4 The Adjoint Transport Equation—The Equation of Neutron Importance

Consideration will be given to the equation which is adjoint to the neutron transport equation. The solutions to an adjoint equation will be seen to be, in a sense, orthogonal to the solutions of the transport equation. Moreover, the former has a clear physical significance as the "importance" of neutrons within a particular system. For these and other reasons, the solutions to the adjoint transport equation are widely used in perturbation theory, kinetic parameters, variational methods, and other calculations related to the behavior of the nuclear reactor.

The subject requires the introduction of some mathematical definitions, identities, and transport operators which are presented in Sects. 4.1 and 4.2. In Sect. 4.3, the derivation of the equation of neutron importance (the adjoint to the transport equation) is carried out along with boundary conditions. The adjoint multi-group neutron diffusion equation and the adjoint P_1 equations are discussed in Sect. 4.4. Finally, the kinetic parameters of a one-node and two-node reactor are derived in Sect. 4.5 using the concept of adjoint flux.

4.1 Mathematical Definitions [1, 2]

Define the inner product between the two functions ψ, φ:

$$(\psi, \varphi) = \int \psi(\xi)\varphi(\xi)d\xi \tag{4.1}$$

where ξ stands for all possible variables defining the function ψ, φ.

Requirement (1)

If ψ, φ are well-behaved functions and M is a *self-adjoint* operator, then

$$(\psi, M\varphi) = (\varphi, M\psi) \tag{4.2}$$

R. M. Kuridan, *Neutron Transport*, Graduate Texts in Physics,
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The eigenfunctions of operators are orthogonal and the eigenvalues are always real. In neutron transport equation, operators are real, hence no need for conjugates, however they are not self-adjoint.

Requirement (2)

If L is *not self-adjoint*, then there is L^* which is the adjoint of L and operates on the adjoint eigenfunction φ^* and the operator L operates on the eigenfunction φ then both satisfy:

$$\left(\varphi^*, L\varphi\right) = \left(\varphi, L^*\varphi^*\right) \tag{4.3}$$

The eigenfunctions of the adjoint operator, L^*, are then orthogonal to those of the operator L. Thus, if φ is an eigenfunction, then

$$L\varphi = \lambda\varphi$$

and if φ^* is also an eigenfunction, then

$$L^*\varphi^* = \eta\varphi^*$$

where λ and η are eigenvalues. Using the requirement (4.3) and substituting for $L\varphi, L^*\varphi^*$
$(\varphi^*, \lambda\varphi) = (\varphi, \eta\varphi^*)$ and factorizing $\lambda(\varphi^*, \varphi) = \eta(\varphi, \varphi^*)$, we get

$$(\lambda - \eta)\left(\varphi, \varphi^*\right) = 0$$

$(\varphi^*, \varphi) = (\varphi, \varphi^*)$ because they are commutative (from Eq. (4.1)) therefore, it is either $(\varphi, \varphi^*) = 0$ and hence $(\lambda \neq \eta)$and $\varphi, \varphi^*$are orthogonal. If $(\varphi, \varphi^*) \neq 0$ then $\lambda = \eta$.

Transport operators:

Recall the general steady-state form of the neutron transport equation in a multiplying medium from Chap. 1

$$\begin{aligned}
&-\underline{\Omega}\cdot\nabla\varphi\left(\underline{r}, E, \underline{\Omega}\right) - \Sigma_t\left(\underline{r}, E, \underline{\Omega}\right)\varphi\left(\underline{r}, E, \underline{\Omega}\right)\\
&+\int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\varphi\left(\underline{r}, E', \underline{\Omega'}\right)\Sigma_s\left(\underline{r}, E' \to E, \underline{\Omega'} \to \underline{\Omega}\right)\\
&+\frac{\chi(E)}{4\pi}\int_{4\pi} d\underline{\Omega'}\int_0^\infty dE'\varphi\left(\underline{r}, E', \underline{\Omega'}\right)\nu\left(E'\right)\Sigma_f\left(\underline{r}, E'\right) = 0
\end{aligned} \tag{4.4}$$

Then it can be written using the operator notation in the form $L\,\phi\left(\underline{r}, E, \underline{\Omega}\right) = 0$ such that

$$L\varphi\left(\underline{r}, E, \underline{\Omega}\right) = -\underline{\Omega}\cdot\nabla\varphi\left(\underline{r}, E, \underline{\Omega}\right) - \Sigma_t\left(\underline{r}, E, \underline{\Omega}\right)\varphi\left(\underline{r}, E, \underline{\Omega}\right) +$$

$$\int_0^\infty dE' \int_{4\pi} d\underline{\Omega}' \Sigma_s(\underline{r}, E' \to E, \underline{\Omega}' \to \underline{\Omega})\varphi(\underline{r}, E', \underline{\Omega}')$$
$$+ \frac{\chi(E)}{4\pi} \int_{4\pi} d\underline{\Omega}' \int_0^\infty dE' \upsilon(E') \Sigma_f(\underline{r}, E')\varphi(\underline{r}, E', \underline{\Omega}' \qquad (4.5)$$

4.2 Derivation of the Equation of Neutron Importance (The Adjoint Function)

Generally speaking, neutrons may have different effects on the fission process depending on their energy, position, and direction. For example, a thermal neutron introduced anywhere in the core will be more capable of inducing fission in a U235 nucleus than a fast neutron. Furthermore, if the thermal neutron is placed near the edge of the core, it will be exposed more to leakage than if placed in the middle. A neutron moving in the direction toward the interior of the core should be expected to have a more chance to interact than a neutron moving in the outward direction. Therefore, the neutron energy, position, and direction play a major role in defining the concept of neutron importance [3].

Consider a neutron at position $\underline{r}$ with energy E and direction $\underline{\Omega}$ in $dV, dE, d\Omega$, respectively, at time t having an importance $\varphi^*(\underline{r}, E, \underline{\Omega}, t)$ (notice that it does not have the units of neutron flux). In a time interval Δt, the neutron moves a distance $d\underline{r} = \underline{\upsilon}\Delta t = \underline{\Omega}\upsilon\ \Delta t$ or $|d\underline{r}| = \upsilon\ \Delta t$ and its importance at $t + \Delta t$ becomes $\varphi^*(\underline{r} + \underline{\Omega}\upsilon\ \Delta t, E, \underline{\Omega}, t + \Delta t)$. Figure 4.1 shows the change of importance.

4.2.1 The Equation of Conservation of Importance

The importance at time t is equal to the total importance of the collided and un-collided neutrons during Δt as shown in the diagram in Fig. 4.2. That is:

$$\varphi^*(\underline{r}, E, \underline{\Omega}, t) \equiv [1 - \upsilon\Delta t \Sigma_t(\underline{r}, E)]\, \varphi^*(\underline{r} + \upsilon\underline{\Omega}\Delta t, E, \underline{\Omega}, t + \Delta t) + [\upsilon\Delta t \Sigma_t(\underline{r}, E)]$$

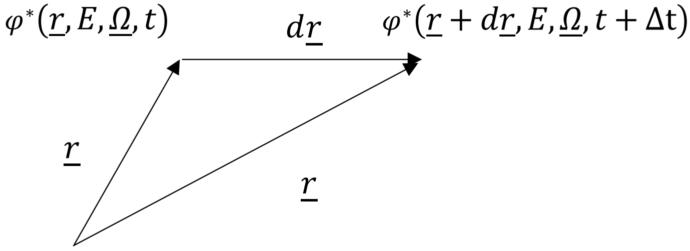

Fig. 4.1 The change of importance during Δt

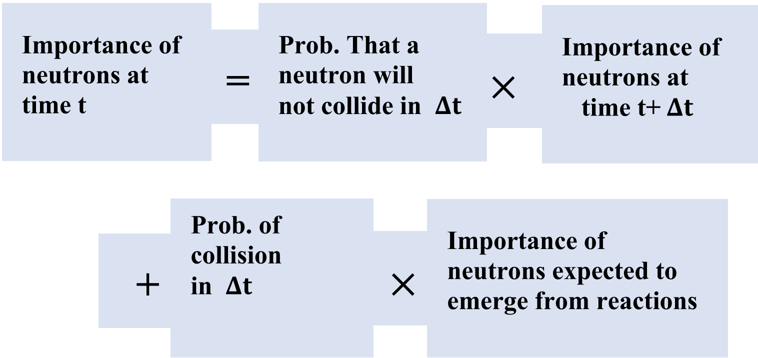

Fig. 4.2 The balance of neutron importance during Δt

$$\times\left[\int_0^\infty dE' \int_{4\pi} d\underline{\Omega}' \frac{\Sigma_s\left(\underline{r}, E \to E', \underline{\Omega} \to \underline{\Omega}'\right)}{\Sigma_t\left(\underline{r}, E\right)} \varphi^*\left(\underline{r}, E', \underline{\Omega}', t\right)\right] + \left[\upsilon \Delta t \Sigma_t\left(\underline{r}, E\right)\right]$$
$$\times\left[\int_0^\infty dE' \int_{4\pi} d\underline{\Omega}' \frac{\nu(E)\Sigma_f\left(\underline{r}, E\right)\chi(E \to E')}{4\pi \Sigma_t\left(\underline{r}, E\right)} \varphi^*\left(\underline{r}, E', \underline{\Omega}', t\right)\right] \tag{4.6}$$

Simplifying the first term first

$$\varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) \equiv \varphi^*\left(\underline{r} + \upsilon\underline{\Omega}\Delta t, E, \underline{\Omega}, t + \Delta \mathrm{t}\right)$$
$$- \upsilon\ \Delta t \Sigma_t\left(\underline{r}, E\right)\varphi^*\left(\underline{r} + \upsilon\underline{\Omega}\Delta t, E, \underline{\Omega}, t + \Delta \mathrm{t}\right)$$

