

Lecture 1: The Transport Equation

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1.1 Derivation of the Transport Equation

Neutrons move about in a reactor in complicated, zigzag paths due to repeated collisions with nuclei. As a consequence of this motion, neutrons that were originally in one part of reactor and moving in a particular direction with a particular energy appear at the later time in another part of the system, moving in another direction with some other energy. The neutrons in this case are said to be transported from the first region and energy to the second, and the study of this phenomenon is known as transport theory.

Transport theory is relatively simple in principle, and an exact equation governing transport phenomena can be derived. This is called the Boltzman equation, or the transport equation.

The transport (Boltzman) equation is a balance of neutrons for the differential time element dt , for the differential volume and speed element d^3rd^3v , of the 6 dimensional phase space, about the point (r, v) at time t . The equation is for the function $n(\mathbf{r}, \mathbf{v}, t)$, where $n(\mathbf{r}, \mathbf{v}, t)d^3rd^3vdt$ is the number of neutrons, at time t , in the volume d^3rd^3v about (\mathbf{r}, \mathbf{v}) . The neutron balance comes as a result of considering the change in the number of neutrons, in the considered space volume, in the time span dt .

On the one hand, the change is simply:

$$dn = \left[\frac{\partial n(\mathbf{r}, \mathbf{v}, t)}{\partial t} \right] d^3rd^3vdt \quad (1.1)$$

On the other hand, the change is a sum of differential changes originating in number of processes:

1. The net (positive or negative) number of neutrons leaking during dt out of the volume, namely d^3r , of the phase space element.
2. The number of neutron collisions with the medium nuclei in the time span dt , causing:
 - (a) Disappearance of the colliding neutrons from the velocity-part d^3v phase in the space-part d^3r phase
3. The number of neutron collisions with the medium nuclei in the time span dt , causing:
 - (a) Appearance of colliding neutrons in the velocity-part d^3v phase in the space-part d^3r phase
4. The number of neutrons created, during dt , at d^3rd^3v about the space point (\mathbf{r}, \mathbf{v}) by the sources (fission source is included).

The statement/equation that describes the net change (i.e. balance) is as follows:

$$dn = -(1) - (2) + (3) + (4) \quad (1.2)$$

We can now write a mathematical expression for each of these contributions (i.e. gains and losses).

1.1.1 Leakage process, term (1)

For the illustration of the process, let us suppose that d^3r is Cartesian, $d^3r = dxdydz$, a differential cube about the spatial point $\mathbf{r} = (x, y, z)$. Upon the square $dydz$, at $x + \Delta x/2$, we build an inclined, $|\mathbf{v}|dt$ long, box in the \mathbf{v} direction as shown in Fig. 1.1.

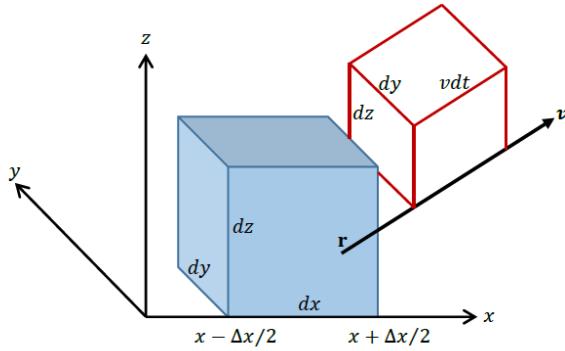


Figure 1.1: Illustration of the leakage process.

The volume of the inclined box is $(v_x dt) dy dz$. Consider the neutrons belonging to the velocity element d^3v about \mathbf{v} as they cross the $\Delta y \Delta z$ square at $x + \Delta x/2$. After a time span of dt these neutrons will be enclosed in the box we have built. Their number is the product of the box volume and the neutron density at the point $(x + \Delta x/2, y, z)$:

$$n(x + \Delta x/2, y, z, \mathbf{v}, t) (v_x dt) dy dz d^3v \equiv \text{Number of } \mathbf{v} d^3v \text{ neutrons crossing during } dt \text{ the } \Delta y \Delta z \text{ square at } x + \Delta x/2 \quad (1.3)$$

Similarly:

$$n(x - \Delta x/2, y, z, \mathbf{v}, t) (v_x dt) dy dz d^3v \equiv \text{Number of } \mathbf{v} d^3v \text{ neutrons crossing during } dt \text{ the } \Delta y \Delta z \text{ square at } x - \Delta x/2 \quad (1.4)$$

The net number of neutrons which, during dt , leak through the two parallel squares $\Delta y \Delta z$, out of the $\Delta x \Delta y \Delta z$ element, is the number exiting at $x + \Delta x/2$ minus the number entering at $x - \Delta x/2$.

Thus from the definition of derivation, the x-wise leakage is:

$$\begin{aligned} \text{x-wise leakage} &= \\ &\frac{\partial n(\mathbf{r}, \mathbf{v}, t)}{\partial x} (v_x dt) dx dy dz d^3v \end{aligned} \quad (1.5)$$

Likewise, the leakage considerations for the y direction, through the $\Delta x \Delta z$ pair of squares, and for the z direction, through the $\Delta x \Delta y$ pair of squares, yield parallel expressions. It all sums up in:

$$\begin{aligned} \text{Leakage} &= \\ &\left[\frac{\partial n}{\partial x} v_x + \frac{\partial n}{\partial y} v_y + \frac{\partial n}{\partial z} v_z \right] dt d^3 r d^3 v \end{aligned} \quad (1.6)$$

The definition of directional derivative of a scalar field $n(x, y, z)$ in the direction

$\mathbf{v}(x, y, z) = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$ is defined as:

$$\mathbf{v} \cdot \mathbf{grad}n = \left[\frac{\partial n}{\partial x} v_x + \frac{\partial n}{\partial y} v_y + \frac{\partial n}{\partial z} v_z \right] = \mathbf{v} \cdot \nabla n \quad (1.7)$$

This gives the rate of change of the field n in the direction of \mathbf{v} . Therefore, the leakage term expression becomes:

$$\begin{aligned} \text{Leakage} &= \\ &\mathbf{v} \cdot \nabla n(\mathbf{r}, \mathbf{v}, t) dt d^3 r d^3 v \end{aligned} \quad (1.8)$$

1.1.2 Loss due to collisions, term (2)

The outcome of a neutron collision with a nucleus is either

- absorption,
- or a change in the magnitude of the velocity,
- or a change in the direction of the velocity

In any case it is a disappearance from the element $d^3 v$. Denoting the neutron macroscopic cross section for collision as Σ , then when it moves a differential length of Δl , the probability for collision is $\Sigma \Delta l$ ($\Sigma \Delta l$ equals $1 - e^{-\Sigma \Delta l}$ for very small Δl values).

For neutrons that travel with velocity \mathbf{v} for a time period dt , the differential length is $\Delta l = v dt$, where $v = |\mathbf{v}|$. Thus the number of neutrons, of those in the phase space element $d^3 r d^3 v$ about (\mathbf{r}, \mathbf{v}) , that will collide in the time span dt is as dictated by Eq. 1.9. Collisions remove neutrons from element $d^3 v$ so the letter is also the number of neutrons removed, during dt , from $d^3 r d^3 v$.

$$\begin{aligned} \text{Neutrons removed by collision} &= \\ &\Sigma(\mathbf{r}, \mathbf{v}, t) n(\mathbf{r}, \mathbf{v}, t) v dt d^3 r d^3 v \end{aligned} \quad (1.9)$$

1.1.3 Gain due to collisions, term (3)

Neutrons that belong to $d^3 r d^3 v'$ about the phase space point $(\mathbf{r}, \mathbf{v}')$, and colliding, may move, as a result of their collision, to the element $d^3 r d^3 v$ about (\mathbf{r}, \mathbf{v}) . The number of these neutrons would be determined by three factors:

- Number of collisions (same as in term-2): $\Sigma(\mathbf{r}, \mathbf{v}', t) n(\mathbf{r}, \mathbf{v}', t) v' dt d^3 r d^3 v'$
- Average number of neutrons emerging from a collision $C(\mathbf{r}, \mathbf{v}', t)$
- The probability that a collision entry at \mathbf{v}' will lead to an exit at \mathbf{v} , i.e. $P(\mathbf{r}, \mathbf{v}' \rightarrow \mathbf{r}, \mathbf{v})$

A neutron may enter \mathbf{v} , having been in any \mathbf{v}' and therefore an integration over all \mathbf{v}' is required as follows:

$$\begin{aligned} \text{Neutrons moved in collision} &= \\ &\left[\int \Sigma(\mathbf{r}, \mathbf{v}', t) n(\mathbf{r}, \mathbf{v}', t) C(\mathbf{r}, \mathbf{v}', t) P(\mathbf{r}, \mathbf{v}' \rightarrow \mathbf{r}, \mathbf{v}) v' d^3 v' \right] dt d^3 r d^3 v \end{aligned} \quad (1.10)$$

1.1.4 Source, term (4)

If there is an external source, supplying at time t , in unit time, volume and velocity, $S(\mathbf{r}, \mathbf{v}, t)$ neutrons to point $(\mathbf{r}, \mathbf{v}, t)$, then to the element $d^3 r d^3 v$ it supplies, during dt , the following amount of neutrons:

$$\begin{aligned} \text{External source} &= \\ S(\mathbf{r}, \mathbf{v}, t) dt d^3 r d^3 v \end{aligned} \quad (1.11)$$

Returning now to the beginning, we equate Eq. 1.1 to Eq. 1.2. Note that the $dt d^3 r d^3 v$ factors appearing everywhere were all dropped.

$$\begin{aligned} \frac{\partial n(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla n(\mathbf{r}, \mathbf{v}, t) + v \Sigma(\mathbf{r}, \mathbf{v}, t) n(\mathbf{r}, \mathbf{v}, t) - \\ - \left[\int \Sigma(\mathbf{r}, \mathbf{v}', t) n(\mathbf{r}, \mathbf{v}', t) C(\mathbf{r}, \mathbf{v}', t) P(\mathbf{r}, \mathbf{v}' \rightarrow \mathbf{r}, \mathbf{v}) v' d^3 v' \right] - S(\mathbf{r}, \mathbf{v}, t) = 0 \end{aligned} \quad (1.12)$$

Eq. 1.12 is the transport equation. **H.W. It is not the most general equation, why?.**

Lecture 2: The Continuity Equation

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2.1 Transport equation as defined in the literature

$$\frac{\partial n(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \nabla n(\mathbf{r}, \mathbf{v}, t) + v \Sigma(\mathbf{r}, \mathbf{v}, t) n(\mathbf{r}, \mathbf{v}, t) - \left[\int \Sigma(\mathbf{r}, \mathbf{v}', t) n(\mathbf{r}, \mathbf{v}', t) C(\mathbf{r}, \mathbf{v}', t) P(\mathbf{r}, \mathbf{v}' \rightarrow \mathbf{r}, \mathbf{v}) v' d^3 v' \right] - S(\mathbf{r}, \mathbf{v}, t) = 0 \quad (2.1)$$

This part of our lecture would focus on converting Eq. 2.1 to a more common representation of the transport equation as often defined in the literature. To do so we need to introduce additional terminology as would be apparent in the following sub-sections.

2.1.1 Directional flux

The density distribution $vn(\mathbf{r}, \mathbf{v}, t)$, where $v = |\mathbf{v}|$, is the fundamental dependent variable in terms of which we may find convenient to use. We therefore define:

$$\begin{aligned} \text{Directional flight} &= \\ \varphi(\mathbf{r}, \mathbf{v}, t) &\equiv vn(\mathbf{r}, \mathbf{v}, t) \end{aligned} \quad (2.2)$$

Prior to substituting Eq. 2.2 to Eq. 2.1 we note that:

$$\begin{aligned} \mathbf{v} = \hat{\Omega}v &\Rightarrow \mathbf{v} \cdot \nabla n(\mathbf{r}, \mathbf{v}, t) = \hat{\Omega}v \cdot \nabla n(\mathbf{r}, \mathbf{v}, t) \\ \hat{\Omega} \cdot \nabla vn(\mathbf{r}, \mathbf{v}, t) &= \hat{\Omega} \cdot \nabla \varphi(\mathbf{r}, \mathbf{v}, t) \end{aligned} \quad (2.3)$$

The substitution results in:

$$\begin{aligned} \frac{1}{v} \frac{\partial \varphi(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \hat{\Omega} \cdot \nabla \varphi(\mathbf{r}, \mathbf{v}, t) + \Sigma(\mathbf{r}, \mathbf{v}, t) \varphi(\mathbf{r}, \mathbf{v}, t) - \\ - \left[\int \Sigma(\mathbf{r}, \mathbf{v}', t) \varphi(\mathbf{r}, \mathbf{v}', t) C(\mathbf{r}, \mathbf{v}', t) P(\mathbf{r}, \mathbf{v}' \rightarrow \mathbf{r}, \mathbf{v}) d^3 v' \right] - S(\mathbf{r}, \mathbf{v}, t) = 0 \end{aligned} \quad (2.4)$$

The units of $[\varphi]$ are in *neutrons/sec/cm²*, which are a direct product of velocity *cm/sec* and neutron density *neutrons/cm³*. By itself φ has no special meaning and it is just more mathematically useful. However the product of $\Sigma\varphi$ is meaningful since it describes the reaction rates of neutrons in the medium.

2.1.2 Velocity domain

Instead of characterizing the velocity domain by the three velocity components, it is more customary to use three other variables. We can express the velocity vector \mathbf{v} by $\hat{\Omega}v$, where:

- The magnitude of the velocity is v . The value of v corresponds to a E value, which is the kinetic energy of the neutron.
- The unit vector of the velocity \mathbf{v} is $\hat{\Omega}$. As a unit vector it depends on two variables, i.e. the polar and azimuthal angles.

We can again re-write the transport equation by introducing $\varphi(\mathbf{r}, E, \hat{\Omega}, t)$ and $\Sigma(\mathbf{r}, E, \hat{\Omega}, t)$ as:

$$\begin{aligned} & \frac{1}{v} \frac{\partial \varphi}{\partial t} + \hat{\Omega} \cdot \nabla \varphi + \Sigma(\mathbf{r}, E, \hat{\Omega}, t) \varphi(\mathbf{r}, E, \hat{\Omega}, t) - \\ & - \left[\int \int \Sigma(\mathbf{r}, E', \hat{\Omega}', t) \varphi(\mathbf{r}, E', \hat{\Omega}', t) C(\mathbf{r}, E', \hat{\Omega}', t) P(\mathbf{r}, E', \hat{\Omega}' \rightarrow \mathbf{r}, E, \hat{\Omega}) d^2 \hat{\Omega}' dE' \right] - \\ & - S(\mathbf{r}, E, \hat{\Omega}, t) = 0 \end{aligned} \quad (2.5)$$

2.1.3 Scattering

Since in a scattering reaction $C = 1$, the integrand in Eq. 2.5 may be simplified to:

$$\begin{aligned} & \text{Integrand in scattering term} = \\ & \Sigma_s(E', \hat{\Omega}') \varphi(\mathbf{r}, E', \hat{\Omega}', t) P(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) \end{aligned} \quad (2.6)$$

Frequently the scope of scattering (elastic and inelastic) does not include events whose outcome depends separately both on $\hat{\Omega}$ and on $\hat{\Omega}'$, the entry and the exit directions. More often the outcome is a function of the angle between $\hat{\Omega}$ and $\hat{\Omega}'$, equivalently the scalar product $\hat{\Omega} \cdot \hat{\Omega}'$. Consequently we write:

$$\begin{aligned} & \text{The scattering term} = \\ & \int d^2 \hat{\Omega}' \int dE' \Sigma_s(E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega}) \varphi(\mathbf{r}, E', \hat{\Omega}', t) \end{aligned} \quad (2.7)$$

2.1.4 Fission

The integral in Eq. 2.5 also includes fission, but the latter reaction is dealt separately from scattering due to different characteristics of the reaction. The different terms in the integral can be modified as follows:

- The fission cross section, $\Sigma(E', \hat{\Omega}') \Rightarrow \Sigma_f(E')$.
- The average number of neutrons emitted in a fission event with energy E' , $C(E', \hat{\Omega}') \Rightarrow \nu(E')$.

- The fission spectrum – probability that a fission neutron would have an energy within dE about E ,
 $P(E', \hat{\Omega}' \rightarrow E, \hat{\Omega}) \Rightarrow \chi(E)$
- Removing $\hat{\Omega}'$ from the above terms expresses the experimental fact that the fission outcome is independent of the entry direction.
- Removing $\hat{\Omega}$ from the χ expression reflects the experimental fact that fission neutrons are emitted isotropically in all directions.
- Removing E' from χ reflects the experimental fact that the energy distribution of fission neutrons is only slightly dependent on the energy E' of the fission initiating neutron. This is certainly not a valid assumption for neutrons with very high energies, i.e. 1000 MeV.

The transport equation takes the following form:

$$\begin{aligned} & \frac{1}{v} \frac{\partial \varphi}{\partial t} + \hat{\Omega} \cdot \nabla \varphi + \Sigma_t(\mathbf{r}, E) \varphi(\mathbf{r}, E, \hat{\Omega}, t) \\ &= \int d^2 \hat{\Omega}' \int dE' \Sigma_s(E' \rightarrow E, \hat{\Omega}' \rightarrow \hat{\Omega}) \varphi(\mathbf{r}, E', \hat{\Omega}', t) + \\ &+ \chi(E) \int d^2 \hat{\Omega}' \int dE' \Sigma_f(E') \varphi(\mathbf{r}, E', \hat{\Omega}', t) + S(\mathbf{r}, E, \hat{\Omega}, t) \end{aligned} \quad (2.8)$$

Frequently the fission term in Eq. 2.8 is called source, in distinction to S , which is the external source.

2.2 Boundary conditions

Suppose the spatial domain within which the neutron transport equation is to be solved has volume V surrounded by a surface Γ . To solve the transport equation, the flux at $t = 0$, $\varphi(\mathbf{r}, E, \hat{\Omega}, 0)$ must be specified. In addition, the flux distribution entering V across Γ must be known. More specifically, if e_Γ is the outward normal to Γ , then $\varphi(\mathbf{r}, E, \hat{\Omega}, t)$ for $\hat{\Omega} \cdot e_\Gamma < 0$ and $\mathbf{r} \in \Gamma$ are required as boundary data. Thus,

$$\varphi(\mathbf{r}, E, \hat{\Omega}, t) = \tilde{\varphi}(\mathbf{r}, E, \hat{\Omega}, t) \quad , \quad \hat{\Omega} \cdot e_\Gamma < 0, \mathbf{r} \in \Gamma \quad (2.9)$$

The frequent case in which $\tilde{\varphi}$ is identically zero is referred to as the vacuum boundary condition:

$$\varphi(\mathbf{r}, E, \hat{\Omega}, t) = 0 \quad , \quad \hat{\Omega} \cdot e_\Gamma < 0, \mathbf{r} \in \Gamma \quad (2.10)$$

Eqs. 2.9 and 2.10 are explicit boundary conditions in that the incoming flux is explicitly known. Implicit boundary conditions are routinely used in transport calculations in order to take advantage of symmetries satisfied by the physical problem. Implicit boundary conditions are relationships between the incoming and outgoing fluxes. They take three forms:

Albedo boundary conditions

The incoming flux on a boundary is set equal to a known isotropic albedo, $\alpha(E)$, times the outgoing flux on the same boundary in the direction corresponding to spectral reflection,

$$\varphi(\mathbf{r}, E, \hat{\Omega}, t) = \alpha(E) \varphi(\mathbf{r}, E, \hat{\Omega}', t) \quad , \quad \hat{\Omega} \cdot e_\Gamma < 0, \mathbf{r} \in \Gamma \quad (2.11)$$

Here, $\hat{\Omega}$ is the reflection angle corresponding to an incidence angle $\hat{\Omega}'$:

$$\hat{\Omega} \cdot e_\Gamma = -\hat{\Omega}' \cdot e_\Gamma \text{ and } (\hat{\Omega} \times \hat{\Omega}') \cdot e_\Gamma = 0 \quad (2.12)$$

The special case of $\alpha = 1$, is referred to as a reflective boundary condition since all outgoing particles are reflected back.

White boundary conditions

All particles passing out of V over a surface increment return with an isotropic distribution:

$$\varphi(\mathbf{r}, E, \hat{\Omega}, t) = \frac{1}{4\pi} \int_{\hat{\Omega}' \cdot e_\Gamma > 0} d\hat{\Omega}' \hat{\Omega}' \cdot e_\Gamma \varphi(\mathbf{r}, E, \hat{\Omega}', t) \quad , \quad \hat{\Omega} \cdot e_\Gamma < 0, \mathbf{r} \in \Gamma \quad (2.13)$$

Periodic boundary conditions

The flux distribution on the boundary is equal to the flux distribution on another boundary in a periodic lattice:

$$\varphi(\mathbf{r}, E, \hat{\Omega}, t) = \varphi(\mathbf{r} + \mathbf{r}_l, E, \hat{\Omega}', t) \quad (2.14)$$

2.3 The neutron continuity equation

The transport equation is an exact mathematical balance of neutron collisions and movement in a medium. It is an integral-differential equation for the directional flux φ . The latter is a function of seven variables: 3 for space, 2 for solid angle, 1 for energy and 1 for time. Under certain conditions, to be defined later, one can switch from the transport equation onto a diffusion equation.

This equation too is a balance, although approximate, for the neutron domain. It is an equation for the scalar flux ϕ , in which the concept of a current \mathbf{J} appears. There are five variables: 3 for space, 1 for energy, 1 for time. The solid angle is excluded.

This lecture will focus on the methodical derivation of the diffusion equation from the transport equation. This is normally a complex-delicate matter. As far as rigour is concerned we will deal with a methodical derivation for the energy independent version of the transport equation. This restriction is to serve the aim

of simplicity in presentation. Most of the aspects and approximations spill over into the energy dependent case. There are important exceptions which we will note.

We start by writing down the transport equation for the movement of single energy (one-group) neutrons, namely we omit the variable E from the functions Σ_t , Σ_s and φ . In this stage of the derivation the fission term would be included under the s sources. Thus:

$$\frac{1}{v} \frac{\partial \varphi}{\partial t} + \hat{\Omega} \cdot \nabla \varphi (\mathbf{r}, \hat{\Omega}, t) + \Sigma_t (\mathbf{r}, t) \varphi (\mathbf{r}, \hat{\Omega}, t) - \int d\hat{\Omega}' \Sigma_s (\hat{\Omega}' \rightarrow \hat{\Omega}) \varphi (\mathbf{r}, \hat{\Omega}', t) - s (\mathbf{r}, \hat{\Omega}, t) = 0 \quad (2.15)$$

2.3.1 Angle-integrated flux

The details of the angular dependence of the flux are not necessary and we only need the angle-integrated flux. Therefore, we start by simply integrating the transport equation (Eq. 2.15) over all solid angles $\hat{\Omega}$, i.e. $0 \leq \vartheta \leq 2\pi, 0 \leq \theta \leq \pi$. We do it term by term. We start by defining the scalar flux and source:

$$\phi (\mathbf{r}, t) = \int d\hat{\Omega} \varphi (\mathbf{r}, \hat{\Omega}, t) \quad (2.16)$$

$$S (\mathbf{r}, t) = \int d\hat{\Omega} s (\mathbf{r}, \hat{\Omega}, t) \quad (2.17)$$

Obtaining for the (1), (3) and (5) terms:

$$\int d\hat{\Omega} \frac{1}{v} \frac{\partial \varphi}{\partial t} \Rightarrow \frac{1}{v} \frac{\partial \phi}{\partial t} \quad (2.18)$$

$$\int d\hat{\Omega} \Sigma_t (\mathbf{r}, t) \varphi (\mathbf{r}, \hat{\Omega}, t) \Rightarrow \Sigma_t \phi (\mathbf{r}, t) \quad (2.19)$$

$$\int d\hat{\Omega} s (\mathbf{r}, \hat{\Omega}, t) \Rightarrow S (\mathbf{r}, t) \quad (2.20)$$

As for the second term in Eq. 2.15 we need to perform the following integration: $\int d\hat{\Omega} \hat{\Omega} \cdot \nabla \varphi (\mathbf{r}, \hat{\Omega}, t)$. In order to perform the integration we first prove the following identity:

$$\hat{\Omega} \cdot \nabla \varphi = \nabla \cdot \hat{\Omega} \varphi \quad (2.21)$$

where φ is a scalar. In Cartesian coordinates:

1. $\nabla \equiv \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$
2. $\hat{\Omega} \cdot \nabla \varphi \equiv [\Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k}] \cdot \left[\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right]$
3. Opening the brackets we note that $\mathbf{n} \cdot \mathbf{m} = 0$ if $n \neq m$ and $\mathbf{n} \cdot \mathbf{m} = 1$ otherwise. Therefore, $\hat{\Omega} \cdot \nabla \varphi = \frac{\partial \varphi}{\partial x} \Omega_x + \frac{\partial \varphi}{\partial y} \Omega_y + \frac{\partial \varphi}{\partial z} \Omega_z$
4. The derivative operations (e.g. $\frac{\partial}{\partial x}$) would not operate on the components Ω . The derivatives belong to the space domain where Ω belongs to the velocity domain.
5. We thus write $\hat{\Omega} \cdot \nabla \varphi = \frac{\partial \Omega_x \varphi}{\partial x} + \frac{\partial \Omega_y \varphi}{\partial y} + \frac{\partial \Omega_z \varphi}{\partial z} = \nabla \cdot \hat{\Omega} \varphi$

Returning now to the integral to be performed, we have instead to evaluate $\int d\hat{\Omega} [\nabla \cdot \hat{\Omega} \varphi]$. Basically we need to perform 2 operations on $\hat{\Omega} \varphi$, i.e. make a scalar product with $\nabla \cdot$ and integrate over $\hat{\Omega}$. These operations are in two different domains, space and velocity and therefore the order of execution is unimportant. We thus reformulate our equation to the following form:

$$\int d\hat{\Omega} [\nabla \cdot \hat{\Omega} \varphi (\mathbf{r}, \hat{\Omega}, t)] = \nabla \cdot \int d\hat{\Omega} \hat{\Omega} \varphi (\mathbf{r}, \hat{\Omega}, t) \quad (2.22)$$

On the RHS of Eq. 2.22 we have a new quantity - *the neutron current*, which is defined as:

$$\mathbf{J}(\mathbf{r}, t) = \int d\hat{\Omega} \hat{\Omega} \varphi (\mathbf{r}, \hat{\Omega}, t) \quad (2.23)$$

This new quantity deserves some attention as it describes a vector sum $\hat{\Omega} \varphi (\hat{\Omega})$. Namely, direction $\hat{\Omega}$ is weighted with the directional flux in that direction.

The current, unlike the flux, has an immediate physical meaning. Looking at the plot from the previous lecture, if the square $\Delta y \Delta z$ is perpendicular to \mathbf{v} , then $v_y = v_z = 0; v_x = |\mathbf{v}|$. Recall that $\hat{\Omega} = \mathbf{v}/|\mathbf{v}|, |\mathbf{v}|n$ then has a meaning as follows:

$$\begin{aligned} &\text{Number of neutrons crossing in a sec a cm}^2 \text{ area perpendicular to } \hat{\Omega} = \\ &|\mathbf{v}|n = \varphi (\hat{\Omega}) \end{aligned} \quad (2.24)$$

Having weighted with $\varphi (\hat{\Omega})$ each $\hat{\Omega}$, namely activating the definition of the current \mathbf{J} , one obtains a *net amount*: neutrons per cm^2 per sec (dimensions of φ). $|\mathbf{J}(\mathbf{r})|$ is the number of neutrons crossing, in the \mathbf{J} direction, in a sec., at \mathbf{r} , a cm^2 area placed perpendicular to \mathbf{J} .

In summary:

$$\int d\hat{\Omega} \hat{\Omega} \cdot \nabla \varphi = \nabla \cdot \mathbf{J}(\mathbf{r}, t) \quad (2.25)$$

The last term we need to deal with is: $\int d\hat{\Omega} \int d\hat{\Omega}' \Sigma_s (\hat{\Omega}' \cdot \hat{\Omega}) \varphi (\hat{\Omega}')$ But, the order integration may be changed, namely we will consider:

$$\int d\hat{\Omega}' \left[\int d\hat{\Omega} \Sigma_s (\hat{\Omega}' \cdot \hat{\Omega}) \right] \varphi (\hat{\Omega}') \quad (2.26)$$

Consider the integral $\left[\int d\hat{\Omega} \Sigma_s (\hat{\Omega}' \cdot \hat{\Omega}) \right]$. For the purpose of integration $\int d\hat{\Omega}$ will be described by its polar angle θ and azimuth angle ϑ . The scattering we usually deal with do not depend on ϑ , solely on θ . Namely the scattering to the entire envelope of the cone, given by the head angle of 2θ is described by:

$$\begin{aligned} \Sigma_s (\mu) &= 2\pi \Sigma_s (\hat{\Omega}' \cdot \hat{\Omega}) \\ \mu &= \hat{\Omega}' \cdot \hat{\Omega} = \cos \theta \end{aligned} \quad (2.27)$$

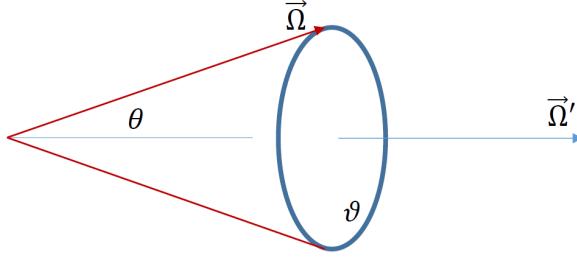


Figure 2.1: Illustration of the scattering process.

Substituting Eq. 2.27 into the integral in $\left[\int d\hat{\Omega} \Sigma_s (\hat{\Omega}' \cdot \hat{\Omega}) \right]$ leads to the following relation:

$$\frac{1}{2\pi} \int d\hat{\Omega} \Sigma_s (\mu) = \int_0^{2\pi} d\vartheta \int_{-1}^{+1} \Sigma_s (\mu) d\mu = \int_{-1}^{+1} \Sigma_s (\mu) d\mu \quad (2.28)$$

Define the total scattering cross section at \mathbf{r}

$$\Sigma_s (\mathbf{r}) = \int_{-1}^{+1} \Sigma_s (\mathbf{r}, \mu) d\mu \quad (2.29)$$

Substituting the results from Eq. 2.29 into Eq. 2.26, we finally obtain:

$$\int d\hat{\Omega}' \Sigma_s \varphi (\hat{\Omega}') = \Sigma_s \int d\hat{\Omega}' \varphi (\hat{\Omega}') = \Sigma_s (\mathbf{r}) \phi (\mathbf{r}) \quad (2.30)$$

Summing up all the partial results, the $d\hat{\Omega}$ integration of Eq. 2.15 yields:

$$\frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) + \Sigma_t(\mathbf{r}, t) \phi(\mathbf{r}, t) - \Sigma_s(\mathbf{r}, t) \phi(\mathbf{r}, t) - S(\mathbf{r}, t) \quad (2.31)$$

Eq. 2.31 is known as the *neutron continuity equation*, since it is just the mathematical statement of neutron balance. We can replace the Σ_s and Σ_t terms in Eq. 2.31 by Σ_a , which is $\Sigma_a = \Sigma_t - \Sigma_s$.

The resulting equation is thus:

$$\frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) + \Sigma_a(\mathbf{r}, t) \phi(\mathbf{r}, t) - S(\mathbf{r}, t) \quad (2.32)$$

Eq. 3.1 is an exact outcome of the transport equation. It is an equation for the scalar flux ϕ and the current \mathbf{J} . This makes an equation in 4 unknowns: the flux and the three components of the current. We therefore need more ties between ϕ and \mathbf{J} to enable a solution.

Lecture 3: The Diffusion Equation

Lecturer: Dr. Dan Kotlyar

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Eq. 3.1 is an exact outcome of the transport equation. It is an equation for the scalar flux ϕ and the current \mathbf{J} . This makes an equation in 4 unknowns: the flux and the three components of the current. We therefore need more ties between ϕ and \mathbf{J} to enable a solution.

$$\frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) + \Sigma_a(\mathbf{r}, t) \phi(\mathbf{r}, t) - S(\mathbf{r}, t) \quad (3.1)$$

3.1 The one-speed diffusion equation

Since there is no simple equation for $\phi(\mathbf{r}, t)$, we shift our attention instead to developing an equation for the current density $\mathbf{J}(\mathbf{r}, t)$, which is defined according to Eq. 2.23.

For this purpose we propose multiplying the transport equation 2.15 by $\hat{\Omega}$ and then integrating once again over angle $d\hat{\Omega}$. Here our efforts will yield an approximate relation between ϕ and \mathbf{J} .

The approximate nature of the derivation with term (2) is the most complicated here and therefore we leave its treatment to the last. Terms (1), (3) and (5) are relatively straightforward:

$$\int d\hat{\Omega} \hat{\Omega} \left[\frac{1}{v} \frac{\partial \varphi}{\partial t} \right] = \frac{1}{v} \frac{\partial}{\partial t} \int d\hat{\Omega} \hat{\Omega} [\varphi] = \frac{1}{v} \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} \quad (3.2)$$

$$\int d\hat{\Omega} \hat{\Omega} \Sigma_t(\mathbf{r}, t) \varphi(\mathbf{r}, \hat{\Omega}, t) = \Sigma_t \int d\hat{\Omega} \hat{\Omega} \varphi = \Sigma_t(\mathbf{r}, t) \mathbf{J}(\mathbf{r}, t) \quad (3.3)$$

$$\int d\hat{\Omega} \hat{\Omega} s(\mathbf{r}, \hat{\Omega}, t) \quad (3.4)$$

If s includes external sources the dependence of s on $\hat{\Omega}$ is a matter of fact and the integration has to be performed, perhaps numerically. In the case of s being the fission source the fact that fission neutrons are emitted isotropically means $s(\hat{\Omega}) = const$. This, in turn means:

$$\int d\hat{\Omega} \hat{\Omega} s(\mathbf{r}, \hat{\Omega}, t) = const \int d\hat{\Omega} \hat{\Omega} \quad \text{for an isotropic source} \quad (3.5)$$

The last equality is because a summation of $\hat{\Omega}$ over all angles in the unit sphere is zero: to every $\hat{\Omega}$ there will be a $-\hat{\Omega}$.

The integration $d\hat{\Omega}'\hat{\Omega}$ of the scattering term is described as follows:

$$\int d\hat{\Omega}'\hat{\Omega} \left[\int d\hat{\Omega}'\Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) \right] \varphi(\hat{\Omega}') = \int d\hat{\Omega}' \left[\int d\hat{\Omega}'\Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) \right] \varphi(\hat{\Omega}') \quad (3.6)$$

We firstly want to evaluate the integral in the [] brackets. In order to do so we consider the plot illustrated in Fig. 3.1.

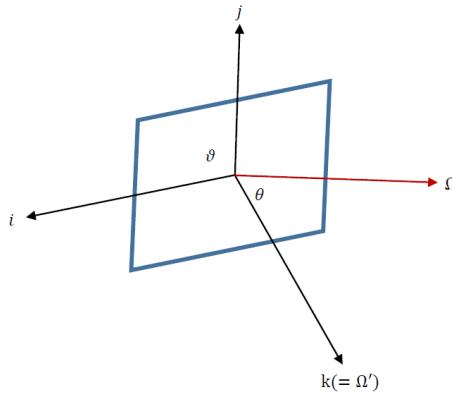


Figure 3.1: Illustration of the directional flight.

The axes z becomes united with the direction of neutron entering the collision, i.e. $\hat{\Omega}' = k$. $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the direction of the axes x, y, z respectively. The definition of the spatial angle is as follows:

$$\hat{\Omega} = \sin \theta \cos \vartheta \mathbf{i} + \sin \theta \sin \vartheta \mathbf{j} + \cos \theta \mathbf{k} \quad (3.7)$$

We will also use previously developed relations

$$\begin{aligned} \hat{\Omega}' \cdot \hat{\Omega} &= \mathbf{k} \cdot \hat{\Omega} \\ \Sigma_s(\mathbf{k} \cdot \hat{\Omega}) &= \frac{1}{2\pi} \Sigma_s(\mu) \end{aligned} \quad (3.8)$$

Substituting these into the integral within the [] brackets of Eq. 3.6 produces the following result:

$$\int_{-1}^{+1} d\mu \int_0^{2\pi} d\vartheta (\sin \theta \cos \vartheta \mathbf{i} + \sin \theta \sin \vartheta \mathbf{j} + \cos \theta \mathbf{k}) \frac{1}{2\pi} \Sigma_s(\mu) \quad (3.9)$$

The integration $\int_0^{2\pi}$ makes \mathbf{i} and \mathbf{j} terms drop out. Thus Eq. 3.9 is simplified to the following form:

$$\int_{-1}^{+1} d\mu \int_0^{2\pi} d\vartheta \cos \theta \mathbf{k} \frac{1}{2\pi} \Sigma_s(\mu) = \int_{-1}^{+1} d\mu \mu \hat{\Omega}' \frac{1}{2\pi} \Sigma_s(\mu) \int_0^{2\pi} d\vartheta = \int_{-1}^{+1} d\mu \mu \hat{\Omega}' \Sigma_s(\mu) \quad (3.10)$$

We now define $\bar{\mu}$, the average scattering cosine, by:

$$\bar{\mu} = \frac{\int_{-1}^{+1} d\mu \mu \Sigma_s(\mu)}{\Sigma_s} \quad (3.11)$$

Thus makes Eq. 3.10 read as:

$$\int_{-1}^{+1} d\mu \mu \hat{\Omega}' \Sigma_s(\mu) = \bar{\mu} \Sigma_s \hat{\Omega}' \quad (3.12)$$

Returning now to Eq. 3.6:

$$\begin{aligned} \int d\hat{\Omega}' \left[\int d\hat{\Omega} \hat{\Omega} \Sigma_s(\hat{\Omega}' \cdot \hat{\Omega}) \right] \varphi(\hat{\Omega}') &= \int d\hat{\Omega}' \bar{\mu} \Sigma_s \hat{\Omega}' \varphi(\hat{\Omega}') \\ \bar{\mu} \Sigma_s \int d\hat{\Omega}' \hat{\Omega}' \varphi(\hat{\Omega}') &= \bar{\mu} \Sigma_s(\mathbf{r}, t) \mathbf{J}(\mathbf{r}, t) \end{aligned} \quad (3.13)$$

Terms (3) and (5) in Eq. 2.15 are usually combined together by defining the *transport cross section* $\Sigma_{tr}(\mathbf{r}, t)$ as follows:

$$\Sigma_{tr}(\mathbf{r}, t) = \Sigma_t(\mathbf{r}, t) - \bar{\mu} \Sigma_s(\mathbf{r}, t) \quad (3.14)$$

The derivation has been strict so far. Perhaps the only exception to strictness was in term (5): the assumption of an isotropic source is good for a fission source, and many other external sources. A forced deviation from strictness is introduced as one deal with term (2): $\nabla \cdot \int d\hat{\Omega} \hat{\Omega} [\hat{\Omega} \varphi(\hat{\Omega})]$. The needed integration, $\int d\hat{\Omega} \hat{\Omega} [\hat{\Omega} \varphi(\hat{\Omega})]$, cannot be expressed in terms of just the scalar flux ϕ and current \mathbf{J} .

A major approximation is made. The consequences of this approximation may be better discussed at the end of the formal derivation. Suffice it to say here that it precludes treating by the diffusion approximation high absorption and/or proximity to interfaces where material properties change considerably. The restriction introduced is as follows:

$$\varphi(\hat{\Omega}) \cong \frac{1}{4\pi} [A + 3\mathbf{B} \cdot \hat{\Omega}] \quad \text{higher powers of } \hat{\Omega} \text{ are neglected} \quad (3.15)$$

This is a vector function described as Taylor series. This is a first- $\hat{\Omega}$ -order approximation for the directional flux $\varphi(\hat{\Omega})$; our first order of business is to ascertain that Eq. 3.15 retains two important entities, i.e. the scalar flux ϕ and the current \mathbf{J} .

Retain ϕ by integrating Eq. 3.15:

$$\int d\hat{\Omega} \varphi(\hat{\Omega}) = \frac{1}{4\pi} A \int d\hat{\Omega} + \frac{3}{4\pi} \mathbf{B} \cdot \int d\hat{\Omega} \hat{\Omega} \quad (3.16)$$

But $\int d\hat{\Omega} = 4\pi$ and $\int d\hat{\Omega} \hat{\Omega} = 0$, therefore:

$$\int d\hat{\Omega} \varphi(\hat{\Omega}) = \phi = A \quad (3.17)$$

Retain \mathbf{J} by multiplying Eq. 3.15 with $\hat{\Omega}$ and then integrating over the spatial angle:

$$\int d\hat{\Omega} \hat{\Omega} \varphi(\hat{\Omega}) = \frac{1}{4\pi} \phi \int d\hat{\Omega} \hat{\Omega}^0 + \frac{3}{4\pi} \int d\hat{\Omega} \hat{\Omega} (\hat{\Omega} \cdot \mathbf{B}) \quad (3.18)$$

We deal with the remaining integrals by decomposing \mathbf{B} and $\hat{\Omega}$, each to its Cartesian coordinates:

$$\frac{3}{4\pi} \int d\hat{\Omega} \hat{\Omega} (\hat{\Omega} \cdot \mathbf{B}) = \frac{3}{4\pi} \int d\hat{\Omega} [\Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k}] [\Omega_x B_x + \Omega_y B_y + \Omega_z B_z] \quad (3.19)$$

Executing the [] multiplications will generate 9 terms: cross-terms/mixed (e.g. $\Omega_x \Omega_y B_y$), and "pure" terms (e.g. $\Omega_z^2 B_z$). Integration of $d\hat{\Omega}$ over cross-terms yields zero, since on each integrand there will be one with the same magnitude but a different sign ("+" instead of a "-", and vice versa). **This exercise is included in your home assignment.** What remains is:

$$\frac{3}{4\pi} \int d\hat{\Omega} \hat{\Omega} (\hat{\Omega} \cdot \mathbf{B}) = \frac{3}{4\pi} \int d\hat{\Omega} [\Omega_x^2 B_x \mathbf{i} + \Omega_y^2 B_y \mathbf{j} + \Omega_z^2 B_z \mathbf{k}] \quad (3.20)$$

A consideration of Eq. 3.7 shows the components of $\hat{\Omega}$ as functions of θ and ϑ . For example:

$$\int d\hat{\Omega} \Omega_x^2 \mathbf{i} = \int_0^{2\pi} d\vartheta \int_{-1}^{+1} d\mu \mu^2 \mathbf{i} = \frac{4\pi}{3} \mathbf{i} \quad (3.21)$$

Home exercise—show that

$$\begin{aligned} \int d\hat{\Omega} \Omega_y^2 \mathbf{j} &= \frac{4\pi}{3} \mathbf{j} \\ \int d\hat{\Omega} \Omega_z^2 \mathbf{k} &= \frac{4\pi}{3} \mathbf{k} \end{aligned} \quad (3.22)$$

Combining Eqs. 3.20–3.22 we get:

$$\int d\hat{\Omega} \hat{\Omega} \varphi(\hat{\Omega}) = \frac{3}{4\pi} \frac{4\pi}{3} [B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}] = \mathbf{J} \quad (3.23)$$

So, the diffusion approximation is given by:

$$\varphi(\hat{\Omega}) \cong \frac{1}{4\pi} [\phi + 3\mathbf{J} \cdot \hat{\Omega}] \quad \text{higher powers of } \hat{\Omega} \text{ are neglected} \quad (3.24)$$

We now perform $\int d\hat{\Omega} \hat{\Omega} [\hat{\Omega} \cdot \nabla \varphi]$. In our previous lecture we showed that the operators $\hat{\Omega} \cdot \nabla$ and $\nabla \cdot \hat{\Omega}$ are interchangeable, thus:

$$\int d\hat{\Omega} \hat{\Omega} [\hat{\Omega} \cdot \nabla \varphi] = \int d\hat{\Omega} \hat{\Omega} [\nabla \cdot \hat{\Omega} \varphi] \quad (3.25)$$

If we now substitute the approximation from Eq. 3.24 into Eq. 3.25 it leads to:

$$\begin{aligned} \int d\hat{\Omega} \hat{\Omega} [\nabla \cdot \hat{\Omega} \varphi] &\cong \frac{1}{4\pi} \int d\hat{\Omega} \hat{\Omega} [\nabla \cdot \hat{\Omega} (\phi + 3\mathbf{J} \cdot \hat{\Omega})] = \\ &= \frac{1}{4\pi} \int d\hat{\Omega} (\Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k}) \left(\Omega_x \frac{\partial}{\partial x} + \Omega_y \frac{\partial}{\partial y} + \Omega_z \frac{\partial}{\partial z} \right) \{\phi + 3(\Omega_x J_x + \Omega_y J_y + \Omega_z J_z)\} \end{aligned} \quad (3.26)$$

If we only focus on the curly brackets in Eq. 3.26 and consider only the $3(\Omega_x J_x + \Omega_y J_y + \Omega_z J_z)$ term, multiplying these with all other terms will generate 27 terms, *all possible combinations in the Cartesian components of $\hat{\Omega}$* .

Home exercise—show that all the 27 components of $\hat{\Omega}$ in this equation equal to zero.

Opening the curly brackets in Eq. 3.26 with ϕ generates 9 members. As previously shown, all cross-terms are equal to zero and we finally obtain:

$$\int d\hat{\Omega} \hat{\Omega} [\nabla \cdot \hat{\Omega} \varphi] = \frac{1}{4\pi} \frac{4\pi}{3} \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) \phi = \frac{1}{3} \nabla \phi \quad (3.27)$$

Summing up all results in:

$$\frac{1}{v} \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} + \frac{1}{3} \nabla \phi(\mathbf{r}, t) + \Sigma_{tr}(\mathbf{r}, t) \mathbf{J}(\mathbf{r}, t) = 0 \quad (3.28)$$

$$\frac{1}{v} \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} + \frac{1}{3} \nabla \phi(\mathbf{r}, t) + \Sigma_{tr}(\mathbf{r}, t) \mathbf{J}(\mathbf{r}, t) = 0 \quad (3.29)$$

As the discussion progresses we become more reactor engineering oriented. A simple comparison of magnitudes will convince us that the $\frac{1}{v} \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t}$ term is negligible, compared with the $\Sigma_{tr}(\mathbf{r}, t) \mathbf{J}(\mathbf{r}, t)$ term. Suppose $\partial \mathbf{J}$ is in the \mathbf{J} direction, this will render the current augmentation maximal. We discuss the magnitude of the quantity:

$$\left[\frac{1}{v\delta t} \frac{\delta J}{J} + \Sigma_{tr} \right] J \quad (3.30)$$

Take a LWR as an example. Σ_{tr} is typically of the order of a cm^{-1} ; for thermal neutrons v is about 200,000 cm/sec ; therefore in order that the first term in Eq. 3.30 will, in one second, be equal to the Σ_{tr} order of magnitude, a $\frac{\delta J}{J}$ of the order of 1000% is required. This does not happen. Neglecting $\frac{1}{v} \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t}$ leaves us with:

$$\frac{1}{3} \nabla \phi(\mathbf{r}) + \Sigma_{tr}(\mathbf{r}) \mathbf{J}(\mathbf{r}) = 0 \quad (3.31)$$

This is more frequently written as:

$$\begin{aligned} \mathbf{J}(\mathbf{r}) &= -\frac{1}{3\Sigma_{tr}(\mathbf{r})} \nabla \phi(\mathbf{r}) = -D(\mathbf{r}) \nabla \phi(\mathbf{r}) \\ D(\mathbf{r}) &\equiv \frac{1}{3\Sigma_{tr}(\mathbf{r})} \end{aligned} \quad (3.32)$$

Eq. 3.32, coupled with the definition of the diffusion coefficient D , is what is known as (experimental) Fick's Law.

To conclude:

1. By integrating the transport equation (2.15) over all spatial angles $d\hat{\Omega}$ we obtained the exact representation of the balance/continuity equation:

$$\frac{1}{v} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{J} + \Sigma_a \phi = S \quad (3.33)$$

2. By multiplying the transport equation with $\hat{\Omega}$ and then integrating over $d\hat{\Omega}$ we obtain the approximate relation between the current and the scalar flux.

$$\mathbf{J} = -D\nabla\phi \quad (3.34)$$

When these two equations are solved as a pair (the second is actually three equations in a 3D problem), we solve what is termed the P_1 equations, or P_1 approximation. Since the approximation of linearly anisotropic angular dependence in Eq. 3.24 in one-dimensional plane geometry is equivalent to expanding the angular flux in Legendre polynomials in $\mu = \cos\theta$ and retaining only the $l = 0$ and $l = 1$ terms: $\varphi(x, \mu) = \phi(x) \frac{1}{2}P_0(\mu) + \mathbf{J}(x) \frac{3}{2}P_1(\mu)$. If \mathbf{J} is dropped between these two, via the $\nabla \cdot \mathbf{J}$ operation on the second equation, one obtains the diffusion equation:

$$\frac{1}{v} \frac{\partial\phi}{\partial t} - \nabla \cdot (D\nabla\phi) + \Sigma_a\phi = S \quad (3.35)$$

Inside a homogeneous medium $D(\mathbf{r}) = D$ and the diffusion equation is further simplified:

$$\begin{aligned} \frac{1}{v} \frac{\partial\phi}{\partial t} - D\nabla^2\phi + \Sigma_a\phi &= S \\ \nabla^2 &\equiv \nabla \cdot \nabla \end{aligned} \quad (3.36)$$

where $\nabla^2\phi$ is the *Laplacian of ϕ* .

