Progress in Nuclear Energy 51 (2009) 719–726

Contents lists available at ScienceDirect

Progress in Nuclear Energy

journal homepage: www.elsevier.com/locate/pnucene

Exact solution for the non-linear two point kinetic model of reflected reactors

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Keywords:
Two point kinetic
Reflected reactors
Analytical solution
Reactivity feedback
Basic exponential function
Coupled differential equation

A B S T R A C T

A new method to solve the coupled two point kinetic model with a time varying reactivity for reflected reactors is presented using analytical method. The technique is based on an explicit analytical solution based on a set of basis functions in the form of exponential functions. This set of basis functions for the transient solution are developed by multiplying each term by the corresponding eigenvalues and eigenfunctions of the two-point kinetic matrix. The developed method was found to be very fast, accurate and has the ability to reproduce all the features of the transients including prompt jump; the stiffness of the equations is also overcome. The formulism is applicable equally well to a non-linear problems, where the reactivity depends on the neutron population through temperature feedback. It was evident that the presented method is proved to be an excellent solution for cases in which the reactivity is represented by a series of steps and improves accuracy and efficiency for more general cases of time varying reactivity and reactivity feedback.

1. Introduction

Reflected reactors constitute one of the most important classes of nuclear reactors. Since the dimensions of a critical core of a given composition depend on the fraction of the neutrons that leak out, these dimensions can be reduced if some of the leaking neutrons are reflected back into the core. A reflector has the added benefit of making the neutron flux distribution in the core more uniform by increasing the neutron population in the outer region due to reflected neutrons which otherwise would have escaped.

Numerous theories and models have been developed for two-region reflected systems (Avery, 1958; Wasserman, 1959; Carroll et al., 1961; Cohn, 1962; Wasserman and Johnson, 1963; Kawai, 1965; Murley, 1967; Karam, 1967; Stacey, 1969; Shinkawa et al., 1978; Yamane et al., 1980; Spriggs and Busch, 1994; Spriggs et al., 1997; Aboanber and El Mhlawy, 2008) to describe the time-dependent behavior of multiplying systems comprising of an arbitrary number of regions, where each region is characterized by a multiplication factor and a neutron lifetime. The mathematical treatment of this lumped model for a two-region system consisting of a simple core surrounded by a non-multiplying, source-free reflector generalized the point kinetics approximation to a coupled, multi-point kinetic equation where the coupling between the cores is described by coupling constants (Avery, 1958; Kolesov et al., 1975; Difilippo and Waldman, 1976; Shinkawa et al., 1978; Nishina and Yamane, 1985; Hashimoto et al., 1990; Kobayashi, 1991; Van Dam, 1996; Spriggs et al., 1997; Yoshioka et al., 1998; Jatuff et al., 2003; Pavel, 2005). Such models are very useful because they are sufficiently simple to give insight and physical understanding for the response of neutron population to an external reactivity input in a two point nuclear reactor, with a formalism that is tractable by mostly analytical tools.

The present work discusses the effect of fuel temperature for a two point kinetic reactor model due to increasing neutron population in reactor core which in turn produces changes in reactivity and/or reactivity feedback. In fact, reactivity of a nuclear reactor which is a degree of off-criticality (Ott and Neuhold, 1985) is a function fuel type, enrichment, core fuel loading, amount of poison and temperature of different regions, etc. As fuel type loading is decided initially, but reactor operation is constantly accompanied by changing temperature of its different component, so their effects are constantly taken into account and have an important bearing on operation and safety of nuclear reactor. This effect of changing parameters such as fuel temperature on reactivity and feedback effect is discussed by many authors (e.g. Lamarsh, 1982; Stacey, 2001; Van Dam, 2007; Khan et al., 2007; Nicolino et al., 2008).

In the study considered here, the two-point reactor kinetic equations are a system of coupled non-linear ordinary differential equations. Included in the system are equations which describe the neutron level for both core and reflector, time-dependent reactivity, an arbitrary number of delayed-neutron groups, and any

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doi:10.1016/j.pnucene.2009.03.003

thermodynamic variables that enter into the reactivity equation. These equations are used to describe the power as neutronic properties of the internal elements of a nuclear reactor that changes with time. This would include the motion of control rods, the motion of fuel material in an accident scenario, the loss of coolant as the reactor undergoes blow-down accident and additional material motion. The equations are formulated as a set (usually of eight) of ordinary differential equations that can exhibit a rather stiff solution (widely spaced eigenvalues). Except for a few special cases, it is not possible to obtain closed solutions to these equations in terms of elementary functions because of a time-dependent reactivity and the stiffness. While the time dependence makes it difficult to obtain an analytical solution; the stiffness of the kinetics equations, however, restricts the time step to a small increment, making the numerical solution very inefficient. Based on the Analytical Inversion Method, the analytical approach has been introduced and directly applied on Padé approximations for one and two point kinetic models (Aboanber and Nahla, 2002; Aboanber and El Mhlawy, 2009). In these works, method was based on a generalization of the analytical procedures for inverting polynomials of the two-point kinetic matrix, which has been included in the solution of the one and two-point kinetic equations. The method provided a fast and accurate computational technique with step reactivity and a large time increment (time step) compared to the other conventional methods.

The aim of this work is to introduce a new method to solve the coupled two point kinetic model of coupling reactor theory with time-dependent reactivity and temperature feedback using an explicit analytical solution based on a set of basis functions in the form of exponential functions. This set of basis functions for the transient solution are developed by multiplying each term by the corresponding eigenvalues and eigenfunctions of the two-point kinetic matrix. The presence of temperature feedback is useful in providing an estimate of the transient behavior of a reactor power and of other system variables in the reactor core, which are fairly tightly coupled. To verify the validity of the studied solution, it was applied to ramp input of reactivity, oscillator input of reactivity for different transient in addition to reactivity feedback as a function of neutron flux in the core region.

