Percolation on Bethe lattice - computation of $\theta(p)$

Let p be the probability for a site to be occupied. Let $Q(p)$ be the probability that site 0 is NOT connected to infinity along a specific branch.

Then we have one of two possibilities (assuming $z = 3$):

- site 1 is empty, and this will happen with probability $1 - p$
- site 1 is occupied – probability of this is 1 and
  is not connected to ∞ via branch $(1, 11)$ – probability of this is $Q(p)$ and
  is not connected to ∞ via branch $(1, 12)$ – probability of this is again $Q(p)$

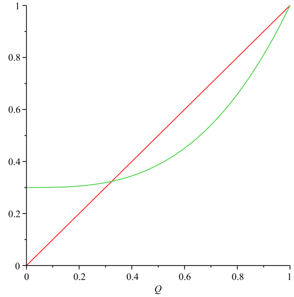
This yields

$$Q(p) = 1 - p + pQ^2(p)$$

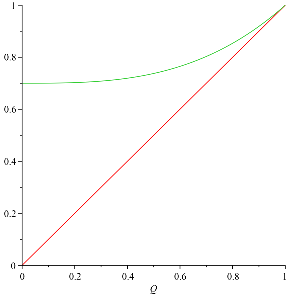
For general z, this becomes

$$Q(p) = 1 - p + pQ^{z-1}(p)$$

Graphs of $f(Q) = Q$ and $g(Q) = 1 - p + pQ^{z-1}(p)$ intersect in one or two places, depending on p, as shown below for $z = 3$:



$p = 0.7$



$p = 0.3$

The boundary between these two cases is the case of the red curve being tangent to the green curve. This will happen when the slopes at $Q = 1$ are the same, thus

$$\frac{\partial g}{\partial Q}\bigg|_{Q=1} = \frac{\partial f}{\partial Q}\bigg|_{Q=1}$$

$$p(z-1)Q^{z-2}\Big|_{Q=1} = 1$$

$$p(z-1) = 1$$

Solution of this equation is the critical probability for bond percolation on Bethe lattice,

$$p_c^{site} = \frac{1}{z-1}$$

Note that for $p \leq p_c^{site}$, the equation $Q(p) = 1 - p + pQ^{z-1}(p)$ has one solution $Q(p) = 1$, while for $p > p_c^{site}$ it has two solutions. Let us find them for $z = 3$.

$$Q(p) = 1 - p + pQ^2(p)$$

$$pQ^2(p) - Q(p) + 1 - p = 0$$

$$Q(p) = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} = \frac{1 \pm \sqrt{1 - 4p + 4p^2)}}{2p}$$

$$Q(p) = \frac{1 \pm \sqrt{(2p-1)^2}}{2p} = \frac{1 \pm |2p-1|}{2p}$$

If $p > 1/2$,

$$Q(p) = \frac{1 \pm |2p-1|}{2p} = \frac{1 \pm (2p-1)}{2p} = 1 \texttt{ or } \frac{2-2p}{2p}$$

$$Q(p) = 1 \texttt{ or } \frac{1-p}{p}$$

Since at $p = 1$ we are sure $Q(p) = 0$, we must reject the first of these and only keep the second one,

$$Q(p) = \frac{1-p}{p}$$

If $p \leq 1/2$,

$$Q(p) = \frac{1 \pm |2p-1|}{2p} = \frac{1 \pm (1-2p)}{2p} = \frac{2-2p}{2p} \texttt{ or } 1$$

$$Q(p) = \frac{1-p}{p} \texttt{ or } 1$$

Since at $p = 0$ we are sure $Q(p) = 1$, we must reject the second of these and only keep the first one,

$$Q(p) = 1$$

Thus we have

$$Q(p) = \begin{cases} 1 & \text{for } p \leq 1/2, \\ \dfrac{1-p}{p} & \text{for } p > 1/2. \end{cases}$$

