ON SOME ASPECTS OF K-THEORETIC POSITIVITY

MATT LARSON

In these notes, we will discuss some positivity phenomena arising from the K-theory of projective varieties. This can be summarized as showing that some integer is non-negative (or non-positive) by realizing it as the Euler characteristic of a vector bundle on some variety, and then showing that the cohomology of that vector bundle is concentrated in a particular degree.

This technique has a number of combinatorial application. The first combinatorial applications of this technique, in some form, are due to Stanley [Sta75, Sta80]. Our main example will be a paper of Speyer [Spe09], which bounds the complexity of matroid polytope subdivisions.

We will work over a field k, which we will often need to assume has characteristic 0. We will assume that it is algebraically closed for convenience. The reader will lose little by assuming that $k = \mathbb{C}$. Let $[n] = \{1, \ldots, n\}$, and let $\binom{[n]}{r}$ denote the set of subsets of $[n]$ of size r.

1. The Grassmannian and matroids

1.1. **The Grassmannian.** For a natural number n and $0 \leq r \leq n$, the *Grassmannian* $\mathrm{Gr}(r, n)$ is the space of r-dimensional subspaces of k^n. This can be given the structure of a projective variety, as follows. For an r-dimensional subspace L of k^n, choose an $r \times n$ matrix M whose row span is L. If M' is another such matrix, then there is $g \in GL_r(k)$ such that $M' = gM$.

The *Plücker coordinates* of L are the $\binom{n}{r}$ determinants of the maximal minors of M. Up to scaling by a non-zero constant, they are independent of the choice of M: if we replace M by gM for some $g \in GL_r(k)$, then all of the Plücker coordinates are scaled by $\det g$. We therefore obtain a well-defined element $[L]$ of projective space $\mathbb{P}^{\binom{n}{r}-1}$.

It turns out that the subset $\{[L] : L \in \mathrm{Gr}(r, n)\}$ is a Zariski-closed subset of $\mathbb{P}^{\binom{n}{r}-1}$, i.e., it is the vanishing locus of a set of polynomial equations. These equations are generated by a very explicit set of equations called *Plücker relations*. One can also show that the linear subspace L can be recovered from its Plücker coordinates, and we can identify $\mathrm{Gr}(r, n)$ with this locus in $\mathbb{P}^{\binom{n}{r}-1}$. For a proof of the above statements, see [Ful97].

Example 1.1. The Grassmannian $\mathrm{Gr}(1, n)$ is $\mathbb{P}^{n-1}$.

Example 1.2. The Grassmannian $\mathrm{Gr}(2, 4)$ is a hypersurface in $\mathbb{P}^5$. If the coordinates on $\mathbb{P}^5$ are $p_{12}, p_{13}, p_{14}, p_{23}$, and p_{34}, then $\mathrm{Gr}(2, 4)$ is the vanishing locus of $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$.

Example 1.3. There is an isomorphism from $\mathrm{Gr}(r, n)$ to $\mathrm{Gr}(n-r, n)$, given by taking $[L] \in \mathrm{Gr}(r, n)$ to the subspace $L^\perp = \{v \in k^n : \langle v, w\rangle = 0 \text{ for all } w \in L\}$, where $\langle -, -\rangle$ is the standard inner product on k^n.

The Grassmannian admits a transitive action on $GL_n(k)$. In terms of the $r \times n$ matrix M whose row span is L, an element $g \in GL_n(k)$ replaces L by the row span of Mg. This transitive action implies that $\mathrm{Gr}(r, n)$ is smooth. Because GL_n is connected, it also implies that $\mathrm{Gr}(r, n)$ is irreducible. One can show that the dimension of $\mathrm{Gr}(r, n)$ is $r(n - r)$.

Inside of GL_n, there are two particularly important subgroup. The *Borel* subgroup B is the space of upper triangular matrices. The *torus* T is the space of diagonal matrices. If L is the row span of an $r \times n$

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matrix M and $t = (t_1, \dots, t_n) \in T$, then the point $t \cdot [L]$ represents the row span of the matrix obtained by scaling the n columns of M. This torus is isomorphic to $\mathbb{G}_m^n$, i.e., the nth power of the multiplicative group. Much of these notes will focus on the geometry and combinatorics of torus-orbits on Grassmannians.