Substituting from the chain rule of derivatives in the first term on the right-hand side

$$\varphi^*\left(\underline{r} + d\underline{r}, E, \underline{\Omega}, t + \Delta \mathrm{t}\right) = \varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) + \frac{\partial \varphi^*}{\partial t}\Delta \mathrm{t} + \underline{\nabla} \cdot \underline{\Omega}\varphi^*\left|d\underline{r}\right| \tag{4.7}$$

where the second term on the right side is the partial change in importance due to time change and the third term is due to the position change.

$$\varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) \equiv \varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) + \frac{\partial \varphi^*}{\partial t}\Delta t + \underline{\nabla} \cdot \underline{\Omega}\varphi^*\left|d\underline{r}\right|$$
$$- \upsilon\ \Delta t \Sigma_t\left(\underline{r}, E\right)\left[\varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) + \frac{\partial \varphi^*}{\partial t}\Delta t + \underline{\nabla} \cdot \underline{\Omega}\varphi^*\left|d\underline{r}\right|\right]$$

Subtracting $\varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right)$ from both sides and multiplying by $\frac{1}{\upsilon\ \Delta t} = \frac{1}{|d\underline{r}|}$, we get

$$0 \equiv \frac{1}{\upsilon}\frac{\partial \varphi^*}{\partial t} + \underline{\nabla} \cdot \underline{\Omega}\phi^* - \Sigma_t\left(\underline{r}, E\right)\varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) - \Sigma_t\left(\underline{r}, E\right)\frac{\partial \varphi^*}{\partial t}\Delta t$$

$$+ \underline{\nabla} \cdot \underline{\Omega} \varphi^* \left| d\underline{r} \right|$$

The last two terms are too small ($\partial\varphi^* \Delta t$) and can be ignored, $\underline{\nabla} \cdot \underline{\Omega} \varphi^* = \frac{\partial \varphi^*}{\partial x} i + \ldots$ and $\left| d\underline{r} \right| = \upsilon\ \Delta t$, the last equation reduces to

$$0 \equiv \frac{1}{\upsilon} \frac{\partial \varphi^*}{\partial t} + \underline{\nabla} \cdot \underline{\Omega} \varphi^* - \Sigma_t\left(\underline{r}, E\right) \varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) \tag{4.8}$$

The above division $\frac{1}{\upsilon\ \Delta t} = \frac{1}{\left| d\underline{r} \right|}$ applies to other terms in Eq. (4.7) as well. Substituting (4.8) into (4.7) and $\Sigma_t\left(\underline{r}, E\right)$ is canceled out and knowing that the fission spectrum is independent from the initial energy $\chi\left(E \to E'\right) = \chi\left(E'\right)$, we get

$$\begin{aligned} -\frac{1}{\upsilon} \frac{\partial \varphi^*}{\partial t} &= \underline{\Omega}.\underline{\nabla} \varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) - \Sigma_t\left(\underline{r}, E\right) \varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right) \\ &+ \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'} \Sigma_s\left(\underline{r}, E \to E', \underline{\Omega} \to \underline{\Omega'}\right) \varphi^*\left(\underline{r}, E', \underline{\Omega'}, t\right) \\ &+ \frac{\nu(E) \Sigma_f\left(\underline{r}, E\right)}{4\pi} \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'} \chi\left(E'\right) \varphi^*\left(\underline{r}, E', \underline{\Omega'}, t\right) \end{aligned} \tag{4.9}$$

This is the adjoint equation or the equation of neutron importance. Notice that the adjoint φ^* represents the importance per neutron. Now compare to the forward neutron transport equation:

$$\begin{aligned} +\frac{1}{\upsilon} \frac{\partial \varphi}{\partial t} &= -\underline{\Omega} \cdot \nabla \varphi\left(\underline{r}, E, \underline{\Omega}, t\right) - \Sigma_t\left(\underline{r}, E, \underline{\Omega}\right) \varphi\left(\underline{r}, E, \underline{\Omega}, t\right) \\ &+ \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'} \Sigma_s\left(\underline{r}, E' \to E, \underline{\Omega'} \to \underline{\Omega}\right) \varphi\left(\underline{r}, E', \underline{\Omega'}, t\right) \\ &+ \frac{\chi(E)}{4\pi} \int_{4\pi} d\underline{\Omega'} \int_0^\infty dE' \nu\left(E'\right) \Sigma_f\left(\underline{r}, E'\right) \varphi\left(\underline{r}, E', \underline{\Omega'}, t\right) \end{aligned} \tag{4.10}$$

The time and space operators in the adjoint equation are both with a negative sign as if time is revered and motion in space is in the opposite direction. The energy transfer in scattering and fission is reversed (Fig. 4.3). As a consequence of characterizing the adjoint system as a reversable process, the neutron transport equation gained the name forward. The loss term via absorption and scattering out of the differential energy dE (the total reaction rate) is the same in both equations which makes it the only term which is self-adjoint.

An interesting interpretation of the adjoint flux $\varphi^*\left(\underline{r}, E, \underline{\Omega}, t\right)$—besides being the importance of a neutron at position r in dV and with energy E in dE and direction $\underline{\Omega}$ in $d\ \underline{\Omega}$ at time t—is through a time reversed neutron cycle such

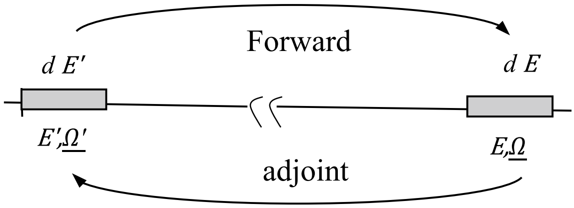

Fig. 4.3 Neutron scattering in both the forward and adjoint cases

that fast adjoint phantom causes fissions which yield k slow adjoint neutrons per fast neutron absorbed. These neutrons are moderated upward in energy until they become fast, then they cause more fissions yielding more slow adjoint phantom neutrons.

Therefore, we have derived the adjoint operators making up the adjoint operator L^*. They are all not self-adjoint except the loss term operator.

In the steady-state case Eq. 4.10 becomes

$$\begin{aligned}0 &= \underline{\Omega} \cdot \nabla\varphi^*(\underline{r}, E, \underline{\Omega}) - \Sigma_t(\underline{r}, E)\varphi^*(\underline{r}, E, \underline{\Omega}) \\ &+ \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\,\Sigma_s(\underline{r}, E \to E', \underline{\Omega} \to \underline{\Omega'})\varphi^*(\underline{r}, E', \underline{\Omega'}) \\ &+ \frac{\nu(E)\Sigma_f(\underline{r}, E)}{4\pi}\int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\,\chi(E')\varphi^*(\underline{r}, E', \underline{\Omega'}, t)\end{aligned} \tag{4.11}$$

and Eq. 4.11 becomes

$$\begin{aligned}0 &= -\underline{\Omega} \cdot \nabla\varphi(\underline{r}, E, \underline{\Omega}) - \Sigma_t(\underline{r}, E, \underline{\Omega})\varphi(\underline{r}, E, \underline{\Omega}) \\ &+ \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\,\varphi(\underline{r}, E', \underline{\Omega'})\Sigma_s(\underline{r}, E' \to E, \underline{\Omega'} \to \underline{\Omega}) \\ &+ \frac{\chi(E)}{4\pi}\int_{4\pi} d\underline{\Omega'} \int_0^\infty dE'\varphi(\underline{r}, E', \underline{\Omega'})\nu(E')\Sigma_f(\underline{r}, E')\end{aligned} \tag{4.12}$$

So, the adjoint operator is written as

$$\begin{aligned}L^*\varphi^*(\underline{r}, E, \underline{\Omega}) &= \underline{\Omega} \cdot \nabla\varphi^*(\underline{r}, E, \underline{\Omega}) - \Sigma_t(\underline{r}, E)\varphi^*(\underline{r}, E, \underline{\Omega}) \\ &+ \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\,\Sigma_s(\underline{r}, E \to E', \underline{\Omega} \to \underline{\Omega'})\varphi^*(\underline{r}, E', \underline{\Omega'}) \\ &+ \frac{\nu(E)\Sigma_f(\underline{r}, E)}{4\pi}\int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\,\chi(E')\varphi^*(\underline{r}, E', \underline{\Omega'})\end{aligned}$$

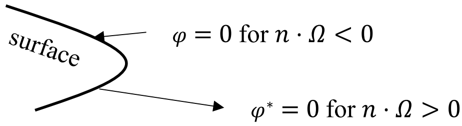

Fig. 4.4 Boundary surface conditions

And the forward operator in the steady state is

$$L\varphi(\underline{r}, E, \underline{\Omega}) = -\underline{\Omega} \cdot \nabla\varphi(\underline{r}, E, \underline{\Omega}) - \Sigma_t(\underline{r}, E, \underline{\Omega})\varphi(\underline{r}, E, \underline{\Omega}) + \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\Sigma_s(\underline{r}, E' \to E, \underline{\Omega'} \to \underline{\Omega})\varphi(\underline{r}, E', \underline{\Omega'}) + \frac{\chi(E)}{4\pi}\int_{4\pi} d\underline{\Omega'}\int_0^\infty dE'\nu(E')\Sigma_f(\underline{r}, E')\varphi(\underline{r}, E', \underline{\Omega'})$$

4.2.2 Boundary Condition

(i) $\varphi^*(\underline{r}, E, \underline{\Omega}, t) = 0$ for $\underline{n} \cdot \underline{\Omega} > 0$ which means a zero importance for the lost outgoing neutrons. The motion outward is represented by $\underline{n} \cdot \underline{\Omega} > 0$ since $\underline{n} \cdot \underline{\Omega} = cos\,\theta$ is positive > 0 and $\underline{n}$ is the out normal to the boundary surface (Fig. 4.4) [4].

(ii) Notice that the b.c. for the forward case $\varphi(\underline{r}, E, \underline{\Omega}, t) = 0$ for $\underline{n} \cdot \underline{\Omega} < 0$ (zero incoming angular flux or the non-reentrant surface condition).