Section 2 of this work introduces a description of the mathematical model formulation for the solution of the two-point kinetic equations with time varying reactivity and investigates the basic procedures of the developed method. Numerical results are presented and discussed in Section 3, while the conclusion comes finally in Section 4.

2. Mathematical model formulation

The two-point kinetic equations model the time-dependent behavior of a reflected nuclear reactor. Computational solutions of these two-point kinetic equations provide insight into the dynamics of nuclear reactor operation and are useful, for example, in understanding the power fluctuation experienced during start-up or shutdown when the control rods are adjusted. The two-point kinetic equations are a system of differential equations for the neutron density of the two regions (core and reflector) and for the delayed neutron precursor concentrations. The neutron density and delayed neutron precursor concentrations determine the time-dependent behavior of the power level of a nuclear reactor and are influenced, as mentioned before, for example, by control rod position, fuel temperature, motion of fuel material in an accident scenario, loss of coolant as the reactor undergoes a blow-down accident and additional material motion.

In the space-average approximation, the general point kinetic model, which describes the time-dependent behavior of multiplying systems and comprises of an arbitrary number of regions each characterized by a multiplication factor and a neutron lifetime, can be stated as a set of coupled differential equations (Avery, 1958). In this work, a two-point reactor kinetic model uses lumped variables and parameters for a two-region system consisting of a simple core surrounded by a non-multiplying source-free reflector with G groups of delayed neutrons is investigated. This model was stated by Avery and reduced by Cohn (1962) to the following set of coupled differential equations:

$$\left.\begin{aligned}\frac{dN_c}{dt} &= \frac{\rho(t)-\beta-f_{cr}f_{rc}}{\Lambda_c}N_c+\frac{f_{rc}}{\tau_r}N_r+\sum_{i=1}^{G}\lambda_i C_i+Q_c\\ \frac{dN_r}{dt} &= \frac{f_{cr}}{\Lambda_c}N_c-\frac{N_r}{\tau_r}+Q_r\\ \frac{dC_i}{dt} &= \frac{\beta_i}{\Lambda_c}N_c-\lambda_i C_i\end{aligned}\right\} \quad (1)$$

The two-point reactor kinetic model with reactivity feedback is a combination of the conventional two-point reactor kinetic equation (1), a linear newtonian temperature feedback from the fuel for the reactivity and an equation for heat transfer via Newton cooling. The two latter are described in terms of the fuel temperature coefficient of reactivity and the inverse of the mean time for heat transfer to the coolant as the following:

$$\left.\begin{aligned}\frac{dT(t)}{dt} &= \Re N_c(t)-\gamma(T-T_c)\\ \rho(t) &= I(t)+b[T(t)-T_0]\end{aligned}\right\} \quad (2)$$

where

$N_c(t)$: number of neutrons in the core region,

$N_r(t)$: number of neutrons in the reflector region,

$C_i(t)$: concentration of the ith precursor group ($i=1,2,\ldots,G$),

Q_c, Q_r: intrinsic/external neutron source rate in core and reflector, respectively,

$\rho(t)$: net reactivity as a function of time,

β, β_i,: effective and delayed neutron fractions for the ith precursor group,

λ_i: decay constant of the ith precursor group,

G: number of delayed-neutron groups,

$T(t)$,T_0: core-averaged fuel temperature at time t and zero, respectively,

T_c: core-averaged coolant temperature independent of time.

Λ_c: neutron generation time for an infinite core,

τ_r: neutron lifetime in the reflector region,

f_{cr}: fraction of fission neutrons produced that escapes to the reflector and partly returns to the core,

f_{rc}: fraction of reflector neutrons following to the core, being equal to the reflection coefficient or albedo of the reflector; it should be noted here that all neutrons entering the reflector are either absorbed or leak out or return to the core, the fractional part of the latter being equal to the albedo,

$\Re = e\Sigma_f/C_f$: reciprocal of the reactor heat capacity where, e is the average energy release per fission reaction, Σ_f is the macroscopic fission cross section and C_f is the total fuel heat capacity.

$1/\gamma = C_f/\zeta_f$: is interpreted as the mean time for heat transfer to the coolant, where ζ_f fuel-to-coolant heat-transfer constant,

b: fuel temperature coefficient of reactivity, and

$I(t)$: impressed reactivity.

By anticipating a very short time scale for the excursion, ignoring heat loss when the mean time for heat transfer ($1/\gamma$) is very large compared to the time scale of the excursion and using the adiabatic model as:

$$\left.\begin{aligned}\frac{dT(t)}{dt} &= \Re N_c(t)\\ \rho(t) &= I(t)+b[T(t)-T_0]\end{aligned}\right\} \tag{3}$$

So, the reactivity $\rho(t)$ is represented in a generalized notation as:

$$\rho(t) = I(t)+Y(t) \tag{4}$$

where, $Y(t)$ is a function representing the reactivity feedback. For example, $I(t)$ may have the form $\sin(\omega t)$, $\exp(\omega t)$, or a polynomial in t, while $Y(t)$ may be a function of temperature, power level, density or other variables. Assume a shutdown effect proportional to integrated neutron density (which in turn is proportional to fission energy release for a given τ_c). Since all calculations started from initial equilibrium with $N_c(0)=1$ neutron/cm^3 the compensated reactivity $\rho(t)$ in equation (3) can be represented as:

$$\rho(t) = I(t)+P(t)\int_0^t N_c(t')dt' \tag{5}$$

where $P(t)$ is the shutdown coefficient of the reactor core system ranging from $\sim 10^{-13}$ cm^3/sec for slow systems to $\sim 10^{-7}$ cm^3/sec for fast systems. However, it will not be necessary to specify the explicit form of $\rho(t)$ until a specific problem is considered.

