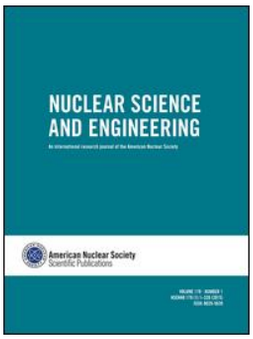

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The Application of Inhour Modes to the Description of Nonseparable Reactor Transients*

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For a large power reactor it appears possible to describe nonseparable space-time kinetics transients in terms of a particular set of spatial harmonics to be called inhour modes. These modes are defined as a subset of the period modes obtained by assuming a separable time variation $e^{\omega t}$ for all variables in the source-free, time-dependent neutron and neutron-precursor equations. Their use is appropriate whenever details of the neutron energy and angular behavior are not required. Inhour modes are shown to occur in clusters of seven, the seven eigenvalues of a given cluster being obtained as the roots of an inhour equation appropriate to the cluster. The neutron flux shapes associated with a particular cluster of seven modes are all approximately the same. It is shown that if these shapes are assumed to be identical, certain useful orthogonality relations and certain identities involving the roots of the inhour formula for a given cluster are obtained. Use of these results simplifies the extension of the conventional equations of reactor kinetics to the nonseparable case. Inhour modes are also useful in analyzing certain experiments involving subcritical assemblies. As an illustration, application to the source-jerk and pulsed-source experiments is made.

INTRODUCTION

The capability of computing detailed transient behavior in a reactor has increased considerably in recent years. To some extent this advance is associated with the greater capacity of digital computing equipment; computations formerly performed with analogue equipment are now run with much greater generality and accuracy on digital machines. However, to a greater extent, the advance has resulted from an improved understanding of the heat transfer, pressure drop, fluid flow and neutron kinetic phenomena which interact in the course of a reactor transient.

Of course, along with increased capability has come an obligation for responsible use. It is clearly inadvisable to go to great lengths to achieve a detailed description of one part of an over-all phenomenon when another essential interacting part is being treated grossly. In this connection, the question of when the conventional 'point' reactor kinetics equations must be replaced by spacetime equations and the problem of just how to describe nonseparable space-time neutron behavior in a practical manner have become increasingly important. For example, the accidental withdrawal of a control rod located near one side of a large, critical, pressurized water-moderated power reactor will lead not only to a rise in over-all power level but also to a tilting of the power shape. Feedback effects due to resultant temperature and density changes cannot be properly accounted for unless both the power rise and the power tilting are described.

If the prediction of detailed, nonseparable space-time neutron behavior is desired, it cannot in most practical situations be achieved by computing for the time-dependent diffusion equations a straightforward difference solution in both the space and time variables. Programs which determine such a solution have been written[1,2,3]. However, they usually deal with only one space dimension, in which case their application to practical situations

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[1] R. L. CROWTHER and W. H. WOLF, *Trans. Am. Nucl. Soc.* 6, 210 (1963).

is limited. For large power reactors, extension to two and three space dimensions is likely to result in long running times since, if detail is desired, the number of spatial mesh points involved tends to become excessively large and since the convergence of spatial solutions for large cores is extremely slow. Accordingly solutions to space-time problems have generally been obtained in some approximate manner, the particular approximation generally being tailored to the particular problem at hand.

Most approximation schemes that have been developed for the solution of space-time kinetics problems can be categorized as either 'nodal' or 'modal'. A nodal treatment[4,5] splits the reactor into sections and ascribes to each section a separate kinetic behavior determined partly by its own characteristics and partly by a coupling through neutron leakage to neighboring sections. A modal analysis describes the space-time behavior as a linear combination of full-core flux shapes, the coefficients of combination being functions of time. The flux shapes combined may be solutions to the Helmholtz equation[6,7], to the period-eigenvalue reactor equations[8,9], or they may be particular shapes chosen to describe best the expected behavior of the reactor transient at hand[10]. In this last case both time and space-time syntheses are possible (see Ref. 10).

If a modal expansion is to be used, the choice of basic flux shapes involves an assessment of the advantages and shortcomings of the alternatives just named. The Helmholtz modes are attractive in that they are complete, easily computed and possess useful orthogonality properties. On the other hand, a geometrical situation involving small heterogeneities—for example a cross-shaped control rod or water hole—will require the use of a great many modes, and the resultant computations may be long. The period-eigenvalue modes (which we shall henceforth call 'ω-modes'—see Eq. 3) can be more readily tailored to the heterogeneities of the problem at hand since those modes appropriate to the particular geometry in question can in principle be determined. In addition, through application of orthogonality relationships involving associated adjoint modes, the equations that determine the time-dependent coefficients can be simplified. Unfortunately, however, in two- and three-dimensional cases it is very difficult to determine any ω-modes other than the fundamental (everywhere positive) one. The specially chosen trial-function approach[10] remains as a very real practical possibility. However it has the disadvantage that the trial functions have no built-in orthogonality properties, and hence their use leads to rather complex equations for determining the behavior of the time-dependent coefficients in situations where feedback is involved. The variant of the trial-function method known as space-time synthesis[10], in which one- (or two-) dimensional spatial trial functions are combined into two- (or three-) dimensional space-time solutions appears also to be particularly powerful. However, especially in the case of feedback, it results in even more complicated equations specifying the behavior of the coefficients (which now depend on one space dimension as well as on time). Thus all the modal methods have advantages and disadvantages which must be balanced in choosing which method to apply to a specific case or to develop as a general-purpose tool.

The modal methods just enumerated are not the must naturally suited to deal with transients in which delayed-neutron behavior is important. For such situations some generalization of the conventional kinetics equations (which have the form given for example by Eqs. 13 and 14 of Ref. 11) appears advantageous. For the separable space-time case, a single input quantity—reactivity—which can frequently be specified by predetermined coefficients, determines the behavior of all six delayed-neutron precursor groups as well as the neutron flux. If this simplicity of description can be retained for nonseparable situations, numerical computations ought to be somewhat simplified.

Two formal generalizations of the point equations have been described in the literature. The so-called 'adiabatic' approach[12] results in kinetics equations having the conventional form but con-

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[3]J. W. RIESE and G. COLLIER, "VARI-QUIR: A Two-Dimensional Time-Dependent Multi-Group Diffusion Code," WANL-TNR-133 (1963).

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[9]A. M. WEINBERG, *Am. J. Phys.* **20**, 401 (1952).

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[11]A. F. HENRY, "Computation of Parameters Appearing in the Reactor Kinetics Equations," WAPD-142 (December 1955).

[12]A. F. HENRY, "Application of Reactor Kinetics to the Analysis of Experiments," *Nucl. Sci. Eng.* **3**, p. 52 (1958).

taining input parameters which depend on the instantaneous shape of the flux in the reactor. In applying this method to nonseparable problems, some modal or nodal scheme for determining this shape must be applied[13]. Thus the adiabatic formalism does not itself completely specify how to solve a nonseparable space-time problem. The other generalization[14,15] consists of defining an inhour formula and a corresponding set of flux modes for each spatial harmonic. Nonseparable flux shapes and delayed-neutron-precursor concentrations can be expanded in these modes.

The present paper is concerned with a further exploration of this latter generalization. It will be argued that generalizing the inhour equation in this fashion is a practical and useful procedure only when the effects caused by higher-order energy and angular modes can be neglected. Thus the formalism will be useful primarily for large reactors in which spatial harmonics die out more slowly than those associated with directional and energy effects. Under these circumstances generalized reactivity coefficients that excite particular spatial harmonics can be defined and used along with the conventional coefficients that excite the fundamental to treat nonseparable problems in which feedback effects are important. The generalization to be explored is also useful in the analysis of certain experiments—for example rod-jerk or pulsed-neutron experiments performed to estimate the degree of subcriticality of a reactor. It will be shown that the assumptions about the independence of flux shapes and period—assumptions almost invariably made whenever the inhour formalism is actually used—lead to certain identities involving the roots of the inhour formula and useful in analysing kinetic phenomena in which there is no feedback.

In succeeding sections we shall first review the ω-mode formalism, next define and discuss what we shall call "inhour modes" and finally describe some applications of this type of modal analysis.

