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Solution of two-point kinetics equations for reflected reactors using Analytical Inversion Method (AIM)

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A B S T R A C T

The development of sophisticated kinetics theory model of a reflected reactors which comprising core and reflector regions has been considered. Within each region, the theory is space-independent and assumes one energy group. A detailed investigation of both mathematical and numerical solutions has been conducted and applied to the two-point kinetics for reflected reactors using the Analytical Inversion Method (AIM), which permits a fast inversion of polynomials by going temporarily to the complex plane. This method has been tested using two types of reflected reactor systems: the experimental zero-power reactor (PROTEUS) and a large tightly coupled system with a small homogeneous core surrounded by a series of non-multiplying reactors (AGN-201). In addition, an economical general approximate expression to the exponential function for the two-point kinetics matrix is derived and Padé rational approximations of different types are applied. In order to ensure the validity and stability of this method, the numerical results obtained with this algorithm are tested on a set of four kinetic problems, step, ramp, zigzag and oscillatory of reflected reactors reactivity change under considerations.

1. Introduction

The accurate prediction of reactor behavior is costly because it involves the determination of a multidimensional power distribution throughout a large and often geometrically complicated core. Even if it is possible to determine the group diffusion theory parameters that accurately predict the average group fluxes within small, explicitly represented regions, such as homogenized fuel cells, control rods, burnable poison lumps, etc., the solution of the corresponding group diffusion equations is expensive. The detailed description of a thermal reactor can easily involve a million mesh points and several energy groups. For these reasons, it is still worth trying to develop improved schemes for the solution of the one and/or two-point kinetic equations, particularly because any physical or mathematical intuition gained at this rather elementary level can usually be applied to much realistic models.

Two-point reflected reactors constitute one of the most important classes of nuclear reactors' models. Since the dimension of a critical core of a given composition depends on the fraction of the neutrons that leak out, these dimensions can be reduced if some of the leaking neutrons are reflected back into the core. A reflector has the added benefit of making the neutron flux distribution in the core more uniform by increasing the neutron population in the outer region due to reflected neutrons which otherwise would have accepted.

In the region of analytical methods many works contain too much mathematics makes it not easy to calculate and compare experimental data. In order to enable an analytical treatment in favour of physical insight, a relatively simple two-point model was adopted by Van Dam (1996), based on the original pioneer approaches of Avery (1958) and Cohn (1962) which have been widely used to analyze coupled reactors and the strength of coupling between cores or nodes in large reactors (McFarlane et al., 1984; Brumbach et al., 1988). In 1997, Spriggs et al. presented a simplified methodology for determining the coupling parameters and the neutron lifetimes based on simple probability relationships that describe the aggregate migration of neutrons between the core and the reflectors. A new version of the reflected core inhour equation that differs in form from the inhour equations proposed by Avery, Cohn, Spriggs and others was derived and analyzed by Aboanber and El Mhlawy (2008).

The goal of the present work is to introduce a mathematical treatment of reflected reactors by solving the time dependent-kinetics diffusion equation in both core and reflector for *m*-groups of delayed neutrons using Analytical Inversion Method (AIM) previously adopted by Aboanber and Nahla (2002). This method was created and applied to the one point kinetics model with excellent

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results being obtained. This developed method is based on a generalization of the analytical procedures for inverting polynomials of the two-point kinetics matrix, which has been included in the solution of the two-point kinetics equations. In addition, an approximate expression for the exponential function namely Padé rational approximations has been suggested for the exponential of the point kinetics matrix.

Padé approximations have been used several times in the nuclear engineering literature, in the traditional context forms or in the modified forms. For further details, the reader should consult Porsching (1966), da Nobrega (1971), Turnage (1973), Kang and Hansen (1973), Devooght and Mund (1975), Hennart and Torres (1976), Hennart (1977) and Aboanber (2006). Ehle (1973) has shown that only three diagonal lines of the Padé table, ($P_{pq}(z)$ with $p=q$, $q-1$, $q-2$) correspond to A-stable, practical considerations; however, limiting their use to the second order elements would involve either storage of complex quantities or evaluation of higher powers of the operator. Recently Aboanber (2006) has discussed the $A(\alpha)$ stability for different types of Padé approximations with success application to the point kinetics equations.

The presented work is organized as follows: the formation of the two-point kinetics equations in a matrix form and the theory of analytical inversion is analyzed and discussed in Section 2. The generalization of the Analytical Inversion Method for inverting the two-point kinetics matrix and different forms of Padé approximations to rational matrix functions is introduced in Section 3. Analytical inversion for two-point kinetics as well as the modified inhour equation is derived in Section 4. Numerical results are interpreted and discussed in Section 5.

2. Theory of Analytical Inversion Method for two-point reactor

It was recognized that the nuclear reactor was mathematically a very complex device, particularly if one were to treat the kinetic effects on a spatial basis throughout the reactor. Soodak and Campbell (1950) first indicated what the response of the reactor would be if all of the spatial effects were lumped and the reactor was considered as a simple entity, all portions behaving the same way at the same time. This theory was recently confirmed by many authors, e.g. Stacey (2001). The equations for the response of the neutron level in a reflected reactor to any change in reactivity were described in the space-average approximation by a coupled system of stiff differential equations for two-point kinetics equations, i.e. the core region and the reflector region, and it may be written as

$$\begin{aligned}\frac{dN_c}{dt} &= \frac{\rho_0(t)-\beta-K_{cr}K_{rc}}{\Lambda_c}N_c+\frac{K_{rc}}{\ell_r}N_r+\sum_i \lambda_i N_{pi}+Q_c\\ \frac{dN_r}{dt} &= \frac{K_{cr}}{\Lambda_c}N_c-\frac{N_r}{\ell_r}+Q_r\\ \frac{dN_{pi}}{dt} &= \frac{\beta_i}{\Lambda_c}N_c-\lambda_i N_{pi}\end{aligned} \tag{1}$$

where N_c, N_r is the total number of neutron populations in core and reflector, respectively, N_{pi} is the population of ith group of delayed neutron precursors, K_{cr} is the probability that a disappearing core neutron will reappear in the reflector, K_{rc} is the probability that a disappearing reflector neutron will reappear in the core, $\rho_0(t) = 1-(1/K)$ and K is the static multiplication factor, Λ_c is the neutron generation time for an infinite core and ℓ_r is the neutron lifetime in the reflector. The other symbols have their usual meaning (Aboanber and El Mhlawy, 2008). The quantities $N_c(t)$, $N_r(t)$, $N_{pi}(t)$, $F(t)$ and $\rho_0(t)$ are, in general, functions of time while β_i, λ_i and Λ_c are assumed constants.