The solution of these equations for the neutron level in the core and the reflector regions and for precursor concentrations as a function of time after change in reactivity was quickly recognized by introducing the vector of $(G+2)$ unknown state variables $\Psi(t)$ as:

$$\Psi(t) = \mathrm{col}[N_c(t)\ \ C_1(t)\ \cdots\ C_G(t)\ \ N_r(t)]$$

Thus, the coupled system of differential equation (1) can be reduced to the following matrix form:

$$\frac{d\Psi(t)}{dt} = F(t,N_c,N_r)\Psi(t)+S(t) \tag{6}$$

where, $S(t)$ is the source term defined as:

$$S(t) = \mathrm{col}[S(t)\quad 0\cdots 0],$$

and $F(t,\ N_c,\ N_r)$ is the $(G+2)\times(G+2)$ matrix operator often consisting of group cross sections and delayed group constants defined as:

$$F(t,N_c,N_r) = \begin{pmatrix} \alpha_1 & \lambda_1 & \lambda_2 & \cdots & \cdots & \lambda_G & \alpha_2\\ \mu_1 & -\lambda_1 & 0 & 0 & \cdots & 0 & 0\\ \mu_2 & 0 & -\lambda_2 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mu_G & 0 & 0 & 0 & \ldots & -\lambda_G & 0\\ \xi_1 & 0 & 0 & 0 & 0 & 0 & \xi_2 \end{pmatrix}$$

where

$$\alpha_1 = \frac{\rho(t,N_c)-\beta-f_{cr}f_{rc}}{\Lambda_c},\ \ \alpha_2 = \frac{f_{rc}}{\tau_r},\ \ \xi_1 = \frac{f_{cr}}{\Lambda_c},$$
$$\xi_2 = \frac{1}{\tau_r}\ \text{and}\ \ \mu_i = \frac{\beta_i}{\Lambda_c}$$

The matrix $F(t,N_c,N_r)$ is usually named the two-point kinetic matrix, where $\rho(t)$ and $S(t)$ are varying with time and/or neutron density in addition to temperature in the case of feedback.

The system of coupled kinetic equation (1) can be linearized to obtain the source transfer function by conventional means. Then, the transfer functions are obtained by taking the quotients of output and input of Laplace transforms:

$$C(s) = \frac{1}{N_{co}}\cdot\frac{\delta N_c(s)}{\delta\rho(s)} = \frac{1}{\Lambda_c s-\rho+\dfrac{f_{cr}f_{rc}\tau_r s}{1+\tau_r s}+\displaystyle\sum_i\frac{\beta_i s}{s+\lambda_i}} \tag{7}$$

$$R(s) = \frac{1}{N_{ro}}\cdot\frac{\delta N_r(s)}{\delta\rho(s)} = \frac{C(s)}{(1+\tau_r s)} \tag{8}$$

Where $C(s)$ and $R(s)$ are the reactivity transfer functions of core and reflector, respectively; s is the Laplace transform variable. From the general theory of dynamical systems, the response is governed by the poles of the transfer function and the inhour equation is simply the denominator of the transfer function (7) put to zero. For this situation, the inhour equation for a reflected system is obtained by:

$$\rho = \omega\Lambda_c+\frac{f_{cr}f_{rc}\tau_r\omega}{(1+\tau_r\omega)}+\sum_{i=1}^{G}\frac{\beta_i\omega}{\omega+\lambda_i} \tag{9}$$

It is interesting to note that the physical effect of the reflector on the core reactivity transfer function in equivalent to adding an extra delayed-neutron group as a pseudo group, where the parameters of this group are defined as $\lambda_r = \tau_r^{-1},\ \beta_r = f_{rc}f_{cr}$. Furthermore, the core prompt-neutron generation time and the reflector prompt-neutron generation time are defined respectively as:

$$\Lambda_c = \frac{\tau_c}{k_{eff}(1-f)},\ \ \Lambda_r = \frac{\tau_r}{k_{eff}(1-f)}$$

Where f is the total fraction of core neutrons returned to the core after having leaked into the reflector ($=f_{cr}f_{rc}$). Accordingly, the inhour equation (9) can be written compactly as:

$$\rho = \omega\Lambda_c+\frac{f\tau_r\omega}{(1+\tau_r\omega)}+\sum_{i=1}^{G}\frac{\beta_i\omega}{\omega+\lambda_i} \tag{10}$$

In most reflected systems, the reflector lifetime will be sufficiently small such that $\tau_r\omega_j<1$ for the first G roots of the reflected-core inhour equation (Spriggs et al., 1997). When this occurs, the core and reflector generation times can be readily combined to yield an equivalent one-region model given by:

$$\rho = \omega\Lambda_m+\sum_{i=1}^{G}\frac{\beta_i\omega}{\omega+\lambda_i} \tag{11}$$

Where the system mean dynamic prompt-neutron generation time, Λ_m is defined by $\Lambda_m = \Lambda_c+f\Lambda_r$. Equation (11) is identical to the standard one-region inhour equation. The eigenvalues of the point kinetic matrix are obtained by solving the inhour equation (10) for two-point and/or the equivalent form of equation (11) for one-region kinetic model at different values of reactivity insertion. It is well known that the eigenvalues, ω_n, are distinct.