THE ω-MODE FORMALISM

For present purposes it is sufficient to start with the time-dependent transport equation with time-independent cross sections and a single fissionable isotope. With these restrictions, $N(\underset{\sim}{r},\underset{\sim}{\Omega},u,t)$, the number density of neutrons at $\underset{\sim}{r}$ having direction $\underset{\sim}{\Omega}$ and lethargy u at time t and $C_i(\underset{\sim}{r},u,t)$, the associated i-th group delayed-neutron-precursor concentration are specified by

$$\left.\begin{aligned}
&-\underset{\sim}{\Omega}\cdot\nabla v(u)N(\underset{\sim}{r},\underset{\sim}{\Omega},u,t) - \Sigma(\underset{\sim}{r})v(u)N(\underset{\sim}{r},\underset{\sim}{\Omega},u,t) + \\
&+ \int du' \int d\underset{\sim}{\Omega}' \left[\Sigma_s(\underset{\sim}{r},u',u,\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega}) + \right. \\
&\left. + \nu\Sigma_f(\underset{\sim}{r},u')f_p(u)(1-\beta)\right]v(u')N(\underset{\sim}{r},\underset{\sim}{\Omega}',u',t) + \\
&+ \sum_i f_i(u)\lambda_i C_i(\underset{\sim}{r},t) + Q(\underset{\sim}{r},\underset{\sim}{\Omega},u,t) = \frac{\partial}{\partial t}N(\underset{\sim}{r},\underset{\sim}{\Omega},u,t) \\
&\qquad \frac{\partial}{\partial t}C_i(\underset{\sim}{r},t) = \\
&= \int du' \int d\underset{\sim}{\Omega}' \nu\Sigma_f(\underset{\sim}{r},u')\beta_i v(u')N(\underset{\sim}{r},\underset{\sim}{\Omega}',u',t) - \lambda_i C_i(\underset{\sim}{r},t).
\end{aligned}\right\} \quad (1)$$

Here $f_p(u)$ and $f_i(u)$ represent prompt and i-th-group delayed-neutron spectra; the rest of the notation is standard (see Ref. 12).

In operator form these equations may be written

$$HN + \sum_i f_i\lambda_i C_i + Q = \frac{\partial N}{\partial t}$$

$$\frac{\partial(f_i C_i)}{\partial t} = M_i N - \lambda_i f_i C_i \quad (2)$$

where for convenience the second of Eqs. 1 has been multiplied by f_i. A comparison of Eq. 1 and 2 defines the H and M_i, which, it should be noted, are independent of time.

Replacing $\frac{\partial N}{\partial t}$ by ωN and $\frac{\partial C_i}{\partial t}$ by ωC_i and neglecting Q leads to the eigenvalue equations which specify the "ω-modes". In matrix form, these are

$$\begin{pmatrix} H & \lambda_1 \dots \lambda_6 \\ M_1 & -\lambda_1 \dots 0 \\ \vdots & \vdots \\ M_6 & 0 \dots -\lambda_6 \end{pmatrix}\begin{pmatrix} N \\ f_1C_1 \\ \vdots \\ f_6C_6 \end{pmatrix} = \omega\begin{pmatrix} N \\ f_1C_1 \\ \vdots \\ f_6C_6 \end{pmatrix} \text{ or } \mathcal{H}\Psi = \omega\Psi. \quad (3)$$

The corresponding adjoint modes are obtained by assuming that solutions to the source-free equation adjoint to Eq. 1 behave as $e^{-\omega^* t}$. The result is

$$\begin{pmatrix} H^* & M_1^* \dots M_6^* \\ \lambda_1 & -\lambda_1 \quad 0 \\ \vdots & \vdots \quad \vdots \\ \lambda_6 & 0 \quad -\lambda_6 \end{pmatrix}\begin{pmatrix} N^* \\ (f_1C_1)^* \\ \vdots \\ (f_6C_6)^* \end{pmatrix} \equiv \mathcal{H}^*\Psi^* = \omega^*\Psi^*. \quad (4)$$

We shall restrict attention to reactors of finite size containing neutrons of finite energy, and impose boundary conditions of zero incoming neutron density and zero outgoing neutron importance. We

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[14] E. R. COHEN, "Some Topics in Reactor Kinetics," 2nd U. N. Int. Conf. on the Peaceful Uses of Atomic Energy, Geneva, A/Conf. P/629 (1958).

[15] T. GOZANI, *Nukleonik* **5**, 55 (1963).

shall further restrict attention to those solutions to Eq. 3 and 4 having discrete (but not necessarily real) eigenvalues, ω and ω^*, and shall assume the corresponding eigenfunctions are linearly independent. Then, in the standard manner, applying the definition of H^* and using a bar to denote complex conjugate we obtain

$$\langle \Psi_a^* | \mathcal{H} | \Psi_b \rangle - \langle \overline{\Psi_b | \mathcal{H}^* | \Psi_a^*} \rangle = 0 = (\omega_b - \overline{\omega_a^*}) \langle \Psi_a^* | \Psi_b \rangle \tag{5}$$

where we have used the Dirac notation to draw attention to the fact that, if (as is the case here) $\langle \Psi_a^* |$ is a row vector with eigenvalue $\overline{\omega_a^*}$ (and the complex conjugate of the column, $|\Psi_a^*\rangle$) and $|\Psi_b\rangle$ is a column vector with eigenvalue ω_b, the inner product is defined as both a product of vectors in the usual sense and an integral over phase space. For *components* of the $\langle \Psi_a^* |$ and $|\Psi_b\rangle$ the same inner-product notation will be used but in this case implies an integral over phase space only. Thus,

$$\langle \Psi_a^* | \Psi_b \rangle = \langle \overline{\Psi_b | \Psi_a^*} \rangle \equiv$$

$$\equiv \int dv \int d\Omega \int du \left(\{ \overline{N_a^*}, \overline{(f_1 C_1)_a^*}, \ldots \overline{(f_6 C_6)_a^*} \} \begin{Bmatrix} N_b \\ (f_1 C_1)_b \\ \vdots \\ (f_6 C_6)_b \end{Bmatrix} \right)$$

$$= \int dv \int d\Omega \int du \left[\overline{N_a^*} N_b + \sum_{i=1}^{6} \overline{(f_i C_i)_a^*} (f_i C_i)_b \right]$$

$$= \langle N_a^* | N_b \rangle + \sum_{i=1}^{6} \langle (f_i C_i)_a^* | (f_i C_i)_b \rangle$$

$$= \langle \overline{N_b | N_a^*} \rangle + \sum_{i=1}^{6} \langle \overline{(f_i C_i)_b | (f_i C_i)_a^*} \rangle . \tag{6}$$

For every $|\Psi_b\rangle$ we shall assume there is at least one vector (call it $\langle \Psi_b^* |$) such that $\langle \Psi_b^* | \Psi_b \rangle$ is finite and non-zero. If for a given $|\Psi_b\rangle$ there are a number of vectors that yield this condition, the assumed linear independence of the $\langle \Psi_a^* |$ permits taking linear combinations such that all combinations save one are orthogonal to $|\Psi_b\rangle$.

These assumptions along with Eq. 5 imply that the eigenvalues of Eq. 4 are the complex conjugates of those of Eq. 3 and permit an orthogonality relation to be written. Thus

$$\langle \Psi_a^* | \Psi_b \rangle = \delta_{ab} . \tag{7}$$

For any particular eigenvalue, Eqs. 3 and 4 can be used to relate the components $f_i C_i$ and $(f, C_i)^*$ respectively to the N and N^*. We obtain

$$\left.\begin{aligned} [f_i(u) C_i(\underset{\sim}{r})]_b &= \frac{M_i N_b(\underset{\sim}{r}, \underset{\sim}{\Omega}, u)}{\omega_b + \lambda_i} \\ [f_i(u) C_i(\underset{\sim}{r})]_a^* &= \frac{\lambda_i N_a^*(\underset{\sim}{r}, \underset{\sim}{\Omega}, u)}{\omega_a + \lambda_i} \end{aligned}\right\} \tag{8}$$

so that Eqs. 6 and 7 lead to

$$\langle N_a^* | N_b \rangle + \sum_{i=1}^{6} \frac{\lambda_i \langle N_a^* | M_i | N_b \rangle}{(\lambda_i + \omega_a)(\lambda_i + \omega_b)} = \delta_{ab} . \tag{9}$$

Finally, writing out the first line of Eq. 3, $H|N_a\rangle + \sum_i \lambda_i |f_i C_i\rangle a = \omega_a |N_a\rangle$ operating by $\langle N_a^* |$, assuming $\langle N_a^* | N_a \rangle \neq 0$, and using Eq. 8 along with the definitions

$$\left.\begin{aligned} \left(\frac{\beta_i}{\Lambda}\right)_a &\equiv \frac{\langle N_a^* | M_i | N_a \rangle}{\langle N_a^* | N_a \rangle} \\ \left(\frac{\rho}{\Lambda}\right)_a &\equiv \frac{\langle N_a^* | H + \sum_i M_i | N_a \rangle}{\langle N_a^* | N_a \rangle} \end{aligned}\right\} \tag{10}$$

we get the general relationship

$$\left(\frac{\rho}{\Lambda}\right)_a = \omega_a + \sum_{i=1}^{6} \left(\frac{\beta_i}{\Lambda}\right)_a \left(\frac{\omega_a}{\omega_a + \lambda_i}\right) . \tag{11}$$

Equation 11 has the form of the inhour formula and has usually been considered the most straightforward generalization of that relationship[15]—a generalization which permits finding time constants associated with all the higher modes $|\Psi_a\rangle$ as well as with the fundamental. There is however a feature of Eq. 11 that is generally overlooked in the usual application of the inhour formula to the fundamental shape. This oversight is sometimes important for the fundamental[16]; it is increasingly important for higher harmonics. Specifically, because of Eq. 3, the neutron-density function $|N_a\rangle$ is in general a different function of $\underset{\sim}{r}$, $\underset{\sim}{\Omega}$ and μ for each ω_a. Hence $\left(\frac{\rho}{\Lambda}\right)_a$ and the $\left(\frac{\beta_i}{\Lambda}\right)_a$ as determined by Eq. 10 depend implicitly on ω_a. As a result, for any *particular* value of $\left(\frac{\rho}{\Lambda}\right)_a$ and the $\left(\frac{\beta_i}{\Lambda}\right)_a$ (corresponding, for example, to an eigenvalue ω_a' of Eq. 3, six of the seven roots, ω_a, of Eq. 11, will *not* be eigenvalues of Eq. 3. Only the one root, ω_a', (used in the computation of $\left(\frac{\rho}{\Lambda}\right)_a$ and the $\left(\frac{\beta_i}{\Lambda}\right)_a$) will be an eigenvalue of Eq. 3.

