Nonlinear relationships: bottom-up approach

In the previous lecture, we introduced some nonlinear relationships between two quantities, often used in mathematical modeling:

- exponential $y = a\exp(bx)$
- power law $y = ax^b$
- logarithmic $y = a\log(bx)$
- logistic

What is the origin of these relationships? We will try to use bottom-up approach to answer this question: we will look at some simple models, endow them with simple laws, and recover the aforementioned relationships.

Unlimited growth: models with continuous time

Consider a population of imaginary creatures called *monofera* (sing. *monoferum*). They breed by budding in the following fashion: in time interval Δt, each monoferum produces a single offspring with probability $\lambda \Delta t$.

This means that if the population of monofera at time t is $N(t)$, then at time $t + \Delta t$ the expected value of their population will be

$$N(t + \Delta t) = N(t) + \lambda \Delta t N(t),$$

hence

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = \lambda N(t).$$

Taking the limit $\Delta t \to 0$, we obtain

$$\frac{dN}{dt} = \lambda N,$$

that is, *differential equation* for unknown function $N(t)$.

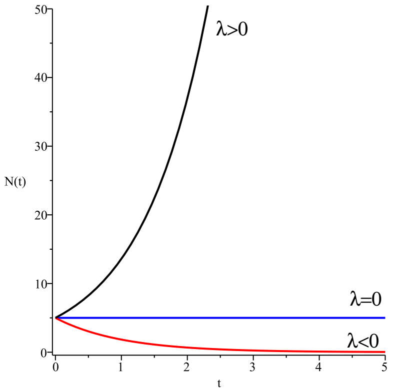
If we assume that $N(0) = N_0$ is given, we have so-called *initial value problem* (IVP)

$$\boxed{\begin{aligned} \frac{dN}{dt} &= \lambda N \\ N(0) &= N_0, \end{aligned}}$$

where λ and N_0 are given numbers.

Solution:

$$\boxed{N(t) = N_0 e^{\lambda t}}$$



Power law exponential growth

$$\boxed{\begin{aligned} &\frac{dN}{dt} = \lambda N^r, \quad r \neq 1 \\ &N(0) = N_0 \end{aligned}}$$

Solution (separation of variables)

$$N^{-r} dN = \lambda dt$$

$$\int_{N_0}^{N} N^{-r} dN = \lambda \int_0^t dt$$

$$\left. \frac{N^{-r+1}}{-r+1} \right|_{N_0}^{N} = \lambda t$$

$$N^{1-r} - N_0^{1-r} = \lambda(1-r)t$$

Solving for N and simplifying we obtain

$$\boxed{N(t) = \left[N_0^{1-r} + \lambda(1-r)t\right]^{\frac{1}{1-r}}}$$

Note that for $r = 2$ we obtain *hyperbolic solution*

$$N(t) = \left[N_0^{-1} - \lambda t\right]^{-1}$$

while for $r = \frac{1}{2}$ we obtain *quadratic solution*,

$$N(t) = \left[\sqrt{N_0} + \frac{1}{2}\lambda t\right]^2$$

and for $r = -1$ we obtain

$$N(t) = \sqrt{N_0^2 + 2\lambda t}$$

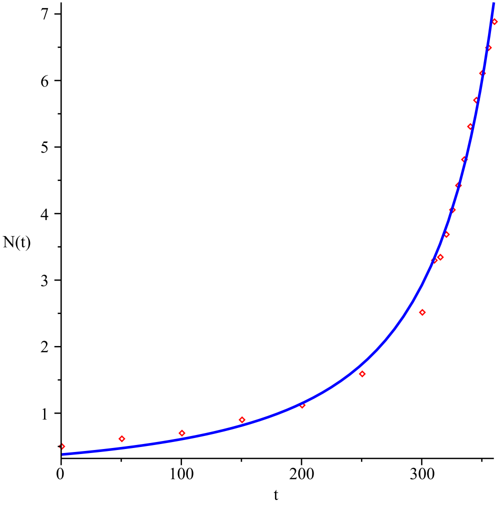
Example: world population

Year	t	Population [billions]
1650	0	0.510
1700	50	0.625
1750	100	0.710
1800	150	0.910
1850	200	1.129
1900	250	1.600
1950	300	2.524
1960	310	3.306
1965	315	3.354
1970	320	3.696
1975	325	4.066
1980	330	4.432
1985	335	4.822
1990	340	5.318
1995	345	5.714
2000	350	6.118
2005	355	6.503
2010	360	6.894

We fit

$$N(t) = \left[N_0^{1-r} + \lambda(1-r)t\right]^{\frac{1}{1-r}}$$

to data, obtaining $N_0 = 0.378$, $r = 1.51$, $\lambda = 0.00695$.