The inhour equation (10) can be simplified (Aboanber, 2003; Aboanber and El Mhlawy, 2008) to:

$$\Lambda_c\sum_{K=0}^{G+1}\omega^{G+2-K}\sum_{\left[\frac{G+1}{K}\right]}\left(\lambda\ldots^K\ldots\lambda\right)+\sum_{i=1}^{G+1}\sum_{K=0}^{G}\omega^{G+1-K}$$
$$\left(\beta_i\sum_{\left[\frac{G+2}{K}\right]}\left(\lambda\ldots^K\ldots\lambda\right)_{i=\neq}\right)-\sum_{K=0}^{G+1}\omega^{G+1-K}$$
$$\left(\rho\sum_{\left[\frac{G+1}{K}\right]}\left(\lambda\ldots^K\ldots\lambda\right)\right) = 0 \tag{12}$$

Table 1
Parameters for models analysis.

			ρ_∞	Λ_c	τ_r	f_{rc}	f_{cr}	$f_{rc}f_{cr}$
PROTEUS	Case I	One Point	0.0	2 (ms)	0.0 (ms)	0.0	0.0	0.0
		Two Point	0.400	0.4 (ms)	4 (ms)	0.5	0.8	0.4
AGN-201	Case II	One Point	0.0	59.7 (μs)	0.0	0.0	0.0	0.0
		Two Point	0.2714	26.0 (μs)	122.7 (μs)	0.4429	0.4872	0.2158
	Case III	One Point	0.0	96.3 (μs)	0.0	0.0	0.0	0.0
		Two Point	0.25670	26.0 (μs)	238.6 (μs)	0.4737	0.4877	0.2310
	Case IV	One Point	0.0	169.3 (μs)	0.0	0.0	0.0	0.0
		Two Point	0.25000	26.0 (μs)	472.3 (μs)	0.4877	0.4880	0.2380
Delayed neutron parameters		T	55.7	22.7	6.22	2.30	0.610	0.230
		a_i	0.0328	0.2190	0.1960	0.3950	0.1150	0.0420

$\beta = 0.00723$ (PROTEUS), $\beta = 0.0075$ (AGN-201).

Where, the notation $i = \neq$ implies that one does not include the ith term in the multiplication, i.e. $i = 1$, λ_1 is not included in the multiplication. Equation (12) is considered as the fundamental algebraic equation which for a given set of delayed neutron parameters, neutron lifetime and neutron generation time for an infinite core can be solved for the roots ω.

As mentioned before, the system of equation (1) presented a general point kinetic model for a two-region system consisting of a core surrounded by non-multiplying source-free reflector. This system of equations are exactly analogous to the bare-reactor equations with the exception that the production term for the pseudo-group "precursors" is not multiplied by static multiplication factor as in the case of the true delayed neutrons. The reactivity ρ is defined in the usual way as $(k_{\text{eff}} - 1)/k_{\text{eff}}$. In most reflected reactors, the effective multiplication factor k_{eff} is controlled by changing k_c (multiplication factor of the bare core) by means of inserting or removing control rods. Nevertheless, there are many reactors still in operation that control k_{eff} by adding or removing reflector (Spriggs and Busch, 1994).

Depending on these pervious fundamental equations, the method of determining a general analytical solution for the two-point kinetic equations can be deduced. First, let us consider equation (6) in the absence of the external source, so that the solution can be formed as:

$$\Psi(t) = \sum_{k=1}^{G+1} A_k \exp(\omega_k t)\mathbf{U}_k \tag{13}$$

where A_k are constant matrix depending on the initial condition; ω_k are the eigenvalues of the matrix $\mathbf{F}(t, N_c, N_r)$ which coincide to the roots of the inhour equation (9), and U_k are the corresponding eigenvectors, which is given by:

$$\mathbf{F}(t, N_c, N_r)\mathbf{U}_k = \omega_k \mathbf{U}_k \text{ and } [\mathbf{F}(t, N_c, N_r)]^T \mathbf{V}_k = \omega_k \mathbf{V}_k$$

The eigenvectors $\mathbf{U_k}$ and $\mathbf{V_k}$ of the matrices $\mathbf{F}(t, N_c, N_r)$ and $\mathbf{F}[(t, N_c, N_r)]^T$ form a biorthonormal set when properly normalized, so $V_i^T \mathbf{U}_k = \delta_{ik}$.

So, the eigenvectors $\mathbf{U}_k$ and $\mathbf{V}_k$ are easily estimated from their defining equations as:

$$[\omega \mathbf{I} - \mathbf{F}(t, N_c, N_r)]\mathbf{U}_k = 0,$$

$$\left[\omega \mathbf{I} - \mathbf{F}(t, N_c, N_r)^T\right]\mathbf{V}_k = 0,$$

and the eigenvectors are:

$$\mathbf{U}_k = \begin{bmatrix} \dfrac{1}{(\lambda_1 + \omega_k)} \\ \vdots \\ \dfrac{\mu_G}{(\lambda_G + \omega_k)} \\ \dfrac{\xi_1}{(\lambda_{G+1} + \omega_k)} \end{bmatrix}, \quad \mathbf{V}_k = \text{col} \begin{bmatrix} \dfrac{1}{(\lambda_1 + \omega_k)} \\ \vdots \\ \dfrac{\lambda_G}{(\lambda_G + \omega_k)} \\ \dfrac{\alpha_2}{(\lambda_{G+1} + \omega_k)} \end{bmatrix}.$$

Or, expressed in the normalized form as:

$$\mathbf{V}_k = \frac{1}{\left[1 + \sum_{i=1}^{G+1} \dfrac{\mu_i \lambda_i}{(\lambda_i + \omega_k)^2}\right]} \begin{bmatrix} \dfrac{1}{(\lambda_1 + \omega_k)} \\ \vdots \\ \dfrac{\lambda_G}{(\lambda_G + \omega_k)} \\ \dfrac{\alpha_2}{(\lambda_{G+1} + \omega_k)} \end{bmatrix},$$

Where $[1 + \sum_{i=1}^{G+1} \mu_i \lambda_i / (\lambda_i + \omega_k)^2]^{-1}$ is the normalization factor.

