Summary of previous lectures

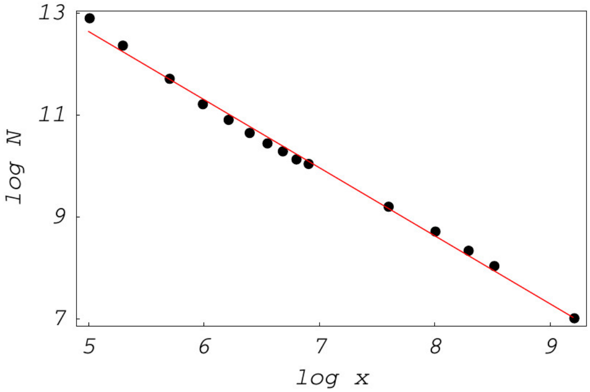
In our quest for "microscopic" explanation of common relationships between data we found the following:

- Phenomena exhibiting exponential growth can arise in systems behaving like "monofera model". Note: pure exponential growth is rare in nature, but exponential decay is common: mass of radioactive substances, atmospheric pressure, temperature of cooled object, etc.
- Phenomena exhibiting logistic growth can arise in systems behaving like "monofera in restricted space" model Note: logistic growth can be observed in the spread of innovations, substitution of technologies, growth of bacterial populations, spread of rumors, etc.

Logarithmic function is an inverse of the exponential function, thus we will not dwell on it for now. As far as *power laws* are concerned, we have not presented any model which would reproduce such behaviour, and we will do it shortly. Before we do this, however, we will examine some examples of power laws.

Examples of power laws: income distribution

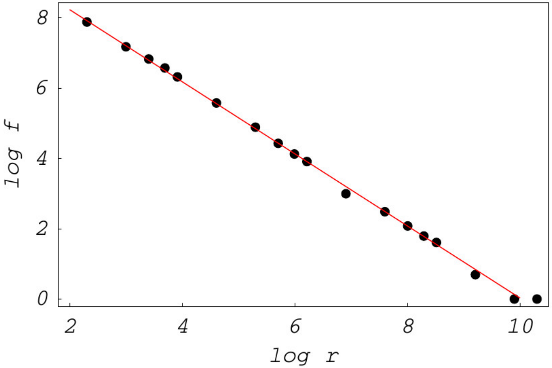
The first classic example of a power law comes from the book of Italian economist and sociologist, Vilfredo Pareto (1848–1923). The figure below (reproduced from B. Boccara, *Modeling Complex Systems*, Springer 2010) shows distribution of income of a category of British taxpayers for the year 1893–1894. x is the annual income and $N(x)$ the number of taxpayers whose annual income is greater than x.



$$\boxed{N(x) \sim x^{-1.5}}$$

Examples of power laws: Zipf'slaw

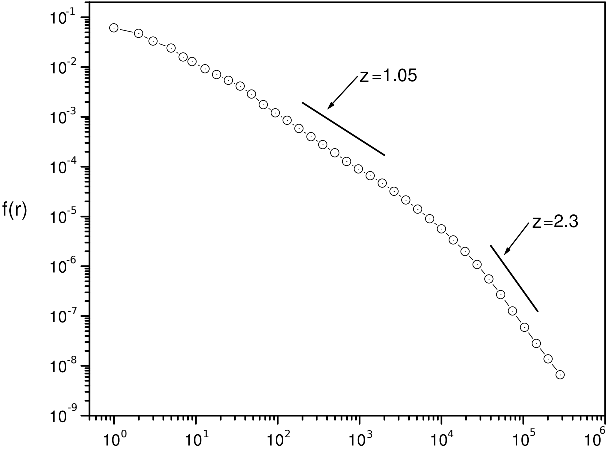
George Kingsley Zipf (1902-1950) discovered that the frequency f of a word is inversely proportional to its rank r, a word of rank r is the r-th word in the list of all words ordered with decreasing frequency (figure reproduced from B. Boccara, *Modeling Complex Systems*, Springer 2010).



$$\boxed{f \sim r^{-1.05}}$$

Modern modification of Zipf's law

In 2001, M. Montemurro studied the word-frequency distribution of English words using a large corpus consisting of 2606 books in English (*Physica A*, 300:567578). He found that words for which the rank is below 3000-4000 obey Zipf's law regardless of the text length. Above this limit, there seems to be another power law.



$$\boxed{f(r) \sim r^{-z}}$$

Other examples of power laws

- If s is the population size of a city and $f(s)$ is the number of cities having this size, then

$$f(s) \sim \frac{1}{s^z},$$

where $z = 2.03 \pm 0.05$ for the world, $z = 2.1 \pm 0.1$ for USA.

- Let $N(M)$ be the size of rain events of size M (height of the water column collected) per year and let $N(D)$ be the number of drought periods of duration D per year. Then

$$N(M) \sim M^{-1.36}, \quad N(D) \sim D^{-1.42}.$$

- Other phenomena exhibiting power law behaviour: size of earthquakes, distribution of family names, distribution of votes, distribution of mass extinction events, etc.