The action of $GL_n(k)$ on $\mathrm{Gr}(r, n)$ extends to an action on $\mathbb{P}^{\binom{n}{r}-1}$. This is particularly easy to see for the torus: an element $t = (t_1, \dots, t_n)$ scales the coordinate labeled by a subset S of $\{1, \dots, n\}$ of size r by $\prod_{i \in S} t_i$.

Remark 1.4. The Grassmannian $\mathrm{Gr}(r, n)$ can also be viewed as a moduli space of (essential) arrangements of n hyperplanes in an r-dimensional subspace. Given a subspace L of k^n, we obtain n hyperplanes by intersecting L with the coordinates axes (at least if L is not contained in any coordinate subspace). If we have an arrangement of n hyperplanes in L whose intersection is 0, then we can obtain an embedding of L into k^n by writing each hyperplane as the vanishing locus of a linear form.

1.2. **Torus-orbits on the Grassmannian.** We will now begin a detailed study of the torus-orbits on the Grassmannian, using ideas introduced in [GS87]. The first step is to analyze the stabilizers of points in the Grassmannian. Note that the subgroup of T that consists of multiples of the identity acts trivially on $\mathrm{Gr}(r, n)$, so every point has a stabilizer.

Proposition 1.5. *For each subspace L of k^n, there is a unique finest partition $[n] = S_1 \sqcup \cdots \sqcup S_j$ such that*

$$L = \bigoplus_{i=1}^{j} L \cap k^{S_i}.$$

The stabilizer of $[L] \in \mathrm{Gr}(r, n)$ in the torus T is $\mathbb{G}_m^j$, with the jth factor scaling k^{S_j}.

We say that the partition $[n] = S_1 \sqcup \cdots \sqcup S_j$ is the partition into *connected components*. If $0 < r < n$, then a general point of $\mathrm{Gr}(r, n)$ has a 1-dimensional stabilizer in T, i.e., there is only a single connected component.

The key tool to study torus-orbits in $\mathrm{Gr}(r, n)$ will be the theory of *moment polytopes*. We first recall the general theory, see [Ful93, Chapter 3], [Sot03, Section 8], or [EFLS24, Section 6.1]. Let H be a torus with character lattice M. Let V be a representation of H. Then V has a unique decomposition $V \simeq \oplus_{i=1}^N V_i$ into H-eigenspaces, where V_i is the subspace of V where H acts by some character $a_i \in M$.

There is an action of H on $\mathbb{P}V$. Given a point $x \in \mathbb{P}V$ with representative $v \in V$, let

$$\mathcal{A}_x = \{a_i : v_i \neq 0 \text{ in the expression } v = \sum_{i=1}^N v_i\}.$$

Then the normalization of the torus-orbit closure $\overline{H \cdot x}$ is the toric variety corresponding to the normal fan of $\mathrm{Conv}(\mathcal{A}_x) \subset M \otimes_{\mathbb{Z}} \mathbb{R}$, with respect to the affine lattice generated by $\mathcal{A}_x$, i.e., the translation of the sublattice of M generated by $\{a_i - a_j : a_i, a_j \in \mathcal{A}_x\}$ so that it contains $\mathcal{A}_x$. The polytope $\mathrm{Conv}(\mathcal{A}_x) \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ is called the *moment polytope* of $\overline{H \cdot x}$. In particular, by [CLS11, Corollary 3.A.6], there is a bijection between k-dimensional faces of the moment polytope and k-dimensional H-orbits in $\overline{H \cdot x}$.

We apply this to points of the Grassmannian in its Plücker embedding. The character lattice of the torus T is identified with $\mathbb{Z}^n$, and the eigenspaces of T acting on the ambient space of the Plücker embedding are the coordinate lines. The character corresponding to a subset $S \in \binom{[n]}{r}$ is $e_S := \sum_{i \in S} e_i$. The torus-fixed pointed of $\mathbb{P}^{\binom{n}{r}-1}$ are in bijection with $\binom{[n]}{r}$.