If it were necessary in all cases to discard six of the seven roots of Eq. 11 obtained for fixed values of $\left(\frac{\rho}{\Lambda}\right)_a$ and the $\left(\frac{\beta_i}{\Lambda}\right)$, the inhour relationship

[16] A. F. HENRY, *Trans. Am. Nucl. Soc.* **6**, No. 2, p. 212 (1963).

would be quite useless, and all periods would have to be found by direct solution of Eq. 3. Fortunately, as will be discussed further below, some (but not all) of the eigenfunctions of Eq. 3 fall into clusters of seven, all vectors in a given cluster having density functions $|N_a>$ which are approximately the same function of $\underset{\sim}{r}$, Ω and μ. Thus the seven $|N_a>$ of a given cluster lead through Eq. 10 to seven values of $\left(\frac{\rho}{\Lambda}\right)_a$ (and of the $\left(\frac{\beta_i}{\Lambda}\right)_a$) which are approximately the same, so that any one of the seven sets, $\left(\frac{\rho}{\Lambda}\right)_a$; $\left(\frac{\beta_i}{\Lambda}\right)_a$, belonging to a cluster will yield approximately the same roots for Eq. 11 as any other. For those eigenfunctions of Eq. 3 that are members of a cluster we shall make use of a double subscript. Thus we define a subset $|\Psi_{nJ}>$ of the vectors $|\Psi_a>$ (and $<\Psi_{mK}|$ of the $<\Psi_b^*|$) with $n = 0,1,2\ldots\infty$; $J = 0,1,\ldots 6$ so that the seven neutron-density shapes, $|N_{nJ}>$, for fixed n and $J = 0,1\ldots 6$ are all approximately the same functions of position, angle and lethargy. Under these circumstances, $\left(\frac{\rho}{\Lambda}\right)_{nJ}$ and the $\left(\frac{\beta_i}{\Lambda}\right)_{nJ}$ in Eq. 11 will be approximately independent of J for a fixed n. If this slight J-dependence is neglected, the generalized inhour formula Eq. 11 will produce roots, ω_{nJ}, in clusters of seven, each root of a given cluster being approximately an eigenvalue of Eq. 3. We shall refer to the eigenvectors $|\Psi_{nJ}>$ as "inhour modes". Since they are a subset of the modes $|\Psi_b>$, Eqs. 5 through 11 are still valid with the subscript replacements $a \rightarrow nJ$; $b \rightarrow mK$. (δ_{ab} becomes $\delta_{nm}\,\delta_{JK}$.)

Not all the eigenfunctions of Eq. 3 can be identified with a cluster. Specifically, those eigenfunctions representing higher angular and energy modes are excluded. To find from Eq. 11 an eigenvalue ω corresponding to one of these it is *essential* that the $\left(\frac{\rho}{\Lambda}\right)_a$ and $\left(\frac{\beta_i}{\Lambda}\right)_a$ corresponding to that particular eigenvalue be used. But, to find the function $|N_a>$ needed to compute these quantities, the eigenvalue in question will already have to be known from having solved Eq. 3. Hence there is no reason to apply Eq. 11 at all. It is because of this situation that we shall restrict use of Eq. 11 to values of $\left(\frac{\rho}{\Lambda}\right)_a$ and the $\left(\frac{\beta_i}{\Lambda}\right)_a$ computed using inhour modes $|\Psi_{nJ}>$. In fact in most practical situations Eq. 11 will be useful only when the neutron-density part of the vector $|\Psi_{nJ}>$ is such that $|N_{nJ}> = |N_{nK}>$; $J, K = 0,1\ldots 6$.

INHOUR MODES

We have characterized a cluster of inhour modes as a set of seven vectors, $|\Psi_{nJ}>$, $J = 0, 1\ldots 6$, all having a first element, $|N_{nJ}>$, which, except for normalization, is approximately the same function of $\underset{\sim}{r}$, Ω and μ. This characterization is made unsatisfyingly imprecise by the word "approximately". To illustrate how in principle this vagueness can be removed and also to show for a specific case how the quantities $\left(\frac{\rho}{\Lambda}\right)$ and $\left(\frac{\beta}{\Lambda}\right)$ in the inhour formula depend on period, we shall consider the simple case of a two-energy-group P-1 approximation with isotropic scattering applied to a bare slab.

If subscripts 1 and 2 refer to a fast and thermal group and if ϕ and J refer to the P-0 and P-1 components of the vector flux, the appropriate one-dimensional two-group P-1 equations are

$$\left\{\begin{array}{l} \dfrac{\partial J_1}{\partial x} + (\Sigma_1 + \dfrac{\omega}{v_1})\phi_1 = \nu\Sigma_{f_2}\phi_2(1-\beta) + \sum_i C_i\lambda_i \\ \dfrac{\partial \phi_1}{\partial x} + 3(\Sigma_{t_1} + \dfrac{\omega}{v_1})J_1 = 0 \\ \dfrac{\partial J_2}{\partial x} + (\Sigma_2 + \dfrac{\omega}{v_2})\phi_2 = \Sigma_{r_1}\phi_1 \\ \dfrac{\partial \phi_2}{\partial x} + 3(\Sigma_{t_2} + \dfrac{\omega}{v_2})J_2 = 0 \\ \omega C_i = \beta_i\nu\Sigma_{f_2}\phi_2 - \lambda_i C_i \qquad i = 1, 2, \ldots 6 \end{array}\right. \tag{12}$$

where Σ_{t_i} is the total absorption plus scattering cross section of group i

Σ_{r_1} is the removal cross section for group 1

$$\Sigma_1 \equiv \Sigma_{t_1} - \Sigma_{s_1} + \Sigma_{r_1} = \Sigma_{a_1} + \Sigma_{r_1}$$

$$\Sigma_2 \equiv \Sigma_{t_2} - \Sigma_{s_2} = \Sigma_{a_2}$$

For a bare core ϕ_i and J_i are combinations of $e^{iB_n x}$ and $e^{-iB_n x}$ where $B_n = \frac{n\pi}{2L}$, $n = 1, 2\ldots\infty$, L being the extrapolated half-thickness. Either one of these linearly independent solutions when inserted into Eq. 12 yields a critical determinant which can be reduced to the "inhour" form

$$1 - \frac{\left[\Sigma_1 + \dfrac{B_n^2}{3\left(\Sigma_{t_1} + \frac{\omega}{v_1}\right)}\right]\left[\Sigma_2 + \dfrac{B_n^2}{3\left(\Sigma_{t_2} + \frac{\omega}{v_2}\right)}\right]}{\Sigma_{r_1}\nu\Sigma_{f_2}} = \left\{\frac{\dfrac{1}{v_1}\left[\Sigma_2 + \dfrac{B_n^2}{3\left(\Sigma_{t_2} + \frac{\omega}{v_2}\right)}\right] + \dfrac{1}{v_2}\left[\Sigma_1 + \dfrac{B_n^2}{3\left(\Sigma_{t_1} + \frac{\omega}{v_1}\right)}\right] + \dfrac{\omega}{v_1 v_2}}{\Sigma_{r_1}\nu\Sigma_{f_2}}\right\}\omega + \sum_{i=1}^{6}\frac{\omega\beta_i}{\omega + \lambda_i} \tag{13}$$

or
$$\rho = \Lambda\omega + \sum_i \frac{\omega\beta_i}{\omega + \lambda_i} \,. \tag{13a}$$

Equation 13 illustrates why we restrict the use of the inhour form to values of ω belonging to a cluster. For a given value of B_n^2 (spatial mode) there are ten roots of Eq. 13. For the six of lowest magnitude (those approximately equal to the negative of the λ_i) the quantities ρ and Λ are essentially constant, and very exact values of these roots can be obtained by neglecting the $\frac{\omega}{v_i}$ terms in ρ and Λ. The ω of next lowest magnitude can also be found approximately in this manner, although, in this case, some iterative correction to ρ and Λ may be required. However the remaining three roots of Eq. 13, which are associated physically with angular and energy transients, depend essentially on the fact that ρ and Λ are functions of ω. For these roots the inhour form seems inappropriate.