4.2.3 Final and Initial Conditions

The time operator is negative as if going backward in time, hence we use the final condition instead of the initial condition.

$$\varphi^*(\underline{r}, E, \underline{\Omega}, t)\big|_{t=t_f} = \varphi^*(\underline{r}, E, \underline{\Omega}, t_f)$$

Which is the start point then by integrating backward in time to find the adjoint at earlier times. On the other hand, the initial condition for the forward case is

$$\varphi(\underline{r}, E, \underline{\Omega}, t)\big|_{t=t_0} = \varphi(\underline{r}, E, \underline{\Omega}, t_0)$$

Example Show that the streaming term operator L^*_{st} is the adjoint to L_{st}.

We have derived that

$$L_{st}\varphi(\underline{r}, E, \underline{\Omega}) = -\underline{\Omega} \cdot \nabla\varphi(\underline{r}, E, \underline{\Omega})$$

$$L_{st}^{*}\varphi^{*}(\underline{r}, E, \underline{\Omega}) = \underline{\Omega} \cdot \nabla\varphi^{*}(\underline{r}, E, \underline{\Omega})$$

therefore

$$(\varphi^{*}, L_{st}\varphi) = \iiint \varphi^{*}(\underline{r}, E, \underline{\Omega})(-\underline{\Omega} \cdot \nabla\varphi(\underline{r}, E, \underline{\Omega}))dVdEd\Omega$$

$$(\varphi, L_{st}^{*}\varphi^{*}) = \iiint \varphi(\underline{r}, E, \underline{\Omega})(\underline{\Omega} \cdot \nabla\varphi^{*}(\underline{r}, E, \underline{\Omega}))dVdEd\Omega$$

In other words, we have to prove that the above should satisfy requirement (2): $(\varphi^{*}, L_{st}\varphi) = (\varphi, L_{st}^{*}\varphi^{*})$ or $(\varphi^{*}, L_{st}\varphi) - (\varphi, L_{st}^{*}\varphi^{*}) = \Delta = 0$.

Substituting from above and rearranging using $(\underline{\Omega} \cdot \nabla\varphi = \nabla.\ \underline{\Omega}\varphi)$ and the adjoint $\underline{\Omega} \cdot \nabla\varphi^{*} = \nabla \cdot \underline{\Omega}\varphi^{*}$, we have to show that $\Delta = 0$

$$\begin{aligned}\Delta &= -\iiint \varphi^{*}(\nabla.\underline{\Omega}\varphi) + \varphi(\nabla.\underline{\Omega}\varphi^{*})dVdEd\Omega \\ &= -\iiint \varphi^{*}(\underline{\Omega}.\nabla\varphi) + \varphi(\underline{\Omega}.\nabla\varphi^{*})dVdEd\Omega \\ &= -\iiint (\underline{\Omega}\varphi^{*}.\nabla\varphi) + (\underline{\Omega}\varphi.\nabla\varphi^{*})dVdEd\Omega\end{aligned}$$

Using the identity $(fg)' = fg' + gf'$, hence, accordingly

$$\begin{aligned}\nabla.\underline{\Omega}\varphi\varphi^{*} &= \underline{\Omega}\varphi.\nabla\varphi^{*} + \varphi^{*}\nabla.\underline{\Omega}\varphi \\ &= \underline{\Omega}\varphi.\nabla\varphi^{*} + \varphi^{*}\underline{\Omega}.\nabla\varphi \\ &= \underline{\Omega}\varphi.\nabla\varphi^{*} + \underline{\Omega}\varphi^{*}.\nabla\varphi\end{aligned}$$

Therefore, we can substitute in the integrand by

$$\Delta = -\iiint \nabla.\ \underline{\Omega}\varphi\varphi^{*}dVdEd\Omega$$

where $\underline{\Omega}\varphi$ and $\nabla\varphi^{*}$ are vectors and they are commutative. Using the divergence theorem to convert the volume integral into the surface integral

$$\int \nabla.\underline{\Omega}\varphi dV = \int\limits_{\underline{n}.\underline{\Omega}>0} \underline{n}.\underline{\Omega}\varphi dS + \int\limits_{\underline{n}.\underline{\Omega}>0} \underline{n}.\underline{\Omega}\varphi dS$$

in spherical coordinates $\underline{n}.\underline{\Omega} > 0(cos\,\theta+)and\underline{n}.\underline{\Omega} < 0\ (cos\,\theta-)$, where $\underline{n}.\underline{\Omega} = cos\theta$. Therefore, the integral covers up and down or right and left, i.e. inward and outward directions. $\underline{n}$ is the normal to the differential surface on the volume from which the neutron current passes. Applying the divergence theorem, we get

$$\Delta = -\iiint \nabla.\underline{\Omega}\varphi\varphi^* dVdEd\Omega = -\iiint \underline{n}.\underline{\Omega}\varphi\varphi^* dSdEd\Omega$$
$$= -\iint dEd\Omega \left\{ \int_{\underline{n}.\underline{\Omega}<0} \varphi^*(\underline{n}.\underline{\Omega}\varphi)dS + \int_{\underline{n}.\underline{\Omega}>0} \varphi(\underline{n}.\underline{\Omega}\varphi^*)dS \right\}$$

The first term represents the inward current which is zero (b.c) and the second term represents the outward importance equal to zero because it is lost (b.c). Therefore $\Delta = 0$ which satisfies the requirement.

The adjoint operator L^* will not be self-adjoint even if only one term in the operator is not self-adjoint or even only the sign is opposite to the sign of the corresponding term.

Now we are in the position to examine the adjoint equations to some familiar forms of the neutron transport equation.

4.3 The Adjoint Neutron Diffusion Equation

We recall the equations from Chap. 1 without going into details of their derivations.

4.3.1 The Energy-Dependent Neutron Diffusion Equation (DE)

$$\begin{aligned} &-\nabla\cdot D(\underline{r},E)\nabla\phi(\underline{r},E) + \Sigma_t(\underline{r},E)\phi(\underline{r},E) \\ &\qquad - \int_0^\infty dE'\Sigma_s(\underline{r},E'\to E)\phi(\underline{r},E') \\ &\qquad - \chi(E)\int_0^\infty dE'v(E')\Sigma_f(\underline{r},E')\phi(\underline{r},E') = 0 \end{aligned} \tag{4.13}$$

$$\begin{aligned} L_{diff}\phi(\underline{r},E) &= -\nabla\cdot D(\underline{r},E)\nabla\phi(\underline{r},E) + \Sigma_t(\underline{r},E)\phi(\underline{r},E) \\ &\quad - \int_0^\infty dE'\Sigma_s(\underline{r},E'\to E)\phi(\underline{r},E') \\ &\quad - \chi(E)\int_0^\infty dE'v(E')\Sigma_f(\underline{r},E')\phi(\underline{r},E') \end{aligned}$$

where $\phi(\underline{r}, E) = \int_{4\pi} \varphi(\underline{r}, E, \underline{\Omega}) d\underline{\Omega}$. By comparison with the transport equation (TE) we may deduce the adjoint operator for the (DE) such that

1. The transfer terms' scattering and fission are similar.
2. The directional dependence is absent in the DE because it is based on the assumption of isotropic scattering and flux.
3. The boundary conditions of the outgoing zero importance and the non-reentrant surface made the streaming term not self-adjoint in the TE.
4. In the DE, the conditions in (3) are not applicable.

Therefore, the streaming term is self-adjoint, hence we can write

$$\begin{aligned} L^*_{diff}\phi^*(\underline{r}, E) = &-\nabla \cdot D(\underline{r}, E)\nabla\phi^*(\underline{r}, E) + \Sigma_t(\underline{r}, E)\phi^*(\underline{r}, E) \\ &- \int_0^\infty dE' \Sigma_s(\underline{r}, E \to E')\phi^*(\underline{r}, E') \\ &- v(E)\Sigma_f(\underline{r}, E)\int_0^\infty dE' \chi(E')\phi^*(\underline{r}, E') \end{aligned}$$

4.3.2 The One-Speed Neutron Diffusion Equation

Recall from Chap. 2 the one-speed (one-group) neutron diffusion equation

$$-\nabla \cdot D(\underline{r})\nabla\phi(\underline{r}) + \Sigma_a(\underline{r})\phi(\underline{r}) = v\Sigma_f(\underline{r})\phi(\underline{r}) \tag{4.14}$$

The operator form is

$$L_{1speed}\phi(\underline{r}) = -\nabla \cdot D(\underline{r})\nabla\phi(\underline{r}) + \Sigma_a(\underline{r})\phi(\underline{r}) - v\Sigma_f(\underline{r})\phi(\underline{r})$$

And the adjoint form can just be written down as

$$L^*_{1speed}\phi^*(\underline{r}) = -\nabla \cdot D(\underline{r})\nabla\phi^*(\underline{r}) + \Sigma_a(\underline{r})\phi^*(\underline{r}) - v\Sigma_f(\underline{r})\phi^*(\underline{r})$$

Notice that the boundary condition applied to the transport equation at the left boundary $\varphi(\underline{r}_0, \underline{\Omega}) = 0$ for $\underline{\Omega} > 0$ and $\varphi^*(\underline{r}_0, \underline{\Omega}) = 0$ for $\underline{\Omega} < 0$ or $\varphi^*(\underline{r}_0, \underline{\Omega}) = 0$ for $\underline{\Omega} < 0$ and $\varphi^*(\underline{r}_0, \underline{\Omega}) = 0$ for $\underline{\Omega} < 0$. The unit vector $\underline{\Omega}$ causes the streaming term in the TE to be of opposite sign. This is not the case in the diffusion theory where the flux is isotropic. Notice also $j(\underline{r}, \underline{\Omega}) = \underline{\Omega}\varphi(\underline{r}, \underline{\Omega})$ and $J(\underline{r}) = \underline{\Omega}\phi(\underline{r})$ are both vectors and $\varphi(\underline{r}, \underline{\Omega})$ and $\phi(\underline{r})$ are both scalars. The boundary conditions to the DE are

$$J_-(\underline{r}_s, E) = 0 \text{ and } J^*_+(\underline{r}_s, E) = 0$$

Proof Applying requirement II to DE and its adjoint:

$$\left(\phi^*, -\nabla\cdot D\nabla\phi + \Sigma_a\phi - v\Sigma_f\phi\right) - \left(\phi, -\nabla\cdot D\nabla\phi^* + \Sigma_a\phi^* - v\Sigma_f\phi^*\right)$$

The absorption and fission terms are clearly self-adjoint and no need to show the proof. We prove only the streaming term:

$$\int -\phi^*\nabla\cdot D\nabla\phi + \int \phi\nabla\cdot D\nabla\phi^*$$

Integration by parts for the two integrals, first integral is

$$u = -\phi^*;\quad dv = \nabla\cdot D\nabla\phi$$

$$du = -\nabla\phi^*;\quad v = D\nabla\phi$$

Apply the identity $\int -\phi^*\nabla\cdot D\nabla\phi = uv - \int vdu = -\phi^* D\nabla\phi + \int D\nabla\phi\nabla\phi^* dV$

$$-\phi^* D\nabla\phi + \phi^* D\nabla\phi = 0$$

and the second integral is likewise. Therefore, the streaming term is self-adjoint and hence the one-speed diffusion equation.