The solution of these equations for the neutron level in the core and the reflector regions and for precursor concentrations as a function of time after change in reactivity was quickly recognized by introducing the vector of $(m+2)$ unknown state variables $\Psi(t)$ as

$$\Psi(t) = \mathrm{col}\left[N_c(t) \quad N_{P_1}(t)\ldots N_{P_m}(t) \quad N_r(t)\right].$$

The coupled system of differential equations (1) has been reduced to the matrix form:

$$\frac{d\Psi(t)}{dt} = A(t)\Psi(t)+F(t) \tag{2}$$

where $F(t)$ is the source term defined as

$$F(t) = \mathrm{col}[F(t) \quad 0 \quad \ldots \quad 0],$$

and $\mathbf{A}(t)$ is the $(m+2)\times(m+2)$ matrix operator often consisting of group cross-sections and delayed group constants written as

$$\mathbf{A}(t) = \begin{pmatrix} \alpha_1 & \lambda_1 & \lambda_2 & \ldots & \ldots & \lambda_m & \alpha_2\\ \mu_1 & -\lambda_1 & 0 & 0 & \ldots & 0 & 0\\ \mu_2 & 0 & -\lambda_2 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\ \mu_m & 0 & 0 & 0 & \cdots & -\lambda_m & 0\\ \xi_1 & 0 & 0 & 0 & 0 & 0 & \xi_2 \end{pmatrix}.$$

where

$$\alpha_1 = \frac{\rho_0(t)-\beta-K_{cr}K_{rc}}{\Lambda_c},\quad \alpha_2 = \frac{K_{rc}}{\ell_r},\quad \xi_1 = \frac{K_{cr}}{\Lambda_c},\quad \xi_2 = \frac{1}{\ell_r} \text{ and } \mu_i = \frac{\beta_i}{\Lambda_c}$$

The system of equations (1) and (2) presented a general point kinetic model for a two-region system consisting of a core surrounded by non-multiplying source-free reflector, that describe the time dependent behavior of multiplying systems, and composed of an arbitrary number of regions each one characterized by a multiplication factor and a neutron lifetime. This system of equations are exactly analogous to the bare-reactor equations with the exception that the production term for the pseudo-group "precursors" is not multiplied by K as in the case with the true delayed neutrons.

The matrix $\mathbf{A}(t)$ is usually named the two-point kinetics matrix, where $\rho_0(t)$ and $F(t)$ varying with time and/or neutron density and temperature in the case of feedback. Eq. (2) is usually solved in a series of time steps; the assumption being that $\rho_0(t)$ and $F(t)$ are constants and equal to their average values during the time step under consideration.

The exact solution of equation (2) under the assumption of constant $\mathbf{A}$ is given by

$$\Psi_{n+1}(t) = \exp(h\mathbf{A})\Psi_n(t)+\mathbf{A}^{-1}[\exp(h\mathbf{A})-\mathbf{I}]F(t),\quad n = 0,1,2,\ldots \tag{3}$$

where h is the step time interval $h = t_{n+1}-t_n$, $\mathbf{I}$ is the unit matrix, $\Psi_n(t)$ is the value of the vector Ψ at time t_n and $\Psi_{n+1}(t)$ is the value of the vector Ψ at time t_{n+1}. Note that the last term of Eq. (3), a matrix term multiplying $F(t)$, is always well defined even if the two-point kinetics matrix $\mathbf{A}$ is singular.

The mathematical treatment of the system of equations (3) is a relatively simple one, its solution can be obtained by calculating all eigenvalues of the matrix $\mathbf{A}$ and performing straightforward computations. However, this system is an expensive scheme, especially when the reactivity varies with time, since the calculations of the eigenvalues amounts to solving an $(m+2)$th-order algebraic equations, i.e. the inhour formula should be carried out at every time step for all its roots.

Mathematically, the eigenvectors of the two-point kinetics matrix $\mathbf{A}$ can be obtained from the solution of the famous inhour

equation which represents the basis of reactor kinetics governing time constants in reactor transients as dependent on reactivity, $\rho_0(t)$, for a two-point model. Let us denote the eigenvector of the two-point kinetics matrix by $\mathbf{U}_n$, thus the following equation defines the eigenvalue problem $\mathbf{A}\mathbf{U}_n = \omega_n \mathbf{U}_n$ where ω_n are the roots of the inhour formula for two-point region given by

$$\omega \Lambda_c - \rho_0(t) + \frac{K_{cr} K_{rc} \ell_r \omega}{(1+\ell_r \omega)} + \sum_{i=1}^{m} \frac{\beta_i \omega}{\omega + \lambda_i} = 0. \tag{4}$$

It is well known that the eigenvalues, ω_n, are distinct, and thus the eigenvectors $\mathbf{U}_n$ are complete.

Eq. (3) contains the exponential function of the two-point kinetic matrix **A**. Exact solution of this equation is prohibitive So, a form that expected to be quite suitable for generating an approximate solution of the more general problem (time dependent problem) is suggested and studied. Recall that the exponential function $e^{\mathbf{A}}$ admits the power series, thus

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} = 1 + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots \tag{5}$$

Direct appeal to the series definition, expression (5), is impractical since the number of computations needed for each additional term makes the computing time prohibitive (Porsching, 1966). For this and to avoid the instabilities associated with the computational effort involved in using the explicit methods, Eq. (3) requires a particular class of approximations for the exponential function namely Padé approximations. Table 1 presented different types of Padé approximations in both explicit and implicit form as well as the associated errors.

