Percolation

The process the movement of fluids through porous materials is familiar to us. Some well known examples of this process include coffee percolation and filtration of water through soil and permeable rocks.

We will refer to such processes as *percolation* (from Latin percolare, “to filter” or “trickle through”)

We will consider a family of mathematical models known as *percolation models*, which exhibit features of real percolation processes, but are highly idealized and simplified.

These models are studied using methods of combinatorics and probability theory, as well as methods of statistical physics.

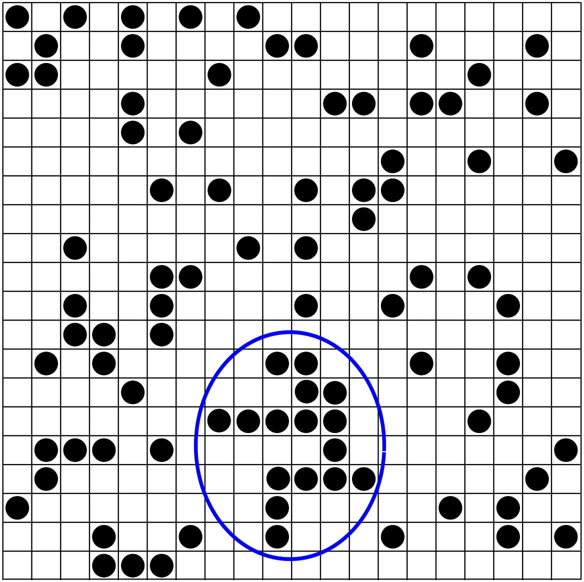
Percolation

Imagine a plane with square grid, extending to infinity in all directions. We will denote it by $\mathbb{Z}^2$. Such a grid covers the plane with squares to be called *cells*.

Assume that a cell can be in two states, *open* or *closed*. Moreover, assume that the state of each cell is random and independent of states of other cells. We will assume that each cell is open with probability p or closed with probability $1-p$, where p is a given parameter, $p \in [0,1]$.

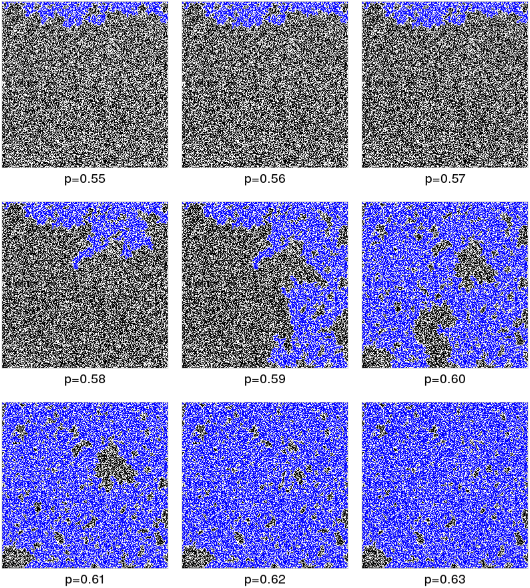
This means that if we are interested in a part of the $\mathbb{Z}^2$ with N cells, we can expect that, on average, pN of them are open, and $(1-p)N$ of them are closed.

The next figure shows an example of a 50×50 lattice with $p = 0.25$. Open cells are dark circles, closed cells are white.



Groups of connected open cells are called *clusters*.
We define two neighbouring cells as those having common side. If they only share a common corner, they are not neighbours.
Two cells belong to the same cluster if one can connects them by a chain of neighbouring cells.
Suppose now that the $\mathbb{Z}^2$ plane is a model of some inhomogeneous rock, where open cells represent grains with high permeability, and closed cells - grains with low permeability. If, in the previous image, the top row exposed to water, will the water percolate to the bottom row?
Obviously, this is only possible if there is a cluster joining these two rows (top and bottom). Intuitively, existence of such a cluster is rather unlikely for small p, but as p increases, it becomes more and more probable. The next slide shows simulations of this process on a lattice 250×250 sites, for varying p.

Percolation on a lattice 250×250



Top rows remains in contact with water (shown in blue color). Open cells are black, closed are white.

Critical probability

We can see that in the first five cases, the water did not reach the bottom row, while in the remaining cases it did. If we repeated this experiment on a very large lattice and many times, we would find that a cluster joining top and bottom row exists for $p > 0.59275$, but does not exist for $p < 0.59275$!
The value $p_c \approx 0.59275$ is called the *critical probability*.

Critical probability – formal definition

Mathematically precise definition of p_c is obtained if we consider infinite plane $\mathbb{Z}^2$. The most important quantity investigate in the theory of perccolation is the *probability of percolation* $\theta(p)$, defined as a probability that a given cell belongs to an infinite cluster of open cells.

Because of translational invariance, we can choose the center of coordinate system as a point of reference, and define$\theta(p)$ as probability that the center of the coordinate system belongs to an infinite cluster of open cells.

