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Rigorous Derivation of Multi-Point Reactor Kinetics Equations with Explicit Dependence on Perturbation

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Nodal kinetic equations for coupled reactors, namely multi-point reactor kinetics equations whose dependent variables are the fission sources of each reactor are derived rigorously, using kinetics parameters with the explicit dependence on a perturbation, from the time dependent multi-group diffusion equation. Exact expressions for the coupling coefficients, neutron life time and the change of the coupling coefficients due to the perturbation are given. Since the present equations are exact independent of the strength of the coupling, these equations can be used for any reactor by dividing a core into appropriate subregions. Some analytical expressions of the coupling coefficients, neutron life time and other kinetics parameters are given for some simple geometries.

***KEYWORDS:* coupled reactors, coupling coefficients, kinetics equations, reactivity equations, reactivity, point reactor, multi-point reactor, perturbation, effective delayed neutron fraction, neutrons lifetime**

I. INTRODUCTION

There are many works on coupled reactors[1]~[6], however, they are treated with some approximations. Recently, it was shown that nodal static and kinetics equations for coupled reactors whose dependent variables are fission neutron sources could be derived rigorously from the multi-group diffusion[7] or transport equations[8] without any approximation.

In the present work, nodal kinetics equations for coupled reactors, namely multi-point reactor kinetics equations whose dependent variables are fission source of each region are derived rigorously for perturbed system, and the exact expressions for the change of the coupling coefficients due to the perturbation are given explicitly like in the usual exact perturbation theory[9]. In the previous work[7][8], the Green's function for the perturbed system is used to bring the final equations to be the same form as those given by Avery[1]. In the former formulation, there is no explicit dependence of the kinetics parameters on the perturbation, since it is absorbed in the change of the Green's function. In the present formulation, the Green's function for the unperturbed system is used, and the change of the kinetics parameters is expressed explicitly in terms of the change of the cross sections.

In case of a large core where the coupling between distant regions in a core is weak, there may arise a flux tilt, or the flux distribution may change globally, even if control rods are moved locally[10][11]. In the present method, the coupling coefficients between any region can be calculated exactly by dividing a core into appropriate subregions, and we can know the relation of the fiux tilt and the strength of the coupling between regions, which may help the physical understanding of large reactors.

If we treat the whole system as one region, the multi-point reactor kinetics equations are reduced to the point-reactor kinetics equa-

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tions. The difference of the present kinetics equations from the usual point-reactor kinetics equations is in that the dependent variables in the present method have the physical meaning of the fission source and the number of delayed neutron precursors, whereas in the usual method[12][13], a dependent variable is the amplitude function for the flux whose physical meaning depends on the choice of the weighting function, and it can not be the fission source for any choice of the weighting function[8].

In Chap. II are derived multi-point reactor kinetics equations for a perturbed system and the expressions for the changes of the coupling coefficients due to the perturbation. In Chap. III are derived the coupling coefficients and other kinetics parameters explicitly for some simple geometries, and a short discussion is given in Chap. IV.

Although the formulations are given here for the case of the multi-group diffusion equation, the application of the present method to the multi-group transport equation is straightforward simply by replacing the diffusion operator by the transport operator.

II. KINETICS EQUATIONS FOR PERTURBED SYSTEM

For simplicity, we use the boundary condition of the zero flux at the outermost boundary S of the whole system V, namely

$$\phi_g(\boldsymbol{r})=0, \qquad \text{at } \boldsymbol{r}\in S, \tag{1}$$

where $\phi_g(\boldsymbol{r})$ is the total flux of g-th group.

We assume that the following steady state multi-group diffusion equation for an unperturbed system of coupled reactors has a non-zero solution with a criticality factor k_0;

$$A\phi_g(\boldsymbol{r})=\frac{1}{k_0}B\phi_g(\boldsymbol{r}), \tag{2}$$

where the operators A and B are defined by

$$A\phi_g(\boldsymbol{r})=(-\nabla D_g\nabla+\Sigma_{rg})\phi_g(\boldsymbol{r}) -\sum_{g''}\Sigma_s(g\leftarrow g'')\phi_{g''}(\boldsymbol{r}), \tag{3}$$

$$B\phi_g(\boldsymbol{r})=\chi_g\sum_{g''}\nu\Sigma_{fg''}(\boldsymbol{r})\phi_{g''}(\boldsymbol{r}) =\chi_g F\phi_g(\boldsymbol{r}), \tag{4}$$

where D_g is the diffusion coefficient, Σ_{rg} the removal cross section, $\nu\Sigma_{fg}$ the fission cross section multiplied by the number of fission neutrons, χ_g the average fission neutron spectrum for the prompt and delayed neutrons of the g-th group as given by Eq. (33) later and $\Sigma_s(g\leftarrow g'')$ the scattering cross section from the g''-th group to the g-th group. In Eq. (4), the fission operator F is defined by

$$F\phi_g(\boldsymbol{r})=\sum_{g''}\nu\Sigma_{fg''}(\boldsymbol{r})\phi_{g''}(\boldsymbol{r}). \tag{5}$$

Using the adjoint operator A^+ of the operator A of Eq. (3),

$$A^+=-\nabla D_g\nabla+\Sigma_{rg}-\sum_{g''}\Sigma_s(g''\leftarrow g), \tag{6}$$

we define an importance function $G_m(\boldsymbol{r}, g)$ by

$$A^+G_m(\boldsymbol{r}, g)=\nu\Sigma_{fg}(\boldsymbol{r})\delta_m(\boldsymbol{r}), \tag{7}$$

where $\delta_m(\boldsymbol{r})$ is defined by

$$\delta_m(\boldsymbol{r})=\begin{cases}1 & \boldsymbol{r}\in V_m\\ 0 & \boldsymbol{r}\notin V_m.\end{cases} \tag{8}$$

Here, the notation V_m denotes the m-th region of the reactor, and it is assumed that the region V_m is chosen such that there is a non-zero fission cross section in the region. This importance function must satisfy the following boundary condition;

$$G_m(\boldsymbol{r}, g)=0, \qquad \text{at } \boldsymbol{r}\in S, \tag{9}$$

in order to make the boundary term vanish which appears in the partial integration of the differential operator term of Eq. (7).