The diffusion approximation in Eq. 3.34 implies that a spatial variation in the neutron flux (or density) will give rise to a current of neutrons flowing from regions of high to low density. Physically this is understandable since the collision rate in high neutron density regions will be higher with the corresponding tendency for neutrons to scatter more frequently away toward lower densities. The rate at which such diffusion occurs depends on the diffusion coefficient, which is inversely proportional to the transport cross section Σ_{tr} . It is convenient to introduce the concept of a *transport mean free path*:

$$\lambda_{tr} \equiv (\Sigma_{tr})^{-1} = (\Sigma_t - \bar{\mu}\Sigma_s)^{-1} \quad (3.37)$$

The transport mfp can be regarded as a corrected mfp accounting for anisotropies in the scattering collision process. Since $\bar{\mu}$ is almost always positive – biased in the direction of forward scattering, the transport λ_{tr} will always be somewhat larger than the actual mfp, $(\Sigma_t)^{-1}$. This essentially accounts for the fact that neutrons experiencing forward scattering tend to be transported somewhat further in a sequence of collision than those being isotropically scattered.

3.2 Difficulties with the diffusion approximation

The diffusion equation has been, for 50 years now, and with considerable success, the main theoretical tool of LWR design and operation. Yet it is a low order approximation of the transport equation, and may thus lead to inaccurate evaluation as a result of this. It is not the intention here to discuss properly the problems

encountered with the use of the diffusion equation. Many papers and chapter in books were written on the subject. For example a good textbook material for this topic is included in chapters 4 and 9 in A.F.Henry's "Nuclear-Reactor Analysis".

Here, we will briefly point out main issues and perhaps even possible solutions.

3.2.1 Boundaries

In transport the boundary between material and vacuum is accurately mapped by the directional flux. For example, if the boundary is flat and θ , the polar angle of $\hat{\Omega}$, is zero in the direction perpendicular to the boundary, then the directional flux $\varphi(\theta)$ is exactly zero for $\theta \leq \frac{\pi}{2}$, reflecting the fact that material-escaping neutrons will not collide in vacuum, whereby change direction and re-enter the material. In diffusion theory the scalar flux ϕ is incapable of tracing such a situation. Instead one uses a mathematical ploy. At the boundary ϕ is assumed to become linear, continuing into vacuum with the slope it had at the boundary itself. This ploy will bring the flux to zero at a point, a distance δ into the vacuum. δ is called the extrapolation distance. Diffusion calculation then may assume that δ cm away from the boundary the flux ϕ is zero. We stress again the point that leaking neutrons make the real flux strictly non zero at any distance from the boundary. A simple calculation shows that the extrapolation distance is:

$$\delta = 2D \quad (3.38)$$

Where D relates to the material bordering the vacuum. In reactor calculations most frequently the extrapolation distance may be neglected: D is of the order of a cm and the reactor dimension is on the order of meters. As the dimension of the calculated body shortens account of the extrapolation distance must be taken.

3.2.2 Absorption and interfaces

The directional flux $\varphi(\hat{\Omega})$ may be mathematically expanded in an infinite spherical harmonics series, of which Eq. 3.24 is just the beginning. In Eq. 3.24 linear is the highest $\hat{\Omega}$ order. It means that the directional flux may depend only weakly on the direction. The latter condition precludes strong absorption and/or proximity to interfaces with sharp material discontinuities. In reactor calculations the net of homogenized assemblies usually presents moderate absorption and non singular absorbers. However any attempt to learn about flux variation in the near vicinity, let alone the interior, of a control or burnable absorber rod must turn away from diffusion to transport.

3.2.3 Energy dependence

There is complete parallelism between the derivation of the diffusion equation, as presented above, and the derivation for the energy dependent case. It is in the definition of Σ_{tr} that this parallelism breaks down. The energy dependent parallels of (3) and (4) are:

$$(3) \rightarrow \Sigma(E) \mathbf{J}(E) \quad (3.39)$$

$$(4) \rightarrow \int dE' \mu(E', E) \Sigma_s(E', E) \mathbf{J}(E') \quad (3.40)$$

We would like to derive energy-dependent parallels to Eq. 3.32, namely to Σ_{tr} , D and Fick's law. This starts with a definition of Σ_{tr} by combining Eqs. 3.39 and 3.40.

$$\begin{aligned} \Sigma_{tr}(E) \mathbf{J}(E) &= \Sigma(E) \mathbf{J}(E) - \int dE' \mu(E', E) \Sigma_s(E', E) \mathbf{J}(E) \\ \Sigma_{tr}(E) &= \Sigma(E) - \int dE' \mu(E', E) \Sigma_s(E', E) \left[\frac{\mathbf{J}(E')}{\mathbf{J}(E)} \right] \end{aligned} \quad (3.41)$$

Where $\Sigma_s(E', E)$ is the scattering cross section for a neutron entry with kinetic energy E' and exit with E ; $\mu(E', E)$ is a cosine of half the head angle of a cone, onto the envelope of which the exiting neutron emerges if its direction upon entry has been the main axis of the cone. The problem is in the [] brackets. As written it is meaningless and requires the multi-group version of Eq. 3.39. One common procedure for avoiding this difficulty is to neglect the anisotropic contribution to energy transfer in a scattering collision by setting:

$$\mu(E', E) \Sigma_s(E', E) = \mu(E) \Sigma_s(E) \delta(E' - E) \quad (3.42)$$

So that the quotient $\frac{\mathbf{J}(E')}{\mathbf{J}(E)}$ is equal to 1 and the definition of D , and Fick's law, follow:

$$\begin{aligned} D(\mathbf{r}, E) &= \frac{1}{3\Sigma_{tr}(\mathbf{r}, E)} \\ \mathbf{J}(\mathbf{r}, E) &= -D(\mathbf{r}, E) \nabla \phi(\mathbf{r}, E) \end{aligned} \quad (3.43)$$

Lecture 4: The Diffusion Approximation

Lecturer: Dr. Dan Kotlyar

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4.1 Exercise: one-group diffusion theory

Fig. 4.1 shows one-group (i.e. energy-wise) spatial distribution neutron flux in one-dimensional (function of x) adjacent slabs. Using the one-group theory, match each region and interface with an appropriate property.

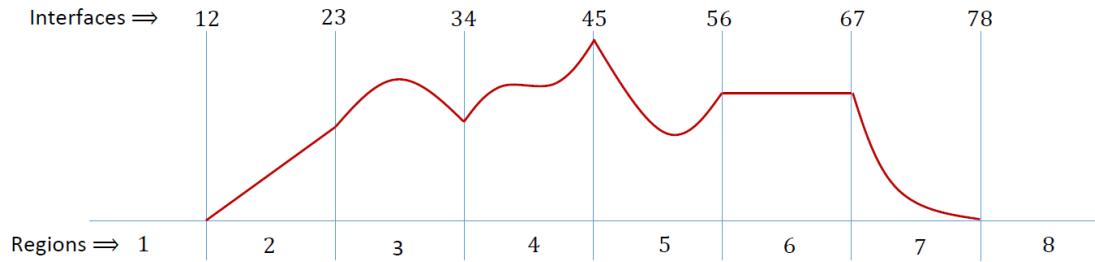


Figure 4.1: Description of the one-group neutron flux in adjacent slabs.

Properties of regions:

1. $\nu\Sigma_f = \Sigma_a$
2. $\nu\Sigma_f > \Sigma_a$
3. $\nu\Sigma_f < \Sigma_a$
4. $\Sigma_a \rightarrow \infty$
5. Vacuum

Properties of interfaces:

1. Absorber sheet
2. Source sheet
3. Transparent sheet
4. Reflective

4.1.1 Solution: one-group diffusion theory

Several reminders that will help us with this exercise:

1. If we are solving the following equation: $-D \frac{d^2\phi}{dx^2} + \Sigma_a \phi = \nu \Sigma_f \phi$, which can be described as follows:
 $\phi'' + \frac{\nu \Sigma_f - \Sigma_a}{D} \phi = 0$. where we can denote $\frac{\nu \Sigma_f - \Sigma_a}{D} = B^2$

Based on the roots of characteristic equation, we can obtain the solutions of this equation.

$$\lambda^2 + B^2 = 0 \Rightarrow \lambda = \alpha + i\beta. \text{ In this case: } \lambda = \pm \sqrt{-B^2} = \pm \sqrt{\frac{\Sigma_a - \nu \Sigma_f}{D}}$$

$\nu \Sigma_f > \Sigma_a$: the solution is $\phi(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x)$. This creates a concave shape.

$\nu \Sigma_f < \Sigma_a$: the solution is $\phi(x) = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x)$. This creates a convex shape.

$$\nu \Sigma_f = \Sigma_a: \phi'' = 0 \Rightarrow \phi = C_1 + C_2 \times x$$

2. We have previously seen that for a plane source, the flux decays with distance:

$$\phi \propto e^{-|\frac{x}{L}|}$$

We also know that in a non-vacuum medium: $J \propto -\frac{d\phi}{dx}$:

$J > 0$: This is a source of neutrons, it happens when $\frac{d\phi}{dx} < 0$.

$J < 0$: This is a sink of neutrons, it happens when $\frac{d\phi}{dx} > 0$.

$J = 0$: This is an ideal reflector, it happens when $\frac{d\phi}{dx} = 0$.

3. For non-void regions, the following condition applies on the boundary: $-D_i \frac{d\phi_i}{dx} = -D_{i+1} \frac{d\phi_{i+1}}{dx}$

Returning now to the exercise we conclude that the interfaces are:

1. Absorbers (sinks): 12, 34 and 78. The currents are directed into the interface.
2. Sources: 45, 56 and 67. The currents are in the outwards direction from the interface.
3. Transparent: 23. The different gradient are due to different diffusion coefficients in the adjacent regions.
4. Reflective:

As for the regions we can conclude:

1. $\nu \Sigma_f = \Sigma_a$: 2. The flux is linear.
2. $\nu \Sigma_f > \Sigma_a$: 3 and 4. The eigenvalue is above unity which imposes a convex shape. The shape in region 4 is not purely 'cos' since interface 45 is a source (outgoing current is positive).
3. $\nu \Sigma_f < \Sigma_a$: 5 and 7. The eigenvalue is below unity which imposes a concave shape.
4. Vacuum: 6.
5. $\Sigma_a \rightarrow \infty$: 1 and 8. The flux diminishes immediately.

4.2 Thermal diffusion length experiment

In order to assess how various parameters (e.g. flux) will behave in the reactor, the cross sections must be known. In this section, a method to measure the diffusion length L will be described ($L^2 = D/\Sigma_a$).

Thermal neutron diffusion length can be determined experimentally. This is done by measuring the axial flux (through an activity of a sample) in a long (relative to the mean-free-path) block of material as shown in Fig. 4.2.

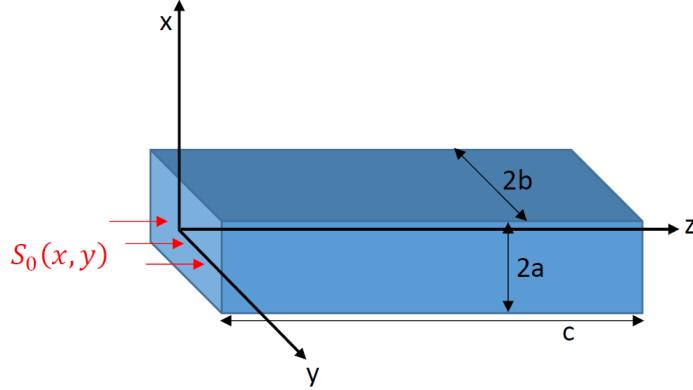


Figure 4.2: Rectangular geometry for pile experiment.

Fig. 4.2 presents a rectangular parallel-piped of length c and a cross sectional area of $2a \times 2b$. As isotropic neutron source $S_0(x, y)$ is placed at $z = 0$.

You are required to evaluate the diffusion length, L .

4.2.1 Solution

The partial differential equation that describes the behavior of the neutron flux is:

$$-D\nabla^2\phi + \Sigma_a\phi = 0 \quad (4.1)$$

If we divide eq. 4.1 by D and use the ∇^2 in Cartesian coordinates:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{L^2}\phi = 0 \quad (4.2)$$

with the following boundary conditions:

$$j^+(x, y, 0) = \frac{1}{2}S_0(x, y) \quad (4.3)$$

and the flux at the extrapolation length equals to zero:

$$\begin{aligned}\phi(\pm a, y, z) &= 0 \\ \phi(x, \pm b, z) &= 0 \\ \phi(x, y, c) &= 0\end{aligned}\tag{4.4}$$

where a , b and c are the extrapolated length, e.g. $a + 0.71\lambda_{tr}$.

We use the separation of variables method to solve this problem:

$$\phi = X(x)Y(y)Z(z)\tag{4.5}$$

we'll substitute eq. 4.5 into eq. 4.2 and divide by ϕ :

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{L^2}\tag{4.6}$$

This equation can only be satisfied if each term on the LHS of eq. 4.6 equals to a constant:

$$\begin{aligned}\frac{X''}{X} &= -k_1^2 \\ \frac{Y''}{Y} &= -k_2^2 \\ \frac{Z''}{Z} &= k_3^2\end{aligned}\tag{4.7}$$

where k_3^2 must be positive ($1/L^2$ is positive) and equals to:

$$k_3^2 = \frac{1}{L^2} + k_1^2 + k_2^2\tag{4.8}$$

The general solution:

$$\begin{aligned}X(x) &= A_1 \sin k_1 x + C_1 \cos k_1 x \\ Y(y) &= A_2 \sin k_2 y + C_2 \cos k_2 y \\ Z(z) &= A_3 \exp(-k_3 z) + C_3 \exp(+k_3 z)\end{aligned}\tag{4.9}$$

Since the solution in x and y should be symmetric A_1 and A_2 must be equal to zero. In order to find the constant C_3 , we will use $\phi(x, y, c) = 0$.

$$\begin{aligned}Z(z = c) &= 0 = A_3 \exp(-k_3 c) + C_3 \exp(+k_3 c) \Rightarrow C_3 = -A_3 \exp(-2k_3 c) \\ Z(z) &= A_3 e^{(-k_3 z)} \left[1 - e^{-2k_3(c-z)} \right]\end{aligned}\tag{4.10}$$

The symmetric and extrapolated, i.e. $\phi(\pm a) = 0$ and $\phi(\pm b) = 0$, boundary conditions mandates that:

$$\begin{aligned} k_{1n} &= \frac{\pi}{2a} (2n + 1) \quad n = 0, 1, \dots \\ k_{2m} &= \frac{\pi}{2b} (2m + 1) \quad m = 0, 1, \dots \\ k_{3nm}^2 &= \frac{1}{L^2} + k_{1n}^2 + k_{2m}^2 \end{aligned} \quad (4.11)$$

where k can only take discrete values. The final solution is therefore:

$$\phi(x, y, z) = \sum_{n,m=0}^{\infty} A_{nm} \cos(k_{1n}x) \cos(k_{2m}y) e^{(-k_{3nm}z)} \left[1 - e^{-2k_{3nm}(c-z)} \right] \quad (4.12)$$

The constants A_{nm} are not required to determine the diffusion length. Also, one can easily notice that k_{3nm} increases with n and m and the asymptotic neutron flux along the z -axis, which is far away from $z = 0$ (next to the source) is:

$$\phi(0, 0, z) \cong \phi(z) = A_{00} e^{(-k_{300}z)} \left[1 - e^{-2k_{300}(c-z)} \right] \quad (4.13)$$

The diffusion length is then provided by the following expression:

$$\frac{1}{L^2} = k_{300}^2 - \left(\frac{\pi}{2a} \right)^2 - \left(\frac{\pi}{2b} \right)^2 \quad (4.14)$$

So far, we have derived the the neutron flux expression and now we need to relate the neutron flux to some measured quantity.

Measurement of the neutron-flux by foil-activation techniques:

1. Choose a sample with high activation cross-section for thermal neutrons (e.g. In¹¹⁵) and a sensible half-life of the activated product (e.g. In^{116m} 54 minutes).
2. Place the sample foils in the pile. The sample will capture neutrons and produce the radioactive product (e.g. $n + \text{In}^{115} \rightarrow \text{In}^{116m}$).
3. The production rate of In^{116m} is proportional to the flux, i.e. $R_{act} \propto \phi$.
4. Although the derivation will not be shown here, it easy to deduct the activity of the samples and thereafter to relate these values to the flux distribution.

Now that the flux distribution is measured through the activity, one could easily create a horizontal curve of the measure thermal flux and compare with the expected distribution:

$$X(x) = A \cos\left(\frac{\pi}{2a}x\right) \quad (4.15)$$

where a equals to the dimensions of the pile (i.e. $2a$) with the addition of the extrapolated value (i.e. $0.71\lambda_{tr}$).

Now that we know the transport mean free path, we need to fit the vertical data along the z -axis. Before doing so, we will take another look on eq. 4.13. When c is large enough, the expression in the square parenthesis can be neglected, as shown in Fig. 4.3, and the neutron flux can be approximated by eq. 4.16.

$$\phi(z) = A_{00}e^{(-k_{300}z)} \quad (4.16)$$

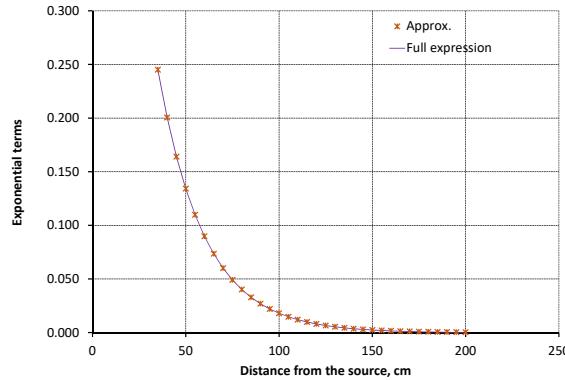


Figure 4.3: Illustration of the neutron flux exponential decay along the z -axis (for a graphite pile).

In the last stage, a semi-log vertical plot can be used (Fig. 4.14) to find k_{300} (fit the measured data with linear interpolation).

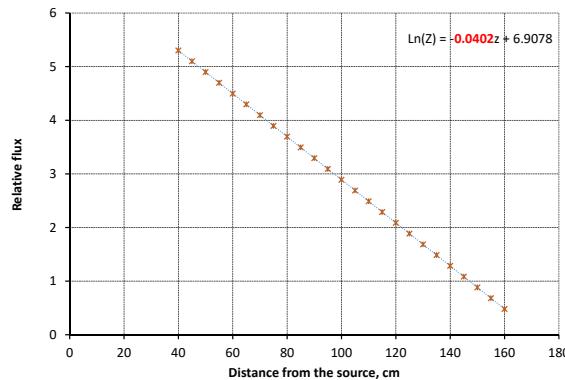


Figure 4.4: Log-linear relative neutron flux along the z -axis.

If we apply *log* on both sides of eq. 4.14, we obtain eq. 4.17. The slope is k_{300} and hence we can finally obtain the diffusion length.

$$\begin{aligned} \ln\left(\frac{\phi(z)}{\phi(z_0)}\right) &= -k_{300}(z) \\ \ln(\phi(z)) - \ln(\phi(z_0)) &= -k_{300}(z) \end{aligned} \tag{4.17}$$

Lecture 5: Unbalanced Core

Lecturer: Dr. Dan Kotlyar

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5.1 Flux dependence on the reactivity

A normal state of affairs in a power station is when $k = 1$ and the neutron population is time invariant. Strictly k is continually changing but the change is extremely slow, compared with the period of a year (within the cycle, which might be 2 years). In calculating the balanced core, every so often control rods, and/or burnable poisons, and/or soluble boron, change position or concentration to compensate for the changing k . This is the practice in both calculation and operation: periodical repositioning of the control mechanism. Long duration changes come under the heading of fuel depletion; Intermediate duration changes carry no special name. The short duration changes of interest presently are changes occurring within fractions of a second to a few minutes and may have a marked influence on the normal operation of the station. We shall refer to these latter changes as the kinetics of an **unbalanced core**.

One of the important parameters determining the kinetics of the unbalanced core is the reactivity ρ , defined as:

$$\rho = 1 - \frac{1}{k} \quad (5.1)$$

In practice the interesting situations are those where k is close to 1. One then has

$$k \cong 1 \rightarrow \rho \cong k - 1 \quad (5.2)$$

In the balanced core $\rho = 0$ and the flux is time constant. If $\rho \neq 0$, there will be a temporal flux dependency. The diffusion, one group, homogeneous and bare, core equation, describing a non-balanced state is:

$$\frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} = (D\nabla^2 - \Sigma_a + \nu\Sigma_f) \phi(\mathbf{r}, t) \quad (5.3)$$

$\phi(\mathbf{r}, t)$ is the time-and-space dependent flux, zero on the extrapolated boundaries of the core. We change somewhat the appearance of this equation by using the material buckling definition:

$$B_m^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} \quad (5.4)$$

This makes Eq. 5.3 read

$$\frac{1}{vD} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} = (\nabla^2 + B_m^2) \phi(\mathbf{r}, t) \quad (5.5)$$

We use the separation of variables to solve this problem:

$$\phi(\mathbf{r}, t) = T(t)\Psi(\mathbf{r}) \quad (5.6)$$

Substitution of Eq. 5.6 into 5.5 and division by $\phi(\mathbf{r}, t)$ leads to:

$$\frac{1}{vD} \frac{\dot{T}}{T} - \frac{\nabla^2 \Psi}{\Psi} = B_m^2 \quad (5.7)$$

The 'dot' operator is shorthand for d/dt . B_m^2 is a constant, therefore separation of variables holds, provided each of the two L.H.S. terms in the second of Eq. 5.7 is a constant, and provided the sum of these two constants is B_m^2 .

we write the spatial part of Eq. 5.7 as follows:

$$\frac{\nabla^2 \Psi}{\Psi} = -B_g^2 \quad (5.8)$$

B_g^2 is a constant, called the geometrical buckling. Eq. 5.8 is the Helmholtz Equation. Mathematically it is solvable for infinitely many B_g^2 values. Only one value of these yields an eigenfunction Ψ which is positive all over the reactor \mathbf{r} space. This fundamental solution will dominate since the higher order spatial modes die out very rapidly (1000 sec^{-1}). In a one dimensional geometry $B_g^2 = \frac{\pi}{a}$, where a is the extrapolated thickness of the reactor.

The substitution of Eq. 5.8 in 5.7 yields

$$\frac{\dot{T}(t)}{T(t)} = vD (B_m^2 - B_g^2) \quad (5.9)$$

The solution of this equation is:

$$T(t) = T(0) \exp(vD(B_m^2 - B_g^2)t) \quad (5.10)$$

Next we handle the exponent in Eq. 5.10:

$$D(B_m^2 - B_g^2) = D \left(\frac{\nu \Sigma_f - \Sigma_a}{D} - B_g^2 \right) = \nu \Sigma_f - (\Sigma_a + DB_g^2) \quad (5.11)$$

The combination of $\tilde{\Sigma}_a = (\Sigma_a + DB_g^2)$ is well familiar and describes the loss term, namely the absorption+leakage.

Relating to criticality now:

$$k = \frac{\nu\Sigma_f}{\Sigma_a} \frac{1}{1 + L^2 B_g^2} = \frac{\nu\Sigma_f}{\tilde{\Sigma}_a} \quad (5.12)$$

Therefore,

$$D(B_m^2 - B_g^2) = \tilde{\Sigma}_a (k - 1) \quad (5.13)$$

and returning to Eq. 5.10,

$$T(t) = T(0) \exp(v\tilde{\Sigma}_a (k - 1) t) \quad (5.14)$$

In order that an assessment of Eq. 5.14 can be made we make first use of the neutron average lifetime l . The mean free path of the neutron, from the moment of birth to either absorption or leakage, is $1/\tilde{\Sigma}_a$. This distance is traveled in an average velocity v , therefore

$$l = \frac{1}{v\tilde{\Sigma}_a} \quad \text{average neutron lifetime} \quad (5.15)$$

Eq. 5.14 can now be rephrased:

$$T(t) = T(0) \exp\left(\frac{k-1}{l}t\right) \quad (5.16)$$

or

$$n(t) = n(0) \exp\left(\frac{k-1}{l}t\right) \quad (5.17)$$

One has in Eq. 5.17 an estimate of the temporal flux dependence in a core. This dependence is mainly on two factors: the neutron lifetime l and the deviation from criticality, i.e. $k - 1$. This result is actually depressing: suppose the criticality is of the order of $+1.001$, a very small value; a typical lifetime in the typical thermal reactor is 0.001 seconds or shorter: these make the exponent in Eq. 5.17 of the order of 1. It means that the flux level would increase by a factor of about e in one second. Such circumstances would render the operation of a power station absolutely prohibitive.

Fortunately our analysis was a crude estimation of the time-period. We have simplified (and suffered) on the count of not considering the delayed neutrons. Fraction of the fission neutrons are emitted with seconds to minutes delay from the fission-event time. These are the delayed neutrons. They have a marked influence on the seemingly abrupt temporal behavior of the flux.

5.2 Delayed Neutrons

Not all fission neutrons are what we term immediate, namely liberated practically at the time the fission event occurs: a fraction β (not to be confused with β decay) of the fission-liberated neutrons is released with a delay, relative to the fission event itself. The β fraction depends on the nuclide undergoing fission:

1. $\beta(U^{235}) = 0.0064$
2. $\beta(U^{233}) = 0.0026$
3. $\beta(Pu^{239}) = 0.0021$

We see that the relative amount of these delayed neutrons is small, a few tenths of one percent, yet their influence, as shall be seen, is considerable. These delayed neutrons are liberated in fission product chains. Often the fission fragments, and the nuclei they breed in a chain, make their way towards the stability zone by β^- emissions. In a β^- emission a neutron in the nucleus becomes a proton, and an electron is being ejected from the atom. Sometimes the road to stability entails an extra neutron emission that follows immediately this β^- emission. The delayed neutron is delayed relative to the fission event by a time length, the average of which is the so-called average emission time τ for that particular β^- emission.

If the decay constant for this radioactive decay is λ , then the average emission time is given by $\tau = 1/\lambda$.

Experimentally, one in general discerns between 6 groups of delayed neutrons. In Table 5.1 and 5.2 the first two groups originate in two specific distinct emissions; the other four groups are the result of, a somewhat arbitrary, aggregation of emissions. Unlike the nuclei-dependent total fraction β , the group relative fractions β_i/β ($i=1,\dots,6$) are almost nuclide-independent.

Table 5.1: Delayed neutrons U^{235} from thermal fission.

Group No.	Decay constant, λ_i, sec^{-1}	τ_i, sec	β_i/β
1	0.0124	80.6	0.033
2	0.0305	32.8	0.219
3	0.1110	9.40	0.196
4	0.3010	3.30	0.395
5	1.1400	0.88	0.115
6	3.0100	0.33	0.042

Table 5.2: Delayed neutrons Pu^{239} from thermal fission.

Group No.	Decay constant, λ_i, sec^{-1}	τ_i, sec	β_i/β
1	0.0128	78.1	0.035
2	0.0301	33.2	0.298
3	0.1240	8.06	0.211
4	0.3250	3.08	0.326
5	1.1200	0.89	0.086
6	2.6900	0.37	0.044

Delayed neutron emitters are fission fragments that are highly unstable because of excess neutrons. The usual mode decay in such cases is beta emission, often in chains of four or five successive events. Occasionally, a branch is followed that leads to an unusually highly excited nucleus; the result is the immediate ejection of a neutron, leaving a stable nuclide. An example is Br^{87} , shown in Fig. 5.1, which probably accounts for the delayed-neutron of group 1. Note that the lifetime for neutron emission is controlled by the beta-decay half-life of the parent nuclide.

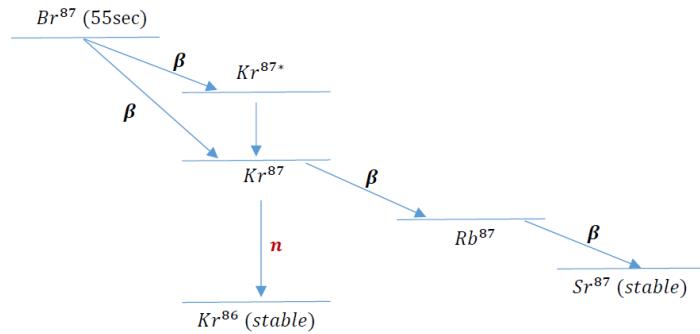


Figure 5.1: Origin of delayed neutrons from Br^{87} .

Lecture 6: The Kinetic Equations

Lecturer: Dr. Dan Kotlyar

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6.1 On the mellowing influence of the delayed neutrons

Prior to a well rigorous discussion of the influence of the delayed neutrons, which is enabled only by replacing the simple-minded core diffusion equation, i.e. Eq. 5.3, by the delayed-neutrons-dependent core kinetic equations. This is taken up in the next section. We precede such a discussion with simplistic explanations of how the delayed neutrons are so influential.

In Fig. 6.1 we plot the erupting flux ϕ as a function of the time t . The fission macroscopic cross section Σ_f is, for the time spans considered, time constant. Suppose the reactivity is positive and suppose we calculated $\phi(t)$ assuming that all neutrons were immediate, namely each neutron considered in the balance was totally available to us as a result of a fission event occurring in the preceding generation.

We know from the preceding section that $\phi(t)$ would be exponential, as drawn in Fig. 6.1.

Hence:

At t: The fission rate, assuming immediacy of all the neutrons = $\Sigma_f\phi(t)$.

We now correct this last expression by accounting for the *delay*:

At t: The contribution of the delayed neutrons to the fission rate = $\beta\Sigma_f\phi(t - \delta)$.

At t: The contribution of the immediate neutrons to the fission rate = $(1 - \beta)\Sigma_f\phi(t)$.

Where δ is a delay time. The meaning of these expressions is clear: in the balance made for time t , a fraction of the neutrons has to be taken as they were "seeded" beforehand, at time $t - \delta$. For a positive reactivity ($k > 1$) the flux will have grown exponentially with time, therefore

$$\phi(t - \delta) < \phi(t) \quad (6.1)$$

Eq. 6.1 permits us to argue that, for the total balance:

$$(1 - \beta)\Sigma_f\phi(t) + \beta\Sigma_f\phi(t - \delta) < \Sigma_f\phi(t) \quad (6.2)$$

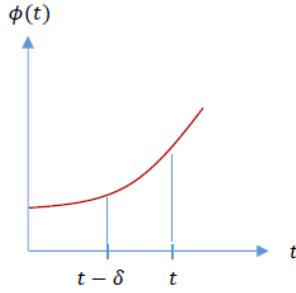


Figure 6.1: The mellowing influence of the delayed neutrons.

We have obtained that the true fission rate is lower than in the curve of Fig. 6.1. For a negative reactivity ($k < 1$) the arguments are similar, leading to the conclusion that the flux decay is slower than normally calculated. In both cases one predicts a mellowing influence of the delayed neutrons on the temporal behavior of the flux. Due to the crude nature of our argument one cannot say, at this point, more.

We offer next an argument why the delayed neutrons have a marked influence on the speed of the flux exponential increase, or decrease. We must warn, again, that the argument to follow is crude too, exaggerating on the influence of the delayed neutrons, and should be taken only qualitatively.

Aided by Table 5.1 we can estimate by how much the delayed neutrons prolong the average lifetime of fission neutrons. Each (of the 6) delayed neutron precursors (fathers) delays the liberation of its neutrons, on the average, by a time τ_i ; therefore l , the average lifetime for the fission neutrons is, taking U²³⁵ for example:

$$l(U^{235}) = (1 - \beta)l^* + \sum_i \beta_i(l^* + \tau_i) = l^* + \sum_i \beta_i \tau_i = 10^{-3} + 0.084 = 0.085 \text{ sec} \quad (6.3)$$

This average lifetime is almost entirely determined by the delayed neutrons, although their total relative amount is less than one percent.

In Eq. 5.17, we now replace the average lifetime l of the new expression (Eq. 6.3):

$$\phi(\mathbf{r}, t) = \Psi(\mathbf{r}) T(0) \exp\left(\frac{k-1}{l}\right) \quad (6.4)$$

We may now repeat the line of argumentation which followed Eq. 5.17, with the difference that now we have a lifetime of the order of 0.1 sec, not 0.001 sec. This will make the flux change per second of the order of $\approx e^{0.01} \approx 1\%$, in other words manageable. Again, this piece of development exaggerates the importance of the delayed neutrons. The only true handling of the delayed neutrons is via a replacement of Eq. 5.3, the core equation, with equations that take proper account of the delayed neutrons.

6.2 The kinetic equations

In this section we interpret the point-reactor model in terms of simplified neutron cycle as shown in 6.2. Prompt neutrons are returning back to the system immediately, and latent neutrons return to the system with various mean times $\tau_i = 1/\lambda_i$. The i^{th} "father" of delayed-neutron precursor (with fraction β_i) is produced at a rate $\beta_i \nu \Sigma_f \phi$

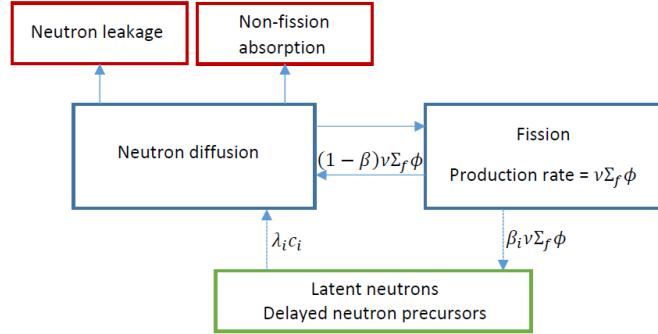


Figure 6.2: Simplified neutron life cycle.

In reality, one cannot proceed so heuristically. We must first set up a set of equations describing the time dependence of the delayed neutrons.

Let $C_i(\mathbf{r}, t)$ denote the i^{th} precursor density, nuclides per cubic cm, at spatial point \mathbf{r} and at time t . Though only the first two entries in Table 5.1 and 5.2 are strictly for a specific precursors, emitting each one neutron, the analysis will assume that $i=1,6$ and that every i stands for a precursor emitting one neutron. By β_i one usually means the fraction of delayed neutrons in group i .

Based on the above, the *formation-destruction* balance for i^{th} precursor is:

$$\frac{\partial C_i(\mathbf{r}, t)}{\partial t} = -\lambda_i C_i(\mathbf{r}, t) + \beta_i [\nu \Sigma_f \phi(\mathbf{r}, t)] \quad i = 1, \dots, 6 \quad (6.5)$$

In Eq. 6.5 we assume that the precursor don't migrate or diffuse before decaying. The production rate of the precursor in group i equals to the production rate of delayed neutrons $\nu_{d,i} \Sigma_f \phi(\mathbf{r}, t)$, where $\nu_{d,i} = \beta_i \nu$.

In considering the core equation, we remember that the equation has to be a statement of balance for the core neutrons. To the fission neutrons liberated by the immediate emitters we have to add the fission neutrons liberated by the delaying precursors. Eq. 5.3 is thus replaced by:

$$\frac{1}{v} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} = \underbrace{[D \nabla^2 - \Sigma_a + (1 - \beta) \nu \Sigma_f]}_{\text{Immediate neutrons}} \phi(\mathbf{r}, t) + \underbrace{\sum_i \lambda_i C_i(\mathbf{r}, t)}_{\text{Delayed neutrons}} \quad (6.6)$$

Eqs. 6.5 and 6.6 are the replacement to Eq. 5.3: the difference between immediate and delayed neutrons has been brought into consideration.

We will apply asymptotic situation in which both the flux and precursor concentrations can be written as separable functions of space and time.

$$\begin{aligned}\phi(\mathbf{r}, t) &= vn(t)\Psi(\mathbf{r}) \\ C_i(\mathbf{r}, t) &= c_i(t)\Psi(\mathbf{r})\end{aligned}\tag{6.7}$$

where the fundamental solution $\Psi = \Psi_1(\mathbf{r})$ is obtained from solving the Helmholtz equation:

$$\nabla^2\Psi_n + B_g^2\Psi_n = 0\tag{6.8}$$

According to this assumption (Eq. 6.7) we have turned to point kinetics (i.e. fixed spatial distribution) for analysis. Experience shows that point kinetics will produce, in the main, a reliable picture. Special problems necessitate improvements in the point kinetics approach, or a direct numerical solution of Eqs. 6.5 and 6.6.

We will use Eqs. 6.5 and 6.6 to get the **point kinetic equations**:

$$\begin{aligned}\frac{dn(t)}{dt} &= \frac{\rho(t) - \beta}{\Lambda}n(t) + \sum_{i=1}^6 \lambda_i c_i(t) \\ \frac{dc_i(t)}{dt} &= \frac{\beta_i}{\Lambda}n(t) - \lambda_i c_i(t) \quad i = 1, \dots, 6\end{aligned}\tag{6.9}$$

In the process we have defined the *mean neutron generation time*:

$$\Lambda = \frac{l}{k}\tag{6.10}$$

The above quantity describes the mean generation time between birth of neutron and subsequent absorption inducing fission. If $k = 1$ then Λ is essentially just the **prompt neutron lifetime** l . We also note that the expression in Eq. 5.1 now is a function of time.

$$\rho(t) = \frac{k(t) - 1}{k(t)}\tag{6.11}$$

$$k = \frac{\nu\Sigma_f/\Sigma_a}{1 + L^2B_g^2}\tag{6.12}$$

H.W. Derive Eqs. 6.9 from Eqs. 6.5 and 6.6

The final product (Eq. 6.9) is a set of seven coupled ordinary differential equations in time that describe both the time-dependence of the neutron population in the reactor and the decay of the delayed neutron precursors.

The solution of this system is not straightforward since reactivity is a function of time and also frequently depends on the neutron population, which leads to a non-linear problem.

6.2.1 Limitations of the point kinetics model

1. One-speed diffusion approximation. The assumption can be dealt explicitly in a much more general fashion.
2. It should be kept in mind that $\bar{\beta}_i$ must be calculated as suitable weighted averages over the relevant mixture of fuel isotopes. In most thermal reactors these averaged delayed neutron fractions will decrease with core life-time since the bred Pu²³⁹ has a lower values.
3. The most serious approximation involved in the point kinetics equations is the assumption that the flux can be adequately represented by a single, time-independent spatial mode $\Psi_1(\mathbf{r})$. This shape function is actually not the fundamental mode (i.e. for critical system) but rather the fundamental (i.e. dominate) mode characterizing the reactor core that has been subjected to a reactivity change.
4. Feedback effects were not included in our current analysis. Change in reactivity will change flux and hence power. The power will vary temperatures, which will have an effect on the reactivity. At the moment we solve the zero-power point reactor model since it ignores the feedback that would occur due to the variations in the reactor power level.

6.2.2 Laplace transform

Let $g(t)$ be defined for all $t \geq 0$. The function may be zero or undefined for $t < 0$. We define:

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty g(t)e^{-st}dt \quad (6.13)$$

We use the notation $G(s)$ for the Laplace transform of $g(t)$.

Example 1:

Let $g(t) = 1$ for $t \geq 0$. Find $G(s)$

$$G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty 1 \times e^{-st}dt = \frac{1}{s} \quad (6.14)$$

Example 2:

Let $g(t) = e^{at}$ for $t \geq 0$. Find $G(s)$

$$G(s) = \int_0^\infty e^{at} \times e^{-st}dt = \frac{1}{s-a} \quad (6.15)$$

The following table summarizes some useful direct and inverse Laplace transforms:

Table 6.1: Laplace transforms.

$g(t) = \mathcal{L}^{-1}\{G(s)\}$	$G(s) = \mathcal{L}\{g(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n, n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}$
$g'(t)$	$sG(s) - g(0)$
$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$

Lecture 8: Solution of the point reactor kinetics equations

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8.1 The Inhour Equation

We finished our last lecture by defining the point kinetic equations (Eq. 8.1). Returning now to our point kinetics model. Probably the easiest way to solve this initial value problem is by using Laplace transform. Eq. 6.9 can be solved for the case of an initially critical reactor in which a reactivity ρ_0 is introduced at t_0 due to the change in various properties.

$$\begin{aligned}\frac{dn(t)}{dt} &= \frac{\rho(t) - \beta}{\Lambda} n(t) + \sum_{i=1}^6 \lambda_i c_i(t) \\ \frac{dc_i(t)}{dt} &= \frac{\beta_i}{\Lambda} n(t) - \lambda_i c_i(t) \quad i = 1, \dots, 6\end{aligned}\tag{8.1}$$

Applying the Laplace transforms leads to:

$$\begin{aligned}sN(s) &= \frac{\rho_0 - \beta}{\Lambda} N(s) + \sum_{i=1}^6 \lambda_i C_i(s) + n(t=0) \\ sC_i(s) &= \frac{\beta_i}{\Lambda} N(s) - \lambda_i C_i(s) + c_i(t=0) \quad i = 1, \dots, 6\end{aligned}\tag{8.2}$$

From eq. 8.2,

$$C_i(s) = \frac{\frac{\beta_i}{\Lambda} N(s) + c_i(0)}{s + \lambda_i}\tag{8.3}$$

If we now substitute eq. 8.3 into eq. 8.2 and solving for $N(s)$ will give us:

$$N(s) = \frac{\Lambda \left[n(0) + \sum_{i=1}^6 \frac{\lambda_i c_i(0)}{s + \lambda_i} \right]}{-\rho_0 + s \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{s + \lambda_i} \right)}\tag{8.4}$$

Eq. 8.4 is written as the ratio of two polynomials in s . The degree of the numerator and denominator are m and $m + 1$ respectively.

If we use the inverse Laplace transform on eq. 8.4, the solution of the time-dependent neutron flux will be:

$$n(t) = \sum_{i=0}^6 A_i e^{s_i t} \quad (8.5)$$

The inverse of $N(s)$ created 7 exponential terms.

where s_i are the $m + 1$ roots $s = s_i$ of the denominator of $N(s)$ and A_i are given by:

$$A_i = \frac{\Lambda + \sum_{j=1}^6 \frac{\beta_j}{s_i + \lambda_j}}{1 + k \sum_{i=j}^6 \frac{\beta_j \lambda_j}{(s_i + \lambda_j)^2}} \quad (8.6)$$

In other words, each s_i satisfies the following equation:

$$\rho_0 = s \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{s + \lambda_i} \right) \quad (8.7)$$

The roots of the denominator in eq. 8.4 determine the time-dependence of the neutron and precursor populations. Eq. 8.7 is a 7th order equation, known as the inverse hour, or more in short the inhour equation. The solutions which are best visualized graphically as shown in Fig. 8.1.

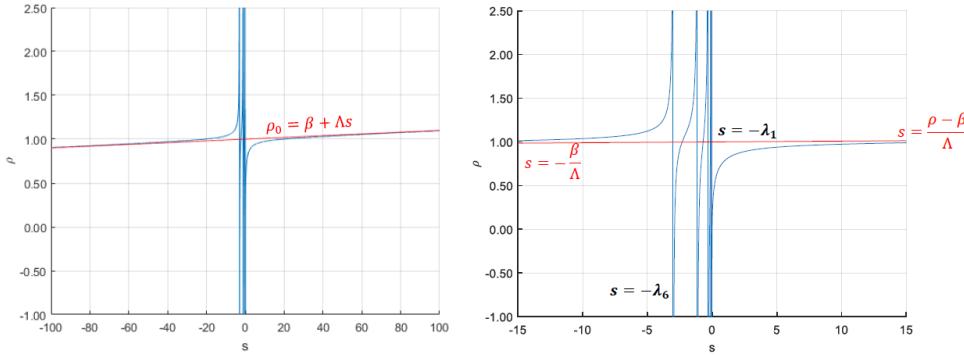


Figure 8.1: Up to scale plot of the inhour equation for six group of delayed neutrons.

For very large s , either positive or negative eq. 8.7 has the following asymptote:

$$\rho_0 = \beta + \Lambda s \quad (8.8)$$

For which:

$$s = \frac{\rho_0 - \beta}{\Lambda} \quad (8.9)$$

It is seen from eq. 8.7 that $\rho_0 \rightarrow \pm\infty$ as $s \rightarrow -\lambda_i$ and this means that there is a set of m asymptotes $s = -\lambda_i$ and that $m - 1$ branches of the inhour curve are bounded by adjacent asymptotes.

These observations are better seen in Fig. 8.2, which is a qualitative plot of ρ vs. s . Where the right hand-side of eq. 8.7 is plotted in Fig. 8.2. The left-hand side (i.e. ρ_0) would plot as straight horizontal line. The points at which it crosses the right-hand side are the solutions (roots of the equation).

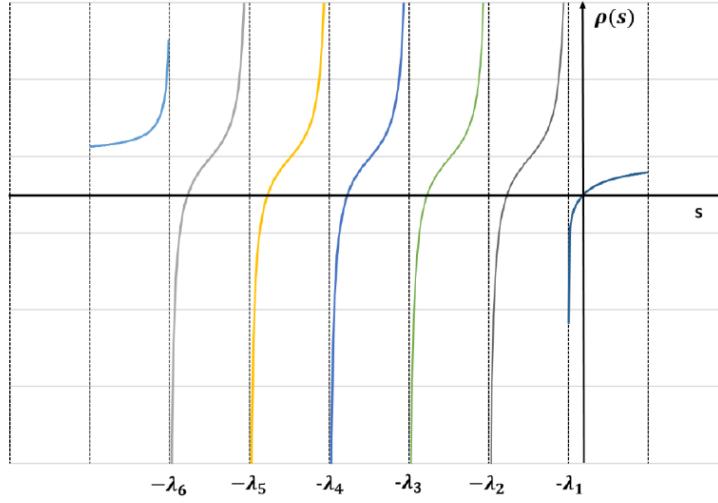


Figure 8.2: Plot of the Inhour equation.

Eq. 8.7 is the equation one usually quotes in the attempt to understand the temporal flux behavior of the unbalanced core. Often the equation is written in terms of the disintegration constants λ_i , instead of the average disintegration time delays τ_i . Historically the equation was named the inhour equation, 'inhour' being shorthand for 'inverse hour', because the period T appears in it by its inverse, $1/T$. One hour defined as the amount of positive reactivity that corresponds to an asymptotic power rise having a time constant of one hour.

The equation is relatively easy to analyze in this form (Eq. 8.7), namely a $\rho(s)$ or $\rho(1/T)$ form. $T(\rho)$ would have 7 solutions, namely 7 periods (s_i) T_i , $i = 0, \dots, 6$ and the temporal part of the flux $n(t)$ would be a linear combination of exponents of these periods (Eq. 8.5).

The periods of the core are $T_i = \frac{1}{s_i}$ and thus:

$$n(t) = \sum_{i=0}^6 A_i e^{\frac{t}{T_i}} \quad (8.10)$$

The $\rho(s)$ plot, has 7 branches. A given reactivity ρ means a horizontal line $\rho = const$, as exemplified in the following figures (8.4 and 8.3). The 7 periods of the core are the 7 intersection points of the horizontal line with the 7-branched $\rho(s)$.

For clarity we discuss separately the $\rho < 0$ and the $\rho > 0$ cases.

$\rho < 0$: The periods are all negative. In increasing order they are:

$$s_6 < s_5 < s_4 < s_3 < s_2 < s_1 < s_0 \text{ for } \rho < 0.$$

All the periods are negative and their corresponding exponents will decay with $\exp(s_i t)$, but $s_0 = 1/T_0$ will persist the longest and hence dominant.

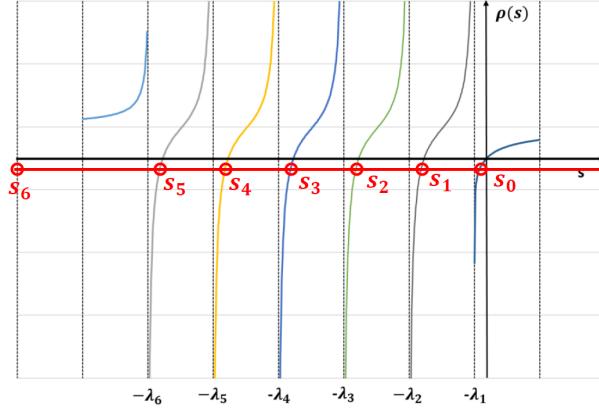


Figure 8.3: Plot of the Inhour equation with negative reactivity insertion.