Enforcing the initial conditions $\Psi(t = 0) = \Psi_0$ we can compute the unknown coefficients matrix A_k as:

$$A_k = V_k^T \Psi_0$$

Consequently, the general analytical solution of the system of equation (6) in the absence of the external sources is re-expressed as:

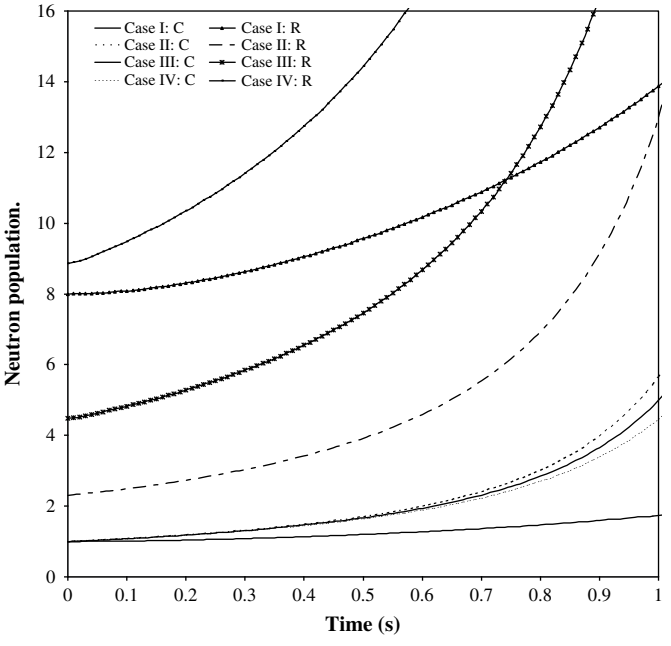

Fig. 1. Variation of neutron population versus time in core and reflector for ramp reactivity.

Table 2
Core and reflector neutron population for ramp reactivity at different times for one and two point kinetic models.

Case	Method	0.1 s (0.1 $/s)		0.5 s (0.1 $/s)		1.0 s (0.1 $/s)		3.0 s (0.1$/s)	
		Core	Reflector	Core	Core	Reflector	Reflector	Core	Reflector
Case I	Exact (One Point)	1.00163	0.0	1.02805	0.0	1.08306	0.0	1.47946	0.0
	Exact (Two Point)	1.00177	8.01315	1.02923	8.23085	1.08644	8.68733	1.50416	12.0234
	Padé 02	−0.00020	0.000175	−0.00049	0.000522	−0.00074	0.000817	−0.00160	0.001663
	Padé 03	−0.00020	0.000187	−0.00049	0.000547	−0.00083	0.000863	−0.00166	0.001663
	Padé 12	0.002396	−0.00232	0.006801	−0.00678	0.008836	−0.00885	0.014759	−0.01472
	Padé 13	0.001797	−0.00176	0.005149	−0.00513	0.006719	−0.00668	0.011169	−0.01114
Case II	Exact (One Point)	1.00930	0.0	1.05610	0.0	1.12878	0.0	1.65270	0.0
	Exact (Two Point)	1.00948	2.32050	1.05630	2.59470	1.12878	2.42800	1.65310	3.79970
	Padé 02	−0.03131	−0.02366	−0.03669	−0.02574	−0.0310	−0.02068	−0.04296	−0.01674
	Padé 03	0.010304	−0.01961	0.013635	−0.01923	0.011608	−0.02732	0.023174	−0.03369
	Padé 12	0.311404	0.375350	0.308209	0.388839	0.297829	0.397387	0.257397	0.476353
	Padé 13	0.119984	−0.01551	0.132847	−0.02352	0.142047	−0.04131	0.205663	−0.09406
Case III	Exact (One Point)	1.00885	0.0	1.05542	0.0	1.12753	0.0	1.64878	0.0
	Exact (Two Point)	1.00990	4.52000	1.05680	4.72970	1.12931	5.05424	1.65350	7.40009
	Padé 02	0.262990	−0.10555	0.289084	−0.13153	0.319310	−0.16167	0.463803	−0.30569
	Padé 03	0.043964	−0.01726	0.048354	−0.02157	0.053041	−0.02629	0.074750	−0.04777
	Padé 12	−2.63356	0.733391	−2.85942	0.954907	−3.12651	1.216761	−4.42963	2.494782
	Padé 13	−0.43855	0.148917	−0.47890	0.188998	−0.52687	0.236613	−0.76033	0.468441
Case IV	Exact (One Point)	1.00795	0.0	1.05407	0.0	1.12555	0.0	1.64035	0.0
	Exact (Two Point)	1.00821	8.93703	1.05452	9.34740	1.12626	9.98322	1.64336	14.5662
	Padé 02	0.416679	−0.18786	0.459358	−0.23019	0.508499	−0.27898	0.735078	−0.50404
	Padé 03	0.080836	−0.04033	0.091037	−0.05040	0.102996	−0.06230	0.151397	−0.11019
	Padé 12	−5.11005	2.474289	−5.66090	3.009788	−6.29449	3.624841	−9.42915	6.669751
	Padé 13	−0.71830	0.322400	−0.78728	0.390697	−0.86623	0.468646	−1.25736	0.855542

$$\Psi(t) = \sum_{k=1}^{G+1} \Psi_0 \exp(\omega_k t) U_k V_k^T \tag{14}$$

For time change reactivity and feedback problems this equation can be developed to suit the variation in the transient at different stepsize as follows:

$$\Psi(t_n + 1) = \sum_{k=1}^{G+1} \exp(\omega_k \Delta t_n) U_k V_k^T \Psi(t_n) \tag{15}$$