The eigenvector of Eq. 12 associated with any particular root, ω_{nJ}, of Eq. 13 and meeting the spatial boundary conditions of the problem is

$$|\Psi_{nJ}\rangle = \begin{bmatrix} \phi_{1_{nJ}} \\ J_{1_{nJ}} \\ \phi_{2_{nJ}} \\ J_{2_{nJ}} \\ C_{1_{nJ}} \\ \vdots \\ C_{6_{nJ}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Sigma_{r_1}}\left[\Sigma_2 + \frac{\omega_{nJ}}{v_2} + \frac{B_n^2}{3\left(\Sigma_{t_2} + \frac{\omega_{nJ}}{v_2}\right)}\right]\sin B_n x \\ -\frac{B_n}{3\left(\Sigma_{t_1} + \frac{\omega_{nJ}}{v_1}\right)}\frac{1}{\Sigma_{r_1}}\left[\Sigma_2 + \frac{\omega_{nJ}}{v_2} + \frac{B_n^2}{3\left(\Sigma_{t_2} + \frac{\omega_{nJ}}{v_2}\right)}\right]\cos B_n x \\ \sin B_n x \\ -\frac{B_n}{3\left(\Sigma_{t_2} + \frac{\omega_{nJ}}{v_2}\right)}\cos B_n x \\ \frac{\beta_1 \nu \Sigma_{f2}}{\lambda_1 + \omega_{nJ}}\sin B_n x \\ \vdots \\ \frac{\beta_6 \nu \Sigma_{f2}}{\lambda_6 + \omega_{nJ}}\sin B_n x \end{bmatrix} \tag{14}$$

where Eqs. 12 have been used to determine the ratio $\phi_{1_{nJ}} : J_{1_{nJ}} : \phi_{2_{nJ}}$, etc. as a function of ω_{nJ}.

If, for a fixed value of n, all roots of Eq. 13 are admitted, ten vectors $|\Psi_{nJ}\rangle$ result. For this slab core example the components of all of these have spatial shapes, $\sin B_n x$ or $\cos B_n x$, independent of ω_{nJ}. However the "energy shapes" ($\phi_{1_{nJ}} : \phi_{2_{nJ}}$ and $J_{1_{nJ}} : J_{2_{nJ}}$) and the "angular shapes" $\frac{\phi_{1nJ}}{\sin B_n x} : \frac{J_{1nJ}}{\cos B_n x}$ and $\frac{\phi_{2nJ}}{\sin B_n x} : \frac{J_{2nJ}}{\cos B_n x}$) of the neutron flux do depend on ω_{nJ}. For the three large roots of Eq. 11 this dependence is very strong. For the seven smallest roots it is weak and can in many cases be neglected. Hence in this two-group P-1 model the seven vectors, $|\Psi_{nJ}\rangle$, $J = 0, 1 \ldots 6$, belonging to a cluster do indeed have neutron flux shapes which are approximately the same function of position, direction and lethargy. They differ from one another significantly only in the ratio of neutron density to delayed-neutron-precursor concentration.

For the model just considered it is easy to identify members of a given cluster of inhour modes. However, we shall describe the procedure as a general method which will be needlessly complicated for the case at hand but which can be extended fairly readily to more complex situations. We begin by integrating over direction and lethargy (but *not* volume) the approximation to the neutron density in phase space given by the model in this case obtaining for the total neutron concentration at point $\underset{\sim}{r}$,

$$N(\underset{\sim}{r},\omega_{nJ}) = \left\{\frac{1}{v_1\Sigma_{r_1}}\left[\Sigma_2 + \frac{\omega_{nJ}}{v_2} + \frac{B_n^2}{3\left(\Sigma_{t_2} + \frac{\omega_{nJ}}{2}\right)}\right] + \frac{1}{v_2}\right\}\sin B_n x$$

Since Eqs. 12 are real (or, more generally, $\mathcal{H}$ of Eq. 3 is real), the eigenvalues ω_{nJ}, if complex, exist in conjugate pairs and the neutron concentration corresponding to the complex conjugate, $\bar{\omega}_{nJ}$, will be $\bar{N}(\underset{\sim}{r},\omega_{nJ}) = N(\underset{\sim}{r},\bar{\omega}_{nJ})$. If the ω_{nJ} *are* com-

plex, we then form the real density function $\frac{1}{2}(\overline{N}+N)$ and refer to this function (or $N(\underset{\sim}{r},\omega_{nJ})$ itself if ω_{nJ} is real) as the total neutron concentration. We then note the number of isolated spatial regions in which this total neutron concentration is either positive or negative. This number serves to classify the eigenvector, Eq. 14, as one having a particular spatial shape. We call it the "shape index" and proceed to search for all other vectors with the same shape index and having eigenvalues with real part more positive than that of the one corresponding to the vector in question. If there are six or fewer such vectors, the vector in question belongs to the cluster $|\Psi_{nJ}>$ and its complex conjugate belongs to the cluster $|\overline{\Psi}_{nJ}>$. (Whether a given vector of a conjugate pair belongs to $|\Psi_{nJ}>$ or to $|\overline{\Psi}_{nJ}>$ is somewhat arbitrary.) If there are more than six such vectors, the vector in question is not an inhour mode and hence belongs to no cluster of seven vectors.

It appears possible to extend this method of identifying inhour modes to more elaborate mathematical models and to general geometrical configurations. For these cases the neutron-density shapes will be slightly different for each of the seven vectors of a cluster. However the shape index merely counts the number of isolated regions where the neutron density is positive or negative. Hence, since we have already restricted attention to eigenvectors with discrete eigenvalues, it can still be used to group these vectors into sets having roughly similar shapes.

One clear difficulty does arise in extending to more than one dimension. To illustrate this, consider a bare cube reactor having a fundamental neutron concentration shape in two dimensions and some harmonic in the third. For this core there will be three clusters of inhour modes having the same set of eigenvalues, and the three-dimensional neutron-density shapes associated with all three clusters will be the same except for their axes of orientation. Hence in this case there will actually be twenty-one (rather than seven) inhour modes having the same shape index. It is likely that an analogous situation will arise for any three-dimensional geometry even when there is no symmetry in the core (although, of course, without symmetry there will be no degeneracy in the eigenvalues). Accordingly in the general case the procedure for classifying inhour modes must be augmented by a qualitative examination to determine which orientation is to be associated with a shape of a given shape index. Modes of the same shape index but corresponding to different axis orientation are so extremely different in shape that no difficulty in distinguishing them is to be expected.

The procedure just outlined for identifying and classifying inhour modes appears to be rigorous for a P_N, multigroup model applied to a homogeneous bare slab with Marshak boundary conditions taken independent of energy. That it can be extended to cores of arbitrary composition in general geometries is a conjecture. A more precise mathematical viewpoint may make the exact natures of this conjecture more clear. To achieve this we shall use the last six of Eqs. 3 to eliminate the C_i and write the result in explicit form. We obtain

$$-\underset{\sim}{\Omega}\cdot\nabla vN-(\Sigma v+\omega)N+\int du'\int d\Omega'\left[\Sigma_s(u',u,\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega},\underset{\sim}{r})+\right.$$
$$\left.+\nu\Sigma_f(u',\underset{\sim}{r})f_p(u)(1-\beta)\right]v(u')N(\underset{\sim}{r},\underset{\sim}{\Omega}',u')+ \qquad (15)$$
$$+\sum_{i=1}^{6}\frac{\lambda_i\beta_i f_i(u)}{\omega+\lambda_i}\int du'\int d\Omega'\nu\Sigma_f(u',\underset{\sim}{r})v(u')N(\underset{\sim}{r},\underset{\sim}{\Omega}',u')=0.$$

Examination of this equation suggests that there are an infinite number of negative eigenvalues ω grouped about each decay constant λ_i. For these, $|\omega|<<\Sigma v$, and we see that if $\Sigma v+\omega$ is replaced by Σv alone, and the delayed-neutron spectra $f_i(u)$ are taken to be the same for all precursor groups, $(f_i(u)=f_d(u);\ i=1,2\ldots 6)$ the whole term, $\sum_i\frac{\lambda_i\beta_i}{\omega+\lambda_i}$, becomes an eigenvalue, and solutions, $N(\underset{\sim}{r},\underset{\sim}{\Omega},u)$, appear in degenerate clusters of six. It is extremely unlikely that removing the degeneracy by accounting for differences in the $f_i(u)$ and by including ω in the term $\Sigma v+\omega$ will change any of the spatial flux shapes $\int du\int d\Omega N(\underset{\sim}{r},u,\underset{\sim}{\Omega})$ sufficiently to prevent its being recognized as belonging to a given cluster of six. Hence even if the spatial index is not a valid quantity for classifying vectors into elements of similar shape, clustering into sets of six appears to be an unambiguous procedure.

It is when we add a seventh vector to the clusters of six that we must bring the spatial index into the picture. This seventh root of an inhour cluster always has magnitude, $|\omega|>>\lambda_i$. Hence the last term in Eq. 15 is expected to have little influence on the solution, and the ω in $(\Sigma v+\omega)N$ serves as the eigenvalue. The eigenfunction associated with this eigenvalue is quite likely to differ considerably in detail from the other six vectors in the cluster, particularly if the cluster is for a high-order spatial harmonic. The most questionable part of the suggested procedure for defining a cluster is then the conjecture that, for a given axis orientation, the spatial index of the seventh vector will be the same as that of the other six. We shall assume it is.