$\rho > 0$: $s_0 = 1/T_0$ is positive, the other periods are negative. In increasing order the periods are:

$$s_6 < s_5 < s_4 < s_3 < s_2 < s_1 < 0 < s_0 \text{ for } \rho > 0.$$

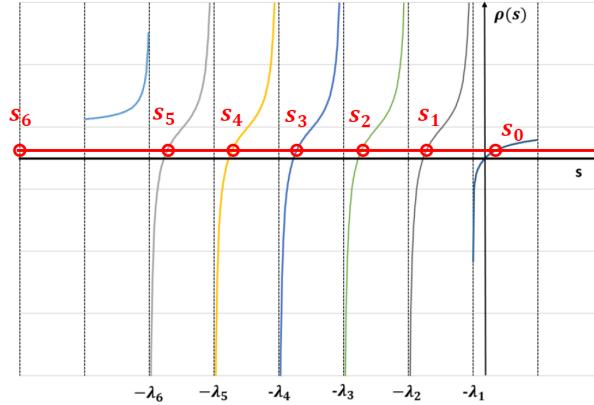


Figure 8.4: Plot of the Inhour equation with positive reactivity insertion.

Again T_0 is the one persisting period. For a positive reactivity ρ : since T_0 remains the only non-decaying period it is called the stable (asymptotic) period. Thus any change in core parameters which translates into positive reactivity will, with time, manifest itself in a temporal flux behavior characterized by a clean exponent, single and increasing.

After a sufficient time, the solution will be dominated by the largest root s_0 ($s_0 > 0$ when $\rho_0 > 0$). Similar argument can be claimed when $\rho_0 < 0$, s_0 will be the most dominant solution (the least negative root).

$$n(t) \cong A_0 e^{s_0 t} = A_0 e^{\frac{t}{T}} \quad (8.11)$$

where $T = s_0^{-1}$ is referred to as the asymptotic period. Measurement of the asymptotic period then provides a means for the experimental determination of the reactivity.

$$\rho_0 = \frac{1}{T} \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{\frac{1}{T} + \lambda_i} \right) \quad (8.12)$$

Lecture 9: One Delayed Neutron Group Solution

Lecturer: Dr. Dan Kotlyar

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9.1 Influence of the delayed neutrons

In order to understand the temporal flux behavior of the unbalanced core, we have focused on analyzing the inhour equation.

$$\rho_0 = \frac{1}{T} \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{\frac{1}{T} + \lambda_i} \right) \quad (9.1)$$

The equation is relatively easy to analyze in this form (Eq. 8.7), namely a $\rho(s)$ or $\rho(1/T)$ form. $T(\rho)$ would have 7 solutions, namely 7 periods (s_i) T_i , $i = 0, \dots, 6$ and the temporal part of the flux $n(t)$ would be a linear combination of exponents of these periods.

We next use the $\rho(T)$ analysis to draw important conclusions about the influence of the delayed neutrons.

$\rho \ll \beta$: Small reactivity insertions.

We can assume that the magnitude of s_0 is small such that:

$$|s_0| \ll \lambda_1 < \dots < \lambda_6 < \beta/\Lambda$$

The inhour equation, is specified for $T = T_0$, namely looked upon at times when only the dominant period remains relevant, we may stipulate $T_0 \gg \tau_i$ (i.e. neglect s_0 relative to λ), obtaining:

$$\rho = \frac{1}{T_0} \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{\lambda_i} \right) \cong \frac{\Lambda + 0.084}{T_0} = \frac{\bar{l}}{T_0} \quad (9.2)$$

The period is thus:

$$T_0 = \frac{\bar{l}}{\rho} \quad (9.3)$$

In other words, the smaller the reactivity the longer the dominant period; this period is proportional to the average delay time $\bar{l} = \Lambda + \sum \beta_i \tau_i$ where the sum part represents the delay time of the delayed neutrons and determines the period. Also of importance: as long as $\rho \ll \beta$ the influence of the delayed neutrons is impressive as mostly they determine the dominant period.

$\rho \gg \beta$: Large reactivity insertions. In the opposite extreme we can assume that $s_0 \gg \lambda_i$ (stable period is close to zero). Stipulating $T_0 \ll \tau_i$, we can obtain:

$$\rho \cong \frac{\Lambda}{T_0} + \beta \cong \frac{\Lambda}{T_0} \quad (9.4)$$

or

$$T_0 = \frac{\Lambda}{\rho} \quad (9.5)$$

Again, the stable period is inversely proportional to the reactivity, but now also proportional to the average life time of the immediate neutrons. In other words, the delayed neutrons influence has all but disappeared. The forgoing analysis shows that β plays a transition role, as far as the reactivity ρ is concerned. Delayed neutrons will have a role as long as $\rho < \beta$, this role will start to diminish as ρ becomes larger than β . We can already discern a major control difference between Uranium and Plutonium fuels: the β value for U²³⁵ is 0.0064, but is only 0.0021 for Pu²³⁹. With an increase of ρ , unsafe conditions will faster set in Pu cores than in U cores.

9.2 Analysis with one-group of delayed neutrons

More inroads can be made with the approximation of one group of delayed neutrons. As we collapse the (already collapsed) six groups into one we first define β , the one group DN (Delayed Neutrons) fraction, and λ , the one group DN decay constant, respectively by

$$\beta = \sum_i \beta_i \quad (9.6)$$

$$\lambda = \frac{\sum_i \beta_i \lambda_i}{\beta} \quad (9.7)$$

Therefore, the following equations are now considered:

$$\begin{aligned} \frac{dn(t)}{dt} &= \frac{\rho - \beta}{\Lambda} n(t) + \lambda c \\ \frac{dc(t)}{dt} &= \frac{\beta}{\Lambda} n(t) - \lambda c \end{aligned} \quad (9.8)$$

Assuming the solutions $n(t)$ and $c(t)$ behave similarly with time:

$$\begin{aligned} n(t) &\propto e^{st} \\ c(t) &\propto e^{st} \end{aligned} \quad (9.9)$$

Therefore,

$$\begin{aligned}\frac{dc(t)}{dt} &= sc(t) \\ \frac{dn(t)}{dt} &= sn(t)\end{aligned}\tag{9.10}$$

We will write the set of equations in a different form:

$$\begin{bmatrix} \frac{\rho-\beta}{\Lambda} - s & \lambda \\ \frac{\beta}{\Lambda} & -(s + \lambda) \end{bmatrix} \begin{bmatrix} n(t) \\ c(t) \end{bmatrix} = 0$$

We find the roots (s_1 and s_2) by using $\det=0$.

H.W.: Show that the resulting 2^{nd} equation is:

$$s^2 - \left(\frac{\rho - \beta}{\Lambda} - \lambda \right) s - \frac{\lambda \rho}{\Lambda} = 0\tag{9.11}$$

For which the solutions are:

$$s_{1,2} = \frac{1}{2} \left(\frac{\rho - \beta}{\Lambda} - \lambda \right) \pm \sqrt{\frac{1}{4} \left(\frac{\rho - \beta}{\Lambda} - \lambda \right)^2 + \frac{\lambda \rho}{\Lambda}}\tag{9.12}$$

Analyzing the solution we can see that:

1. $\rho > 0 \rightarrow$ positive and negative roots
2. $\rho = 0 \rightarrow$ zero and negative roots
3. $\rho < 0 \rightarrow$ both roots are negative

H.W.: Assuming the precursors and the flux behave similarly in time prove that the ratio between the precursor concentration and the flux is constant and equals to:

$$\frac{c(t)}{n(t)} = \frac{\beta}{\Lambda(s_{1,2} + \lambda)} = - \left(\frac{\rho - \beta}{\Lambda} - s_{1,2} \right) / \lambda\tag{9.13}$$

which means that the solutions are in the form:

$$n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}\tag{9.14}$$

and

$$c(t) = A_1 \frac{\beta}{\Lambda(s_1 + \lambda)} e^{s_1 t} + A_2 \frac{\beta}{\Lambda(s_2 + \lambda)} e^{s_2 t} \quad (9.15)$$

Let us now take values that characterize LWR cores:

1. $\beta = 0.0075$
2. $\lambda = 0.08 s^{-1}$
3. $\Lambda = 6 \times 10^{-5} sec$

According to Eq. 9.11:

$$s^2 - \left(\frac{\rho - \beta}{\Lambda} - \lambda \right) s - \frac{\lambda \rho}{\Lambda} = 0$$

In cases where we have small reactivity insertion $\rho \ll \beta$ the roots will be:

1. High magnitude number
2. Low magnitude number

$$\begin{aligned} \frac{\lambda \rho}{\Lambda} &= \frac{0.08 \times 0.0005}{6 \times 10^{-5}} = 0.8 \\ \frac{\rho - \beta}{\Lambda} &= \frac{0.0005 - 0.0075}{6 \times 10^{-5}} = -116.66 \end{aligned}$$

$$\begin{aligned} s^2 + 116s - 0.8 &= 0 \\ s_1 &= -116 \\ s_2 &= 0 \end{aligned}$$

Therefore, we make the following assumptions:

For the high magnitude root: $s_1^2 \gg \frac{\lambda \rho}{\Lambda}$.

Also $\frac{\rho - \beta}{\Lambda} \gg \lambda$.

Therefore the root is:

$$s = \frac{\rho - \beta}{\Lambda} \quad (9.16)$$

For the low magnitude root: $s_2^2 \ll \frac{\lambda \rho}{\Lambda}$.

Hence, the following equation

$$s^2 - \left(\frac{\rho - \beta}{\Lambda} - \lambda \right) s - \frac{\lambda\rho}{\Lambda} = 0$$

reduces to:

$$-\left(\frac{\rho - \beta}{\Lambda} \right) s - \frac{\lambda\rho}{\Lambda} = 0$$

and the root is:

$$s = -\frac{\lambda\rho}{\rho - \beta} \quad (9.17)$$

The constants A_1 and A_2 can be evaluated by requiring that the solution satisfy the initial condition at $t = 0$ that is determined by setting $n(t) = n_0$. One can show that

$$\begin{aligned} A_1 &\cong \frac{n_0\rho}{\rho - \beta} \\ A_2 &\cong -\frac{n_0\beta}{\rho - \beta} \end{aligned} \quad (9.18)$$

where n_0 is the initial neutron population before the reactivity insertion, so the equations that describe the time behavior of $n(t)$ and $c(t)$ are:

$$n(t) = n_0 \left[\frac{\rho}{\rho - \beta} \exp \left(\frac{\rho - \beta}{\Lambda} t \right) - \frac{\beta}{\rho - \beta} \exp \left(-\frac{\lambda\rho}{\rho - \beta} t \right) \right] \quad (9.19)$$

and

$$c(t) = n_0 \left[\frac{\rho\beta}{(\rho - \beta)^2} \exp \left(\frac{\rho - \beta}{\Lambda} t \right) + \frac{\beta}{\Lambda\lambda} \exp \left(-\frac{\lambda\rho}{\rho - \beta} t \right) \right] \quad (9.20)$$

Where n_0 is the initial neutron population before the reactivity insertion. One can easily notice that at $t = 0$ (which also means $\rho = 0$), the relation between the precursors and the neutron population is:

$$c_0 = \frac{\beta}{\Lambda\lambda} n_0 \cong 1600n_0 \quad (9.21)$$

The concentration of the precursors is much larger than the neutron population and hence this large neutron source controls the dynamics of the neutron population.

9.2.1 Step negative reactivity, $\rho < 0$

Let us consider a case, in which a rapid insertion of control rod bank introduces $\rho = -0.05$. Using representative light water reactor parameters:

$$\beta = 0.0075$$

$$\lambda = 0.08 s^{-1}$$

$$\Lambda = 6 \times 10^{-5}$$

Eqs. 9.19-9.20 become:

$$\begin{aligned} n(t) &= n_0 [0.87 \exp(-958t) + 0.13 \exp(-0.068t)] \\ c(t) &= n_0 [0.0113 \exp(-958t) + 1563 \exp(-0.068t)] \end{aligned} \quad (9.22)$$

This solution is illustrated in Fig. 9.1

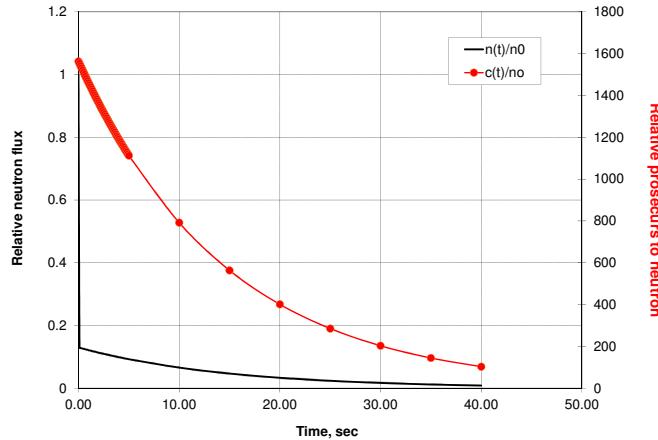


Figure 9.1: Relative neutron population $n(t)$ and delayed neutron precursor $c(t)$ following negative reactivity insertion.

Several observations:

1. Due to the very large negative exponent term (i.e. $\exp(-958t)$), 87% of the initial neutron population diminished within 0.01 sec (black curve in Fig. 9.1).
2. In this period of 0.01 sec, the population of precursors is close to c_0 . Thereafter, $c(t)$ decay with a period of ~ 15 sec ($1/\lambda = 1/0.068$).

Generally speaking, the sharp decay of the first term happens due to the new sub-critical condition of the core.

3. Assuming 100 neutrons exist in a steady state conditions. Following a reactivity insertion of -0.05, the number of neutrons appearing in the following generation would be 95. Since the prompt generation disappears within 6×10^{-5} sec, the neutron population decays with $(0.95)^{\frac{10^5}{6}}$ in one second.
4. The fact that neutrons still exist and (i.e. 13% of n_0) is due to the large population of precursor.

9.2.2 Sub-prompt critical step positive reactivity insertion, $0 < \rho < \beta$

Lets consider a positive reactivity insertion $\rho = 0.0015 < \beta$ (e.g. withdrawal of a control rod).

$$\begin{aligned} n(t) &= n_0 [-0.25 \exp(-100t) + 1.25 \exp(+0.02t)] \\ c(t) &= n_0 [0.3215 \exp(-100t) + 1562.5 \exp(+0.02t)] \end{aligned} \quad (9.23)$$

This solution is illustrated in Fig. 9.2

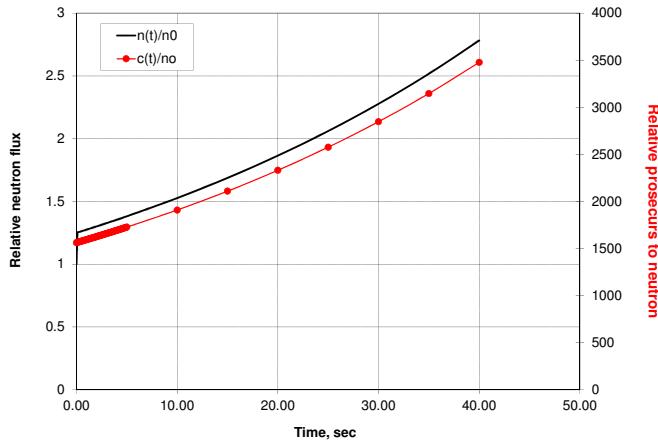


Figure 9.2: Relative neutron population $n(t)$ and delayed neutron precursor $c(t)$ following sub-prompt positive reactivity insertion.

1. $n(t)$ behaves as a step function and $c(t)$ experiences almost no change. The first term decays rapidly and then both terms (neutron population and precursors) increase with a stable period of 50 sec ($1/s=1/0.02$).
2. The sharp change in $n(t)$ is known as the "prompt jump" and it happens due to the super-critical conditions. This is actually the start for unstable rise in flux, however since the reactivity insertion is 0.0015 and 0.0075 fraction of the neutrons are originated from the precursor and are delayed, this instability can not proceed.

9.2.3 Super-prompt critical step positive reactivity insertion, $\rho > \beta$

Lets consider a positive reactivity insertion $\rho = 0.015 > \beta$ that could happen from ejection of a bank of control rods.

$$\begin{aligned} n(t) &= n_0 [2.9 \exp(+66.7t) - 1.90 \exp(-0.23t)] \\ c(t) &= n_0 [5.4 \exp(+66.7t) + 1563 \exp(-0.23t)] \end{aligned} \quad (9.24)$$

This solution is illustrated in Fig. 9.3

If $\rho > \beta$ and the core is super-prompt critical the consequences may be lethal. The term $e^{66.7}$ increases by a factor of 770 in tenth of a second and it will be impossible to take any actions to prevent the core reaching safety limits. Fortunately, there are inherent feedback mechanisms that introduce an immediate negative reactivity following the increase in neutron flux.

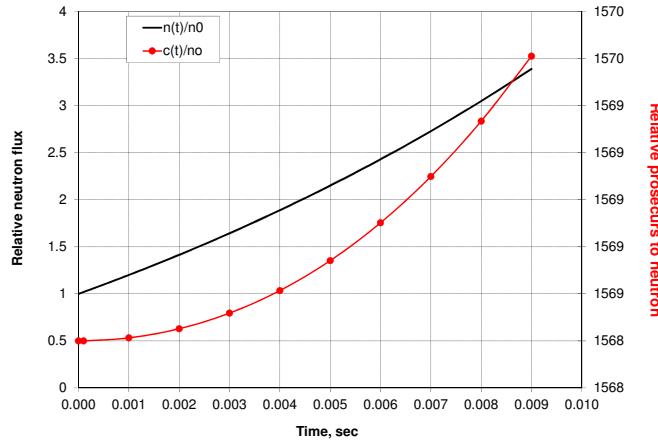


Figure 9.3: Relative neutron population $n(t)$ and delayed neutron precursor $c(t)$ following sub-prompt positive reactivity insertion.

Lecture 10: Time-Dependent Reactivity

Lecturer: Dr. Dan Kotlyar

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10.1 Numerical Solutions of Point Kinetics Equations

In previous lectures, it was assumed that the reactivity is a step function of time (i.e. constant with time). In other words, reactivity is power independent, which is not correct in the general case. The derivations that were applied in previous lectures assumed constant ρ_0 and the focus of this lecture is to analyze the time-dependent reactivity. In order to solve the point kinetic equations we will use numerical methods.

The reactivity term appears in the point kinetic equations and dictates the temporal behavior of the flux/power.

Let us examine the point kinetic equations:

$$\begin{aligned}\frac{dn(t)}{dt} &= \frac{\rho(t) - \beta}{\Lambda} n(t) + \sum_{i=1}^6 \lambda_i c_i(t) \\ \frac{dc_i(t)}{dt} &= \frac{\beta_i}{\Lambda} n(t) - \lambda_i c_i(t) \quad i = 1, \dots, 6\end{aligned}\tag{10.1}$$

We will modify Eq. 10.1 to an equivalent matrix form as described below:

$$\frac{d}{dt} \begin{bmatrix} P(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \\ C_5(t) \\ C_6(t) \end{bmatrix} = \begin{bmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ \frac{\beta_3}{\Lambda} & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ \frac{\beta_4}{\Lambda} & 0 & 0 & 0 & -\lambda_4 & 0 & 0 \\ \frac{\beta_5}{\Lambda} & 0 & 0 & 0 & 0 & -\lambda_5 & 0 \\ \frac{\beta_6}{\Lambda} & 0 & 0 & 0 & 0 & 0 & -\lambda_6 \end{bmatrix} \begin{bmatrix} P(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \\ C_5(t) \\ C_6(t) \end{bmatrix}$$

The analytic solution of the above system is:

$$X_{n+1} = \exp \left(\begin{bmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ \frac{\beta_3}{\Lambda} & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ \frac{\beta_4}{\Lambda} & 0 & 0 & 0 & -\lambda_4 & 0 & 0 \\ \frac{\beta_5}{\Lambda} & 0 & 0 & 0 & 0 & -\lambda_5 & 0 \\ \frac{\beta_6}{\Lambda} & 0 & 0 & 0 & 0 & 0 & -\lambda_6 \end{bmatrix} \Delta t_n \right) X_n$$

where

$$X = \begin{bmatrix} P \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix}$$

and at $t=0$ $X_0 = [P(0) \ C_1(0) \ C_2(0) \ C_3(0) \ C_4(0) \ C_5(0) \ C_6(0)]^T$

The procedure starts at the beginning-of-step where the initial power and concentrations are known (i.e. values at steady-state). The time-step can be divided into sub-steps $\Delta t_n = t_{n+1} - t_n$ and the above equations can be solved sequentially.

10.1.1 Step-function reactivity insertion

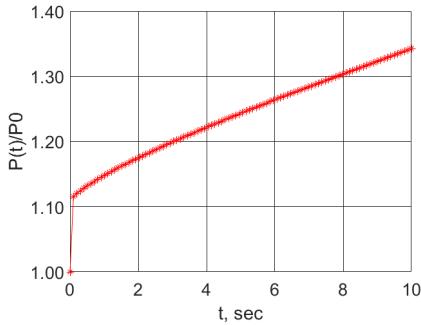
Lets consider a positive and constant reactivity insertion $\rho = 0.10\beta$. Such insertion is possible when the control rod is withdrawn instantaneously from the core. Assuming that all the kinetic parameters are provided, write a Matlab script and obtain the temporal behavior of the power and the precursor concentrations.

Consider a PWR core with $\beta = 0.0075$ and $\Lambda = 6 \times 10^{-5}$

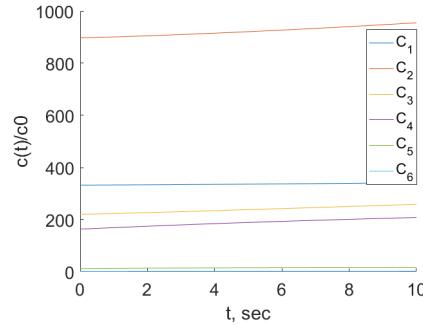
Table 10.1: Delayed neutrons U^{235} from thermal fission.

Group No.	Decay constant, λ_i, sec^{-1}	β_i/β
1	0.0124	0.033
2	0.0305	0.219
3	0.1110	0.196
4	0.3010	0.395
5	1.1400	0.115
6	3.0100	0.042

Use the 'expm' built-in Matlab function to plot the results as shown in Figures 10.1.



(a) Relative flux/power.



(b) Precursors.

Figure 10.1: Temporal behavior of $n(t)$ and $c_i(t)$ for positive reactivity insertion $\rho = 0.1\beta$.

10.1.2 One-group representation

In our previous lecture, we have simplified and introduced the point kinetic equations in a 1-group delayed neutrons. In other words, we have represented all the delayed neutrons as if they had a single mean lifetime. In this subsection we discuss the consequences of such assumption by investigating the inhour equation yet again.

$$\rho_0 = s \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{s + \lambda_i} \right) \quad (10.2)$$

For very small reactivities, in which $s \ll \lambda_i$, eq. 10.2 reduces to:

$$\rho \cong s \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{\lambda_i} \right) \quad (10.3)$$

where the one-group representation is:

$$\rho = s \left(\Lambda + \frac{\beta}{\lambda} \right) \quad (10.4)$$

which indicates that the 1-group λ must satisfy:

$$\frac{1}{\lambda} = \frac{1}{\beta} \sum_{i=1}^6 \frac{\beta_i}{\lambda_i} \quad (10.5)$$

For thermal fission in U²³⁵ the average decay time is $\lambda = 0.0767 \text{ sec}^{-1}$.

We can also examine the near prompt critical state, in which $s \gg \lambda_i$. One can show that the inhour equation is reduced to:

$$\rho_0 \cong \beta + \Lambda s - \sum_{i=1}^6 \frac{\beta_i \lambda_i}{s} \quad (10.6)$$

and in 1-group

$$\rho_0 \cong \beta + \Lambda s - \frac{\beta \lambda'}{s} \quad (10.7)$$

which indicates that the 1-group λ must satisfy:

$$\lambda' = \frac{1}{\beta} \sum_{i=1}^6 \beta_i \lambda_i \quad (10.8)$$

For thermal fission in U^{235} the average decay time is $\lambda' = 0.405 \text{ sec}^{-1}$.

The main issue arises for intermediate positive reactivities. The 1-group flux has one increasing exponential term and one term that diminishes at once. The absence of the intermediate branches of the inhour curve, which are more slowly decay with time, means the one-group model underestimates the time period until the asymptotic term is dominant.

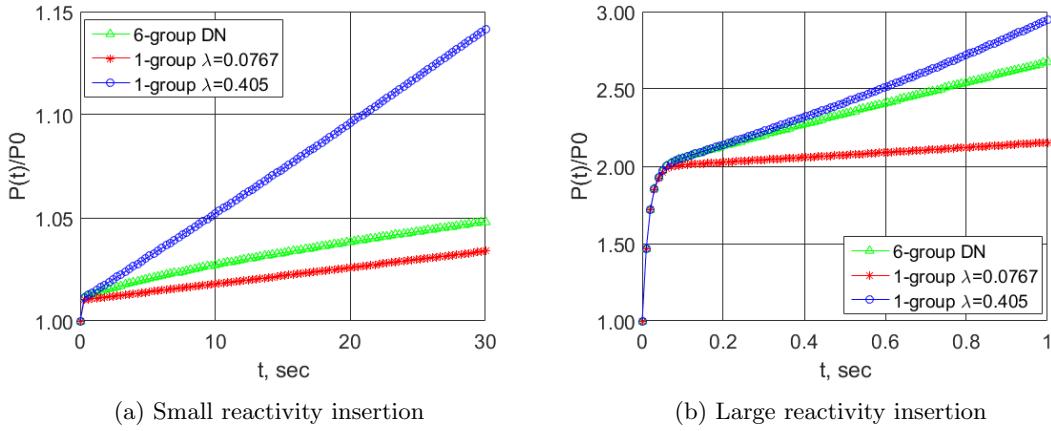


Figure 10.2: Comparison of the flux using 1- and 6-group calculations.

10.1.3 Ramp (linear) reactivity insertion

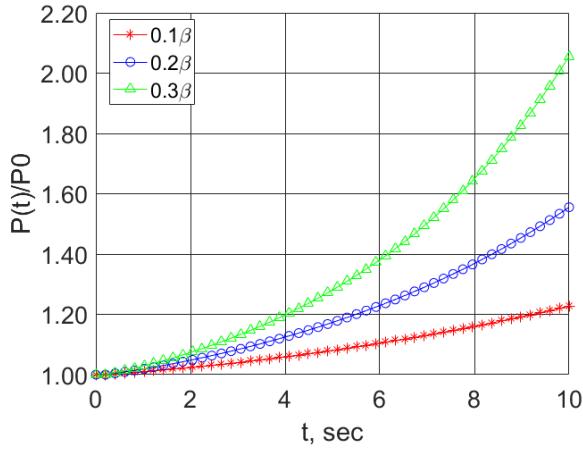
The linear/ramp reactivity change takes place when control rods are moving with constant velocity. The reactivity as a function is described according to the following function:

$$\rho(t) = \rho_0 t \quad (10.9)$$

Fig. 10.3 shows the behavior of the flux following a ramp reactivity insertion. A fitting of the curves will show that the flux can be approximated by the following function:

$$n(t) = n_0 e^{at^2} \quad (10.10)$$

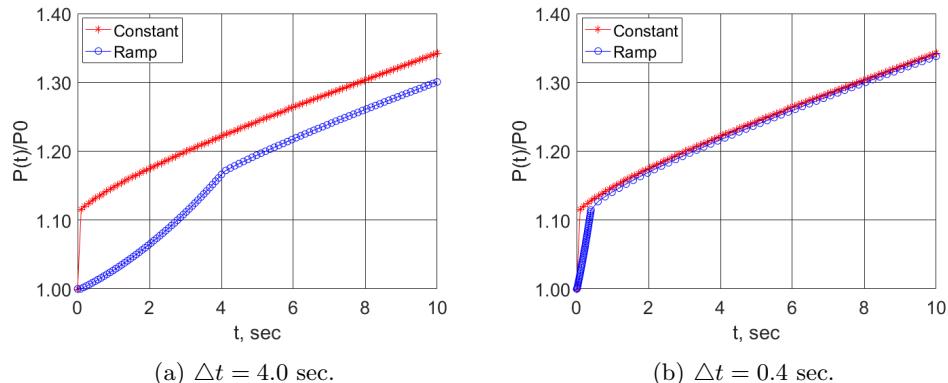
where a is a proper constant. In the step-function reactivity change, the power change is proportional to e^t , however, when the reactivity linearly increases, the power increases as e^{t^2} .

Figure 10.3: Temporal behavior following different ρ ramp reactivity insertions.**Ramp vs. step reactivity insertion:**

You are required to write a program that computes the change in the relative power following a step vs. ramp reactivity change during a transient period of 10 seconds.

1. **Step reactivity:** 0.1β is inserted at $t=0$.
2. **Ramp reactivity:** 0.1β is linearly inserted during 4.0 sec, i.e. only at $t = 4$ the reactivity reaches 0.1β . Afterwards the reactivity is remained constant.
3. **Ramp reactivity:** 0.1β is inserted 10 times faster, i.e. during 0.4 sec.

Fig. 10.4 shows the flux behavior following the step vs. ramp reactivity insertion. It can be seen that step change of reactivity is a good approximation for the case with the ramp insertion of reactivity only if the insertion rate is very fast. These two solutions are almost identical (Fig. 10.4b). However, when the insertion is slow, the two solutions will differ and the step-change solution predicts higher power values during the first seconds of the reactivity initiated transients.

Figure 10.4: Temporal behavior of $n(t)$ for ramp ($\rho = \frac{0.1\beta}{\Delta t}t, t < \Delta t$) and step ($\rho = 0.1\beta$) reactivity insertion.

Lecture 11: Reactivity feedback

Lecturer: Dan Kotlyar

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11.1 Numerical solution of the point-kinetic equations

Up until this point we assumed either constant or time-dependent reactivity insertion, $\rho(t)$. The reactivity term appears in the point kinetic equations and dictates the temporal behavior of the flux/power. The procedure starts at the beginning-of-step where the initial power and concentrations are known (i.e. values at steady-state). The time-step can be divided into sub-steps $\Delta t_n = t_{n+1} - t_n$ and the above equations can be solved sequentially.

$$\begin{bmatrix} P(N+1) \\ C_1(N+1) \\ C_2(N+1) \\ C_3(N+1) \\ C_4(N+1) \\ C_5(N+1) \\ C_6(N+1) \end{bmatrix} = \exp \left(\begin{bmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ \frac{\beta_3}{\Lambda} & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ \frac{\beta_4}{\Lambda} & 0 & 0 & 0 & -\lambda_4 & 0 & 0 \\ \frac{\beta_5}{\Lambda} & 0 & 0 & 0 & 0 & -\lambda_5 & 0 \\ \frac{\beta_6}{\Lambda} & 0 & 0 & 0 & 0 & 0 & -\lambda_6 \end{bmatrix} \Delta t_n \right) \begin{bmatrix} P(N) \\ C_1(N) \\ C_2(N) \\ C_3(N) \\ C_4(N) \\ C_5(N) \\ C_6(N) \end{bmatrix}$$

11.1.1 Sinusoidal reactivity change

The sinusoidal change of reactivity can be used to determine kinetic parameters by evaluating the transfer function. If the power level is very low, such that no reactivity feedback is present, we can show that by applying the Laplace transform on the PK equations, the following transfer function is obtained:

$$G(s) = \frac{\delta \tilde{N}(s)}{\delta \rho(s)} = \frac{1}{s \left(\Lambda + \sum_{i=1}^6 \frac{\beta_i}{s+\lambda_i} \right)} \quad (11.1)$$

where $\delta \rho(s)$ is the Laplace transform of small reactivity perturbation $\delta \rho(t)$ and $\delta \tilde{N}(s)$ is the Laplace transform of the corresponding relative power response $\delta n(t)/n_0$. For very small reactivity changes the transfer function becomes:

$$G(s) \cong \frac{1}{\sum_{i=1}^6 \frac{\beta_i s}{s+\lambda_i}} \quad (11.2)$$

The transfer can be measured over a range of different frequencies s . Then, using fitting procedures, the kinetic parameters, e.g. β , can be determined.

In the case of a sinusoidal reactivity change:

$$\rho(t) = \rho_0 \sin(wt) \quad (11.3)$$

where ρ_0 and w are the amplitude and the frequency of the reactivity oscillations. Such reactivity change may be induced by using an oscillator piston that is capable of moving periodically between two positions located inside and outside the core.

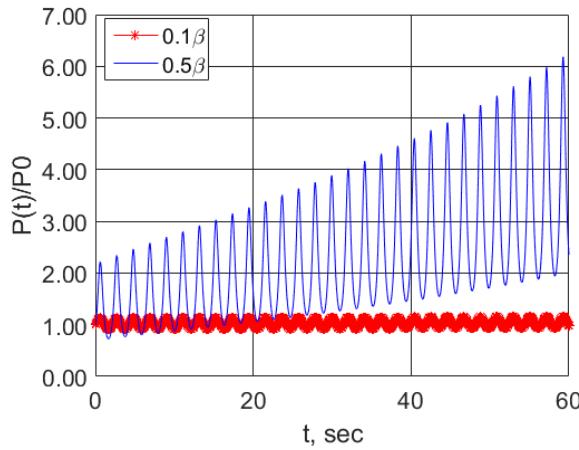


Figure 11.1: Flux/power oscillations following sinusoidal reactivity with a $w = 3 \text{ sec}^{-1}$ frequency.

11.2 Prompt-Jump approximation

In previous section we saw that following positive reactivity insertion ($0 < \rho < \beta$) the neutron population changes sharply on the generation time scale. If we are not interested in the details of the prompt neutron kinetics during the prompt jump, we can simplify the equations by assuming that the prompt jump happens instantaneously in response to any reactivity change.

This is called *prompt jump approximation* and is taken into a consideration by neglecting the time derivative dn/dt in the point reactor kinetics equation:

$$0 = (\rho(t) - \beta) n(t) + \Lambda \sum_{i=1}^6 \lambda_i c_i(t) \quad (11.4)$$

The delayed neutron precursor concentration does not respond immediately to a change in reactivity (i.e. slow behavior of $c(t)$), eq. 11.4 is valid with the same $c(t)$ before and after the reactivity insertion:

$$\begin{aligned} 0 &= (\rho_0 - \beta) n_0 + \Lambda \lambda c(t) && \text{Before} \\ 0 &= (\rho_1 - \beta) n_1 + \Lambda \lambda c(t) && \text{After} \end{aligned} \quad (11.5)$$

This model predicts that a reactivity jump from ρ_0 to ρ_1 causes an instantaneous change in reactor power from n_0 to n_1 .

$$\frac{n_1}{n_0} = \frac{\beta - \rho_0}{\beta - \rho_1} \quad (11.6)$$

The prompt jump approximation is frequently used in numerical studies of the point reactor kinetics equations since it eliminates the very short time scale due to Λ . Use of Eq. 11.4 to eliminate $n(t)$ in the equation that describes the rate of change in the precursor concentration, which in one delayed precursor group approximation takes on the simple form:

$$\frac{dc(t)}{dt} = \frac{-\lambda c(t)}{1 - \beta/\rho(t)} \quad (11.7)$$

Using the one group approximation after solving for $c(t)$ and then substitute the solution into the $dn(t)/dt$ expression to obtain:

$$[\rho(t) - \beta] \frac{dn(t)}{dt} + \left[\frac{d\rho(t)}{dt} + \lambda\rho(t) \right] n(t) = 0 \quad (11.8)$$

For a given reactivity $\rho(t)$, the neutron population can be found according to:

$$n(t) = n_0 \exp \left[\int_0^t \frac{\dot{\rho}(t) + \lambda\rho(t)}{\beta - \rho(t)} dt \right] \quad (11.9)$$

11.3 Description of the feedback

The reactivity in reality depends on the power/flux. This is an indirect dependence since the reactivity depends on the macroscopic cross sections (i.e. nuclide densities and microscopic cross sections). Nuclide densities and cross-sections depend on the temperature, which depends on power.

Let's not forget that cross-sections are not only space and time dependent but also energy dependent. Neutron energy spectrum will also depend on the temperature distribution and hence power.

Variation in temperature is the basic mechanism determining the inherent stability of a nuclear reactor with respect to variations in power.

In the current lecture we will adopt a simplified approach to account for the dynamic nature of the system. To avoid the complexity of a full spatial solution, we will replace it with a lumped (averaged) model. The core is usually characterized by several average temperatures:

1. Average fuel temperature
2. Average moderator temperature
3. Average coolant temperature

This involves solving a heat conduction-convection problem. In this lecture we will focus on studying how the results of the average temperatures (T_F and T_M) can be used to describe reactivity feedback.

We will describe the reactivity $\rho(t)$ as a sum of two contributors:

$$\rho(t) = \delta\rho_{ext}(t) + \delta\rho_f(P) \quad (11.10)$$

$\delta\rho$ is a measure from equilibrium (from $\rho = 0$). $\delta\rho_{ext}(t)$ is the external reactivity insertion (e.g. control rod adjustment) and $\delta\rho_f(P)$ denotes the change in reactivity corresponding to inherent feedback, which depends on the power level.

For a steady-state, in which the reactor operates at P_0 :

$$0 = \rho_0 + \rho_f(P_0) \quad (11.11)$$

Eq. 11.11 mandates that sustaining a critical core, one should provide external reactivity to compensate the negative feedback $\rho_f(P_0)$.

We define our incremental reactivities as:

$$\begin{aligned} \delta\rho_{ext}(t) &= \rho_{ext}(t) - \rho_0 \\ \delta\rho_f(P) &= \rho_f(P) - \rho_f(P_0) \end{aligned} \quad (11.12)$$

It is also useful to use the incremental power definition:

$$p(t) = P(t) - P_0 \quad (11.13)$$

The feedback mechanism in reactivity is presented in a closed-loop block diagram as shown in Fig.

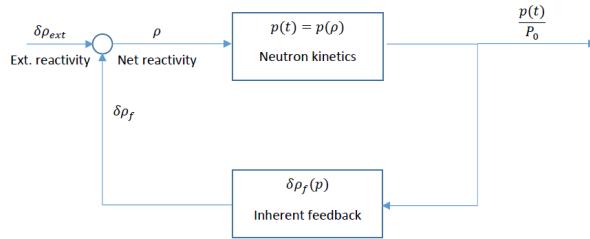


Figure 11.2: Closed-loop block mechanism with feedback.

Fig. 11.2 shows that the increment in power affects the reactivity, which again changes the power.

Such a model explicitly presents the effect of temperature feedback on reactor stability. For a design with a positive α_T , any increase in temperature would produce a further increase in reactivity, which will increase the power even further. The temperature will then rise even more and it will continue to diverge.

Therefore, one should design a system with negative temperature reactivity α_T coefficients.

The following figure presents the criticality as a function of the moderator-to-fuel ratio. In other words, the question we need to ask ourselves is how far should we place the fuel rods from each other. This is not a straightforward question, since we also need to include heat transfer analysis in order to properly address this question. However, let us now ask a simple question: should the spacing between the rods be such that the quantity V_m/V_f is from the right of the peak or from the left?

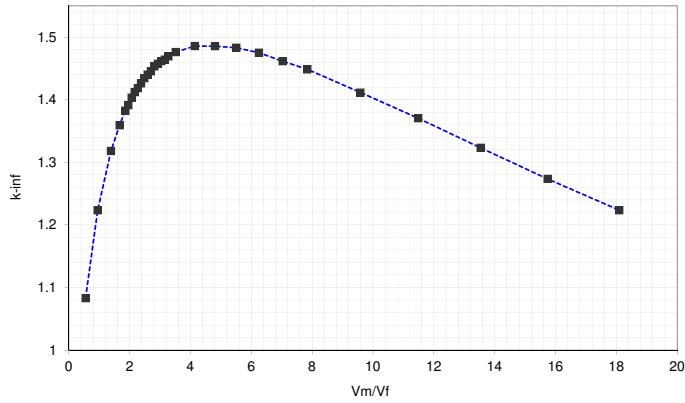


Figure 11.3: Criticality as a function of the moderator-to-fuel ratio.

11.4 Temperature feedback

Reactivity strongly depends on temperature and has a crucial effect on the dynamics and stability of a nuclear reactor. The reactivity on various parameters should be known and can be linearly estimated by applying the linear form:

$$\Delta\rho(T_F, T_M, \dots) \cong \frac{\partial\rho}{\partial T_F} \Delta T_F + \frac{\partial\rho}{\partial T_M} \Delta T_M + \dots \quad (11.14)$$

Temperature reactivity coefficients α_T are defined as:

$$\alpha_T = \frac{\partial\rho}{\partial T} \quad (11.15)$$

The temperature reactivity coefficient is defined as:

$$\alpha_T = \frac{\partial\rho}{\partial T} = \frac{\partial}{\partial T} \left(\frac{k-1}{k} \right) = \frac{1}{k^2} \frac{\partial k}{\partial T} \cong \frac{1}{k} \frac{\partial k}{\partial T} \quad (11.16)$$

In order to gain some insight on the physical behavior of the reactivity coefficients, we will use the 5-factor formula

$$k = k_\infty P_{NL} = \eta f \epsilon p P_{NL} \quad (11.17)$$

Differentiating eq.11.17 results in:

$$\alpha_T = \frac{1}{\eta} \frac{\partial \eta}{\partial T} + \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial T} + \frac{1}{f} \frac{\partial f}{\partial T} + \frac{1}{p} \frac{\partial p}{\partial T} + \frac{1}{P_{NL}} \frac{\partial P_{NL}}{\partial T} \quad (11.18)$$

For each effect (e.g. change in fuel temperature, moderator temperature) the reactivity coefficient will be different. The most dominant temperature effects in most reactors are the change in absorption following the fuel temperature increase, which leads to resonance broadening (i.e. Doppler effect) and the change in the neutron spectrum following the moderator temperature increase (density decrease), which leads to lower moderation power.

It must be pointed out that the change in the fuel temperature will merely affect the neutron spectrum. However, the change in moderator density will influence on the slowing down process and hence change the resonance absorption. Therefore, there is a direct relation between the coefficients and yet it is accustomed to separate these influences and analyze these coefficients separately.

11.4.1 Doppler effect

In case of a positive reactivity insertion, the power will increase and hence the fuel temperature will rise as well. The negative fuel coefficient introduced by the Doppler broadening (Fig. 11.4) will start adding negative reactivity immediately and hence considered to be more important than the moderator coefficient, which kicks in later on. The coefficient is known as the *Doppler reactivity Coefficient- DC*.

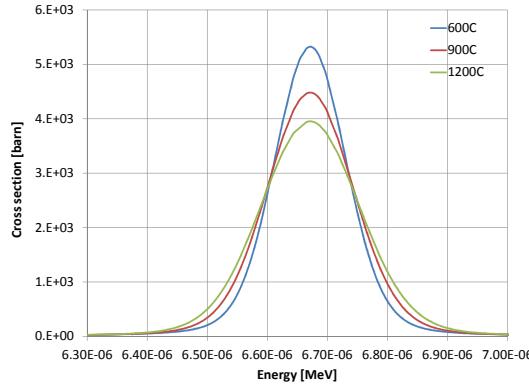


Figure 11.4: Resonance broadening with temperature.

Its value can be estimated from:

$$\begin{aligned}\alpha_T^F &= \frac{\partial \rho}{\partial T_F} = \frac{1}{k^2} \frac{\partial k}{\partial T_F} \cong \frac{1}{k} \frac{\partial k}{\partial T_F} \\ &= \ln(p) \frac{1}{I} \frac{\partial I}{\partial T_F}\end{aligned}\tag{11.19}$$

The relation between the resonance escape probability, p , and the resonance integral, I , is given by the following expression:

$$p = \exp\left(-\frac{N_F I}{\xi \Sigma_s}\right)\tag{11.20}$$

The effective resonance integral I for UO_2 at 300K is given by the empirical expression:

$$I_{\text{UO}_2}(300K) = 4.45 + 84.5 \sqrt{\frac{A}{M}}, \text{ barns}\tag{11.21}$$

The temperature dependence of the resonance integral is given by the following experimental correlation:

$$I_{\text{UO}_2}(T) = I_{\text{UO}_2}(300K) \left[1 + \beta \left(\sqrt{T} - \sqrt{300} \right) \right]\tag{11.22}$$

where the expansion coefficient β for UO_2 equals to:

$$\beta = 6.1 \times 10^{-3} + 4.7 \times 10^{-3} \frac{A}{M}\tag{11.23}$$

where A is the surface area (cm^2) and M is the mass (kg) of the fuel.

Using the above equations yields the following expression for DC :

$$\alpha_T^F = -\ln\left(\frac{1}{p(300K)}\right) \frac{\beta}{2\sqrt{T_F}}\tag{11.24}$$

Reactivity coefficients are usually measured through $\frac{pcm}{K}$, where pcm stands for percent-milli.

$$\begin{aligned}1 \triangle k/k &= 100\% \triangle k/k = 10^5 pcm \\ 1pcm &= 10^{-5} \triangle k/k\end{aligned}\tag{11.25}$$

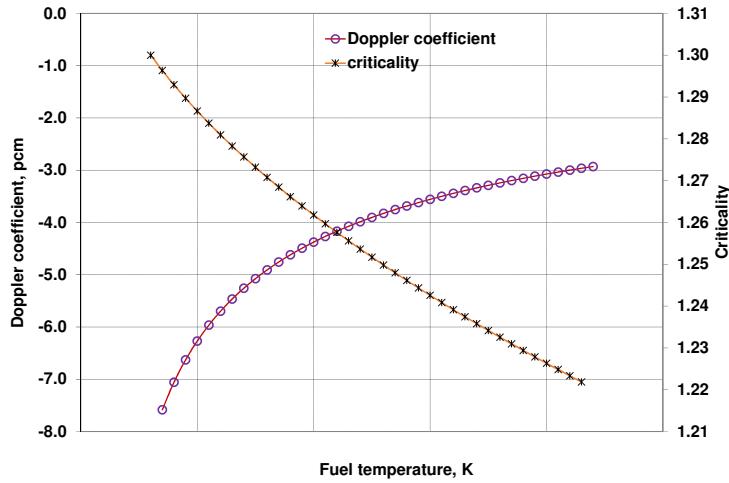


Figure 11.5: Doppler coefficient as a function of temperature.

11.4.2 Moderator effect- MTC

In most LWRs, the coolant and moderator are the same, in such a case any change in the coolant temperature and hence density will affect the moderation power.

The most dominant reactivity effect in a reactor moderated by water is the change in the moderator density, due to the thermal expansion of the fluid or due to boiling (BWRs). The moderation power is decreased due to the decrease in moderator's density and causes corresponding increase in resonance absorption. The reactivity coefficient can be estimated from the following expression:

$$\alpha_T^M = \frac{1}{k} \frac{\partial k}{\partial T_M} = \frac{1}{p} \frac{\partial}{\partial T_M} \left[\exp \left(-\frac{N_F I}{\xi N_M \sigma_s} \right) \right] = \ln \left(\frac{1}{p} \right) \frac{\partial N_M}{\partial T_M} \quad (11.26)$$

For PWRs:

1. DC \sim -4 to -1 pcm/K
2. MTC \sim -70 to -10 pcm/K

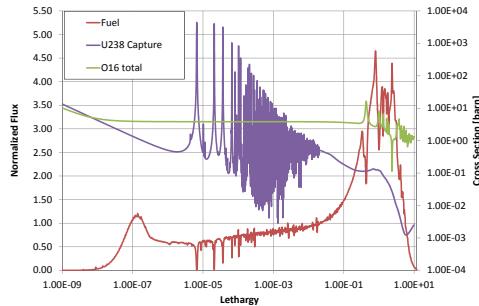


Figure 11.6: Neutron spectrum.

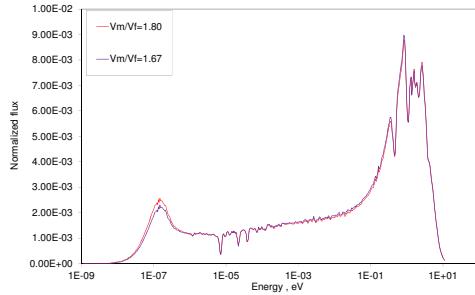


Figure 11.7: Change in the neutron spectrum due to the change in moderator temperature.

11.4.3 Soluble boron reactivity control (chemical shim)

This is a classical long-term reactivity control in PWRs needed to compensate for the change in reactivity due to fuel depletion. Coolant chemistry and volume control system adjusts the level of soluble boron with time as needed (typically diluting borated water and reducing the boron concentration with time). Boron is introduced into water coolant in the form of H_3BO_3 .