The Green's function defined by

$$A^+G(g, \boldsymbol{r}; g', \boldsymbol{r}')=\delta(\boldsymbol{r}-\boldsymbol{r}')\delta_{gg'}, \tag{10}$$

with the boundary condition of Eq. (1), has the relation with the importance function as

$$G_m(\boldsymbol{r}, g)=\int_{V_m}d\boldsymbol{r}'\sum_{g'}\nu\Sigma_{fg'}(\boldsymbol{r}') \cdot G(g, \boldsymbol{r}; g', \boldsymbol{r}'). \tag{11}$$

Now, we assume that a perturbation of changing cross sections is introduced in the reactor described by Eq. (2), and the operators A, B and F are changed to A', B' and F' by δA, δB and δF, respectively, namely

$$A'=A+\delta A,\ B'=B+\delta B,\ F'=F+\delta F. \tag{12}$$

With this perturbation, the system deviates from the critical state, and we assume that the flux changes according to the following time dependent multi-group diffusion equation with delayed neutrons,

$$\frac{1}{v_g}\cdot\frac{\partial\phi_g(\boldsymbol{r},t)}{\partial t}=\left(-A'+\frac{(1-\beta)}{k_0}B^{p'}\right)\phi_g(\boldsymbol{r},t)+\sum_i\chi^d_{ig}\lambda_i C_i(\boldsymbol{r},t). \quad (13)$$

Here B^p is the fission operator for prompt neutrons defined by

$$B^{p'}\phi_g(\boldsymbol{r},t)=\chi^p_g F'\phi_g(\boldsymbol{r},t)=\chi^p_g(F+\delta F)\phi_g(\boldsymbol{r},t), \quad (14)$$

and $C_i(\boldsymbol{r},t)$, λ_i and χ^d_{ig} are the density of the delayed neutron precursor, its decay constant, and the energy spectrum of delayed neutrons, of i-th delayed neutron group, respectively; χ^p_g is the prompt fission neutron spectrum, v_g the mean neutron velocity of g-th group and β the delayed neutron fraction. The delayed neutron precursor density satisfies the following equation

$$\frac{\partial C_i(\boldsymbol{r},t)}{\partial t}=\frac{1}{k_0}\beta_i F'\phi_g(\boldsymbol{r},t)-\lambda_i C_i(\boldsymbol{r},t), \quad (15)$$

where β_i is the fraction of the i-th delayed neutron group, and the sum of β_i is equal to β.

Operating $\int_V d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)$ to Eq. (13), and $\int_V d\boldsymbol{r}\sum_g\phi_g(\boldsymbol{r},t)$ to Eq. (7), and making a difference of the resulting two equations, we obtain

$$\begin{aligned}&\int_V d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\frac{1}{v_g}\cdot\frac{\partial\phi_g(\boldsymbol{r},t)}{\partial t}\\&=-\int_V F\phi_g(\boldsymbol{r},t)d\boldsymbol{r}\\&\quad-\int_V d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\delta A\phi_g(\boldsymbol{r},t)\\&\quad+\frac{1}{k_0}\int_V d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\\&\qquad\cdot(1-\beta)\chi^p_g(F+\delta F)\phi_g(\boldsymbol{r},t)\\&\quad+\int_V d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\sum_i\chi^d_{ig}\lambda_i C_i(\boldsymbol{r},t). \quad (16)\end{aligned}$$

We define the time dependent fission source with the perturbed fission operator by

$$S(\boldsymbol{r},t)=F'\phi_g(\boldsymbol{r},t)=(F+\delta F)\phi_g(\boldsymbol{r},t), \quad (17)$$

the time dependent coupling coefficients for prompt and delayed neutrons by

$$k^P_{mn}(t)=\frac{\int_{V_n}d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\chi^p_g F'\phi_g(\boldsymbol{r},t)}{\int_{V_n}d\boldsymbol{r}F'\phi_g(\boldsymbol{r},t)}, \quad (18)$$

and

$$k^d_{imn}(t)=\frac{\int_{V_n}d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\chi^d_{ig}C_i(\boldsymbol{r},t)}{\int_{V_n}d\boldsymbol{r}C_i(\boldsymbol{r},t)}, \quad (19)$$

The respectively time constant and delayed neutron fractions are defined by

$$l_m(t)=\frac{\int_V d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\frac{1}{v_g}\cdot\frac{\partial\phi_g(\boldsymbol{r},t)}{\partial t}}{\int_{V_m}d\boldsymbol{r}\frac{\partial F'\phi_g(\boldsymbol{r},t)}{\partial t}}, \quad (20)$$

and

$$\beta_{mn}(t)=\frac{\int_{V_n}d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\beta\chi^p_g F'\phi_g(\boldsymbol{r},t)}{\int_{V_n}d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\chi^p_g F'\phi_g(\boldsymbol{r},t)}, \quad (21)$$

respectively.

