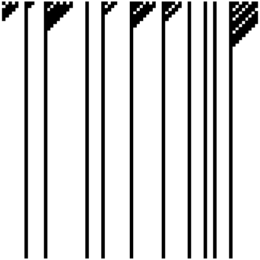
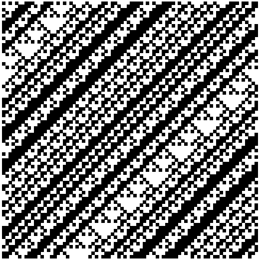
Some other elementary rules

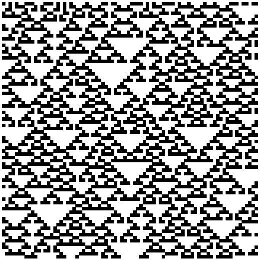
172:



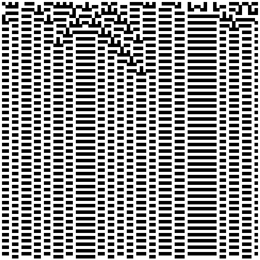
154:



22:

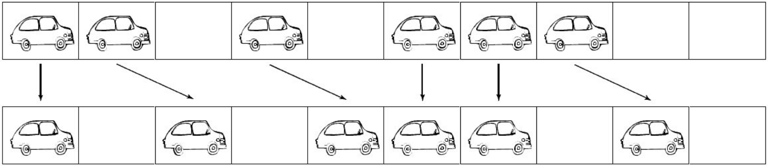


37:



Traffic model

Are CA useful in practice? Consider the following traffic model:



Suppose that $s_t(i) = 1$ if there is a car at cell i, otherwise $s_t(i) = 0$. Then we have CA with local rule

$$f(0,0,0) = 0,\ f(0,0,1) = 0,\ f(0,1,0) = 0,\ f(0,1,1) = 1,$$
$$f(1,0,0) = 1,\ f(1,0,1) = 1,\ f(1,1,0) = 0,\ f(1,1,1) = 1,$$

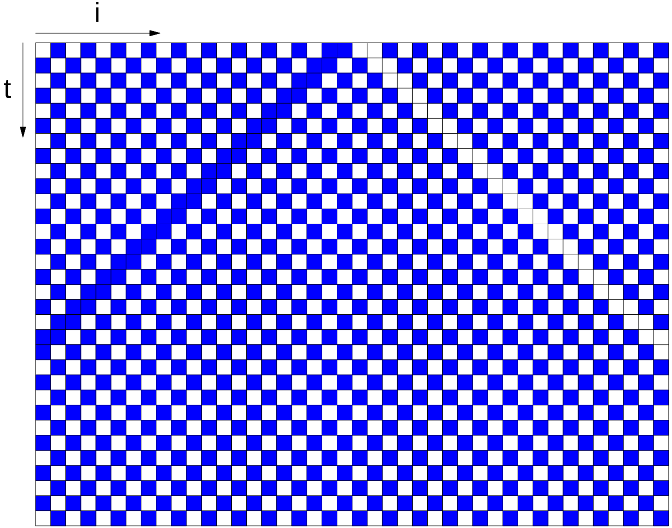
or equivalently

$$s_{t+1}(i) = s_t(i-1) + s_t(i)s_t(i+1) - s_t(i-1)s_t(i).$$

This is rule with $W(f) = 184$.

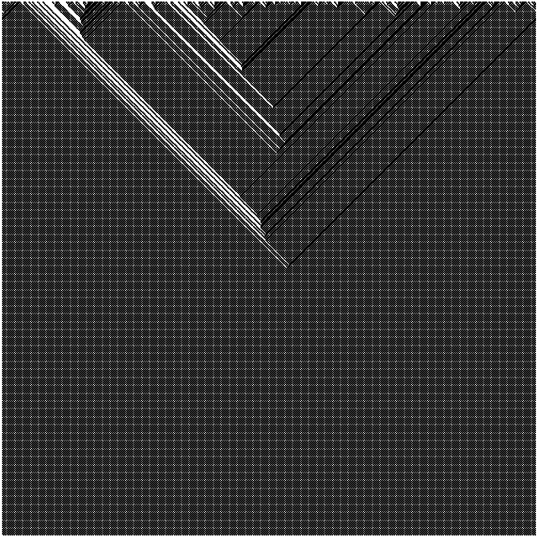
Spatio-temporal diagram for rule 184

If the initial configuration for rule 184 contains only one pair 11 followed by 00, we obtain:



Spatio-temporal diagram for rule 184, 500 sites

Larger simulation, $M = 500$, 500 time steps.