In what follow, the method of determining a general approximate expression of the exponential function for the two-point kinetics equations will be deduced. First let us introduce the following fact that for any function $f(\mathbf{A})$ for which $f(\omega_i)$ is bounded for all i, and the following expression introduced by Clark and Hansen (1964) holds for any matrix **A** satisfying $\mathbf{A}\mathbf{U}_i = \omega_i \mathbf{U}_i$ and $\mathbf{A}^T\mathbf{V}_i = \omega_i \mathbf{V}_i$ where $\mathbf{U}_i^T\mathbf{V}_i = 1$, i.e. normalized to unity. Consequently, the following relation will be

$$\exp(\mathbf{A}) = f(\mathbf{A}) + \sum_{i=0}^{m+1} [\exp(\omega_i) - f(\omega_i)] \mathbf{U}_i \mathbf{V}_i^T. \tag{6}$$

Since two matrices with the same eigenvalues and eigenvectors are identical, Eq. (6) is still true if the following relation is used:

$$\exp(\mathbf{A})\mathbf{U}_k = \exp(\omega_k)\mathbf{U}_k, \quad \text{for all } \mathbf{U}_k.$$

Using the fact that the eigenvalues of the matrix **A** and $\mathbf{A}^T$ form a biorthonormal set when properly normalized, then $\mathbf{V}_i^T\mathbf{U}_k = \delta_{ik}$. Acting on both sides of Eq. (6) by $\mathbf{U}_k$ yields

$$\begin{aligned} \exp(\mathbf{A})\mathbf{U}_k &= f(\omega_k)\mathbf{U}_k + \sum_{i=0}^{m+1} [\exp(\omega_i) - f(\omega_i)] \mathbf{U}_i \delta_{ik} \\ &= f(\omega_k)\mathbf{U}_k + [\exp(\omega_k) - f(\omega_k)]\mathbf{U}_k = \exp(\omega_k)\mathbf{U}_k. \end{aligned} \tag{7}$$

Let us introduce the factor h into the expression terms of Eq. (6) to suit our particular needs for application on the two-point kinetics model; Eq. (6) then becomes

$$\exp(h\mathbf{A}) = f(h\mathbf{A}) + \sum_{i=0}^{m+1} [\exp(h\omega_i) - f(h\omega_i)] \mathbf{U}_i \mathbf{V}_i^T, \tag{8}$$

Table 1
Different types of rational Padé approximations for the two-point exponential matrix

Padé	Explicit form	Implicit form	$\varepsilon = a + ib$, Error
$f_{0,1}$	$\frac{\mathbf{I}}{(\mathbf{I} - h\mathbf{A})}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}$	$h, \frac{-h^2\mathbf{A}^2}{2}$
$f_{0,2}$	$\frac{\mathbf{I}}{\left(\mathbf{I} - h\mathbf{A} + \frac{h^2\mathbf{A}^2}{2}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1}$	$\frac{h}{2}(1+\mathrm{i}), \frac{h^3\mathbf{A}^3}{6}$
$f_{0,3}$	$\frac{\mathbf{I}}{\left(\mathbf{I} - h\mathbf{A} + \frac{h^2\mathbf{A}^2}{2} - \frac{h^3\mathbf{A}^3}{6}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1}[\mathbf{I} - e\mathbf{A}]^{-1}$	$a = (0.1867)h,\ b = (0.4808)h,\ e = (0.6265)h, \frac{-h^4\mathbf{A}^4}{24}$
$f_{1,1}$	$\frac{\left(\mathbf{I} + \frac{h\mathbf{A}}{2}\right)}{\left(\mathbf{I} - \frac{h\mathbf{A}}{2}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}\left\{\mathbf{I} + \varepsilon\mathbf{A}\right\}$	$\frac{h}{2}, \frac{-h^3\mathbf{A}^3}{12}$
$f_{1,2}$	$\frac{\left(\mathbf{I} + \frac{h\mathbf{A}}{3}\right)}{\left(\mathbf{I} - \frac{2h\mathbf{A}}{3} + \frac{h^2\mathbf{A}^2}{6}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1}\left\{\mathbf{I} + \frac{h\mathbf{A}}{3}\right\}$	$\frac{h}{3}\left(1 + \frac{\mathrm{i}}{\sqrt{2}}\right)\frac{h^4\mathbf{A}^4}{72}$
$f_{1,3}$	$\frac{\left(\mathbf{I} + \frac{h\mathbf{A}}{4}\right)}{\left(\mathbf{I} - \frac{3h\mathbf{A}}{4} + \frac{h^2\mathbf{A}^2}{4} - \frac{h^3\mathbf{A}^3}{24}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1}[\mathbf{I} - e\mathbf{A}]^{-1}\left\{\mathbf{I} + \frac{h\mathbf{A}}{4}\right\}$	$a = (0.1846)h,\ b = (0.2745)h,\ e = (0.3808)h, \frac{-h^5\mathbf{A}^5}{480}$
$f_{2,2}$	$\frac{\left(\mathbf{I} + \frac{h\mathbf{A}}{2} + \frac{h^2\mathbf{A}^2}{12}\right)}{\left(\mathbf{I} - \frac{h\mathbf{A}}{2} + \frac{h^2\mathbf{A}^2}{12}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1}\left\{\mathbf{I} + \frac{h\mathbf{A}}{2} + \frac{h^2\mathbf{A}^2}{12}\right\}$	$\frac{h}{4}\left(1 + \frac{\mathrm{i}}{\sqrt{12}}\right)\frac{h^5\mathbf{A}^5}{720}$
$f_{2,3}$	$\frac{\left(\mathbf{I} + \frac{2h\mathbf{A}}{5} + \frac{h^2\mathbf{A}^2}{20}\right)}{\left(\mathbf{I} - \frac{3h\mathbf{A}}{5} + \frac{3h^2\mathbf{A}^2}{20} - \frac{h^3\mathbf{A}^3}{60}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1}[\mathbf{I} - e\mathbf{A}]^{-1}\left\{\mathbf{I} + \frac{2h\mathbf{A}}{5} + \frac{h^2\mathbf{A}^2}{20}\right\}$	$a = (0.6256)h,\ b = (0.1849)h,\ e = (0.2749)h, \frac{-h^6\mathbf{A}^6}{7200}$
$f_{3,3}$	$\frac{\left(\mathbf{I} + \frac{h\mathbf{A}}{2} + \frac{h^2\mathbf{A}^2}{10} + \frac{h^3\mathbf{A}^3}{120}\right)}{\left(\mathbf{I} - \frac{h\mathbf{A}}{2} + \frac{h^2\mathbf{A}^2}{10} - \frac{h^3\mathbf{A}^3}{120}\right)}$	$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1}[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1}[\mathbf{I} - e\mathbf{A}]^{-1}\left\{\mathbf{I} + \frac{h\mathbf{A}}{2} + \frac{h^2\mathbf{A}^2}{10} + \frac{h^3\mathbf{A}^3}{120}\right\}$	$a = (0.1424)h,\ b = (0.1358)h,\ e = (0.2153)h, \frac{-h^7\mathbf{A}^7}{100800}$

where $\mathbf{U}_i$ and $\mathbf{V}_i$ are the unchanged eigenvectors of the matrix $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$, respectively, since $(h\mathbf{A})\mathbf{U}_i = (h\omega_i)\mathbf{U}_i$, and $(h\mathbf{A}^{\mathrm{T}})\mathbf{V}_i = (h\omega_i)\mathbf{V}_i$.