Power laws in growth and diffusion phenomena

In the above examples, the independent variable in the power law was the size or rank of something (earthquake, city, rainfall, etc.). There are also power laws in which the independent variable is the time.

- In a *random walk*, standard deviation of the position of the walker after time t is proportional to $t^{1/2}$.
- In so-called *forest fires models*, the total number of trees burnt after time t grows as $t^{1.59}$. (W. von Niessen and A. Blumen, *J. Phys. A.* **19** L289, 1986)
- In the *general epidemic process* studied by P. Grassberger, individuals distributed on a lattice can spread infection by direct contact and become immune after recovery. The total number of recovered individuals grows as $t^{0.807\pm0.01}$ (P. Grassberger *Math. Biosc,* **63** 167, 1983)

We will now consider some simple models in which power laws with time appear. These are generally based on spatially-extended discrete dynamical systems.

Spatially-extended discrete dynamical systems

Spatially-extended discrete dynamical systems are systems in which space, time, and the states of the system are all discrete. Typically, they have the following properties

- Space is represented by a regular lattice in one, two or three dimensions
- Each site, or *cell* of the lattice can be in one of many possible states (our of a *finite* number of possibilities)
- The system evolves over a succession of time steps

Examples: cellular automata, lattice gases, interacting particle systems, etc.

Elementary cellular automata

Let $s_t(i)$ denote the state of cell i at time t, where $t, i \in \mathbb{Z}$, $s_t(i) \in \{0, 1\}$. Furthermore, let $f : \{0, 1\}^3 \to \{0, 1\}$.
Dynamical system defined by

$$s_{t+1}(i) = f\left(s_t(i-1), s_t(i), s_t(i+1)\right)$$

is called *elementary cellular automaton* with local rule (function) f.
Elementary CA are usually identified by their Wolfram number $W(f)$, defined as

$$W(f) = \sum_{x_1, x_2, x_3 = 0}^{1} f(x_1, x_2, x_3) 2^{(2^2 x_1 + 2^1 x_2 + 2^0 x_3)}.$$

Rule 18

Let us construct a CA rule which models the following artificial society of monofera. They are distributed on a lattice, and we assume $s_t(i) = 0$ if cell i is empty, otherwise $s_t(i) = 1$.
Suppose they obey the following laws:

- A site which is occupied becomes empty in the next time step (they have short lifespan, just one time step).
- A site which is empty becomes occupied if **exactly one** if its neighbours is occupied

Let us now denote by the cell in question in blue color, and its right and left neighbours in black color. The new state of the cell will be in red color. The possible transitions are

- $\star 1 \star \to 0$ (star means any state)
- $100 \to 1$, $001 \to 1$, $101 \to 0$, $000 \to 0$.

Rule 18

This is a CA with the local rule $f : \{0,1\}^3 \to \{0,1\}$ defined as

$$\{0,0,0\} \to 0,\ \{0,0,1\} \to 1,\ \{0,1,0\} \to 0,\ \{0,1,1\} \to 0,$$
$$\{1,0,0\} \to 1,\ \{1,0,1\} \to 0,\ \{1,1,0\} \to 0,\ \{1,1,1\} \to 0,$$

$$\begin{array}{cccccccccccccc}
\cdots & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \cdots \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 & \cdots & \boxed{0} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} & \cdots &
\end{array}$$

Its Wolfram number is 18, because

$$W(f) = \sum_{x_1,x_2,x_3=0}^{1} f(x_1,x_2,x_3) 2^{(2^2 x_1 + 2^1 x_2 + 2^0 x_3)}.$$

$$W(f) = 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 + 0 \cdot 2^5 + 0 \cdot 2^6 + 0 \cdot 2^7 = 18$$

Local rules can be defined by different methods. The one which we have already encountered is called *rule table*. We simply list values of the function for all configurations of the neighbourhood,

$$f(0,0,0) = 0,\ f(0,0,1) = 1,\ f(0,1,0) = 0,\ f(0,1,1) = 0,$$
$$f(1,0,0) = 1,\ f(1,0,1) = 0,\ f(1,1,0) = 0,\ f(1,1,1) = 0.$$

Note that the above implies that only 001 and 100 can produce 1. If variables of f are x_0, x_1, x_2, we can write

$$f(x_0, x_1, x_2) = \begin{cases} 1, & \text{for } x_0 = 0, x_1 = 0, x_2 = 1 \\ 1, & \text{for } x_0 = 1, x_1 = 0, x_2 = 0 \\ 0, & \text{otherwise} \end{cases}$$

First condition will be satisfied when $(1 - x_0)(1 - x_1)x_2 = 1$. Second condition will be satisfied when $x_0(1 - x_1)(1 - x_2) = 1$. We thus have

$$f(x_0, x_1, x_2) = (1 - x_0)(1 - x_1)x_2 + x_0(1 - x_1)(1 - x_2)$$

After expansion and simplification,
$f(x_0, x_1, x_2) = x_0 + x_2 - x_2x_1 - x_1x_0 - 2x_2x_0 + 2x_2x_1x_0.$