Notation

Let $s_t(i)$ be the state of the cell i at time t.
We will typically assume that the initial configuration is constructed randomly, and it is drawn from Bernoulli distribution, meaning that

$$Pr\left(s_0(i) = 1\right) = p \text{ for all } i,$$

$$Pr\left(s_0(i) = 0\right) = 1 - p \text{ for all } i.$$

If $p = 1/2$, we will call this symmetric Bernoulli distribution.
Let us also define probability of occurrence of block of symbols $a_1 a_2 \ldots a_k$ at time t by

$$P_t(a_1 a_2 \ldots a_k) = Pr\Big(s_t(i) = a_1, s_t(i+1) = a_1, \ldots, s_t(i+k) = a_k\Big)$$

The above will be called block probability. Note that the block probability is independent of i, because both the initial configuration and the CA rule is are translationally-invariant.

Simulations

- In simulations, we will be typically using lattice of size M with *periodic boundary condition*, that is, we will take $i = 0, 1, \ldots M-1$, and assume that the left neighbour of $s_t(0)$ is $s_t(M-1)$, while the right neighbour of $s_t(M-1)$ is $s_t(0)$.
- In many cases, periodic boundary conditions approximate (in some respects) infinite system. Detailed discussion of this topic is unfortunately beyond the scope of this course.
- Moreover, if M is very large,

$$P_t(a_1, a_2, \ldots a_k) \approx \frac{N_t(a_1, a_2, \ldots a_k)}{M},$$

where $N_t(a_1, a_2, \ldots a_k)$ is the number of occurrences of the block $a_1, a_2, \ldots a_k$ after t iterations of the rule.

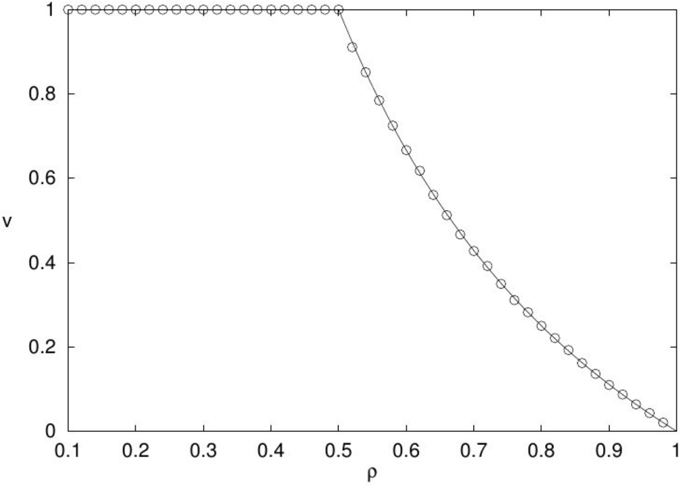
Now, let us go back to rule 184. Define density of 1's to be

$$\rho_t = P_t(1)$$

Note that $\rho_t = \rho_{t+1}$ for all t, because the number of cars does not change. We can thus drop the index t for ρ.
If we think of this rule as a system of moving cars, a car at site i can move only if site $i+1$ is empty. Such car has velocity 1 (one site per time step).
If, on the other hand, site $i+1$ is occupied, the velocity is zero.
The expected value of the velocity of cars at time t is, therefore,

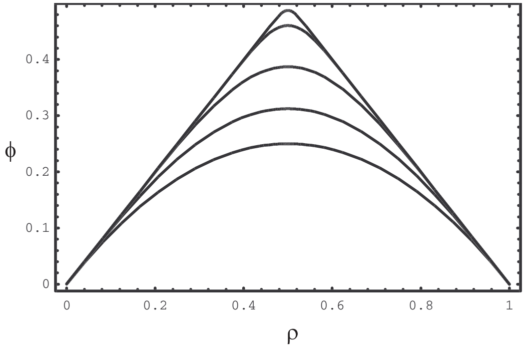
$$v_t = 1 \cdot \frac{P_t(10)}{P_t(1)} + 0 \cdot \frac{P_t(10)}{P_t(1)} = \frac{P_t(10)}{\rho}$$