Note that this means that the world population seems to satisfy

$$\frac{dN}{dt} = 0.00695 \cdot N^{1.51},$$

or approximately

$$\frac{dN}{dt} \sim N^{3/2}$$

$$\boxed{\text{rate of growth} \sim (\text{population})^{3/2}}$$

It is an interesting example of so-called "power law". For obvious reasons, this can only be valid for a limited time.
Interestingly, in 1968 N. Keyfitz in his "Introduction to the Mathematics of Population" concluded that

$$\boxed{\text{rate of growth} \sim (\text{population})^{2}}$$

This clearly indicates that the population growth is slowing down.

Exponential growth with constant harvesting

Let us return to our monofera. We will again assume that in time interval Δt, each monoferum produces a single offspring with probability $\lambda \Delta t$, but we will assume that we are interested only in the number of monofera living a certain "monocity". We also assume that in time interval Δt, somebody "harvests" $h\Delta t$ monofera, where h is called *harvesting rate*. We then obtain

$$N(t+\Delta t) = N(t) + \lambda \Delta t N(t) - h\Delta t,$$

hence

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = \lambda N(t) - h.$$

Taking the limit $\Delta t \to 0$, we obtain

$$\frac{dN}{dt} = \lambda N - h.$$

If we assume that the initial population $N(0) = N_0$ is given, this yields initial value problem

$$\boxed{\begin{aligned} \frac{dN}{dt} &= \lambda N - h \\ N(0) &= N_0, \end{aligned}}$$

where λ, h and N_0 are given numbers.
Solution can be obtained by separation of variables,

$$\boxed{N(t) = N_0 e^{\lambda t} - \frac{h}{\lambda}(e^{\lambda t} - 1)}$$

Extinction time

Is it possible for the population to reach $N(t) = 0$ (extinction)?

$$N_0 e^{\lambda t} - \frac{h}{\lambda}(e^{\lambda t} - 1) = 0$$

$$\left(N_0 - \frac{h}{\lambda}\right) e^{\lambda t} + \frac{h}{\lambda} = 0$$

$$e^{\lambda t} = \frac{-\frac{h}{\lambda}}{N_0 - \frac{h}{\lambda}}$$

$$t = \frac{1}{\lambda} \ln \frac{1}{1 - \frac{\lambda N_0}{h}}$$

Clearly, when $h > \lambda N_0$, the population becomes extinct at time

$$\boxed{t_e = \frac{1}{\lambda} \ln \frac{1}{1 - \frac{\lambda N_0}{h}}}$$

Monofera in space

Consider now a population $P(t)$ of monofera which live in *restricted space*, and therefore cannot multiply *ad infinitum*.
More precisely, they live in a world where there are exactly M spaces available.
Initially, P_0 of these spaces are occupied, the rest is empty.
In time interval $(t, t + \Delta t)$, each monoferum produces one offspring with probability $\lambda \Delta t$ and attempts to place it in a spot randomly selected among all sites. If the selected spot is already occupied, the newborn monoferum dies.
If several monofera attempt to "colonize" the same cell simultaneously, only one of these new beasts survive.

This means that each empty cell can receive new life from each of $P(t)$ monofera, with probability $\frac{\lambda}{M}\Delta t$. Let us define $r = \frac{\lambda}{M}$.
Probability of not receiving new life from a given monoferum is $1 - r\Delta t$
Probability of not receiving new life from any monoferum is $(1 - r\Delta t)^{P(t)}$
Probability of receiving new life from at least one monoferum is $1 - (1 - r\Delta t)^{P(t)}$
Note that when $r\Delta t$ is small

$$1 - (1 - r\Delta t)^{P(t)} \approx 1 - (1 - P(t)r\Delta t) = P(t)r\Delta t$$

The expected value of the number of empty cells which became occupied in time Δt is

$$(M - P(t))P(t)r\Delta t$$

The expected value of the number of creatures at time $t + \Delta t$ is

$$P(t + \Delta t) = P(t) + (M - P(t))\, P(t) r \Delta t$$

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = r(M - P(t))P(t)$$

Taking the limit $\Delta t \to 0$ we obtain

$$\boxed{\frac{dP}{dt} = r(M - P)P}$$

This is called *logistic differential equation*.