IDENTITIES ARISING FROM THE USE OF INHOUR MODES

The chief purpose of focusing attention on inhour modes is that they appear to provide a simple extension to the nonseparable case of the standard methods for treating reactor transients involving feedback phenomena. It is apparent from their definition that these modes are not capable of describing situations in which angular and energy transients are important since all vectors associated explicitly with angular or energy harmonics have been excluded from consideration. Hence the only angular or energy transients that inhour modes are capable of describing are those that arise from the fact that the different spatial shapes (which the inhour modes *do* describe) may have associated with them different angular distributions and energy spectra. The consequence of this from a physical viewpoint is that inhour modes are best applied to transients in the range of tens-of-milliseconds to minutes. For example, flux tilts due to control-rod motion or coolant-flow conditions in large thermal reactors appear to be well suited to analysis by this method. The relatively slow change in concentration of delayed-neutron-precursor concentrations from an initial to a final distribution can affect significantly the space-time power behavior of such transients, and this phenomenon can be adequately accounted for by inhour modes.

Accordingly we shall consider a subspace of all functions which can be expressed as a linear combination of inhour modes. We shall first derive certain general properties of operators formed from eigenfunctions and adjoint eigenfunctions in the subspace and then apply these operators to the development of certain identities involving roots of the inhour equation.

A. Operators Formed from Inhour Modes

By definition, any vector $|\phi>$ in the subspace of inhour modes can be expanded in the $|\Psi_{mK}>$ as

$$|\Phi> = \sum_{m=0}^{\infty} \sum_{K=0}^{6} a_{mK} |\Psi_{mK}> . \tag{16}$$

Consider the operator $\sum_{nJ} |\Psi_{nJ}> < \Psi^*_{nJ}|$. If this sum of dyads operates on Eq. 16, and 7 with $a \rightarrow nJ$; $b \rightarrow mK$ is used, we find

$$\sum_{n,J} |\Psi_{nJ}> < \Psi^*_{nJ}|\Phi> = \sum_{n,J} a_{nJ} |\Psi_{nJ}> = |\Phi>. \tag{17}$$

Therefore, with respect to all vectors that can be expressed as a sum of the $|\Psi_{nJ}>$, the operator $\sum_{n,J} |\Psi_{nJ}><\Psi^*_{nJ}|$ is the identity operator, $\mathcal{I}$. Writing out the vectors $|\Psi_{nJ}>$ and the operator $<\Psi^*_{nJ}|$ as in Eq. 6 and using Eq. 8, we get

$$\mathcal{I} = \sum_{nJ} \begin{bmatrix} |N_{nJ}> \\ \dfrac{M_1|N_{nJ}>}{\omega_{nJ}+\lambda_1} \\ \vdots \\ \dfrac{M_6|N_{nJ}>}{\omega_{nJ}+\lambda_6} \end{bmatrix} \left[<N^*_{nJ}| \frac{\lambda_1 <N^*_{nJ}|}{\omega_{nJ}+\lambda_1} \cdots \frac{\lambda_6 <N^*_{nJ}|}{\omega_{nJ}+\lambda_6} \right]$$

$$= \sum_{nJ} \begin{bmatrix} 1 & \dfrac{\lambda_1}{\omega_{nJ}+\lambda_1} & \dfrac{\lambda_2}{\omega_{nJ}+\lambda_2} & \cdots & \dfrac{\lambda_6}{\omega_{nJ}+\lambda_6} \\ \dfrac{M_1}{\omega_{nJ}+\lambda_1} & \dfrac{M_1\lambda_1}{(\omega_{nJ}+\lambda_1)^2} & \dfrac{M_1\lambda_2}{(\omega_{nJ}+\lambda_1)(\omega_{nJ}+\lambda_2)} & \cdots & \dfrac{M_1\lambda_6}{(\omega_{nJ}+\lambda_1)(\omega_{nJ}+\lambda_6)} \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{M_6}{\omega_{nJ}+\lambda_6} & \dfrac{M_6\lambda_1}{(\omega_{nJ}+\lambda_6)(\omega_{nJ}+\lambda_1)} & \dfrac{M_6\lambda_2}{(\omega_{nJ}+\lambda_6)(\omega_{nJ}+\lambda_2)} & \cdots & \dfrac{M_6\lambda_6}{(\omega_{nJ}+\lambda_6)^2} \end{bmatrix} |N_{nJ}><N^*_{nJ}| . \tag{18}$$

If the operator in this expanded form operates on some particular vector $|\Phi> \equiv \text{col}\,[|\phi_0> |\phi_1> \ldots |\phi_6>]$, we find

$$\sum_{nJ}\begin{bmatrix} |N_{nJ}><N^*_{nJ}|\phi_0> + \sum_{i=1}^{6}\frac{\lambda_i|N_{nJ}><N^*_{nJ}|\phi_i>}{\omega_{nJ}+\lambda_i} \\ \frac{M_i|N_{nJ}><N^*_{nJ}|\phi_0>}{\omega_{nJ}+\lambda_i} + \sum_{i=1}^{6}\frac{M_1\lambda_i|N_{nJ}><N^*_{nJ}|\phi_i>}{(\omega_{nJ}+\lambda_1)(\omega_{nJ}+\lambda_i)} \\ \vdots \\ \frac{M_6|N_{nJ}><N^*_{nJ}|\phi_0>}{\omega_{nJ}+\lambda_6} + \sum_{i=1}^{6}\frac{M_6\lambda_i|N_{nJ}><N^*_{nJ}|\phi_i>}{(\omega_{nJ}+\lambda_6)(\omega_{nJ}+\lambda_i)} \end{bmatrix} = \mathcal{J}\Phi = \begin{bmatrix} |\phi_0> \\ |\phi_1> \\ \vdots \\ |\phi_6> \end{bmatrix} . \quad (19)$$

If the vector $|\Phi>$ could be assumed to be completely arbitrary—in particular, if vectors having all components except one equal to zero could be admitted—certain auxiliary identities involving some of the operators $|N_{nJ}><N^*_{nJ}|$ could be proved. However, the operator $\mathcal{J}$, (Eq. 18) is the identity operator only for functions belonging to the subspace of inhour modes, and, in general, all the vectors in this subspace have finite components. Hence before proceeding further it will first be necessary to restrict attention to an approximate set of inhour modes.

B. *Identities Involving Roots of the Inhour Formula*

In standard applications of the conventional reactor kinetics equation (those corresponding to the fundamental spatial mode in the present generalization) it is always assumed that the flux shapes associated with all seven modes of the fundamental inhour cluster are the same. If this assumption were not made, computations would become extremely complex. For example, quantities like temperature coefficients of reactivity would become dependent on the instantaneous period of reactor power level.

In making the first-order extension of the conventional kinetics equations to the nonseparable case, it seems reasonable again to assume that all seven vectors in an inhour cluster have the same first element,—i.e., to assume $|N_{n\bar{J}}> = |N_{nK}>$ for all J and K. The assumption becomes increasingly suspect as the order of the spatial harmonic (value of n) increases. For small reflected cores it can be a poor approximation even in the fundamental mode[15,16]. (Note that the inhour formula as conventionally used yields, for some cases, some roots which are incorrect[16]. However, in unreflected homogeneous systems it is an excellent assumption, and for any large reactor it should be quite satisfactory for the first several clusters of inhour modes.)

The assumption that $|N_{nJ}> = |N_{nK}>$ for all J,K (and its analogue $<N^*_{mJ}| \sim <N^*_{mK}|$) give rise to a number of useful mathematical simplifications. First of all, the general orthogonality relation, $<\Psi^*_{nJ}|\Psi_{mK}> = \delta_{nm}\,\delta_{jK}$, yields several analogous relationships involving the flux shapes alone. Thus, if the shapes $|N_{nK}>$ are the same for all K, we may form a linear combination such that

$$\sum_{K=0}^{6} a_K|\Psi_{mK}> = \begin{pmatrix} |N_{mK}> \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad a_K \neq 0,\ K = 0,1\ldots 6.$$

Then by Eq. 7, with $a \rightarrow nJ$; $b \rightarrow mK$,

$$<\Psi^*_{nJ}|\sum_{K=0}^{6} a_K|\Psi_{mK}> = a_{\bar{J}}\,\delta_{nm} = <N^*_{nJ}|N_{mK}> = <N^*_{n\bar{J}}|N_{mJ}>.$$