We define also the direct changes of coupling coefficients due to the perturbation of δA and δF by

$$\Delta k^A_{mn}(t)=\frac{\int_{V_n}d\boldsymbol{r}\sum_g G_m(\boldsymbol{r},g)\delta A\phi_g(\boldsymbol{r},t)}{\int_{V_n}d\boldsymbol{r}F'\phi_g(\boldsymbol{r},t)}, \quad (22)$$

$$\Delta k^F_m(t)=\frac{\int_{V_m}d\boldsymbol{r}\delta F\phi_g(\boldsymbol{r},t)}{\int_{V_m}d\boldsymbol{r}F'\phi_g(\boldsymbol{r},t)}. \quad (23)$$

Using Eqs. (17)~(23), Eq. (16) can be written

$$\begin{aligned}l_m(t)\frac{dS_m(t)}{dt}&=-(1-\Delta k^F_m(t))S_m(t)\\&\quad+\sum_{n=1}^N\left[\frac{1}{k_0}(1-\beta_{mn}(t))k^p_{mn}(t)-\Delta k^A_{mn}(t)\right]S_n(t)\\&\quad+\sum_{n=1}^N\sum_i k^d_{imn}(t)\lambda_i C_{in}(t), \quad (24)\end{aligned}$$

where N is the number of reactors or regions in a reactor, and

$$S_m(t)=\int_{V_m} S(\boldsymbol{r}, t)d\boldsymbol{r},$$

$$C_{im}(t)=\int_{V_m} C_i(\boldsymbol{r}, t)d\boldsymbol{r}. \quad (25)$$

The $S_m(t)$ and $C_{im}(t)$ are the fission neutrons produced in a unit time and the delayed neutron precursor in the region V_m, respectively.

Integrating Eq. (15) over V_m and using the notation of Eqs. (25), we obtain the equation for the precursor of the delayed neutrons;

$$\frac{dC_{im}(t)}{dt}=\frac{1}{k_0}\beta_{im}(t)S_m(t)-\lambda_i C_{im}(t), \quad (26)$$

where $$\beta_{im}(t)=\frac{\int_{V_m} d\boldsymbol{r}\beta_i F'\phi_g(\boldsymbol{r}, t)}{\int_{V_m} d\boldsymbol{r}F'\phi_g(\boldsymbol{r}, t)}. \quad (27)$$

Equations (24) and (26) are the rigorous nodal kinetics equations for coupled reactors or multi-point reactor kinetics equations.

If the perturbations δA and δB are independent of time, the time dependence of the parameters defined by Eqs. (18)~(23) can be eliminated as follows. Assuming that the flux and precursor density have the form

$$\phi_g(\boldsymbol{r}, t)=\phi_{g\omega}(\boldsymbol{r})e^{\omega t}$$
$$C_i(\boldsymbol{r}, t)=C_{i\omega}(\boldsymbol{r})e^{\omega t}, \quad (28)$$

the nodal flux and precursor of Eqs. (25) can be written in the form

$$S_m(t)=\int_{V_m} S(\boldsymbol{r}, t)d\boldsymbol{r}=S_{m\omega}e^{\omega t},$$

$$C_{im}(\boldsymbol{r}, t)=\int_{V_m} C_i(\boldsymbol{r}, t)d\boldsymbol{r}=C_{im\omega}e^{\omega t}. \quad (29)$$

Substituting these into Eq. (26) to obtain C_{im}, and then into Eq. (24), we obtain a system of homogeneous linear equations for $S_{m\omega}$,

$$\omega l_m S_{m\omega}=-(1-\Delta k_m^F)S_{m\omega}+\sum_n\left[\frac{1}{k_0}(1-\beta_{mn})k_{mn}^P-\Delta k_{mn}^A\right]S_{n\omega}+\frac{1}{k_0}\sum_n\sum_i k_{imn}^d\frac{\lambda_i\beta_{in}}{\omega+\lambda_i}S_{n\omega}. \quad (30)$$

Here k_{mn}^P, k_{mni}^d, *etc.* are time independent kinetics parameters defined by Eqs. (18), (19) *etc.* where $\phi_g(\boldsymbol{r}, t)$ and $C_i(\boldsymbol{r}, t)$ are replaced by $\phi_{\omega g}(\boldsymbol{r})$ and $C_{i\omega}(\boldsymbol{r})$ of Eqs. (28), respectively.

For example, the time constant of Eq. (20) becomes

$$l_m=\frac{\int_V d\boldsymbol{r}\sum_g G_m(\boldsymbol{r}, g)\frac{1}{v_g}\phi_{g\omega}(\boldsymbol{r})}{\int_{V_m} d\boldsymbol{r}F'\phi_{g\omega}(\boldsymbol{r})}. \quad (31)$$

Substituting Eqs. (28) into Eqs. (13) and (15), we obtain equations for $\phi_{\omega g}(\boldsymbol{r})$ and $C_{i\omega}(\boldsymbol{r})$.

We define the coupling coefficient k_{mn} by

$$k_{mn}=(1-\beta_{mn})k_{mn}^P+\sum_i \beta_{im}k_{imn}^d=\frac{\int_{V_n} d\boldsymbol{r}\sum_g G_m(\boldsymbol{r}, g)\chi_g F'\phi_{\omega g}(\boldsymbol{r})}{\int_{V_n} d\boldsymbol{r}F'\phi_{\omega g}(\boldsymbol{r})}. \quad (32)$$

The χ_g in Eq. (32) is the average fission spectrum for the prompt and delayed neutrons used in Eq. (4), namely

$$\chi_g=(1-\beta)\chi_g^p+\sum_i \beta_i\chi_{ig}^d. \quad (33)$$

Using the coupling coefficient of Eq. (32), Eq. (30) can be rewritten

$$\omega l_m S_{m\omega}=-(1-\Delta k_m^F)S_{m\omega}+\sum_{n=1}^N\left(\frac{1}{k_0}k_{mn}-\Delta k_{mn}^A-\omega\frac{1}{k_0}\sum_i\frac{\bar{\beta}_{imn}}{\omega+\lambda_i}\right)S_{n\omega}, \quad (34)$$

where $\bar{\beta}_{imn}$ is the effective delayed neutron fraction of the i-th group,

$$\bar{\beta}_{imn}=\beta_{im}k_{imn}^d. \quad (35)$$

The condition that the determinant of the coefficients matrix of Eq. (34) vanishes gives the equation which determines the discrete values of ω, namely

$$\det\left|(\omega l_m+1-\Delta k_m^F)\delta_{mn}-\frac{1}{k_0}k_{mn}+\Delta k_{mn}^A+\frac{1}{k_0}\omega\sum_i\frac{\bar{\beta}_{imn}}{\omega+\lambda_i}\right|=0. \quad (36)$$

The solution of the flux and precursor density is expressed as the sum of exponential functions of the form of Eqs. (29) for all these roots of Eq. (36).