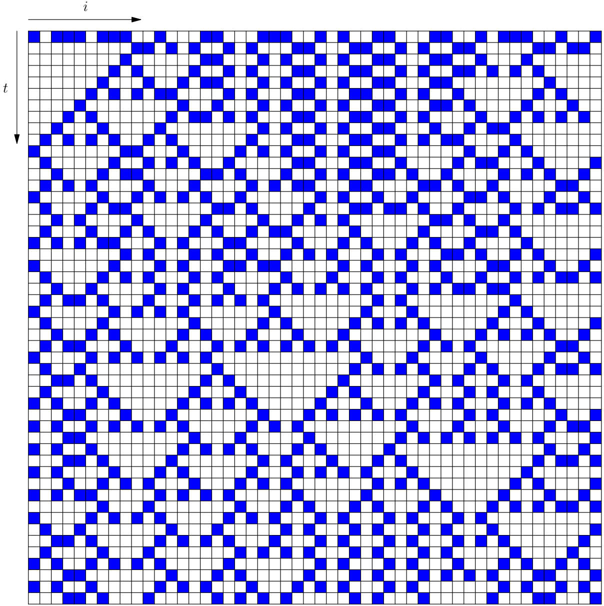
```
// Program for plotting ca rule 18
#include <iostream>
#include "rand.c"
using namespace std;

//definition of the rule
int f(int x0, int x1, int x2)
{
  return x0-x0*x1+2*x0*x2*x1-2*x0*x2+x2-x2*x1;
}

int main()
{
   //initialize RNG
  r250_init(1234);

  int M=50; //number of space sites
  int T=50; //number of iterations


  int world[M]; //array representing the world (1 if occupied, 0 if empty)
  int newworld[M]; //array representing the world (1 if occupied, 0 if empty)
  int i,t;

  //Create initial array with probability p of site in state 1
  double p=0.4;

  for (i=0; i<M; i++) if (dr250()<p) world[i]=1; else world[i]=0;

  for (t=0; t<T; t++) //loop over time
  {
   //first printing the configuration
   for (i=0; i<M; i++) cout <<world[i]; cout <<endl;
   //now we will update the world
   //loop over sites EXCEPT BOUNDARY
   for (i=0; i<M; i++)  newworld[i]=f(world[i-1], world[i], world[i+1]);
   //now we take care of boundaries (periodic)
   newworld[0]=f(world[M-1], world[0], world[1]);
   newworld[M-1]=f(world[M-1], world[M-1], world[0]);

   for (i=0; i<M; i++)  world[i]=newworld[i]; //copy new world to old

  } //end t loop


return 0;
}
```

Output - fragment

```
1011101110010001100011100101001100011001011100100
0000000001101010010100011000110010100110000011011
0000000010000001100010100101001100011001000100000
0000000101000010010100011000110010100110101010000
0000001000100101100010100101001100011000000001000
0000010101011000010100011000110010100100000010100
0000100000000100100010100101001100011010000100010
0001010000000101101010001100011001010000100101010
0010001000010000000010100101001100010010110000000
0101010100101000000010001100011001010110000100000
1000000011000100001010100101001100000010010100000
0100000100101010010000011000110010000101100010000
0010001011000000110100010010100110100100001010100
0101010000100010000101011000110000110100100000110
1000001001010101001000000101001001000011010001001
0100010110000000011010000100011011010010000101011
0010100001000001000010010101000000011010010000000
0100010010100010100101100000100000100001101000000
1010101100010100011000010001010001010010000100000
0000000010100010100100101010001010001101001010000
1000000100010100011011000001010001010000110001001
0100000101010001010000010001000101000100100101011
0010010000010100010001010101010001010110110000000
```

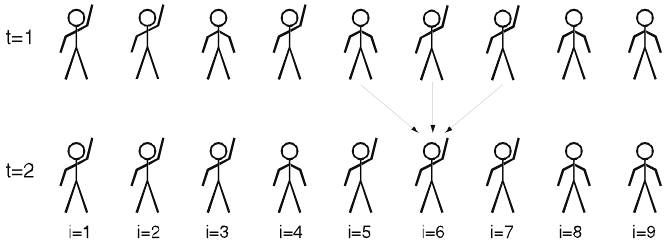
. . .

Nicer output



Example: Majority voting rule

$$s_{t+1}(i) = \text{majority}\{s_t(i-1), s_t(i), s_t(i+1)\}$$



Here

$$\{0,0,0\} \to 0,\ \{0,0,1\} \to 0,\ \{0,1,0\} \to 0,\ \{0,1,1\} \to 1,$$
$$\{1,0,0\} \to 0,\ \{1,0,1\} \to 1,\ \{1,1,0\} \to 1,\ \{1,1,1\} \to 1$$

$$W(f) = 0 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 + 1 \cdot 2^5 + 1 \cdot 2^6 + 1 \cdot 2^7 = 232$$

Spatiotemporal pattern for rule 232