1. **The advantage** of this method over control rods is that the core space, extremely valuable for power production, is not taken by control mechanisms. It also introduces the neutron poison uniformly through the core without causing major changes in power distribution.
2. **The disadvantage**, mostly of concern at the beginning of fuel cycle, is that the high boron concentration may result in a positive moderator temperature reactivity coefficient.

The soluble boron system cannot eliminate the need for control rods or blades: changing the boron content in the water is not an immediate process; therefore the method is not suitable for the quick control response when required.

The usual measure of soluble boron concentration is the ppm (parts per million): $\text{ppm} \equiv \frac{\text{grams of natural boron}}{10^6 \text{ grams of H}_2\text{O}}$

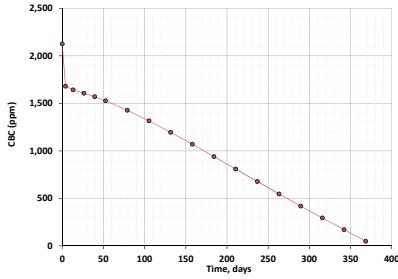
Exercise: Natural boron contains 19.8% B^{10} , absorbing neutrons via (n,α) reaction, and 80.2% B^{11} , is the non-absorbing isotope.

Define:

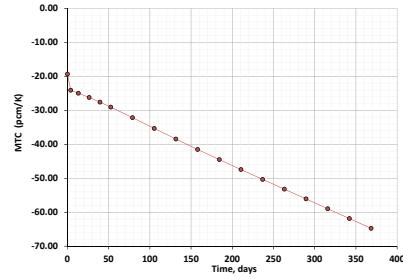
$$\begin{aligned} N(\text{B}^{10}) &= \frac{\text{atoms} - \text{B}^{10}}{\text{cm} \cdot \text{b}} \\ G(\text{H}_2\text{O}) &= \frac{\text{grams} - \text{H}_2\text{O}}{\text{cm}^3} \end{aligned} \quad (11.27)$$

Show that:

$$N(\text{B}^{10}) = 1.104 \times 10^{-8} \times G(\text{H}_2\text{O}) \times \text{ppm} \quad (11.28)$$



(a) Boron letdown curve.



(b) MTC coefficient.

Figure 11.8: Cycle depletion.

Lecture 12: Reactor Dynamics with Feedback

Lecturer: Dr. Dan Kotlyar

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12.1 Point Kinetics with Simple Feedback

The simple model should take into account the effect of the fuel and the moderator temperature on the reactivity. Here, the fluid will be assumed to be in its single phase flow (PWR). The simple model should also take into consideration the time variations of the fuel and coolant temperatures.

In this section, we'll use the point kinetic equations presented in Eq. 10.1.

We will also introduce additional set of equations. The first equation will express the time-dependent reactivity:

$$\rho(t) = \delta\rho_{ext}(t) + \delta\rho_f(u_1, u_2, \dots) \quad (12.1)$$

Where $\delta\rho_{ext}(t)$ is the external reactivity change due to the control system (e.g. control rods, Boron in the water) from ρ_0 , which is zero in steady-state. $\delta\rho_f$ is the reactivity change due to inherent feedback mechanisms.

The feedback mechanisms (temperature reactivity coefficients) do not change too much in respect to the steady-state equilibrium values and hence the temperature reactivity coefficients will be assumed constant here, although these are temperature-dependent in reality. Hence, Eq. 12.1 will be re-written to the following form:

$$\rho(t) = \delta\rho_{ext}(t) + \alpha_T^M (T_M - T_{M,0}) + \alpha_T^F (T_F - T_{F,0}) \quad (12.2)$$

Applying the Newton's relation that describes the heat transfer between coolant and fuel, the heat balance for fuel can be written as:

$$m_F c_F \frac{dT_F}{dt} = P(t) - h(T_F - T_M) \quad (12.3)$$

where, m_F is the mass of the fuel, c_F is the specific heat capacity of the fuel and h is the heat transfer coefficient between fuel and coolant. It can be seen that under steady-state conditions the left-hand side of Eq. 12.3 is zero and we obtain:

$$P(t) = h(T_F - T_M) \quad (12.4)$$

In steady state, the power transferred from fuel to coolant and that carried away by the coolant are equal:

$$P(t) = h(T_F - T_M) = W_M c_M (T_{out} - T_{in}) \quad (12.5)$$

where W_M is the total moderator/coolant mass flow rate (kg/s) through the core, c_M is the specific heat capacity of moderator/coolant. T_{in} and T_{out} are the inlet and outlet temperature respectively. A good approximation for T_M is:

$$T_M = \frac{T_{in} + T_{out}}{2} \quad (12.6)$$

We'll now assume that the moderator (coolant) temperature is not constant. If power transferred from fuel to coolant is greater than that carried away by the coolant, then the difference must be the rate at which the internal energy of the coolant within the core increases:

$$m_M c_M \frac{dT_M}{dt} = h(T_F - T_M) - 2W_M c_M (T_M - T_{in}) \quad (12.7)$$

12.1.1 Initial conditions

In order to solve these dynamic equations it is required to know the initial conditions. These conditions can be found from the steady-state, i.e. equilibrium, for which the time-derivatives are equal to zero $d/dt = 0$. It must be pointed out that we are not searching for the trivial solution (i.e. shut-down reactor, $n=0$), but rather for the solution in which the reactivity is zero but $n \neq 0$.

$$\begin{aligned} C_{i,0} &= \frac{\beta_i}{\lambda_i \Lambda} n_0 \\ T_{F,0} &= T_{in} + \left(\frac{1}{2W_M c_M} + \frac{1}{h} \right) P_0 \\ T_{M,0} &= T_{in} + \frac{P_0}{2W_M c_M} \end{aligned} \quad (12.8)$$

12.2 Derivation of the feedback equations

We will now re-write Eqs. 12.3 and 12.7 in the following manner:

$$\begin{aligned}\frac{d}{dt}T_F(t) &= \frac{1}{M_F c_F} P(t) - \frac{h}{M_F c_F} [T_F(t) - T_M(t)] \\ \frac{d}{dt}T_M(t) &= \frac{h}{M_M c_M} [T_F(t) - T_M(t)] - \frac{2W_M}{M_M} [T_M(t) - T_{in}(t)]\end{aligned}\tag{12.9}$$

In a matrix notation:

$$\frac{d}{dt} \begin{bmatrix} T_F(t) \\ T_M(t) \end{bmatrix} + \begin{bmatrix} b & -b \\ -c & c+d \end{bmatrix} \begin{bmatrix} T_F(t) \\ T_M(t) \end{bmatrix} = \begin{bmatrix} aP(t) \\ dT_{in} \end{bmatrix}$$

The matrix A is as follows:

$$A = \begin{bmatrix} b & -b \\ -c & c+d \end{bmatrix}$$

Using integrating factors we can solve the ODE system:

$$\begin{bmatrix} T_F \\ T_M \end{bmatrix}^{(n+1)} = e^{-A\Delta t} \begin{bmatrix} T_F \\ T_M \end{bmatrix}^{(n)} + A^{-1} [1 - e^{-A\Delta t}] \begin{bmatrix} aP^{(n+1)} \\ dT_{in} \end{bmatrix}$$

12.3 Operator split approach

We now have 7 equations that describe the neutronic part. The solutions of Eqs. 10.1 will give us the power. The power is then transferred to the lumped thermal-hydraulics (TH) model (Eq. 12.7 and 12.3) and then the reactivity is updated in Eq. 12.2. The order is important and may affect the results, however, when time-step is sufficiently small there is no need to have internal iterative procedure.

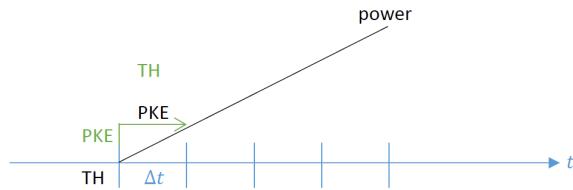


Figure 12.1: Operator splitting approach.

The sequential stages to solve this problem are:

1. Solve PKE and obtain $P^{(n+1)}$
2. Use $P^{(n+1)}$ to obtain $T_F^{(n+1)}$

3. Update the reactivity $\rho(t) = \delta\rho_{ext}(t) + \alpha_F (T_F(t) - T_{F,0}) + \alpha_M (T_M(t) - T_{M,0})$

The equations are synchronized in the following manner:

$$\begin{bmatrix} P \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix}^{(n+1)} = \exp \left(\begin{bmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\beta_2}{\Lambda} & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ \frac{\beta_3}{\Lambda} & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ \frac{\beta_4}{\Lambda} & 0 & 0 & 0 & -\lambda_4 & 0 & 0 \\ \frac{\beta_5}{\Lambda} & 0 & 0 & 0 & 0 & -\lambda_5 & 0 \\ \frac{\beta_6}{\Lambda} & 0 & 0 & 0 & 0 & 0 & -\lambda_6 \end{bmatrix} \Delta t \right) \begin{bmatrix} P \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix}^{(n)}$$

$$\begin{bmatrix} T_F \\ T_M \end{bmatrix}^{(n+1)} = e^{-A\Delta t} \begin{bmatrix} T_F \\ T_M \end{bmatrix}^{(n)} + A^{-1} [1 - e^{-A\Delta t}] \begin{bmatrix} aP^{(n+1)} \\ dT_{in} \end{bmatrix}$$

$$\rho(t) = \rho_C(t) + \alpha_F (T_F^{(n+1)} - T_{F,0}) + \alpha_M (T_M^{(n+1)} - T_{M,0})$$

12.4 Example: \$0.5 reactivity insertion

The students are requested to work in class and produce a program of the coupled solution. Plot the time behavior of the power, temperature of the fuel and moderator and the reactivity. Use the following data:

Table 12.1: Delayed neutrons U^{235} from thermal fission.

Group No.	Decay constant, λ_i, sec^{-1}	β_i/β
1	0.0124	0.033
2	0.0305	0.219
3	0.1110	0.196
4	0.3010	0.395
5	1.1400	0.115
6	3.0100	0.042

$\beta = 0.0075$ and $\Lambda = 0.001$. The power is $2500 \times 10^6 W$. In addition: $h = 4 \times 10^6 \frac{J}{sec K}$, $T_{in} = 550 K$, $c_F = 200 \frac{J}{kg K}$, $c_M = 4000 \frac{J}{kg K}$, $M_F = 40,000 kg$, $M_M = 7,000 kg$, $\alpha_F = -1 \times 10^{-5}$, $\alpha_M = -5 \times 10^{-5}$

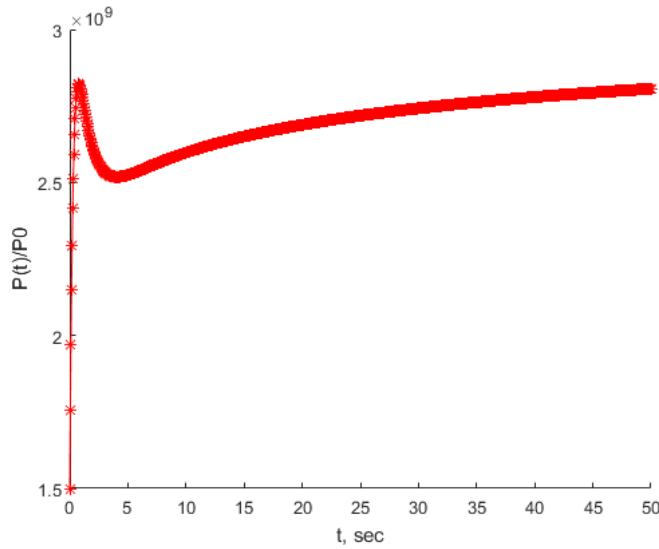


Figure 12.2: Power as a function of time.

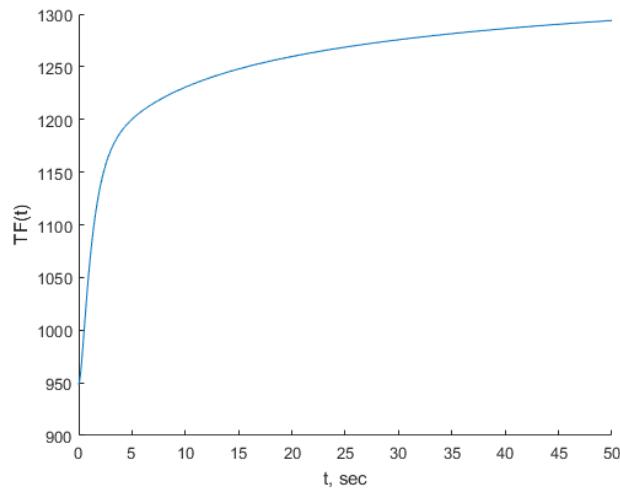


Figure 12.3: Fuel temperature as a function of time.

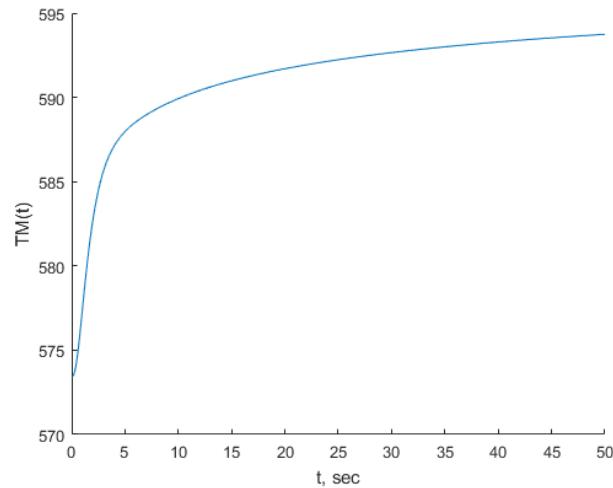


Figure 12.4: Moderator temperature as a function of time.

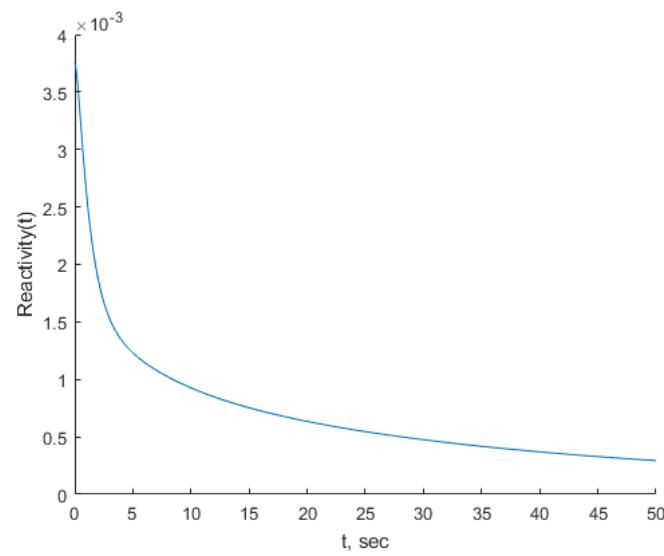


Figure 12.5: Reactivity as a function of time.

Lecture 13: Control Theory and Stability

Lecturer: Dr. Dan Kotlyar

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13.1 Control Theory

A fundamental requirement of a nuclear reactor design is its stability under all operating and accidental conditions.

In this lecture we will focus our study towards the stability of a nuclear system. The concept of stability will be analyzed through the use of the Laplace transform introducing concepts from classical control theory. These concepts will then be used and applied to the point-kinetics model to analyze the neutronic stability of the nuclear system.

The main objectives in the current lecture are:

1. Introducing basic concepts of control theory
2. Analyzing the concept of stability of a system and studying the Lyapunov stability criterion
3. Introducing the concept of feedback system
4. Applying control theory to the point-kinetic model

13.2 Laplace Transform

The Laplace transform is an integral transform that transforms a function of a positive variable t (time) to a function of complex variable s (frequency). The Laplace transform of a function $f(t)$ is a function $F(s)$ defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t)e^{-st}dt \quad (13.1)$$

If the Laplace transform $F(s)$ is rational, it can be expressed as:

$$F(s) = \frac{N(s)}{D(s)} \quad (13.2)$$

The roots of $N(s)$ are defined to be the zeros of the function while the roots of $D(s)$ are defined to be the poles of the function.

13.2.1 Laplace transform properties

The Laplace transform has the following properties:

1. Linearity:

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s) \quad (13.3)$$

2. First derivative

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+) \quad (13.4)$$

3. Integration

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s} \quad (13.5)$$

4. Initial value theorem

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) \quad (13.6)$$

5. Final value theorem

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) \quad (13.7)$$

6. Time delay

$$\mathcal{L}[f(t - \tau)] = e^{-s\tau}F(s) \quad (13.8)$$

13.2.2 Laplace transforms of common functions

Laplace transforms of common functions can be calculated through the definition expressed by Eq. 13.1.

$f(t)$	$F(s)$
C	$\frac{C}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$

13.2.3 Examples

Example 1

Given $f_1(t) = 2e^{-4t} + 4e^{2t}$, calculate $\mathcal{L}[f_1(t)]$, $\mathcal{L}[f'_1(t)]$, $\mathcal{L}\left[\int_0^t f_1(\tau)d\tau\right]$.

Example 2

Given $F_2(s) = \frac{3s+7}{s^2+11s+10}$, calculate $f_2(t)$

Solution 1

$$\mathcal{L}[f_1(t)] = F_1(s) = \frac{2}{s+4} + \frac{4}{s-2} \quad (13.9)$$

$$\mathcal{L}[f'_1(t)] = sF_1(s) - f_1(0^+) = \frac{2s}{s+4} + \frac{4s}{s-2} - 6 \quad (13.10)$$

$$\mathcal{L}\left[\int_0^t f_1(\tau)d\tau\right] = \frac{F_1(s)}{s} = \frac{2}{s(s+4)} + \frac{4}{s(s-2)} \quad (13.11)$$

Solution 3

$$\mathcal{L}^{-1}[F_2(s)] = f_2(t) = \mathcal{L}^{-1}\left[\frac{3s+7}{s^2+11s+10}\right] \quad (13.12)$$

$F_2(s)$ can be decomposed as:

$$F_2(s) = \frac{A}{(s+10)} + \frac{B}{(s+1)} = \frac{23}{9(s+10)} + \frac{4}{9(s+1)} \quad (13.13)$$

The coefficients A and B are found simply by writing the following equation: $3s+7 = A(s+1) + B(s+10)$ and then substituting the roots -1 and -10 to get the coefficients A and B.

$f_2(t)$ is then:

$$f_2(t) = \frac{23}{9}e^{-10t} + \frac{4}{9}e^{-t} \quad (13.14)$$

13.2.4 State-space representation of LTI systems

The state-space description provide the dynamics as a set of coupled first-order differential equations in a set of internal variables known as the **state variables**, together with a set of algebraic equations that combine the state variables into physical output variables.

State variables fully describe the system and its response to any given set of inputs. The initial conditions of the state variables must be known, in order to predict the future system state and outputs for all time $t > t_0$. In general, any system (as described in Fig. 13.1) has r inputs (i.e. $u_1(t), \dots, u_r(t)$) and m output variables (i.e. $y_1(t), \dots, y_m(t)$). If the state variables that describe the system ($x_1(t), \dots, x_n(t)$) are known at some initial time t_0 then any output variable $y_i(t)$ may be computed.

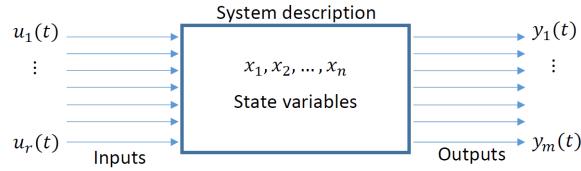


Figure 13.1: Inputs and outputs of a system.

The mathematical description of the system is expressed as a set of n coupled first-order ordinary equations, a.k.a the state equations, in which the time derivative of each state variable is expressed in terms of the state variable ($x_i(t)$) and the systems inputs ($u_i(t)$):

$$\begin{aligned}\dot{x}_1 &= f_1(\mathbf{x}, \mathbf{u}, t) \\ \dot{x}_2 &= f_2(\mathbf{x}, \mathbf{u}, t) \\ &\vdots = \vdots \\ \dot{x}_n &= f_n(\mathbf{x}, \mathbf{u}, t)\end{aligned}\tag{13.15}$$

In vector notation, the set of n equations in eq. 13.15 can be written as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t)\tag{13.16}$$

where \mathbf{F} is a vector function with n components. Since we focus on linear and time-invariant (LTI), we can describe eq. 13.15 by linear differential equations with constant coefficients:

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + \dots + b_{1r}u_r \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + \dots + b_{2r}u_r \\ &\vdots \quad \vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + \dots + b_{nr}u_r\end{aligned}\tag{13.17}$$

Eq. 13.17 can be re-written in a matrix form:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1r} \\ b_{21} & \dots & b_{2r} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

which can be summarized as:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}\tag{13.18}$$

Upper case letters are used to denote matrices and lower case letters denote column vectors.

An important property of the linear state equation is that all system variables may be represented by a linear combination of the state variables and the system inputs. For an arbitrary output variable in a system of order n with r inputs we can write:

$$y(t) = c_1x_1 + \dots + c_nx_n + d_1u_1 + \dots + d_ru_r\tag{13.19}$$

The output equations in a complete and compact form are:

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \quad (13.20)$$

where:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \dots & d_{1r} \\ d_{21} & \dots & d_{2r} \\ \vdots & & \vdots \\ d_{m1} & \dots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

For many physical systems the matrix \mathbf{D} is the null matrix, and the output equation reduces to simple weighted combination of the state variables:

$$\mathbf{y} = \mathbf{Cx} \quad (13.21)$$

13.3 Dynamic systems

A dynamic system is a system that changes over time and that can be described through two sets of time dependent variables: the input variables and the output variables. The former represents the external forces on the system under consideration and the latter represent the system variables of interest. To completely describe a dynamic system, it is necessary to introduce a third vector of time dependent variables: the state variables. The state variables indicate the "state" of the system. In absence of inputs, the time evolution of the output of a dynamic system can be predicted through the use of only state variables, as no external force is applied. The three vectors are indicated as:

- $u(t)$: input variables
- $y(t)$: output variables
- $x(t)$: state variables

A continuous-time dynamic system can be described through the continuous time state equations:

$$\begin{cases} \dot{x} = f(x(t), u(t), t) \\ y = g(x(t), u(t), t) \end{cases} \quad (13.22)$$

The status of a system, by definition, is continuous over time while inputs and outputs can be discontinuous. If the system is linear, i.e., $\dot{x}(t)$ and $y(t)$ are a linear combination of the vector components of $x(t)$ and $u(t)$ (where the linear coefficients are independent on time), the system described in Eq. 13.22 can be expressed as:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (13.23)$$

Where A , B , C and D are time-independent matrices. The solution for $x(t)$ is given by:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \quad (13.24)$$

The system can be solved through the use of the Laplace transform:

$$\begin{cases} sX(s) - x(0) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases} \quad (13.25)$$

Solving for $X(s)$ and $Y(s)$ we obtain:

$$\begin{cases} X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0) \\ Y(s) = [C(sI - A)^{-1}B + D]U(s) + C(sI - A)^{-1}x(0) \end{cases} \quad (13.26)$$

And the Laplace transform functions $X(s)$ and $Y(s)$ can be inverted to calculate $x(t)$ and $y(t)$. The status of the system $X(s)$ is made of two factors: the first one depends on the input $U(s)$ and the second one depends on the initial condition $x(0)$. In case the initial condition is zero ($x(0) = 0$), the output variable becomes:

$$Y(s) = [C(sI - A)^{-1}B + D]U(s) \quad (13.27)$$

Rearranging 13.27, we can define the transfer function $G(s)$ as:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \quad (13.28)$$

The transfer function defines the relation between inputs and outputs of the system, with zero initial conditions. It can be used to calculate the system output in case the system evolves starting from a stationary condition. However, the stability of the systems also depends on the initial conditions.

We now try to solve the continuous time state equations, assuming the system is linear:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (13.29)$$

We introduce the following change of variables:

$$\begin{aligned} \hat{x} &= Tx \\ x &= T^{-1}\hat{x} \end{aligned} \quad (13.30)$$

Substituting the new variables, we obtain:

$$\begin{cases} T\dot{x} = TAx + TBu \\ y = CT^{-1}\hat{x} + Du \end{cases} \quad (13.31)$$

Defining and introducing the following new matrices:

$$\begin{aligned}\hat{A} &= TAT^{-1} \\ \hat{B} &= TB \\ \hat{C} &= CT^{-1} \\ \hat{D} &= D\end{aligned}\tag{13.32}$$

We can simplify the system expressed by Eq. 13.29 to:

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ y = \hat{C}\hat{x} + \hat{D}u \end{cases}\tag{13.33}$$

The transformation matrix T can be chosen in order to have \hat{A} diagonal:

$$\hat{A} = \text{diag}\{s_1, s_2, \dots, s_n\}\tag{13.34}$$

The solution of the system then becomes:

$$\hat{x}(t) = e^{\hat{A}t}\hat{x}_0\tag{13.35}$$

Changing back the variable from $\hat{x}(t)$ to $x(t)$, leads to the following solution:

$$x(t) = T^{-1}\hat{x} = T^{-1}\text{diag}\{e^{s_1 t}, e^{s_2 t}, \dots, e^{s_n t}\}T x_0\tag{13.36}$$

The solution of the system (Eq. 13.36) is a combination of different exponential terms. If any of the exponent coefficients s_i is real and positive (or have positive real part), the solution of $x(t)$ diverges to infinity and the system is unstable. However, if all s_i have a negative real part, the system is stable.

13.4 Stability criteria

Consider an autonomous system defined by the nominal variables $\tilde{u}(t)$, $\tilde{x}(t)$ and \tilde{x}_0 . The system is then perturbated and evolves according the state variable $x(t)$ starting from the perturbated initial condition x_0 . The system is said to be Lyapunov stable if, for every $\epsilon > 0$, there exist a $\delta > 0$ such that:

$$\begin{aligned}\|x_0 - \tilde{x}_0\| &\leq \delta \\ \|x(t) - \tilde{x}(t)\| &\leq \epsilon, \quad \forall t\end{aligned}\tag{13.37}$$

Lyapunov stability means that solutions starting from a condition "close enough" to the equilibrium (within a distance δ), remain "close enough" forever (within a distance ϵ). For a Lyapunov stable system this must be true for any ϵ that one may want to choose. In addition to the Lyapunov stability condition, the system is asymptotically stable if:

$$\lim_{t \rightarrow \infty} \|x(t) - \tilde{x}(t)\| = 0\tag{13.38}$$

The system is asymptotically stable if, after the perturbation, its state variable tends to the value of the unperturbed state variable. In absence of inputs, the output of a dynamic is given by (dropping the first term of Eq. 13.26):

$$Y(s) = C(sI - A)^{-1}x(0) \quad (13.39)$$

A necessary and sufficient condition for asymptotic stability is that the matrix A has all eigenvalues with negative real part. The eigenvalues of the matrix A represent the exponents of the solution for $x(t)$ (Eq. 13.35). To study the asymptotic stability is then sufficient to calculate the eigenvalues of the matrix A through the following equation:

$$\det(sI - A) = 0 \quad (13.40)$$

The Lyapunov concept of asymptotic stability can also be applied to transfer functions.

13.4.1 Examples

Example 1

Consider two nuclear reactors each one having transfer functions describing the reactor power (output variable, $Y(s)$), as a function of the position of control rods in the core (input variable, $U(s)$):

$$G_1(s) = -\frac{(s + 24)}{(s + 3)(s + 4)} \quad (13.41)$$

$$G_2(s) = \frac{(s - 24)}{(s + 3)(s + 4)} \quad (13.42)$$

Although this is an arbitrary selected example, $G(s)$ could be simply the change in power due to the change in the position of the control rod, i.e. $G(s) = \frac{\Delta \text{Power}}{\Delta \text{Height}}$. We can see that the poles for both reactors are identical $P_{1,2} = -3, -4$, which indicate that for both the asymptotic behavior is the same. However, the zeros are different for these systems, i.e. $Z_1 = -24$ and $Z_2 = 24$, which indicate that the systems will behave differently during the transient.

In both nuclear reactors the control rods are inserted by a single unit length. Draw the power transient (power as a function of time) and the asymptotic value of power for each reactor.

Solution 1

Recalling the definition of transfer function (Eq. 13.28), the output of the system can be calculated as:

$$Y(s) = G(s)U(s) \quad (13.43)$$

In our case, the input variable as a function of time is:

$$u(t) = 1 \quad (13.44)$$

As the transfer function is defined in the frequencies domain, we have to calculate the Laplace transform of $u(t)$:

$$U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \quad (13.45)$$

The output variables for the two reactors become:

$$Y_1(s) = -\frac{(s+24)}{s(s+3)(s+4)} \quad (13.46)$$

$$Y_2(s) = \frac{(s-24)}{s(s+3)(s+4)} \quad (13.47)$$

The asymptotic value of $y(t)$ can be calculated from $Y(s)$ using the final value theorem (Eq. 13.7)

$$\lim_{t \rightarrow \infty} y_1(t) = \lim_{s \rightarrow 0} sY_1(s) = -2 \quad (13.48)$$

$$\lim_{t \rightarrow \infty} y_2(t) = \lim_{s \rightarrow 0} sY_2(s) = -2 \quad (13.49)$$

The two reactors reach the same power level at the end of the transient. To be able to sketch the power transient, we can calculate the value of $y(t)$ and its derivative at $t = 0$ using the initial value theorem (Eq. 13.6):

$$\lim_{t \rightarrow 0} y_1(t) = \lim_{s \rightarrow \infty} sY_1(s) = 0 \quad (13.50)$$

$$\lim_{t \rightarrow 0} y_2(t) = \lim_{s \rightarrow \infty} sY_2(s) = 0 \quad (13.51)$$

$$\lim_{t \rightarrow 0} y'_1(t) = \lim_{s \rightarrow \infty} s^2 Y_1(s) = -1 \quad (13.52)$$

$$\lim_{t \rightarrow 0} y'_2(t) = \lim_{s \rightarrow \infty} s^2 Y_2(s) = 1 \quad (13.53)$$

We can see from the value of the derivative in $t = 0$ that the transient for the two reactors will behave differently. The functions $y_1(t)$ and $y_2(t)$ can be determined by operating the inverse of the Laplace transform:

$$\begin{aligned} y_1(t) &= \mathcal{L}^{-1}\{Y_1(s)\} = \mathcal{L}^{-1}\left\{-\frac{(s+24)}{s(s+3)(s+4)}\right\} = \\ &= \mathcal{L}^{-1}\left\{\frac{7}{(s+3)} - \frac{5}{(s+4)} - \frac{2}{s}\right\} = 7e^{-3t} - 5e^{-4t} - 2 \end{aligned} \quad (13.54)$$

$$\begin{aligned} y_2(t) &= \mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\left\{\frac{(s-24)}{s(s+3)(s+4)}\right\} = \\ &= \mathcal{L}^{-1}\left\{-\frac{9}{(s+3)} + \frac{7}{(s+4)} + \frac{2}{s}\right\} = 9e^{-3t} - 7e^{-4t} - 2 \end{aligned} \quad (13.55)$$

We can now plot the $y_1(t)$ and $y_2(t)$ (Figures 13.2a, 13.2b).

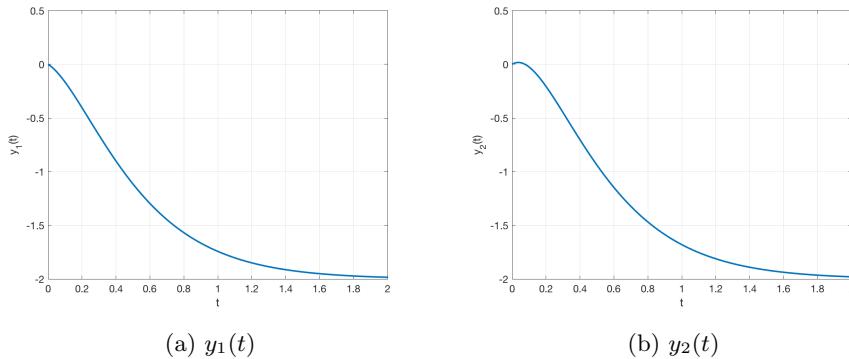


Figure 13.2: Neutronic population response for different transfer functions

Both reactors are characterized by the same asymptotic behavior and both are asymptotically stable, as both transfer functions are characterized by the same poles (roots of the denominator). However, it can be noted that the transient of these two reactors differ. Following a rod insertion, in the first reactor the power decreases monotonically, while the second reactor is characterized by an increase in power at the beginning of the transient. A system presenting this behavior is called non-minimum-phase systems. This analysis show that often the asymptotically stability is necessary but not sufficient, as in the nuclear field all the systems are required to be stable during every transient. An example of a non-minimum-phase system is the Chernobyl reactor. Following a control rods insertion, the graphite in the control rods causes a power increase in the first seconds of the transient.

Lecture 14: Stability of Feedback Systems

Lecturer: Dr. Dan Kotlyar

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The question that we need to raise relates to a given system at equilibrium. Will the system return to its equilibrium if its state is perturbed?

The reactor dynamics equations are nonlinear since the reactivity depends on power. A linear system is therefore only a 1st order approximation and yet important conclusions about the nonlinear system can be derived by studying the linear approximation.

14.1 Block diagrams

A useful tool that can be used to describe a dynamical system consists of block diagrams. A block diagram represents a flow of information in which every variable is represented by an arrow and every system is represented by a block containing its transfer function. Every block has an input variable and an output variable. The transfer function of the block represents the function operated on the input variable that calculates the output variable.

For example, a system having an input $U(s) = 1$ and an output $Y(s) = 1/(s + 1)$ can be represented as:

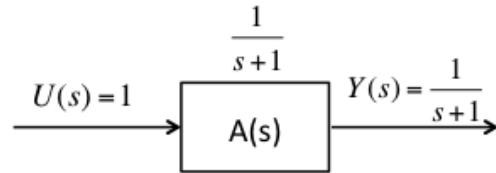


Figure 14.1: Block Diagram Example

As shown in Section 13.3, the transfer function is defined as the ratio between the output and the input:

$$A(s) = \frac{Y(s)}{U(s)} = \frac{1}{(s + 1)} \quad (14.1)$$

14.2 Feedback systems

A feedback system is a dynamical system having a forward loop and a feedback loop. The block diagram representing a feedback system is shown in Fig. 14.2. The block $A(s)$ represents the forward-loop transfer function and the block $B(s)$ represents the feedback transfer function. The function $E(s)$ is an error signal that represents the difference between the input and the output of the feedback element. Here, we assume the validity of superposition of solutions.

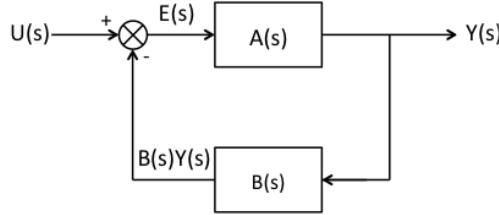


Figure 14.2: Block Diagram of a feedback system

Applying Eq. 14.1 we obtain:

$$Y(s) = A(s)E(s) \quad (14.2)$$

In control-system terminology, $E(s)$ is an error signal that represents the difference between the input and the output of the feedback element

$$E(s) = U(s) - B(s)Y(s) \quad (14.3)$$

Substituting $E(s)$ (Eq. 14.3) into Eq. 14.2 we can write the overall transfer function (closed loop transfer function) as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{A(s)}{1 + A(s)B(s)} \quad (14.4)$$

The product $A(s)B(s)$ is called the open loop transfer function. If $B(s)$ is positive, we say that the system has a negative feedback. In case $B(s)$ is negative, the system has a positive feedback.

The transient response of the system is characterized by the poles of $G(s)$, which are the roots of the characteristic equation:

$$1 + A(s)B(s) = 0 \quad (14.5)$$

the transient response is a linear combination of exponential terms $e^{s_i t}$ where s_i are the roots of Eq. 14.5. If one of the roots s_i has a positive real part, the response to the input will approach to ∞ as $t \rightarrow \infty$ and the system is unstable. If all s_i have negative real part, the response to the system will decay to zero as $t \rightarrow \infty$. In this case, the system will decay to its original value and it is asymptotically stable. If no root has a positive real part but at least one root has a zero real part, we have a borderline case that is called "critical". The time response of a critical system is characterized by a combination of decaying exponential and constant terms or sustained oscillations.

14.2.1 The nuclear reactor as a feedback system

The neutronic population as a function of a reactivity change can be described as a feedback system. The forward-transfer function is determined by the reactor kinetics, while the closed-loop transfer functions are determined by the neutronic and thermodynamic feedbacks. In a standard Pressurized Water Reactor, the main reactivity feedbacks are:

- Fuel Temperature (Doppler Effect)
- Coolant/Moderator Temperature
- Coolant/Moderator Void Fraction
- Reactor Pressure

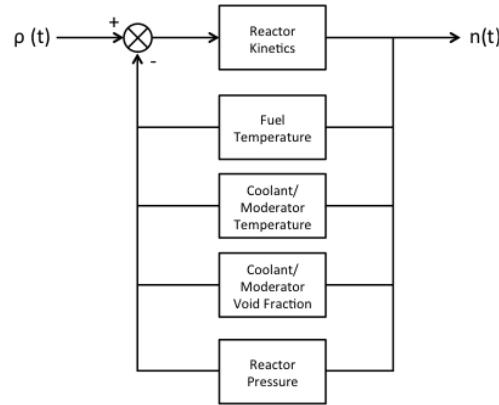


Figure 14.3: Main PWR reactivity feedbacks

In our simple model, we will only account for the influence of the fuel and the moderator temperature coefficient. In previous lecture we have incorporated the temperature feedback through the following equations:

$$\begin{aligned} \frac{d}{dt}T_F(t) &= \frac{1}{M_F c_F} P(t) - \frac{h}{M_F c_F} [T_F(t) - T_M(t)] \\ \frac{d}{dt}T_M(t) &= \frac{h}{M_M c_M} [T_F(t) - T_M(t)] - \frac{2W_M}{M_M} [T_M(t) - T_{in}(t)] \end{aligned} \quad (14.6)$$

The neutronic equations in the 1-group representation are:

$$\begin{aligned} \frac{dP(t)}{dt} &= \frac{\rho - \beta}{\Lambda} P(t) + \lambda c(t) \\ \frac{dc(t)}{dt} &= \frac{\beta}{\Lambda} P(t) - \lambda c(t) \end{aligned} \quad (14.7)$$

Now we will introduce normalized values:

$$\begin{aligned}
 x_1 &= \frac{P - P_0}{P_0} \\
 x_2 &= \frac{c - c_0}{c_0} \\
 x_3 &= \frac{T_F - T_{F,0}}{T_{F,0}} \\
 x_4 &= \frac{T_M - T_{M,0}}{T_{M,0}}
 \end{aligned} \tag{14.8}$$

following the substitution of the normalized values into eqs. 14.7 and 14.6 we can obtain the normalized point reactor dynamic equations:

$$\begin{aligned}
 \frac{dx_1}{dt} &= -\frac{\beta}{\Lambda}x_1 + \frac{\beta}{\Lambda}x_2 + \frac{\rho}{\Lambda}x_1 \quad \text{change in power} \\
 \frac{dx_2}{dt} &= +\lambda x_1(t) - \lambda x_2(t) \quad \text{change in delayed neutrons precursor} \\
 \frac{dx_3}{dt} &= \frac{P_0}{M_F C_F T_{F,0}} x_1(t) - \frac{h}{M_F C_F} x_3(t) + \frac{h T_{M,0}}{M_F C_F T_{F,0}} x_4(t) \quad \text{change in fuel temperature} \\
 \frac{dx_4}{dt} &= \frac{h T_{F,0}}{M_M C_M T_{M,0}} x_3(t) - \frac{2C_M W_{M,0} + h}{M_M C_M} x_4(t) + \frac{2W_{M,0} T_{in,0}}{M_M T_{M,0}} u_1(t) - \frac{2W_{M,0}(T_{M,0} - T_{in,0})}{M_M T_{M,0}} u_2(t) - \\
 &\quad \frac{2W_{M,0}}{M_M} u_2(t) x_4(t) + \frac{2W_{M,0} T_{in,0}}{M_M T_{M,0}} u_1(t) u_2(t) \quad \text{change in moderator/coolant temperature} \\
 \rho &= \rho_{ex} + \alpha_F T_{F,0} x_3 + \alpha_M T_{M,0} x_4 \quad \text{change in reactivity/coolant temperature}
 \end{aligned} \tag{14.9}$$

In the process, we have defined new inputs:

$$\begin{aligned}
 u_1 &= \frac{T_{in} - T_{in,0}}{T_{in,0}} \quad \text{dimensionless coolant inlet T} \\
 u_2 &= \frac{W_M - W_{M,0}}{W_{M,0}} \quad \text{dimensionless coolant flow rate}
 \end{aligned} \tag{14.10}$$

The derived model is represented by a set of non-linear ordinary differential equations and one additional algebraic equation that describes the total reactivity.

14.3 Reactor dynamics

This section will present the stability analysis of a system undergoing small perturbations.

The point dynamic model can be written as:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t), \quad \mathbf{x}(0) = 0 \quad (14.11)$$

when using the one-DN-group approximation the state variable is: $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T$

The vector \mathbf{u} is the input vector, which is $\mathbf{u} = [\rho_{ex} \ u_1 \ u_2]$

The equilibrium point of the system is represented by vectors \mathbf{x}_0 and \mathbf{u}_0 , which satisfy $\mathbf{F}(\mathbf{x}_0, \mathbf{u}_0, t) = 0$.

Due to the chosen formalization, the state variables and inputs are all zero in equilibrium:

$$\begin{aligned} \mathbf{x}_0 &= [0 \ 0 \ 0 \ 0]^T \\ \mathbf{u}_0 &= [0 \ 0 \ 0]^T \end{aligned} \quad (14.12)$$

The most important question related to the operation of nuclear reactors is their behavior around a fixed point. In other words, is the system of interest stable around the fixed point (i.e. in proximity to equilibrium).

The behavior of a dynamic system around any point can be investigated using the first method of Lyapunov. We can note that the kinetic and T/H models are not linear. For example, the component of the input vector ρ is multiplied by a component of the state vector $x_1(t)$ or the non-linear product of $u_1(t)u_2(t)$. In order to be able to apply the concepts of linear control theory, we need to linearize the equations.

For the current model the linearization can be easily achieved by dropping the product terms: $\rho x_1(t)$, $u_1 u_2$, $u_2 x_4$ and the equations are transformed into linear equations with constant coefficients.

As was shown from the previous lecture, the stability of the system can be analyzed by examining the eigenvalue of the matrix \mathbf{A} from the state equations: $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$.

$$\mathbf{A} = \begin{bmatrix} -\frac{\beta}{\Lambda} & \frac{\beta}{\Lambda} & 0 & 0 \\ \lambda & -\lambda & 0 & 0 \\ \frac{P_0}{M_F C_F T_{F,0}} & 0 & -\frac{h}{M_F C_F} & \frac{h T_{M,0}}{M_F C_F T_{F,0}} \\ 0 & 0 & \frac{h T_{F,0}}{M_M C_M T_{M,0}} & -\frac{2 W_{M,0} C_M + h}{M_M C_M} \end{bmatrix}$$

As can be seen from the previous lecture, the stability of the system can be analyzed by studying:

$$\det(SI - A) = 0 \quad (14.13)$$

The solution of eq. 14.13 using the operational data given in lecture 12 results the matrix A having a zero eigenvalue and 3 negative eigenvalues (-7.5767, -2.4649, -0.4636).

14.4 Analysis of point dynamics in the frequency domain

We will now use the Laplace transform to analyze the point-dynamic model in the frequency domain. The transfer function $G(s)$, $H_F(s)$ and $H_M(s)$ describe the reactor response, the fuel and the moderator/coolant feedback. Please notice that the reactivity that is set as the input to the forward loop is the **sum** of the external and feedback reactivity.

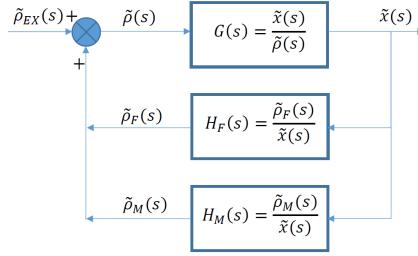


Figure 14.4: Block diagram for a perturbed system with feedback.

We will now derive these transfer functions from eqs. 14.9. Performing the Laplace transform on the point dynamic equations leads to:

$$\begin{aligned}
 s\tilde{x}_1(s) &= -\frac{\beta}{\Lambda}\tilde{x}_1(s) + \frac{\beta}{\Lambda}\tilde{x}_2(s) + \frac{\tilde{\rho}(s)}{\Lambda} \\
 s\tilde{x}_2(s) &= +\lambda\tilde{x}_1(s) - \lambda\tilde{x}_2(s) \\
 s\tilde{x}_3(s) &= \frac{P_0}{M_F C_F T_{F,0}}\tilde{x}_1(s) - \frac{h}{M_F C_F}\tilde{x}_3(s) + \frac{hT_{M,0}}{M_F C_F T_{F,0}}\tilde{x}_4(s) \\
 s\tilde{x}_4(s) &= \frac{hT_{F,0}}{M_M C_M T_{M,0}}\tilde{x}_3(s) - \frac{2C_M W_{M,0} + h}{M_M C_M}\tilde{x}_4(s) + \frac{2W_{M,0} T_{in,0}}{M_M T_{M,0}}\tilde{u}_1(s) - \frac{2W_{M,0}(T_{M,0} - T_{in,0})}{M_M T_{M,0}}\tilde{u}_2(s) \\
 \tilde{\rho}(s) &= \tilde{\rho}_{ex}(s) + \alpha_F T_{F,0}\tilde{x}_3 + \alpha_M T_{M,0}\tilde{x}_4
 \end{aligned} \tag{14.14}$$

Obtain $G(s) = \frac{\tilde{x}_1(s)}{\tilde{\rho}(s)}$: from the 1st and 2nd equations:

$$G(s) = \frac{1}{\Lambda s \left(1 + \frac{1}{\Lambda} \frac{\beta}{s + \lambda} \right)} \tag{14.15}$$

Obtain $H_F(s)$ and $H_M(s)$:

Please note that the mass flow rate and inlet temperature are assumed to be constant here, i.e. $\tilde{u}_1(s) = \tilde{u}_2(s) = 0$. First, we need to obtain the relation between $\tilde{x}_3(s)$ and $\tilde{x}_4(s)$:

$$\tilde{x}_4(s) = \frac{hT_{F,0}}{T_{M,0}(M_M C_M s + 2C_M W_{M,0} + h)}\tilde{x}_3(s) \tag{14.16}$$

Substituting eq. 14.16 into the 3rd equation of eq. 14.14:

$$\tilde{x}_3(s) = \frac{P_0}{T_{F,0} \left(M_F C_F s + h - \frac{h^2}{M_M C_M s + 2C_M W_{M,0} + h} \right)} \tilde{x}_1(s) \quad (14.17)$$

We also know that the fuel reactivity feedback is: $\alpha_F T_{F,0} \tilde{x}_3$ and thus:

$$H_F(s) = \frac{\alpha_F T_{F,0} \tilde{x}_3}{\tilde{x}_1(s)} = \frac{\alpha_F P_0}{\left(M_F C_F s + h - \frac{h^2}{M_M C_M s + 2C_M W_{M,0} + h} \right)} \quad (14.18)$$

We can easily obtain the moderator/coolant feedback transfer function:

$$H_M(s) = \frac{\alpha_M h}{\alpha_F (M_M C_M s + 2C_M W_{M,0} + h)} H_F(s) \quad (14.19)$$

The block diagram presented in Fig. 14.4 could be easily replaced by:

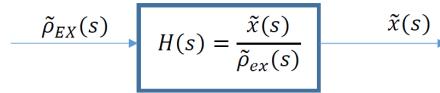


Figure 14.5: A simplified block diagram for a system with feedback.

In order to obtain the transfer function we can use the following relations:

$$\begin{aligned} H(s) &= \frac{\tilde{x}_1(s)}{\tilde{\rho}_{ex}(s)} \\ \tilde{\rho}(s) &= \tilde{\rho}_{ex}(s) + H_F(s)\tilde{x}_1(s) + H_M(s)\tilde{x}_1(s) \\ \Rightarrow H(s) &= \frac{\tilde{x}_1(s)}{\tilde{\rho}(s) - H_F(s)\tilde{x}_1(s) - H_M(s)\tilde{x}_1(s)} \end{aligned} \quad (14.20)$$

Therefore the transfer function of the simplified block-diagram is:

$$H(s) = \frac{\tilde{x}_1(s)}{\tilde{\rho}_{ex}(s)} = \frac{G(s)}{1 - [H_F(s) + H_M(s)] G(s)} \quad (14.21)$$

The transient response of the system is characterized by the poles of $H(s)$, which are the roots of the characteristic equation:

$$1 - [H_F(s) + H_M(s)] G(s) = 0 \quad (14.22)$$

As we mentioned before, a reactor without feedback (i.e. open-loop) is intrinsically unstable. However, a nuclear system can be described as a closed-system due to presence of multiple feedback mechanisms. The roots of the characteristic equation, which are the poles of the transfer function will describe the system's behavior in proximity to equilibrium. For example a system with a transfer function of:

$$H(s) = \frac{1}{s - a}$$

will be characterized by an exponential time behavior, i.e. $h(t) = \mathcal{L}^{-1}\{H(s)\} = e^{at}$. The transfer function has a pole $s = a$. If the real part of a is greater than zero, the function will diverge with time and the system is *unstable*. The borderline case when the real part of a equals zero may cause the system to oscillate around an equilibrium point, but not to diverge with time. Lastly, if all the real parts are less than zero, the system is *stable*, since any perturbation will damp out with time.