In the case of only one region, the multi-point reactor kinetics equations of Eqs. (24) and (26) for $N=1$ are reduced to the point-reactor kinetics equations as

$$l_1(t)\frac{dS_1(t)}{dt}=-(1-\Delta k_1^F(t))S_1(t)+\left[\frac{1}{k_0}(1-\beta_{11}(t))k_{11}^P(t)-\Delta k_{11}^A(t)\right]S_1(t)+\sum_i k_{i11}^d(t)\lambda_i C_{i1}(t), \quad (37)$$

$$\frac{dC_{i1}(t)}{dt}=\frac{1}{k_0}\beta_{i1}(t)S_1(t)-\lambda_i C_{i1}(t). \quad (38)$$

If we define the reactivity $\rho(t)$ due to the perturbation by

$$\rho(t)=\frac{k_{11}^P(t)}{k_0}-1+\Delta k_1^F(t)-\Delta k_{11}^A(t), \quad (39)$$

Eq. (37) can be written as

$$l_1(t)\frac{dS_1(t)}{dt}=\left(\rho(t)-\frac{1}{k_0}\beta_{11}(t)k_{11}^P(t)\right)S_1(t)+\sum_i k_{i11}^d(t)\lambda_i C_{i1}(t). \quad (40)$$

Equation (34) for $N=1$ is

$$\omega l_1 S_{1\omega}=\left(-1+\frac{k_{11}}{k_0}+\Delta k_1^F-\Delta k_{11}^A-\frac{\omega}{k_0}\sum_i\frac{\bar{\beta}_{i11}}{\omega+\lambda_i}\right)S_{1\omega}. \quad (41)$$

Using the time independent reactivity of the same form of Eq. (39), we obtain a reactivity equation for the point-reactor from Eq. (41) as

$$\rho=\omega\left(l_1+\frac{1}{k_0}\sum_i\frac{\bar{\beta}_{i11}}{\omega+\lambda_i}\right). \quad (42)$$

Equations (40) and (42) have the similar forms as the usual point reactor kinetics equations and the usual reactivity equation[9], respectively, however there is a difference in the definition of kinetics parameters. Further, the dependent variables $S_1(t)$ and $S_{1\omega}$ have the physical meaning of the number of the fission neutrons in the present method, whereas in the usual point reactor kinetics equations, $S_1(t)$ corresponds to the number of neutrons, when the weighting function is chosen to be constant[13]. The present equations hold exactly for any geometry independent of the strength of the perturbation.

III. Some Examples for Simple Geometries

Let us derive the coupling coefficients and kinetics parameters for some simple geometries explicitly, which will help the physical understanding of the coupled reactors and the structure of the present theory. We can also confirm that the results are consistent with the solution obtained directly from the diffusion equation of Eq. (2) or Eqs. (13) and (15).

1. Single Core Reactor

Now, we consider the kinetics parameters of Eq. (37) for the simple case explicitly, the case of one group and one-dimensional slab geometry. We assume that a homogeneous reactor is placed at $-a/2\leqq x\leqq a/2$, and the flux vanishes at $x=\pm a/2$. A perturbation is assumed to be uniform in the reactor, namely, $\delta A=\Delta\Sigma_a=$constant and $\delta B=\Delta\nu\Sigma_f=$constant in the reactor, where Σ_a is the absorption cross section. With this perturbation, the diffusion equation of Eq. (13) becomes

$$\frac{1}{v}\cdot\frac{\partial\phi(x,t)}{\partial t}=\left[D\frac{d^2}{dx^2}-(\Sigma_a+\Delta\Sigma_a)+\frac{(1-\beta)}{k_0}(\nu\Sigma_f+\Delta\nu\Sigma_f)\right]\phi(x,t)+\sum_i\lambda_i C_i(x,t). \quad (43)$$

We assume that the reactor is just critical before the perturbation, and the flux and precursor density are in the fundamental mode of spatial distribution. Since cross sections change uniformly, the flux and precursor density remain the same fundamental mode also after the perturbation, and they can be written in the form

$$\phi(x,t)=\phi(t)\cos Bx$$
$$C_i(x,t)=C_i(t)\cos Bx, \quad (44)$$

where $B=\pi/a$.

Green's function of Eq. (10) for this slab geometry is

$$G(x;x')=\frac{\sinh\kappa(x+a/2)\sinh\kappa(a/2-x')}{D\kappa\sinh\kappa a}, \quad x\leqq x', \quad (45)$$

where $\kappa^2=\Sigma_a/D$. The Green's function

$G(x;x')$ for $x \gtrless x'$ is obtained by exchanging x and x' in Eq. (45). Using Eq. (45), the importance function of Eq. (11) for $N=1$ becomes

$$G_1(x)=\int_{-a/2}^{a/2} dx'\nu\Sigma_f G(x;x') = \frac{k_\infty}{\cosh\frac{\kappa a}{2}}\left(\cosh\frac{\kappa a}{2}-\cosh\kappa x\right), \quad (46)$$

where k_∞ is the infinite multiplication factor defined by $k_\infty=\nu\Sigma_f/\Sigma_a$.