Let $v = \lim_{t\to\infty} v_t$. From computer experiments (taking large, but finite t):



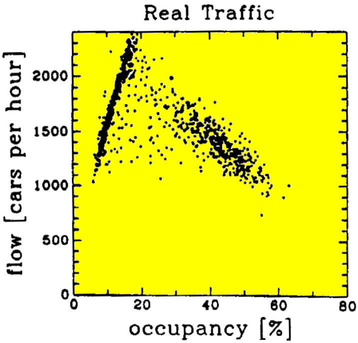
Free moving phase, $\rho < 0.5$, $v = 1$
Jammed phase, $\rho > 0.5$, $v < 1$.

Define now the flow at time t as $\phi_t = \rho v_t$. Graph of the flow ϕ_t versus density ρ is called *fundamental diagram*.
Fundamental diagram for rule 184, for t=0, 1, 5, 50, 5000 is shown below. For large t, it becomes "tent shaped".



Real traffic data from a highway in Japan

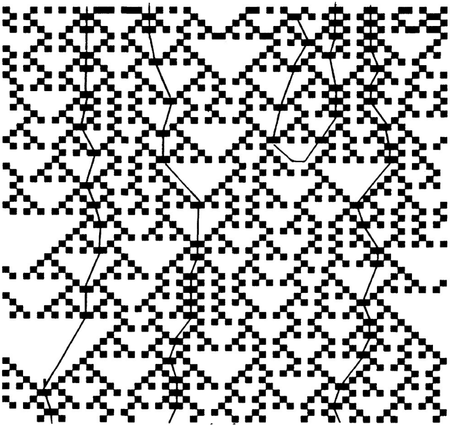
This is a fundamental diagram for a real traffic (on Japanese highway).



One can see that it is also tent-shaped, although somewhat slanted. Rule 184 reproduces the shape correctly (of course, better CA models exist).

Power laws in CA

An early example of a power law in CA has been found in rule 18 by P. Grassberger (Phys. Rev. A **28** 3666 1983). Grassberger noticed that in rule 18 number of pairs 11 decreases with time.



Plot from Grassberger's paper

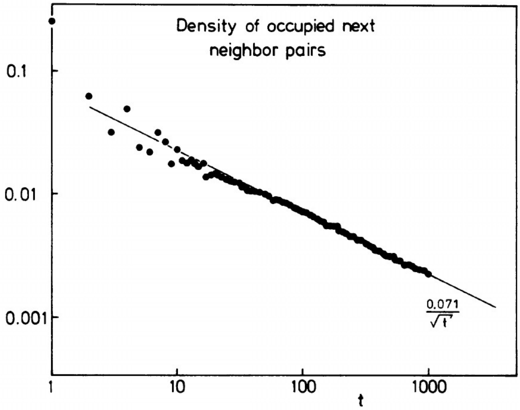


FIG. 3. Average density of occupied pairs after random start (60 000 sites). For large t, values averaged over t are plotted in order to suppress fluctuations.

$$P_t(11) \sim t^{-1/2}$$

The way to perform these calculations is to run simulation of rule 18 on a lattice with M sites, where $s_t(i)$ is the state of site i at time t. Then we simply compute the quantity

$$N_t(11) = \sum_{i=0}^{M} s_t(i) s_t(i+1)$$

We expect to obtain

$$\boxed{N_t(11) = At^{-1/2}}$$

where A is some constant. Taking log of both sides of this we obtain

$$\ln N_t(11) = \ln A - \frac{1}{2} \ln t,$$

which means that if we fit linear function to data $(\ln t, \ln N_t(11))$, its slope should be $-1/2$.

Rule 18 is the simplest model in which power law of the type $x(t) \sim t^{-1/2}$ appears. Some other well know examples are

- Rule 184 (traffic rule discussed earlier). If one starts with symmetric Bernoulli initial configuration, then

$$P_t(11) \sim t^{-1/2}$$

$$P_t(00) \sim t^{-1/2}$$

- Rule 14: If one starts with symmetric Bernoulli initial configuration, then,

$$P_0(1) = 0.5$$

$$P_1(1) = 0.375$$

$$P_0(1) - P_t(1) \sim t^{-1/2} \text{ for large } t$$

Rule 14

In rule 14, the origin of the power law is now fully understood. Local function of the elementary cellular automaton rule 14 is defined as $f(0,0,1) = f(0,1,0) = f(0,1,1) = 1$, and $f(x_0, x_1, x_2) = 0$ for all other triples $(x_0, x_1, x_2) \in \{0,1\}^3$. This is equivalent to

$$f(x_0, x_1, x_2) = x_1 + x_2 + x_1 x_0 x_2 - x_1 x_2 - x_0 x_2 - x_1 x_0.$$

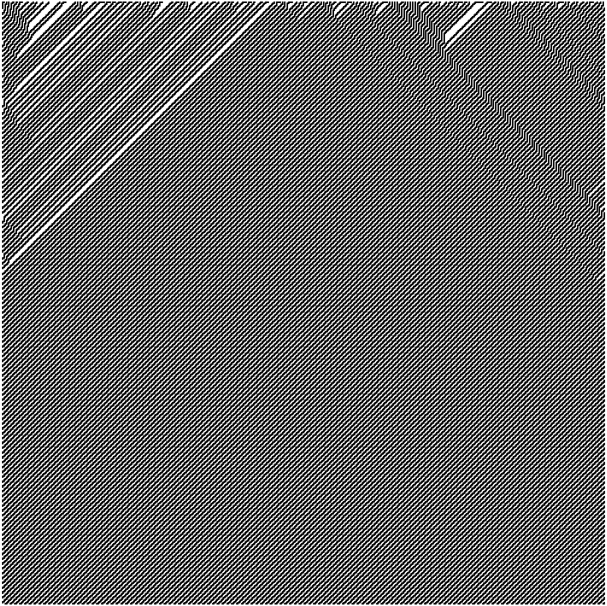
Theorem (H. Fukś and J. Haroutunian, *J. of cellular automata*, 4:99–110, 2009) *If in the initial configuration all cells are independently in state 1 with probability* $1/2$ *or in state 0 with probability* $1/2$*, then the expected density of ones after* t *iterations of rule 14 is*

$$\rho_t = \frac{1}{2}\left(1 - \frac{2t-1}{4^t} C_{t-1}\right),$$

where C_n *is the* n*-th Catalan number,*

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}.$$

Rule 14 - spatiotemporal pattern for 500 sites



Stirling's formula for large n

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

yields

$$C_n = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} \approx \frac{1}{n+1}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2} = \frac{1}{n+1}\frac{2^{2n}}{\sqrt{\pi n}} = \frac{1}{n+1}\frac{4^n}{\sqrt{\pi n}}$$

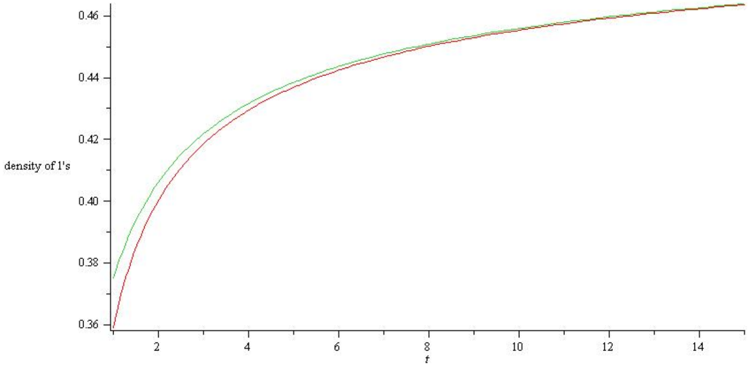
Now

$$\rho_t = \frac{1}{2}\left(1 - \frac{2t-1}{4^t}C_{t-1}\right) \approx \frac{1}{2}\left(1 - \frac{2t-1}{4^t}\cdot\frac{1}{t}\cdot\frac{4^{t-1}}{\sqrt{\pi(t-1)}}\right)$$

For large t we can replace $t-1$ by t and $2t-1$ by $2t$, thus

$$\rho_t \approx \frac{1}{2}\left(1 - \frac{2t}{4^t}\cdot\frac{1}{t}\cdot\frac{4^{t-1}}{\sqrt{\pi t}}\right) = \frac{1}{2} - \frac{1}{4\sqrt{\pi}}t^{-1/2}$$

Graph of the exact density (green) and the one obtained via Stirling's approximation (red) for rule 14.



For rule 14, therefore, the existence of the power law $x(t) \sim t^{-1/2}$ is fully understood. Similar explanation can be given for rule 184 (H. Fukś *Phys. Rev. E*, 60:197–202, 1999). No rigorous explanation is currently known for rule 18.

Other exponents in CA

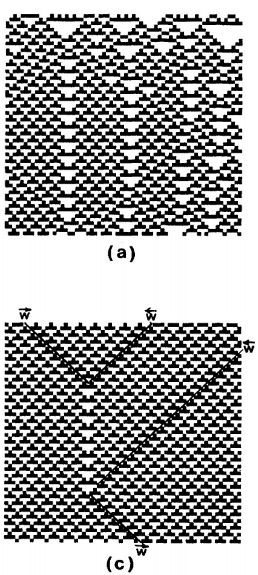
Are there any rules with exponents different that $-1/2$? Consider rule 54, defined by
$f(0,0,1) = f(1,0,0) = f(0,1,0) = f(1,0,1) = 1$, and
$f(x_0, x_1, x_2) = 0$ for all other triples $(x_0, x_1, x_2) \in \{0,1\}^3$.
Alternatively,

$$f(x_0, x_1, x_2) = x_0 - 2x_0x_1 + x_2 + x_1 - x_0x_2 + 2x_0x_2x_1 - 2x_2x_1.$$

This is one of the most complex elementary CA rules. Its spatiotemporal patterns reveal existence of four types of "particles", shown in the next two slides and denoted as g_o, g_e, $\overleftarrow{w}$ and $\overrightarrow{w}$.

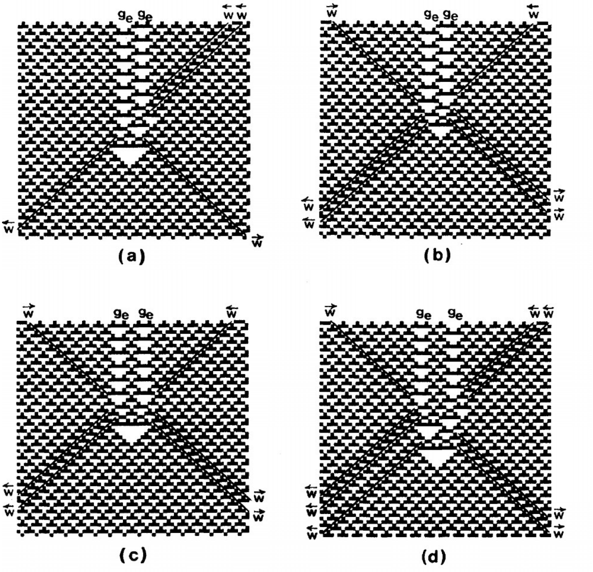
“Particles” in rule 54

Figure from N. Boccara, J. Nasser and M. Roger, *Phys. Rev. A* 44:2 866 (1991)



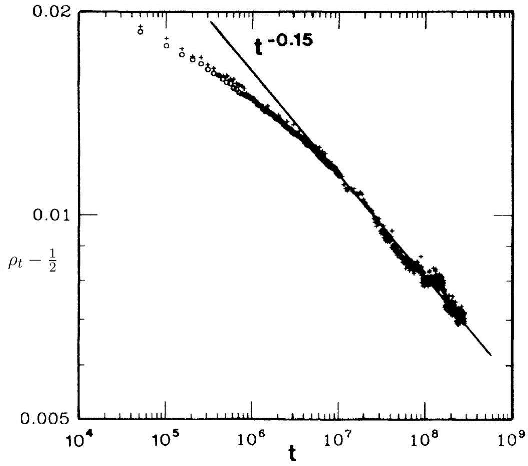
“Particles” in rule 54

Figure from N. Boccara, J. Nasser and M. Roger, *Phys. Rev. A* 44:2 866 (1991)



Power law in rule 54

Number of particles of type g_e as a function of time decreases as $t^{-0.15}$. Density of ones ρ_t approaches $1/2$ also as $t^{-0.15}$, as shown below. (adapted from N. Boccara, J. Nasser and M. Roger, *Phys. Rev. A* 44:2 866 (1991))



No rigorous explanation known.

Rule 54 – lattice of 1000 sites