Eq. (8) has a form that permits to approximate $\exp(h\,\mathbf{A})$ in an economical fashion. It is interesting to note that if $f(h\omega_i)$ is a good approximation to $\exp(h\omega_i)$, then we are justified in dropping the ith-term from the summation. It will have a very small coefficient that is

$$[\exp(h\omega_i) - f(h\omega_i)] \ll 1. \tag{9}$$

Since $\exp(h\omega_i) \approx f(h\omega_i)$, so, to a high degree of accuracy, we have

$$\exp(h\mathbf{A}) \cong g(h\mathbf{A}) = f(h\mathbf{A}) + \sum_i^{\neq}[\exp(h\omega_k) - f(h\omega_k)]\mathbf{U}_k\mathbf{V}_k^{\mathrm{T}}, \tag{10}$$

where the sum $\sum_i^{\neq}$ is over only k for which inequality (9) does not hold. The eigenvectors $\mathbf{U}_k$ and $\mathbf{V}_k$ are easily estimated from their defining equation as

$$(\omega\mathbf{I} - \mathbf{A})\mathbf{U}_k = 0,$$

and the eigenvectors are

$$\mathbf{U}_k = \mathrm{col}\Big[1 \quad \frac{\mu_1}{(\lambda_1+\omega_k)} \quad \cdots \quad \frac{\mu_m}{(\lambda_m+\omega_k)} \quad \frac{\xi_1}{(\lambda_{m+1}+\omega_k)}\Big].$$

Similarly for $\mathbf{V}_k$:

$$\left(\omega\mathbf{I} - \mathbf{A}^{\mathrm{T}}\right)\mathbf{V}_k = 0,$$

and the eigenvectors are

$$\mathbf{V}_k = \mathrm{col}\Big[1 \quad \frac{\lambda_1}{(\lambda_1+\omega_k)} \quad \cdots \quad \frac{\lambda_m}{(\lambda_m+\omega_k)} \quad \frac{\alpha_2}{(\lambda_{m+1}+\omega_k)}\Big].$$

Or, in the normalized form:

$$\mathbf{V}_k = \chi_k\mathrm{col}\Big[1 \quad \frac{\lambda_1}{(\lambda_1+\omega_k)} \quad \cdots \quad \frac{\lambda_m}{(\lambda_m+\omega_k)} \quad \frac{\alpha_2}{(\lambda_{m+1}+\omega_k)}\Big]$$

where χ_k is the normalization factor, which satisfy the normalization condition $\mathbf{U}_i^{\mathrm{T}}\mathbf{V}_i = 1$ and given by

$$\chi_k = \left[1 + \sum_{i=1}^{m+1}\frac{\mu_i\lambda_i}{(\lambda_i+\omega_k)^2}\right]^{-1} < 1.$$

3. The Padé approximations and rational matrix functions

The accurate evaluation of the matrix exponential is itself a difficult problem. Unless the time step size is small, the power series defining the exponential converges too slowly for practical use. What is done here is to replace the exponential in approximation (10) by certain rational matrix functions to approximate it. To achieve that purpose, a particular class of approximations for the exponential function, namely the Padé rational approximations, is considered. For any of these approximations in which the degree of the polynomial's denominator is larger than unity we have a full square matrix of order $(m+2)$ to be inverted. That task is being normally tried to be avoided, particularly for the case of varying reactivity when such inversion needs to be done at every time step.

However, we have developed Padé approximation for two-point kinetics to obtain a simple analytical expression for such inverses by going temporary to the complex plane. As a result, the same numbers of arithmetic operations that are sufficient to multiply the inverse of the polynomial of matrix $\mathbf{A}$ by a vector are equal to those that required multiplying the polynomial by itself. This fact makes the computational effort involved in using implicit methods of any order equal to that used for explicit methods of the same order (Taylor series expansion). However, the instabilities associated with the latter are avoided.

Generally, the presented method is based on an expression for the inverse of $[\mathbf{I} - \varepsilon\mathbf{A}]$, where ε is a scalar complex number. When one tries to invert a general polynomial of the matrix $\mathbf{A}$, which can be expressed as a product of factors having the form $[\mathbf{I} - \varepsilon\mathbf{A}]$, the utility of the developed method is evident for general $P_k(\mathbf{A})$ polynomial matrix. To explain this utility in more detail, consider the following matrix polynomial:

$$P_k(\mathbf{A}) = \sum_{n=0}^{k+1} c_n\mathbf{A}^n,$$

with c_n being real number and $c_0 = 1$, the matrix polynomial $P_k(\mathbf{A})$ can be factorized as

$$P_k(\mathbf{A}) = \prod_{n=1}^{k+1}[\mathbf{I} - \varepsilon_n\mathbf{A}],$$

where ε_n is a general complex number. Then

$$[P_k(\mathbf{A})]^{-1} = \prod_{n=1}^{k+1}[\mathbf{I} - \varepsilon_n\mathbf{A}]^{-1}.$$

The method of factorization considered above is of great advantage and has direct applicability to the Padé approximations of the exponential matrix of expression (10) listed in Table 1 (Aboanber and Nahla, 2002).