Substituting the flux and importance function of Eqs. (44) and (46) into Eqs. (18)~(23), we obtain the kinetics parameters

$$k_{11}^P=k_{11i}^d=\frac{\int_V dx G_1(x)F'\phi(x,t)}{\int_V dx F'\phi(x,t)} = \frac{k_\infty}{(1+L^2B^2)}, \quad (47)$$

$$l_1=\frac{\int_V dx G_1(x)\frac{1}{v}\cdot\frac{\partial}{\partial t}\phi(x,t)}{\int_V dx\frac{\partial}{\partial t}F'\phi(x,t)} = \frac{\int_{-a/2}^{a/2} dx G_1(x)\frac{1}{v}\cos Bx}{\int_{-a/2}^{a/2} dx(\nu\Sigma_f+\Delta\nu\Sigma_f)\cos Bx} = \frac{l_\infty\nu\Sigma_f}{(1+L^2B^2)(\nu\Sigma_f+\Delta\nu\Sigma_f)} = \frac{l_{\text{eff}}}{1+\Delta\nu\Sigma_f/\nu\Sigma_f}, \quad (48)$$

$$\bar\beta_{i11}=\frac{\beta_i k_\infty}{1+L^2B^2}, \quad (49)$$

where l_∞ is the neutron life time in the infinite system and l_{eff} the effective neutron life time in the finite and unperturbed system defined by

$$l_\infty=\frac{1}{\Sigma_a v} \quad \text{and} \quad l_{\text{eff}}=\frac{l_\infty}{(1+L^2B^2)}. \quad (50)$$

The changes of coupling coefficients due to the perturbation are

$$\Delta k_1^F=\frac{\int_V dx\delta F\phi(x,t)}{\int_V dx F'\phi(x,t)}=\frac{\Delta\nu\Sigma_f}{\nu\Sigma_f+\Delta\nu\Sigma_f}, \quad (51)$$

$$\Delta k_{11}^A=-\frac{\int_V dx G_1(x)\delta A\phi(x,t)}{\int_V dx F'\phi(x,t)} = \frac{\nu\Sigma_f\Delta\Sigma_a}{\Sigma_a(1+L^2B^2)(\nu\Sigma_f+\Delta\nu\Sigma_f)}. \quad (52)$$

For this simple case, we can show easily that the kinetics equation of Eq. (37) with parameters of Eqs. (47)~(52) agree with the diffusion Eq. (43). Namely, substituting Eqs. (44) into Eq. (43), Eq. (43) becomes

$$\frac{1}{v}\cdot\frac{d\phi(t)}{dt}=\left[-DB^2-(\Sigma_a+\Delta\Sigma_a)+\frac{(1-\beta)}{k_0}(\nu\Sigma_f+\Delta\nu\Sigma_f)\right]\phi(t)+\sum_i\lambda_i C_i(t). \quad (53)$$

Using Eqs. (17) and (25), we obtain

$$S_1(t)=\frac{2}{B}(\nu\Sigma_f+\Delta\nu\Sigma_f)\phi(t). \quad (54)$$

Dividing Eq. (53) by $\Sigma_a(1+L^2B^2)$ and using Eq. (54), Eq. (53) can be rewritten as

$$\frac{1}{v\Sigma_a(1+L^2B^2)(\nu\Sigma_f+\Delta\nu\Sigma_f)}\cdot\frac{dS_1(t)}{dt} = \left[-\frac{1}{(\nu\Sigma_f+\Delta\nu\Sigma_f)} - \frac{\Delta\Sigma_a}{\Sigma_a(1+L^2B^2)(\nu\Sigma_f+\Delta\nu\Sigma_f)} + \frac{(1-\beta)}{k_0(1+L^2B^2)\Sigma_a}\right]S_1(t) + \frac{1}{(1+L^2B^2)\Sigma_a}-\sum_i\lambda_i C_i(t), \quad (55)$$

which can be shown to be identical with Eq. (37) with parameters of Eqs. (47)~(52).

It seems strange at the first glance that the neutron life time given by Eq. (48) is independent of the perturbed cross section of $\Delta\Sigma_a$. The reason is seen in Eq. (55) which is derived directly from the diffusion equation that the coefficient of the differential term of the left hand side of Eq. (55) which corresponds to the neutron life time does not include $\Delta\Sigma_a$. In the previous formulations[7], the perturbation is included in the importance function, and the equation corresponds to Eq.

(55) is

$$\frac{1}{v(\Sigma_a+\Delta\Sigma_a)(1+L'^2B^2)(\nu\Sigma_f+\Delta\nu\Sigma_f)}\cdot\frac{dS_i(t)}{dt}$$
$$=\left[-\frac{1}{(\nu\Sigma_f+\Delta\nu\Sigma_f)}+\frac{(1-\beta)}{k_0(1+L'^2B^2)(\Sigma_a+\Delta\Sigma_a)}\right]S_i(t)$$
$$+\frac{1}{(1+L'^2B^2)(\Sigma_a+\Delta\Sigma_a)}\sum_i\lambda_iC_i(t), \tag{56}$$

where $L'^2=D/(\Sigma_a+\Delta\Sigma_a)$, which is obtained by replacing Σ_a by $\Sigma_a'=\Sigma_a+\Delta\Sigma_a$ and dividing Eq. (53) by $\Sigma_a'(1+L'^2B^2)$. Thus the neutron life time of the previous definition changes as a function of $\Delta\Sigma_a$. Since no approximation is made to derive both kinetics equations, the definition of the other kinetics parameters is different such that the resulting kinetics equations are consistent. The present definition may be convenient for the practical application, since the life time for the unperturbed system can be used for the perturbation of $\Delta\Sigma_f=0$.

2. Two Region Treatment of a Core

Next, as a simple case of tight coupling, we consider the previous bare homogeneous core with two regions by dividing it at the center of the core. The diffusion equation for this geometry is also Eq. (43). For simplicity, we consider here only the unperturbed system. We call the region $-a/2\leqq x\leqq 0$ as the region V_1, and $0\leqq x\leqq a/2$ as the region V_2.