Therefore (if $|N_{mJ}> = |N_{mK}>$),

$$<N^*_{nJ}|N_{mJ}> \sim \delta_{nm} . \quad (20)$$

Similarly, by forming linear combinations such as

$$\sum_{K=0}^{6} b_K|\Psi_{mK}> = \begin{pmatrix} 0 \\ M_1|N_{mJ}> \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad b_K \neq 0,\ K = 0,1\ldots 6$$

we can show that (if $|N_{mJ}> = |N_{mK}>$),

$$<N^*_{nJ}|M_i|N_{m\bar{J}}> \sim \delta_{nm};\ i = 1,2\ldots 6 . \quad (21)$$

Finally, from the first line of Eq. 3 operated on by $<N^*_{nJ}|$ we get, using Eq. 8,

$$<N^*_{nJ}|H|N_{mK}> + \sum_i \frac{<N^*_{nJ}|M_i|N_{mK}>}{\omega_{mK}+\lambda_i} = <N^*_{nJ}|N_{mK}>\omega_{mK} \quad (22)$$

which, when Eqs. 20 and 21 are applied, shows that, (if $|N_{mK}> = |N_{mK}>$)

$$<N^*_{nJ}|H|N_{mJ}> \sim \delta_{nm} . \quad (23)$$

The vector $|\Phi>$ for which Eq. 19 is valid can be any linear combination of inhour modes. If we now restrict attention to the approximate modes for which $|N_{mJ}> = |N_{mK}>$, Eq. 19 is again valid when the operators, $|N_{nJ}><N^*_{nJ}|$ in $\mathcal{I}$ are constructed from these approximate functions and when $|\Phi>$ is composed of these approximate inhour modes. But we have just seen that the approximate inhour modes permit the construction of vectors for which all the components except one vanish. It follows that, with $|\Phi> = \text{col}\,(|\phi_0>, 0, 0 \ldots 0)$

$$\left.\begin{aligned} &\sum_{nJ} |N_{nJ}><N^*_{nJ}| = I \\ &\sum_{nJ} \frac{M_i |N_{nJ}><N^*_{nJ}|}{(\omega_{nJ} + \lambda_i)} = 0, \quad i = 1, 2 \ldots 6 \end{aligned}\right\} \tag{24a}$$

where I and 0 are the identity and null operators in $(\underset{\sim}{r}, \underset{\sim}{\Omega}, u)$ space for all vectors that can be expressed as a linear combination of the $|N_{nJ}>$. With other choices for $|\Phi>$ it follows from Eq. 19 that

$$\left.\begin{aligned} &\sum_{nJ} \frac{\lambda_i |N_{nJ}><N^*_{nJ}|}{\omega_{nJ} + \lambda_i} = 0, \quad i = 1,2 \ldots 6 \\ &\sum_{nJ} \frac{M_j \lambda_i |N_{nJ}><N^*_{nJ}|}{(\omega_{nJ}+\lambda_i)(\omega_{nJ}+\lambda_j)} = 0 + \delta_{ij} I, \quad i,j = 1,2 \ldots 6 \end{aligned}\right\} \tag{24b}$$

where here I and 0 are the identity and null operators in $(\underset{\sim}{r}, \underset{\sim}{\Omega}, u)$ space for all vectors that can be expressed as linear combinations of the form $\sum_n a_n M_i |N_{nJ}>$, $i = 1,2 \ldots 6$.

If now we operate with the first of the operators Eq. 24 on $|N_{mK}>$ and apply Eq. 20, we find that

$$\sum_{nJ} |N_{n\bar{J}}><N^*_{nJ}|N_{mK}> = |N_{mK}> =$$

$$= \sum_J |N_{mJ}><N^*_{mJ}|N_{mK}> = |N_{mK}> \sum_J <N^*_{mJ}|N_{mJ}>.$$

It follows from Eqs. 9 and 10 that

$$\sum_J \left[\frac{1}{1 + \sum_i \dfrac{\lambda_i \left(\dfrac{\beta_i}{\Lambda}\right)_m}{(\lambda_i + \omega_{m\bar{J}})^2}}\right] = 1 \tag{25}$$

where we have simplified the notation using $\left(\frac{\beta_i}{\Lambda}\right)_{mJ} \to \left(\frac{\beta_i}{\Lambda}\right)_m$ for the shape of $<N^*_{mJ}|$ and $|N_{mJ}>$ independent of J.

From the first operator of Eq. 24b we get, operating on $M_i|N_{mK}>$ and using Eq. 21,

$$\sum_J \frac{\lambda_i |N_{mJ}><N^*_{mJ}|M_i|N_{mK}>}{\omega_{mJ} + \lambda_i} = 0 =$$

$$= \sum_J \frac{\lambda_i \left(\frac{\beta_i}{\Lambda}\right)_m |N_{mK}><N^*_{mJ}|N_{mJ}>}{\omega_{mJ} + \lambda_i}$$

so that

$$\sum_J \frac{1}{\omega_{mJ} + \lambda_i} \left[\frac{1}{1 + \sum_k \dfrac{\lambda_k \left(\dfrac{\beta_k}{\Lambda}\right)_m}{(\lambda_k + \omega_{mJ})^2}}\right] = 0, \; i = 1,2 \ldots 6. \tag{26}$$

We get the same identity from the second operator of Eq. 24a by operating on the left by $<N^*_{mK}|$ and on the right into $|N_{mK}>$. Finally, from the last of the operators Eq. 24b we get

$$\sum_{nJ} \frac{<N^*_{mK}|M_j \lambda_i |N_{nJ}>}{(\omega_{nJ} + \lambda_i)(\omega_{nJ} + \lambda_j)} <N^*_{nJ}|N_{mK}> =$$

$$= \sum_J \frac{\lambda_i \left(\frac{\beta_j}{\Lambda}\right)_m <N^*_{mK}|N_{mK}>}{(\omega_{mJ} + \lambda_i)(\omega_{mJ} + \lambda_j)} <N^*_{mJ}|N_{mJ}> =$$

$$= <N^*_{mK}|N_{mK}> \delta_{ij}$$

or

$$\sum_J \frac{1}{(\omega_{mJ} + \lambda_i)(\omega_{mJ} + \lambda_j)} \left[\frac{1}{1 + \sum_k \dfrac{\lambda_k \left(\dfrac{\beta_k}{\Lambda}\right)_m}{(\lambda_k + \omega_{mJ})^2}}\right] = \tag{27}$$

$$= \delta_{ij}; \quad i,j = 1,2 \ldots 6.$$

Combination of the sum rules Eq. 25-27 with the inhour formula Eq. 11 leads to one other useful identity. When the shape of $<N^*_{mJ}|$ and $|N_{mJ}>$ are the same for all J, the inhour formula becomes

$$\left(\frac{\rho}{\Lambda}\right)_m = \omega_{mJ} + \sum_i \frac{\omega_{mJ} \left(\frac{\beta_i}{\Lambda}\right)_m}{\omega_{mJ} + \lambda_i} \tag{28}$$

which, when combined with Eq. 26 multiplied by $\left(\frac{\beta_i}{\Lambda}\right)_m$ and summed over i, gives

$$\sum_J \frac{\frac{1}{\omega_{mJ}}\left(\frac{\rho}{\Lambda}\right)_m - 1}{1 + \sum_k \dfrac{\lambda_k \left(\dfrac{\beta_k}{\Lambda}\right)_m}{(\lambda_k + \omega_{mJ})^2}} = 0. \tag{29}$$

Thus, with the help of Eq. 24, we find that

$$\sum_J \frac{\left(\frac{\rho}{\Lambda}\right)_m}{\omega_{mJ} + \sum_k \frac{\lambda_k \omega_{mJ} \left(\frac{\beta_k}{\Lambda}\right)_m}{(\lambda_k + \omega_{mJ})^2}} = 1 . \qquad (30)$$

APPLICATIONS

In principle, inhour modes can be applied to any kinetic phenomenon for which transients in the angular and energy distribution of the neutron flux are not significant. In practice, however, they appear to be useful only when it is a valid approximation to assume that the neutron-density shape $|N_{mK}>$ is the same for all vectors of a cluster. Accordingly, in illustrating applications of inhour modes we shall use a "restricted" set of inhour modes. That is, we shall assume the $|N_{mK}>$ are the same for all K and hence that Eqs. 20, 21, 23, 25, 26, 27 and 30 apply.

Illustrative applications to two classes of problems will be made, the first involving feedback phenomena and the second concerned with transients during which all cross sections remain constant. Specifically we shall first consider a generalization of the conventional reactor kinetics equations with arbitrary feedback to the nonseparable space-time problem. Then two experiments, the source-jerk and the pulsed-source technique, will be examined.

A. *Extension of the Kinetics Equations to Nonseparable Transients*

In the development so far we have neglected feedback effects and considered all cross sections to be independent of time. We now extend Eqs. 2 to the general case by writing

$$\left.\begin{aligned} (H + \delta H(t))\,|N(t)> + \sum_i \lambda_i |f_i C_i(t)> + \\ + |Q(t)> = \frac{\partial}{\partial t}|N(t)> \\ (M_i + \delta M_i(t))\,|N(t)> - \lambda_i |f_i C_i(t)> = \frac{\partial}{\partial t}|f_i C_i(t)> \end{aligned}\right\} \qquad (31)$$

where H and M_i are the same (time-independent) operators appearing in Eqs. 2 and 3, while $\delta H(t)$ and $\delta M_i(t)$ represents arbitrary time–dependent perturbations.

