Additional remarks on Boltzmann distribution

A. Dagulescu and V. Yakovenko (2000) considered a model of a closed closed economic system in which the total amount of money is conserved.

They found that, in a large system of N economic agents, money plays the role of energy and the probability that an agent has an amount of money between m and $m + dm$ is $p(m)dm$ with

$$p(m) = Ce^{m/T},$$

where C is a normalizing factor and T is an effective "temperature". This, of course, has the same form as the Boltzmann distribution. The "temperature" T is the total amount of money M divided by N, $T = M/N$.

Many other parallels have been found between physical systems and economic systems ("econophysics"). Some economic models actually exhibit phase transitions (see, for example, Gyorgy Szabo, Christoph Hauert and "Phase transitions and volunteering in spatial public goods game" Phys. Rev. Lett. 89 (2002) 118101).

Criticality in phase transitions

In phase transition or in percolation, the critical point occurs at a single value of the parameter (such as p_c or T_c). The "signature" property of the critical point is that the distribution of cluster sizes (or a related quantity) follows a power law only at the critical point, but not elsewhere.

So if we have a natural system exhibiting a power law, is it in a critical state? If so, what is pushing it to be *exactly* at the critical point?

Perhaps some complex system are always sitting at the critical point? Are there any simple models with this property?

In 1980's, Bak, Tang and Wiesendfeld proposed a model exhibiting this feature.

The sandpile model

Consider a finite $L \times L$ square lattice.

- The state of a cell is a nonnegative integer representing the number of sand grains stacked on top of one another in that cell.
- In the initial state, each cell contains a random number of sand grains equal to 0, 1, 2 or 3.
- To cells sequentially chosen at random we add one sand grain.
- When the number of sand grains in one cell becomes equal to 4, we stop adding sand grains, remove the 4 grains and equally distribute them among the 4 nearest-neighboring cells. That is, the number of sand grains in the cell which reached the threshold value 4 at time t becomes equal to 0 at time $t+1$, and the number of sand grains in the 4 nearest-neighboring cells increases by one unit.
- This process is repeated as long as the number of sand grains in a cell reaches the threshold value.
- When the number of sand grain in a boundary cell reaches the value 4, the same process applies but the grains falling off the lattice are discarded.

Self-organized criticality

A sequence of toppling events occurs when nearest-neighboring cells of a cell reaching the threshold value contain 3 sand grains. Such a sequence is called an avalanche. An avalanche is characterized by its size, that is, the number of consecutive toppling events, and its duration equal to the number of update steps.

The sandpile model illustrates the basic idea of self-organized criticality. For a large system, adding sand grains leads at the beginning to small avalanches.

But as time increases, the system reaches a stationary state where the amount of added sand is balanced by the sand leaving the system along the boundaries. In this stationary state, there are avalanches of all sizes, up to the size of the entire system.

Numerical simulations show that the number $N(s)$ of avalanches of size s behaves as

$$N(s) \sim s^{-\tau},$$

where, in two-dimensional systems, the exponent τ is close to 1.

Forest fire model of Bak, Chen and Tang

Another model which was expected to exhibit self-organized criticality had been proposed by Bak, Chen and Tang. Their model is a three-state cellular automaton.

- State 0 represents an empty site, state 1 a green tree, and state 2 a burning tree.
- At each time step, a green tree has a probability p to grow at an empty site,
- a green tree becomes a burning tree if there is, at least, one burning tree in its *von Neumann neighborhood* (that is, among four nearest neighbours)
- sites occupied by burning trees become empty.

Detailed studies of this model demonstrated that it does not exhibit self-organized criticality.

