The variations of core neutron density for nonlinear one-point and two-point kinetics equations duo to reactivity-compensated response to ramp function additions of reactivity exhibit a similar behavior for both system. Furthermore the characteristic damped-oscillatory approach to equilibrium core neutron density (power level) at which the rate of reactivity compensated due to (adiabatic) temperature increase just balances the rate of external reactivity addition. For nonlinear one-point and two-point kinetics systems, heat loss in practical systems would, of course, result in correspondingly higher power(or normalized neutron density). In the case of two-point reflected reactor, neutrons diffuse through the reflector and some are returned to the core region. Furthermore, when the reflector is in place, neutrons that would otherwise be lost are returning to the core. The ratio of the neutron flux in the reflector region to the neutron flux in the core region was recorded in our case 8: 1 at the initial time $t = 0$. For step reactivity, this ratio is changed from slight decreasing to 7.987: 1 at 1 s followed by slight increasing to 7.9918: 1 at 3 s up to the initial ratio with time due to neutrons that would otherwise be lost and are returned and added to the existing neutron flux just inside the boundary between the core and reflector region. For time varying reactivity, the ratio of the neutron flux in the reflecting region to the neutron flux in the core region was recorded a slight decrease with time to 7.993: 1 after 3 s due to a slight increase of the neutron flux in the core region, this fact were illustrated in the presented tables. Of course, the ratio of the number of neutrons reflected back into the reactor to the number entering the reflector (albedo) measured the efficiency of a reflector. The value of the albedo depends on the composition and thickness of the reflector, which will be discussed in the future work.

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