With $|N(t)>$ expanded in the (time-independent) shape functions $|N_{mJ}>$ associated with inhour modes of Eq. 3, we may write

$$|N(t)> = \sum_m T_m(t)\,|N_{mJ}> . \qquad (32)$$

If this expression is inserted into Eq. 31, if the resultant equations are operated on by $\frac{<N^*_{nJ}|}{<N^*_{nJ}|N_{nJ}>}$, and if the orthogonality relationships Eqs. 20, 21 and 23 are used along with the definitions Eq. 10 and $Q_n(t) \equiv \frac{<N^*_{nJ}|Q(t)>}{<N^*_{nJ}|N_{nJ}>}$; $C_{ni} \equiv \frac{<N^*_{nJ}|f_i C_i>}{<N^*_{nJ}|N_{nJ}>}$, we obtain

$$\left.\begin{aligned} \left(\frac{\rho-\beta}{\Lambda}\right)_n T_n + \sum_m \frac{<N^*_{nJ}|\delta H|N_{mJ}>}{<N^*_{nJ}|N_{nJ}>} T_m + \sum_i \lambda_i C_{ni} + Q_n = \dot{T}_n \\ \left(\frac{\beta_i}{\Lambda}\right)_n T_n + \sum_m \frac{<N^*_{nJ}|\delta M_i|N_{mJ}>}{<N^*_{nJ}|N_{nJ}>} T_m - \lambda_i C_{ni} = \dot{C}_{ni} . \end{aligned}\right\} \qquad (33)$$

Because of the orthogonality relations applied in obtaining Eqs. 33, the coupling between modes has been considerably reduced. First of all, coupling through the $\dot{T}$ terms has been removed entirely. In addition, the terms involving δM_1 can probably always be neglected. Finally, the terms involving δH are likely to be important only for the fundamental mode ($n=0$). For this mode $\left(\frac{\rho}{\Lambda}\right)_n = \left(\frac{\rho}{\Lambda}\right)_0 = 0$ so that the terms in δH contribute significantly. However for $n \neq 0$, $\left(\frac{\rho}{\Lambda}\right)_n$ will frequently be of significent magnitude that, in comparison, the whole sum of δH terms may legitimately be ignored. As a result, driving terms ($<N^*_{nJ}|\delta H|N_{nJ}>$) and coupling terms ($<N^*_{nJ}|\delta H|N_{mJ}>$) need be considered only for the fundamental-mode equation ($n=0$). Thus solving for the time-dependent coefficients of an expansion in inhour modes presents difficulties little more severe than those associated with the usual point reactor computations.

One important practical problem remains, namely that of determining the inhour modes in the first place. Finding the fundamental (everywhere-positive) mode is straightforward. For symmetric reactors, the first harmonic results if a zero-flux (rather than a zero-current) boundary condition is used about the plane of symmetry. But it is difficult to determine higher harmonics. Unless those modes required for the problem at hand can be found inexpensively, the trial-function approach[10] to solving space-time problems appears to be superior to an expansion in inhour modes.

B. *Experiments Involving Neutron Sources*

For physical situations involving fixed subcritical assemblies and arbitrary time-dependent neutron sources, the appropriate descriptive equations are

$$\mathcal{H}\,|\Psi(t)\rangle + \begin{pmatrix} Q(t)\rangle \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{\partial}{\partial t}\,|\Psi(t)\rangle \qquad (34)$$

where the matrix operator, $\mathcal{H}$, is the same as in Eq. 3 and the column vector representing the source introduces neutrons only. If $|\Psi(t)\rangle$ is expanded in terms of the inhour modes $|\Psi_{mK}\rangle$ of the operator $\mathcal{H}$, so that

$$|\Psi(t)\rangle = \sum_{m,K} a_{mK}(t)\,|\Psi_{mK}\rangle \qquad (35)$$

then substitution into Eq. 34, use of Eq. 3, operation by the row vector $\langle\Psi^*_{nJ}|$ and use of Eq. 7 and 21 leads to

$$a_{nJ}(t)\,\omega_{nJ} + \langle N^*_{nJ}|Q(t)\rangle = \dot{a}_{nJ}(t)\ . \qquad (36)$$

Thus,

$$a_{nJ}(t) = a_{nJ}(0)\,e^{\omega_{nJ}t} + e^{\omega_{nJ}t}\int_0^t \langle N^*_{nJ}|Q(\tau)\rangle\, e^{-\omega_{nJ}\tau}\,d\tau\ . \qquad (37)$$

If neutrons are counted by a counter which has a cross section $\Sigma_c(\underset{\sim}{r},\underset{\sim}{\Omega},u)$ for neutron capture and an efficiency ϵ_n for recording counts belonging to the n-th spatial mode $|N_{nJ}\rangle$, the total counting rate at time t is given by

$$C(t) = \sum_{nJ} a_{nJ}(t)\langle\epsilon_n v\Sigma_c|N_{nJ}\rangle \qquad (38)$$

where v is the velocity of the neutrons being detected.

Equations 37 and 38 represent the analysis in terms of inhour modes of the general time-dependent source problem in a subcritical assembly, the material properties of which are constant in time. They are valid for complex ω_{nJ} and $|N_{nJ}\rangle$. However since available evidence is that only the very high order spatial modes are complex[17], we shall utilize only real modes in the following specific applications.

1. *The Source-Jerk Experiment*

In the source-jerk experiment[18] an external source of constant strength is inserted into a subcritical assembly, left there until equilibrium flux and delayed precursor distributions are obtained, and then suddenly withdrawn.

To analyse the experiment we assume $|Q(t)\rangle$ to be a constant in the range $-\infty < t < 0$ so that, all ω_{nJ} being negative, the $a_{nJ}(0)$ in Eq. 37 are given by

$$a_{nJ}(0) = -\frac{\langle N^*_{nJ}|Q\rangle}{\omega_{nJ}}\ . \qquad (39)$$

Since $|Q(t)\rangle = 0$ for $t \geq 0$, Eqs. 37 and 38 give for the count rate subsequent to withdrawal of the source,

$$C(t) = \sum_{nJ} -\frac{\langle\epsilon_n v\Sigma_c|N_{nJ}\rangle\langle N^*_{nJ}|Q\rangle}{\omega_{nJ}}\,e^{\omega_{nJ}t}\ . \qquad (40)$$

If we now multiply and divide by $\langle N^*_{nJ}|N_{nJ}\rangle$ $\left(= \dfrac{1}{1+\sum_i \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i+\omega_{nJ})^2}}\right.$, by Eqs. 9 and 10 $\left.\right)$ and define

$$S_n \equiv \frac{\langle\epsilon_n v\Sigma_c|N_{nJ}\rangle\langle N^*_{nJ}|Q\rangle}{\langle N^*_{nJ}|N_{nJ}\rangle} \qquad (41)$$

(which by the assumption that the shapes $|N_{nJ}\rangle$ and $\langle N^*_{nJ}|$ are independent of J is itself independent of J) we obtain

$$C(t) = \sum_n S_n \sum_{J=0}^{6}\left[\frac{e^{\omega_{nJ}t}}{-\omega_{nJ}\left\{1+\sum_i\left(\frac{\beta_i}{\Lambda}\right)_n\frac{\lambda_i}{(\lambda_i+\omega_{nJ})^2}\right\}}\right]. \qquad (42)$$

The identities Eqs. 25 and 30, along with the inhour formula Eq. 28 can be used to simplify this equation. We first note that at $t = 0$, application of Eq. 30 leads to

$$C(0) = \sum_n S_n \frac{-1}{\left(\frac{\rho}{\Lambda}\right)_n}\ . \qquad (43)$$

This in itself is the generalization of a well known relationship. Furthermore, since we are expanding in real $\langle N^*_{nJ}|$ and $|N_{nJ}\rangle$, $\left(\frac{\rho}{\Lambda}\right)_m$ and $\left(\frac{\beta}{\Lambda}\right)_m$ are real. As a result the roots of the inhour formula Eq. 28 are real and, (the assembly being subcritical) for a given spatial mode m, one root (call it ω_{n0}) is very large while the other six are close in magnitude to the λ_i. Thus at a short time δt after time zero, the $e^{\omega_{n0}\delta t}$ in Eq. 41 are negligibly small and the $e^{\omega_{nJ}\delta t}$ $(J \neq 0)$ can be written $1 + \omega_{nJ}\,\delta t$. Thus we may write

[17] P. B. DAITCH and D. B. EBEOGLU, *Nucl. Sci. Eng.* **17**, 212 (1963).

[18] JANKOWSKI, KLEIN and MILLER, *Nucl. Sci. Eng.* **2**, 288 (1957).

$$C(\delta t) = \sum_n S_n \sum_{\bar{J}=1}^{6} \frac{1 + \omega_{n\bar{J}}\,\delta t}{-\omega_{n\bar{J}} \left\{1 + \sum_i \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i + \omega_{n\bar{J}})^2}\right\}}$$

$$= \sum_n S_n \left[\sum_{J=0}^{6} \frac{1 + \omega_{nJ}\,\delta t}{-\omega_{nJ} \left\{1 + \sum_i \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i + \omega_{nJ})^2}\right\}} - \right.$$

$$\left. - \frac{1 + \omega_{n0}\,\delta t}{-\omega_{n0} \left\{1 + \sum_i \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i + \omega_{n0})^2}\right\}} \right] \qquad (44)$$

$$= \sum_n S_n \left[-\frac{1}{\left(\frac{\rho}{\Lambda}\right)_n} - \delta t + \frac{1}{\omega_{n0}} + \delta t \right]$$

where Eqs. 25 and 30 have been used to obtain the first two terms, and the fact that $|\omega_{n0}| >> \left(\frac{\beta_i}{\Lambda}\right)_n \lambda_i$ justifies the second two. For the root ω_{n0} of large magnitude the inhour formula Eq. 30 gives

$$\left(\frac{\rho}{\Lambda}\right)_n = \omega_{n0} + \left(\frac{\beta}{\Lambda}\right)_n \quad . \qquad (45)$$

Thus

$$\frac{C(0) - C(\delta t)}{C(\delta t)} = \frac{\sum_n S_n \left[\left(\frac{\beta}{\Lambda}\right)_n - \left(\frac{\rho}{\Lambda}\right)_n\right]^{-1}}{\sum_n - S_n \left(\frac{\rho}{\beta}\right)_n^{-1} \left[\left(\frac{\beta}{\Lambda}\right)_n - \left(\frac{\rho}{\Lambda}\right)_n\right]^{-1}} \qquad (46)$$

where $\left(\frac{\rho}{\beta}\right)_n = \left(\frac{\rho}{\Lambda}\right)_n \left(\frac{\beta}{\Lambda}\right)_n^{-1}$.