Forest fire model of Drossel and Schwabl

To fix this, Drossel and Schwabl proposed an improved model, by introducing the following extra subrule: A green tree with no burning tree in its von Neumann neighborhood has a probability $f << p$ to be struck by lightning and become a burning tree. Peter Grassberger studied the model in detail, and found the following:

- If s is the number of trees burnt in one fire, in the limit $p/f \to \infty$, the fire size distribution behaves as $s^{1-\tau}$ with $\tau = 2.15 \pm 0.02$.
- For a finite value of p/f , the above power law behavior is cut off at $s_{max} \sim (p/f)^{\lambda}$ with $\lambda = 1.08 \pm 0.02$.
- The average cluster radius $\langle R^2 \rangle^{1/2}$ scales as $(p/f)^{\nu}$ with $\nu = 0.584 \pm 0.01$.

These results, however, remain controversial. Some recent studies suggest that the above results do not describe the true asymptotic regime, just transient phenomena which will disappear (or will be significantly different) at much larger lattice sizes.

Some researchers believe that power laws occurring in such phenomena as distribution of earthquakes, distribution of mass extinction events, or sizes of rainfall evens can be explained by self-organized criticality.
This remains a highly controversial claim, and much more research is needed before the question is settled.
One should also stress that rigorous mathematical results on self-organized criticality are scarce.

To illustrate the difficulty of detecting a power law in real-world (or model) data, consider a hypothetical case where two observed quantities are related by

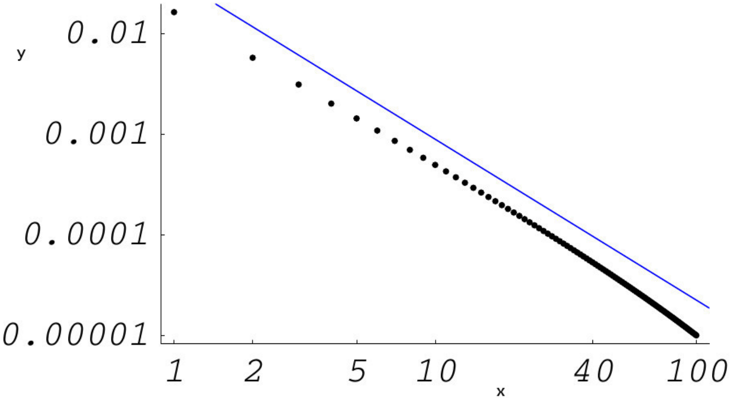
$$y = Cx^{-\alpha}(1-\epsilon)^{-x},$$

where $C > 0$ and ϵ is a small positive number. This relationship can be easily mistaken for a pure power law.
Suppose we generate 100 datapoints so that $x \in \{10, 20, 30, \dots 1000\}$, and compute corresponding y using

$$y = 0.5x^{-1.5}(1-0.0005)^{-x}.$$

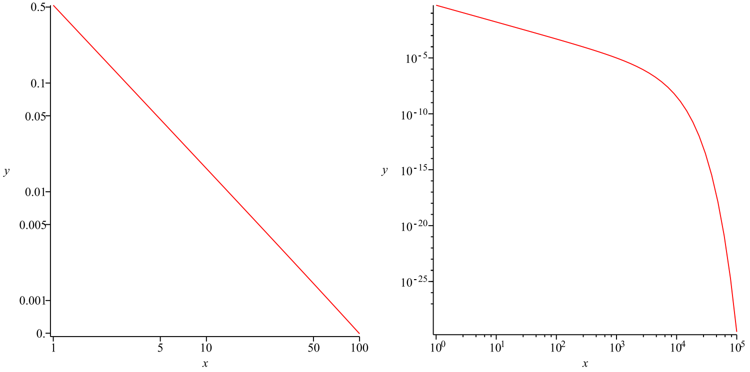
We then plot the data in log-log coordinates.

Here is what we obtain (black points).



One can easily believe that the above is a power law. If you fit a straight line (blue, slightly shifted for clarity), one obtains slope -1.6, suggesting that $y = Cx^{-1.6}$

Yet this is not true. Below we see the graph of $y = 0.5x^{-1.5}(1-0.0005)^{-x}$ for different ranges of x.



One can see that the power law which seems to exist for $x < 100$ is only an illusion. conclusion: There is always a danger that the power law observed in some experimental data does not hold on a larger scale.