4.3.3 The Multi-Group Neutron Diffusion Equation

The steady-state multi-group DE with no up-scattering

$$\begin{aligned} &-\nabla\cdot D_g(\underline{r})\nabla\phi_g(\underline{r}) + \Sigma_{Rg}(\underline{r})\phi_g(\underline{r}) = \frac{1}{k}\chi_g \sum\nolimits_{g'=1}^{G} \nu_{g'}\Sigma_{fg'}(\underline{r})\phi_{g'}(\underline{r}) \\ &\quad + \sum_{g'\neq g;\, g'>g}^{G} \Sigma_{sg'g}(\underline{r})\phi_{g'}(\underline{r}),\ g = 1, 2, 3, \cdots, G \end{aligned}$$

where

$$\Sigma_{Rg} = \Sigma_{tg} - \Sigma_{sgg} = \Sigma_{ag} + \Sigma_{sg} - \Sigma_{sgg} = \Sigma_{ag} + \sum_{g'=1}^{G}\Sigma_{sgg'}(\underline{r}) - \Sigma_{sgg}$$

In a matrix form:

$$\underline{\mathbf{A8}} = \frac{1}{k}\underline{\mathbf{F8}}$$

where $\underline{\mathbf{A}}$ and $\underline{\mathbf{F}}$ and $\underline{\mathbf{8}}$ are given explicitly as

$$\underline{\mathbf{A}} = \begin{bmatrix} -\nabla\cdot D_1\nabla + \Sigma_{R1} & 0 & 0 & \ldots & 0 \\ -\Sigma_{s12} & -\nabla\cdot D_2\nabla + \Sigma_{R2} & 0 & \ldots & 0 \\ -\Sigma_{s13} & -\Sigma_{s23} & -\nabla\cdot D_3\nabla + \Sigma_{R3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Sigma_{s1G} & -\Sigma_{s2G} & -\Sigma_{s3G} & \ldots & -\nabla\cdot D_G\nabla + \Sigma_{RG} \end{bmatrix}$$

$$\underline{\mathbf{F}} = \begin{bmatrix} \chi_1 v_1 \Sigma_{f1} & \chi_1 v_2 \Sigma_{f2} & \chi_1 v_3 \Sigma_{f3} & \cdots & \chi_1 v_G \Sigma_{fG} \\ \chi_2 v_1 \Sigma_{f1} & \chi_2 v_2 \Sigma_{f2} & \chi_2 v_3 \Sigma_{f3} & \cdots & \chi_2 v_G \Sigma_{fG} \\ \chi_3 v_1 \Sigma_{f1} & \chi_3 v_2 \Sigma_{f2} & \chi_3 v_3 \Sigma_{f3} & \cdots & \chi_3 v_G \Sigma_{fG} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_G v_1 \Sigma_{f1} & \chi_G v_2 \Sigma_{f2} & \chi_G v_3 \Sigma_{f3} & \cdots & \chi_G v_G \Sigma_{fG} \end{bmatrix}; \quad \underline{\mathbf{\Phi}} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_G \end{bmatrix}$$

Reversing the energy transfer terms, we can readily obtain the adjoint form

$$-\nabla\cdot D_g(\underline{r})\nabla\phi_g^*(\underline{r}) + \Sigma_{Rg}(\underline{r})\phi_g^*(\underline{r}) = \frac{1}{k}\nu_g\Sigma_{fg}(\underline{r})\sum\nolimits_{g'=1}^{G}\chi_{g'}\phi_{g'}^*(\underline{r})$$
$$+\sum_{g\neq g';g>g'}^{G}\Sigma_{sgg'}(\underline{r})\phi_{g'}^*(\underline{r}),\ g = 1, 2, 3, \cdots, G$$

And the matrix form is (notice the same eigenvalue)

$$\underline{\mathbf{A^*\Phi^*}} = \frac{1}{k}\underline{\mathbf{F^*\Phi^*}}$$

where $\underline{\mathbf{A^*}}$ and $\underline{\mathbf{F^*}}$ are just the transpose of the matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{F}}$, respectively, and $\underline{\mathbf{\Phi^*}} \neq \underline{\mathbf{8}}$, thus

$$\underline{\mathbf{A}}^* = \begin{bmatrix} -\nabla.D_1\nabla + \Sigma_{R1} & -\Sigma_{s12} & -\Sigma_{s13} & \cdots & -\Sigma_{s1G} \\ 0 & -\nabla.D_2\nabla + \Sigma_{R2} & -\Sigma_{s23} & \cdots & -\Sigma_{s2G} \\ 0 & 0 & -\nabla.D_3\nabla + \Sigma_{R3} & \cdots & -\Sigma_{s3G} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -\nabla.D_G\nabla + \Sigma_{RG} \end{bmatrix}$$

$$\underline{\mathbf{F}}^* = \begin{bmatrix} \chi_1 v_1 \Sigma_{f1} & \chi_2 v_1 \Sigma_{f1} & \chi_3 v_1 \Sigma_{f1} & \cdots & \chi_G v_1 \Sigma_{f1} \\ \chi_1 v_2 \Sigma_{f2} & \chi_2 v_2 \Sigma_{f2} & \chi_3 v_2 \Sigma_{f2} & \cdots & \chi_G v_2 \Sigma_{f2} \\ \chi_1 v_3 \Sigma_{f3} & \chi_2 v_3 \Sigma_{f3} & \chi_3 v_3 \Sigma_{f3} & \cdots & \chi_G v_3 \Sigma_{f3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_1 v_G \Sigma_{fG} & \chi_2 v_G \Sigma_{fG} & \chi_3 v_G \Sigma_{fG} & \cdots & \chi_G v_G \Sigma_{fG} \end{bmatrix}; \quad \underline{\mathbf{\Phi}}^* = \begin{bmatrix} \phi_1^* \\ \phi_2^* \\ \phi_3^* \\ \vdots \\ \phi_G^* \end{bmatrix}$$

4.3.4 The Two-Group Neutron Diffusion Equation

The matrix form is straightforward from the multi-group DE and its adjoint. The forward equation is

$$\begin{bmatrix} -\nabla.D_1\nabla + \Sigma_{R1} & 0 \\ -\Sigma_{s12} & -\nabla.D_2\nabla + \Sigma_{R2} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{k}\begin{bmatrix} \chi_1\nu_1\Sigma_{f1} & \chi_1\nu_2\Sigma_{f2} \\ \chi_2\nu_1\Sigma_{f1} & \chi_2\nu_2\Sigma_{f2} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

And the adjoint equation is

$$\begin{bmatrix} -\nabla.D_1\nabla + \Sigma_{R1} & -\Sigma_{s12} \\ 0 & -\nabla.D_2\nabla + \Sigma_{R2} \end{bmatrix}\begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix} = \frac{1}{k}\begin{bmatrix} \chi_1\nu_1\Sigma_{f1} & \chi_2\nu_1\Sigma_{f1} \\ \chi_1\nu_2\Sigma_{f2} & \chi_2\nu_2\Sigma_{f2} \end{bmatrix}\begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix}$$

Most of the fission neutrons are born in the fast energy range, therefore $\chi_1 = 1$ and $\chi_2 = 0$. Also, from the definition of the removal of x-section $\Sigma_{R1} = \Sigma_{a1} + \Sigma_{s12}$ and $\Sigma_{R2} = \Sigma_{a2} + \Sigma_{s21}$, however, $\Sigma_{s21} = 0$ because we have ignored up-scattering, therefore $\Sigma_{R2} = \Sigma_{a2}$ and we can rewrite the equations again

$$\begin{bmatrix} -\nabla.D_1\nabla + \Sigma_{R1} & 0 \\ -\Sigma_{s12} & -\nabla.D_2\nabla + \Sigma_{a2} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{k}\begin{bmatrix} \nu_1\Sigma_{f1} & \nu_2\Sigma_{f2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$
$$\begin{bmatrix} -\nabla.D_1\nabla + \Sigma_{R1} & -\Sigma_{s12} \\ 0 & -\nabla.D_2\nabla + \Sigma_{a2} \end{bmatrix}\begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix} = \frac{1}{k}\begin{bmatrix} \nu_1\Sigma_{f1} & 0 \\ \nu_2\Sigma_{f2} & 0 \end{bmatrix}\begin{bmatrix} \phi_1^* \\ \phi_2^* \end{bmatrix}$$

Calculation of fluxes and adjoints can be performed using either deterministic methods or Monte Carlo methods through well known codes. One can use a one-dimensional or two-dimensional code such as ONED or 2DB to obtain fluxes and adjoints of a reflected core and plot them to see the major differences between them. Referring to Fig. 4.5, we firstly notice the differences between fast and thermal fluxes, especially at the reflector peak and the higher fast flux in the core. Secondly, the differences between the fast and thermal adjoints where the adjoint of thermal neutrons is higher than the adjoint of fast neutrons indicating higher importance since it is more likely to induce fission as we have discussed earlier [5, 6, 7].