The coupling coefficients of Eq. (18) can be calculated using Eq. (11) and Green's function of Eq. (45) as

$$k_{12}=\frac{\nu\Sigma_f\int_0^{a/2}dx'F'\phi(x',t)\int_{-a/2}^{0}dxG(x';x)}{\int_0^{a/2}dx'F'\phi(x',t)}$$
$$=\frac{BL\left(\cosh\frac{\kappa a}{2}-1\right)}{2\sinh\frac{\kappa a}{2}}\cdot\frac{k_\infty}{(1+L^2B^2)}, \tag{57a}$$

$$k_{22}=\frac{\nu\Sigma_f\int_0^{a/2}dx'F'\phi(x',t)\int_0^{a/2}dxG(x';x)}{\int_0^{a/2}dx'F'\phi(x',t)}$$
$$=\left(1-\frac{BL\left(\cosh\frac{\kappa a}{2}-1\right)}{2\sinh\frac{\kappa a}{2}}\right)\frac{k_\infty}{(1+L^2B^2)}. \tag{57b}$$

The coupling coefficients $k_{21}=k_{12}$ and $k_{11}=k_{22}$, because the reactor is symmetric with respect to $x=0$.

Since the reactor is assumed to be critical without perturbation, $\Delta k_1^F=\Delta k_{11}^A=\omega=0$, and Eq. (36) for $N=2$ becomes

$$\det\begin{vmatrix}1-\frac{1}{k_0}k_{11}, & -\frac{1}{k_0}k_{12}\\ -\frac{1}{k_0}k_{21}, & 1-\frac{1}{k_0}k_{22}\end{vmatrix}=0. \tag{58}$$

From Eq. (58) with the coupling coefficients of Eqs. (57), we obtain the criticality factor for the whole system of the unperturbed system,

$$k_0=k_{22}+k_{21}=\frac{k_\infty}{1+L^2B^2}, \tag{59}$$

which agree with the criticality factor derived directly from Eq. (43) without perturbation, which is equal to k_{11} of Eq. (47).

If the size of the reactor a is much larger than the diffusion length, namely $\kappa a=a/L>3$, the coupling coefficient of Eq. (57a) becomes

$$k_{12}\doteqdot\frac{BL}{2}k_0=\frac{\pi L}{2a}, \tag{60}$$

with $k_0=1$ and $B=\pi/a$. Namely, the strength of the coupling between the two parts of the bare homogeneous reactor becomes weak proportionally with the inverse of the thickness a of the core.

3. Two Cores System

We consider here the system where two separate cores of the infinite slab are in an infinite moderator. In order to obtain simple analytical expressions, the thickness T of the slab is assumed to be very thin and its absorption and fission cross sections of the core can be expressed by the δ-function. Namely, we assume that the very thin cores are at $x=-d/2$ and $x=d/2$ with the distance d, and the unperturbed system of Eq. (2) is expressed by the following one group diffusion equation:

$$\left[-D\frac{d^2}{dx^2}+\Sigma_{ar}+\Sigma_{ac}T(\delta(x+d/2)+\delta(x-d/2))\right]\phi(x)=\frac{1}{k_0}\nu\Sigma_f T(\delta(x+d/2)+\delta(x-d/2))\phi(x), \tag{61}$$

where the suffixes c and r denote the core and reflector, respectively. The boundary condition for the flux is that $\phi(x)\to 0$ for $x\to\pm\infty$. We use the index $m=1$ for the core at $x=-d/2$ and $m=2$ for the core at $x=d/2$.

Analytical solution of Eq. (61) can be easily obtained. The solution for $-d/2\leqq x\leqq d/2$ has the form with an arbitrary constant A,

$$\phi(x)=A\cosh\kappa x,\quad -d/2\leqq x\leqq d/2,\qquad \kappa^2=\Sigma_{ar}/D, \tag{62}$$

since the system is symmetric at $x=0$. The flux for $x\leqq -d/2$ or $d/2\leqq x$ satisfying the boundary condition at $x=\pm\infty$ has a form

$$\phi(x)=\begin{cases}A\cosh\kappa d/2\,e^{\kappa(x+d/2)}, & -\infty<x\leqq -d/2\\ A\cosh\kappa d/2\,e^{-\kappa(x-d/2)}, & d/2\leqq x<\infty,\end{cases} \tag{63}$$

since the flux of Eq. (63) must be continuous with that of Eq. (62) at $x=\pm d/2$. Integrating Eq. (61) from $x=d/2-0$ to $x=d/2+0$, we obtain the discontinuity condition for the current that the flux must satisfy at $x=d/2$,

$$-D\frac{d\phi(x)}{dx}\Big|_{x=d/2-0}^{x=d/2+0}=\left(\frac{1}{k_0}\nu\Sigma_f-\Sigma_{ac}\right)T\phi(d/2). \tag{64}$$

Substituting Eqs. (62) and (63) into Eq. (64), we obtain

$$D\kappa(\cosh\kappa d/2+\sinh\kappa d/2)A=A\left(\frac{1}{k_0}\nu\Sigma_f-\Sigma_{ac}\right)T\cosh\kappa d/2, \tag{65}$$

which gives the criticality factor for the unperturbed system,

$$k_0=\frac{k_\infty}{1+\alpha\exp(\kappa d/2)/\cosh(\kappa d/2)}, \tag{66}$$

where $k_\infty=\nu\Sigma_f/\Sigma_{ac}$, $\alpha=\dfrac{\Sigma_{ar}L_r}{\Sigma_{ac}T}$, $L_r=1/\kappa$. (67)

In the case that there is only one core at $x=d/2$, the flux is

$$\phi(x)=\begin{cases}A\,e^{\kappa(x-d/2)}, & -\infty<x\leqq d/2\\ A\,e^{-\kappa(x-d/2)}, & d/2\leqq x<\infty.\end{cases} \tag{68}$$

Substituting this flux into the discontinuity condition of the current of Eq. (64), we obtain the criticality factor k_0 for the case of single core,

$$k_0=\frac{k_\infty}{1+2\alpha}. \tag{69}$$

This criticality factor can be also obtained from the diffusion equation for the system in which the core of the finite thickness of T is in an infinite moderator by assuming that the diffusion coefficient of the core is very large, which means that the flux distribution in the core is flat. We can see that the criticality factor of Eq. (66) for the two cores tends to that of Eq. (69) for the single core as the distance d tends to infinity as expected.