4. Analytical inversion for two-point kinetics

In this section the investigated Analytical Inversion Method (AIM) is applied to the two-point kinetics matrix $\mathbf{A}$. As we mentioned above, the method is aimed to develop an expression for the inverse of $[\mathbf{I} - \varepsilon\mathbf{A}]$; thus for a real ε the following expression is introduced:

$$[\mathbf{I} - \varepsilon\mathbf{A}]^{-1} = \gamma^{-1}\mathbf{B} + \mathbf{C} \tag{11}$$

where

$$\gamma = \left[1 - \frac{\varepsilon\rho_0}{\Lambda_c} + \varepsilon\sum_{i=1}^{m+1}\frac{\mu_i}{1+\varepsilon\lambda_i}\right],$$

$$\mathbf{C} = \mathrm{Diag}\Big[0 \quad \frac{1}{1+\varepsilon\lambda_1} \quad \frac{1}{1+\varepsilon\lambda_2} \quad \frac{1}{1+\varepsilon\lambda_3} \quad \cdots \quad \frac{1}{1+\varepsilon\lambda_{m+1}}\Big],$$

and

$$\mathbf{B} = RS^{\mathrm{T}},$$

where

$$R = \mathrm{col}\Big[1 \quad \frac{\varepsilon\mu_1}{1+\varepsilon\lambda_1} \quad \frac{\varepsilon\mu_2}{1+\varepsilon\lambda_2} \quad \cdots \quad \frac{\varepsilon\mu_m}{1+\varepsilon\lambda_m} \quad \frac{\varepsilon\xi_1}{1+\varepsilon\lambda_{m+1}}\Big],$$

and

$$S = \mathrm{col}\Big[1 \quad \frac{\varepsilon\lambda_1}{1+\varepsilon\lambda_1} \quad \frac{\varepsilon\lambda_2}{1+\varepsilon\lambda_2} \quad \cdots \quad \frac{\varepsilon\lambda_m}{1+\varepsilon\lambda_m} \quad \frac{\varepsilon\alpha_2}{1+\varepsilon\lambda_{m+1}}\Big].$$

where m is the number of the delayed neutron groups which is considered here as six groups and $\lambda_{m+1} = \ell_r^{-1}$ for the reflected region group. Eq. (11) is easily to be solved when γ is a scalar quantity that depends on the reactivity. Similarly, we can define

$$[\mathbf{I} - \bar{\varepsilon}\mathbf{A}]^{-1} = \bar{\gamma}^{(-1)}\mathbf{D} + \mathbf{E},$$

where $\bar{\gamma}$, $\mathbf{D}$ and $\mathbf{E}$ are complex conjugates of γ, $\mathbf{B}$, and $\mathbf{C}$, respectively.

On the contrary, expression (11) is of no great advantage by itself, since we can directly solve the system of equations implied

by the inverse with the same computational effort that is required in the above case. However, the utility of the Analytical Inversion Method is evident when ε_n is in a complex conjugate pairs form. In this case, the following pairs of conjugated factors are considered:

$$[\mathbf{I}-\varepsilon\mathbf{A}]^{-1}[\mathbf{I}-\bar{\varepsilon}\mathbf{A}]^{-1}=\left[I-2\mathrm{Re}(\varepsilon)\mathbf{A}+|\varepsilon|^2\mathbf{A}^2\right]^{-1}.$$

Since they form a real matrix, thus they have a real inverse. This expression can be expanded for two-point kinetics in the form:

$$\begin{aligned}[\mathbf{I}-\varepsilon\mathbf{A}]^{-1}[\mathbf{I}-\bar{\varepsilon}\mathbf{A}]^{-1} &= \gamma^{-1}\overline{\gamma}^{(-1)}\mathbf{BD}+\gamma^{-1}\mathbf{BE}+\overline{\gamma}^{(-1)}\mathbf{CD}+\mathbf{CE}\\ &= (\gamma\overline{\gamma})^{-1}\mathbf{Q}+\gamma^{-1}\mathbf{W}+\overline{\gamma}^{(-1)}\mathbf{Y}+\mathbf{Z}. \end{aligned} \tag{12}$$

For generality, assume that $\varepsilon=\alpha+\mathrm{i}\eta$ and $\bar{\varepsilon}=\alpha-\mathrm{i}\eta$, where α and η are real constants and $\mathrm{i}=\sqrt{-1}$, then

$$\begin{aligned}\gamma\overline{\gamma} &= 1-\frac{2\alpha\rho_0}{\Lambda_c}+\left(\frac{r\rho_0}{\Lambda_c}\right)^2+\sum_{j=1}^{m+1}2\mu_j p_j\left(\alpha+r^2\lambda_j\right)\\ &\quad-\frac{r^2\rho_0}{\Lambda_c}\sum_{j=1}^{m+1}2\mu_j p_j(1+\alpha\lambda_j)+r^2\left(\sum_{j=1}^{m+1}\mu_j p_j\right)^2\\ &\quad+2\alpha r^2\left(\sum_{j=1}^{m+1}\mu_j p_j\right)\left(\sum_{j=1}^{m+1}\mu_j p_j\lambda_j\right)+r^4\left(\sum_{j=1}^{m+1}\mu_j p_j\lambda_j\right)^2,\end{aligned}$$

where $r^2=\alpha^2+\eta^2$, $s^2=\alpha^2-\eta^2$, and $p_j^{-1}=(1+2\alpha\lambda_j+r^2\lambda_j^2)$, $j=1,\ldots,m+1$.

The **Q** matrix element can be represented as

$$\mathbf{Q}=\mathbf{BD}=q_{k,\ell},$$

where $q_{1,1}=1+\sum_{i=1}^{m+1}r^2\mu_i\lambda_i P_i$, $q_{1,\ell+1}=(\bar{\varepsilon}\lambda_\ell/(1+\bar{\varepsilon}\lambda_\ell))q_{1,1}$, $\ell=1,2,\ldots,m$, $q_{1,\ell+2}=(\bar{\varepsilon}\eta_1/(1+\bar{\varepsilon}\lambda_{G+1}))q_{1,1}$, $\ell=m$, $q_{k+1,1}=(\varepsilon\mu_k/(1+\varepsilon\lambda_k))q_{1,1}$, $k=1,2,\ldots,m$, $q_{k+2,1}=(\varepsilon\mu_k/(1+\varepsilon\lambda_{k+1}))q_{1,1}$, $k=m$, and $q_{k+1,\ell+1}=(r^2\mu_k\lambda_\ell/(1+\varepsilon\lambda_k)(1+\bar{\varepsilon}\lambda_\ell))q_{1,1}$.

The matrix **W** is defined as

$$\mathbf{W}=\mathbf{BE}=w_{k,\ell},$$

where $w_{1,1}=0$, $w_{k+1,1}=0$, $k=1,2,\ldots,m+1$, $w_{1,\ell+1}=\varepsilon\lambda_\ell P_\ell$, $\ell=1,2,\ldots,m$, $w_{1,\ell+2}=\varepsilon\alpha_2 P_{\ell+1}$, $\ell=m$, and $w_{k+1,\ell+1}=(\varepsilon^2\mu_k\lambda_\ell/(1+\varepsilon\lambda_k))P_\ell$.