Solution

Let us solve IVP

$$\boxed{\begin{aligned} &\frac{dP}{dt} = r(M-P)P \\ &P(0) = P_0, \end{aligned}}$$

where r, M and P_0 are given numbers.
We first separate variables

$$\frac{dP}{P(M-P)} = r dt$$

We will use decomposition into simple fractions:

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

$$\frac{dP}{P} + \frac{dP}{M-P} = rM\, dt$$

$$\int_{P_0}^{P} \frac{dP}{P} + \int_{P_0}^{P} \frac{dP}{M-P} = \int_0^t rM\,dt$$

Assuming that $0 \le P(t) \le M$ for all $t > 0$ we obtain

$$\ln P\big|_{P_0}^{P} - \ln(M-P)\big|_{P_0}^{P} = rMt\big|_0^t$$

$$\ln P - \ln P_0 - \ln(M-P) + \ln(M-P_0) = rMt$$

$$\ln \frac{P(M-P_0)}{P_0(M-P)} = rMt \implies \frac{P(M-P_0)}{P_0(M-P)} = e^{rMt}$$

$$P_0(M-P)e^{rMt} = P(M-P_0)$$

$$P_0 M e^{rMt} = P(M-P_0) + P_0 P e^{rMt}$$

$$P = \frac{P_0 M e^{rMt}}{M - P_0 + P_0 e^{rMt}}$$

$$P(t) = \frac{P_0 M e^{rMt}}{M - P_0 + P_0 e^{rMt}} \cdot \frac{e^{-rMt}}{e^{-rMt}}$$

We obtain the final solution

$$\boxed{P(t) = \frac{P_0 M}{(M - P_0)e^{-rMt} + P_0}}$$

If we solved slightly more general IVP, namely

$$\boxed{\begin{aligned} \frac{dP}{dt} &= r(M-P)P \\ P(t_0) &= P_0, \end{aligned}}$$

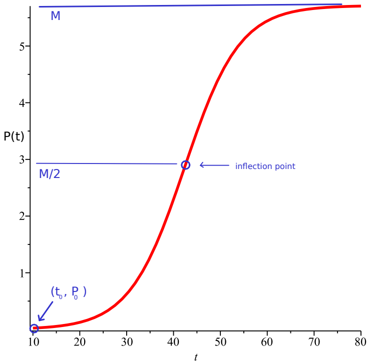
we would obtain

$$\boxed{P(t) = \frac{P_0 M}{(M - P_0)e^{-rM(t-t_0)} + P_0}},$$

which is the general form of the logistic curve discussed earlier.

Logistic curve

The solution is, therefore, logistic curve with with four parameters, t_0, P_0, M and r:



$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-rM(t-t_0)}},$$

Properties of the logistic curve

Assume, for simplicity, $t_0 = 0$, and let $M > 0$, $r > 0$. Then

$$P(t) = \frac{P_0 M}{(M - P_0)e^{-rMt} + P_0} = \frac{M}{(\frac{M}{P_0} - 1)e^{-rMt} + 1} = \frac{M}{me^{-rMt} + 1}$$

where $m = \frac{M}{P_0} - 1$.

$$P'(t) = \frac{d}{dt}\left(\frac{M}{me^{-rMt} + 1}\right) = \frac{-M}{(me^{-rMt} + 1)^2}\left(-rmMe^{-rMt}\right)$$

$$= \frac{rmM^2 e^{-rMt}}{(me^{-rMt} + 1)^2} > 0$$

Function $P(t)$ is, therefore, always increasing.

Inflection point

Using Maple, we obtain after simplification

$$P''(t) = \frac{M^3 m r^2 \mathrm{e}^{-rMt}\left(m\mathrm{e}^{-rMt} - 1\right)}{\left(1 + m\mathrm{e}^{-rMt}\right)^3}$$

At the inflection point, $P''(t) = 0$, which can only happen if

$$m\mathrm{e}^{-rMt} - 1 = 0.$$

The above has solution $t = \frac{\ln m}{rM}$, and this is the t-coordinate of the inflection point, to be called t_i,

$$t_i = \frac{\ln m}{rM}$$

$$P(t_i) := P_i = \frac{M}{me^{-rMt_i} + 1} = \frac{M}{1+1} = \frac{M}{2}$$

Inflection point is, therefore, at $\left(\frac{\ln m}{rM}, \frac{M}{2}\right)$.