4.4 The Adjoint One-Speed P_1 Equations

$$\nabla \cdot J(\underline{r}) + (\Sigma_t - \Sigma_s)\phi(\underline{r}) = S_0(\underline{r})$$
$$\frac{1}{3}\nabla\phi(\underline{r}) + (\Sigma_t - \overline{\mu}_0\Sigma_s)J(\underline{r}) = S_1(\underline{r})$$

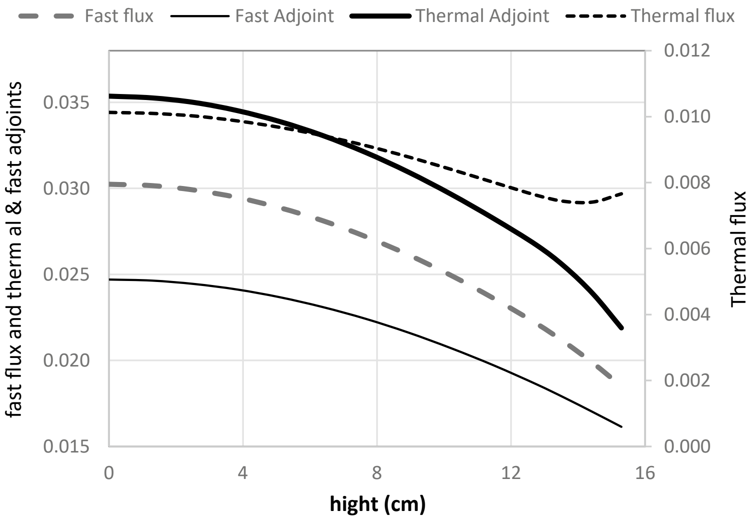

Fig. 4.5 Relative two-group flux and adjoint for the Tajoura Nuclear Research Center Research Reactor (TNRCRR)

which reduces to

$$\nabla \cdot J(\underline{r}) + \Sigma_a \phi(\underline{r}) = S(\underline{r}) \tag{4.15}$$

$$\frac{1}{3}\nabla\phi(\underline{r}) + \Sigma_{tr} J(\underline{r}) = S_1(\underline{r}) \tag{4.16}$$

They satisfy the boundary condition (from Chap. 3)

$$J(0) = -\frac{\phi(0)}{2}; \quad J(a) = \frac{\phi(a)}{2}$$

or

$$\underline{n}.\underline{J} = \frac{\phi}{2}$$

We can use the zero reentrant partial current boundary condition ($J_- = 0$) and zero outgoing adjoint ($J_+^* = 0$) since the partial currents are angle integrated so we arrive at the adjoint form of Eq. 4.15 and Eq. 4.16 as

$$-\nabla \cdot J^*(\underline{r}) + \Sigma_a \phi^*(\underline{r}) = S^*(\underline{r}) \tag{4.17}$$

$$-\frac{1}{3}\nabla\phi^*\left(\underline{r}\right) + \Sigma_{tr}J^*\left(\underline{r}\right) = S_1^*\left(\underline{r}\right) \tag{4.18}$$

They satisfy the boundary condition

$$J^*(0) = +\frac{\phi^*(0)}{2}; \quad J^*(a) = -\frac{\phi^*(a)}{2} \text{ and } \underline{n}.\underline{J^*} = -\frac{\phi^*}{2}$$

Proof Use requirement (2) and the following steps: (i) assume isotropic sources ($S_1 = 0$) and (S = 0); (ii) multiply Eq. 4.15 by ϕ^* and Eq. 4.16 by J^*; (iii) multiply (4.17) by ϕ and (4.18) by (J); (iv) subtract iii from ii; and (v) add the resulting equations. Prove that $\Delta = 0$.

Steps (i), (ii), (iii)

$$\phi^*\left[\nabla\cdot J\left(\underline{r}\right) + \Sigma_a\phi\left(\underline{r}\right)\right]; \, J^*\left[\nabla\phi\left(\underline{r}\right) + 3\Sigma_{tr}J\left(\underline{r}\right)\right]$$

and

$$\phi\left[-\nabla\cdot J^*\left(\underline{r}\right) + \Sigma_a\phi^*\left(\underline{r}\right)\right]; \, J\left[-\nabla\phi^*\left(\underline{r}\right) + 3\Sigma_{tr}J^*\left(\underline{r}\right)\right]$$

Subtract

$$\begin{gathered}\phi^*\left[\nabla\cdot J\left(\underline{r}\right) + \Sigma_a\phi\left(\underline{r}\right)\right] - \phi\left[-\nabla\cdot J^*\left(\underline{r}\right) + \Sigma_a\phi^*\left(\underline{r}\right)\right] \\ J^*\left[\nabla\phi\left(\underline{r}\right) + 3\Sigma_{tr}J\left(\underline{r}\right)\right] - J\left[-\nabla\phi^*\left(\underline{r}\right) + 3\Sigma_{tr}J^*\left(\underline{r}\right)\right]\end{gathered}$$

Add the two equations

$$\begin{aligned}&\int \phi^*[\nabla\cdot J] + J^*[\nabla\phi] + \phi\left[\nabla\cdot J^*\right] + J\left[\nabla\phi^*\right]dV \\ &\quad = \int \nabla\cdot[\phi^*J] + \nabla\cdot[\phi J^*]dV \\ &\quad = \int \nabla\cdot[\phi^*J] + \nabla\cdot[\phi J^*]dV\end{aligned}$$

Using the divergence theorem

$$\int \nabla\cdot[\phi^*J] + \nabla\cdot[\phi J^*]dV = \int\left[\left(\underline{n}\cdot\underline{J}\right)\phi^* + \left(\underline{n}\cdot J^*\right)\phi\right]dS$$

Using the boundary conditions above

$$\int\left[\frac{\phi\phi^*}{2} - \frac{\phi^*\phi}{2}\right]dS = 0$$

4.5 Derivation of the Kinetic Parameters

4.5.1 Transport Model

In this section, we establish the methodology for a generic definition of the kinetic parameters used in the kinetic equations by weighing such parameters on the adjoint (importance function). The kinetic parameters are going to be derived from the neutron transport and the adjoint equations [8]. The transport equation is written explicitly for the prompt and delayed neutrons as (see Chap. 1)

$$\begin{aligned}\frac{1}{\upsilon}\frac{\partial\varphi}{\partial t} &= -\underline{\Omega}\cdot\nabla\varphi\left(\underline{r}, E, \underline{\Omega}, t\right) - \Sigma_t\left(\underline{r}, E, \underline{\Omega}\right)\varphi\left(\underline{r}, E, \underline{\Omega}, t\right)\\ &+ \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\,\Sigma_s\left(\underline{r}, E' \to E, \underline{\Omega'} \to \underline{\Omega}\right)\varphi\left(\underline{r}, E', \underline{\Omega'}, t\right)\\ &+ (1-\beta)\chi(E) \times \int_{4\pi} d\underline{\Omega'} \int_0^\infty dE'\nu\left(E'\right)\Sigma_f\left(\underline{r}, E'\right)\varphi\left(\underline{r}, E', \underline{\Omega'}, t\right)\\ &+ \sum_{i=1}^{6}\lambda_i C_i\left(\underline{r}, t\right)\chi_i(E)\end{aligned} \tag{4.19}$$

The last term is the number of delayed neutrons produced from precursor decay and the preceding term represents the prompt fission neutrons. The familiar precursor equation from previous knowledge of reactor kinetics is

$$\frac{\partial}{\partial t}C_i\left(\underline{r}, t\right) = \int_{4\pi} d\underline{\Omega'} \int dE'\beta_i\nu\left(E'\right)\Sigma_f\left(\underline{r}, E'\right)\varphi\left(\underline{r}, E', \underline{\Omega'}, t\right) - \lambda_i C_i\left(\underline{r}, t\right) \tag{4.20}$$

The adjoint equation for a critical unperturbed reference reactor can be written in analogy to the above transport Eq. 4.19 as

$$\begin{aligned}&-\underline{\Omega}\cdot\nabla\varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right) + \Sigma_t\left(\underline{r}, E\right)\varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)\\ &- \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\,\Sigma_s\left(\underline{r}, E \to E', \underline{\Omega} \to \underline{\Omega'}\right)\varphi_0^*\left(\underline{r}, E', \underline{\Omega'}\right)\\ &= \frac{1}{k}\nu(E)\Sigma_f\left(\underline{r}, E\right)\int_0^\infty dE'\Big[(1-\beta)\chi\left(E'\right)\\ &+ \sum_{i=1}^{6}\beta_i\chi_i\left(E'\right)\int d\underline{\Omega'}\varphi_0^*\left(\underline{r}, E', \underline{\Omega'}\right)\Big]\end{aligned} \tag{4.21}$$

The overall system is subject to the boundary conditions discussed before. The derivation proceeds as follows:

Equation 4.19 is multiplied by $\varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)$ and integrated over $\underline{r}, E, \underline{\Omega}$ and Eq. 4.21 is multiplied by $\varphi\left(\underline{r}, E, \underline{\Omega}, t\right)$ and integrated over $\underline{r}, E, \underline{\Omega}$ then the two are subtracted giving

$$\begin{aligned}
\frac{\partial}{\partial t}\int \frac{\varphi_0^*\varphi}{\upsilon} dVdEd\underline{\Omega} &= -\int \underline{\Omega}\cdot\nabla\left(\varphi_0^*\varphi\right)dVdEd\underline{\Omega} \\
&+ \frac{k-1}{k}\int\int dE'd\underline{\Omega}'\varphi_0^*\left(\underline{r}, E', \underline{\Omega}'\right)\left[X'\right] \\
&\times \iiint \nu(E)\Sigma_f\left(\underline{r}, E\right)\varphi\left(\underline{r}, E, \underline{\Omega}, t\right)dVdEd\underline{\Omega} \\
&- \int_{4\pi} d\underline{\Omega}'\int_0^\infty dE'\varphi_0^*\left(\underline{r}, E', \underline{\Omega}'\right)\left[\sum_{i=1}^{6}\beta_i\chi_i\left(E'\right)\right] \\
&\qquad \times \iiint \nu(E)\Sigma_f\left(\underline{r}, E\right)\varphi\left(\underline{r}, E, \underline{\Omega}, t\right)dVdEd\underline{\Omega} \\
&\qquad + \sum_{i=1}^{6}\lambda_i\int \varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)C_i\left(\underline{r}, t\right)\chi_i(E)dVdEd\underline{\Omega}
\end{aligned} \tag{4.22}$$

where $X' = \left[(1-\beta)\chi\left(E'\right) + \sum_{i=1}^{6}\beta_i\chi_i\left(E'\right)\right]$.