Given $[L] \in \mathrm{Gr}(r, n)$, we see that the set $\mathcal{A}_{[L]}$ is $\{e_S : p_S([L]) \neq 0\}$, where p_S is the Plücker coordinate corresponding to S. We see that the moment polytope of $\overline{T \cdot [L]}$ is

$$\mathrm{Conv}(e_S : p_S([L]) \neq 0).$$

However, $\overline{T \cdot [L]}$ is contained in $\mathrm{Gr}(r,n)$. Although there is a unique T-fixed curve between any two distinct T-fixed points in $\mathbb{P}^{\binom{n}{r}-1}$, most of these curves do not lie in $\mathrm{Gr}(r,n)$.

Proposition 1.6. *The T-fixed curve in $\mathbb{P}^{\binom{n}{r}-1}$ between the T-fixed points corresponding to distinct subsets S_1, S_2 in $\binom{[n]}{r}$ lies in $\mathrm{Gr}(r,n)$ if and only if $S_2 = (S_1 \setminus i) \cup j$ for some $i \in S_1, j \in S_2$.*

Remark 1.7. There are finitely many 2-dimensional T-fixed subvarieties of $\mathrm{Gr}(r,n)$. However, $\mathrm{Gr}(2,4)$ has infinitely many 3-dimensional T-fixed subvarieties.

In particular, every edge in the moment polytope of $\overline{T \cdot [L]}$ must connected two vertices e_{S_1} and e_{S_2} with $S_2 = (S_1 \setminus i) \cup j$. This happens if and only if every edge is parallel to $e_i - e_j$, i.e., it is parallel to a root of type A_{n-1}. The proves the following.

Theorem 1.8. [GS87] *All edges of the moment polytope of a torus-orbit closure in $\mathrm{Gr}(r,n)$ are parallel to a vector of the form $e_i - e_j$.*

Motivated by this, we define *matroids* as a generalization of moment polytopes of torus-orbits in Grassmannians. Matroids are certain subsets of $\binom{[n]}{r}$ that model the set of non-vanishing Plücker coordinates of a linear subspace.

Definition 1.9. *A subset $\mathcal{B}$ of $\binom{[n]}{r}$ is a matroid of rank r on ground set $[n]$ if the polytope*

$$\mathrm{Conv}(e_S : S \in \mathcal{B})$$

has all edges parallel to a vector of the form $e_i - e_j$.

A matroid is usually called M, and we say that the set $\mathcal{B}$ appearing in Definition 1.9 is its set of *bases*.

For a matroid M, the polytope $\mathrm{Conv}(e_S : S \in \mathcal{B})$ is called the *matroid polytope* of M. It is denoted $P(\mathrm{M})$. From Theorem 1.8, we deduce that each linear subspace $L \subseteq k^n$ of dimension r gives rise to a matroid on ground set $[n]$ of rank r. A matroid arising in this way is called *realizable* (over k).

Example 1.10. For $0 \le r \le n$, the *uniform matroid* $U_{r,n}$ is the matroid whose bases are all subsets of $[n]$ of size r. A general linear subspace realizes $U_{r,n}$.

Example 1.11. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

If k has characteristic 2, then the row span of A in k^7 realizes a matroid called the *Fano matroid* F_7. If k has characteristic different from 2, then it realizes a matroid called the *non-Fano matroid* F_7^+, which is the same as the Fano matroid except that $\{5,6,7\}$ is a basis. The Fano matroid is realizable over fields of characteristic 2, and the non-Fano matroid is realizable over fields of characteristic different from 2.

Example 1.12. Given two matroids M_1 and M_2 of ranks r_1, r_2 on ground sets $\{1, \ldots, n\}$ and $\{n+1, \ldots, m\}$, their *direct sum* $\mathrm{M}_1 \oplus \mathrm{M}_2$ is the matroid of rank $r_1 + r_2$ whose bases are $\{B_1 \cup B_2 : B_i \text{ basis of } \mathrm{M}_i\}$. The matroid $F_7 \oplus F_7^+$ is not realizable over any field.