The crucial problem of the percolation theory is existence of the critical probability p_c, defined as the value of p such that

$$\theta(p) \begin{cases} = 0 & \text{for } p < p_c \\ > 0 & \text{for } p > p_c. \end{cases}$$

The existence of the critical probability, and the fact that $0 < p_c < 1$, have been demonstrated in 1950's. The following result has been established at that time:

Theorem

The probability of existence of an infinite cluster, to be denoted by $\psi(p)$, is given by

$$\psi(p) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p > p_c. \end{cases}$$

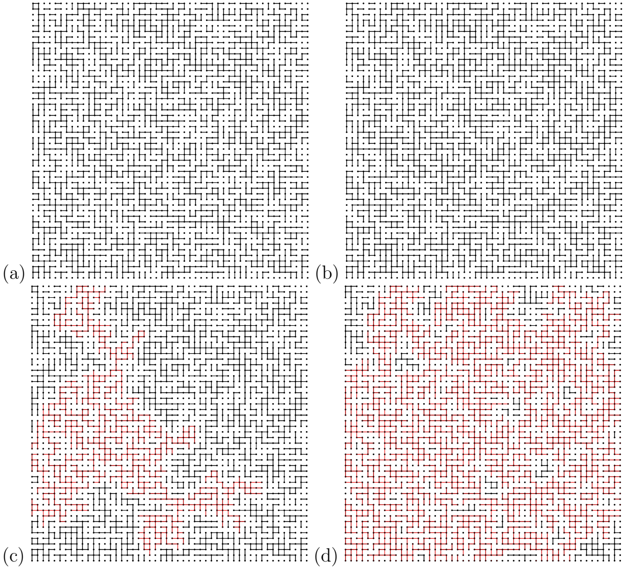
This means that the infinite cluster appears suddenly, when the probability p exceeds its critical value p_c.

Proof of this amazing fact can be found in the book of G. Grimmett, *Percolation*, Springer, 1999.

Bond percolation

The aforementioned phenomenon is usually called *site percolation*, to distinguish it from yet another model, called *bond percolation*. In bond percolation, instead of open and closed cells, we consider open or closed bond, that is, links between nodes of a lattice. AS before, we assume that bonds are open with probability p and closed with probability $1 - p$, independently of each other. Also in this model the infinite cluster appears when p reaches certain threshold value p_c, although this value is different from the corresponding value for site percolation. For the square lattice the critical probability is exactly equal to $1/2$, and a formal proof of this fact is known. For site percolation on a square lattice we only know that $p_c \approx 0.59275$, but the computing the exact value remain an open (an very difficult) problem.

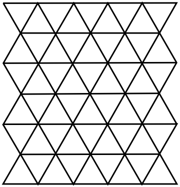
Bond percolation – example

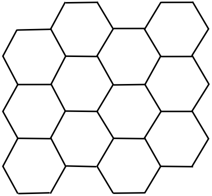


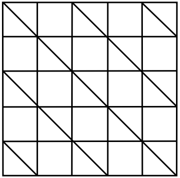
Bond percolation for $p = 0.48$ (a) and $p = 0.52$ (b). Below the same configurations with the largest cluster shown in red.

Other lattice types

In addition to square lattice percolation, percolation on other lattices can be considered, for example, on triangular (a), hexagonal (b), or “bow-tie” lattices(c).

(a) 

(b) 

(c) 

Moreover, percolation in dimensions higher than 2 has been considered.

Critical probabilities for bond and site percolation

Independently of lattice type, the critical probability for bond percolation never exceeds the he critical probability for site percolation on the same lattice. The following theorem establishes this fact.

Theorem

Let G be an infinite lattice of arbitrary type such that the number of bonds at every node is finite and equal to Δ. Then the critical probabilities for bond percolation p_c^{bond} and site percolation p_c^{site} on G satisfy

$$\frac{1}{\Delta - 1} \leq p_c^{bond} \leq p_c^{site} \leq 1 - (1 - p_c^{bond})^{\Delta}.$$

For most common lattices the stronger inequality holds, $p_c^{bond} < p_c^{site}$.

Some known exact results for common lattices:

lattice	p_c
square, bond percolation	$1/2$
triangular, site percolation	$1/2$
triangular, bond percolation	$2\sin(\pi/18)$
honeycomb, bond percolation	$1-2\sin(\pi/18)$
,,bow-tie", bond percolation	$1-p_c-6p_c^2+6p_c^3-p_c^5=0$

Note that the case of site percolation on the square lattice is remarkably absent in the above table. The current best estimate of the critical probability for site percolation on the square lattice is $p_c = 0.59274605079210(2)$.
Numerical values of p_c are usually obtained using so-called *Hoshen-Kopelman algorithm*, which we will not discuss here.

A very intriguing question in percolation theory is the behaviour of the lattice exactly at the critical point p_c. We know, for example, that an infinite cluster exists for $p > p_c$, and does not exist for $p < p_c$, but what if $p = p_c$? Does the infinite cluster exist in this case?

It turns out that the answer is “no” for two-dimensional lattice, and when lattice dimensionality exceeds 18. For lattices of dimension d such that $2 < d \leq 18$ the answer is not known. Most researchers believe that the answer is no for all dimensions greater than 1.

So what happens at $p = p_c$? Let $P(|C| \geq n)$ be the probability that the center of coordinate system belngs to a cluster of size n or larger. There exists strong and convincing evidence that

$$P(|C| \geq n) \approx n^{-1/\delta} \quad \text{when } n \to \infty,$$

where δ is a number depending on the dimensionality of lattice, called *critical exponent*. Its exact value is a subject of speculations and conjectures, sometimes highly controversial. Statistical physicists believe that for bond percolation on a square lattice $\delta = 91/5$, but there is no formal mathematical proof of this. Many other power laws exists near the critical point, and values of corresponding exponents have been postulated. Significant progress has been achieved in this area for dimensions greater that 18, but not for low dimensions.

Bethe lattice

One simple lattice on which percolation threshold can be computed exactly is the Bethe lattice or Cayley tree. It is an infinite connected cycle-free graph where each node is connected to z neighbours, where z is called the *coordination number.*