Recall

Previously, we saw the inverse $\mathbf{M}$ of a matrix $\mathbf{A}$:

$$\mathbf{AM} = \mathbf{MA} = \mathbf{I}_n$$

where $\mathbf{A}$ is an $n \times n$ matrix; that is, $\mathbf{A}$ must be square to have an inverse.

In a 3×3 matrix, this equates to solving each of

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{A}\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{A}\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which can be formulated as the process of transforming an extended augmented matrix through row reductions:

$$\left[\mathbf{A} \mid \mathbf{I}\right] \longrightarrow \left[\mathbf{I} \mid \mathbf{M}\right]$$

Fundamental Theorem of Invertible Matrices

Let $\mathbf{A}_{n \times n}$ be a matrix over a field $\mathbb{F}$. Then the following are equivalent (TFAE):

1. $\mathbf{A}$ is invertible.
2. The reduced row echelon form (rref) of $\mathbf{A}$ is $\mathbf{I}$.
3. The $\text{rank}(A) = n$.
4. The columns of $\mathbf{A}$ are linearly independent.
5. The rows of $\mathbf{A}$ are linearly independent.
6. The *homogeneous system* $\mathbf{A}\vec{x} = \vec{0}$ has only the zero solution $\vec{x} = \vec{0}$.
7. $\mathbf{A}\vec{x} = \vec{b}$ will have a unique solution for all $\vec{b} \in \mathbb{R}^n$.
8. The span of the columns of $\mathbf{A} = \mathbb{R}^n$.
9. The columns and rows of $\mathbf{A}$ form a basis of $\mathbb{R}^n$.
10. The $\det(\mathbf{A}) \neq 0$.
11. $\mathbf{A}$ is the product of elementary matrices.

Proof of (6). Consider $\mathbf{A}\vec{x} = \vec{0}$. If $\mathbf{A}$ has an inverse $\mathbf{M}$, then

$$\begin{aligned} \mathbf{A}\vec{x} &= \vec{0} \\ \mathbf{MA}\vec{x} &= \mathbf{M}\vec{0} \\ \mathbf{I}\vec{x} &= \vec{0} \\ \vec{x} &= \vec{0} \end{aligned}$$

Proof of (7). Suppose $\vec{b} \in \mathbb{R}^n$ is an arbitrary vector. Next consider the system

$$\begin{aligned} \mathbf{A}\vec{x} &= \vec{b} \\ \mathbf{M}\mathbf{A}\vec{x} &= \mathbf{M}\vec{b} \\ \vec{x} &= \mathbf{M}\vec{b} \end{aligned}$$

so the system has a unique solution for any $\vec{b}$.

Let us take a closer look at some of the implications of this theorem. Suppose $\mathbf{A}$ is a 3×3 matrix. Suppose each of the following systems

$$\mathbf{A}\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{A}\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{A}\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

has a solution, called $\vec{x}_1, \vec{x}_2, \vec{x}_3$, respectively.

Suppose we want to find the solution of $\mathbf{A}\vec{x} = \begin{bmatrix} 1 \\ e \\ \pi \end{bmatrix}$. Note that

$$\begin{aligned} \begin{bmatrix} 1 \\ e \\ \pi \end{bmatrix} &= 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= 1\mathbf{A}\vec{x}_1 + e\mathbf{A}\vec{x}_2 + \pi\mathbf{A}\vec{x}_3 \\ &= A\Big(1\vec{x}_1 + e\vec{x}_2 + \pi\vec{x}_3\Big) \end{aligned}$$

So the solution is given by $\vec{x}_1 + e\vec{x}_2 + \pi\vec{x}_3$.

Elementary Matrices

An **elementary matrix**, $\mathbf{E}$, is a matrix we obtain by performing exactly one row operation on an identity matrix.

For example, in the 2×2 case,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \circlearrowleft R_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are all elementary matrices.

Consider multiplying a 2×2 matrix $\mathbf{A}$ by the second of these elementary matrices:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \mathbf{A} &= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a & b \\ -3a + c & -3b + d \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{-3R_1 + R_2} \begin{bmatrix} a & b \\ -3a + c & -3b + d \end{bmatrix}$$

We find that multiplying $\mathbf{A}$ by an elementary matrix has the same effect as performing the same single row operation on $\mathbf{A}$ as was performed to obtain the elementary matrix. This shows us that to perform one row operation on a matrix $\mathbf{A}$ we just need to multiply $\mathbf{A}$ on the left by an appropriate elementary matrix.