Summary:

1. The transient response is a linear combination of exponential terms $\exp(s_j t)$, where the roots s_j are the solution of eq. 14.22.
2. If one or more of the roots has a positive real part, the response to a perturbation will grow indefinitely and the system is unstable.
3. If all the roots have negative real parts, the response to a perturbation will ultimately decay to zero as $t \rightarrow \infty$. The system is said to be asymptotically stable.
4. If no root has a positive real part but at least one root has a zero real part, we have a borderline, whose time response contains decaying transients together with constant terms or sustained oscillations. This is also known as a "critical" system.

In general stability of nuclear reactors can be analyzed using 2 approaches:

1. Time-domain: reactor dynamic equations are solved and stability is evaluated based on the power output. Straightforward approach that incorporates all the non-linear effects and thus accounts for states far from the equilibrium point. However, the approach can be computationally expensive.
2. Frequency-domain: reactor dynamic equations are linearized around the equilibrium point and stability is analyzed according to the poles of the transfer function. This method is valid only for small perturbations to predict instability threshold, but can not predict the system's behavior away from the equilibrium point. This method is computationally inexpensive.

14.5 Example:

The following set of differential equations describe a BWR model, in which the variation in void (and hence coolant density) is taken into consideration. The following set of equations describe the dynamics of a BWR system:

$$\begin{aligned}
 \frac{dx_1(t)}{dt} &= \frac{\rho(t) - \beta}{\Lambda} x_1(t) + \lambda x_2(t) + \frac{\rho(t)}{\Lambda} \quad \text{change in power} \\
 \frac{dx_2(t)}{dt} &= \frac{\beta}{\Lambda} x_1(t) - \lambda x_2(t) \quad \text{change in precursor} \\
 \frac{dx_3(t)}{dt} &= 25.04x_1(t) - 0.23x_3(t) \quad \text{change in fuel temperature} \\
 \frac{dx_4(t)}{dt} &= x_5(t) \\
 \frac{dx_5(t)}{dt} &= k_0 x_3(t) - 6.82x_4(t) - 2.25x_5(t) \quad \text{change in void}
 \end{aligned} \tag{14.23}$$

The first 2 equations describe the neutronic model (excess neutron population and precursor normalized to nominal values). The 3rd equation describes the excess fuel temperature and the last 2 equations describe the time dependence of the excess void reactivity feedback. The total reactivity change is given by:

$$\rho(t) = x_4(t) - 2.52 \times 10^{-5} x_3(t) \tag{14.24}$$

Additional data include: $\beta = 0.0056$, $\Lambda = 4 \times 10^{-5}$ s, $\lambda = 0.08$ s⁻¹

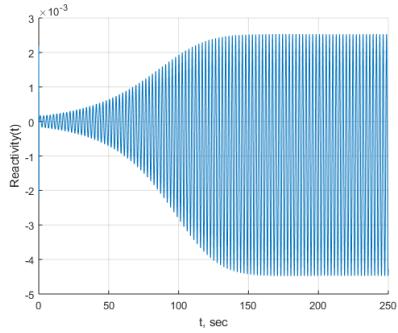
You are required to:

1. Describe the state variables and their initial values.
2. Investigate the stability of the linear system. Set $k_0 = -3.7 \times 10^{-3}$

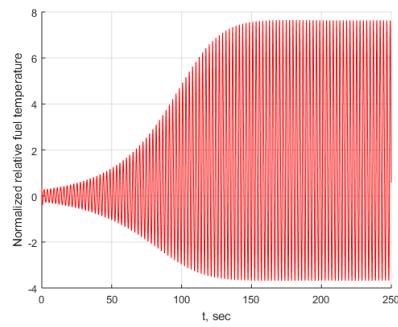
For an external reactivity perturbation:

$$\rho_{ex}(t) = 0.0022 \cos(10\pi t) \quad 0 \leq t < 0.1 \text{ s} \quad \text{otherwise zero} \tag{14.25}$$

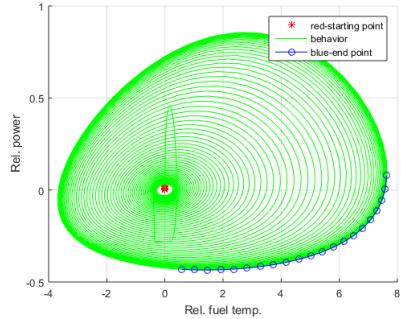
The time-dependent solution is:



(a) Excess reactivity.



(b) Excess fuel temperature.



(c) Power vs. T-fuel representation.

Figure 14.6: Temporal behavior of the system for $k_0 = -3.7 \times 10^{-3}$.

The parameter k_0 defines the linear stability of the reactor point model. When $k_0 = -3.7 \times 10^{-3}$, the system is at its critical state. In other words, decreasing this factor will dampen the oscillations and the system will return to its equilibrium point as shown in Fig. 14.7.

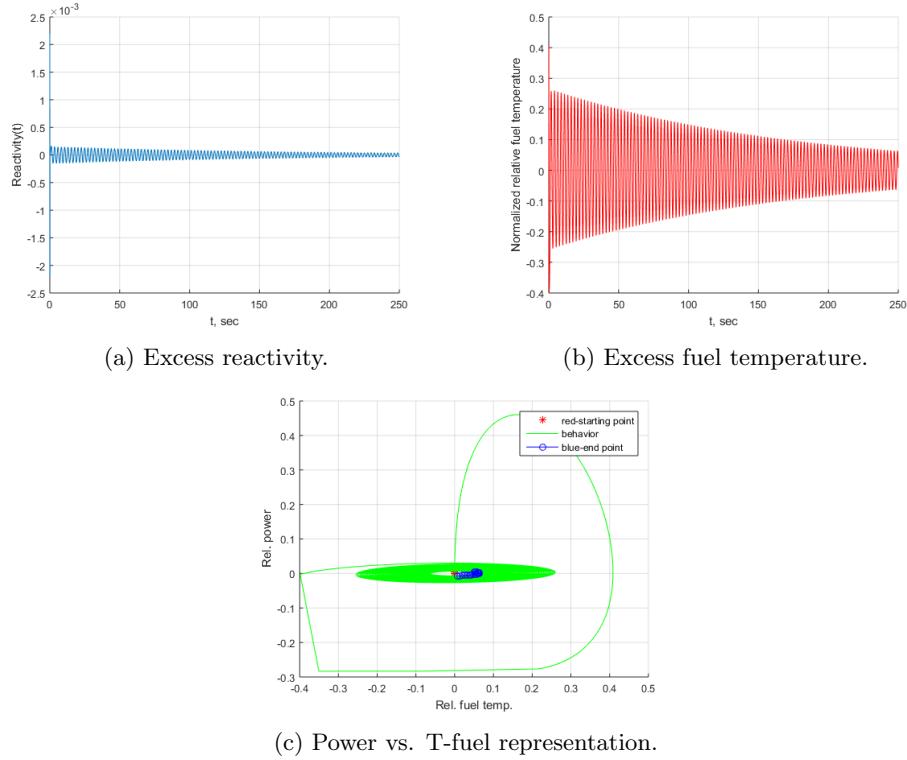


Figure 14.7: Temporal behavior of the system for $k_0 = -3.5 \times 10^{-3}$.

Lecture 15: Isotope Depletion

Lecturer: Dr. Dan Kotlyar

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15.1 Time-dependent isotopic variation

Isotopic depletion stands for a number of processes:

1. Fuel burn-up produces actinides. In a LWR this is the result of absorption in U^{235} and U^{238} .
2. Fuel burn-up produces also fission products, via the fission process.
3. The fuel composition will change also through radioactive α and β emissions.

We will focus our attention on the following aspects:

1. A mathematical derivation for the time dependent density of the i^{th} isotopes in a chain. In other words, we will introduce the Bateman equations. Many programs exist worldwide to solve a large number of such equations. For example, ORIGEN was developed in United States. Such solvers are usually coupled with a neutronic solver. For example, SERPENT, which is being developed in VTT (Finland) and has a built-in depletion solver that is able to track thousands of isotopes very efficiently by adopting suitable mathematical procedures.
2. The actinide chains of U^{238} and U^{235} detailing somewhat the depletion of U^{235} and the buildup of Pu^{239} . The practical time spans for these depletion and buildup is months to years.

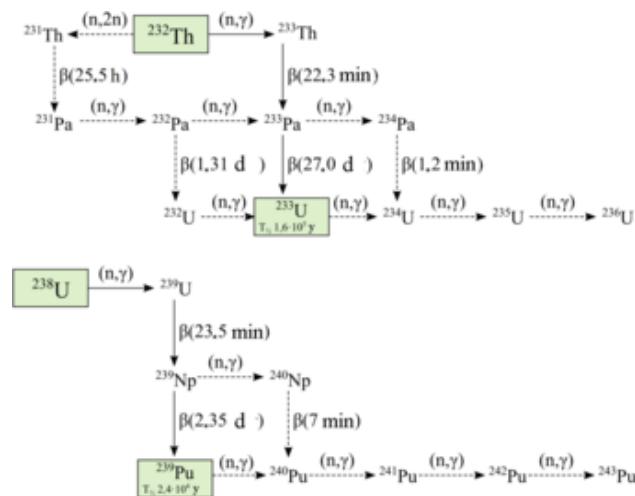


Figure 15.1: U^{238} and Th^{232} transmutation chains.

3. An exposition of the important Xe^{135} chain. Xe^{135} poisoning is a process of hours and may have a marked influence on the daily routine in a power station. **However, this will be covered only in the next lecture.**
4. In addition we have to devote a few words to radioactive decays, which may be covered in a simplified manner for the current purposes in the next few lines:
 - (a) In α -decay a nucleus ${}_Z X^A$ becomes ${}_{Z-2} X^{A-4}$.
 - (b) In $\beta-$ or $\beta+$ decay a nucleus ${}_Z X^A$ becomes, respectively, ${}_{Z+1} X^A$ or ${}_{Z-1} X^A$.
 - (c) In γ -decay an excited nucleus falls to its ground state, emitting an electromagnetic wave of a very short wavelength.
 - (d) It must be noted that there are some exceptions, in which the decay to ground state (from the meta-stable state takes relatively long time). Such transition, for example, is happening when Am^{241} captures a neutron as shown in Fig. 15.2. Am^{242m} decays γ with $t_{1/2} = 141$ yr and Am^{242g} decays $\beta-$ with $t_{1/2} = 16$ hr to Cm^{242} , which then decays to Pu^{238} .

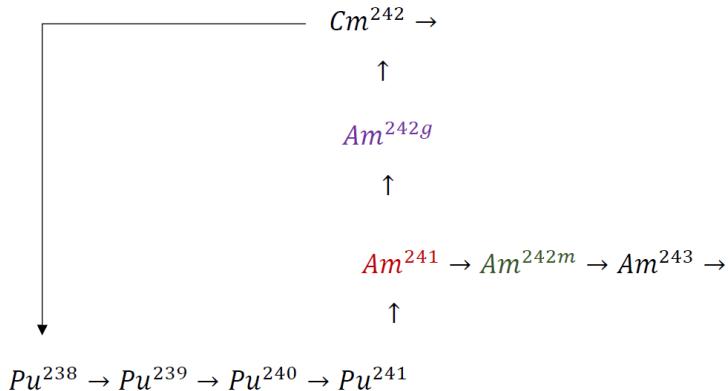


Figure 15.2: Pu-Am transmutation chain.

In a radioactive emission (α , β , or γ), of an ensemble of nuclei of one identity j , the decay time constant λ_j is the probability that a nucleus, in one second, undergoes the emission of relevance.

The time decrease of the density of a radioactive ensemble, decaying according to the constant λ_j and left to diminish with no supply, is:

$$N_j(t) = N_j(0)e^{-\lambda_j t} \quad (15.1)$$

The half life $t_{1/2}$ is the time it takes for N_0 will diminish to $0.5N_0$. $t_{1/2} = \frac{\ln 2}{\lambda_j}$.

15.2 Fuel depletion

The term Fuel Depletion usually means the isotopic changes taking place in the fuel while it is in core. These changes are primarily due to:

1. The capture reaction (n,γ).
2. The fission reaction (n,f).
3. α and β radiation reactions.

Fuel depletion is mostly a phenomenon of only neutron absorptions (primarily capture+fission). The radioactive radiations may generally be disregarded: α emissions for one reason, β emissions for an opposite reason. The core lifetime of a fuel batch is typically 3-4 years; α radiations have a typical half life $t_{1/2}$ of thousands or millions of years, therefore they may be omitted in any discussion of fuel isotopic changes while the fuel is in the core. β radiation is, excluding the β emission in Pu^{241} , a much faster process than neutron absorption. A more detailed explanation of the said above follows.

To get a feel for the quantities involved we have adopted thermal (0.625eV) values for the average σ of several actinides, and a 3×10^{13} value for the total thermal flux in a typical LWR. Table 15.1 is the outcome of this exercise. One could conclude from the table that indeed α and β emissions are, respectively, too long or too short to be of any importance to fuel depletion during the 30–40 years of core life.

Table 15.1: Radiation vs. absorption half-lives for various isotopes.

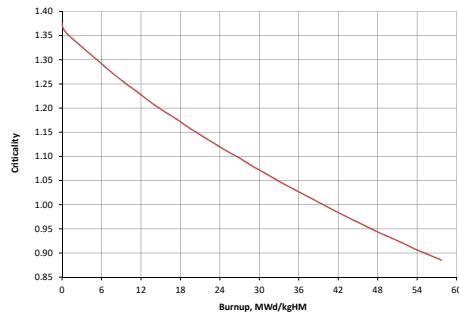
Isotope	Radiation	radiation $t_{1/2}$	absorption $t_{1/2}$
Th^{232}	α	1.4×10^{10}	58
U^{233}	α	1.6×10^5	1.3
U^{235}	α	7.0×10^8	1.1
U^{238}	α	4.5×10^9	270
Pu^{239}	α	2.4×10^4	0.7
Pa^{233}	β	7.4×10^{-2}	18
Np^{239}	β	4.6×10^{-5}	330

15.3 Fuel depletion analysis (fuel cycle)

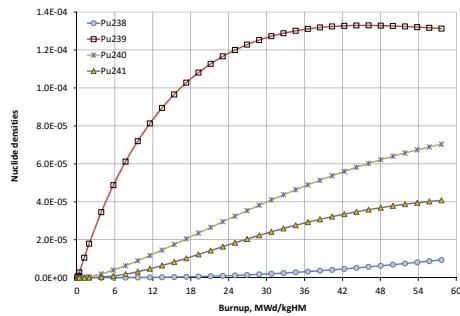
Fuel depletion analysis are focusing on predicting the long-term variation in reactor fuel composition caused by exposure to neutron flux during operation. These changes play a crucial role on the operating life of a reactor. One must ensure that sufficient excess reactivity is provided to achieve a desired operational period at a certain power level.

However, fuel cycle analysis should also focus on the variation in power density due to the burnup distribution variation.

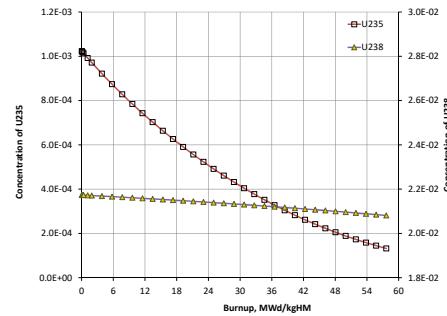
The following figures present the variation in criticality and various isotopic densities as a function of burnup.



(a) Criticality vs. burnup.



(b) Concentration of plutonium isotopes vs. burnup.



(c) Concentration of uranium isotopes vs. burnup.

Figure 15.3: Time-dependent variation.

15.4 Bateman equations

The purpose of burn-up calculations is to determine fuel material compositions as a function of time. Each heterogeneous reactor core is divided into material zones. The compositions for all materials are known at the beginning Of life (BOL). Assuming only one burnable region with a vector with $\mathbf{N}(t)$ elements $N_j(t)$ (the atom number density), the objective of the burn-up calculation is to obtain the vector $\mathbf{N}(t)$ as a function of time. The burn-up problem can be formulated (Bateman equations) as follows:

$$\frac{dN_j(t)}{dt} = -\lambda_j N_j(t) - \phi \sigma_j N_j(t) + \sum_{\substack{k=1 \\ k \neq j}}^M (\lambda_k BR_{k \rightarrow j} N_k(t) + \phi \sigma_{k \rightarrow j} N_k(t)) \quad (15.2)$$

$N_j(t)$	Atomic density of nuclide j
$\frac{dN_j(t)}{dt}$	Atomic density change of nuclide j
M	Number of considered isotopes in the chain
$BR_{k \rightarrow j}$	Branching ratio from nuclide k to j
σ_j	Average absorption cross cross-section of nuclide j
$\sigma_{k \rightarrow j}$	Average cross-section of the nuclide k that leads to j
ϕ	flux

It must be pointed out that for the equation that describes a fission product, $\sigma_{k \rightarrow j}$ will simply be $\gamma^{k \rightarrow j} \times \sigma_k$, where γ is the fission yield (probability that a fission product j will be formed from a fission event in nuclide k) and σ_k is the fission cross-section of the nuclide k , as shown below.

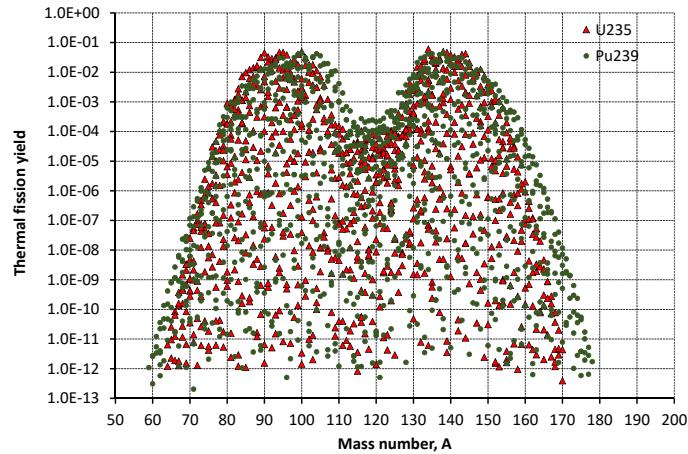


Figure 15.4: Thermal fission yield for U^{235} and Pu^{239} .

Assuming that the one-group cross sections and the total flux are known and constant, the Bateman equations, which describe the rate of change of various nuclides, are given by a system of linear homogeneous, first-order differential equations with constant coefficients.

These equations can be re-written in the following matrix form:

$$\frac{d\mathbf{N}(t)}{dt} = [\underline{\Lambda} + \underline{\Sigma}\phi] \mathbf{N}(t) \quad (15.3)$$

Where the decay matrix $\underline{\Lambda}$ is:

$$\underline{\Lambda} = \begin{bmatrix} -\lambda_1 & BR_{2 \rightarrow 1}\lambda_2 & \dots & BR_{i \rightarrow 1}\lambda_i & \dots & BR_{M \rightarrow 1}\lambda_M \\ BR_{1 \rightarrow 2}\lambda_1 & -\lambda_2 & \dots & BR_{i \rightarrow 2}\lambda_i & \dots & BR_{M \rightarrow 2}\lambda_M \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ BR_{1 \rightarrow M}\lambda_1 & BR_{2 \rightarrow M}\lambda_2 & \dots & BR_{i \rightarrow M}\lambda_i & \dots & -\lambda_M \end{bmatrix}$$

the transmutation matrix $\underline{\Sigma}$ is:

$$\underline{\Sigma} = \begin{bmatrix} -\sigma_1 & \sigma_{2 \rightarrow 1} & \dots & \sigma_{i \rightarrow 1} & \dots & \sigma_{M \rightarrow 1} \\ \sigma_{1 \rightarrow 2} & -\sigma_2 & \dots & \sigma_{i \rightarrow 2} & \dots & \sigma_{M \rightarrow 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{1 \rightarrow M} & \sigma_{2 \rightarrow M} & \dots & \sigma_{i \rightarrow M} & \dots & -\sigma_M \end{bmatrix}$$

The expression $[\underline{\Lambda} + \underline{\Sigma}\phi]$ is a $M \times M$ matrix which will be represented by matrix $\underline{\underline{A}}$.

$$\frac{d\mathbf{N}(t)}{dt} = \underline{\underline{A}}\mathbf{N}(t) \quad (15.4)$$

The solution is given by:

$$\mathbf{N}(t) = \exp [\underline{\underline{A}} \triangle t] \mathbf{N}(0) \quad (15.5)$$

The above solution can be obtained by applying various matrix exponential algorithms. Among such methods are the Pade approximation, Jordan decomposition, Krylov subspace and perhaps the most efficient Chebyshev rational approximation.

The size of the matrix will affect the calculation time as can be seen from the following figure.

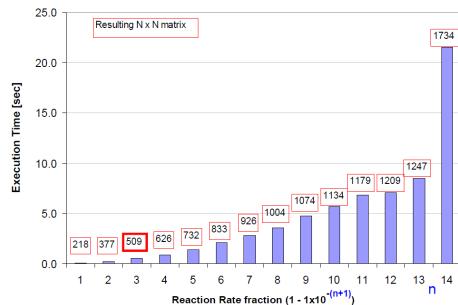


Figure 15.5: Execution time vs. fraction of the calculated matrix.

Question: Why do we need so many isotopes (on the order of thousands)? In reality deterministic codes usually use only ~ 300 isotopes and some much less (~ 100).

Answer: From a neutronic point of view, it is possible to achieve a good accuracy in determining the flux and the eigenvalue with only ~ 100 isotopes, although the original chains will have to be modified (e.g. eliminate short living isotopes and so on). However, if one wants to obtain the decay heat curves, which are required for analyzing accident scenarios, thousands isotopes must be included and tracked.

15.4.1 Post-irradiation characteristics

The potential biological hazard of a nuclear waste is measured by either ingested or inhaled radiotoxicities in Sieverts (Sv). The ingested/inhaled radiotoxicity for a given isotope is determined by the activity (Bq) multiplied by the isotope effective dose for ingestion or inhalation coefficients (Sv/Bq units), which are isotope-dependent. These coefficients account for radiation type and tissue weighting factors, metabolic and biokinetic information of an adult based on ICRP 72 report. Typically, ingestion radiotoxicity is considered to be more significant than inhalation radiotoxicity for long-term disposal. This is because the greatest biological hazard to humans occurs when the isotope is absorbed in nearby ground water and eventually enters the food chain. Therefore, all the radiotoxicity values used in this lecture are calculated as ingested radiotoxicity to maintain consistency and expressed in relative terms, i.e. divided by the ingested radiotoxicity of the equivalent freshly mined natural uranium.

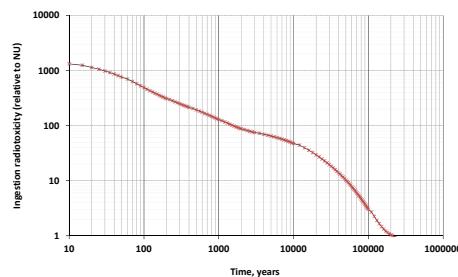


Figure 15.6: Relative (to NU) Radiotoxicity.

The decay heat is simply the product of activity and proportion factor, i.e. the Q -value. Fig. 15.7 shows that the decay heat is mainly generated by the decay of fission products for the first 60 years, with the contribution dominated by barium (Ba^{137m}) and yttrium (Y^{90}) as decay products of cesium (Cs^{137}) and strontium (Sr^{90}). After about 60 years, the decay heat is mostly from actinide elements, with the most significant ones being plutonium and americium. Beyond 200 years, the decay heat is generated almost entirely by the plutonium and americium, out to at least 10^4 years.

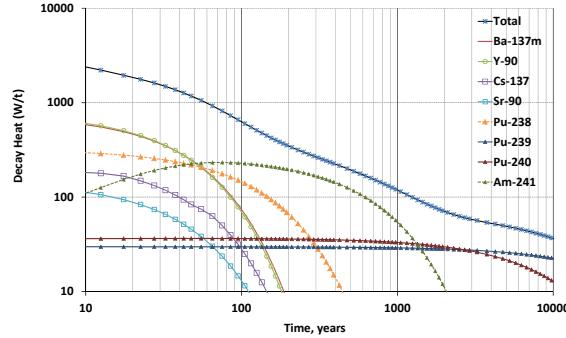


Figure 15.7: Decay heat as a function of time.

15.5 Burnup coupling methodology

The depletion equations use time dependent fluxes, although still assumed to be constant for each time step, to determine the evolution of nuclide inventories with time. However, nuclide inventories depend on the flux, which by itself requires a prior knowledge of the nuclide inventories. There are several approaches to solve this non-linear problem. First, the solution requires discretizing the full time scale into time steps, in which the parameters of interest (i.e. reaction rates and nuclide densities) are to be computed. At each time step, the procedure requires solving 2 independent problems. The first is the neutron transport eigenvalue equation that provides reaction rates. In order to progress in time, the Bateman equations must be solved to obtain the new composition.

Many coupling schemes were developed and used. Perhaps the most known one but not necessarily the most popular is the **explicit Euler method**.

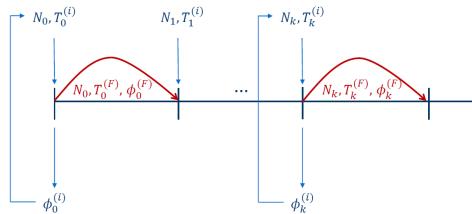


Figure 15.8: Time discretization and operator splitting approach.

The neutron transport solution is obtained at the beginning-of-step (BOS) for a predetermined fuel inventory. Then, the space and energy dependent microscopic reaction rates are assumed to be constant during the depleted time step. Knowing these reaction rates allows obtaining the concentration at the end-of-step (EOS) in a single calculation step.

The explicit neutronic-burnup-thermal hydraulic coupling scheme is used in many deterministic codes as well as in Monte Carlo based codes. In this method, the flux, power and temperature distributions are evaluated at fixed time points. Such method may be numerically unstable as can be seen from Fig. 15.9. This figure shows the fuel temperature distribution for sequential time-points. The model relies on 3D fuel assembly that was divided into multiple axial layers. The transport solution (i.e. reaction rates) was obtained by a Monte-Carlo code. The coupling scheme relied on the one described in Fig. 15.8.

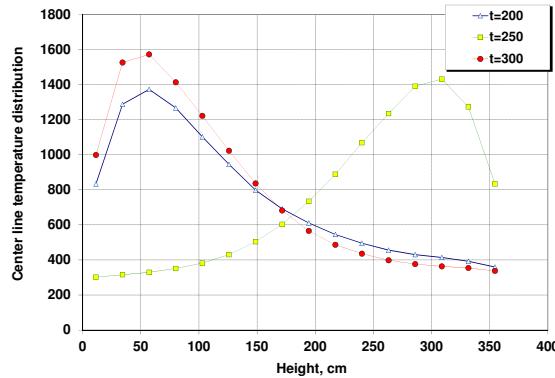


Figure 15.9: Example of axial oscillations using the Explicit Euler approach.

The actual reason of the problematic numerical stability is the presence of very strong feedback between the neutron flux and the nuclide and thermal-hydraulic fields.

15.6 Example

A PWR core is loaded with UO₂ fuel. The core is loaded with 100 ton_{HM} with 4.5% weight enrichment. The thermal power produced by the core is 3400 MW_{th} ($\frac{P}{V} = 104 \frac{W}{cm^3}$).

1. Write the transmutation chain that leads to Pu²³⁹.
2. Write the differential equations that describe the rate of change for each component.
3. Solve and find the explicit solution/expression for Pu²³⁹.
4. What would be the concentration of Pu²³⁹ after 60 days.

Use the following data: $\sigma_f(U^{235}) = 7.705 \text{ b}$ $\sigma_a(U^{238}) = 2.982 \text{ b}$ $\sigma_c(U^{238}) = 2.595 \text{ b}$ $\sigma_a(Pu^{239}) = 3.318 \text{ b}$

Lecture 16: Depletion: Fission Product Poisoning

Lecturer: Dr. Dan Kotlyar

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16.1 Fission products

In our previous lecture we have described how the concentration of various nuclides changes with time by introducing the Bateman equations:

$$\frac{dN_j(t)}{dt} = -\lambda_j N_j(t) - \phi \sigma_j N_j(t) + \sum_{\substack{k=1 \\ k \neq j}}^M (\lambda_k B R_{k \rightarrow j} N_k(t) + \phi \sigma_{k \rightarrow j} N_k(t) + \phi \sigma_f^k \gamma^{k \rightarrow j} N_k(t)) \quad (16.1)$$

Please note that in Eq. 16.1 we have added a term that describes the production of a fission product.

Every fission product absorbs, therefore poisoning is present by the mere phenomenon of fission. The amount of poisoning depends, among other, on the neutron spectrum. Absorption, in general, decreases with energy, therefore poisoning in a fast reactor, whose spectrum is much harder than that of a LWR, is considerably less severe than in a LWR. Some chains necessitate special attention, as they lead to noticeable poisoning.

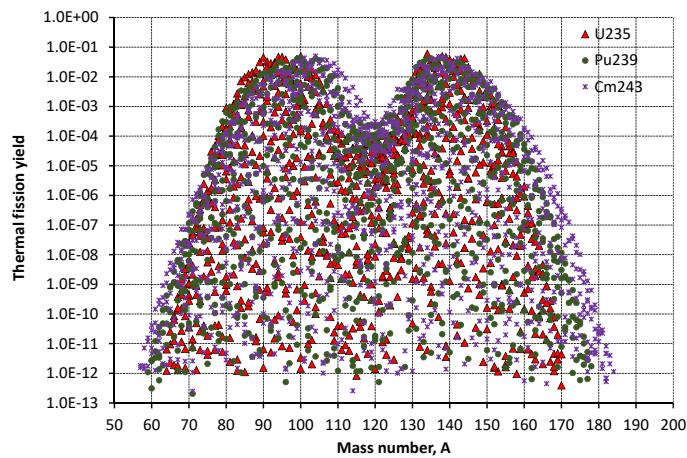


Figure 16.1: Fission product yields.

Fission products' characteristics:

1. In most cases 2 fission products fragments are produced in a fission event. The probability of a pair to form-up depends on the element that undergoes fission and the energy of the incident neutron. There is a small probability for a ternary fission event, in which additional light fragment is created.
2. In most cases, fission products are unstable and have either extra neutrons or protons and hence tend to decay by either $\beta-$ or $\beta+$.
3. Fission products typically have large thermal absorption cross section and hence the build-up of these are extremely important in thermal spectra systems (e.g. LWR). In fast spectrum systems, the absorption of fission products is much less pronounced.

16.2 Samarium and Xenon poisoning

Our current lecture focuses on the dominant fission product: Sm^{149} and Xe^{135} , which have very large absorption cross sections. The build-up of these isotopes introduces reactivity change, which could also determine spatial power distribution.

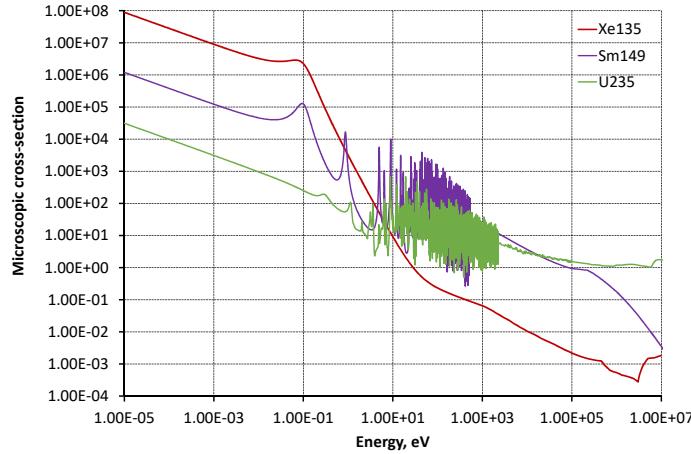


Figure 16.2: Microscopic cross sections of Sm^{149} and Xe^{135} as a function of energy.

16.2.1 Samarium Poisoning

The general chain of Sm¹⁴⁹ is shown in Fig. 16.3. As can be seen, Nd¹⁴⁹ is the fission product which then decays $\beta-$ to Pm¹⁴⁹, which subsequently decays to Sm¹⁴⁹. It must be pointed out that the half life of Sm¹⁴⁹ is very long, i.e. $2 \times 10^{15} \text{ yr}$. and therefore its decay will not be taken into account.

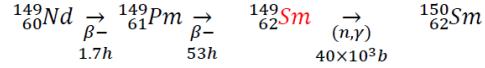


Figure 16.3: Sm¹⁴⁹ chain.

The fission yield γ^{Nd} is:

1. 0.0066 (U²³³)
2. 0.0113 (U²³⁵)
3. 0.0190 (P²³⁹)

In writing the differential equations that describe the change in the concentration of Sm¹⁴⁹ we will assume that Pm¹⁴⁹ is produced in the fission event rather than Nd¹⁴⁹ since its half-life is relatively short. However, we will use the fission yield for Nd¹⁴⁹ to describe Pm¹⁴⁹ Promethium production rate.

The production-destruction rate differential equations are:

$$\begin{aligned} \frac{dP(t)}{dt} &= \gamma^{Nd}\Sigma_f\phi - \lambda_P P(t) \\ \frac{dS(t)}{dt} &= \lambda_P P(t) - \sigma_a^S \phi S(t) \end{aligned} \tag{16.2}$$

Where P stands for Pm¹⁴⁹ and S for Sm¹⁴⁹.

When the flux is assumed to be constant then the solution of these equations is:

$$\begin{aligned} P(t) &= \frac{\gamma^{Nd}\Sigma_f\phi}{\lambda_P} (1 - e^{-\lambda_P t}) + P(0)e^{-\lambda_P t} \\ S(t) &= S(0)e^{-\sigma_a^S \phi t} + \frac{\gamma^{Nd}\Sigma_f}{\sigma_a^S} (1 - e^{-\sigma_a^S \phi t}) - \frac{\gamma^{Nd}\Sigma_f\phi - \lambda_P P(0)}{\lambda_P - \sigma_a^S \phi} (e^{-\sigma_a^S \phi t} - e^{-\lambda_P t}) \end{aligned} \tag{16.3}$$

At the beginning of life in a fresh core, $P(0)$ and $S(0)$ is zero and these build-up to equilibrium values (Eq. 16.4) by setting the time-derivatives in Eq. 16.2 to zero;

$$\begin{aligned} P_{eq} &= \frac{\gamma^{Nd}\Sigma_f\phi}{\lambda_P} \\ S_{eq} &= \frac{\lambda_P P_{eq}}{\sigma_a^S \phi} \end{aligned} \tag{16.4}$$

Substituting S_{eq} into P_{eq} results in:

$$S_{eq} = \frac{\gamma^{Nd}\Sigma_f}{\sigma_a^S} \quad (16.5)$$

Observations:

1. Pm^{149} in equilibrium depends on the flux.
2. Sm^{149} is produced from Pm^{149} and disappear by absorbing neutrons. Both production-destruction processes depend on the flux that is canceled and hence Sm^{149} in equilibrium does not depend on flux.
3. The time it takes to reach equilibrium depends on σ_a^S , ϕ and λ_P . In LWRs it takes a few hundred hours to reach equilibrium.

When the reactor is shut-down then $P(0) = P_{eq}$, $S(0) = S_{eq}$ and the flux is zero and hence Eq. 16.3 reduces to:

$$\begin{aligned} P(t) &= P_{eq}e^{-\lambda_P t} \\ S(t) &= S_{eq} + P_{eq}(1 - e^{-\lambda_P t}) \rightarrow S_{eq} + P_{eq} \end{aligned} \quad (16.6)$$

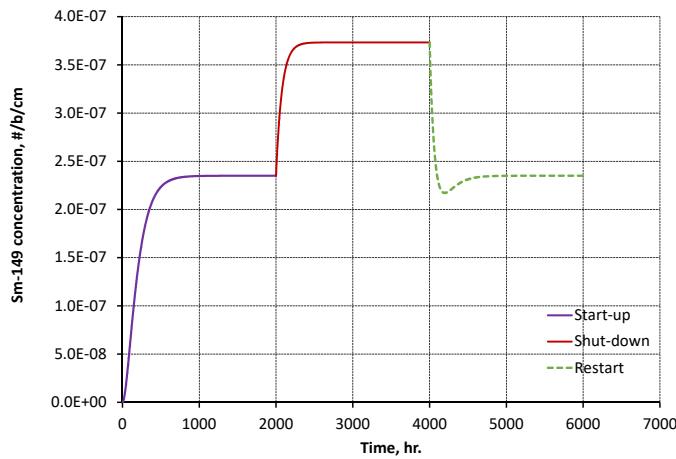


Figure 16.4: Concentration of Sm^{149} under different flux levels.

Sm^{149} concentration will increase to $S_{eq} + P_{eq}$ since Pm^{149} decays into Sm^{149} . If the reactor is restarted, Sm^{149} burns out until Pm^{149} builds-up and hence eventually Sm^{149} reaches an equilibrium, as illustrated in Fig. 16.4.

Perturbation theory estimates the reactivity worth of Sm^{149} is:

$$\rho_S(t) = -\frac{\Sigma_a^S(t)}{\Sigma_a^{tot}} = -\frac{S(t)\sigma_a^S}{\Sigma_a^{tot}} \quad (16.7)$$

Example:

Estimate Sm reactivity worth in an infinite critical homogeneous reactor when $\nu = 2.6$ and $\gamma = 0.0113$.

Solution:

We will substitute the equilibrium expression, $\frac{\gamma^{Nd}\Sigma_f}{\sigma_a^S}$, for Sm into Eq. 16.7.

$$\rho_S(t) = -\frac{\gamma^{Nd}\Sigma_f}{\sigma_a^S} \frac{\sigma_a^S}{\Sigma_a^{tot}} = -\gamma^{Nd} \frac{\Sigma_f}{\Sigma_a} \quad (16.8)$$

Since the system is critical, we can use the following relation:

$$k = \frac{\nu\Sigma_f}{\Sigma_a} = 1 \quad (16.9)$$

Hence:

$$\rho_S(t) = -\gamma^{Nd} \frac{\Sigma_f}{\Sigma_a} = -\frac{\gamma^{Nd}}{\nu} = 0.00435 \frac{\Delta k}{k} \quad (16.10)$$

16.2.2 Xenon Poisoning

Xe^{135} is a thermal absorber with $\sigma = 2.6 \times 10^6$ b. Xe^{135} is directly produced from fission and its fission yield here will be denoted as γ^{Xe} but it is also a decay product of I^{135} that is produced from Te^{135} , which is a direct fission product.

The transmutation scheme for Xe^{135} is presented below:

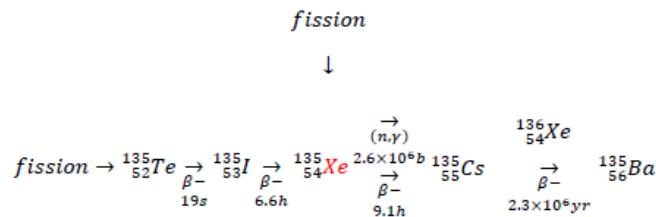


Figure 16.5: Xe^{135} chain.

The fission yield γ^{Te} is:

1. 0.051 (U^{233})
2. 0.061 (U^{235})
3. 0.055 (P^{239})

and $\gamma^{Xe} = 0.003$ for U^{235} .

We will assume that I^{135} is a direct fission product with fission yield of γ^{Te} . The production-destruction rate differential equations are:

$$\begin{aligned}\frac{dI(t)}{dt} &= \gamma^{Te} \Sigma_f \phi - \lambda_I I(t) \\ \frac{dX(t)}{dt} &= \gamma^{Xe} \Sigma_f \phi + \lambda_I I(t) - (\lambda_X + \sigma_a^X \phi) X(t)\end{aligned}\quad (16.11)$$

These equations have the following solutions:

$$\begin{aligned}I(t) &= \frac{\gamma^{Te} \Sigma_f \phi}{\lambda_I} (1 - e^{-\lambda_I t}) + I(0) e^{-\lambda_I t} \\ X(t) &= \frac{(\gamma^{Te} + \gamma^{Xe}) \Sigma_f \phi}{\lambda_X + \sigma_a^X \phi} \left[1 - e^{-(\lambda_X + \sigma_a^X \phi)t} \right] + \frac{\gamma^{Te} \Sigma_f \phi - \lambda_I I(0)}{\lambda_X - \lambda_I + \sigma_a^X \phi} \left[e^{-(\lambda_X + \sigma_a^X \phi)t} - e^{-\lambda_I t} \right] + X(0) e^{-(\lambda_X + \sigma_a^X \phi)t}\end{aligned}\quad (16.12)$$

At $t=0$, when the fresh core is started $X(0) = I(0) = 0$. Similarly to Pm and Sm, I^{135} and Xe^{135} also reach an equilibrium (i.e. $d/dt = 0$):

$$\begin{aligned}I_{eq} &= \frac{\gamma^{Te} \Sigma_f \phi}{\lambda_I} \\ X_{eq} &= \frac{(\gamma^{Te} + \gamma^{Xe}) \Sigma_f \phi}{\lambda_X + \sigma_a^X \phi}\end{aligned}\quad (16.13)$$

I^{135} and Xe^{135} reach equilibrium with time constants $1/\lambda_I \cong 0.1h$ and $1/(\lambda_X + \sigma_a^X \phi) \cong 30h$ respectively.

The reactivity worth of Xe^{135} is:

$$\begin{aligned}\rho_X(t) &= -\frac{\Sigma_a^X(t)}{\Sigma_a^{tot}} = -\frac{X(t) \sigma_a^X}{\Sigma_a^{tot}} \\ \rho_X^{eq} &= -\frac{(\gamma^{Te} + \gamma^{Xe}) \Sigma_f \phi \sigma_a^X}{(\lambda_X + \sigma_a^X \phi) \Sigma_a} \cong -\frac{\gamma^{Te} + \gamma^{Xe}}{\nu \left(1 + \frac{\lambda_X}{\sigma_a^X \phi} \right)} = -\frac{0.026}{1 + 0.756 \times 10^{13}/\phi}\end{aligned}\quad (16.14)$$

One can see from Eq. 16.14 that for higher fluxes the negative reactivity worth is larger, which means that Xe^{135} absorbs more. **What will happen in the core is the power is increased (in terms of Xe reactivity worth)?**

16.2.3 Xenon peak following a shutdown

When a reactor is shutdown from an equilibrium xenon condition, the equations for I^{135} and Xe^{135} are:

$$\begin{aligned} I(t) &= I_{eq} e^{-\lambda_I t} \\ X(t) &= X_{eq} e^{-\lambda_X t} + I_{eq} \frac{\lambda_I}{\lambda_I - \lambda_X} (e^{-\lambda_X t} - e^{-\lambda_I t}) \end{aligned} \quad (16.15)$$

For a flux above a certain value, the concentration of Xe^{135} will increase following a shutdown since I^{135} will decay to Xe^{135} . This means that restarting a core may be impossible since its reactivity worth will simply be too high.

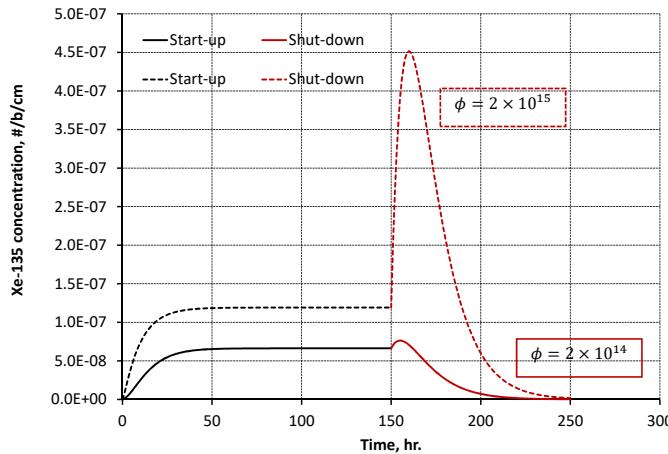


Figure 16.6: Xe^{135} concentration as a function of time.

16.2.4 Effect of power-level changes

- Following a start-up of a fresh core, both I^{135} and Xe^{135} increase and these isotopes reach equilibrium after 30 hours.
- Following a shut-down, I^{135} decays rapidly into Xe^{135} , which therefore has a peak. The peak originates due to the fact that there is a source (i.e. I^{135}) that creates Xe^{135} but since the flux is zero the disappearance rate is slower than the production one.
- In LWRs, there is roughly 3% (although in this example it is more 4%) reactivity penalty due to the presence of Xe^{135} , although its atomic density is extremely small.
- The concentration of Xe^{135} decreases following a shut-down period because the disappearance rate due to neutron absorption is the dominant process. However, Xenon reaches equilibrium again after roughly 2 days.

5. One can see that the reactivity worth of Xenon during the shutdown period reaches a peak with an enormous reactivity penalty, which makes it impossible to restart the reactor.

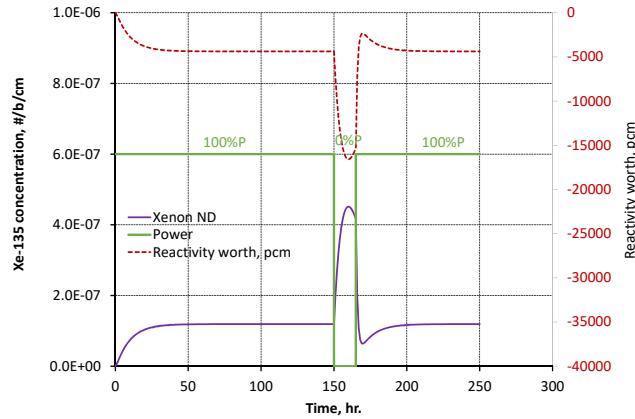


Figure 16.7: Xe^{135} concentration and reactivity worth as a function of time.

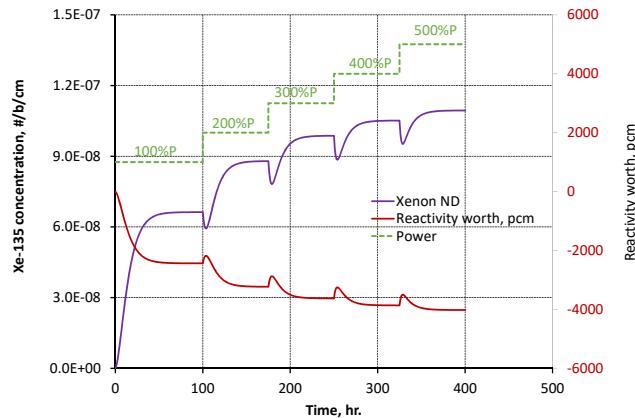


Figure 16.8: Xe^{135} concentration and reactivity worth as a function of time.

Fig. 16.8 shows that Xenon in equilibrium ($t \rightarrow \infty$) depends on the power/flux level. In addition, the increase in power causes the concentration of Xe^{135} to decrease due to the increase in absorption.

16.3 Examples

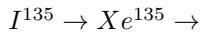
1. Critical core is fueled with UO_2 and operates with constant flux $\phi = 5 \times 10^{13}$ for 2 weeks.

- (a) Estimate xenon reactivity worth in equilibrium.
- (b) Estimate xenon reactivity worth following a shutdown of 6 hours.

Use the following data:

$$\sigma_a^X = 2.6 \times 10^6 \text{ b}, t_{1/2}^I = 6.6 \text{ h}, t_{1/2}^X = 9.1 \text{ h}, \gamma^{Te} = 0.061, \gamma^{Xe} = 0.003, \nu = 2.46.$$

2. A bare, homogeneous 1-D slab reactor with width 'a' operates at full power level for 20 days. Assume the following simplified chain:



Both Iodine and Xenon are produced in fission events. Iodine decay to Xenon which subsequently can either decay or absorb neutrons.

Use void boundary conditions and neglect the extrapolation length. Use the following parameters: Σ_f , λ_I , λ_{Xe} , γ_I , γ_{Xe} , $I(t)$, $X(t)$ and a . In addition, the flux peak is ϕ_0 .