In its most general form, equation (15) is more suitable than equation (14) at adapting the stepsize $\Delta t_n = t_{n+1} - t_n$. When this procedure is applied step-by-step to all time intervals, the analytical solution of the two-point reactor kinetic equations is obtained. This method is computationally efficient for the stiff systems encountered for the two point as well as one point kinetic equations.

3. Numerical results and discussion

To apply and verify the efficiency of the introduced technique, several different computational examples are considered in this section. The general solution for neutron population in the core and the reflector as well as the precursor concentrations has been obtained using the designed FORTRAN code. This code is applied to the ramp reactivity input and oscillatory reactivity changes. Whenever the reactivity is given, including the case in which the reactivity feedback is a function of neutron density, the developed code can provide a straightforward procedure for computing the neutron density. The reactor kinetics parameters for the investigated four different cases of the reactor and the delayed neutron data are shown in Table 1. The first case is a representative case for the experimental zero power PROTEUS (Van Dam, 1996; Jatuff et al., 2003). The other three cases consist of the core plus graphite reflector; the core plus graphite reflector and lead shield; the core plus graphite reflector and the lead and water shields for AGN-201 reactor (Spriggs et al., 1997).

3.1. Ramp reactivity input

To check the validity and accuracy of the new adopted technique, comparisons were made to the few cases for which Padé approximations exist. The comparison was also made between neutron population of one point and two point kinetic systems. For ramp variation of reactivity, two examples are presented and discussed in the following sections:

The first example is taken such that the response of the neutron population in the core and the reflector for the investigated four representative cases to the linear variation of reactivity $\rho(t) = \rho_0 + \beta(1-f)t$ from initial equilibrium is shown in Fig. 1. Using the coupled two-point kinetic equations for simple reflected systems, it was shown that there exist two distinct time constants associated with the decay or growth of prompt neutron chain (Spriggs and Busch, 1994). One of the time constants is always negative while the other time constants becomes positive at reactivities greater than $\rho = \beta(1-f)$, where f is the fraction of core neutrons that return to the core region after having leaked into the reflector. By definition, this reactivity must correspond to the point of prompt critical in a reflected system. Fig. 1 shows the variations of neutron population in the core and in the reflector for the different representative cases versus time for linear variation of reactivity. From the reactivity relation, the decreased value of f is accompanied by an increase in the neutron population in both core and reflector region as a result of increasing the reactivity.

The second example is taken as a linear time variation of reactivity $\rho(t) = at$ from initial equilibrium for assumed values of ramp reactivity input at a slow rate of 0.1 $\$\,s^{-1}$. The values of the neutron population in the core and the reflector are reported in Table 2 at different time for the representative cases. The code of the developed method was run for one and two point kinetic systems and the neutron population in both core and reflector are compared to the different type of Padé approximations (Aboanber and El Mhlawy, 2009). The results of comparison agree very well with the four different types of Padé approximations. The minimum and the maximum relative percentage error was recorded as 0.00017% and −9.4%, respectively. Table 2 show that in the case of two point