The resulting product of the matrix **C** and **D** define the matrix **Y** as

$$\mathbf{Y}=\mathbf{CD}=y_{k,\ell},$$

where $y_{1,1}=0$, $y_{1,\ell+1}=0$, $\ell=1,2,\ldots,m+1$, $y_{k+1,1}=\bar{\varepsilon}\mu_k P_k$, $k=1,2,\ldots,m$, $y_{k+2,1}=\bar{\varepsilon}\eta_1 P_{k+1}$, $k=m$ and $y_{k+1,\ell+1}=(\bar{\varepsilon}^2\mu_k\lambda_\ell/(1+\bar{\varepsilon}\lambda_\ell))P_k$.

The matrix $\mathbf{Z}=\mathbf{CE}=z_{k,\ell}$ is a diagonal matrix where

$$z_{\ell+1,\,\ell+1}=P_\ell,\ \ell=1,2,\ldots,m+1,$$
otherwise $z_{k,\ell}=0$, $k=1,2,\ldots,m+1$.

Consequently, the expression (12) can be rewritten as

$$[\mathbf{I}-\varepsilon\mathbf{A}]^{-1}[\mathbf{I}-\bar{\varepsilon}\mathbf{A}]^{-1}=(\gamma\overline{\gamma})^{(-1)}\mathbf{F}+\mathbf{Z}. \tag{13}$$

The matrix **F** is the combination of the matrices defined as

$$\mathbf{F}=\mathbf{Q}+\overline{\gamma}\mathbf{W}+\gamma\mathbf{Y}=f_{k,\ell},$$

where

$$f_{1,1}=1+r^2\sum_{j=1}^{m+1}\mu_j\lambda_j p_j,$$

$$\begin{aligned}f_{1,\ell+1} &= \lambda_\ell p_\ell\left\{2\alpha+r^2\left(\lambda_\ell-\frac{\rho_0}{\Lambda_c}+\sum_{j=1}^{m+1}\mu_j p_j\right)+r^2\left(2\alpha+r^2\lambda_\ell\right)\right.\\ &\quad\left.\times\sum_{j=1}^{m+1}\mu_j\lambda_j p_j\right\},\end{aligned}$$

$$\begin{aligned}f_{k+1,\ell+1} &= \mu_k\lambda_\ell p_k p_\ell\left\{2s^2+r^2\left(1+2\alpha(\lambda_k+\lambda_\ell)+r^2\lambda_k\lambda_\ell\right.\right.\\ &\quad-\frac{\rho_0}{\Lambda_c}\left(2\alpha+r^2(\lambda_k+\lambda_\ell)\right)+\left(2\alpha+r^2(\lambda_k+\lambda_\ell)\right)\sum_{j=1}^{m+1}\mu_j p_j\\ &\quad\left.\left.+\left(2s^2+r^2+2\alpha r^2(\lambda_k+\lambda_\ell)+r^4\lambda_k\lambda_\ell\right)\sum_{j=1}^{m+1}\mu_j\lambda_j p_j\right)\right\},\\ &\quad \ell,k=1,2,\ldots,m+1.\end{aligned}$$

Finally, the matrix **Z** is rewritten as

$$\mathbf{Z}=\mathrm{diag}[0\quad p_1\quad p_2\quad\ldots\quad p_{m+1}].$$

5. Numerical results and discussions

In order to examine the usefulness of the presented method, a computer code was developed to solve the two-point kinetics equations based on the Analytical Inversion Method (AIM). In addition, different types of Padé approximations have been developed for the two-point kinetic matrix to obtain a simple analytical expression for such inverses by going temporary to the complex plane. The emphasis was on the accuracy of the solutions as opposed to the computational speed. The speed of the code is strongly dependent on the implicit and/or explicit approximations to the polynomial of the two-point kinetic matrix **A** and also on the numbers of arithmetic operations which in turn depend on the type of Padé approximations. All of the considered transients in this section were initiated by perturbing a steady-state situation using different types of reactivity perturbations; it was assumed that when these perturbations occur, the flux and the precursor concentrations were at their steady-state value. These initial conditions were generally obtained from static calculations. To achieve our comparison, four different model problems have been set up to simulate various types of transients defined by $\rho_0(t)$. They included step, ramp, periodic and zigzag reactivity variations. Table 2 shows the analytical expressions of reactivity variations of the four model problems. Each problem was applied to both PROTEUS (Van Dam,

Table 2
Analytical expressions of reactivity insertions for four model problems

Problem number	Reactivity insertion	Parameters
1	$\rho(t)=\beta(1.0-f)$	f = Fraction of core neutron returned to the core after having leaked into the reflector.
2	$\rho=\rho_0+\gamma t$	$\gamma=0.1\$$, $\gamma=0.03\$$.
3	$\rho(t)=\rho_0+\mu\sin(\frac{T\pi}{180})$	T for one full cycle ($0\le t\le 2T$), where $\mu=0.5\$$, $0.6\$$, $0.7\$$ and $1\$$.
4	$\rho(t)=\begin{cases}+1\$/\mathrm{s} & 0:0.5\ \mathrm{s}\\ -1\$/\mathrm{s} & 0.5:1\ \mathrm{s}\\ +1\$/\mathrm{s} & 1:1.5\ \mathrm{s}\\ +0.5\$/\mathrm{s} & 1.5:3\ \mathrm{s}\end{cases}$	Padé approximation is applied to compute the response of a reactor core to a zigzag ramp input of reactivity.