- (a) Write the differential set of equations that describe the rate of change in Iodine and Xenon.
 - (b) Parametrically calculate and qualitatively draw (by hand) Xenon spatial distribution within the slab.
 - (c) The reactor operates for another year. Qualitatively draw Xenon distribution and compare with the distribution at the start of operation. Describe your estimation.
3. Assuming the core initially fueled with 134.408 kg of U enriched to 93%. The fuel is UO_2 and depleted under a constant flux of $\phi = 5 \times 10^{13}$.

The fuel is depleted for one year. Assume that fission and capture microscopic cross-sections of U^{235} are 365 and 85 barn respectively. Calculate the average burnup, i.e. $\frac{MWd}{kg}$

Lecture 17: Multigroup diffusion

Lecturer: Dr. Dan Kotlyar

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17.1 Background

For practical nuclear reactor design problems, the 1-group diffusion model is very inaccurate and hence inappropriate. In the 1-group diffusion theory all neutrons are assumed to have a mono-energetic velocity. However, the neutrons in a reactor have a range from 10 MeV to 0.01 eV. The following figure shows 2 typical spectra (i.e. thermal and fast):

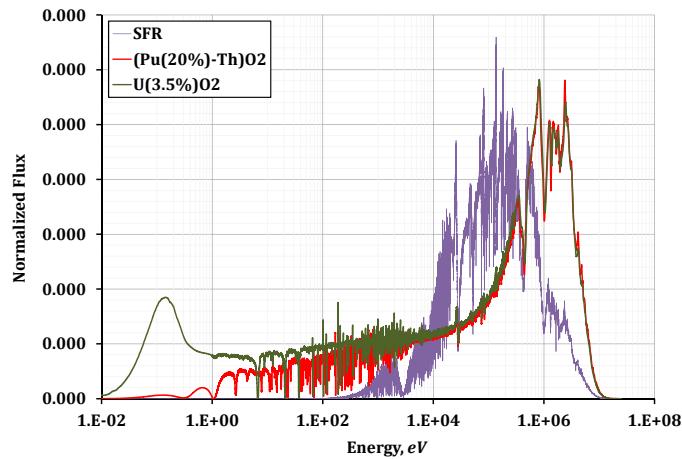


Figure 17.1: Typical thermal (i.e. LWR) and fast spectra.

As we already know the probability (i.e. cross sections) of various isotopes and reactions is very sensitive to the energy of the incident neutron. The following figure shows examples of this sensitive behavior for various nuclides and reactions.

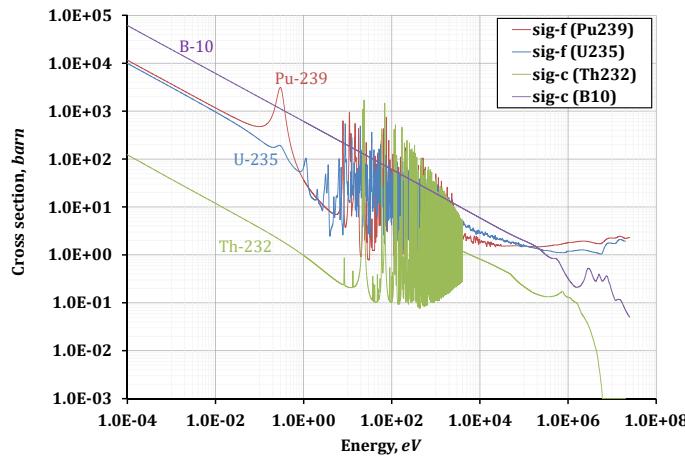


Figure 17.2: Typical thermal (i.e. energy-dependent cross sections).

As one can easily see, cross sections are drastically vary in the MeV to 0.01 eV energy range.

What is done in reality:

Accurate cell calculations are typically done with hundreds of neutron energy groups to obtain cell-averaged few-group cross-sections. These, few-group cross-sections are then used in the full core calculations.

Few-group calculation are only successful if they are generated by detailed multi-group cell calculations.

17.2 The Multigroup equations

Practical reactor calculations require to include the energy treatment as well. However, it will not be practical to include a continuous energy treatment and hence our analysis will introduce the multigroup approach (i.e. groups of energy intervals). To form the multigroup neutron diffusion equations we first divide the energy range for the neutrons up into groups as shown in Fig. 17.3.

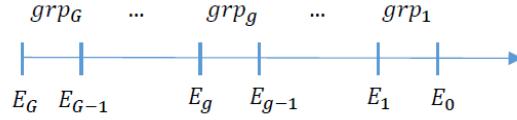


Figure 17.3: Neutron energy groups.

In each energy group g ($E \in [E_g, E_{g-1}]$) the flux is $\phi_g = \phi_g(\mathbf{r}, t)$. The multigroup equation describes the rate of change of neutrons in any group g as follows:

$$\frac{1}{v_g} \frac{\partial \phi_g}{\partial t} = \begin{aligned} & \text{rate of change} \\ & \nabla \cdot D_g \nabla \phi_g \quad \text{leakage} \\ & -\Sigma_{ag} \phi_g \quad \text{absorption} \\ & -\Sigma_{sg} \phi_g \quad \text{scattering out of group } g \\ & + \sum_{g'=1}^G \Sigma_{sg'g} \phi'_{g'} \quad \text{scattering into group } g \\ & + \chi_g \sum_{g'=1}^G \nu'_g \Sigma_{fg'} \phi'_{g'} \quad \text{fission} \\ & + S_g^{ext} \quad \text{external source} \end{aligned}$$

In the fission event a compound nucleus is formed and then sequentially undergoes fission (equivalent to a decay of radioactive isotopes). The neutrons are produced in the decay event without any knowledge of their parents (e.g. U²³⁵, U²³³, Pu²³⁹). We therefore write the fission production as a sum that is independent of the index g . χ_g is the fraction appearing in group g . Note that:

$$\sum_{g=1}^G \chi_g = 1 \tag{17.1}$$

Again, χ_g is the probability that a fission neutron will be born with an energy in group g . However, the fission cross section, $\Sigma_{fg'}$ and the number of emitted neutrons ν'_g are dependent on the energy of the incident neutron g' .

The group-transfer cross section $\Sigma_{sg'g}$ is $\Sigma_s(g' \rightarrow g)$.

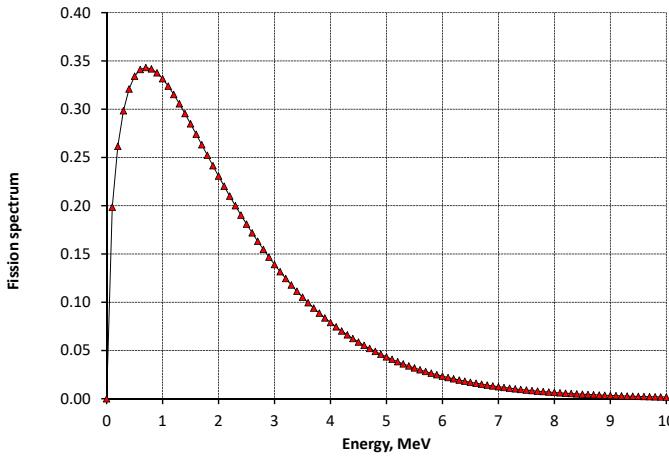


Figure 17.4: Fission neutron energy spectrum.

17.2.1 Fission term

In order to clarify things even more lets break the fission term to its components. Assuming we have only 4 groups (i.e. 1 thermal and 3 fast):

$$\sum_{g=1}^4 \nu'_g \Sigma_{fg'} \phi'_g = \nu_4 \Sigma_{f4} \phi_4 + \nu_3 \Sigma_{f3} \phi_3 + \nu_2 \Sigma_{f2} \phi_1 + \nu_1 \Sigma_{f1} \phi_1 \quad (17.2)$$

Most fission events in LWR are mostly from fission that are induced from **thermal neutrons** and about **3%** of the neutrons are produced from fast fission events.

1. The fraction of neutrons born in the thermal energy is 0 (i.e. $\chi_4 = 0$)- there are no fission source neutrons. but χ_3, χ_2, χ_1 are non-zero.
2. The summation term contains contributors from all 4 terms, but the most dominant comes from the thermal group (i.e. group 4).
3. However, this sum of fission neutrons will only show-up for the group with non-zero χ , i.e. χ_3, χ_2, χ_1 .
4. *Question:* What is the source term for group 4?

17.2.2 Out-scattering term

In one-group theory, the neutrons were all assumed to have a mono-energetic velocity or in other words, the cross-sections were assumed to be energy independent.

Therefore, the scattering was not considered in the loss-gain balance simply because the neutron of interest was not removed but rather deflected. Hence only absorption and leakage processes were considered.

However, the neutrons are now sub-divided into energy groups. A scattering event, therefore, may either remove or produce neutrons at a certain energy group.

In the equation that describes the balance of neutrons in a group, we have the following term:

$$\Sigma_{sg}\phi_g = (\Sigma_{sg1} + \Sigma_{sg2} + \dots + \Sigma_{sgG})\phi_g = \left(\sum_{g=1}^G \Sigma_{sgg'} \right) \phi_g \quad (17.3)$$

This scattering removal term is illustrated in the following figure:

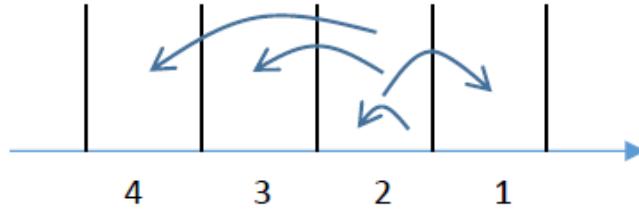


Figure 17.5: Scattering out of group-2.

17.2.3 In-scattering term

The scattering source term is:

$$\left(\sum_{g'=1}^G \Sigma_{sg'g} \right) \phi'_g = \Sigma_{s1g}\phi_1 + \Sigma_{s2g}\phi_2 + \dots + \Sigma_{sGg}\phi_G \quad (17.4)$$

Neutrons are scattered into group g from all groups. Please notice that we also have a term that represents the scattering from group g to group g .

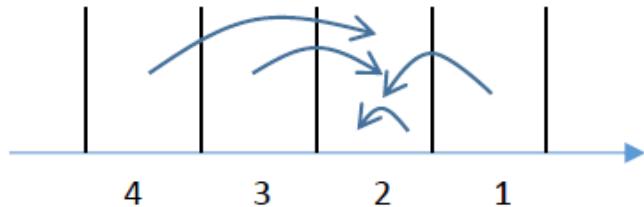


Figure 17.6: Scattering into group-2.

If we now consider only 4 groups and write the balance for group 2, we get:

$$\begin{aligned} \frac{1}{v_2} \frac{\partial \phi_2}{\partial t} &= \nabla \cdot D_2 \nabla \phi_2 - \Sigma_{a2} \phi_2 \\ &- (\Sigma_{s21} \phi_2 + \Sigma_{s22} \phi_2 + \Sigma_{s23} \phi_2 + \Sigma_{s24} \phi_2) \\ &+ (\Sigma_{s12} \phi_1 + \Sigma_{s22} \phi_2 + \Sigma_{s32} \phi_3 + \Sigma_{s42} \phi_4) \\ &+ \chi_2 \sum_{g'=1}^4 \nu'_g \Sigma_{fg'} \phi'_g + S_2^{ext} \end{aligned} \quad (17.5)$$

17.2.4 Removal cross section

We saw that the self in- and out-scatter terms both appear in Eq. 17.5 and hence we will define the following term:

$$\Sigma_{rg} \equiv \Sigma_{tg} - \Sigma_{sgg} = \Sigma_{ag} + \Sigma_{sg} - \Sigma_{sgg} = \Sigma_{ag} + \sum_{\substack{g'=1 \\ g' \neq g}}^G \Sigma_{sgg'} \quad (17.6)$$

Incorporating this into our multigroup equation results in:

$$\begin{aligned} \frac{1}{v_g} \frac{\partial \phi_g}{\partial t} &= \nabla \cdot D_g \nabla \phi_g - \Sigma_{rg} \phi_g \\ &+ \sum_{\substack{g'=1 \\ g' \neq g}}^G \Sigma_{sg'g} \phi_{g'} \\ &+ \chi_g \sum_{g'=1}^G \nu'_g \Sigma_{fg'} \phi'_g + S_g^{ext} \end{aligned} \quad (17.7)$$

17.3 Few-group constants

While the multigroup solution can obtain an accurate solution it requires to know the group-wise constants (i.e. cross sections). These cross sections, in reality, are continuous functions of energy. In various data libraries (e.g. JEFF, ENDF) there are hundreds of thousands (500,000) energy groups for nuclides such as U²³⁸.

In order to obtain the few-group cross-sections we need to collapse the $\sigma(E)$ with the flux $\phi(E)$.

A bit *problematic* since we are searching for this flux!

To find the few-group cross-sections we take a step back and use the energy continuous diffusion equation:

$$\begin{aligned} \frac{1}{v} \frac{\partial \phi}{\partial t} &= \nabla \cdot D \nabla \phi - \Sigma_a \phi - \Sigma_s \phi \\ &+ \int_0^\infty \Sigma_s(E' \rightarrow E) \phi(E') dE' \\ &+ \chi(E) \int_0^\infty \nu(E') \Sigma_f(E') \phi(E') dE' + S^{ext} \end{aligned} \quad (17.8)$$

Please note that $\phi = \phi(\mathbf{r}, E, t)$. We define the group flux as:

$$\phi_g(\mathbf{r}, t) \equiv \int_{E_g}^{E_{g-1}} \phi(\mathbf{r}, E, t) dE \quad (17.9)$$

We will now eliminate the energy variable in the energy-dependent diffusion by integrating Eq. 17.8 over the g^{th} energy group. These will generate rigorous definition of group-averaged cross sections.

$$\frac{\partial}{\partial t} \left(\int_{E_g}^{E_{g-1}} \frac{1}{v(E)} \phi(E) dE \right) = \frac{\partial}{\partial t} \left(\frac{\phi_g}{v_g} \right) \quad (17.10)$$

where the neutron speed characterizing group g is:

$$\frac{1}{v_g} \equiv \frac{1}{\phi_g} \int_{E_g}^{E_{g-1}} \frac{1}{v(E)} \phi(E) dE \quad (17.11)$$

The coefficients $\frac{1}{v_g}$ are determined as the flux weighted integral over the group energy range. We will see similar pattern as we go over the various terms.

$$\nabla \cdot \left(\int_{E_g}^{E_{g-1}} D(E) \nabla \phi(E) dE \right) \equiv \nabla \cdot (D_g \nabla \phi_g) \quad (17.12)$$

Hence:

$$D_g = \frac{\int_{E_g}^{E_{g-1}} D(E) \nabla \phi(E) dE}{\int_{E_g}^{E_{g-1}} \nabla \phi(E) dE} \quad (17.13)$$

Note that the weights are not simply fluxes but rather the $\nabla \phi$.

The absorption term is just:

$$\int_{E_g}^{E_{g-1}} \Sigma_a(E) \phi(E) dE \equiv \Sigma_{ag} \phi_g \quad (17.14)$$

i.e.

$$\Sigma_{ag} = \frac{1}{\phi_g} \int_{E_g}^{E_{g-1}} \Sigma_a(E) \phi(E) dE \quad (17.15)$$

The scattering removal term has the same form, however, the scattering (source) term requires more work:

$$\begin{aligned} \int_{E_g}^{E_{g-1}} \left(\int_0^\infty \Sigma_s(E' \rightarrow E) \phi(E') dE' \right) dE = \\ \int_{E_g}^{E_{g-1}} \left(\sum_{g'=1}^G \int_{E_{g'}}^{E_{g'-1}} \Sigma_s(E' \rightarrow E) \phi(E') dE' \right) dE = \\ \sum_{g'=1}^G \int_{E_g}^{E_{g-1}} \left(\int_{E_{g'}}^{E_{g'-1}} \Sigma_s(E' \rightarrow E) \phi(E') dE' \right) dE = \sum_{g'=1}^G \Sigma_{sg'g} \phi_{g'} \end{aligned} \quad (17.16)$$

We define the group-transfer cross section as:

$$\Sigma_{sg'g} \equiv \frac{1}{\phi_g} \int_{E_g}^{E_{g-1}} dE \int_{E_{g'}}^{E_{g'-1}} \Sigma_s(E' \rightarrow E) \phi(E') dE' \quad (17.17)$$

The fission term is:

$$\begin{aligned} \int_{E_g}^{E_{g-1}} \chi(E) \left(\int_0^\infty \nu(E'') \Sigma_f(E'') \phi(E'') dE'' \right) dE = \\ \chi_g \int_0^\infty \nu(E') \Sigma_f(E') \phi(E') dE' = \chi_g \sum_{g'=1}^G \int_{E_g}^{E_{g-1}} \nu(E') \Sigma_f(E') \phi(E') dE' \equiv \chi_g \sum_{g'=1}^G \nu_{g'} \Sigma_{fg'} \phi_{g'} \end{aligned} \quad (17.18)$$

where:

$$\begin{aligned} \nu_{g'} \Sigma_{fg'} &\equiv \frac{1}{\phi_g} \int_{E_g}^{E_{g-1}} \nu(E') \Sigma_f(E') \phi(E') dE' \\ \chi_g &\equiv \int_{E_g}^{E_{g-1}} \chi(E) dE \end{aligned} \quad (17.19)$$

While the multigroup solution is accurate it requires prior knowledge of $\phi(E)$ and this is just the function we are trying to find. However, we will aim to approximate the energy dependent flux within any group and then to obtain the average constants within the group. In a few lectures, we will develop the energy dependent flux in various energy regions.

Lecture 18: Solution of the multigroup diffusion

Lecturer: Dr. Dan Kotlyar

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18.1 Few-group constants

The group-wise constants would need to be calculated for each region of the reactor. The constants should be recalculated should any change occur (e.g. with and without control rods or simply accounting for the change in the isotopic composition). It must be emphasized that flux spectrum calculations must be performed to generate multigroup constants (e.g. D_g , Σ_{ag} , Σ_{sg} , etc.).

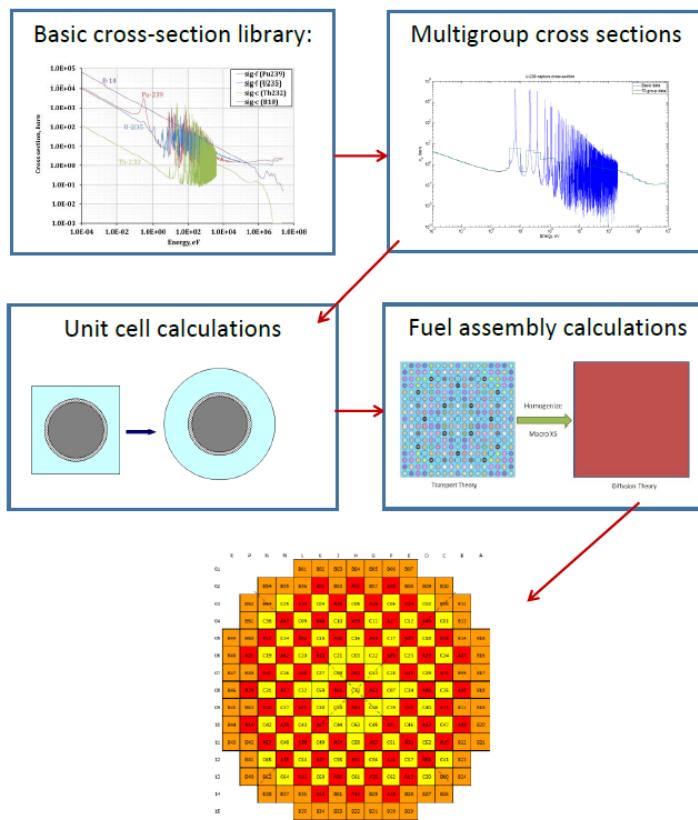


Figure 18.1: Energy condensation stages.

18.2 Approximations

Neutrons are born in the fast energy range and typically down-scattered towards the low energy regions. In other words, neutrons lose their energy. There is an exception, however, when thermal neutrons can gain energy. However, if we wisely choose the energy cutoffs, neutrons will not experience up-scattering. If the thermal cutoff is 1eV then it is reasonable to assume that there is no up-scattering.

Hence:

$$\Sigma_{sg'g} \cong 0 \quad \text{for } g' > g \quad (18.1)$$

The group in scattering term is:

$$\sum_{g'=1}^G \Sigma_{sg'g} \phi_{g'} = \sum_{g'=1}^{g-1} \Sigma_{sg'g} \phi_{g'} + \Sigma_{sgg} \phi_g \quad (18.2)$$

The $\Sigma_{sgg} \phi_g$ term can be used in the removal term. Sometimes we make the assumption that neutrons can scatter down only to the next lowest group (a.k.a directly coupled), therefore:

$$\sum_{g'=1}^G \Sigma_{sg'g} \phi_{g'} = \Sigma_{sg-1,g} \phi_{g-1} + \Sigma_{sgg} \phi_g \quad (18.3)$$

If we recall from the lecture of kinematics of scattering, the relation between the exit and initial energies is:

$$E' = \left(\frac{A-1}{A+1} \right)^2 E = \alpha E \quad (18.4)$$

If we maintain the group separation such that:

$$\frac{E_g}{E_{g-1}} = \alpha \quad (18.5)$$

then the exit energy cannot be below the next group one. However, for Hydrogen this is a bit of a problem since $\alpha = 0$, but if the ratio $E_g/E_{g-1} > 150$ then the error is less than 1% and direct coupling can be achieved.

18.3 Criticality calculations

We will focus on problems for which we omit the time-dependence and external sources, i.e. criticality calculations. In such a case the multigroup equation becomes:

$$-\nabla \cdot D_g \nabla \phi_g + \Sigma_{rg} \phi_g - \sum_{\substack{g'=1 \\ g' \neq g}}^G \Sigma_{sg'g} \phi_{g'} = \frac{1}{k} \chi_g \sum_{g'=1}^G \nu_{g'} \Sigma_{fg'} \phi_{g'} \quad (18.6)$$

The described above problem is very similar to the one-group equation, but now we need to sweep through the energy groups.

We rewrite Eq. 18.6 in a matrix form:

$$\underline{\underline{M}} \underline{\phi} = \frac{1}{k} \underline{F} \underline{\phi} \quad (18.7)$$

Where the matrix $\underline{\underline{M}}$ is:

$$\underline{\underline{M}} = \begin{bmatrix} -\nabla \cdot D_1 \nabla + \Sigma_{r1} & 0 & 0 & 0 & 0 & \dots \\ -\Sigma_{s12} & -\nabla \cdot D_2 \nabla + \Sigma_{r2} & 0 & 0 & 0 & \dots \\ -\Sigma_{s13} & -\Sigma_{s23} & -\nabla \cdot D_3 \nabla + \Sigma_{r3} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The matrix $\underline{\underline{M}}$ is a lower triangular matrix (why?).

$$\underline{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \end{bmatrix}$$

and the matrix \underline{F} is:

$$\underline{F} = \begin{bmatrix} \chi_1 \nu_1 \Sigma_{f1} & \chi_1 \nu_2 \Sigma_{f2} & \chi_1 \nu_3 \Sigma_{f3} & \dots \\ \chi_2 \nu_1 \Sigma_{f1} & \chi_2 \nu_2 \Sigma_{f2} & \chi_2 \nu_3 \Sigma_{f3} & \dots \\ \chi_3 \nu_1 \Sigma_{f1} & \chi_3 \nu_2 \Sigma_{f2} & \chi_3 \nu_3 \Sigma_{f3} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Note that in our notation we neglect the up-scattering and hence the upper triangular part of matrix $\underline{\underline{M}}$ is zero. Please also note that the span is over the energy groups (and not space).

In the case of a directly coupled (scattering-wise) matrix, the matrix $\underline{\underline{M}}$ has the following **bi-diagonal** form:

$$\underline{\underline{M}} = \begin{bmatrix} \textcolor{red}{x} & & & & \\ \textcolor{red}{x} & \textcolor{red}{x} & & & \\ x & \textcolor{red}{x} & \textcolor{red}{x} & & \\ x & x & \textcolor{red}{x} & \textcolor{red}{x} & \\ x & x & x & \textcolor{red}{x} & \textcolor{red}{x} \end{bmatrix}$$

If one chooses to subdivide the thermal group in such a way that **up-scattering** is allowed then:

$$\underline{\underline{M}} = \begin{bmatrix} \textcolor{red}{x} & & & & \\ \textcolor{red}{x} & \textcolor{red}{x} & & & \\ x & \textcolor{red}{x} & \textcolor{red}{x} & \textcolor{blue}{x} & \textcolor{blue}{x} \\ x & x & \textcolor{red}{x} & \textcolor{red}{x} & \textcolor{blue}{x} \\ x & x & x & \textcolor{red}{x} & \textcolor{red}{x} \end{bmatrix}$$

18.4 Two-group diffusion theory

The discretization into 2 groups is very common and is very illustrative. Instead of using continuous energy representation as shown in Fig. 18.2, we divide the energy to fast and thermal neutrons. The cutoff is sufficiently high and thus up-scattering can be ignored.

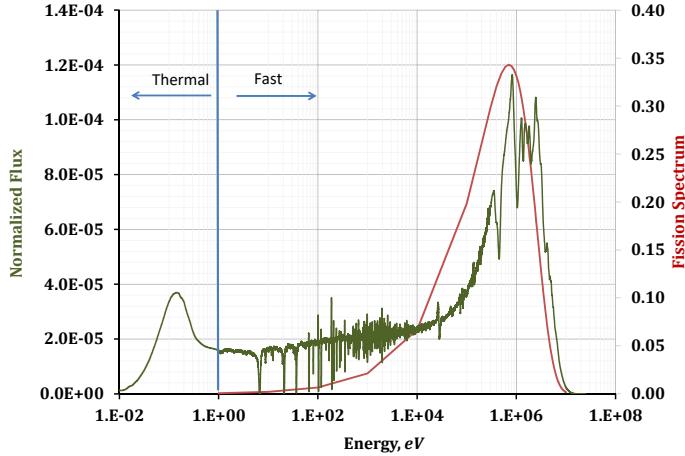


Figure 18.2: Energy dependent neutron flux and fission spectrum.

Lets us now focus on the fission spectrum: Neutrons are born in the fast group and hence:

$$\chi_1 = \int_{1\text{eV}}^{10\text{MeV}} \chi(E) dE = 1 \quad (18.8)$$

Whereas for the thermal group χ is:

$$\chi_1 = \int_{10^{-4}\text{eV}}^{1\text{eV}} \chi(E) dE = 0 \quad (18.9)$$

Therefore, the general fission source is:

$$S_g = \chi_g \sum_{g'=1}^G \nu_{g'} \Sigma_{fg'} \phi_{g'} \quad (18.10)$$

$$\begin{aligned} S_1 &= \nu_1 \Sigma_{f1} \phi_1 + \nu_2 \Sigma_{f2} \phi_2 && \text{Fast group-1} \\ S_2 &= 0 && \text{Thermal group-2} \end{aligned} \quad (18.11)$$

Since up-scattering is omitted:

$$\Sigma_{s21} = 0 \rightarrow \Sigma_{s2} = \Sigma_{s22} \quad (18.12)$$

Therefore:

$$\Sigma_{r2} = \Sigma_{t2} - \Sigma_{s22} = \Sigma_{a2} \quad (18.13)$$

In steady-state, the time-dependency is omitted and instead a k is used:

$$\begin{aligned} -\nabla \cdot D_1 \nabla \phi_1 + \Sigma_{r1} \phi_1 &= \frac{1}{k} [\nu_1 \Sigma_{f1} \phi_1 + \nu_2 \Sigma_{f2} \phi_2] \\ -\nabla \cdot D_2 \nabla \phi_2 + \Sigma_{a2} \phi_2 &= \Sigma_{s12} \phi_1 \end{aligned} \quad (18.14)$$

Notice how thermal neutrons are produced from fast through the scattering events and thermal are the source for fast neutrons.

18.5 Two-group criticality

Let us consider a case of a bare reactor and our aim is to develop a criticality condition. Our assumption here is that both fast and thermal fluxes can be characterized by the same spatial shape $\psi(\mathbf{r})$:

$$\begin{aligned} \phi_1(\mathbf{r}) &= \phi_1 \psi(\mathbf{r}) \\ \phi_2(\mathbf{r}) &= \phi_2 \psi(\mathbf{r}) \end{aligned} \quad (18.15)$$

Defining the buckling term as before and using the Helmholtz equation:

$$\nabla^2 \psi(\mathbf{r}) + B^2 \psi(\mathbf{r}) = 0 \quad (18.16)$$

we can obtain the following matrix form:

$$\begin{bmatrix} \left(D_1 B^2 + \Sigma_{r1} - \frac{\nu_1 \Sigma_{f1}}{k} \right) & -\frac{\nu_2 \Sigma_{f2}}{k} \\ -\Sigma_{s12} & D_2 B^2 + \Sigma_{a2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 0$$

In other words we have:

$$\underline{\underline{A}} \underline{\phi} = 0 \quad (18.17)$$

which has a non-trivial solution only if $|\underline{\underline{A}}|$. The following relation can be obtained:

$$k = \frac{\nu_1 \Sigma_{f1}}{\Sigma_{r1} + D_1 B^2} + \frac{\Sigma_{s12}}{\Sigma_{r1} + D_1 B^2} \frac{\nu_2 \Sigma_{f2}}{\Sigma_{a2} + D_2 B^2} = 1 \text{ for a critical core} \quad (18.18)$$

Term 1 is the fast fission contribution (this term can be lumped in the overall expression). The second term can be re-written in the following form:

$$k_2 = \frac{\Sigma_{s12}/\Sigma_{r1}}{1 + L_1^2 B^2} \frac{\nu_2 \Sigma_{f2}/\Sigma_{a2}}{1 + L_2^2 B^2} = 1 \text{ for a critical core} \quad (18.19)$$

The ratio Σ_{s12}/Σ_{r1} is simply p , the resonance escape probability. This ratio describes the neutrons that scatter down to group 2 over the number that leave group 1.

The ratio $\nu_2 \Sigma_{f2}/\Sigma_{a2}$ is ηf : number of neutrons born to the number of absorbed.

Hence:

$$k = \eta_2 f_2 p P_{NL1} P_{NL2} \quad (18.20)$$

$$\epsilon = 1 + \frac{k_1}{k_2} \quad (18.21)$$

Finally, we get:

$$k = k_1 + k_2 = \epsilon k_2 = \eta_2 f_2 p \epsilon P_{NL1} P_{NL2} \quad (18.22)$$

18.6 Solution of the multigroup equation

A bare slab (void, no reflector) of 40 cm with homogeneously distributed fuel is given. The group constants are given in the following table:

Table 18.1: Group cross sections.

Group constant	Group 1	Group 2	Group 3
χ	0.580	0.430	0.000
$\nu \Sigma_f \text{ (cm}^{-1}\text{)}$	0.010	0.020	0.180
$\Sigma_a \text{ (cm}^{-1}\text{)}$	0.005	0.030	0.120
$\Sigma_s^{g \rightarrow g+1} \text{ (cm}^{-1}\text{)}$	0.083	0.124	0.000
D	2.160	1.720	0.350

1. Write the time-dependent diffusion equations and the initial and boundary conditions.
2. Write the set of equations and the boundary conditions to obtain the criticality and evaluate its value.
3. Calculate the spectrum.

Comment: there are no up-scattering and the system is directly coupled (scattering only to adjacent groups).

Solution

1. The differential equation is:

$$\frac{1}{v_g} \frac{\partial \phi_g}{\partial t} - D_g \nabla^2 \phi_g + \Sigma_{tg} \phi_g = \sum_{g'=1}^G [\chi_g \nu \Sigma_{fg'} + \Sigma_{g'g}] \phi_{g'} \quad (18.23)$$

where ϕ is space, time and energy dependent $\phi(x, E, t)$

The boundary conditions are:

$$\phi(+\tilde{a}/2) = \phi(-\tilde{a}/2) = 0 \quad (18.24)$$

The initial conditions are:

$$\phi(x, E, t = 0) = \phi_0(x, E) \quad (18.25)$$

2. In order to find the multiplication factor, we first need to assume that energy and space are separable:

$$\phi(x, E) = R(x)\Psi(E) \quad (18.26)$$

The slab is homogeneous (space independent) and thus the cross sections are also space independent.

We can use the Helmholtz equation

$$\nabla^2 R(x) = -B_x^2 R(x) \quad (18.27)$$

The general spatial solution is:

$$R(x) = C_1 \sin(B_x x) + C_2 \cos(B_x x) \quad (18.28)$$

Applying the BC we can see that C_1 must be equal to zero and that $B_x = \frac{n\pi}{40}$. Since we are interested in the fundamental solution

$$B_x = \frac{\pi}{40} \quad (18.29)$$

The final solution will be:

$$\phi(x, E) = C \cdot R(x) \Psi(E) = C \cdot \cos(B_x x) \cdot \Psi(E) \quad (18.30)$$

How would we find the constant C ?

We used the Helmholtz relation through the buckling. If we now substitute $\nabla^2 R(x) = -B_x^2 R(x)$ into the general diffusion equation, the spatial dependence can be canceled and we will get:

$$[D_g B_x^2 + \Sigma_{tg}] \Psi_g = \sum_{g'=1}^G \left[\frac{1}{k} \chi_g \nu \Sigma_{fg'} + \Sigma_{g'g} \right] \Psi_{g'} \quad (18.31)$$

Let us now re-write into a matrix form:

$$\left(\underline{\underline{A}} - \frac{1}{k} \underline{\underline{F}} \right) \Psi = 0 \quad (18.32)$$

Where the matrix $\underline{\underline{A}}$ is:

$$\underline{\underline{A}} = \begin{bmatrix} D_1 B_x^2 + \Sigma_{a1} + \Sigma_{s1} & \textcolor{red}{0} (= \Sigma_s^{2 \rightarrow 1}) & \textcolor{red}{0} (= \Sigma_s^{3 \rightarrow 1}) \\ -\Sigma_s^{1 \rightarrow 2} & D_2 B_x^2 + \Sigma_{a2} + \Sigma_{s2} & \textcolor{red}{0} (= \Sigma_s^{3 \rightarrow 2}) \\ \textcolor{blue}{0} (= -\Sigma_s^{1 \rightarrow 3}) & -\Sigma_s^{2 \rightarrow 3} & D_3 B_x^2 + \Sigma_{a3} \end{bmatrix}$$

Comments:

- (a) The red terms are up-scattering terms
- (b) The blue term is a down scattering from group 1 (fast) to 3 (thermal). Since we have assumed directly coupled system, this term is zero.
- (c) Σ_{s1} does not include the self-group scattering term, i.e. $\Sigma_{s1} = \Sigma_s^{1 \rightarrow 2} + 0 (= \Sigma_s^{1 \rightarrow 3})$

The matrix $\underline{\underline{F}}$ has the following form:

$$\underline{\underline{F}} = \begin{bmatrix} \chi_1 \nu \Sigma_{f1} & \chi_1 \nu \Sigma_{f2} & \chi_1 \nu \Sigma_{f3} \\ \chi_2 \nu \Sigma_{f1} & \chi_2 \nu \Sigma_{f2} & \chi_2 \nu \Sigma_{f3} \\ 0 & 0 & 0 \end{bmatrix}$$

We will find the eigenvalue from the following equation:

$$\left| \underline{\underline{A}} - \frac{1}{k} \underline{\underline{F}} \right| = 0 \quad (18.33)$$

We need to find the buckling term, which is:

$$B_x = \frac{\pi}{40} = 6.1685 \times 10^{-3}, \text{ cm}^{-2} \quad (18.34)$$

After substituting the values for the various parameters, the matrices are:

$$\underline{\underline{A}} = \begin{bmatrix} 0.1013 & 0 & 0 \\ -0.083 & 0.1646 & 0 \\ 0 & -0.124 & 0.1222 \end{bmatrix}$$

and

$$\underline{\underline{F}} = \begin{bmatrix} 5.5 \times 10^{-3} & 0.0116 & 0.1044 \\ 4.3 \times 10^{-3} & 8.6 \times 10^{-3} & 0.072 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left| \underline{\underline{A}} - \frac{1}{k} \underline{\underline{F}} \right| = 0$$

$$-\frac{-50939008k^2 + 579916363k + 970920}{25 \times 10^{10}k^2} = 0 \quad (18.35)$$

$$k_1 = 1.1401$$

$$k_2 = -0.0017 \quad \text{not a physical solution}$$

3. Now that we know the eigenvalue we will substitute it to get the eigenvector, which is our spectrum: Ψ_1, Ψ_2, Ψ_3 .

$$\left(\underline{\underline{A}} - \frac{1}{1.1401} \underline{\underline{F}} \right) \underline{\Psi} = 0 \quad (18.36)$$

$$\begin{bmatrix} +0.0962\Psi_1 & -0.0102\Psi_2 & -0.0916\Psi_3 \\ -0.0868\Psi_1 & +0.1571\Psi_2 & -0.0632\Psi_3 \\ +0\Psi_1 & -0.1240\Psi_2 & +0.12222\Psi_3 \end{bmatrix}$$

Assume $\Psi_1 = 1$ we can then find: $\Psi_2 \cong 0.9332$ and $\Psi_3 \cong 0.9463$.

Lecture 19: Numerical solution of the multigroup problem

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19.1 Multigroup equations

This section will focus on solving the following multigroup equations. Assuming that there are no up-scattering, the equations take the following form:

$$\begin{aligned}
 -\nabla \cdot D_1 \nabla \phi_1 + \Sigma_{r1} \phi_1 &= \frac{1}{k} \chi_1 S \\
 -\nabla \cdot D_2 \nabla \phi_2 + \Sigma_{r2} \phi_2 &= \Sigma_{s12} \phi_1 + \frac{1}{k} \chi_2 S \\
 -\nabla \cdot D_3 \nabla \phi_3 + \Sigma_{r3} \phi_3 &= (\Sigma_{s13} \phi_1 + \Sigma_{s23} \phi_2) + \frac{1}{k} \chi_3 S \\
 &\vdots && \vdots && \vdots \\
 -\nabla \cdot D_G \nabla \phi_G + \Sigma_{rG} \phi_G &= (\Sigma_{s1G} \phi_1 + \dots + \Sigma_{sG-1,G} \phi_{G-1}) + \frac{1}{k} \chi_G S
 \end{aligned} \tag{19.1}$$

The source term S is the fission term, which is defined as:

$$S(\mathbf{r}) \equiv \sum_{g'=1}^G \nu_{g'} \Sigma_{fg'} \phi_{g'}(\mathbf{r}) \tag{19.2}$$

During our previous lecture we have assumed that the energy and space variables are separable:

$$\phi(\mathbf{r}, E) = \phi_g \Psi(\mathbf{r}) \tag{19.3}$$

In other words, we assume that the spatial dependence in each group is the same in each group. However, this is no longer the case in Eq. 19.1.

19.2 Solution procedure

1. We begin by guessing a fission source $S(\mathbf{r}) = S^{(0)}(\mathbf{r})$ and the criticality $k=k^{(0)}$
2. Next we calculate the flux in the first (i.e. fast) group:

$$-\nabla \cdot D_1 \nabla \phi_1^{(1)} + \Sigma_{r1} \phi_1^{(1)} = \frac{1}{k^{(0)}} \chi_1 S^{(0)}(\mathbf{r}) \quad (19.4)$$

We now have the flux spatial distribution in group 1.

3. We proceed to the next sequential group:

$$-\nabla \cdot D_2 \nabla \phi_2^{(1)} + \Sigma_{r2} \phi_2^{(1)} = \Sigma_{s12} \phi_1^{(1)} + \frac{1}{k^{(0)}} \chi_2 S^{(0)}(\mathbf{r}) \quad (19.5)$$

4. Since we have obtained $\phi_1^{(1)}$ from the previous stage, we can now solve this equation and all the other sequential group equations to find:

$$\phi_1^{(1)}(\mathbf{r}), \phi_2^{(1)}(\mathbf{r}), \dots, \phi_G^{(1)}(\mathbf{r}) \quad (19.6)$$

5. We can now recalculate the new fission source:

$$S^{(1)}(\mathbf{r}) \equiv \sum_{g'=1}^G \nu_{g'} \Sigma_{fg'} \phi_{g'}^{(1)}(\mathbf{r}) \quad (19.7)$$

6. The corresponding criticality is:

$$k^{(1)} = k^{(0)} \frac{\int S^{(1)}(\mathbf{r}) dr}{\int S^{(0)}(\mathbf{r}) dr} \quad (19.8)$$

We can perform the source iteration by solving down the multi-group equations in a descendant order (i.e. lower energies). Since we assumed that there are no up-scattering, the flux in the higher energy groups determines the source term in the lower energy groups.

It must be pointed out that such procedure will have to be modified to include up-scattering terms.

Mathematically, one can show that there will always be a maximum k that is real and positive with a corresponding eigenfunction that is unique and non-negative.

19.3 General iterative strategy

1. Initial guess of $\underline{S}^{(0)}$ and $k^{(0)}$
2. In order to solve the set of non-homogeneous equations:

$$\underline{\underline{M}} \underline{\phi}^{(n+1)} = \frac{1}{k^{(n)}} \underline{S}^{(n)} \quad (19.9)$$

- (a) Solve the diffusion equation for each energy group g :

$$\underline{\underline{M}}_g \underline{\phi}_g^{(n+1)} = \frac{1}{k^{(n)}} \underline{S}_g^{(n)} + \underline{\underline{R}}_{g-1} \underline{\phi}_{g-1}^{(n+1)} \quad (19.10)$$

In the stage we will obtain the S .

- (b) For multi-regional problems, we will need to use iterative techniques, in which the previous flux vector (spatial) estimate $\underline{\phi}_g^{(n-1)}$ as the first guess in solving Eq. 19.10. We need to iteratively solve Eq. 19.10 by using power iteration methods as we did in our previous lectures. In order to speed-up the solution, various relaxation techniques can be applied:

$$\underline{\phi}_g^{(n+1)} = (1 - w) \underline{\phi}_g^{(n)} + w \underline{\phi}_g^{(n+1)} \quad (19.11)$$

3. After obtaining the flux estimate $\underline{\phi}^{(n+1)}$, we can now estimate the next iteration-wise multiplication factor:

$$k^{(n+1)} = k^{(n)} \frac{\int S^{(n+1)}(\mathbf{r}) dr}{\int S^{(n)}(\mathbf{r}) dr} \quad (19.12)$$

where $S^{(1)}(\mathbf{r})$ is:

$$S^{(1)}(\mathbf{r}) = \sum_{g=1}^G \nu \Sigma_f^g(\mathbf{r}) \phi_f^g(\mathbf{r}) \quad (19.13)$$

4. We check the convergence by comparing either the

Integral quantity:

$$\left| \frac{k^{(n+1)} - k^{(n)}}{k^{(n+1)}} \right| < \epsilon_k \quad (19.14)$$

or *local quantities*:

$$\max \left| \frac{S_{gi}^{(n+1)} - S_{gi}^{(n)}}{S_{gi}^{(n+1)}} \right| < \epsilon_S \quad (19.15)$$

5. If the convergence is not achieved, the fission source iterations procedure continues. Again, if up-scattering can not be neglected then the procedure above must be modified by using iterative methods to solve for the fluxes in the thermal groups. Also, please note that extrapolation techniques for predicting $S^{(n+1)}$ are often used to make the procedure more efficient.

19.3.1 Matlab Psuedo-code

Below is a block of example code highlighting how one could solve Eq. 19.17. The block below should not be taken as a literal solution method, since some of the internal sections have been left out. However, it should serve as an instructive look into the process of solving this problem.

```

1 %% Set up iteration control
2 L = 100 % maximum allowable iterations
3 n = 0 % iteration index
4 epsK = 1E-10 % convergence criteria for eigenvalue
5 epsF = 1E-6 % convergence criteria for fission source
6 delK = 1.0 % initial error on K
7 delF = 1.0 % initial error on flux
8 %% Initial k and flux guess
9 kOld = 1.0
10 fluxOld = ones(G, M) % m number of spatial meshes
11 fluxNew = ones(G, M)
12 %% Build cross section matrices
13 nsf = xsmatrix('nuSigmaFission') % G x M
14 chi = xsmatrix('chi') % G x M
15 removal = xsmatrix('removal') % G x M
16 transport = xsmatrix('transport') % G x M
17 scatter = scatterMatrix() % G^2 x M
18 scatterSource = zeros(G, M)
19 %% begin the iteration procedure
20
21 while (n < L && (delK >= epsK && delF >= epsF))
22 %% Compute updated sources
23   oldSource = chi .* nsf .* fluxOld
24   for gPrime=1:G
25     scatterSource(gPrime, :) = updateScatter(gPrime, fluxOld, scatter)
26   end
27   source = oldSource / kOld + scatterSource
28
29 %% Solve the group flux equations
30   for gPrime=1:G
31     fluxNew(gPrime, :) = solveGroups(fluxOld, gPrime, kOld,
32         removal(gPrime, :), transport(gPrime, :), source)
33   end
34   fissSource = chi .* nsf .* fluxNew
35   kNew = kOld * fissSource / oldSource
36 %% Check for convergence
37   delK = abs(1-kOld/kNew)
38   delF = max((sum(fissSource) - oldSource))/fissSource
39
40 %% Update k and flux
41   kOld = kNew
42   fluxOld = fluxNew
43   n = n + 1
44
45 end

```

One could further break this algorithm up with outer iterations on the eigenvalue, and inner iterations on the group fluxes. The inner iterations would be bound by some iteration limit and flux convergence criteria, and the eigenvalue would remain constant through this process. If the flux iterations converged or exceeded the allowable iteration limit, the new eigenvalue is computed. If the eigenvalue converges or the outer iteration exceeds the iteration limit, the process is terminated.

19.4 Alternative Scattering Coupling Schemes

In the previous section, it was assumed that neutrons do not scatter into higher energy groups, i.e.

$$\Sigma_{s,g'g} = 0 \text{ for } g' > g \quad (19.16)$$

We are going to follow the same solution method where we update the fast-group flux first, then progress down the group solving for the flux in each group. Discounting upscatter, the general solution for the flux in group g at iteration n is

$$-\nabla \cdot D_g \nabla \phi_g^{(n+1)} + \Sigma_{r2} \phi_g^{(n+1)} = \sum_{g'=1}^{g-1} \Sigma_{sg'g} \phi_{g'}^{(n+1)} + \frac{1}{k^{(n)}} \chi_g S^{(n)}(\mathbf{r}) \quad (19.17)$$

For the case of $g = 1$, the summation over higher energy groups drops out and Eq. 19.4 is recovered.

If many groups are used, typically more than 10, reactor physicist often restrict the number of groups a neutron can scatter into. The rationale being it is unlikely a neutron will scatter from the MeV range down to the eV range. *Direct coupling* is the most extreme of such schemes, where a neutron can only scatter into adjacent groups, i.e.

$$\Sigma_{s,g'g} = 0 \text{ for } g-2 > g' > g+2 \quad (19.18)$$

Figure 19.1 demonstrates the contributions of in-scatter for the two nearest energy groups into group g .

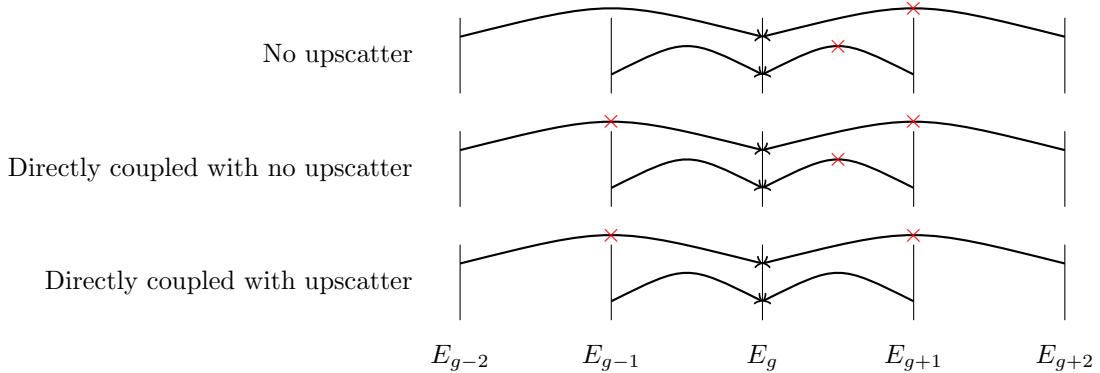


Figure 19.1: Various schemes for counting in-scatter contributions for group g

The task now becomes modifying Eq. 19.17 to account for up-scatter, while still following the same solution method. The right hand side of Eq. 19.17 involves using *known* group fluxes for higher energy groups to solve for the current *unknown* group flux. Therefore, at iteration n and group g , $\phi_{g'}^{(n+1)}$ for $g' > g$ and

$\phi_{g'}^{(n)}$ for $g' \leq g$ are known. This allows a more generalized form of Eq. 19.17 to be written accounting for full up- and down-scatter as

$$-\nabla \cdot D_g \nabla \phi_g^{(n+1)} + \Sigma_{rg} \phi_g^{(n+1)} = \sum_{g'=1}^{g-1} \Sigma_{sg'g} \phi_{g'}^{(n+1)} + \sum_{g'=g+1}^G \Sigma_{sg'g} \phi_{g'}^{(n)} + \frac{1}{k^{(n)}} \chi_g S^{(n)}(\mathbf{r}) \quad (19.19)$$

Lecture 20: Group constants

Lecturer: Dr. Dan Kotlyar

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20.1 Spectrum calculations

Multigroup theory is the primary tool in reactor analysis. However, the diffusion equation in each group requires prior knowledge of the energy averaged constants, i.e. Σ_{ag} , Σ_{sg} , D_g , $\nu\Sigma_{fg}$ and the group to group scattering cross sections $\Sigma_{sgg'}$.