The total reaction rate terms will cancel each other by subtraction and the scattering terms are like so since integration over E and E' will remove the difference between the two terms and the direction of energy transfer wouldn't matter. The boundary conditions on the exterior of the reactor volume, i.e. $\varphi\left(\underline{r}_s, E, \underline{\Omega}, t\right) = 0$ for $\underline{n}\cdot\underline{\Omega} < 0$ and $\varphi^*\left(\underline{r}_s, E, \underline{\Omega}, t\right) = 0$ for $\underline{n}\cdot\underline{\Omega} > 0$ can be applied if the integration over $\underline{r}$ extends to include the whole reactor core. As a result, the steaming term will disappear.

$$\int \underline{\Omega}\cdot\nabla\left(\varphi_0^*\varphi\right)dVdEd\underline{\Omega} = \int \underline{\Omega}\cdot\left(\varphi\nabla\varphi_0^* + \varphi_0^*\nabla\varphi\right)dVdEd\underline{\Omega} = 0 \tag{4.23}$$

where we have used the identity $\nabla\left(\varphi_0^*\varphi\right) = \varphi\nabla\left(\varphi_0^*\right) + \varphi_0^*\nabla(\varphi)$.

However, if this integration is limited to a subregion of the reactor as in the case of core and reflector regions, there is no reason for this term to disappear. For example, at the core–reflector interface, a fraction of neutrons leaving the core will re-enter the core at a later time, hence their importance is non-zero.

The shape function:

The angular flux is separated into an amplitude factor and a shape function:

$$\varphi\left(\underline{r}, E, \underline{\Omega}, t\right) = n(t)\psi\left(\underline{r}, E, \underline{\Omega}, t\right) \tag{4.24}$$

For the following cases, where the point reactor model is accurate, hence, the shape function is not strongly varying function with time and it can be calculated using eigenvalue calculations:

1 The transients have died out.
2 The reactor is not anywhere near or above prompt critical.
3 The reactor is near critical and the perturbations are small.

Hence, we can write

$$\psi\left(\underline{r}, E, \underline{\Omega}, t\right) \cong \psi\left(\underline{r}, E, \underline{\Omega}\right) \tag{4.25}$$

4.5.2 One-Node Model

The core and reflector are considered as one node; therefore, the integration is over the reactor volume. First, we examine the left side of Eq. 4.22 by substituting for the flux in Eq. 4.24

$$\begin{aligned}\frac{\partial}{\partial t}\int \frac{\varphi_0^*\varphi}{\upsilon} dVdEd\underline{\Omega} &= \frac{\partial}{\partial t}\int \frac{n(t)\psi\varphi_0^*}{\upsilon} dVdEd\underline{\Omega} \\ &= \frac{\partial n}{\partial t}\int \frac{\psi\varphi_0^*}{\upsilon} dVdEd\underline{\Omega} + n(t)\frac{\partial}{\partial t}\int \frac{\psi\varphi_0^*}{\upsilon} dVdEd\underline{\Omega}\end{aligned}$$

Using the approximation in Eq. 4.25, the integration in the last term in the substitution above is constant with time and hence the derivative is zero

$$\frac{\partial}{\partial t}\int \frac{\varphi_0^*\varphi}{\upsilon} dVdEd\underline{\Omega} = \frac{\partial n}{\partial t}\int \frac{\psi\varphi_0^*}{\upsilon} dVdEd\underline{\Omega}$$

Substituting in Eq. 4.22 and dividing the equation by $\int \frac{\psi\varphi_0^*}{\upsilon} dVdEd\underline{\Omega}$ and using Eq. 4.23 and defining $d\xi = dVdEd\underline{\Omega}$, we get

$$\begin{aligned}\frac{\partial n}{\partial t} &= \frac{k-1}{k} n(t) \frac{\int\int dE' d\underline{\Omega'}\varphi_0^*\left(\underline{r}, E', \underline{\Omega'}\right)\left[X'\right]\iiint \nu(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E, \underline{\Omega}\right)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi} \\ &- n(t) \frac{\int_{4\pi} d\underline{\Omega'}\int_0^\infty dE'\varphi_0^*\left(\underline{r}, E', \underline{\Omega'}\right)\left[\sum_{i=1}^6 \beta_i\chi_i\left(E'\right)\right]\iiint \nu(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E, \underline{\Omega}\right)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi} \\ &+ \frac{\sum_{i=1}^6 \lambda_i \int \varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)C_i\left(\underline{r}, t\right)\chi_i(E)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi}\end{aligned} \tag{4.26}$$

From (4.26), we can designate $\rho = \frac{k-1}{k}$ and

$$\frac{1}{\Lambda} = \frac{\int\int dE' d\underline{\Omega'}\varphi_0^*\left(\underline{r}, E', \underline{\Omega'}\right)\left[X'\right]\iiint \nu(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E, \underline{\Omega}\right)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi}$$

$$\frac{\beta_{eff}}{\Lambda} = \frac{\int_{4\pi} d\underline{\Omega}' \int_0^\infty dE' \varphi_0^*\left(\underline{r}, E', \underline{\Omega}'\right)\left[\sum_{i=1}^{6} \beta_i \chi_i\left(E'\right)\right] \iiint \nu(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E, \underline{\Omega}\right)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi}$$

$$c_i(t) = \frac{\int \varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)C_i\left(\underline{r}, t\right)\chi_i(E)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi}$$

The precursor Eq. 4.20 is also multiplied by $\varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)\chi_i(E)$ and integrated over all variables and divided by $\int \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}$, and we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} \frac{\int \varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)\chi_i(E)C_i\left(\underline{r}, t\right)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi} \\
&= n(t) \frac{\int \varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)\beta_i\chi_i(E)d\xi \int_{4\pi} d\underline{\Omega}' \int dE' \nu\left(E'\right)\Sigma_f\left(\underline{r}, E'\right)\psi\left(\underline{r}, E', \underline{\Omega}'\right)}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi} \\
&- \lambda_i \frac{\int \varphi_0^*\left(\underline{r}, E, \underline{\Omega}\right)\chi_i(E)C_i\left(\underline{r}, t\right)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi}
\end{aligned} \tag{4.27}$$

where we define

$$\left(\frac{\beta_i}{\Lambda}\right)_{eff} = \frac{\int_{4\pi} d\underline{\Omega}' \int dE' \varphi_0^*\left(\underline{r}, E', \underline{\Omega}'\right)\beta_i\chi_i\left(E'\right) \iiint \upsilon(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E, \underline{\Omega}\right)d\xi}{\int \frac{\psi\varphi_0^*}{\upsilon} d\xi}$$

Rewriting Eqs. 4.26 and 4.27 using the notations defined above and collecting terms, we obtain the single node or point kinetic equation

$$\begin{aligned}
\frac{\partial n}{\partial t} &= \frac{\rho - \beta_{eff}}{\Lambda} n(t) + \sum_{i=1}^{6} \lambda_i c_i\left(\underline{r}, t\right) \\
\frac{\partial c_i\left(\underline{r}, t\right)}{\partial t} &= \left(\frac{\beta_i}{\Lambda}\right)_{eff} n(t) - \lambda_i c_i\left(\underline{r}, t\right)
\end{aligned} \tag{4.28}$$

4.5.3 Diffusion Model

From part A above in the section on the diffusion equation, the non-steady-state form is

$$\begin{aligned}
\frac{1}{\upsilon}\frac{\partial \phi}{\partial t} &= \nabla \cdot D\left(\underline{r}, E\right)\nabla\phi\left(\underline{r}, E\right) - \Sigma_t\left(\underline{r}, E\right)\phi\left(\underline{r}, E\right) \\
&+ \int_0^\infty dE' \Sigma_s\left(\underline{r}, E' \to E\right)\phi\left(\underline{r}, E'\right)
\end{aligned}$$

$$+\frac{\chi(E)}{k}\int_0^\infty dE'\nu(E')\Sigma_f(\underline{r},E')\phi(\underline{r},E')$$

And the adjoint diffusion equation for a critical reference reactor is

$$\begin{aligned}0 &= \nabla\cdot D(\underline{r},E)\nabla\phi_0^*(\underline{r},E) - \Sigma_t(\underline{r},E)\phi_0^*(\underline{r},E)\\ &+\int_0^\infty dE'\Sigma_s(\underline{r},E\to E')\phi_0^*(\underline{r},E')\\ &+\nu(E)\Sigma_f(\underline{r},E)\int_0^\infty dE'\chi(E')\phi_0^*(\underline{r},E')\end{aligned}$$