Table 3
Core and reflector neutron flux for different types of Padé approximations versus time for cases at step reactivity

Cases	Time	1.0 s		2.0 s		3.0 s	
	Case	N_c	N_r	N_c	N_r	N_c	N_r
One point		2.359369	0.0	3.293477	0.0	4.300094	0.0
Two-point Case I PROTEUS	Padé 01	2.360458	18.852160	3.294808	26.328348	4.302004	34.381181
	Padé 02	2.360463	18.852194	3.294817	26.328424	4.302021	34.381313
	Padé 03	2.360464	18.852201	3.294820	26.328440	4.302024	34.381341
	Padé 11	2.359930	18.847977	3.293640	26.319062	4.299989	34.365135
	Padé 12	2.360109	18.849366	3.294034	26.322148	4.300666	34.370468
	Padé 13	2.360190	18.850011	3.294216	26.323603	4.300980	34.372979
	Padé 22	2.359900	18.847643	3.293685	26.319299	4.300103	34.365892
	Padé 23	2.360053	18.848905	3.293905	26.321104	4.300443	34.368666
	Padé 33	2.399467	19.164551	3.370075	26.930232	4.433630	35.433409
Exact[a]		2.360462	18.852188	3.294816	26.328411	4.302018	34.381290
One point		8.283412	0.0	16.506964	0.0	31.962611	0.0
Two-point Case II AGN-201 6G	Padé 01	8.037142	18.477463	15.887071	36.524756	30.492807	70.103826
	Padé 02	8.037720	18.479336	15.888860	36.529945	30.497487	70.116652
	Padé 03	8.036096	18.475054	15.883928	36.517526	30.484702	70.085181
	Padé 11	7.951514	18.279497	15.631902	35.935944	29.838258	68.594851
	Padé 12	7.982254	18.351603	15.721555	36.144880	30.065525	69.122735
	Padé 13	7.994987	18.380356	15.759177	36.230345	30.161593	69.341629
	Padé 22	7.954929	18.289450	15.640213	35.959184	29.857065	68.645981
	Padé 23	7.970358	18.323634	15.685032	36.059689	29.970520	68.901976
	Padé 33	8.187736	18.822552	16.495920	37.922234	32.277403	74.202191
Exact[a]		8.03884	18.48136	15.89231	36.53678	30.50650	70.13526

[a] Aboanber and El Mhlawy (2008).

1996) and AGN-201 (Spriggs et al., 1997) reactors with nuclear parameters taken from our previous work (Aboanber and El Mhlawy, 2008).

The first computational example simulates a step reactivity insertion provides simple computational solutions for comparing the different types of Padé approximations among each other, although it does not model an actual physical nuclear reactor problem. Table 3 displays the neutron flux of both core and reflector regions as a function of time for different types of Padé approximations. The exact solutions of this problem are made as a linear combination of $(m+1)$ exponential functions with time constant ω_i $(i=1,\ldots, m+1)$ (Aboanber and El Mhlawy, 2008). The standard six groups of delayed neutron precursors with fraction β_i and decay constant λ_i $(i=1,\ldots,6)$ in addition to the reflector region with $\beta_7 = K_{cr}K_{rc}$ and $\lambda_7 = \ell_r^{-1}$ have been used. The most accurate results are reported for the Padé approximations 01, 02, 03, 11 and 12 which preformed consistently for all problems at different transients and times. An excellent accuracy for this case is provided by Padé 01, 02 and 03 with a global truncation error of $O(h^2)$, $O(h^3)$ and $O(h^4)$, respectively. The number of exact digits is even higher than that provided by the other approximations (Table 3).

Table 4
Core and reflector neutron flux for different types of Padé approximations versus time for cases at ramp reactivity

Padé		$\gamma = 0.1$				$\gamma = 0.03$			
		1.0 s		5.0 s		1.0 s		5.0 s	
		Core	Reflector	Core	Reflector	Core	Reflector	Core	Reflector
Case I	Padé 01	1.004589	8.025706	1.023429	8.130975	1.001374	8.007697	1.006950	8.038924
	Padé 02	1.004589	8.025706	1.023429	8.130976	1.001374	8.007698	1.006950	8.038924
	Padé 03	1.004589	8.025707	1.023429	8.130976	1.001374	8.007698	1.006950	8.038924
	Padé 11	1.004582	8.025676	1.023422	8.130945	1.001367	8.007667	1.006943	8.038893
	Padé 12	1.004585	8.025683	1.023425	8.130952	1.001370	8.007675	1.006947	8.038901
Case II	Padé 01	1.079573	2.481048	1.510469	3.462984	1.022950	2.351674	1.123878	2.582216
	Padé 02	1.079573	2.481079	1.510470	3.463034	1.022950	2.351703	1.123878	2.582250
	Padé 03	1.079554	2.481003	1.510441	3.462918	1.022932	2.351632	1.123858	2.582170
	Padé 11	1.078345	2.478098	1.508981	3.459448	1.021759	2.348805	1.122622	2.579204
	Padé 12	1.078931	2.479595	1.509595	3.461019	1.022340	2.350290	1.123212	2.580710
Case III	Padé 01	1.062303	4.750160	1.372101	6.109915	1.018191	4.555805	1.096182	4.899294
	Padé 02	1.062299	4.750004	1.372104	6.109736	1.018191	4.555672	1.096173	4.899110
	Padé 03	1.062325	4.750260	1.372138	6.110078	1.018216	4.555916	1.096200	4.899374
	Padé 11	1.063325	4.754963	1.373289	6.115428	1.019193	4.560524	1.097216	4.904150
	Padé 12	1.062893	4.752672	1.372842	6.113041	1.018764	4.558248	1.096782	4.901847
Case IV	Padé 01	1.043100	9.233049	1.242614	10.929479	1.012687	8.973202	1.066074	9.429103
	Padé 02	1.043100	9.233120	1.242614	10.929567	1.012687	8.973271	1.066074	9.429176
	Padé 03	1.043096	9.233018	1.242610	10.929441	1.012683	8.973172	1.066070	9.429071
	Padé 11	1.043093	9.232858	1.242606	10.929282	1.012680	8.973012	1.066067	9.428911
	Padé 12	1.043089	9.233029	1.242602	10.929462	1.012675	8.973181	1.066062	9.429083

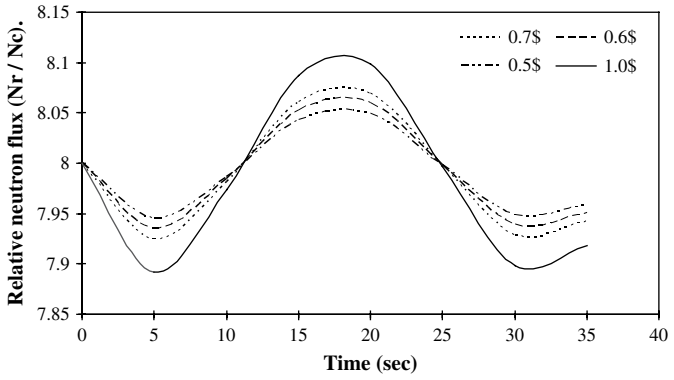

Fig. 1. Oscillatory reactivity variation for relative neutron flux versus time for PROTEUS Case I.