Now, we derive the kinetics parameters for the coupled reactor of Eq. (61). Equation (7) for the importance function of $m=2$ is

$$\left[-D\frac{d^2}{dx^2}+\Sigma_{ar}+\Sigma_{ac}T(\delta(x+d/2)+\delta(x-d/2))\right]G_2(x)=\nu\Sigma_f T\delta(x-d/2). \tag{70}$$

Integrating Eq. (70) from $x=d/2-0$ to $x=d/2+0$ and from $x=-d/2-0$ to $x=-d/2+0$, we obtain the conditions for $G_2(x)$,

$$-D\frac{dG_2(x)}{dx}\Big|_{x=d/2-0}^{x=d/2+0}+\Sigma_{ac}TG_2(d/2)=\nu\Sigma_f T, \tag{71}$$

$$-D\frac{dG_2(x)}{dx}\Big|_{x=-d/2-0}^{x=-d/2+0}+\Sigma_{ac}TG_2(-d/2)=0, \tag{72}$$

$G_2(x)$ must be continuous at $x=-d/2$ and $x=d/2$, namely

$$G_2(x)|_{x=-d/2-0}=G_2(x)|_{x=-d/2+0},\qquad G_2(x)|_{x=d/2-0}=G_2(x)|_{x=d/2+0}. \tag{73}$$

The solution of Eq. (70) can be written in the form

$$G_2(x)=\begin{cases} A\,e^{-\kappa(x-d/2)}, & d/2\leqq x\leqq\infty \\ C\,e^{-\kappa x}+E\,e^{\kappa x}, & -d/2\leqq x\leqq d/2 \\ F\,e^{\kappa(x+d/2)}, & -\infty\leqq x\leqq -d/2, \end{cases} \quad (74)$$

with appropriate constants A, C, E and F. Substituting Eq. (74) into Eqs. (71)~(73), we obtain the constants of Eq. (74),

$$A=\frac{k_\infty}{\beta}[(1+2\alpha)\exp(\kappa d)-\exp(-\kappa d)]$$

$$C=-\frac{k_\infty}{\beta}\exp(-\kappa d/2)$$

$$E=\frac{k_\infty}{\beta}(1+2\alpha)\exp(\kappa d/2),\quad F=\frac{2k_\infty\alpha}{\beta}$$

$$\beta=(1+2\alpha)^2\exp(\kappa d)-\exp(-\kappa d). \quad (75)$$

The coupling coefficient of Eq. (18) in a core of the unperturbed system is

$$k_{22}^p=\frac{\int_{V_2}dx\,G_2(x)\nu\Sigma_f T\delta(x-d/2)\phi(x)}{\int_{V_2}dx\nu\Sigma_f T\delta(x-d/2)\phi(x)}$$
$$=G_2(d/2)=A, \quad (76)$$

and the coupling coefficient between two cores is

$$k_{21}^p=\frac{\int_{V_1}dx\,G_2(x)\nu\Sigma_f T\delta(x+d/2)\phi(x)}{\int_{V_1}dx\nu\Sigma_f T\delta(x+d/2)\phi(x)}$$
$$=G_2(-d/2)=F, \quad (77)$$

which are independent of the flux $\phi(x)$, because the width of the core is very thin and the flux distribution in a core is constant. Since the system is symmetric, $k_{11}=k_{22}$ and $k_{12}=k_{21}$. The coupling coefficient k_{12} of Eq. (77) which represents the number of the fission neutrons produced in the core V_1 by the neutrons produced in the core V_2 decreases by the factor $\exp(-\kappa d)$, when κd is large, which seems reasonable, because the neutrons produced in the core V_2 decreases as $\exp(-\kappa x)$ as seen in Eq. (68).

In order to exist a non-zero solution for Eq. (34) of $N=2$, Eq. (58) for the present system must hold. Using the coefficients of Eqs. (76) and (77) in Eq. (58), we obtain the criticality factor for the unperturbed system,

$$k_0=k_{22}+k_{21}$$
$$=\frac{2k_\infty\cosh(\kappa d/2)}{(1+2\alpha)\exp(\kappa d/2)+\exp(-\kappa d/2)}, \quad (78)$$

which is the same as Eq. (66) derived directly from the diffusion equation. This confirms that the coupling coefficients of Eqs. (76) and (77) are exact and consistent with the original diffusion equation of Eq. (61).

The criticality factor of the core V_2 alone of the previous homogeneous reactor of Sec. III-2 is $k_0=k_\infty/(1+4L^2B^2)$ with the buckling $B^2=(\pi/a)^2$ for the whole reactor used in Eq. (57b), when the core V_2 is isolated from the core V_1, which is much different from the criticality factor k_{22} of Eq. (57b). On the other hand, the coupling coefficient k_{22} of Eq. (76) is nearly the same as the criticality factor k_0 of Eq. (69) for the isolated single core, when $3<\kappa d$. This is because the flux distribution in the region V_2 of the homogeneous reactor joined to the region V_1 in Sec. III-2 of Eq. (57b) is largely different from that in which the region V_2 is isolated, and the leakage probability and hence the fission probability in the region V_2 by the neutrons born in the region V_2 changes largely.