We follow the same derivation steps performed based on the transport equation:

$$\begin{aligned}\frac{\partial}{\partial t}\int\frac{\phi_0^*\phi}{\upsilon}dVdE &= \int[\phi_0^*(\underline{r},E)\nabla\cdot D\nabla\phi(\underline{r},E)]dVdE\\ &-\int[\phi(\underline{r},E)\nabla\cdot D\nabla\phi_0^*(\underline{r},E)]dVdE\\ &+\frac{k-1}{k}\int\int dE'\phi_0^*(\underline{r},E')[X']\iiint\nu(E)\Sigma_f(\underline{r},E)\phi(\underline{r},E,t)dVdE\\ &-\int_0^\infty dE'\phi_0^*(\underline{r},E')\left[\sum_{i=1}^{6}\beta_i\chi_i(E')\right]\iiint\nu(E)\Sigma_f(\underline{r},E)\phi(\underline{r},E,t)dVdE\\ &+\sum_{i=1}^{6}\lambda_i\int\phi_0^*(\underline{r},E)C_i(\underline{r},t)\chi_i(E)dVdE\end{aligned}$$

Integrating by parts the first two terms on the right-hand side following the same steps in the proof of the one-speed diffusion equation being self-adjoint leads to $\phi^*D\nabla\phi - \phi^*D\nabla\phi = 0$. The second term is also zero likewise. The above equation becomes

$$\begin{aligned}&\frac{\partial}{\partial t}\int\frac{\phi_0^*\phi}{\upsilon}dVdE\\ &=\frac{k-1}{k}\int\int dE'\phi_0^*(\underline{r},E')[X']\iiint\nu(E)\Sigma_f(\underline{r},E)\phi(\underline{r},E,t)dVdE\\ &-\int_0^\infty dE'\phi_0^*(\underline{r},E')\left[\sum_{i=1}^{6}\beta_i\chi_i(E')\right]\iiint\nu(E)\Sigma_f(\underline{r},E)\phi(\underline{r},E,t)dVdE\\ &+\sum_{i=1}^{6}\lambda_i\int\phi_0^*(\underline{r},E)C_i(\underline{r},t)\chi_i(E)dVdE\end{aligned}$$

We finally obtain the kinetic parameters based on the diffusion theory:

$$\rho = \frac{k-1}{k}$$

$$\frac{1}{\Lambda} = \frac{\int\int dE'\phi_0^*\left(\underline{r}, E'\right)\left[X'\right]\int\int\int \nu(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E\right)dVdE}{\int \frac{\psi\phi_0^*}{\upsilon}dVdE}$$

$$\frac{\beta_{eff}}{\Lambda} = \frac{\int_0^\infty dE'\phi_0^*\left(\underline{r}, E'\right)\left[\sum_{i=1}^6 \beta_i\chi_i\left(E'\right)\right]\int\int\int \nu(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E\right)dVdE}{\int \frac{\psi\phi_0^*}{\upsilon}dVdE}$$

$$c_i(t) = \frac{\int \phi_0^*\left(\underline{r}, E\right)C_i\left(\underline{r}, t\right)\chi_i(E)dVdE}{\int \frac{\psi\phi_0^*}{\upsilon}dVdE}$$

where we define

$$\left(\frac{\beta_i}{\Lambda}\right)_{eff} = \frac{\int dE'\phi_0^*\left(\underline{r}, E'\right)\beta_i\chi_i\left(E'\right)\int\int\int \nu(E)\Sigma_f\left(\underline{r}, E\right)\psi\left(\underline{r}, E\right)dVdE}{\int \frac{\psi\phi_0^*}{\upsilon}dVdE}$$

3 *Two-dimensional multi-group form for the diffusion model*

We can further obtain the multi-group form of the kinetic parameters for a two-dimensional reactor

$$\frac{1}{\Lambda} = \frac{\sum_{i=1}^N \sum_{j=1}^M \sum_{g'=1}^G X'_{g'}\phi_{0_{g'}}^{*i,j}\Delta_{g'}\sum_{g=1}^G \nu_g\Sigma_{fg}^{i,j}\psi_g^{i,j}\Delta_g\Delta_i\Delta_j}{\sum_{i=1}^N \sum_{j=1}^M \sum_{g=1}^G \frac{1}{\upsilon_g}\phi_{0_g}^{*i,j}\psi_g^{i,j}\Delta_g\Delta_i\Delta_j}$$

and $X'_{g'} = \left[(1-\beta)\chi_{g'} + \sum_{i=1}^{6} \beta_i\chi_i^{g'}\right]$.

$$\frac{\beta_{eff}}{\Lambda} = \frac{\sum_{i=1}^N \sum_{j=1}^M \sum_{g'=1}^G \phi_{0_{g'}}^{*i,j}\Delta_{g'}\sum_{i=1}^6 \beta_i\chi_i^{g'}\sum_{g=1}^G \nu_g\Sigma_{fg}^{i,j}\psi_g^{i,j}\Delta_g\Delta_i\Delta_j}{\sum_{i=1}^N \sum_{j=1}^M \sum_{g=1}^G \frac{1}{\upsilon_g}\phi_{0_g}^{*i,j}\psi_g^{i,j}\Delta_g\Delta_i\Delta_j}$$

$$c_{i=1,6}(t) = \frac{\sum_{i=1}^N \sum_{j=1}^M \sum_{g=1}^G \phi_{0g}^{*i,j}C_{i=1,6}^{i,j}(t)\chi_{i=1,6}^g\Delta_g\Delta_i\Delta_j}{\sum_{i=1}^N \sum_{j=1}^M \sum_{g=1}^G \frac{1}{\upsilon_g}\phi_{0g}^{*i,j}\psi_g^{i,j}\Delta_g\Delta_i\Delta_j}$$

$$\left(\frac{\beta_i}{\Lambda}\right)_{eff} = \frac{\sum_{i=1}^N \sum_{j=1}^M \sum_{g'=1}^G \phi_{0_{g'}}^{*i,j}\beta_{i=1,6}\chi_{i=1,6}^{g'}\Delta_{g'}\sum_{g=1}^G \nu_g\Sigma_{fg}^{i,j}\psi_g^{i,j}\Delta_g\Delta_i\Delta_j}{\sum_{i=1}^N \sum_{j=1}^M \sum_{g=1}^G \frac{1}{\upsilon_g}\phi_{0_g}^{*i,j}\psi_g^{i,j}\Delta_g\Delta_i\Delta_j}$$

So far, we have derived the kinetic equations and the kinetic parameters associated with it. The kinetic parameters are calculated from forward and adjoint transport or diffusion codes. More nodes such as two nodes for the core and reflector can be considered or two core nodes in what is known as the coupled reactor kinetics model, especially for certain reactor core types.

4.5.4 Two-Node Model

In the one-node model, the spatial effects are ignored though the reactor which is large in size where any disturbances at a given point are not felt further away in another point until later. Therefore, multi-nodes are more representative to the spatial effects; however, we will limit ourselves to a two-node model one for the core and one for the reflector.

Example: *A two-node reactor*

The reactor is to be divided into two nodes: core node and a reflector node (Fig. 4.6). The integrals over the core and reflector are expressed, respectively, as $\int_c = \int_{core} dVdEd\underline{\Omega}$ and $\int_r = \int_{reflector} dVdEd\underline{\Omega}$. Dealing with the integration over the core first; Eq. 4.26 becomes

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_c \frac{\varphi_0^*\varphi}{\upsilon} dVdEd\underline{\Omega} = -\int_c \underline{\Omega}\cdot\nabla(\varphi_0^*\varphi)dVdEd\underline{\Omega} \\
&+\frac{k-1}{k}\iiint_c dE'd\underline{\Omega}'\varphi_0^*(\underline{r},E',\underline{\Omega}')[X']\int\int\int \nu(E)\Sigma_f(\underline{r},E)\varphi(\underline{r},E,\underline{\Omega},t)dVdEd\underline{\Omega} \\
&-\int_c\int_{4\pi} d\underline{\Omega}'\int_0^\infty dE'\varphi_0^*(\underline{r},E',\underline{\Omega}')\left[\sum_{i=1}^{6}\beta_i\chi_i(E')\right]\int\int\int \nu(E)\Sigma_f(\underline{r},E)\varphi(\underline{r},E,\underline{\Omega},t)dVdEd\underline{\Omega} \\
&+\sum_{i=1}^{6}\lambda_i\int_c \varphi_0^*(\underline{r},E,\underline{\Omega})C_i(\underline{r},t)\chi_i(E)dVdEd\underline{\Omega}
\end{aligned}$$

Now we follow the same procedure as before in the one-node model by substituting with the shape function and dividing throughout by $\int_c \frac{\psi\phi_0^*}{\upsilon}dVdEd\underline{\Omega}$. However, the streaming term will not disappear since the core is bounded by the reflector and in the previous argument 22 for a void boundary is not valid for this case, therefore we get

$$\begin{aligned}
\frac{\partial n_c}{\partial t} &= -n_c(t)\frac{\int_S\int_{\underline{n}.\underline{\Omega}>0}|\underline{n}\cdot\underline{\Omega}|(\varphi_0^*\psi)dSdEd\underline{\Omega}}{\int_c \frac{\psi\varphi_0^*}{\upsilon}d\xi} \\
&+ n_c(t)\frac{\int_S\int_{\underline{n}.\underline{\Omega}<0}|\underline{n}\cdot\underline{\Omega}|(\varphi_0^*\psi)dSdEd\underline{\Omega}}{\int_c \frac{\psi\varphi_0^*}{\upsilon}d\xi}
\end{aligned}$$