Universality

Let us introduce new variable

$$U(t) = P(t)/M$$

.

$$U(t) = \frac{1}{me^{-rMt} + 1}$$

Since $t_i = \frac{\ln m}{rM}$, $m = e^{rMt_i}$, and hence

$$U(t) = \frac{1}{e^{rMt_i}e^{-rMt} + 1}$$

$$U(t) = \frac{1}{e^{-rM(t-t_i)} + 1}$$

Takeover time

Let t_1 be the time corresponding to $U(t_1) = 0.1$
Let t_2 be the time corresponding to $U(t_2) = 0.9$
Takeover time $T = t_2 - t_1$ is the time required for $U(t)$ to increase from 0.1 to 0.9.

$$0.1 = \frac{1}{e^{-rM(t_1 - t_i)} + 1}$$

$$0.9 = \frac{1}{e^{-rM(t_2 - t_i)} + 1}$$

Now

$$10 = e^{-rM(t_1 - t_i)} + 1$$

$$\frac{10}{9} = e^{-rM(t_2 - t_i)} + 1$$

Therefore

$$9 = e^{-rM(t_1 - t_i)}$$

$$1/9 = e^{-rM(t_2 - t_i)}$$

$$\ln 9 = -rM(t_1 - t_i)$$

$$-\ln 9 = -rM(t_2 - t_i)$$

$$-\frac{\ln 9}{rM} = t_1 - t_i$$

$$\frac{\ln 9}{rM} = t_2 - t_i$$

Hence

$$T = t_2 - t_1 = \frac{2 \cdot \ln 9}{rM}$$

Takeover time is therefore expressed by r and M as

$$\boxed{T = \frac{\ln 81}{rM}}$$

Since $T = \frac{\ln 81}{rM}$, then $rM = \frac{\ln 81}{T}$, thus

$$U(t) = \frac{1}{e^{-\frac{\ln 81}{T}(t-t_i)} + 1} = \frac{1}{e^{-\ln 9 \frac{2(t-t_i)}{T}} + 1}$$

We define

$$\tau = \frac{2(t-t_i)}{T}$$

and then

$$U(\tau) = \frac{1}{e^{-\ln 9 \cdot \tau} + 1} = \frac{1}{1 + 9^{-\tau}}$$

We obtained universal curve

$$\boxed{U(\tau) = \frac{1}{1 + 9^{-\tau}}}$$

Note that:
$\tau = 0$ implies $t = t_i$, $U = 1/2$,
$\tau = -1$ implies $U = 0.1$,
$\tau = +1$ implies $U = 0.9$.

Universal curve: summary

Logistic curve (recall that $r = \frac{\lambda}{M}$)

$$P(t) = \frac{P_0 M}{(M - P_0)e^{-rMt} + P_0}$$

can be transformed by change of variables

$$U(t) = \frac{P(t)}{M}$$

$$\tau = \frac{2(t - t_i)}{T}$$

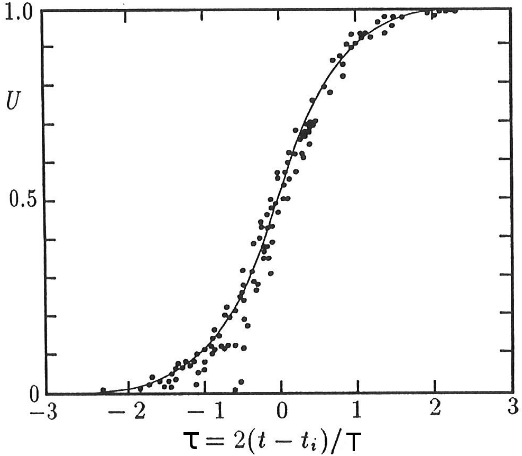
$$T = \frac{2 \cdot \ln 9}{rM}$$

$$t_i = \frac{\ln m}{rM}$$

into universal curve

$$\boxed{U(\tau) = \frac{1}{1 + 9^{-\tau}}}$$

Logistic curve - universality example



Universal plot with normalized time for 17 cases of technology substitution. From Fisher and Pry (1971)

Summary

We considered the following types of growth, known as, correspondingly, exponential, power law exponential, exponential with harvesting, and logistic:

$$\boxed{\begin{aligned}\frac{dN}{dt} &= \lambda N\\ N(0) &= N_0,\end{aligned}} \Longrightarrow \boxed{N(t) = N_0 e^{\lambda t}}$$

$$\boxed{\begin{aligned}\frac{dN}{dt} &= \lambda N^r, \quad r \neq 1\\ N(0) &= N_0\end{aligned}} \Longrightarrow \boxed{N(t) = \left[N_0^{1-r} + \lambda(1-r)t\right]^{\frac{1}{1-r}}}$$

$$\boxed{\begin{aligned}\frac{dN}{dt} &= \lambda N - h\\ N(0) &= N_0,\end{aligned}} \Longrightarrow \boxed{N(t) = N_0 e^{\lambda t} - \frac{h}{\lambda}(e^{\lambda t} - 1)}$$

$$\boxed{\begin{aligned}\frac{dP}{dt} &= r(M-P)P\\ P(0) &= P_0,\end{aligned}} \Longrightarrow \boxed{P(t) = \frac{P_0 M}{(M-P_0)e^{-rMt} + P_0}}$$