The time constant l_2 for the steady state of two slab cores of Eq. (61) is obtained using the importance function of Eq. (74) and the flux of Eqs. (62), (63) regarding as $\phi_\omega(x)$ for $\omega=0$ as

$$l_2=\frac{\frac{1}{v}\int_{-\infty}^{\infty}dx\,G_2(x)\phi(x)}{\int_{V_2}dx\,F'\phi(x)}$$
$$=\frac{l_c(e^{\kappa d}+\kappa d+1)}{\kappa T[e^{\kappa d}+e^{-\kappa d}+2\alpha(e^{\kappa d}+1)+2]}, \quad (79)$$

where $l_c=1/(\Sigma_{ac}v)$ is the neutron life time in the infinite core. Since the system is symmetric, $l_1=l_2$. As the distance d between the cores tends to infinity, this life time approaches as

$$l_2\to\frac{l_c}{\kappa T(1+2\alpha)}, \quad (80)$$

which is the life time of the system of the single core at $x=d/2$. These neutron life

times depend on the diffusion parameters of the moderator.

If a perturbation of $\delta A=\Delta\Sigma_a T\delta(x-d/2)$ and $\delta B=\Delta\nu\Sigma_f T\delta(x-d/2)$ is applied only to the core V_2, the coupling coefficients of Eqs. (22) and (23) due to this perturbation are

$$\Delta k_{22}^A=\frac{\int_{V_2} dx G_2(x)\Delta\Sigma_a T\delta(x-d/2)\phi(x,t)}{\int_{V_2} dx(\nu\Sigma_f+\Delta\nu\Sigma_f)T\delta(x-d/2)\phi(x,t)}=\frac{k_{22}^P\Delta\Sigma_a}{(\nu\Sigma_f+\Delta\nu\Sigma_f)}, \tag{81}$$

$$\Delta k_{12}^A=\frac{\int_{V_2} dx G_1(x)\Delta\Sigma_a T\delta(x-d/2)\phi(x,t)}{\int_{V_2} dx(\nu\Sigma_f+\Delta\nu\Sigma_f)T\delta(x-d/2)\phi(x,t)}=\frac{k_{12}^P\Delta\Sigma_a}{(\nu\Sigma_f+\Delta\nu\Sigma_f)}, \tag{82}$$

$$\Delta k_2^F=\frac{\int_{V_2} dx\Delta\nu\Sigma_f T\delta(x-d/2)\phi(x,t)}{\int_{V_2} dx(\nu\Sigma_f+\Delta\nu\Sigma_f)T\delta(x-d/2)\phi(x,t)}=\frac{\Delta\nu\Sigma_f}{\nu\Sigma_f+\Delta\nu\Sigma_f}. \tag{83}$$

Clearly, $\Delta k_{21}^A=\Delta k_1^F=0$, and the coupling coefficients of Eqs. (71) and (72) do not change by this perturbation.

The time constant for the perturbed system can be calculated by Eq. (31) using the flux for the perturbed system. If the perturbation is small, the time constant for the unperturbed system given by Eq. (79) may be a good approximation for the perturbed system.

IV. DISCUSSIONS

As an example to use Eq. (34), we consider the flux tilt, namely the global change of the spatial flux distribution, which may happen even when a small perturbation is introduced locally. We consider a whole reactor by dividing it into appropriate two regions V_1 and V_2 and assume that a perturbation is introduced only in the region V_2. The whole reactor is assumed to be critical with $k_0=1$ before the perturbation, and we consider the asymptotic time region after the perturbation. Neglecting the delayed neutrons, Eq. (34) for $N=2$ becomes

$$\omega l_1 S_{1\omega}=(-1+k_{11})+S_{1\omega}+(k_{12}-\Delta k_{12}^A)S_{2\omega}, \tag{84a}$$

$$\omega l_2 S_{2\omega}=k_{21}S_{1\omega}+(-1+\Delta k_2^F+k_{22}-\Delta k_{22}^A)S_{2\omega}. \tag{84b}$$

From Eq. (84b), the ratio of the fission source $S_{2\omega}$ in the region V_2 to $S_{1\omega}$ of the region V_1 is

$$\frac{S_{2\omega}}{S_{1\omega}}=\frac{k_{21}}{1-k_{22}+\omega l_2-\Delta k_2^F+\Delta k_{22}^A}. \tag{85}$$

Before the perturbation is introduced, $\Delta k_2^F=\Delta k_{22}^A=\omega=0$, and in the case $k_{12}=k_{21}$, this becomes

$$\frac{S_{2\omega}}{S_{1\omega}}=\frac{k_{21}}{1-k_{22}}=\frac{1-k_{11}}{k_{12}}=\left(\frac{1-k_{11}}{1-k_{22}}\right)^{1/2}. \tag{86}$$

From this, it is seen that the fission source $S_{2\omega}$ is larger than $S_{1\omega}$, if the criticality factor of the region V_2, k_{22} is larger than k_{11}, and $S_{2\omega}=S_{1\omega}$, if $k_{22}=k_{11}$.

If the perturbation is small, the change of the coupling coefficients k_{21}, k_{22} and the term ωl_2 in Eq. (85) will be small compared to the terms Δk_2^F and Δk_{22}^A which vary directly to the change of the cross sections. Therefore, the change of the denominator of Eq. (85) will cause a large change of the ratio of the fission source of Eq. (85) from the value of Eq. (86), when the coupling coefficient k_{21} is small. This is because the influence of the population of neutrons in the region V_2 to the region V_1 is small, if the coupling coefficient k_{21} is small. Therefore, the coupling coefficient k_{21} between two regions gives the indication whether the flux tilt will become large or small.

If we do not mind the complexity of manipulations of equations, we can calculate the coupling coefficient and other kinetic parameters analytically for more realistic models. If we use the numerical solutions for the flux by solving multi-group diffusion or transport equations in multi-dimensions, we can obtain all parameters with reasonable accuracy, that will help to make physical understanding of coupled reactors or loosely coupled large reactors.

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