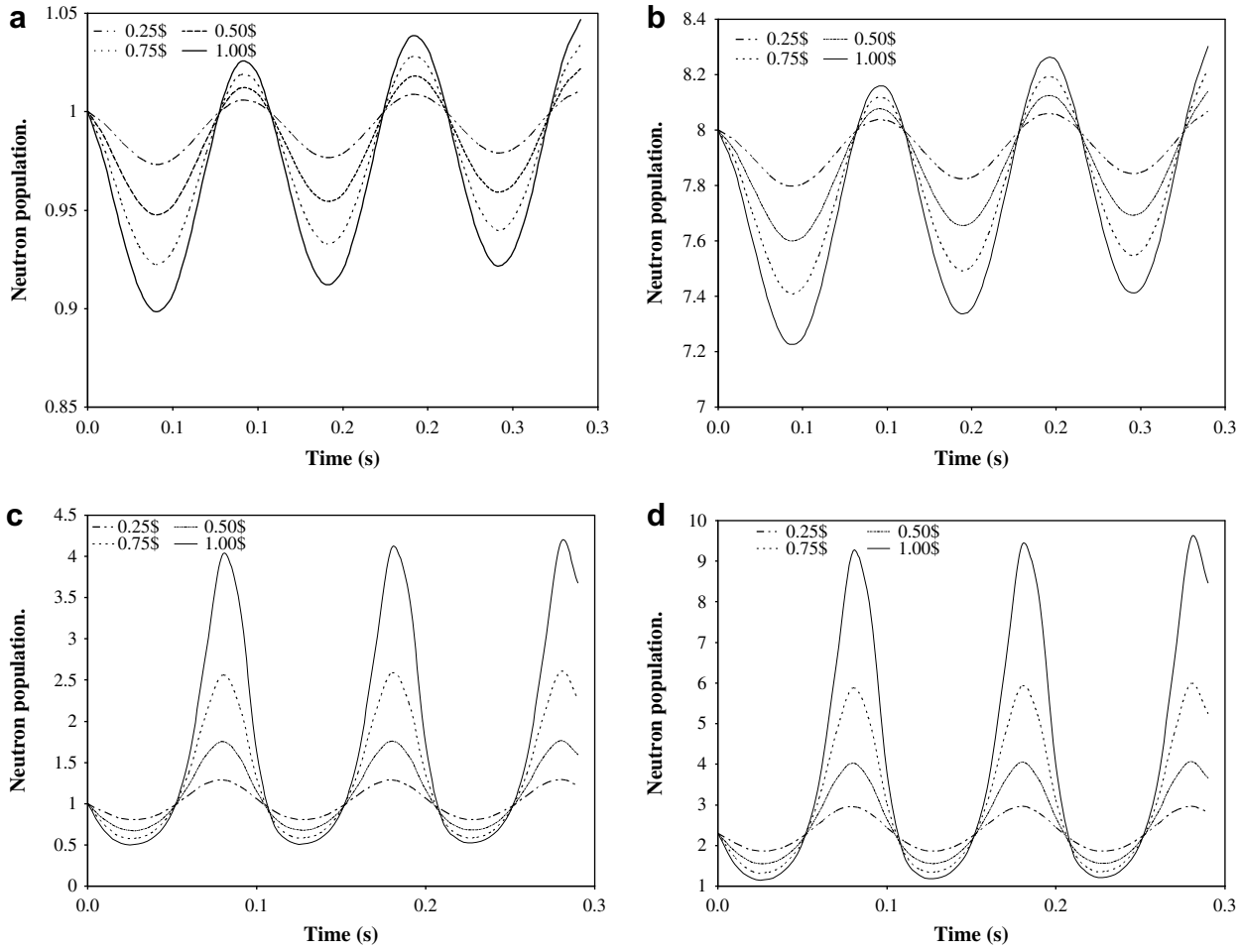

Fig. 2. a) Variation of neutron population versus time in the core with oscillatory reactivity insertion (Case I). b) Variation of neutron population versus time in the reflector with oscillatory reactivity insertion (Case I). c) Variation of neutron population versus time in the core with oscillatory reactivity insertion (Case II). d) Variation of neutron population versus time in the reflector with oscillatory reactivity insertion (Case II).

kinetic model a slight increase for the values of neutron population in the core region over the corresponding values for the one point kinetic model region was recorded. This variation is due to the leakage of neutrons from core to reflector and the return of neutrons from reflector region to core region. It is clear that the variation of neutron population profile in Fig. 1 and Table 2 agrees within graph-reading error with the numerical results presented by Aboanber and Nahla (2002) for the point kinetic model indicating the consistency of the developed method.

3.2. Oscillatory reactivity input

In this case the reactivity $\rho(t)$ is a function of time which is given as:

$$\rho(t) = \rho_0 + \mu \sin\left(\frac{\pi}{180}\right)t, \quad \mu = a\beta$$

where ρ_0 is the constant part of the excess reactivity, $\sin(\pi/180)$ is a given function characterizing the time dependence of the reactivity. The parameter μ is a positive number that represents the magnitude of the variable part of the excess reactivity in dollars. This parameter is assumed sufficiently small compared to unity. That is a real assumption since except in the accidental case, the excess reactivity is always less than one dollar. The wave form oscillations of the flux is such that a modified sinusoidal curve is obtained for different values of the μ parameter for the core region in Fig. 2a and c and for the reflector region in Fig. 2b and d for cases I and II, respectively with reactivity oscillation versus time. Similar pattern for the other cases can be justified.

3.3. Compensated reactivity

The previous two reactivity inputs discussed the neutron kinetics (i. e. the response of neutron population in a nuclear reactor for both core and reflector region to an external reactivity input) under the implicit assumption that the level of the neutron population does not affect the properties of the system that determine the neutron kinetics, most notably the reactivity. This is the situation when the neutron population is sufficiently small that the fission heat does not affect the temperature of the system (i. e. zero power). However, in an operating nuclear reactor the neutron population is large enough that any change in fission heating resulting from change in neutron population will produce changes in temperature, which in turn will produce changes in reactivity, or in reactivity feedback as indicated by equation (5).

The integration over t' in equation (5) may be evaluated using either trapezoidal rule or Simpson's rule for numerical integration. The implicit assumption that the integrand can be represented by a linear expression over the appropriate time interval(s) is clearly valid as soon as Δt is kept small. However, as a subject to this limitation, it is desirable to keep Δt reasonably large to reduce the

Table 3
Comparison of reactor transients with reactivity $\rho = 0.1\,t - 10^{-13}\int_0^t N(t')\mathrm{d}t'$ by three tested techniques (A, B, & C)* for Case II.