Table 4 summarizes an example of time-dependent reactivity (the so-called ramp input) which is considered a linear function of time. This example was specified by the same previous system of kinetic parameters with a ramp reactivity variation defined as $\rho_0(t) = \rho_0(0) + \gamma t$. The effect of $\gamma = 0.1\$/s$ and $\gamma = 0.03\$/s$ reactivity ramp rates are analyzed at $t = 1.0$ s and $t = 5.0$ s. The behavior of the exact solution to this problem is not exponential and cannot be expressed in a closed form except for the case of one delayed neutron family where it is given by a combination of hypergeometric functions (see Hetrick, 1971). Spectral matching, if any, can be made only on the instantaneous roots of the inhour equation. This problem offers no 'prompt jump' effect comparable to the reactivity step and is, therefore, less severe from the point of view of stiffness even for the fast reactor.

Regarding the two-point kinetics results listed in Table 4, the efficient technique to solve the system of equation (1), or in the equivalent matrix form equation (2), consists of using rational schemes where the two-point kinetics matrix is evaluated at different moments within a given time step. The greatest accuracy is obtained when the two-point kinetic matrix is evaluated at different moments. A more precise comparison between the different algorithms can only be made by putting in the perspective accuracy level as a function of computer work involved in each algorithm. The executed time for different approximations is approximately of the same order of magnitude. This later result shows a good performance for different Padé approximation schemes reported for this case.

Finally, from a physical standpoint, addition of each reflector new zone will increase the effective multiplication factors, the reflector return fraction and the neutron flux as shown in Figs. 1–4.

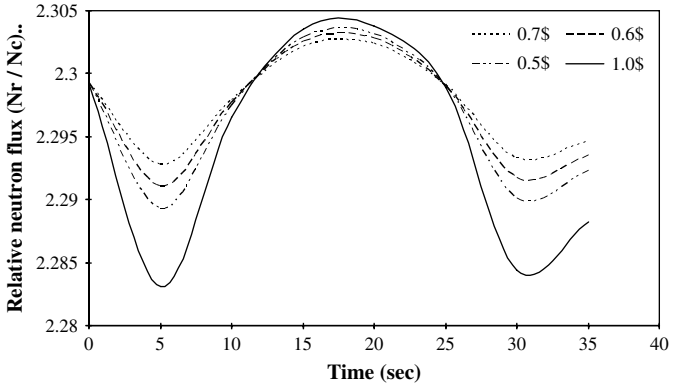

Fig. 2. Oscillatory reactivity variation for relative neutron flux versus time for AGN-201 Case II.

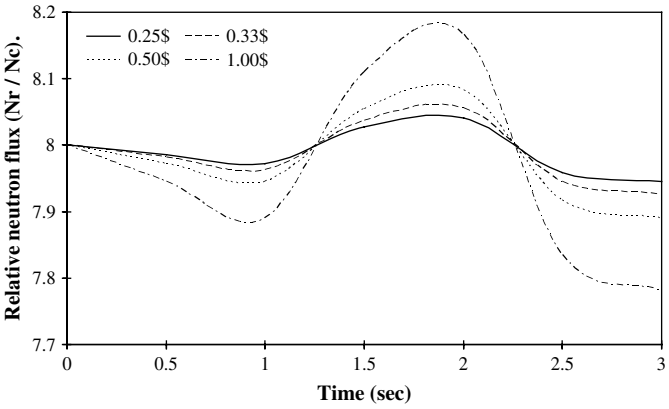

Fig. 3. Relative neutron flux for core and reflector at different value of reactivity (zigzag change) (Case I).

These figures represent the ratio of (N_r/N_c) versus time for both studied reactor types at periodic and zigzag transient problems. As obvious from these figures, the variation behaviors of the relative neutron flux at different parameters for both periodic and zigzag reactivity versus time indicated a good stability pattern.

6. Conclusions

From a physical standpoint, neutrons that reach the reflector region are in majority returned to the core due to scattering collisions which make the size of the critical reactor core smaller than that in case of bare-reactor. Based on this fact the calculations of the neutron flux in both core and reflector regions for two-point kinetics using the developed Analytical Inversion Method (AIM) are the main target of this work. Also, a number of new results concerned the numerical integration of two-point neutron kinetics equation are investigated and interpreted for different types of reactivity variations by means of rational approximation functions. Although these approximations are not by any means the only ones available, the requirement of $A(\alpha)$ stability is mandatory for stiff systems of neutron kinetics. The tested numerical examples of the two-point kinetics at step reactivity insertion have been reported for purpose of comparison among themselves. It was found that no approximation for Padé methods is uniformly better than the other for all investigated problems; however, the choice of the algorithm has been dictated for its ease of implementation. The choice of an algorithm is strongly dependent upon one's objectives: a moderate accuracy with large time steps or a very high accuracy with very

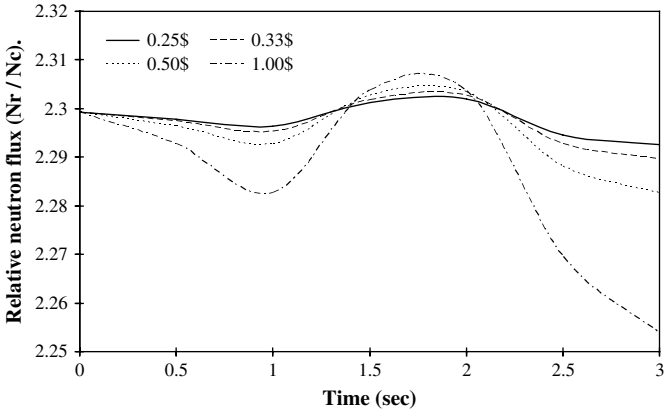

Fig. 4. Relative neutron flux for core and reflector at different value of reactivity (zigzag change) (Case II).

small time steps. Moreover, the success of an algorithm depends to a good deal upon a proper adaptation of the time step during the transient.

Finally, it could be concluded that the Analytical Inversion Method (AIM) with a proper rational approximations to the exponential function, Padé approximations, is an excellent all purpose candidate for one and two-point nuclear reactor kinetics analysis.

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