In order to obtain these constants, we need to know the in-group spectrum $\phi(E)$.

Our goal is to develop approximations to obtain $\phi(E)$ that will be used to obtain the in-group cross-sections. It is obvious, of-course, that the more energy groups we use, even the approximate $\phi(E)$ will produce an accurate results, however the computational requirements will be more considerable.

The energy scale is usually divided into 3 main energy intervals:

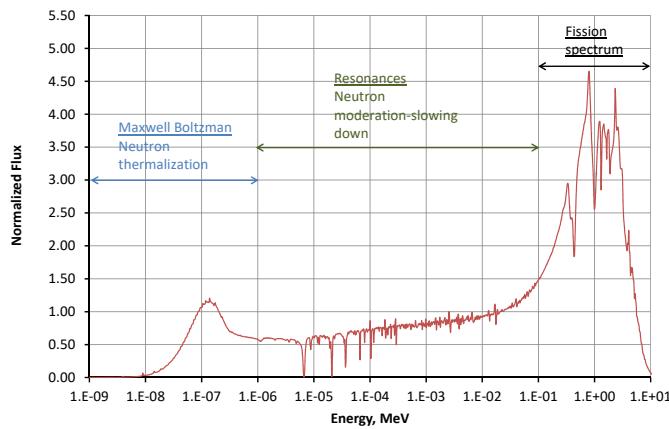


Figure 20.1: Energy sub-division of the neutron spectrum.

1. $E > 1\text{MeV}$, $\phi(E)$ is approximated by the fission spectrum $\chi(E)$. This energy region is characterized by both elastic and inelastic scatterings.
2. Intermediate energy range, $1\text{eV} < E < 1\text{MeV}$. Complicated energy range characterized by resonances which introduce strong absorption.

3. $E < 1\text{eV}$, characterized by the Maxwell Boltzman distribution. Up-scattering is significant.

In this lecture we will start to focus on obtaining the fast neutron spectra and hence the fast-group constants by applying the slowing down theory.

We can not obtain the $\phi(\mathbf{r}, E)$ directly since this is the quantity that we are searching for. Therefore, we will make some assumptions along the way. The first, which may seem like a drastic one, is neglecting the spatial dependence. In other words, we assume that slowing down happens in an infinite medium and hence $\phi(\mathbf{r}, E) \rightarrow \phi(E)$.

In the next lectures we will correct this assumption by re-introducing the spatial dependence through the buckling term B_g^2 . However, this single parameter can not take account for all the spatial effects. We will present various techniques to obtain average spatial and energy dependent cross sections. Since in reality we can have a very heterogeneous design as presented in Fig. 20.2.

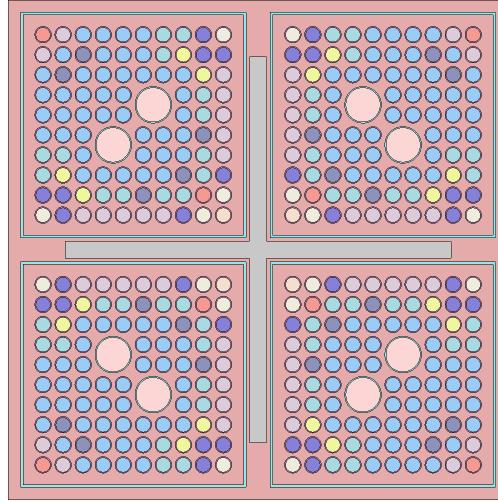


Figure 20.2: Example of a BWR fuel assembly with control rods, water holes and various fuel pin types with burnable poison.

To conclude, generation of few-group constants usually requires to consider the energy and spatial behavior.

20.2 Slowing down in an infinite medium

Let us define the fast group first. Our goal is to obtain $\phi(\mathbf{r}, E)$ for $E > 1\text{eV}$. Up-scattering is ignored. In this section we consider the slowing down process in an infinite and uniform medium. In other words, the spatial dependence does not exist and $\phi(\mathbf{r}, E) \rightarrow \phi(E)$.

The continuity and steady-state equation in an infinite medium becomes:

$$\Sigma_t(E)\phi(E) = \int_0^\infty \Sigma_s(E' \rightarrow E)\phi(E')dE' + S(E) \quad (20.1)$$

Eq. 20.1 is also known as the infinite-medium spectrum equation. Later on we will introduce techniques to include the spatial dependence but at this point it will be ignored.

There are some justification to why one might neglect spatial dependence and focus primarily on the energy dependency. Among these are:

1. In large LWR thermal reactors, leakage is relatively a weak effect compared with energy variation.
2. Heterogeneity effects are also usually second order effects. Unless the design is unconventional (i.e. crazy).

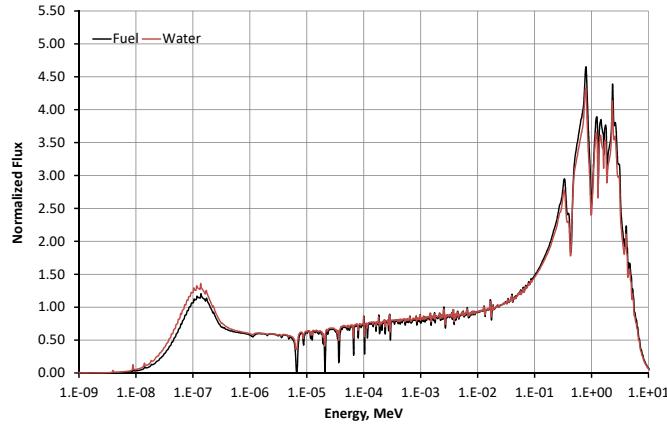


Figure 20.3: Spectrum in fuel and water.

For a certain energy range for which $E \gg KT$ (thermal energy of nuclei) up-scattering can be ignored. In our analysis, at this stage, we will assume dominance of the elastic scattering. According to our previous derivations:

$$\Sigma_s(E' \rightarrow E) = \frac{\Sigma_s(E')}{(1-\alpha)E'} \quad \alpha E' < E < E' \quad (20.2)$$

where, $\alpha = \left(\frac{A-1}{A+1}\right)^2$. We can therefore re-write the neutron slowing down equation to:

$$[\Sigma_s(E) + \Sigma_a(E)] \phi(E) = \int_E^\infty \Sigma_s(E' \rightarrow E) \phi(E') dE' + S(E) \quad (20.3)$$

Please pay attention that the lower boundary of the integral is E since up-scattering is neglected. We restricted our analysis to elastic scattering and hence limited the applicability of our analysis to neutrons with energy below 1 MeV in light moderators and roughly 0.1 MeV for heavier elements. In most reactors, however, the moderating material is H^1 , D^2 or C^{12} .

Inelastic scattering from heavy materials above 0.1 MeV energies can not be neglected. However, such analysis introduces too complicated models to handle analytically and therefore requires to adopt numerical tools.

20.3 Neutron moderation in Hydrogen

20.3.1 Slowing down in the absence of absorption

Hydrogen is used as the moderator in most of the reactors around the world. Absorption still exists in Hydrogen, however the scattering probability is much more pronounced. The absorption to scattering ratio is only slightly above 1%, as seen in Fig. 20.4. Perhaps, the issue with such approximation is that our medium includes heavy isotopes that introduce strong absorption (e.g. U²³⁸).

But even then the absorption in these nuclides takes place on a very narrow energy ranges. Outside these energies, the absorption is indeed negligible as shown in Fig. 20.5. Therefore, such approximation still yields correct quantitative behavior of $\phi(E)$.

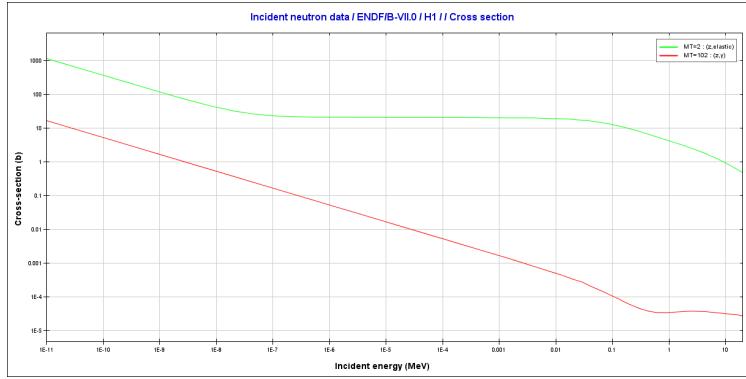


Figure 20.4: Hydrogen elastic scatter and capture cross sections vs. incident neutron energy

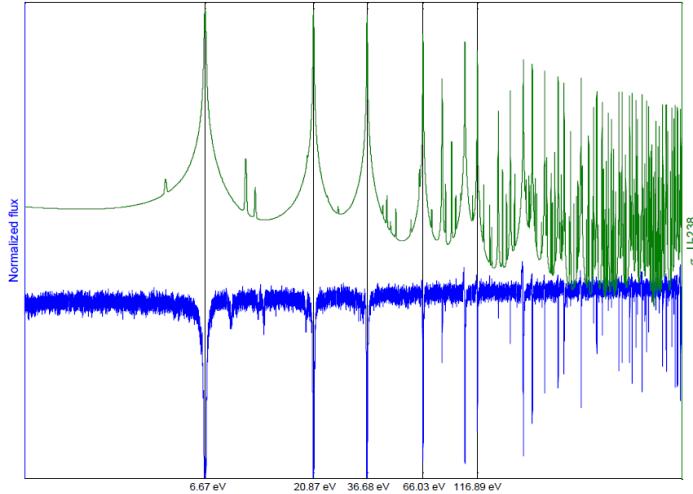


Figure 20.5: Example of weak and strong absorption as a function of energy.

For $H^1 \rightarrow A = 1 \rightarrow \alpha_H = \left(\frac{A-1}{A+1}\right)^2 = 0$: We neglect the absorption term since $\Sigma_a \ll \Sigma_s$, we can write the following relation:

$$\Sigma_s(E)\phi(E) = \int_E^\infty \frac{\Sigma_s(E')\phi(E')}{(1-\alpha)E'} dE' + S(E) \quad (20.4)$$

We will start our analysis by obtaining the neutron spectrum $\phi(E)$ from a mono-energetic source S_0 emitting neutrons of energy E_0 .

In addition, we will define the collision rate density:

$$F(E) = \Sigma_s(E)\phi(E) \quad (20.5)$$

The slowing down equation becomes:

$$F(E) = \int_E^\infty \frac{F(E')}{E'} dE' + S_0\delta(E - E_0) \quad \text{singular source} \quad (20.6)$$

The solution should also include a singular component. We will seek a solution of the form:

$$F(E) = F_c(E) + C\delta(E - E_0) \quad (20.7)$$

We need to determine the constant C and find the distribution $F_c(E)$ for the non-singular problem. Substituting Eq. 20.7 into the slowing down equation (20.6) gives:

$$F_c(E) + C\delta(E - E_0) = \int_E^{E_0} \frac{F_c(E')}{E'} dE' + \frac{C}{E_0} + S_0\delta(E - E_0) \quad (20.8)$$

The component $\frac{C}{E_0}$ is simply an integral of the delta function. Here we used the following property:

$$\begin{aligned} \int f(x)\delta(x - a)dx &= f(a) \\ C \int \frac{\delta(E - E_0)}{E} &= \frac{C}{E_0} \end{aligned} \quad (20.9)$$

The singular components must be equal and hence $C = S_0$.

$F_c(E)$ must satisfy the continues solution:

$$F_c(E) = \int_E^{E_0} \frac{F_c(E')}{E'} dE' + \frac{S_0}{E_0} \quad E \leq E_0 \quad (20.10)$$

One can notice that the source in Eq. 20.10 corresponds to neutrons making a single collision at the source energy E_0 . We can identify the singular component, i.e. $S_0\delta(E - E_0)$ as the uncollided source neutrons.

$$S(E) = S_0 P(E_0 \rightarrow E) = \frac{S_0}{(1 - \alpha)E_0} = \frac{S_0}{E_0} \quad (20.11)$$

Since $F_c(e)$ is the solution to Eq. 20.10, where the source is a stream of once-collided neutrons, $F_c(E)$ can be interpreted as the collision density due to neutrons that had at least one collision (where 'C' stands for collisions).

We will solve Eq. 20.10 by first differentiating (creating ODE) it:

$$\frac{dF_c(E)}{dE} = \frac{d}{dE} \left[\int_E^{E_0} \frac{F_c(E')}{E'} dE' \right] + \frac{d}{dE} \left(\frac{S_0}{E_0} \right) \quad (20.12)$$

Let us now use the following general definition:

$$\frac{d}{dx} \int_b^a F(x, y) dy = F(x, a) \frac{da}{dx} - F(x, b) \frac{db}{dx} + \int_b^a \frac{\partial F(x, y)}{\partial x} dy \quad (20.13)$$

In our case:

$$\frac{d}{dE} \left[\int_E^{E_0} \frac{F_c(E')}{E'} dE' \right] = \frac{F_c(E_0)}{E_0} \cancel{\frac{dE_0}{dE}} - \frac{F_c(E)}{E} \frac{dE}{dE} + \int_E^{E_0} \cancel{\frac{\partial}{\partial E}} \left[\frac{F(E')}{E'} \right] dE' = -\frac{F_c(E)}{E} \quad (20.14)$$

Therefore, we obtain the following equation:

$$\frac{dF_c(E)}{dE} = -\frac{1}{E} F_c(E) \quad (20.15)$$

The general solution of this equation is:

$$F_c(E) = \frac{B}{E} \quad (20.16)$$

We determine B by using Eq. 20.10 evaluated at $E = E_0$:

$$F_c(E_0) = \frac{S_0}{E_0} \Rightarrow B = S_0 \quad (20.17)$$

The final solution for the collision density is:

$$F(E) = \frac{S_0}{E} + S_0 \delta(E - E_0) \quad (20.18)$$

Finally, we obtain the expression for the flux:

$$\phi(E) = \frac{S_0}{\Sigma_s(E)E} + \frac{S_0}{\Sigma_s(E)} \delta(E - E_0) \quad (20.19)$$

To conclude, below the energy of the source E_0 , the flux will be proportional to:

$$\phi(E) \propto \frac{1}{E} \quad (20.20)$$

Very similar analysis can be used to obtain the solution for the energy dependent source $S(E)$ and the solution will be in the form:

$$\phi(E) = \frac{1}{\Sigma_s(E)E} \int_E^\infty S(E')dE' + \frac{S(E)}{\Sigma_s(E)} \quad (20.21)$$

Below the source energies, the flux is again inverse proportional to the energy.

Typically, the scattering cross-section for moderators is only weakly dependent on energy (in the slowing down range), i.e. $\Sigma_s(E) \approx \text{constant}$.

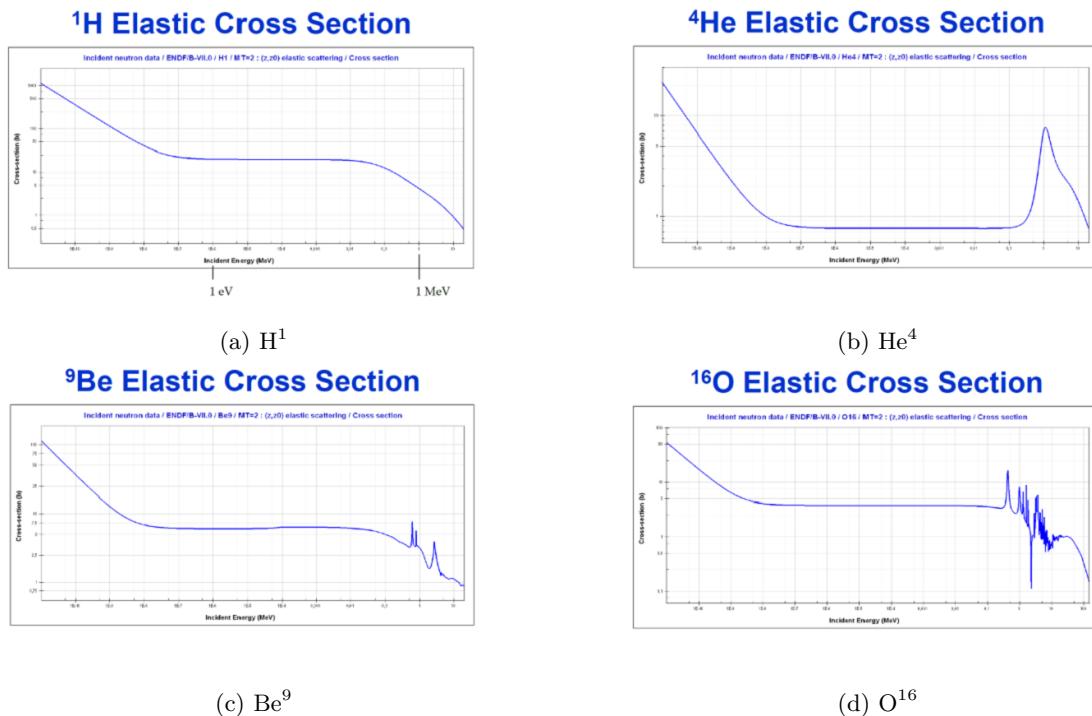


Figure 20.6: Elastic scattering cross sections

20.4 Slowing down density

This quantity describes the rate at which neutrons slow down past a given energy E.

$$q(\mathbf{r}, E) d^3r \equiv \text{number of neutrons slowing down past energy } E \text{ per sec in } d^3r \text{ about } \mathbf{r} \quad (20.22)$$

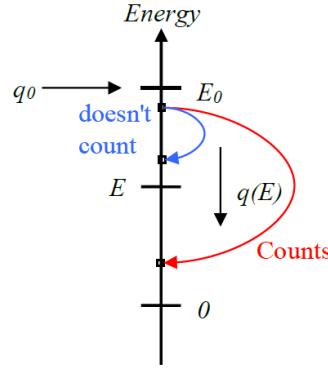


Figure 20.7: Slowing down density.

When the initial energy is $E' \in [E, \infty)$ and the final energy is $E'' \in [0, E]$, the slowing down density is:

$$q(\mathbf{r}, E) = \int_E^\infty dE' \int_0^E \Sigma_s(E' \rightarrow E'') \phi(\mathbf{r}, E') dE'' \quad (20.23)$$

The second term in the integral describes the rate at which neutrons that suffer collision at energy dE' about E' slow down past E.

Applying this definition to calculate $q(E)$ in an infinite, non-absorbing and hydrogen distributed moderator we get:

$$\begin{aligned} q(E) &= \int_0^E dE'' \int_E^\infty \frac{\Sigma_s(E') \phi(E')}{E'} dE' = E \int_E^\infty \frac{F_c(E') + S_0 \delta(E' - E_0)}{E'} dE' \\ &= E \left[\int_E^\infty \frac{F_c(E')}{E'} dE' + \frac{S_0}{E_0} \right] = E \cdot F_c(E) = E \cdot \frac{S_0}{E} = S_0 \end{aligned} \quad (20.24)$$

The scattering cross section as defined in Eq. 20.2 and collision density Eq. 20.10 were used to simplify Eq. 20.23. For this simple problem, the slowing down density is constant and equal to S_0 . This is trivial since in the absence of absorption and leakage, all source neutrons must eventually slow down below E.

Lecture 21: Slowing down in the presence of resonances

Lecturer: Dr. Dan Kotlyar

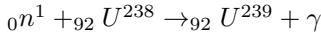
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21.1 Nuclear collision reactions

Collision reactions can be formulated in a very general manner:



where x and y are in- and out-going particles respectively; X and Y are the target and product nucleus respectively. An example of a reaction will be:



The general class of such reactions will be simply denoted as (n,γ) reactions. Nuclear reactions are generally accompanied by either absorption or emission of energy. In order to calculate the energy released or required for a given reaction, we can used the theory of relativity:

$$Q = [(m_x + M_X) - (m_y + M_Y)] c^2 \quad (21.2)$$

When $Q > 0$ then the reaction is exothermic, i.e. release of energy.

If, $Q < 0$ then the reaction is endothermic, i.e. the energy must be supplied (by increasing the kinetic energy) for the reaction to occur.

There are 2 aspects involved in the analysis of neutron cross sections:

1. kinematics of two-particle collisions (which we've covered in our first 2 lectures) and
2. the dynamics of nuclear reactions.

The latter is focused on the fundamental physical mechanisms involved in such collisions. The two main mechanisms of most interest for our applications are:

1. *Potential scattering:* neutron bounces off without actually penetrating the nuclear surface.
2. *Compound nucleus formation:* the neutron is absorbed by the nucleus to form a new nucleus of mass number $A + 1$ which then decays by various paths, e.g. γ , neutron or fission event.

21.1.1 Potential scattering

The neutron scatters elastically off the nuclear potential without ever penetrating the nucleus itself. This is very similar to what would occur between 2 billiard balls. Hence the cross section for such a reaction is essentially just the geometrical cross section of the nucleus. Potential scattering cross sections are almost energy independent between 1 eV up to the MeV range.

21.1.2 Compound nucleus

In such a reaction, the neutron is first absorbed by the nucleus ZX^A to create a new compound nucleus ZX^{A+1} . It would take a thermal neutron (10^5 cm/sec) 10^{-17} sec to cross the nucleus. However, the neutron-nuclear reaction (e.g. fission) occurs on a different time scale of about 10^{-14} . The neutron is absorbed by the nucleus and distributes its kinetic energy and the additional binding energy, before the compound nucleus decays. In other words, the compound nucleus lasts long enough to forget most of the characteristics of the incident neutron and hence the decay process is really independent of the original mode of formation.

21.2 The resonance

The formation of a compound nucleus actually corresponds to a so-called resonance reaction, in which the incident neutron energy matches one of the energy levels in the compound nucleus.

If not for the resonances, the energy dependence, of the microscopic cross sections of all isotopes in all energy ranges, would have been weak. A resonance introduces a very strong and rapid energy variation in the cross section.

If not for the resonances the total microscopic cross section of U^{238} , in the energy range of 1-1000 eV, would have been 10 barns; but throughout this range scattered are about 50 resonances, some of which with total cross section values reaching, at their peak, thousands of barns at operational temperatures of 300-1000 Kelvin as depicted in Fig. 21.1.

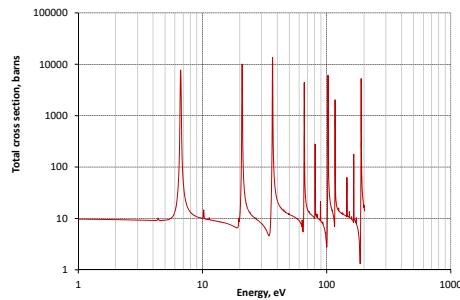


Figure 21.1: Resonances of U^{238} in the range of 1-200 eV.

Such peak values subside fast:

1. to 1/2 the peak value at about a 0.00002eV distance, on both sides, from the resonance center energy;
2. to 1/100 of the peak value at about 0.0002eV on both sides of the peak.

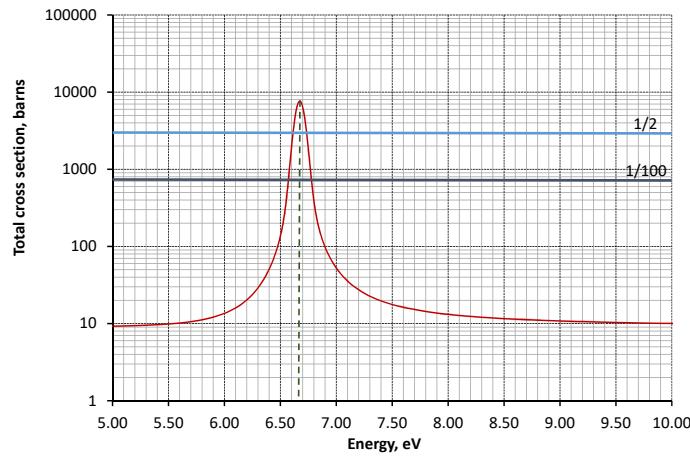


Figure 21.2: Total cross section of U^{238} around 6.67 eV.

A resonance is pictured and/or listed as a cross section hump about certain energy E_0 of the impinging neutron. The resonance, phenomenon and energy form, reflects features of the generation of a compound nucleus out of the neutron and the target it hits. On impact these latter two create a compound nucleus provided:

The kinetic energy of the neutron

$$+ \text{The separation energy (of a neutron from the compound, i.e. } (m_x + M_X - M_{Cn})c^2) = \\ = \text{The energy of a certain level of the compound}$$

At exactly neutron energy equal to the impinging E_0 such a reaction is very strong. Its strength diminishes as one moves away from E_0 , hence the peaked energy dependence of a resonance. For U^{239} , to take an example, the separation energy, mentioned above, is about 6 MeV; the important resonances of U^{238} have impinging- E_0 values of a few eV.

The second stage of a compound nucleus process is the decay of the compound nucleus:

1. The compound nucleus may continue to exist, falling from the extremely high excited level, at which it was generated, to its ground level by emitting a γ ray. If so happens, we have a resonance radiative capture, or simply a *resonance capture*, event.
2. If capture does not occur, the impinging neutron is either dispelled from forming a compound, or is ejected from the compound, and we have a resonance scattering event: no compound is permanently established; the target nucleus remains as is.

3. While capture is solely a resonant phenomenon, scattering has also, at all energies, the non-resonant, mostly energy constant, component $\sigma_{potential}$, or shortly σ_p .

21.2.1 Breit-Wigner formulation

Strict formulas for the resonance phenomenon are the subject matter of nuclear physics. However, we will use the most simplified ones. More specifically, we refer to the Breit-Wigner formulation.

The expressions below detail this formulation for zero absolute temperature, $T = 0$. The dependence of the resonance cross section on temperature will follow as a separate subject.

The capture cross section is:

$$\sigma_\gamma = \sigma_0 \left(\frac{E_o}{E} \right)^{0.5} \frac{\Gamma_\gamma}{\Gamma} \left[\frac{1}{1+x^2} \right] ; \text{barns} \quad (21.3)$$

The scattering cross section is:

$$\sigma_s = \sigma_0 \left(\frac{E_o}{E} \right)^{0.5} \frac{\Gamma_n}{\Gamma} \left[\frac{1}{1+\tau x + x^2} \right] + \sigma_p ; \text{barns} \quad (21.4)$$

where σ_0 is the value of the total cross section $\sigma_t(E)$ at the resonance energy E_0 .

$$\sigma_0 = \frac{2.6 \times 10^6}{E_0} \frac{\Gamma_n}{\Gamma} ; \text{barns} \quad (21.5)$$

The variable x is dimensionless and equals to:

$$x \equiv 2 \frac{E - E_0}{\Gamma} \quad (21.6)$$

where E is the energy in eV at which the cross section is evaluated and E_0 in eV is the resonance energy.

$$\begin{aligned} \Gamma_\gamma &\equiv \text{capture width of the compound level} \\ \Gamma_n &\equiv \text{neutron width of the compound level} \end{aligned} \quad (21.7)$$

$$\Gamma = \Gamma_\gamma + \Gamma_n \quad (21.8)$$

Lastly, the expression τ is:

$$\tau = 2 \sqrt{\sigma_0 \sigma_p \frac{\Gamma_n}{\Gamma}} \quad (21.9)$$

If there is also fission in the resonance, the expression in Eq. 21.8 has also a $[+ \Gamma_f]$ term.

The application of formulas Eqs. 21.3-21.4 is confined to the vicinity of E_0 . It is hard to define what vicinity here means. Suffice it to say that, for the interesting resonances of the heavy elements, vicinity is roughly a few eV.

In the most important of all resonances, namely the low-lying resonances ($E_0 < 100$ eV) of even-mass elements (U^{238} , for example), noticeable contributions to the absorption cross section die out with energy well before the contributions to the next E_0 (next resonance) start to appear.

A plot for the 6.67 eV resonance of U^{238} is shown in Fig. 21.3. Note the logarithmic scale of the ordinate; the peak cross section is very high. Note also the asymmetric form of the scattering cross section. There is a valley in this cross section below E_0 ; the maximum of the scattering cross section is slightly above E_0 .

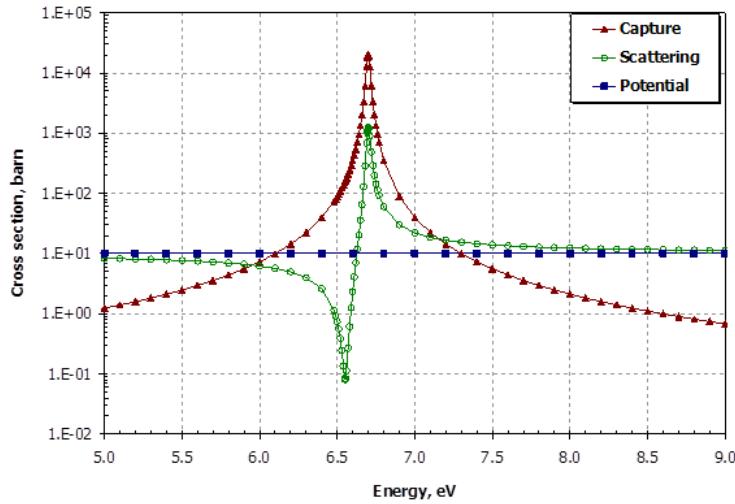


Figure 21.3: Capture and scattering in U^{238} , about 6.67 eV resonance using the Breit-Wigner formulation.

21.3 Resonance broadening by temperature

The Breit-Wigner resonance microscopic capture (n,γ) cross section formula for $T=0$, namely:

$$\sigma_\gamma = \sigma_0 \left(\frac{E_0}{E} \right)^{0.5} \frac{\Gamma_\gamma}{\Gamma} \left[\frac{1}{1 + \left(\frac{E-E_0}{\Gamma/2} \right)^2} \right] \quad (21.10)$$

is a formula for E , the Laboratory neutron kinetic energy, provided the target nucleus is at rest. The hit nuclei, however, are not at rest. In any non zero temperature they vibrate about a certain position. For given neutron and nucleus the energy E in should be E_{rel} , 'rel' standing for relative, *the neutron-nucleus relative energy*.

Assume that the targets (nuclei) velocity is distributed according to $N(\mathbf{V})$, namely

1. $N(\mathbf{V})dV$ is the number of nuclei whose velocities are in a dV vicinity of \mathbf{V} .

2. We are interested in a formula for the average $\bar{\sigma}_\gamma(E)$, such that the total capture rate in the laboratory is conserved.

This in mind, we write the following balance:

$$nvN\bar{\sigma}_\gamma(E) = \int dV N(\mathbf{V})nv_{rel}\sigma_\gamma(v_{rel}) \quad (21.11)$$

where

n	the neutron average density
v	the neutron average velocity
$N = \int dV N(\mathbf{V})$	the total nuclei density
v_{rel}	the neutron-nucleus relative velocity

Note that $\sigma_\gamma(v_{rel})$ is given by Eq. 21.10, simply by replacing v with v_{rel} , or E by E_{rel} .

- On the left of Eq. 21.11: nv is the neutron flux, ϕ , and $N\bar{\sigma}_\gamma(E)$ is the average macroscopic cross section. In other words, flux $\phi \times$ the macroscopic cross section $\Sigma_\gamma(E)$ equals to the total neutron capture rate.
- On the right of Eq. 21.11: the total capture rate is expressed as an integral over all the vibration velocities of the nuclei, namely over $nv_{rel}\sigma_\gamma(v_{rel})$. Since \mathbf{V} can in theory have any value, the integration extends from $-\infty$ to $+\infty$.

The integral in Eq. 21.11 includes the term v_{rel} multiplied by $\sigma_\gamma(v_{rel})$. As we have already seen from Fig. 21.2, $\sigma_\gamma(v_{rel})$ will undergo a sharp change as E_{rel} goes from $E_0 - \delta E$ to $E_0 + \delta E$, where $\delta E \ll E_0$ (almost negligible).

Note that this is true for most of the resonances. But there are few very low energy lying resonances, for which E_0 is very close to zero. For example the $E_0 = 0.3\text{eV}$ resonance of Pu^{239} .

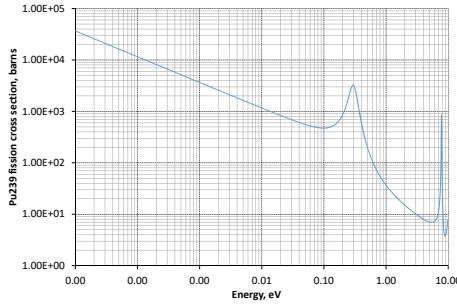


Figure 21.4: Low energy lying fission resonance of Pu^{239} , at about 0.3 eV.

As this left-to-right move takes place, E_{rel} itself, or v_{rel} changes only slightly.

For example, the significant resonance of U^{238} at $E_0 = 6.67\text{ eV}$ will drop by a factor of 2 from both sides of E_0 , while the change in E_{rel} will merely be 0.05%.

It is therefore customary to approximate:

$$v_{rel} = v \quad (21.12)$$

The expression in Eq. 21.11 is then simplified to:

$$\bar{\sigma}_\gamma(E) = \int dV \frac{N(\mathbf{V})}{N} \sigma_\gamma(v_{rel}) \quad (21.13)$$

A further simplification results from the realization that for the resonances considered, namely for E_0 of the order of a few eV or higher, the neutron velocity is of the order of 10,000 m/sec or higher.

The nuclei velocity distribution $\frac{N(\mathbf{V})}{N}$, being a Maxwell distribution, ascribes a high probability to small \mathbf{V} values and a very small probability to high \mathbf{V} values.

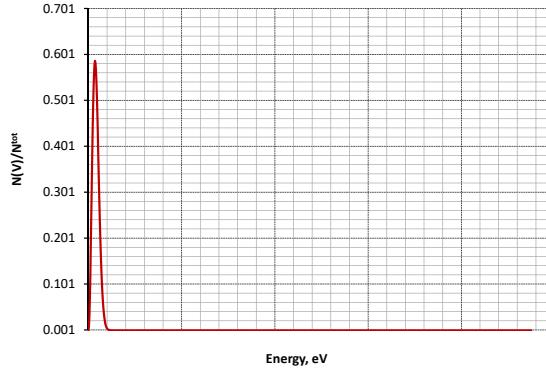


Figure 21.5: Maxwell-Boltzmann distribution.

Our argument is then that in most practical cases:

$$|\mathbf{V}| \ll |\mathbf{v}| \quad (21.14)$$

Proceeding with the current simplification we draw an example, in which $v = 10$ and $V = 1$ as shown in Fig. 21.6.

The neutron is at position A . The target at position B or at C . Since $\mathbf{v}_{rel} = \mathbf{v} - \mathbf{V}$, one obtains:

1. $\mathbf{v}_{rel} = 9$ in the $A \rightarrow C$ direction and
2. $\mathbf{v}_{rel} = 9.95$ in the $A \rightarrow B$ direction.

Split each vibration to its $A - C$ parallel and $A - B$ perpendicular modes, one sees that the perpendicular modes can be neglected. In other words, we neglect the vibration components which are perpendicular to the neutron-target line.

Equating this line with the z-axis, we only deal with the one-dimensional distribution of the velocities V_z . Eq. 21.13 is rephrased to:

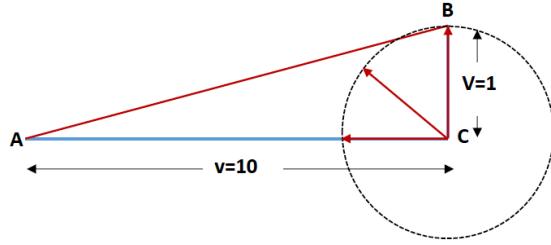


Figure 21.6: Neutron-target parallel and perpendicular vibrations.

$$\bar{\sigma}_\gamma(E) = \int dV_Z \frac{N(\mathbf{V}_Z)}{N} \sigma_\gamma(v_{rel}) \quad (21.15)$$

The normalized distribution in Eq. 21.15 is the one-dimensional Maxwell distribution:

$$\left[\frac{N(\mathbf{V}_Z)}{N} \right] dV_Z = \sqrt{\frac{M}{2\pi kT}} \exp \left[-\frac{MV_Z^2}{2kT} \right] dV_Z \quad (21.16)$$

where

M	nuclear mass, kg
T	temperature, Kelvin
k	Boltzmann constant = 0.86×10^{-4} eV/K
V_Z	nuclei velocity, m/s

What remains to be done are some mathematical manipulations and some approximations, which stem from the substitution of Eq. 21.16 and Eq. 21.10 in Eq. 21.13. The latter substitution reads:

$$\bar{\sigma}_\gamma(E) = \sigma_0 \left(\sqrt{\frac{E_0}{E_{rel}}} \right) \left(\frac{\Gamma_\gamma}{\Gamma} \right) \left(\sqrt{\frac{M}{2\pi kT}} \right) \int_{-\infty}^{+\infty} dV_Z \frac{\exp \left[-\frac{MV_Z^2}{2kT} \right]}{1 + \frac{4}{\Gamma^2}(E_{rel} - E_0)^2} \quad (21.17)$$

We can simplify the above equation by arguing that, outside the integral $E_{rel} \approx E$ and this turns into:

$$\bar{\sigma}_\gamma(E) = \sigma_0 \left(\sqrt{\frac{E_0}{E}} \right) \left(\frac{\Gamma_\gamma}{\Gamma} \right) \left(\sqrt{\frac{M}{2\pi kT}} \right) \int_{-\infty}^{+\infty} dV_Z \frac{\exp \left[-\frac{MV_Z^2}{2kT} \right]}{1 + \frac{4}{\Gamma^2}(E_{rel} - E_0)^2} \quad (21.18)$$

We draw again the neutron-target z axis. Left to our choice, v , the neutron velocity, will be positive when directed towards the right; V_Z , the target velocity, will be positive when directed towards the left, as depicted in Fig.



Figure 21.7: Conventions for the neutron-target line.

For simplicity we will omit the subscript Z .

The relative velocity is:

$$v_{rel} = v + V \quad (21.19)$$

Hence:

$$\begin{aligned} v_{rel}^2 &= v^2 + 2vV + V^2 \quad \text{negligible term} \\ E_{rel} &= \frac{mv_{rel}^2}{2} \cong \frac{mv^2}{2} + mvV = E + V\sqrt{2mE} \end{aligned} \quad (21.20)$$

In Eq. 21.20, m is the neutron mass in kg. Inverting this equation we obtain:

$$V = \frac{E_{rel} - E}{\sqrt{2mE}} \quad (21.21)$$

Next we define new variables x and y in the following manner:

$$\begin{aligned} E_{rel} - E &= \frac{\Gamma}{2} \left[\frac{2}{\Gamma} (E_{rel} - E_0) - \frac{2}{\Gamma} (E - E_0) \right] \\ &= \frac{\Gamma}{2} [y - x] \end{aligned} \quad (21.22)$$

This is simply an identity. Substitution of Eq. 21.22 into Eq. 21.21 results in:

$$V = \frac{\Gamma}{2} \frac{y - x}{\sqrt{2mE}} \quad (21.23)$$

Hence in the integral of 21.18:

$$dV = \frac{\Gamma}{2} \frac{1}{\sqrt{2mE}} dy \quad (21.24)$$

and

$$-\frac{MV^2}{2kT} = -\frac{M}{2kT} \frac{\Gamma^2}{4} \frac{(x-y)^2}{2mE} \quad (21.25)$$

We now substitute Eqs. 21.22–21.25 in the integral in Eq. 21.18. We also note that as V goes from $-\infty$ to $+\infty$ also y for any given x , goes from $-\infty$ to $+\infty$.

All these substitutions plus simplifications amount to:

$$\bar{\sigma}_\gamma(E) = \sigma_0 \left(\sqrt{\frac{E_0}{E}} \right) \left(\frac{\Gamma_\gamma}{\Gamma} \right) \left(\sqrt{\frac{M}{2\pi kT}} \right) \frac{\Gamma}{2} \int_{-\infty}^{+\infty} dy \frac{\exp \left[-\frac{M}{2kT} \frac{\Gamma^2}{4} \frac{(x-y)^2}{2mE} \right]}{\sqrt{2mE}(1+y^2)} \quad (21.26)$$

Most of the latter integral is retrieved as the energy E moves left to right, and close to E_0 . As discussed above, this statement is true in almost all practical cases. We therefore replace E by E_0 inside the integral. As the $\sqrt{2mE}$ term becomes $\sqrt{2mE_0}$ and is moved outside the integral, we get both inside and outside the integral, the quotient $[M/m]$. This is the ratio of target to neutron masses, very closely the target mass number A . In all:

$$\bar{\sigma}_\gamma(E) = \sigma_0 \left(\sqrt{\frac{E_0}{E}} \right) \left(\frac{\Gamma_\gamma}{\Gamma} \right) \left(\sqrt{\frac{A}{4E_0 kT}} \right) \frac{\Gamma}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} dy \frac{\exp \left[-\frac{A}{4E_0 kT} \frac{\Gamma^2}{4} (x-y)^2 \right]}{1+y^2} \quad (21.27)$$

Next define the *Doppler width* Δ :

$$\Delta = \sqrt{\frac{4E_0 kT}{A}} \quad (21.28)$$

'Doppler' is borrowed from sound theory; 'width' refers to the fact that Δ has the dimension of energy. A term most frequent in the current type of analysis is θ , the ratio of the natural width Γ to the Doppler width Δ :

$$\theta = \frac{\Gamma}{\Delta} \quad (21.29)$$

We will now rephrase Eq. 21.27 to a more compact form:

$$\bar{\sigma}_\gamma(E) = \sigma_0 \left(\sqrt{\frac{E_0}{E}} \right) \left(\frac{\Gamma_\gamma}{\Gamma} \right) \Psi(\theta, x) \quad (21.30)$$

where $\Psi(\theta, x)$ is defined as:

$$\Psi(\theta, x) = \frac{1}{2} \frac{\theta}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dy \frac{\exp\left[-\frac{\theta^2}{4}(x-y)^2\right]}{1+y^2} \quad (21.31)$$

Three of the most important properties of the line shape are:

$$\text{as } \theta \rightarrow \infty (T \rightarrow 0) \text{ then } \Psi(\theta, x) \rightarrow \frac{1}{1+x^2} \quad (21.32)$$

The limit $\theta \rightarrow \infty$ is approached as $T \rightarrow 0$. In other words, for the temperature of absolute zero, or motionless molecules, the line shape has the natural form of the Breit-Wigner $\frac{1}{1+x^2}$.

$$\int_{-\infty}^{+\infty} \Psi(\theta, x) dx = \pi \quad (21.33)$$

The above property means that for sufficiently spaced resonances, the area under a resonance is constant, irrespective of the temperature.

$$\Psi(\eta, 0) = \frac{\sqrt{\pi}}{2} \eta \exp(\eta^2) erfc(\eta) \quad (21.34)$$

where $\eta = \frac{\theta}{2}$ and

$$erfc(\eta) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \exp(-t^2) \quad (21.35)$$

The last exercise shows that a temperature increase (a decrease of η , or θ) lowers the peak of a resonance. Since, by Eq. 21.33, the area under the resonance remains unaffected by a temperature change, it follows that a temperature increase broadens the resonance.

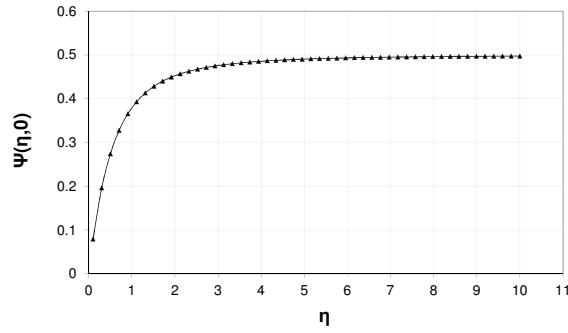


Figure 21.8: Doppler broadening with temperature.

This also means an increase of the absorption cross section with temperature. In other words, the criticality of the core drops with a temperature increase.

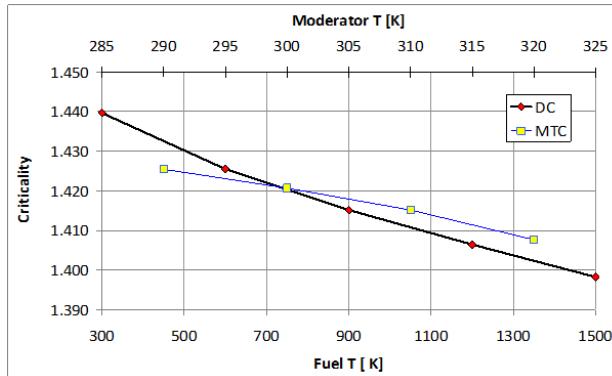


Figure 21.9: Criticality change due to Doppler broadening with temperature.

Lecture 22: Energy dependent flux

Lecturer: Dr. Dan Kotlyar

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22.1 Approximate solutions for $\phi(E)$

This range is usually defined between 0.05 MeV, where there are no fission neutrons, to the top of the thermal energy range, i.e. 1 eV.

We assume that in the this range the effect of the fission spectrum is negligible and that there are no up-scattering:

1. $\chi(E) = 0$
2. $\Sigma_s(E' \rightarrow E) = 0 \quad E' < E$

The discussion here will be limited only to elastic scattering that is isotropic in the CoM.

The most general equation is:

$$\Sigma_t(E)\phi(E) = \sum_j \int_E^{E/\alpha_j} \frac{\Sigma_s^j(E')\phi(E')dE'}{E'(1-\alpha_j)} \quad (22.1)$$

The highest energy E/α_j from which a neutron can scatter by colliding with isotope j and reach the energy E in a single collision.

In our previous lecture, we have developed the flux solution for a specific case, in which only Hydrogen moderated neutrons in a non-absorbing medium.

The flux was shown to be proportional to:

$$\phi(E) \propto \frac{1}{E} \quad (22.2)$$

In the current lecture we will make the derivations more general and we will include additional isotopes. Here we will also include a brief discussion on absorption. Primarily weak absorption will be discussed. In the following lecture a strong absorption introduced by the resonances will be presented in details.

22.1.1 Moderation by hydrogen only

If we have a mixture of j nuclides but only the hydrogen moderates neutrons then we can set all the $A_j \rightarrow \infty, \alpha_j \rightarrow 1$, i.e. no moderation. Where $\alpha_H = 0$.

As a result, the range of integration tends to be zero and therefore the term in the integral is simply:

$$\frac{\Sigma_s^j(E')\phi(E')}{E'} \rightarrow \frac{\Sigma_s^j(E)\phi(E)}{E} \quad (22.3)$$

and hence we can obtain the following relation:

$$\begin{aligned} \Sigma_t(E)\phi(E) &= \int_E^{E/\alpha_H} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'(1-\alpha_H)} + \sum_{j \neq H} \int_E^{E/\alpha_j} \frac{\Sigma_s^j(E')\phi(E')dE'}{E'(1-\alpha_j)} \\ &= \int_E^{E/\alpha_H} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'(1-\alpha_H)} + \sum_{j \neq H} \frac{1}{\alpha_j} \Sigma_s^j(E)\phi(E) \end{aligned} \quad (22.4)$$

The last term in Eq. 22.4 is a direct result of taking the term out of the integral and then integrating over the dE' , i.e. $\int_E^{E/\alpha_j} dE'$.