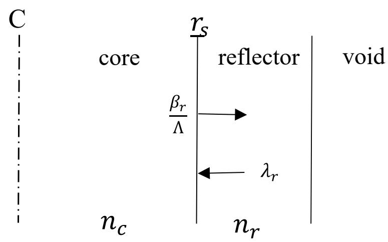

Fig. 4.6 Two-node model

$$
\begin{aligned}
&+\frac{k-1}{k} n_c(t) \frac{\int_c \int \int dE' d\underline{\Omega}' \varphi_0^*(\underline{r}, E', \underline{\Omega}')[X'] \int \int \int \nu(E) \Sigma_f(\underline{r}, E) \psi(\underline{r}, E, \underline{\Omega}) d\xi}{\int_c \frac{\psi \varphi_0^*}{\upsilon} d\xi} \\
&- n_c(t) \frac{\int_c \int_{4\pi} d\underline{\Omega}' \int_0^\infty dE' \varphi_0^*(\underline{r}, E', \underline{\Omega}') \left[\sum_{i=1}^6 \beta_i \chi_i(E')\right] \iiint \nu(E) \Sigma_f(\underline{r}, E) \psi(\underline{r}, E, \underline{\Omega}) d\xi}{\int_c \frac{\psi \varphi_0^*}{\upsilon} d\xi} \\
&+ \frac{\sum_{i=1}^6 \lambda_i \int_c \varphi_0^*(\underline{r}, E, \underline{\Omega}) C_i(\underline{r}, t) \chi_i(E) dV dE d\underline{\Omega}}{\int_c \frac{\psi \varphi_0^*}{\upsilon} d\xi}
\end{aligned}
\tag{4.29}
$$

where S is the surface of the core region and the coefficients except the two streaming terms to the right and left are defined as before. The continuity of the partial currents at the interface is given by

$$J_+^c(\underline{r}_s, E) = J_+^r(\underline{r}_s, E)$$

$$J_-^c(\underline{r}_s, E) = J_-^r(\underline{r}_s, E)$$

where $\underline{r}_s$ is at the core-reflector interface. The condition of continuity can be expressed using the definition of the partial currents

$$
\begin{aligned}
& n_c(t) \int_S \int_{\underline{n}.\underline{\Omega}>0} |\underline{n} \cdot \underline{\Omega}| (\phi_0^* \psi) dS dE d\underline{\Omega} \\
& = n_r(t) \int_S \int_{\underline{n}.\underline{\Omega}<0} |\underline{n} \cdot \underline{\Omega}| (\phi_0^* \psi) dS dE d\underline{\Omega}
\end{aligned}
$$

$$
\begin{aligned}
& n_c(t) \int_S \int_{\underline{n}.\underline{\Omega}<0} |\underline{n} \cdot \underline{\Omega}| (\phi_0^* \psi) dS dE d\underline{\Omega} \\
& = n_r(t) \int_S \int_{\underline{n}.\underline{\Omega}<0} |\underline{n} \cdot \underline{\Omega}| (\phi_0^* \psi) dS dE d\underline{\Omega}
\end{aligned}
$$

The equation for the reflector can be written as

$$
\begin{aligned}
\frac{\partial n_r}{\partial t} &= n_c(t) \frac{\int_S \int_{\underline{n}.\underline{\Omega}>0} |\underline{n} \cdot \underline{\Omega}| (\varphi_0^* \psi) dS dE d\underline{\Omega}}{\int_r \frac{\psi \varphi_0^*}{\upsilon} dV dE d\underline{\Omega}} \\
&\quad - n_r(t) \frac{\int_S \int_{\underline{n}.\underline{\Omega}<0} |\underline{n} \cdot \underline{\Omega}| (\varphi_0^* \psi) dS dE d\underline{\Omega}}{\int_r \frac{\psi \varphi_0^*}{\upsilon} dV dE d\underline{\Omega}}
\end{aligned}
\tag{4.30}
$$

where we have defined the flux in the core

$$\varphi(\underline{r}, E, \underline{\Omega}, t) \cong n_c(t) \psi(\underline{r}, E, \underline{\Omega})$$

And the flux in the reflector is

$$\varphi\big(\underline{r}, E, \underline{\Omega}, t\big) \cong n_r(t)\psi\big(\underline{r}, E, \underline{\Omega}\big)$$

both with the same shape function.

If the above equation is multiplied by

$$\frac{\int_r \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}}{\int_c \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}}$$

and redefining $n_r(t)$ as

$$n_r(t) \leftarrow n_r(t)\frac{\int_r \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}}{\int_c \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}}$$

Equation 4.28 becomes

$$\begin{aligned}\frac{\partial n_r}{\partial t} &= n_c(t)\frac{\int_S \int_{\underline{n}.\underline{\Omega}>0} \left|\underline{n}\cdot\underline{\Omega}\right|\left(\varphi_0^*\psi\right) dS dE d\underline{\Omega}}{\int_c \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}} \\ &\quad - n_r(t)\frac{\int_S \int_{\underline{n}.\underline{\Omega}<0} \left|\underline{n}\cdot\underline{\Omega}\right|\left(\varphi_0^*\psi\right) dS dE d\underline{\Omega}}{\int_r \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}}\end{aligned} \tag{4.31}$$

Then we will be able to define

$$\begin{aligned}\frac{\beta_r}{\Lambda} &= \frac{\int_S \int_{\underline{n}.\underline{\Omega}>0} \left|\underline{n}\cdot\underline{\Omega}\right|\left(\varphi_0^*\psi\right) dS dE d\underline{\Omega}}{\int_c \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}} \\ \lambda_r &= \frac{\int_S \int_{\underline{n}.\underline{\Omega}<0} \left|\underline{n}\cdot\underline{\Omega}\right|\left(\varphi_0^*\psi\right) dS dE d\underline{\Omega}}{\int_r \frac{\psi\varphi_0^*}{\upsilon} dV dE d\underline{\Omega}}\end{aligned}$$

and (4.29) and (4.31) become

$$\begin{aligned}\frac{\partial n_c}{\partial t} &= \frac{\rho - \beta_{eff}}{\Lambda} n_c(t) + \sum_{i=1}^{6} \lambda_i c_i\big(\underline{r}, t\big) - \frac{\beta_r}{\Lambda} n_c(t) + \lambda_r n_r(t) \\ \frac{\partial c_i\big(\underline{r}, t\big)}{\partial t} &= \left(\frac{\beta_i}{\Lambda}\right)_{eff} n_c(t) - \lambda_i c_i\big(\underline{r}, t\big) \\ \frac{\partial n_r}{\partial t} &= \frac{\beta_r}{\Lambda} n_c(t) - \lambda_r n_r(t)\end{aligned} \tag{4.32}$$

The system of Eqs. 4.32 is a two-node or two-point reactor kinetic equations with the added parameters representing the delayed neutrons traveling back from the reflector. This effect will appear at the tail of the transfer function. $\frac{\beta_r}{\Lambda}$ and λ_r can be calculated from the partial currents obtained from diffusion or transport calculation. It can also be extended to a multi-node arrangement [8].

Exercises

1 Prove that the forward and adjoint operators of the total loss L_t, L_t^*; scattering L_s, L_s^*; and fission L_f, L_f^* terms satisfy the requirement 2 and hence they have true adjoints as defined from Eqs. 4.12 and 4.13 as

$$L_t\varphi(\underline{r}, E, \underline{\Omega}) = -\Sigma_t(\underline{r}, E, \underline{\Omega})\varphi(\underline{r}, E, \underline{\Omega})$$

$$L_t^*\varphi^*(\underline{r}, E, \underline{\Omega}) = -\Sigma_t(\underline{r}, E)\varphi^*(\underline{r}, E, \underline{\Omega})$$

$$L_s\varphi(\underline{r}, E, \underline{\Omega}) = \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\Sigma_s(\underline{r}, E' \to E, \underline{\Omega'} \to \underline{\Omega})\varphi(\underline{r}, E', \underline{\Omega'})$$

$$L_s^*\varphi^*(\underline{r}, E, \underline{\Omega}) = \int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\Sigma_s(\underline{r}, E \to E', \underline{\Omega} \to \underline{\Omega'})\varphi^*(\underline{r}, E', \underline{\Omega'})$$

$$L_f\varphi(\underline{r}, E, \underline{\Omega}) = \frac{\chi(E)}{4\pi}\int_{4\pi} d\underline{\Omega'} \int_0^\infty dE'\nu(E')\Sigma_f(\underline{r}, E')\varphi(\underline{r}, E', \underline{\Omega'})$$

$$L_f^*\varphi^*(\underline{r}, E, \underline{\Omega}) = \frac{\nu(E)\Sigma_f(\underline{r}, E)}{4\pi}\int_0^\infty dE' \int_{4\pi} d\underline{\Omega'}\chi(E')\varphi^*(\underline{r}, E', \underline{\Omega'})$$

2 Examine the adjoint of the time-dependent DE by proving $\Delta = 0$.
  (a) Modify the code ONED to obtain its adjoint version ONEDA using the adjoint version of the two-group one-dimensional neutron diffusion equation.
  (b) Run a test problem, e.g. a PWR using both codes. One way to be certain about the correctness of the modification is to check if the eigenvalues in both runs are identical. This is much easier than writing a computer program identical to the ONED code for the adjoint two-group one-dimensional slab, cylinder, and sphere with 10 heterogenous regions.
  (c) Write the kinetic parameters in a numerically integrated form based on two-group one-dimensional diffusion theory for the particular geometry of the test problem.
  (d) Use the results of the two group fluxes and adjoints and the rest of the core data to calculate the kinetic parameters.

4 Write down the energy-dependent P_1 equations and from which try to deduce the adjoint form.

5 Find out the adjoint angular and space-dependent discrete ordinate equation.
6 Derive the last term in Eq. 4.20

$$\nu(E)\Sigma_f\left(\underline{r}, E\right)\int_0^{\infty} dE'\left[(1-\beta)\chi\left(E'\right)+\sum_{i=1}^{6}\beta_i\chi_i\left(E'\right)\right]\int d\underline{\Omega}'\varphi_0^*\left(\underline{r}, E', \underline{\Omega}'\right)$$

7 Follow up the derivation of the kinetic parameters explicitly.
8 Let $J(E)dE$ be the source of neutrons slowing down to the energy interval between E and $E+dE$ from collisions with a moderator of mass A which scatters neutrons isotropically in the center of mass system. $J_A(E)$ can be viewed as an integral operator acting on the flux $\phi(E)$

$$J_A(E) = L\phi(E) = \int_E^{E/\alpha} dE' \frac{\Sigma_s^H\left(E'\right)}{(1-\alpha)E'}\phi\left(E'\right)$$

where symbols are familiar from neutron slowing down theory [7]. Determine the adjoint L^*.

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