Recall that earlier we defined $\theta(p)$ as a probability that a given cell (origin) belongs to an infinite cluster of open cells. For Bethe lattice, this will happen if the origin is open (probability p) and is connected to ∞ via at least one of the outgoing z edges. We have, therefore,

$$\theta(p) = p(1 - Q^z(p)).$$

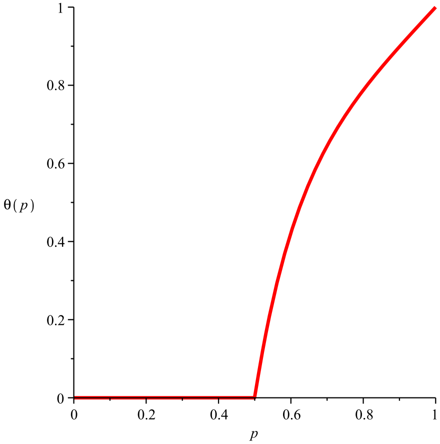
For $z = 3$ this becomes

$$\theta(p) = p(1 - Q^3(p)).$$

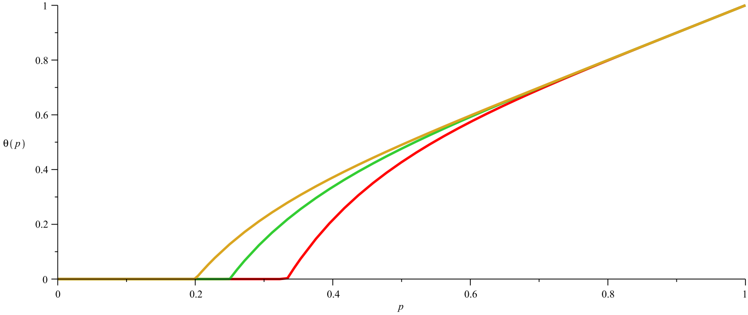
Using our computed value of $Q(p)$,

$$\theta(p) = \begin{cases} 0 & \text{for } p \leq 1/2, \\ p\left(1 - \left(\dfrac{1-p}{p}\right)^3\right) & \text{for } p > 1/2. \end{cases}$$

Graph of $\theta(p)$ vs. p for Bethe lattice site percolation with $z = 3$, for which $p_c = 1/2$.



For larger values of z, similar calculations can be performed. Graph of $\theta(p)$ vs. p for Bethe lattice site percolation with $z = 4, 5, 6$, for which $p_c = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$:



In general, for site percolation on Bethe lattice, $p_c = \frac{1}{z-1}$, and

$$\theta(p) \begin{cases} = 0 & \text{for } p \leq p_c \\ > 0 & \text{for } p > p_c. \end{cases}$$

Coming back to $z = 3$ case for $p > p_c$, we can Taylor expand

$$\theta(p) = p\left(1 - \left(\frac{1-p}{p}\right)^3\right)$$

$$\theta(p) = 6\left(p - \frac{1}{2}\right) - 24\left(p - \frac{1}{2}\right)^2 + 80\left(p - \frac{1}{2}\right)^3 - 224\left(p - \frac{1}{2}\right)^4 + \ldots$$

When p is close to $p_c = 1/2$, the dominant term is the linear term, thus

$$\theta(p) \approx 6\left(p - \frac{1}{2}\right) \text{ if } p > p_c \text{ and } p - p_c << 1.$$

We can say

$$\theta(p) \sim (p - p_c)^\beta \text{ if } p > p_c \text{ and } p - p_c << 1,$$

where, for Bethe lattice site percolation and $z = 3$, we have $\beta = 1$.Once can show that for other values of z, we still have $\beta = 1$.

Critical exponent β

The power law

$$\theta(p) \sim (p - p_c)^\beta \text{ if } p > p_c \text{ and } p - p_c << 1,$$

holds not only for Bethe lattice, but for any other lattice. However, the value of β depends on the lattice.
There is strong evidence that for two-dimensional lattices, $\beta = 5/36$. The proof of this fact has been proposed, but it is not entirely rigorous.

Other power laws in percolation

If n_s is the number of clusters of size s at $p = p_c$, then

$$n_s \sim s^\tau$$

The exponent τ is known as Fisher exponent. It is believed that in two dimensional space $\tau = 187/91$.
Away from p_c, the largest cluster size s_{max} obeys the law

$$s_{max} \sim |p - p_c|^{-1/\sigma}.$$

The value of σ is believed to be, in two dimensional space, $\sigma = 36/91$. Similarly as in the case of surface growth, scaling relationships exist in percolation,

$$\beta = \frac{\tau - 2}{\sigma}$$

It is easy to check that, indeed,

$$\frac{5}{36} = \frac{\frac{187}{91} - 2}{\frac{36}{91}}$$

Notes on applications

Percolation can be used as idealized model for distribution of oil or gas inside porous rock.

Assume that in site percolation unoccupied squares are filled with hard rock, while the occupied squares are pores filled with oil or gas.