The total cross section is defined as:

$$\Sigma_t(E) = \Sigma_a(E) + \Sigma_s^H(E) + \sum_{j \neq H} \Sigma_s^j(E) \quad (22.5)$$

If we use the limit of $\alpha_j \rightarrow 1$ then the Eq. 22.4 becomes:

$$[\Sigma_a(E) + \Sigma_s^H(E)] \phi(E) = \int_E^{E/\alpha_H=\infty} \frac{\Sigma_s^H(E')\phi(E')dE'}{E'} \quad (22.6)$$

The heavy elements contribution from scattering disappeared from both sides. We turn the above equation into a differential equation by taking the derivative from both sides:

$$\frac{d}{dE} \{ [\Sigma_a(E) + \Sigma_s^H(E)] \phi(E) \} = -\frac{\Sigma_s^H(E)\phi(E)}{E} \quad (22.7)$$

Dividing both sides by $[\Sigma_a(E) + \Sigma_s^H(E)]$ and integrating from some lower energy E to some higher E_1 we get:

$$\phi(E) = \frac{[\Sigma_a(E_1) + \Sigma_s^H(E_1)] E_1 \phi(E_1)}{[\Sigma_a(E) + \Sigma_s^H(E)] E} \exp \left(- \int_E^{E_1} \frac{\Sigma_a(E')dE'}{[\Sigma_a(E') + \Sigma_s^H(E')] E'} \right) \quad (22.8)$$

Some observations:

$$\phi(E) = \frac{K}{[\Sigma_a(E) + \Sigma_s^H(E)] E} \exp \left(- \int_E^{E_1} \frac{\Sigma_a(E')dE'}{[\Sigma_a(E') + \Sigma_s^H(E')] E'} \right) \quad (22.9)$$

1. The overall $\frac{1}{E}$ behavior is modified by the presence of $\Sigma_a(E)$
2. The absorption term E is very large in the resonance and the term $\Sigma_a(E)$ in the denominator will cause the flux to dip in the energies of the resonance.
3. At energies just below the resonance, where $\Sigma_a(E)$ becomes small again, the flux will rise again.
4. On practical terms, the effect is very easy to comprehend. When colliding with hydrogen, the neutron can emerge with any energy $[0, E]$, hence there is a small chance they will end-up in the narrow energy range.

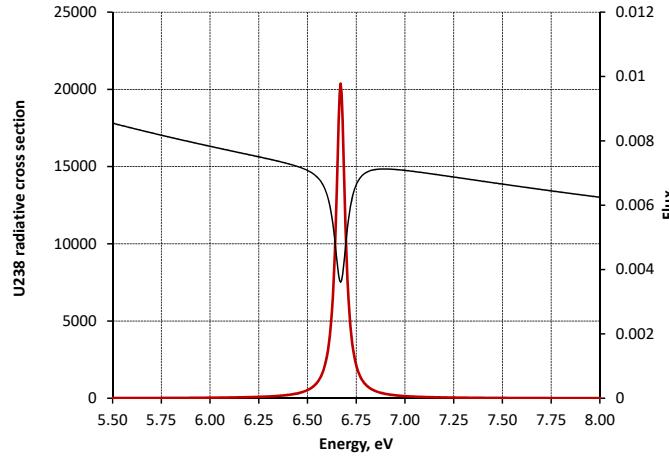


Figure 22.1: Flux in the presence of strong and narrow absorber.

Since the resonance removes neutrons from the system, there will be some lasting effect accounting by the exponential term:

$$\exp \left(- \int_E^{E_1} \frac{\Sigma_a(E') dE'}{[\Sigma_a(E') + \Sigma_s^H(E')] E'} \right) \quad (22.10)$$

Let us assume that $\Sigma_a \rightarrow \infty$ within the resonance between $[E, E + \Delta E]$, where $\Delta E \rightarrow 0$, then Eq. 22.10 simply becomes:

$$\exp \left(- \int_E^{E+\Delta E} \frac{dE'}{E'} \right) = e^{-\ln \frac{E+\Delta E}{E}} = \frac{E}{E + \Delta E} \quad (22.11)$$

If we now take the large U^{238} resonance at 6.67 eV with a width of 0.027 eV, then:

$$\frac{E}{E + \Delta E} = \frac{6.67}{6.697} \cong 0.96 \quad (22.12)$$

Which means that the attenuation represented by the exponential term is only 4%.

In the following figure, one can see the effect of increasing the U/H ratio. The total absorption increases and we can observe larger dips as this ratio is increased. Eventually, every neutron that will reach this energy region will be absorbed by U^{238} because there is so much of it.

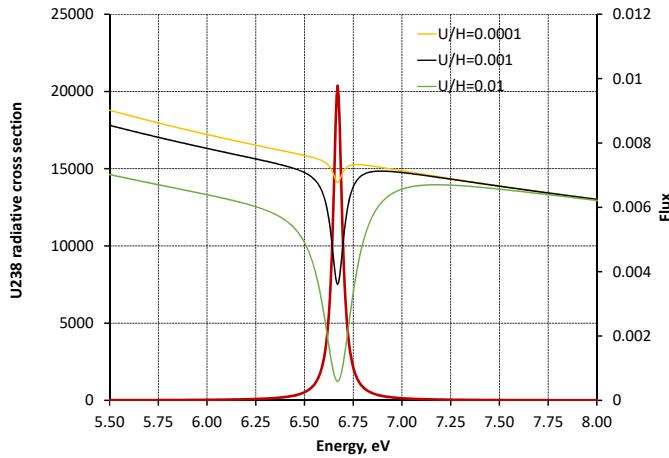


Figure 22.2: Illustration of the Energy self-shielding effect.

According to the exercise only 4% of neutrons slowing down will ever acquire energies in this particular 0.027 eV energy range. With $\Sigma_a \rightarrow \infty$ in this range, all of the neutrons would be absorbed.

This type of saturation is known as the *energy self-shielding*, it occurs when $\Sigma_a \gg \Sigma_s^H$. The following figure presents this effect in a very clear manner:

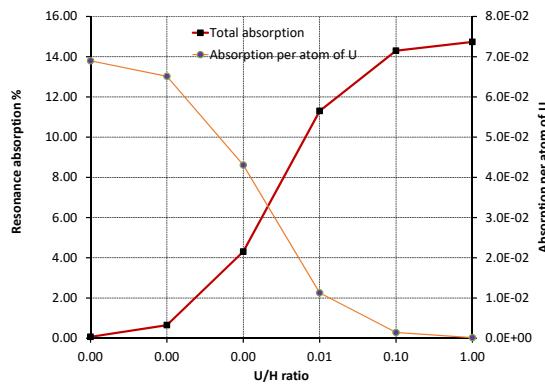


Figure 22.3: Absorption saturation - self shielding effect.

22.2 No absorption

In the previous lecture we have seen that when $\Sigma_a(E) = 0$ then:

$$\Sigma_s(E)\phi(E) = \sum_j \Sigma_s^j(E)\phi(E) = \sum_j \int_E^{E/\alpha_j} \frac{\Sigma_s^j(E')\phi(E')dE'}{E'(1-\alpha_j)} \quad (22.13)$$

When the only moderating material is hydrogen then:

$$\phi(E) = \frac{\Sigma_s^H(E_1)\phi(E_1)E_1}{\Sigma_s^H(E)E} \quad (22.14)$$

If any other moderator k is used only then the expression becomes:

$$\phi(E) = \frac{C_k}{\Sigma_s^k(E)E} \quad (22.15)$$

Where C_k is a constant that depends on the moderating isotope.

22.2.1 Slowing down density and flux relation

A reminder:

$q(\mathbf{r}, E, t)dV$ is the number of neutrons per sec in dV about \mathbf{r} whose energies drop below E at time t .

$$q(E) = \sum_j \int_E^{E/\alpha_j} dE' \int_{\alpha_j E}^E dE'' \frac{\Sigma_s^j(E')\phi(E')}{E'(1-\alpha_j)} \quad (22.16)$$

The second term of this integral is the rate at which neutrons scatter from dE' to some energy below E .

This expression is valid for any mixture and also valid even for cases where absorption exists.

For the case with no absorption: $\phi = \frac{C_k}{\Sigma_s^k(E)E}$. Restricting the sum to a single isotope we can *show that* (H.W.):

$$q_k = C_k \xi_k \quad (22.17)$$

Where ξ_k is the average loss in logarithm.

Hence:

$$\phi(E) = \frac{q_k}{E \xi_k \Sigma_s^k(E)} \quad (22.18)$$

This expression represents the flux in an infinite homogeneous medium with no absorption and with scattering isotropic in CoM.

We will not adopt a mathematically rigorous approach but we will rather claim that if the energy behavior of $\Sigma_s^j(E)$ is the same for all isotopes j then:

$$\phi(E) = \frac{q}{E\bar{\xi}\Sigma_s(E)} \quad (22.19)$$

where

$$\bar{\xi} = \frac{\sum_j \xi_j \Sigma_s^j(E)}{\sum_j \Sigma_s^j(E)} \quad (22.20)$$

and $q = \sum_j q_j$

22.3 Weak absorption

To get further insight we will consider cases with weak absorption. The decrease in $q(E)$ between energies E and $E - dE$ equals to the absorption rate $\sigma_a(E)\phi(E)dE$ in this range and hence:

$$q(E) - [q(E) - dq(E)] = \Sigma_a(E)\phi(E)dE \quad (22.21)$$

or

$$\frac{dq(E)}{dE} = \Sigma_a(E)\phi(E) \quad (22.22)$$

To determine $q(E)$ and $\phi(E)$ we need another equation relating them. Since the absorption is small we will adopt Eq. 22.19 and we substitute Eq. 22.19 into Eq. 22.22:

$$\frac{dq(E)}{dE} = \frac{\Sigma_a(E)q(E)}{E\bar{\xi}\Sigma_s(E)} \quad (22.23)$$

Dividing by $q(E)$ and integrating from E to E_1 yields:

$$q(E) = q(E_1) \exp \left(- \int_E^{E_1} \frac{\Sigma_a(E')dE'}{E'\bar{\xi}\Sigma_s(E')} \right) \quad (22.24)$$

22.4 Slowing down in the vicinity of narrow resonances

This time we want to relate Eq. 22.22 with a different equation that also includes the absorption term. We will use the neutron balance for the case of resonance absorption. We will write a mathematical statement of the fact that the total number of interactions per second (absorption plus scattering) in dE equals to the total rate at which neutrons appear in dE . That is:

$$[\Sigma_a(E) + \Sigma_s(E)] \phi(E) dE = q(E) \frac{d(\ln E)}{\bar{\xi}} = \frac{q(E) dE}{E \bar{\xi}} \quad (22.25)$$

1. $\bar{\xi}$ is the average total loss in logarithm of the neutron energy
2. $d(\ln E)$ is the logarithmic energy width of the energy range in question
3. The ratio $\frac{d(\ln E)}{\bar{\xi}}$ is approximately the fraction of those neutrons slowing down.

Combining Eq. 22.25 and 22.22 yields:

$$\frac{dq(E)}{dE} = \frac{\Sigma_a(E) q(E)}{E \bar{\xi} [\Sigma_a(E) + \Sigma_s(E)]} \quad (22.26)$$

The solution is:

$$q(E) = q(E_1) \exp \left(- \int_E^{E_1} \frac{\Sigma_a(E') dE'}{E' \bar{\xi} [\Sigma_a(E) + \Sigma_s(E)]} \right) \quad (22.27)$$

where E_1 is an arbitrary reference energy just above the resolved-resonance range and $q(E_1)$ is:

$$q(E_1) = \frac{[\Sigma_a(E_1) + \Sigma_s(E_1)] E_1 \bar{\xi} \phi(E_1)}{[\Sigma_a(E) + \Sigma_s(E)] E \bar{\xi}} \quad (22.28)$$

22.5 Example

Consider a solution of boric acid (H_3BO_3) in water such that the ratio of the molecules density H_3BO_3 (molecules/cc) of solution to the molecules density of H_2O of solution is 0.01. The absorption cross sections of the constituents of this mixture are all $1/v$, having values at 0.025 eV of:

1. $\sigma_a^{B(nat)} = 760 \text{ barns}$
2. $\sigma_a^H = 0.315 \text{ barns}$
3. $\sigma_a^O = 0 \text{ barns}$

Also in the range [1,10]eV, the scattering cross sections are constant:

1. $\sigma_s^B = 4 \text{ barns}$
2. $\sigma_s^H = 20 \text{ barns}$
3. $\sigma_s^O = 4.2 \text{ barns}$

Find the ratio of the slowing down density in this mixture at 1eV to that at 10eV under the following conditions:

1. Assuming hydrogen is the only moderator
2. Accounting for the moderation of all constituents
3. **H.W.** Using the narrow-resonance expression (Eq. 22.27) and the approximation $\frac{1}{1+\Sigma_a/\Sigma_s} \cong 1 - \frac{\Sigma_a}{\Sigma_s}$

22.6 Example solution

1. Hydrogen is the only moderator

Using Eq. 22.8

$$\frac{q(1eV)}{q(10eV)} = \exp \left(- \int_1^{10} \frac{\Sigma_a(E') dE'}{[\Sigma_a(E') + \Sigma_s^H(E')] E'} \right)$$

where the absorption term is:

$$\Sigma_a(E) = \sum_j N_j \sigma_{a,j}(E)$$

We know that the absorption cross sections all proportional to the inverse of the velocity:

$$\sigma_{a,j}(E) = \frac{c_j}{\sqrt{E}}$$

We also know their values at 0.025 eV and hence:

$$\begin{aligned} \sigma_{a,j}(0.025) &= \frac{c_j}{\sqrt{0.025}} \\ \sigma_{a,j}(E) &= \frac{\sqrt{0.025}}{\sqrt{E}} \sigma_{a,j}(0.025) \end{aligned}$$

The ratio between the atomic densities is given and equals to:

$$\frac{N(H_3BO_3)}{N(H_2O)} = 0.01$$

The total atomic density N_t of the solution is:

$$N(H_3BO_3) + N(H_2O) = N_t$$

We will divide this equation by $N(H_2O)$ to obtain:

$$\begin{aligned} 0.01 + 1 &= \frac{N_t}{N(H_2O)} \\ \rightarrow \frac{N(H_2O)}{N_t} &= 0.99 \quad \frac{N(H_3BO_3)}{N_t} = 0.01 \end{aligned}$$

Let us now focus on the expression in the integral:

$$\begin{aligned} \frac{\Sigma_a(E')}{[\Sigma_a(E') + \Sigma_s^H(E')] E'} &= \frac{0.01N_t(3\sigma_a^H + \sigma_a^B) + 0.99N_t(2\sigma_a^H)}{[0.01N_t(3\sigma_a^H + \sigma_a^B) + 0.99N_t(2\sigma_a^H) + (0.01N_t \cdot 3\sigma_s^H + 0.99N_t \cdot 2\sigma_s^H)] E'} \\ &= \frac{2.01\sigma_a^H + 0.01\sigma_a^B}{[2.01\sigma_a^H + 0.01\sigma_a^B + 2.01\sigma_s^H] E'} = \frac{2.01 \cdot 0.315 \sqrt{\frac{0.025}{E}} + 0.01 \cdot 760 \sqrt{\frac{0.025}{E}}}{[2.01 \cdot 0.315 \sqrt{\frac{0.025}{E}} + 0.01 \cdot 760 \sqrt{\frac{0.025}{E}} + 2.01\sigma_s^H] E'} \end{aligned}$$

Divide by $\frac{0.025}{E}$ we get:

$$= \frac{1}{E + 30.88E^{\frac{3}{2}}}$$

We will use the following identity:

$$\int \frac{dx}{x + cx^{\frac{3}{2}}} = -2\ln(1 + c\sqrt{x}) + \ln(x)$$

After substituting we get:

$$\frac{q(1eV)}{q(10eV)} = 0.9576$$

Therefore, roughly 4.345% of the neutrons were absorbed in the [1,10]eV energy range.

2. Hydrogen is not the only moderator

We derived the following expression:

$$q(E) = q(E_1) \exp \left(- \int_1^{10} \frac{\Sigma_a(E') dE'}{E' \bar{\xi} \Sigma_s(E')} \right)$$

We need to evaluate $\bar{\xi} \Sigma_s(E')$:

$$\bar{\xi} \Sigma_s = \xi^B \Sigma_s^B + \xi^H \Sigma_s^H + \xi^O \Sigma_s^O$$

We will calculate the logarithmic loss for each isotope according to:

$$\xi = 1 - \frac{(A-1)^2}{2A} \ln \left(\frac{A+1}{A-1} \right)$$

The result is: $\xi^B = 0.1873$, $\xi^O = 0.12$ and $\xi^H = 1$.

The final result will be: $\frac{q(1eV)}{q(10eV)} = 0.95718$.

Lecture 23: Slowing down in the presence of resonances

Lecturer: Dr. Dan Kotlyar

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23.1 The neutron flux in the vicinity of a resonance

As a reminder, the flux solution in the absence of absorbing resonances is given by 23.1 For various moderators, such as: Hydrogen and Carbon, the cross sections are constant $\Sigma_t(E) = \Sigma_s(E) = \Sigma_p(E)$ and $\Sigma_p(E) \cong const$, in the range 1eV -1000eV. Following are few σ_p values for the mentioned range of energies:

Table 23.1: σ_p for various moderators, in the 1-1000 eV.

Moderator	$\sigma_p, barns$
H ¹	20
H ²	3.6
Li ⁷	0.99
Be ⁹	6.2
C ¹²	4.7
O ¹⁶	4.0

In our previous lectures, we have shown that the solution of the slowing down equation in the absence of absorption is:

$$\phi(E) = \frac{q}{E\xi\Sigma_s(E)} \quad (23.1)$$

where:

$$\bar{\xi} = \frac{\sum_j \xi_j \Sigma_s^j(E)}{\sum_j \Sigma_s^j(E)} \quad (23.2)$$

Also,

$$\xi_j = 1 + \frac{\alpha_j}{1 - \alpha_j} \ln \alpha_j \quad (23.3)$$

An approximation for ξ can be derived for $A \gg 10$

$$\xi_j \approx \frac{2}{A + \frac{2}{3}} \quad (23.4)$$

It was shown that ξ_j is the lethargy increment in an isotropic elastic scattering on isotope j . The number q is the slowing down source because it is the constant number of neutrons slowed down, per second per cubic cm, below an energy E which is not in the range of a resonance.

23.2 The practical width

The current section will introduce the notion of a practical width. Despite the fact that the practical width is bigger than Γ , the natural width, the resonance practical bulk occupies only a few percents of the distance between resonances.

Table 23.2: Energy distance between resonances.

Absorber	D	Γ
Th^{232}	7	0.05
U^{238}	18	0.04
Pu^{240}	13	0.00

Under these circumstances we may assume that in-between resonances, and outside the practical widths, the flux energy-form is as if there is no absorption, namely an asymptotic (absorption-free) flux form:

$$\phi(E) = \frac{1}{E} \quad (23.5)$$

The above equation is valid for infinite and source-free medium.

Suppose our infinite mixture contains two elements: 'R' will denote a heavy resonant element, normally U, or Pu, or Th; 'M' will denote a light-weight moderator, normally H^1 , or H^2 , or C or O. The infinite-medium slowing down equation, describing the balance, between the neutrons slowed down into the differential energy dE about E and the neutrons disappearing from that range by colliding there, is:

$$[\Sigma_t^R(E) + \Sigma_p^M] \phi(E) = \int_E^{E/\alpha_M} \frac{\Sigma_p^M}{(1 - \alpha_M)E'} \phi(E') dE' + \int_E^{E/\alpha_R} \frac{\Sigma_s^R(E')}{(1 - \alpha_R)E'} \phi(E') dE' \quad (23.6)$$

On the left the rate is of the disappearance. On the right there are two rates of arrivals at E from higher energies. The first rate is by collisions in the moderator whose scattering cross section is $\Sigma_p^M = \Sigma_s^M$; the second rate is by scatterings in the absorber whose total scattering cross section $\Sigma_s^R(E)$ consists of the constant scattering cross section Σ_p^R and the resonant cross section $\Sigma_s^R(E)$. True understanding of the integrals of Eq. 23.6 depends on a preceding comprehension of the previous lecture, namely the one dealing with the kinematics of elastic scattering.

Prior to presenting approximate solutions to Eq. 23.6 one has to discuss the difference between narrow and wide resonances, in the context of the practical width of the resonance. The natural width of a resonance, Γ , is a well defined entity.

Γ_p , the practical width, is a subjective entity, denoting the importance, or impotence, of a resonance, relative to the background. Usually the practical width means the energy distance between two energy points, equally distanced to the left and right of the resonance energy E_0 , and so distanced that outside this distance the resonance is not important.

$$\begin{aligned} E_R - E_L &= 2(E_0 - E_L) = 2(E_R - E_0) = \Gamma_p \\ E_R - E_0 &= \frac{\Gamma_p}{2} \end{aligned} \quad (23.7)$$

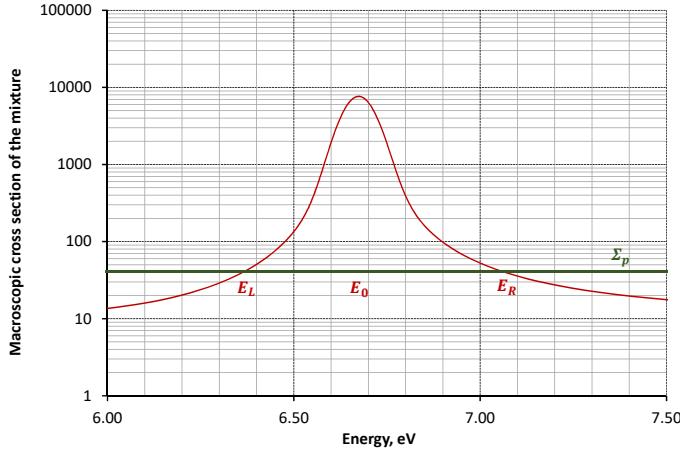


Figure 23.1: The practical width of a resonance.

Since E_R and E_L are so far quite arbitrary, so is Γ_p . We import more meaning to 'important' when we say that our zero-temperature resonance is 'important' in that energy range where the macroscopic cross section of the mixture is greater than the constant background cross section Σ_p , owing to the presence of the resonance.

To estimate the practical width we can use the Breit-Wigner formula to compare the cross section at energy E_R , at which point $\Sigma_p = \Sigma_{Res}$:

$$\begin{aligned} \frac{\Sigma_{peak}}{1 + \left(\frac{E_R - E_0}{\Gamma/2}\right)^2} &= \frac{\Sigma_{peak}}{1 + \left(\frac{\Gamma_p}{\Gamma}\right)^2} = \Sigma_p \\ \Gamma_p &= \Gamma \sqrt{\frac{\Sigma_{peak}}{\Sigma_p} - 1} \end{aligned} \quad (23.8)$$

where in books $\Sigma_{peak} = \Sigma_0$ the peak value. Since we expect an important resonance to obey $\frac{\Sigma_{peak}}{\Sigma_p} \gg 1$:

$$\Gamma_p \cong \Gamma \sqrt{\frac{\Sigma_{peak}}{\Sigma_p}} \quad (23.9)$$

If the number densities of materials 'M' and 'R' are, N_M and N_R respectively, then:

$$\Gamma_p \cong \Gamma \sqrt{\frac{\sigma_{peak}}{\sigma_p^R + \frac{N_M}{N_R} \sigma_p^M}} \quad (23.10)$$

Typically for low-lying resonances in U^{238} $\frac{\Sigma_{peak}}{\Sigma_p} \cong 10^3$, which is much larger than the actual width of the resonance.

H.W.

Consider 2 resonances:

1. U^{238} : $E_0 = 6.7$ eV ; $\Gamma_n = 0.00152$ eV ; $\Gamma = 0.0275$ eV
2. U^{235} : $E_0 = 7.1$ eV ; $\Gamma_n = 0.000125$ eV ; $\Gamma = 0.036$ eV

Use the following data: $\sigma_p^H = 20$ barn ; $\sigma_p^U = 10$ barn (for either U^{235} or U^{238}); $N_H = N_U$ and calculate the practical width for U^{235} and U^{238} .

In the next sections, the approximate solutions to Eq. 23.6 will discern between resonances by comparing their **practical widths** with the **slowing down range** of the moderating elements in the energy vicinity of the resonances discussed. This slowing down range is taken usually to be $E_0 - \alpha E_0 = (1 - \alpha)E_0$, assuming that the neutron entry-energy upon scattering was E_0 , the mid-resonance energy.

A short table of slowing down ranges, for a mid-resonance energy of $E_0 = 7$ eV, follows.

Table 23.3: The slowing down range.

Scattering element	α	$(1 - \alpha)E_0$
H^1	0.00	7.00
H^2	0.11	6.22
C^{12}	0.72	1.99
O^{16}	0.78	1.55
Na^{23}	0.84	1.12
U^{238}	0.98	0.12

The practical width is a mixture dependent entity. The slowing down ranges, compared with the practical ranges, as e.g. the example-practical-ranges evaluated above in the exercise, are large for the typical moderators H, D, C and O, and are small for resonance absorbers like U^{238} .

1. Experience and numerous examples show that, for the moderators, the slowing down range is always quite larger than the practical widths of respective resonances.
2. For the heavy-mass absorbers there is not a one-sided comparison: the slowing down range may be larger or smaller than, or nearly equal to, the practical widths.

23.3 Resonance approximations

The NR (Narrow Resonance), WR (Wide Resonance) and IR (Intermediate Resonance) are shorthand, referring to the measure of the practical width in comparison with the slowing down range of the absorber.

The practical width is always small when compared with the slowing down range of a moderator.

Returning now to Eq. 23.6:

$$[\Sigma_t^R(E) + \Sigma_p^M] \phi(E) = \int_E^{E/\alpha_M} \frac{\Sigma_p^M}{(1 - \alpha_M)E'} \phi(E') dE' + \int_E^{E/\alpha_R} \frac{\Sigma_s^R(E')}{(1 - \alpha_R)E'} \phi(E') dE'$$

The **first integral** on the RHS of Eq. 23.6 is for the moderator in our binary mixture. We will refer to it as the moderator integral.

In the major portion of the integration range of this integral the flux is not influenced by the presence of the resonance; we therefore approximate by assuming the asymptotic flux, throughout the entire integration range of this integral:

$$\int_E^{E/\alpha_M} \frac{\Sigma_p^M}{(1 - \alpha_M)E'} \phi(E') dE' \cong \int_E^{E/\alpha_M} \frac{\Sigma_p^M}{(1 - \alpha_M)E'} \frac{1}{E'} dE' = \frac{\Sigma_p^M}{E} \quad (23.11)$$

We therefore modify our original balance equation to the following form:

$$[\Sigma_t^R(E) + \Sigma_p^M] \phi(E) = \frac{\Sigma_p^M}{E} + \int_E^{E/\alpha_R} \frac{\Sigma_s^R(E')}{(1 - \alpha_R)E'} \phi(E') dE' \quad (23.12)$$

23.3.1 Narrow resonance approximation

Turning next to the integral in Eq. 23.12, the absorber integral, we make an extreme approximation:

The practical width is extremely small, compared with the slowing down range,

$$\langle \Delta E_R \rangle = \frac{1 - \alpha_R}{2} E_0 \gg \Gamma_p \quad (23.13)$$

hence compared with the integration range; therefore the flux in the entire range is the asymptotic flux (as was assumed for the moderator scattering integral) and the cross section in the entire range is the constant, i.e. the potential scattering of the absorber, i.e. Σ_p^R .

In mathematical terms we assume:

$$\begin{aligned} \Sigma_s^R(E') &= \Sigma_p^R \\ \phi(E') &= \frac{1}{E'} \end{aligned} \quad (23.14)$$

The absorber scattering integral becomes:

$$\int_E^{E/\alpha_R} \frac{\Sigma_s^R(E')}{(1 - \alpha_R)E'} \phi(E') dE' \cong \int_E^{E/\alpha_R} \frac{\Sigma_p^R}{(1 - \alpha_R)E'} \frac{1}{E'} dE' = \frac{\Sigma_p^R}{E} \quad (23.15)$$

The flux solution becomes:

$$\phi_{NR}(E) = \frac{\Sigma_p^M + \Sigma_p^R}{\Sigma_t(E)} \frac{1}{E} = \frac{\Sigma_p^M + \Sigma_p^R}{\Sigma_t^R(E) + \Sigma_p^M} \frac{1}{E} \quad (23.16)$$

23.3.2 Wide resonance approximation

Here, we make another extreme approximation that the slowing down range, hence the integration range, is infinitely small in comparison with the practical width:

$$<\Delta E_R> = \frac{1 - \alpha_R}{2} E_0 << \Gamma_p \quad (23.17)$$

In other words, we assume that the absorber is infinitely massive such that neutrons suffer no energy loss in a collision with the absorber. We take the limit as $A_R \rightarrow \infty$ or $\alpha_R \rightarrow 1$ and hence the following scattering integral for the absorber becomes:

$$\lim_{\alpha \rightarrow 1} \int_E^{E/\alpha_R} \frac{\Sigma_s^R(E')}{(1 - \alpha_R)E'} \phi(E') dE' \rightarrow \Sigma_s^R(E) \phi(E) \lim_{\alpha \rightarrow 1} \frac{1}{E} \int_E^{E/\alpha_R} \frac{dE'}{1 - \alpha_R} = \Sigma_s^R(E) \phi(E) \quad (23.18)$$

Hence, Eq. 23.12 turns into:

$$\Sigma_t(E) \phi(E) = \frac{\Sigma_p^M}{E} + \Sigma_s^R(E) \phi(E) \quad (23.19)$$

or

$$\phi_{WR}(E) = \frac{\Sigma_p^M}{\Sigma_t(E) - \Sigma_s^R(E)} \frac{1}{E} = \frac{\Sigma_p^M}{\Sigma_t^R(E) - \Sigma_s^R(E) + \Sigma_p^M} \frac{1}{E} \quad (23.20)$$

23.3.3 Intermediate resonance approximation

Eqs. 23.20 and 23.16 are solutions of the flux equation 23.12, in the resonance range, for two opposite-extreme situations.

Actual cases are somewhere between these two extremes. In doing so we are re-mindful that

$$\Sigma_t(E) = \Sigma_p^M + \Sigma_a^R(E) + \Sigma_s^R(E) \quad (23.21)$$

The NR and WR approximations are as follows:

$$\begin{aligned} E \left[\Sigma_p^M + \Sigma_a^R(E) + \Sigma_s^R(E) \right] \phi(E) &= (\Sigma_p^M + \Sigma_p^R) \quad \text{NR} \\ E \left[\Sigma_p^M + \Sigma_a^R(E) \right] \phi(E) &= (\Sigma_p^M) \quad \text{WR} \end{aligned} \quad (23.22)$$

One may bridge these extremes by introducing the parameter λ :

$$E \left[\Sigma_p^M + \Sigma_a^R(E) + \lambda \Sigma_s^R(E) \right] \phi(E) = (\Sigma_p^M + \lambda \Sigma_p^R) \quad (23.23)$$

or:

$$\phi(E) = \frac{\Sigma_p^M + \lambda \Sigma_p^R}{\Sigma_p^M + \Sigma_a^R(E) + \lambda \Sigma_s^R(E)} \frac{1}{E} \quad (23.24)$$

When λ is 0 or 1 one obtains, respectively, the WR or NR solution. Actual λ values depend on the makeup of the binary mixture and on the resonance in question and are not simple to compute. Some modern libraries bring recommendations for λ of elements in a LWR environment.

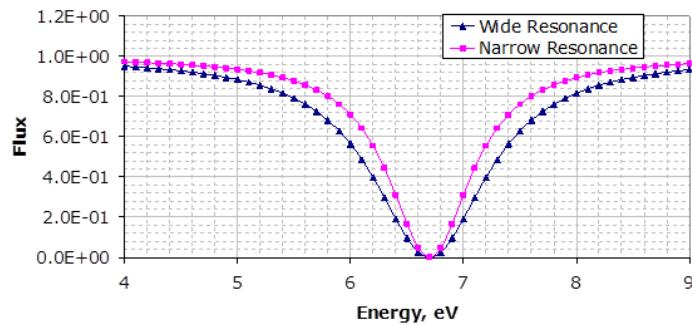


Figure 23.2: Zero-temperature flux in the vicinity of the 6.67 eV resonance of U^{238} . The number density ratio is: U:H is 0.5:1.0

23.4 Resonance Integral

Recall the definition of the multigroup absorption cross section for a given group g:

$$\Sigma_{ag} = \frac{\int_{E_g}^{E_{g-1}} \Sigma_a(E) \phi(E) dE}{\int_{E_g}^{E_{g-1}} \phi(E) dE} \quad (23.25)$$

Since the resonances are quite narrow, over most of the group energy range the flux behaves asymptotically as $\phi(E) = 1/E$ such that:

$$\Sigma_{ag} = \frac{N_R \sum_{i \in g} I_i}{\ln \left(\frac{E_{g-1}}{E_g} \right)} \quad (23.26)$$

I_i is the total absorption rate per nuclei by a resonance absorber and is known as the resonance integral:

$$I = \int \sigma_\gamma(E) \phi(E) dE \quad (23.27)$$

Hence we can generate multigroup constants for absorption directly by using resonance integrals. In order to do so we need to use the approximate flux shapes.

23.5 Example

Consider a binary mixture of Hydrogen and absorber ($A \gg 10$). The absorber has a step-function shaped resonance for capture and scattering (as presented in the figure). The capture and scattering cross sections within the resonance are constants. Calculate the average macroscopic cross section in the energy range $[E_1, E_2]$

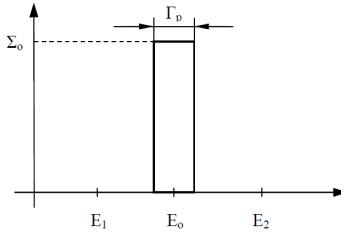


Figure 23.3: Step-wise resonance at E_0

1. Use the WR approximation.
2. Use the NR approximation.
3. Which of the approximation is more suitable for this problem.

Table 23.4: Relevant data.

Σ_a^R capture within the resonance, $1/\text{cm}^2$	200
Σ_s^R scatter within the resonance, $1/\text{cm}^2$	20
Σ_p^M potential scattering (H), $1/\text{cm}^2$	0.5
Σ_p^R potential scattering (Fuel), $1/\text{cm}^2$	0.2
Γ_p practical width, eV	1
E_1 , eV	10
E_2 , eV	20
E_0 , eV	15
ξ , fuel	0.01
ξ , Hydrogen	?

Lecture 24: Resonance Integral and Background dependence

Lecturer: Dr. Dan Kotlyar

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24.1 Resonance absorption

As neutrons slow down from fission energies they have a great chance to be absorbed in many resonances. Typically, fertile materials (e.g. U²³⁸) will have dominant resonances.

The resonance absorption is very important:

1. Determines the multiplication factor of the reactor.
2. Reactor control characteristics. More specifically, the kinetics behaviour of the reactor, i.e. Doppler broadening.
3. Burnup and breeding performance (e.g. U²³⁸ → Pu²³⁹). More absorption in fertile → more breeding.

24.2 Resonance Integral

Assume that we have a general case of neutrons slowing down in an infinite, homogeneous mixture of absorber and moderator.

Let us now assume that the neutrons are produced by a source of strength S_0 at energies E_0 .

Then, the total absorption rate while slowing down from E_0 to E is:

$$\int_E^{E_0} \Sigma_a(E') \phi(E') dE' \quad (24.1)$$

we can now find the resonance escape probability:

$$p(E) = \frac{S_0 - \text{neutron abs. rate}}{S_0} = 1 - \frac{1}{S_0} \int_E^{E_0} \Sigma_a(E') \phi(E') dE' \quad (24.2)$$

We assume that absorption takes place in a number of well-separated resonances lying below the source E_0 .

The flux just above the resonance at energy E_i will be the asymptotic one:

$$\phi(E) = \phi_{asy}(E) = \frac{S_{eff}^{(i)}}{\bar{\xi} \Sigma_s} \frac{1}{E} \quad (24.3)$$

where $S_{eff}^{(i)}$ is the original source strength S_0 reduced by resonance escape probabilities for all higher energies:

$$S_{eff}^{(i)} = S_0 \prod_j p_j \quad E_j > E_i \quad (24.4)$$

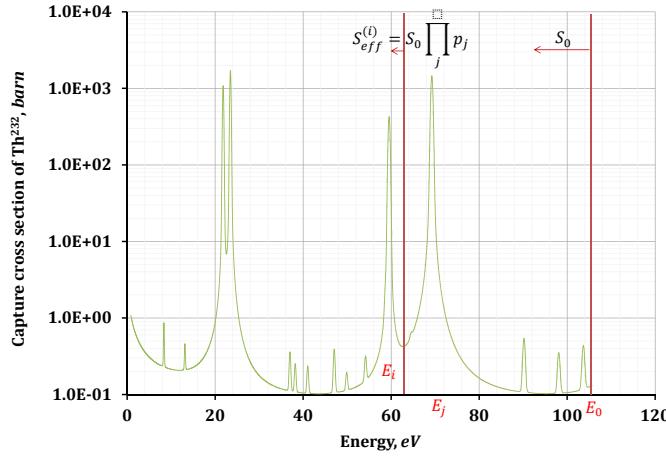


Figure 24.1: Effective source

The resonance escape probability for this particular resonance becomes just:

$$p_i = 1 - \frac{1}{S_{eff}^{(i)}} \int_{E_i} \Sigma_a^R(E) \phi(E) dE \quad (24.5)$$

One can show that:

$$p_i \cong \exp \left[-\frac{I_i}{\xi \Sigma_s} \right] \quad (24.6)$$

where:

$$I_i = \int_{E_i} \Sigma_a^R(E) \phi(E) dE \quad (24.7)$$

or

$$\frac{I_i}{N_R} = \int_{E_i} \sigma_a^R(E) \phi(E) dE \quad (24.8)$$

In order to compute the total escape probability, we only need to add the resonance integrals for each resonance:

$$p_t = \prod_i p_i = \exp \left[-\frac{1}{\xi \Sigma_s} \sum_i I_i \right] \quad (24.9)$$

The evaluation of the resonance integral I is the subject of this section. I should also mention that different literature sources denote various notation of the resonance integral quantity (e.g. I or RI).

24.3 $J(\theta, \beta)$ Integral

Here we will use the WR and NR flux approximations to calculate the resonance integral.

$$I^{NR} = \int_{E_0} \left[\frac{\Sigma_p^M + \Sigma_p^R}{\Sigma_t(E)} \right] \sigma_a^R(E) \frac{dE}{E} \quad (24.10)$$

$$I^{WR} = \int_{E_0} \left[\frac{\Sigma_p^M}{\Sigma_t(E) - \Sigma_s^R(E)} \right] \sigma_a^R(E) \frac{dE}{E} \quad (24.11)$$

Reminder: NR and WR differ in the manner that scattering is treated from the absorber/resonance.

Typically, the density of the absorber is much lower than that of a moderator:

$$N_R \ll N_M \rightarrow \Sigma_p^M \gg \Sigma_p^R.$$

If the absorber concentration is infinitely small compared to the moderator then:

$$\Sigma_t(E) \cong \Sigma_p^M.$$

And the resonance integral becomes an infinite dilution one:

$$I^\infty = \int_{E_0} \sigma_a^R(E) \frac{dE}{E} \quad (24.12)$$

If one compares I^∞ to either I^{NR} or I^{WR} :

$$\frac{I^\infty}{I^{NR}} > 1 \quad \text{or} \quad \frac{I^\infty}{I^{WR}} > 1 \quad (24.13)$$

This is due to the **energy self-shielding**.

If we examine the ϕ_{NR} or ϕ_{WR} expressions in the resonance, we note that flux decreases as $\Sigma_a^R(E)$ increases \rightarrow *flux depression happens*. This flux depression lowers the effective absorption in the resonance and hence I/N_R is similarly decreased.

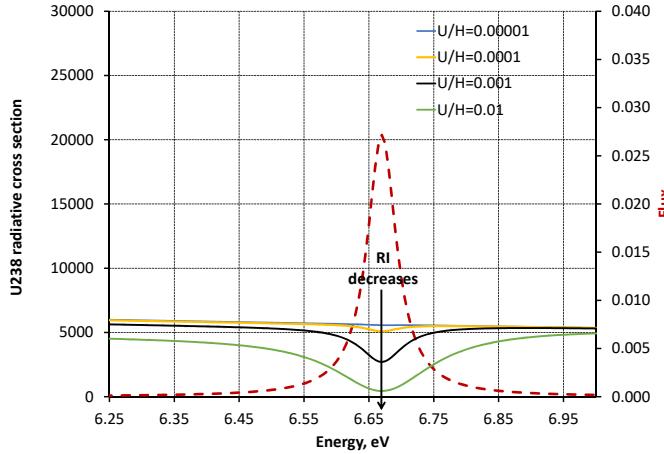


Figure 24.2: The effect of flux depression on the effective absorption in the resonance

24.3.1 Doppler-broadened Breit-Wigner

We will use the Breit-Wigner formulation to describe the capture and scattering cross sections. For simplicity it is customary to neglect the asymmetric (i.e. interference) scattering term.

$$\begin{aligned}\sigma_s^R(E) &\cong \sigma_0 \frac{\Gamma_n}{\Gamma} \Psi(\theta, x) + \sigma_p^R \\ \sigma_a^R(E) &\cong \sigma_0 \frac{\Gamma_\gamma}{\Gamma} \Psi(\theta, x)\end{aligned}\tag{24.14}$$

Then inserting these into our expressions for the resonance integrals yields:

$$\begin{aligned}I^{NR} &= \frac{(\Sigma_p^M + \Sigma_p^R)\Gamma_\gamma}{N_R E_0} J(\theta, \beta) \\ \beta &= \frac{\Sigma_p^M + \Sigma_p^R}{N_R \sigma_0}\end{aligned}\tag{24.15}$$

we can also re-write I^{NR} as:

$$I^{NR} = \frac{\beta \sigma_0 \Gamma_\gamma}{E_0} J(\theta, \beta)\tag{24.16}$$

$$I^{WR} = \frac{\sum_p^M \Gamma}{N_R E_0} J(\theta, \beta') \quad (24.17)$$

$$\beta' = \frac{\sum_p^M \Gamma_\gamma}{N_R \sigma_0 \Gamma}$$

where:

$$J(\theta, \beta) = \int_0^\infty \frac{\Psi(\theta, x)}{\beta + \Psi(\theta, x)} dx \quad (24.18)$$

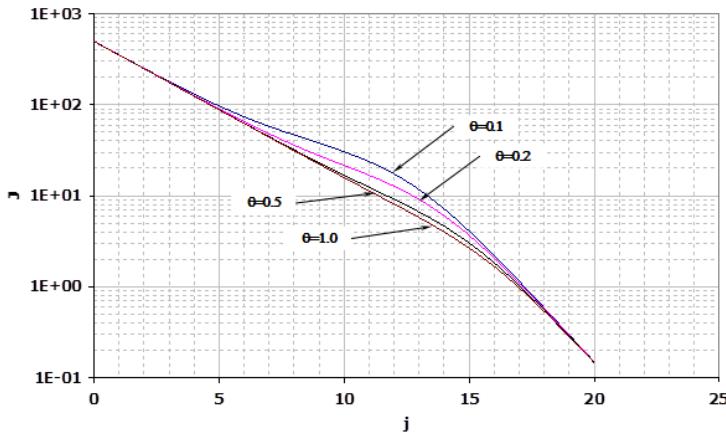


Figure 24.3: The $J(\theta, \beta)$ integral, $\beta = 2^j \times 10^{-5}$

Some observations:

1. J and hence the resonance integral increases as θ decreases.
2. Since T is inverse proportional to θ , it means that: $T \uparrow \rightarrow J \uparrow \rightarrow RI \uparrow$.
3. This is the Doppler broadening, which is the basis for the negative reactivity temperature feedback in LWRs.
4. We find that for most applications ($0.1 < \theta < 1$), unless $1.6 \times 10^{-4} < \beta < 2.6$ there is no temperature dependence of the resonance integral.

24.4 Dilution, Self-shielding, Background

As $T \rightarrow 0, \theta \rightarrow \infty \Rightarrow \Psi(\theta, x) \rightarrow \frac{1}{1+x^2}$.

H.W. prove that:

$$J(\infty, \beta) = \frac{\pi}{2} \sqrt{\frac{1}{\beta(1+\beta)}} \quad (24.19)$$

For $T = 0$ it follows that:

$$I(\infty, \beta) = \frac{\Gamma_\gamma \Sigma_0}{E_0} \frac{\pi}{2} \sqrt{\frac{\beta}{1+\beta}} \quad (24.20)$$

In general, the group cross section for any temperature is roughly similar in its β behavior to the β behavior of the resonance integral I_g . Fig. 24.4 is an illustration of the typical dependence of the resonance integral on β .

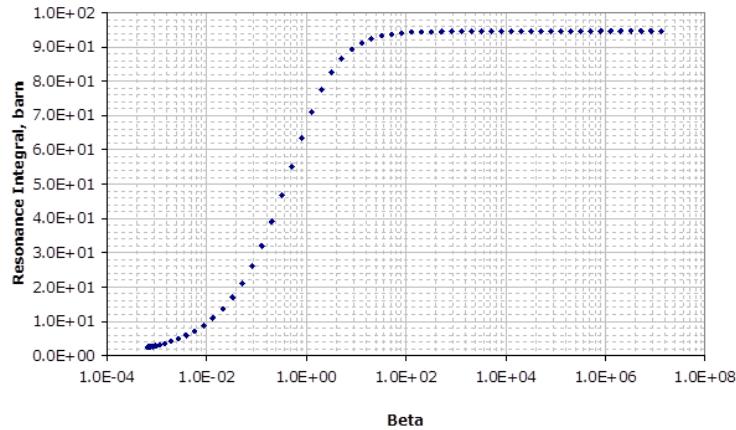


Figure 24.4: The Resonance Integral as a function of the mixture β parameter.

In the present context of a mixture of an absorber and of a moderator, the parameter β may be written in several useful ways. To make matters simpler, but not much different, we will adopt the Narrow Resonance approximation.

$$\beta = \frac{\Sigma_p}{\Sigma_0} \quad ; \quad \Sigma_0 = \Sigma_0^R \quad ; \quad \Sigma_p = \Sigma_p^R + \Sigma_p^M \quad (24.21)$$

Equally, by the above defined meaning:

$$\beta = \frac{N_M \sigma_p^M + N_R \sigma_p^R}{N_R \sigma_0^R} \quad (24.22)$$

N_M and N_R are the densities of the moderator and absorber and 'p' stands for the potential microscopic cross sections.

We now define the background cross section:

$$\sigma_{background} = \sigma_p^R + \frac{N_M}{N_R} \sigma_p^M$$

$$\beta = \frac{\sigma_{background}}{\sigma_0} \quad (24.23)$$

Sometimes $\sigma_{background}$ is named the Bondarenko cross section. Bondarenko was among the first to utilize β in such form (Eq. 24.23). Fig. 24.4 was drawn to exemplify the β dependence of the zero temperature resonance integral, namely the β dependence of (Eq. 24.23). By the same token and an argument brought above, Fig. 24.4 also shows the β dependence of the zero temperature absorption cross section.

β is expressed in terms of N_M/N_R

The more we increase N_M relative to N_R , the more the absorber is diluted in the moderator. In a comparative manner then also:

$$0 < \beta < \infty \rightarrow$$

low < dilution < high

Also often the tabulated quantity, or the plotted ordinate, is $\frac{\sigma(\beta)}{\sigma(\infty)}$; it is then named the self shielding factor. The denominator of this latter expression is the σ for $\beta \rightarrow \infty$, namely the denominator is taken for an infinite dilution of the absorber.

The self shielding factor $\frac{\sigma(\beta)}{\sigma(\infty)}$ is always below 1.

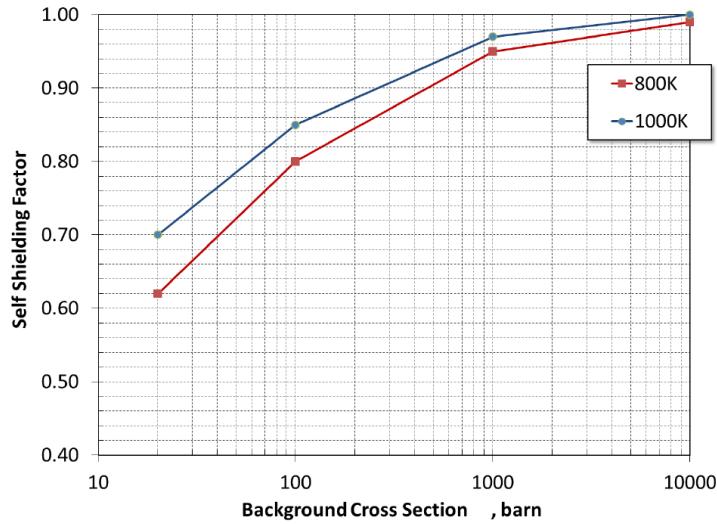


Figure 24.5: Self-shielding factors as a function of the background cross section.

24.5 Summary: multigroup cross-sections generation

The primary tool to generate multi-group cross-section is NJOY, which has the following operating philosophy:

1. Doppler broadening of cross sections for each resonance (for each isotope) for range of temperatures. MCNP libraries for instance come with an increment of 300 K (e.g. 300, 600, 900, ..., 1800).
2. Apply the NR/WR/IR models for range of dilution cross section (10, 100, 1000, 10000, ...)
3. Arrange the multi-group cross sections sets in the specified energy structure.
4. Generate tables of cross-sections vs. temperature and dilution.