We say that a matroid is *connected* if it cannot be written as a direct sum of two matroids on non-empty ground sets. For a matroid M on $[n]$, the dimension of $P(\mathrm{M})$ is $n - c(\mathrm{M})$, where $c(\mathrm{M})$ is the number of connected components of M. In particular, M is connected if and only if its polytope $P(\mathrm{M})$ has dimension $n-1$.

Example 1.13. A face of a matroid polytope is a matroid polytope.

Example 1.14. Give a matroid M on $[n]$, the *dual matroid* $\mathrm{M}^{\perp}$ is the matroid with bases $\{[n] \setminus B : B \text{ basis of M}\}$. If M is realized by $L \subseteq k^n$, then $\mathrm{M}^{\perp}$ is realized by $L^{\perp} \subseteq k^n$, i.e., the image of the point $[L]$ under the isomorphism $\mathrm{Gr}(r,n) \simeq \mathrm{Gr}(n-r,n)$.

1.3. **Normality of torus-orbit closures.** Proposition 1.5 describes the dense torus inside a torus-orbit closure in the Grassmannian, and Theorem 1.8 describes the normal fan. In order to use this to identify the torus-orbit closure (up to isomorphism), we need to know that torus-orbit closures are normal. This follows from the following highly non-obvious property of matroid polytopes. For a polytope P, let aP denote its ath dilate.

Theorem 1.15. [Wel76, Chapter 18.6, Theorem 3] *For any positive integer a, every lattice point in $aP(\mathrm{M})$ is a sum of a lattice points in $P(\mathrm{M})$.*

To prove the normality of torus-orbit closures, we discuss semigroup algebras of lattice polytopes.

Definition 1.16. *Let P be a lattice polytope in $\mathbb{R}^n$. The* semigroup algebra R_P *of P is the graded vector space $\bigoplus_{k \geq 0} \oplus_{p \in kP(\mathrm{M}) \cap \mathbb{Z}^n} k \cdot p$, equipped with the multiplication is induced by $p \cdot q = p + q$.*

Proposition 1.17. [BH93, Theorem 6.1.4] *The semigroup algebra R_P of a lattice polytope is normal.*

If L is a representation of M, then the homogeneous coordinate ring of $\overline{T \cdot [L]}$ is the subring of the semigroup algebra which is generated in degree 1. As Theorem 1.15 implies that the semigroup algebra is generated in degree 1, this implies the homogeneous coordinate ring is equal to the semigroup algebra. As such a semigroup algebra is normal, this implies the following result.

Corollary 1.18. [Spe09, Proposition A.1] *Each torus-orbit closure $\overline{T \cdot [L]}$ in $\mathrm{Gr}(r,n)$ is* projectively normal *in $\mathbb{P}^{\binom{n}{r}-1}$, i.e., the cone over it is normal in $\mathbb{A}^{\binom{n}{r}}$.*

In particular, torus-orbit closures in Grassmannians are normal, and so if $[L_1]$ and $[L_2]$ are points of the Grassmannian over k which realize the same matroid, then there is a toric isomorphism from $\overline{T \cdot [L_1]}$ to $\overline{T \cdot [L_2]}$.

Remark 1.19. Torus-orbit closures in homogeneous spaces of other types do not enjoy the nice properties of torus-orbit closures in the Grassmannian. Outside of type A, torus-orbit closures frequently fail to be normal. Except in a few other cases (such as maximal orthogonal Grassmannians), the vanishing or non-vanishing of the generalized Plücker coordinates of a point does not determine the stabilizer, i.e., there is no description of the stabilizer like in Proposition 1.5. See [ELS25, Example 2.5] for an example of two points in the Lagrangian Grassmannian $LGr(2,4)$ whose torus-orbit closures have the same moment polytope, but different stabilizers.

2. Matroid polytope subdivisions

We will be interested in subdivisions of matroid polytopes into unions of matroid polytopes. It turns out that there is a natural source of these subdivisions arising from the geometry of the Grassmannian, which seems to have first been noticed by Kapranov [Kap93]. The properties of these subdivisions have taken on an increasingly important role in matroid theory, thanks to the development of a technique which allows one to use matroid polytope subdivisions to reduce statements to the case of realizable matroids. In particular, this allows one to use tools from algebraic geometry to prove results about non-realizable matroids.