The average concentration of oil in the rock is represented by p, known in oil industry as *porosity*. In real oil reservoirs, there are some correlations between occupied porer (due to their origin), but percolation models ignore these correlations.

When $p < p_c$, the oil is found in finite clusters, and a randomly placed well will hit only a small cluster (bad investment).

To produce large amount of oil, we need to find a reservoir where $p > p_c$.

We can estimate productivity potential of an oil well by taking samples of rocks, and measure porosity of samples. Typically this is a log of diameter 5 to 10 cm. Is it legitimate to estimate porosity of the oil field by porosity of the sample?
Let us assume we do this on a square lattice with $p > p_c$, and that the well is located in the largest cluster. We take $L \times L$ sample around this point. Let $M(L)$ be the number of points which belong to the same cluster. We can define average number of points connected to our well as $P = \frac{M(L)}{L^2}$.
For $p > p_c$, P will be independent of L.
When $p \approx p_c$, things will be different. Then the largest cluster contains many holes in it, and in these holes there may be other clusters with oil, not reachable through our well. In fact, when $p = p_c$, one has

$$M(L) \sim L^{1.9}.$$

The exponent 1.9 is sometimes called *fractal dimension* of clusters.

This means that at $p = p_c$,

$$P = \frac{M(L)}{L^2} \sim L^{-0.1}$$

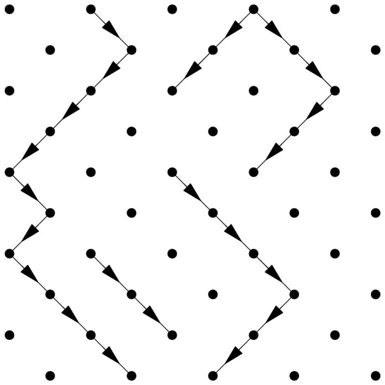
Since P is the average density of extractable oil in a field of size $L \times L$. Suppose now that the oil field has size $L_1 = 100$km, but the sample size is $L_2 = 10$cm, so that $L_1/L_2 = 10^6$.

This means that $P_1/P_2 = \left(10^6\right)^{-0.1} \approx 0.25$. The actual density of extractable oil in the oil field of size 100km is only 25% of that of a sample of size 10cm!

In three dimensions things are much worse: the density of oil in a field of size 100km (with $p = p_c$) is only 10^{-3} of that of a sample of size 10cm!

Directed percolation

Another important process in applications is the directed percolation process. Let us consider the square lattice in which open bonds are randomly distributed with a probability p.Unlike in the usual bond percolation problem, here bonds are directed downwards as indicated by the arrows.



Directed percolation as probabilistic cellular automaton

The directed percolation also has a critical value of p, to be denoted p_c^{DBP}. Using the image of the flowing fluid, above p_c^{DBP}, the fluid has a nonzero probability to reach an infinitely distant last row. Numerical simulations show that $p_c^{DBP} = 0.6445 \pm 0.0005$ for the bond model.

Directed percolation model is a special case of the so-called Domany-Kindzel cellular automaton, defined as follows.

$$
\begin{aligned}
s(i, t+1) &= f\big(s(i,t), s(i+1,t)\big) \\
&= \begin{cases} 0, & \text{if } s(i,t) + s(i+1,t) = 0, \\ 1, & \text{with probability } p_1, \text{ if } s(i,t) + s(i+1,t) = 1, \\ 1, & \text{with probability } p_2, \text{ if } s(i,t) + s(i+1,t) = 2. \end{cases}
\end{aligned}
$$

Directed bond percolation corresponds to $p_1 = p$, $p_2 = 2p - p^2$.
The case $p_1 = p_2 = p$ describes directed site percolation process.
It has critical probability $p_c^{DSP} = 0.7058 \pm 0005$.

Another example: General epidemic process

This is a model of a growth of the cluster of “infected” sites.Initially the cluster consists of the seed site located at the origin. At the next time step, a nearest-neighboring site is randomly chosen. This site is either added to the cluster with a probability p or rejected with a probability $1 - p$. At all subsequent time steps, the same process is repeated, a nearest-neighboring site of any site belonging to the cluster is selected at random, and it is either added to the cluster with a probability p or rejected with a probability$1 - p$. It is clear that there exists a critical probability p_c such that for $p > p_c$, the seed site has a nonzero probability to belong to an infinite cluster.

Example of general epidemic process for $p = 0.6$.