Time (s)	Δt	0.01		0.001		Time (s)	Δt	0.01		0.001	
		One point	Two point	One point	Two point			One point	Two point	One point	Two point
0.1	A	1.4774E+01	1.7883E+01	1.1592E+01	1.3654E+01	0.5	A	6.9178E+10	9.3808E+10	7.4977E+11	1.3518E+12
	B	1.4774E+01	1.7883E+01	1.1592E+01	1.3654E+01		B	1.7716E+12	9.3247E+11	1.1243E+12	1.2848E+12
	C	1.4774E+01	1.7883E+01	1.1592E+01	1.3654E+01		C	1.7716E+12	9.3247E+11	1.1243E+12	1.2848E+12
0.15	A	1.7113E+03	4.0105E+03	9.6315E+02	2.0680E+03	0.6	A	2.9064E+11	5.9673E+11	8.9148E+11	8.7195E+11
	B	1.7113E+03	4.0106E+03	9.6314E+02	2.0680E+03		B	8.9224E+11	1.0944E+12	8.3366E+11	1.0311E+12
	C	1.7113E+03	4.0106E+03	9.6314E+02	2.0680E+03		C	8.9224E+11	1.0944E+12	8.3366E+11	1.0311E+12
0.2	A	1.1493E+07	1.0305E+08	4.4432E+06	3.4225E+07	0.7	A	1.2979E+12	8.1405E+11	1.0972E+12	1.0240E+12
	B	1.1493E+07	1.0305E+08	4.4430E+06	3.4223E+07		B	9.9934E+11	1.0115E+12	1.0794E+12	9.8892E+11
	C	1.1493E+07	1.0305E+08	4.4430E+06	3.4223E+07		C	9.9934E+11	1.0115E+12	1.0794E+12	9.8892E+11
0.25	A	4.8102E+12	3.8685E+13	1.3047E+12	2.7028E+13	0.8	A	1.0455E+12	1.1472E+12	9.9118E+11	1.0239E+12
	B	4.4778E+12	4.1964E+12	1.2995E+12	2.2383E+13		B	1.0317E+12	1.0166E+12	1.0065E+12	1.0205E+12
	C	4.4778E+12	5.2574E+11	1.2995E+12	2.2383E+13		C	1.0317E+12	1.0166E+12	1.0065E+12	1.0205E+12
0.3	A	2.9931E+10	3.4870E+10	7.3800E+10	7.5459E+10	0.9	A	9.4274E+11	1.0009E+12	1.0169E+12	1.0111E+12
	B	4.0067E+10	1.8652E+10	8.3261E+10	8.7893E+10		B	1.0120E+12	1.0159E+12	1.0124E+12	1.0135E+12
	C	4.0067E+10	1.8652E+10	8.3261E+10	8.7893E+10		C	1.0120E+12	1.0159E+12	1.0124E+12	1.0135E+12
0.35	A	3.3686E+10	3.9919E+10	9.0994E+10	1.0609E+11	1.0	A	1.0452E+12	1.0108E+12	1.0133E+12	1.0132E+12
	B	5.8849E+10	3.7035E+10	1.0040E+11	1.3819E+11		B	1.0147E+12	1.0146E+12	1.0138E+12	1.0127E+12
	C	5.8849E+10	3.7035E+10	1.0040E+11	1.3819E+11		C	1.0147E+12	1.0146E+12	1.0138E+12	1.0127E+12
0.4	A	3.9628E+10	4.8141E+10	1.3689E+11	1.8241E+11	2.0	A	1.0090E+12	1.0088E+12	1.0080E+12	1.0080E+12
	B	1.2469E+11	1.6648E+11	1.6359E+11	3.0348E+11		B	1.0085E+12	1.0086E+12	1.0080E+12	1.0080E+12
	C	1.2469E+11	1.6648E+11	1.6359E+11	3.0348E+11		C	1.0085E+12	1.0086E+12	1.0080E+12	1.0080E+12
0.45	A	4.9715E+10	6.2826E+10	2.6460E+11	4.5634E+11	3.0	A	1.0062E+12	1.0061E+12	1.0058E+12	1.0058E+12
	B	5.3157E+11	1.8083E+12	3.6736E+11	1.0072E+12		B	1.0060E+12	1.0061E+12	1.0057E+12	1.0057E+12
	C	5.3157E+11	1.8083E+12	3.6736E+11	1.0072E+12		C	1.0060E+12	1.0061E+12	1.0057E+12	1.0057E+12

A, mean of the summation of the reactor response over the entire interval of integration; B, Simpson's rule; C, Trapezoidal rule.

number of computed points, minimize possible round-off error, etc. Table 3 compared the obtained results of a reactor transients with reactivity feedback by the three numerical integration techniques (A, B and C) for neutron population in the bare core region for one and two point kinetics at different transients and times for case II in Table 3 as an example. At small time step $\Delta t = 0.001$ s, typical results for the three techniques are reported. At large time step a similar behavior of B and C techniques is obtained. Most accurate results are achieved by taking the mean of the summation of the reactor response over the entire interval of integration, i.e. technique A, Table 3. The variations of neutron population pattern in this table exhibited a characteristic damped oscillatory approach to equilibrium power level at which the rate of reactivity compensation due to (adiabatic) temperature increases just to balance the rate of external reactivity addition.

4. Conclusions

Generally, a time-dependent reactivity inserted into a two point kinetic model for reflected reactors is coupled multiplicatively with the neutron density to form a set of linear equations with time dependent coefficients. In the presented work a new developed analytical method based on a set of basis functions in the form of exponential functions is proposed and applied to a variety of kinetic problems. The investigated method has a number of advantages. It can not only employ much larger time increment steps due to the stiffness confinement, but also computes rapidly for a given time step due to its completely analytical formulation. The repeated use of this formulism in successive time intervals has been shown to save considerable computing time. The applied numerical tests of ramp input of reactivity show that the technique is both efficient and accurate to several significant figures for core and reflector regions. The formalism is applicable equally well to non-linear problems, where the reactivity depends on the neutron density through temperature and thermal hydraulic reactivity feedback.

The superiority of the method is evident especially at large transients, leading to saving the CPU time of calculations. Comparatively, under the same conditions the required CPU time for Padé approximations was estimated to be about 1.30 times more than that required for the developed method. Furthermore, the instabilities associated with the computational effort involved in using the numerical methods at large transients are avoided.

It could be concluded that the analytical method for the solution of the two-point kinetic equations constitutes an easy algorithm that provides the results with sufficient accuracy for most applications and is being conceptually and structurally simple. It is more general and more powerful than the other conventional methods. The applicability of the formalism could be further extended to space-time kinetic problems.

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