If the source and counter can be arranged so that $S_0 >> S_{n>0}$, the source-jerk experiment yields (in dollars) the shutdown reactivity of the principal mode in terms of the ratio to counting rates before and just after the source is removed—i.e.,

$$-\left(\frac{\rho}{\beta}\right)_0 = \frac{C(0) - C(\delta t)}{C(\delta t)} \quad . \qquad (46a)$$

2. *The Pulsed-Source Technique*

In the pulsed-source technique applied to subcritical multiplying systems[19], periodic bursts of neutrons are introduced into the core, and the asymptotic background and prompt die-away period of the fundamental spatial mode is observed.

To analyze such an experiment, we first specialize the general expression Eq. 27 for the time-dependent coefficients associated with the expansion of the solution in inhour modes of the assembly. Thus with the conditions $a_{n\bar{J}}(0) = 0$, $|Q(t)> = |Q(t+\Delta)>$ when Δ is the period of the neutron bursts, Eq. 37 becomes in the time interval between N and $N+1$ pulses (when the time is $t = N\Delta + t'$)

$$a_{n\bar{J}}(t') = e^{\omega_{n\bar{J}} t'} \left[\int_0^{\Delta} <N_{n\bar{J}}^* | Q(\tau)> e^{-\omega_{n\bar{J}}\tau}\, d\tau\right] \frac{1 - e^{\omega_{n\bar{J}} N\Delta}}{1 - e^{\omega_{n\bar{J}}\Delta}} -$$

$$- e^{\omega_{n\bar{J}} t'} \int_{t'}^{\Delta} <N_{n\bar{J}}^* | Q(\tau)> e^{-\omega_{n\bar{J}}\tau}\, d\tau \quad . \qquad (47)$$

If an equilibrium cycle has been attained (N so large that $e^{\omega_{nJ} N\Delta} << 1$ for all ω_{nJ}) and if the source burst $|Q(t)>$, is assumed to have a delta-function character at times $n\Delta (n = 0, 1, 2 \ldots N)$ so that $|Q(t)> \equiv |Q> \sum_{n=0}^{\infty} \delta(t - n\Delta)$, then in the interval $0 < t' \leq \Delta$ (i.e. $N\Delta < t \leq (N+1)\Delta$)

$$a_{nJ}(t') = e^{\omega_{nJ} t'} \frac{<N_{nJ}^* | Q>}{1 - e^{\omega_{nJ}\Delta}} \quad . \qquad (48)$$

It follows that during this interval a counter of cross section Σ_c and efficiency ϵ_n for neutrons in the n-th spatial mode will read

$$C(t') = \sum_{nJ} <\epsilon_n v \Sigma_c | N_{nJ}> <N_{nJ}^* | Q> \frac{e^{\omega_{nJ} t'}}{1 - e^{\omega_{nJ}\Delta}} \qquad (49)$$

$$= \sum_n S_n \sum_{J=0}^{6} \left[\frac{1}{1 + \sum_i \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i + \omega_{nJ})^2}}\right] \left[\frac{e^{\omega_{nJ} t'}}{1 - e^{\omega_{nJ}\Delta}}\right]$$

where the definition, Eq. 41 and the same manipulations that led to Eq. 42 have been carried out.

If again we recognize that the ω_{n0} are negative and very large we chose Δ so that $e^{|\omega_{n0}|\Delta} << 1$ and $e^{\omega_{n\bar{J}}\Delta} \simeq 1 + \omega_{nJ}\Delta$ for $J > 0$, Eq. 49 becomes

$$C(t') = \sum_n S_n \left[e^{\omega_{n0} t'} + \right.$$

$$+ \sum_{J=0}^{6} \frac{1 + \omega_{n\bar{J}} t'}{-\omega_{nJ}\Delta \left\{1 + \sum_i \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i + \omega_{nJ})^2}\right\}} - \qquad (50)$$

$$\left. - \frac{1 + \omega_{n0} t'}{-\omega_{n0}\Delta \left\{1 + \sum_i \left(\frac{\beta_i}{\Lambda}\right)_n \frac{\lambda_i}{(\lambda_i + \omega_{n\bar{J}})^2}\right\}} \right] =$$

$$= \sum_n S_n \left[e^{\omega_{n0} t'} - \frac{1}{\Delta} \left(\frac{\rho}{\Lambda}\right)_n^{-1} - \frac{t'}{\Delta} + \frac{1}{\omega_{n0}\Delta} + \frac{t'}{\Delta} \right]$$

[19] B. E. SIMMONS and J. S. KING, "A Pulsed Neutron Technique for Reactivity Determination," *Nucl. Sci. Eng.* **3**, p. 595 (1958).

where again Eqs. 30, 25 and the fact that $|\omega_{n0}| >> \left(\frac{\beta_i}{\Lambda}\right)_n \lambda_i$ have been applied. Finally, using Eq. 45, we find

$$C(t') = \sum_n S_n \left[e^{\omega_{n0} t'} + \left(\frac{1}{\omega_{n0}\Delta}\right)\left(\frac{\rho}{\beta}\right)_n^{-1} \right]. \quad (51)$$

This relationship is the extension to the general case of the result obtained by Garelis and Russell[20] by using a one-group bare slab model (Eq. 28 of their article). These authors go on to note that if $\left(\frac{\beta}{\Lambda}\right)$ is the same for all n (a condition that is rigorously true for the one-group bare slab), this modally independent value of $\left(\frac{\beta}{\Lambda}\right)$ can be found from an integral of the prompt decay signal following a burst ($I_1 \equiv \int_0^\Delta \sum_n S_n e^{\omega_{n0} t'} dt'$) and the background ($N_d \equiv \sum_n S_n \frac{1}{\omega_{n0}\Delta} \left(\frac{\rho}{\beta}\right)_n^{-1}$). This result also can be generalized; for, in terms of the present formalism, if $\left(\frac{\beta}{\Lambda}\right)_n = \left(\frac{\beta}{\Lambda}\right)$ for all n,

$$I_2 \equiv \int_0^\Delta \sum_n S_n e^{\omega_{n0} t'} \times e^{\frac{\beta}{\Lambda} t'} = \int_0^\Delta \sum_n S_n e^{\left[\omega_{n0} + \left(\frac{\beta}{\Lambda}\right)_n\right] t'} =$$

$$= \sum_n S_n \left(\frac{\rho}{\Lambda}\right)_n^{-1} \left[e^{\left(\frac{\rho}{\Lambda}\right)_n \Delta} - 1 \right] \simeq \sum_n -S_n \left(\frac{\rho}{\Lambda}\right)_n^{-1} =$$

$$= I_1 + N_d \Delta .$$

[20] E. GARELIS and J. L. RUSSELL, Jr., *Nucl. Sci. Eng.* **16**, 262 (1963).

Thus, if $\left(\frac{\beta}{\Lambda}\right)$ is independent of n, a numerical search for that function $\exp\left(\frac{\beta}{\Lambda} t'\right)$, which when folded into the experimental data for the prompt decay makes Eq. 52 valid, will yield the value of $\frac{\beta}{\Lambda}$ appropriate to the system. Since the asymptotic part of the prompt decay yields ω_∞, Eq. 45 can be used to determine $\left(\frac{\rho}{\Lambda}\right)_0$, and thence $\left(\frac{\rho}{\beta}\right)_0$, the shutdown of the fundamental mode.

The great advantage of this approach is that the determination of $\left(\frac{\beta}{\Lambda}\right)$ does not require the assumption of the space-time separability. The quantities I_1, I_2 and N_d are sums over all inhour modes. Unfortunately, however, $\left(\frac{\beta}{\Lambda}\right)_n$ is *not* rigorously independent of n except in a one-group, bare core model. Moreover Eq. 51 is based on an expansion in a restricted variety of inhour modes (restricted in that, for a given cluster, $|N_{nJ}>$ is taken to be the same for all J). Thus the method should not be considered rigorous unless the approximations inherent in it are tested for the situation at hand. Its apparent success may be due more to the smallness of the S_n for $n > 0$ than to the lack of dependence of $\left(\frac{\beta}{\Lambda}\right)_n$ on n.

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