Properties of Inverses

For a matrix $\mathbf{A}$, the following are true of its inverse $\mathbf{A}^{-1}$:

1. $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$
2. $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$
3. $\mathbf{A}^{-1}$ is unique
4. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Proof. Suppose $\mathbf{M}_1$ and $\mathbf{M}_2$ act as inverses of $\mathbf{A}$. That is,

$$\begin{aligned} \mathbf{AM}_1 = \mathbf{M}_1\mathbf{A} && = \mathbf{I} \\ \mathbf{AM}_2 = \mathbf{M}_2\mathbf{A} && = \mathbf{I} \end{aligned}$$

Note then that

$$\begin{aligned} \mathbf{M}_1 &= \mathbf{M}_1\mathbf{I} \\ &= \mathbf{M}_1\left(\mathbf{AM}_2\right) \\ &= \left(\mathbf{M}_1\mathbf{A}\right)\mathbf{M}_2 \\ &= \mathbf{IM}_2 \end{aligned}$$

So $\mathbf{M}_1 = \mathbf{M}_2$. ■

Proof. Assume that $\mathbf{A}$ and $\mathbf{B}$ are both invertible.

$$\begin{aligned}(\mathbf{AB})\,\mathbf{B}^{-1}\mathbf{A}^{-1} &= \mathbf{A}\left(\mathbf{BB}^{-1}\right)\mathbf{A}^{-1}\\ &= \mathbf{AIA}^{-1}\\ &= \mathbf{AA}^{-1}\\ &= I\end{aligned}$$

Similarly we can show that $\mathbf{B}^{-1}\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{I}$. ■

Question: Can $(\mathbf{AB})$ be invertible while one of $\mathbf{A}$ or $\mathbf{B}$ is not invertible? Must one be invertible, must both be invertible?

What if $\mathbf{B}$ is not invertible? By the fundamental theorem of invertible matrices, we have that $\mathbf{B}\vec{x} = \vec{0}$ must have a non-trivial solution, $\vec{x}_0$. Then

$$\begin{aligned}(\mathbf{AB})\,\vec{x}_0 &= \mathbf{A}\,(\mathbf{B}\vec{x}_0)\\ &= \mathbf{A}\vec{0}\\ &= \vec{0}\end{aligned}$$

Then the equation $(\mathbf{Ab})\,\vec{x} = \vec{0}$ also has a non-trivial solution. Thus $(\mathbf{AB})$ cannot be invertible.

Elementary Matrices and Invertible Matrices

Consider the following matrix $\mathbf{A}$ and its row reduced echelon form:

$$\mathbf{A} = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix} \xrightarrow{\frac{-1}{2}R_2} \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

By the fundamental theorem of matrices, this shows us that $\mathbf{A}$ is invertible.

Note though that we have said that we can represent row operations as elementary matrices. Corresponding to these row operations then, are the matrices

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \underbrace{\xrightarrow{-3R_1+R_2}}_{\mathbf{E}=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}} \begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix} \underbrace{\xrightarrow{\frac{-1}{2}R_2}}_{\mathbf{F}=\begin{bmatrix}1 & 0\\ 0 & \frac{-1}{2}\end{bmatrix}} \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} \underbrace{\xrightarrow{-2R_1+R_2}}_{\mathbf{G}=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}} \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

If we call the matrices between $\mathbf{A}$ and $\mathbf{I}$ as $\mathbf{B}$ and $\mathbf{C}$, then we have that

$$\begin{aligned}\mathbf{B} &= \mathbf{EA}\\ \mathbf{C} &= \mathbf{FB} = \mathbf{FEA}\\ \mathbf{I} &= \mathbf{GC} = \mathbf{GFEA}\end{aligned}$$

and so $(\mathbf{GFE})$ is the inverse of $\mathbf{A}$:

$$(\mathbf{GFE})\,\mathbf{A} = \mathbf{I}$$

so

$$\mathbf{A} = (\mathbf{GFE})^{-1} = \mathbf{E}^{-1}\mathbf{F}^{-1}\mathbf{G}^{-1}$$

but this assumes that the matrices $\mathbf{E}, \mathbf{F}, \mathbf{G}$ are invertible. This brings us to an important question:

Question: Are elementary matrices invertible? In essence, this equates to asking how to reverse a row operation. Every row operation can be reversed, and so every elementary matrix is invertible.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1}{2} \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{E}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \qquad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{G}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Vector Space

The term vector space shouldn't be confused for what we know as vectors. Consider the set

$$\mathbb{P}_2 = \text{Polynomials of degree} \leq 2$$

That is, elements of $\mathbb{P}_2$ are of the form

$$ax^2 + bx + c$$

where $a, b, c \in \mathbb{R}$.

Let $u, v \in \mathbb{P}_2$. It is apparent then that $u(x) + v(x) \in \mathbb{P}_2$ and for any $k \in \mathbb{R}$, $ku(x) \in \mathbb{P}_2$.

This is very similar to $\mathbb{R}^3$. Consider $\vec{u}, \vec{v} \in \mathbb{R}^3$. Then $\vec{u} + \vec{v} \in \mathbb{R}^3$ and for any $k \in \mathbb{R}^3$, $k\vec{u} \in \mathbb{R}^3$.

In fact, many things besides standard vectors will be considered as vectors. Polynomials, functions, and matrices can all be seen as vectors in their respective vector spaces.