Definition 2.1. *Let P be a polytope in $\mathbb{R}^n$. A* polyhedral subdivision *of P is a collection of polytopes $Q_1, Q_2, \ldots, Q_k$ which are contained in P and of the same dimension as P, such that $P = \cup_{i=1}^{k} Q_i$, the relative interiors of the Q_i are disjoint, and the intersection of any two of the Q_i is a face of both.*

An *interior face* of a polyhedral subdivision is a face of some Q_i which is contained in the relative interior of P. In particular, each of the Q_i is an interior face.

The easiest subdivisions to construct are *regular subdivisions*. These are subdivisions that are constructed by choosing a finite subset S of P which includes all of the vertices and a height function $h\colon S \to \mathbb{R}$, forming the polytope $\hat{P}$ in $\mathbb{R}^n \times \mathbb{R}$ as the convex hull of $(p, h(p))$ for $p \in S$, and then projecting the lower faces.

Definition 2.2. *A* matroid polytope subdivision *of a matroid polytope* $P(\mathrm{M})$ *is a polyhedral subdivision of* $P(\mathrm{M})$ *such that all of the polytopes appearing in the subdivision are matroid polytopes.*

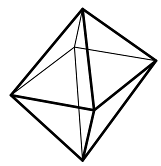

FIGURE 1. A matroid polytope subdivision of $P(U_{2,4})$.

Example 2.3. There is a subdivision of the polytope $P(U_{2,4})$ with two full-dimensional cells, correspond to the top half and the bottom half of the bipyramid in Figure 1.

Because a face of a matroid polytope is a matroid polytope, it suffices to check that the top-dimensional cells are matroid polytopes.

We will now construct a large number of matroid polytope subdivisions. Let R be a discrete valuation ring, with fraction field K, valuation $\nu\colon K^\times \to \mathbb{Z}$, and residue field κ. Let L be an R-submodule of R^n of rank r. This is the same thing as a map $\operatorname{Spec} R \to \mathrm{Gr}(r, n)$, i.e., the germ of a curve in $\mathrm{Gr}(r, n)$. We have a matroid M induced by the subspace $L \otimes_R K$ of K^n.

For each basis B of M, we obtain an integer $\nu(p_B)$ by taking the valuation of the Plücker coordinate corresponding to B. The Plücker coordinates are defined up to scaling, and so these integers are defined up to translation by a global constant. We obtain a regular polyhedral subdivision of $P(\mathrm{M})$ by using the $\nu(p_B)$ as a height function on the vertices of $P(\mathrm{M})$. This subdivision does not change if we translate by heights by a global constant, so it does not depend on the choice of scaling of the Plücker coordinates.

Theorem 2.4. [Spe09, Proposition A.2] *The subdivision of* $P(\mathrm{M})$ *described above is a matroid polytope subdivision.*

Example 2.5. Let R be the discrete valuation ring $k[t]_{(t)}$, and let L be the R-submodule of R^4 generated by the rows of the matrix
$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & t \end{pmatrix}.$$
We have $\mathrm{M} = U_{2,4}$. Using the matrix A to compute the Plücker coordinates of $L \otimes_R K$, we have $\nu(p_{34}) = 1$ and $\nu(p_B) = 0$ for all other bases. Then the induced matroid polytope subdivision is the subdivision in Example 2.3.

This example can be computed very explicitly. The Grassmannian and its Plücker embedding are defined over any base scheme. Inside of $\mathbb{P}^5_R$, projective space over $\operatorname{Spec} R$, $\mathrm{Gr}(2, 4)$ is the hypersurface defined by the equation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. The torus-orbit closure of $[L]$ is 3-dimensional, and it is defined by the additional equation $tp_{12}p_{34} = (1 - t)p_{14}p_{23}$. When we set $t = 0$, the equations defining the torus-orbit closure become $p_{14}p_{23} = 0$ and $p_{12}p_{34} = p_{13}p_{24}$. This has two components, each of which is the toric variety of a square pyramid.

There are two ways to prove Theorem 2.4. In [Spe08, Proposition 2.2], