MATH 3P40 - Mathematics Integrated with Computers and Applications III

Assignment 4

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April 4, 2017

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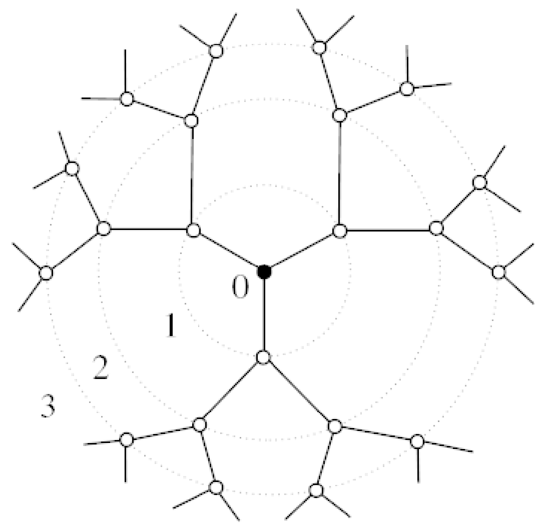
Problem 1

Let p be the probability for a given bond to be open. Let $Q(p)$ be the probability that a given site is not connected to infinity along a specific path, and let $\theta(p)$ be the probability that a given site is a part of an infinite cluster.



Bethe Lattice, z = 3

For any path; a site has two bonds to consider which lead away from the starting site, with two possibilities each:

1. The bond is closed with probability $1-p$

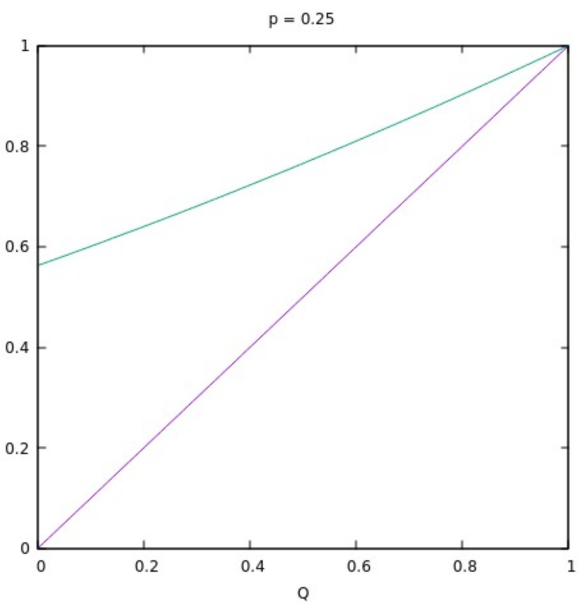
or:

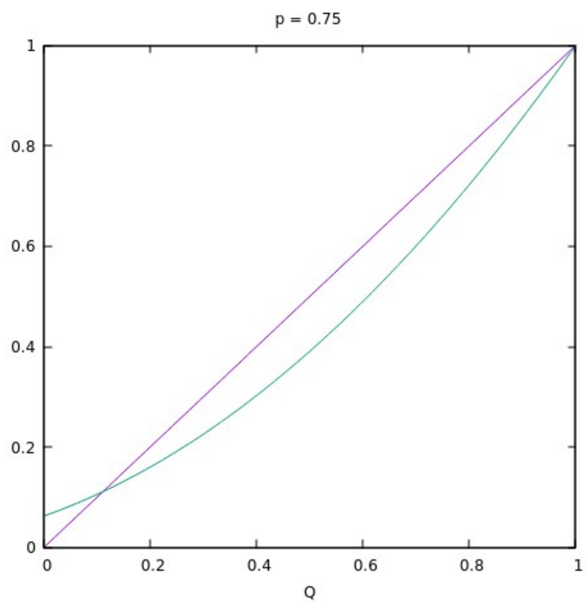
2. The bond is open with probability p and the connected site is not connected to infinity with probability Q.

Q can then be defined as:

$$Q(p)=(1-p+p\cdot Q(p))^2$$

If $f(Q)=Q$ and $g(Q)=(1-p+p\cdot Q)^2$, f and g intersect on the interval $[0:1]$ at $Q=1$ and at $Q<1$ when $p<p_c$ for some critical probability p_c.





This crossover point, p_c can then be found by finding the point when the slopes of f and g are equal at $Q=1$.

$$\left.\frac{\partial f}{\partial Q}\right|_{Q=1}=\left.\frac{\partial g}{\partial Q}\right|_{Q=1}$$

$$\Rightarrow 1=2\cdot p\cdot(1-p+p)$$

$$\Rightarrow 1=2\cdot p$$

$$\Rightarrow p_c=\frac{1}{2}$$

The solutions for Q are given by:

$$Q(p)=(1-p+p\cdot Q(p))^2$$

$$\Rightarrow p^2\cdot Q^2+(2\cdot p-2\cdot p^2-1)\cdot Q+(p^2-2\cdot p+1)=0$$

$$\Rightarrow Q(p)=\frac{-(2\cdot p-2\cdot p^2-1)\pm\sqrt{(2\cdot p-2\cdot p^2-1)-4\cdot p\cdot(p^2-2\cdot p+1)}}{2\cdot p^2}$$

$$\Rightarrow Q(p)=\frac{2\cdot p^2-2\cdot p+1\pm\sqrt{(2\cdot p-1)^2}}{2\cdot p^2}$$

$$\Rightarrow Q(p)=\frac{2\cdot p^2-2\cdot p+1\pm|2\cdot p-1|}{2\cdot p^2}$$

When $p>\frac{1}{2}$:

$$Q(p)=\frac{2\cdot p^2-2\cdot p+1\pm(2\cdot p-1)}{2\cdot p^2}\quad\Rightarrow\quad\begin{matrix}Q(p)=1\\ Q(p)=\frac{(1-p)^2}{p^2}\end{matrix}$$

$Q(p)=\frac{(1-p)^2}{p^2}$ then when $p>\frac{1}{2}$, as it is known that $Q(p)=0$ when $p=1$.

When $p\leq\frac{1}{2}$:

$$Q(p)=\frac{2\cdot p^2-2\cdot p+1\pm(1-2\cdot p)}{2\cdot p^2}\quad\Rightarrow\quad\begin{matrix}Q(p)=1\\ Q(p)=\frac{(1-p)^2}{p^2}\end{matrix}$$

$Q(p)=1$ then when $p\leq\frac{1}{2}$, as it is known that $Q(p)=1$ when $p=0$.

This gives the following final result for Q:

$$Q(p)=\begin{cases} 1 & ; \quad p \leq \frac{1}{2} \\ \frac{(1-p)^2}{p^2} & ; \quad p > \frac{1}{2} \end{cases}$$

θ, the probability that a given site is a part of an infinite cluster, can then be defined in terms of Q as the probability that the site is connected to infinity by at least one of the three paths leading out of it. This gives:

$$\theta(p)=1-Q^3(p)$$

$$\theta(p)=\begin{cases} 0 & ; \quad p \leq \frac{1}{2} \\ 1-\left(\frac{(1-p)^2}{p^2}\right)^3 & ; \quad p > \frac{1}{2} \end{cases}$$

The Taylor expansion of this function for $p>\frac{1}{2}$ at $p>\frac{1}{2}$ is:

$$\theta(p)=24\cdot(p-\frac{1}{2})-288\cdot(p-\frac{1}{2})^2+2336\cdot(p-\frac{1}{2})^3-14592\cdot(p-\frac{1}{2})^4+75648\cdot(p-\frac{1}{2})^5-\ldots$$

When $p-\frac{1}{2}\ll 1$, the linear term of the expansion is the largest contributor. This suggests that:

$\theta(p)\sim(p-p_c)^{\beta}$ where $\beta=1$, when p_c and $p-p_c\ll 1$

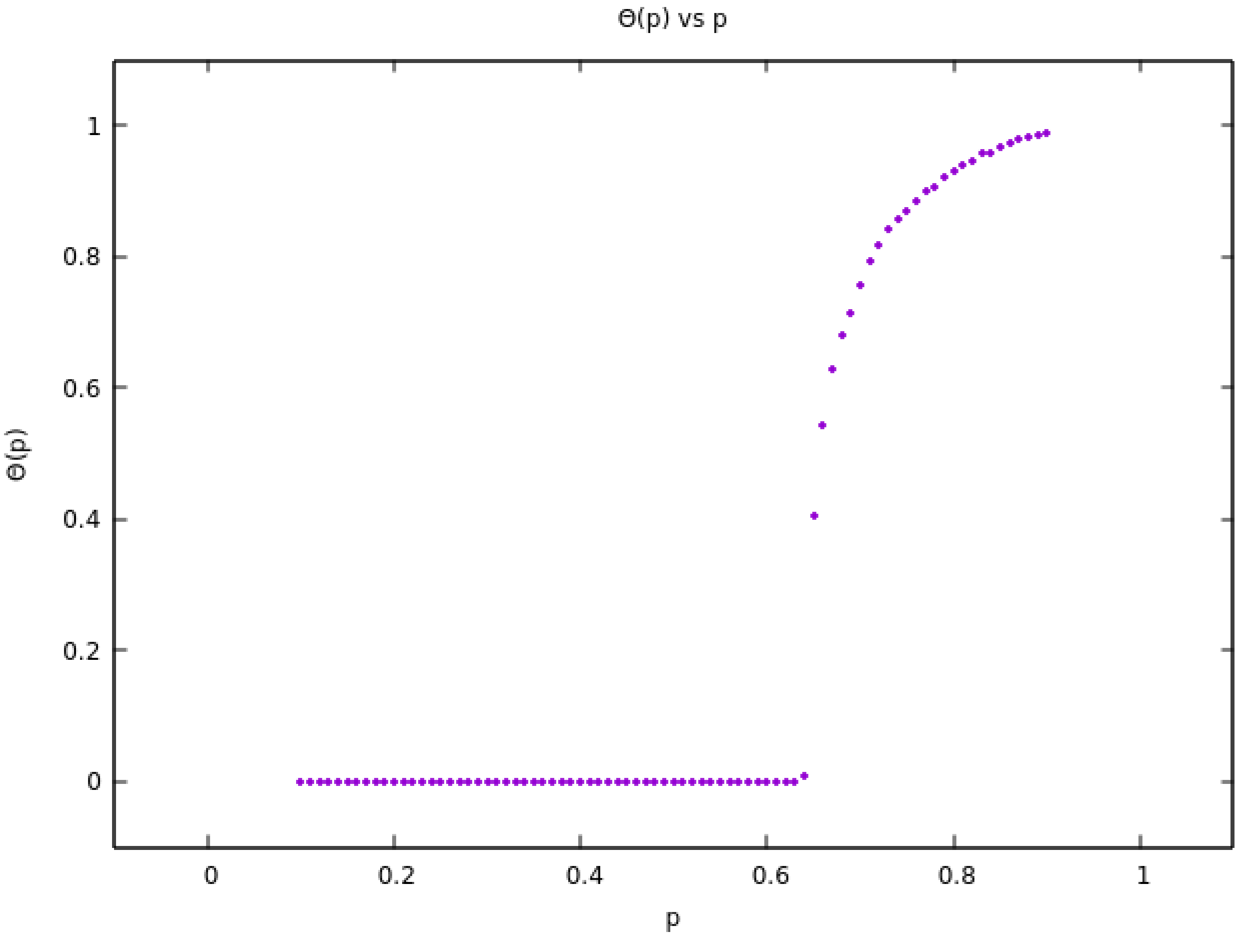
The exact critical values for bond percolation on a Bethe lattice where $z=3$ are therefore:

$$p_c=\frac{1}{2} \quad (Critical\,Probability)$$

$$\beta=1 \quad (Critical\,Exponent)$$

Problem 2

A simulation of the directed bond percolation model for a lattice of size 10000 using probabilities ranging from 0.10 to 0.90 with a step size of 0.01 produced the following plot of the density of wet sites in the bottom row verses p:



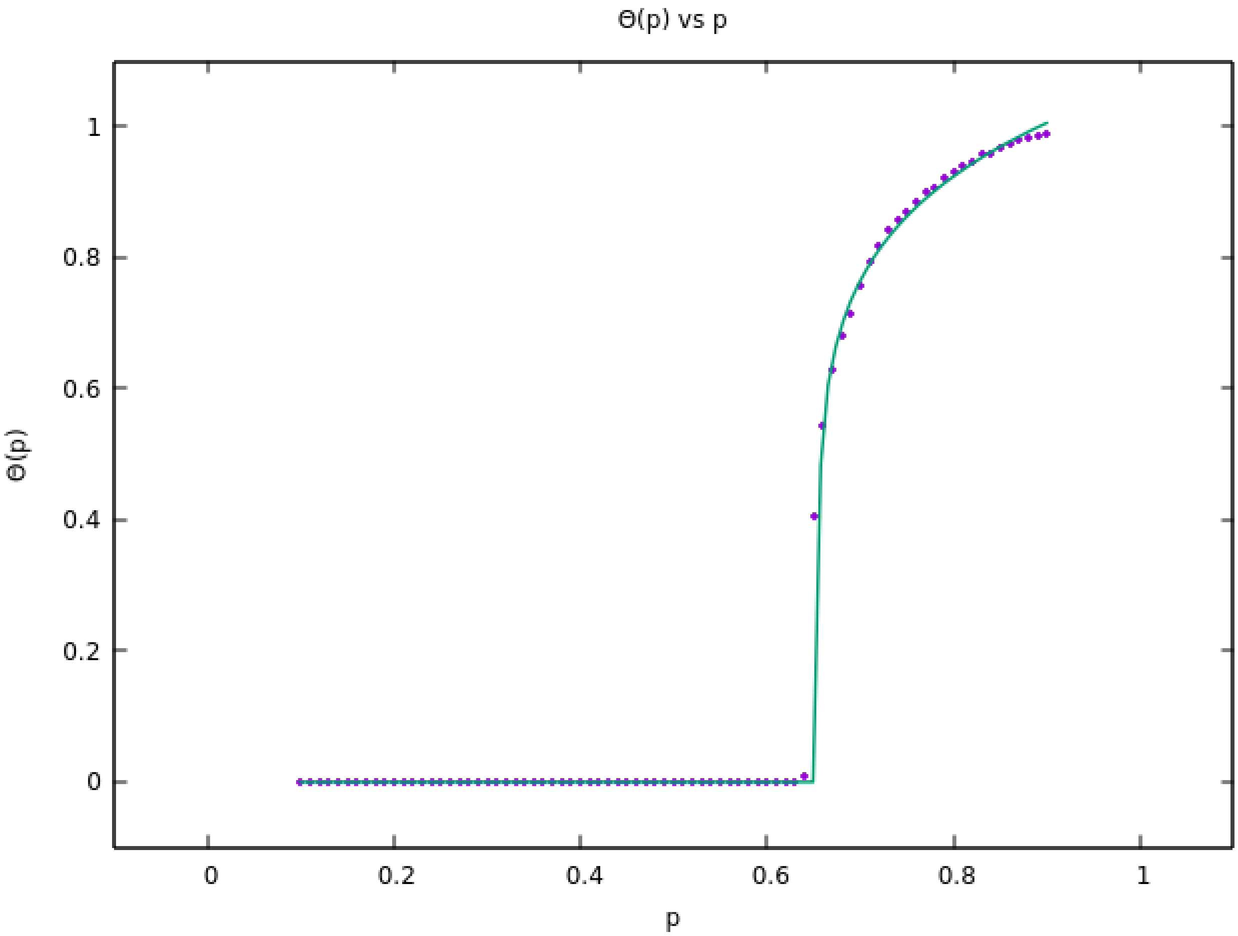
The following piecewise function was then fit to the data using gnuplot:

$$\theta(p)=\begin{cases} 0 & ; \quad x \le p_c \\ a \cdot (x-p_c)^b & ; \quad x > p_c \end{cases}$$

giving the following resulting values:

$$a=1.26234 \pm 0.07151 (5.665\,\%)$$
$$b=0.161657 \pm 0.02973 (18.39\%)$$
$$p_c=0.654959 \pm 0.005166 (0.7887\,\%)$$

Plotting this function with the data produces the following:



The critical probability, p_c, for directed bond percolation on a square lattice is therefore $p_c = 0.654959 \pm 0.005166 (0.7887\,\%)$.

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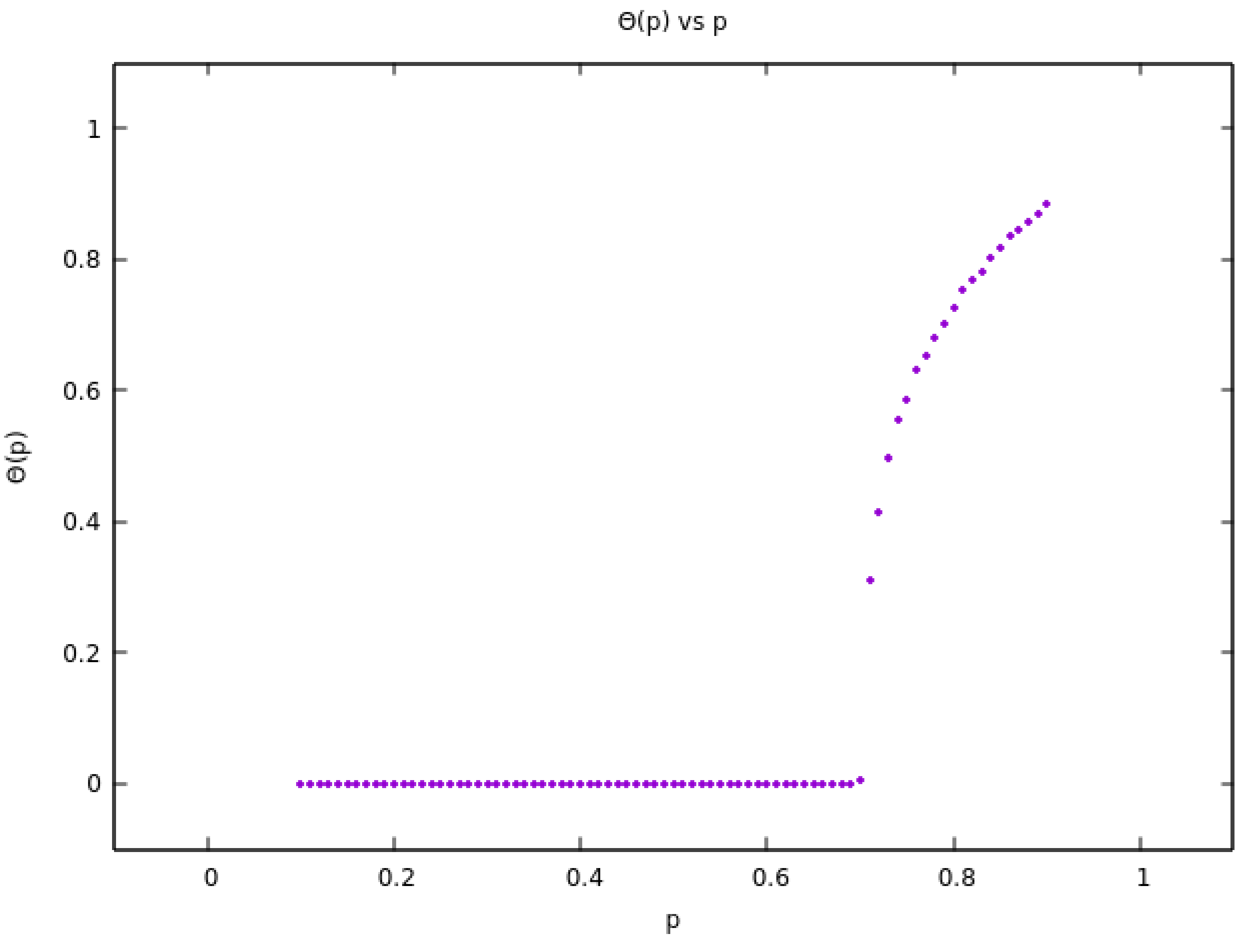
Problem 3

A simulation of the directed site percolation model for a lattice of size 10000 using probabilities ranging from 0.10 to 0.90 with a step size of 0.01 Produced the following plot of the density of wet sites in the bottom row verses p:



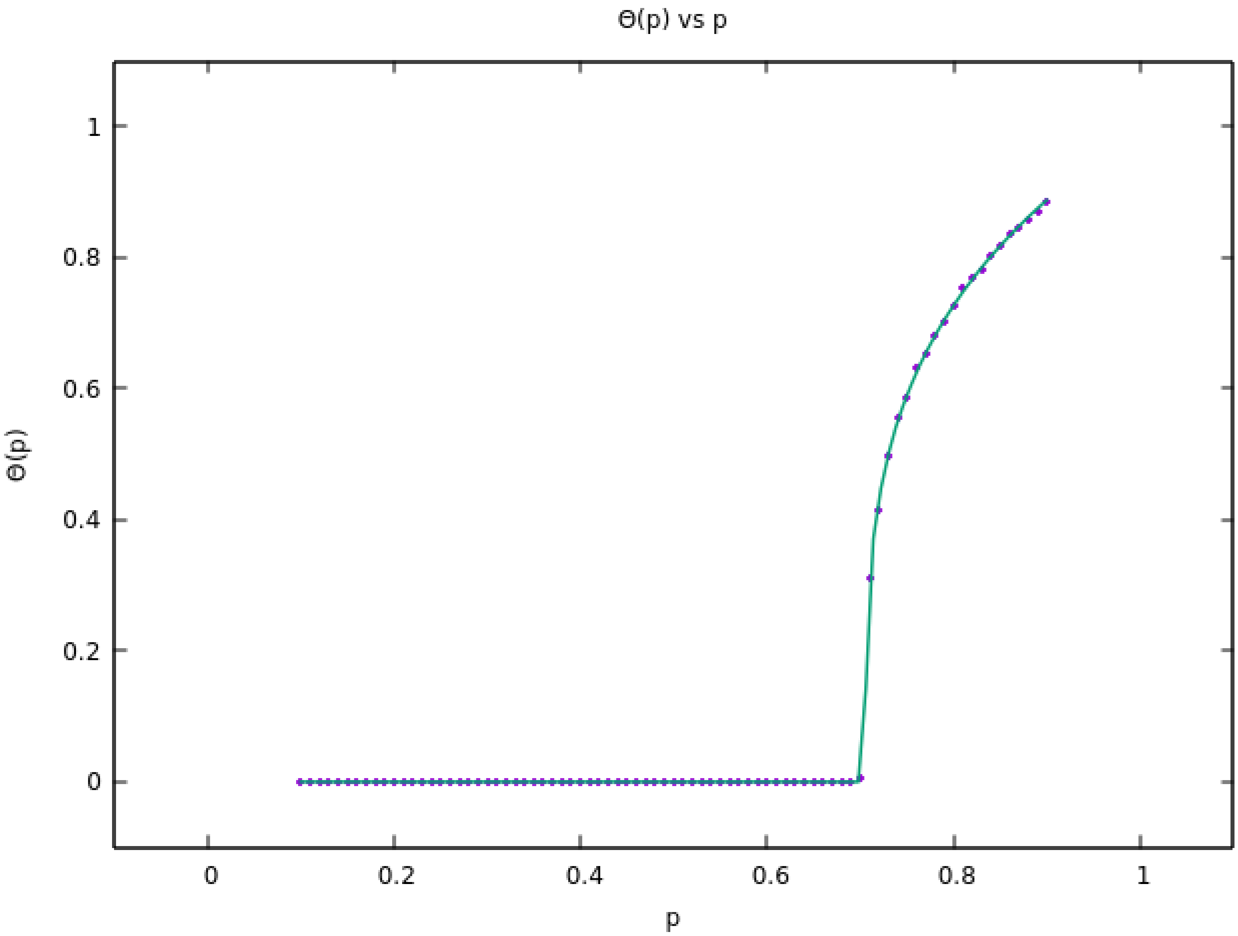
The following piecewise function was then fit to the data using gnuplot:

$$\theta(p)=\begin{cases} 0 & ; \quad x \le p_c \\ a\cdot(x-p_c)^b & ; \quad x > p_c \end{cases}$$

giving the following resulting values:

$$a=1.39823\pm0.01222(0.8741\,\%)$$
$$b=0.27709\pm0.004281(1.545\,\%)$$
$$p_c=0.705789\pm0.0003938(0.05579\,\%)$$

Plotting this function with the data produces the following:



The critical probability, p_c, for directed site percolation on a square lattice is therefore $p_c = 0.705789 \pm 0.0003938 (0.05579\%)$.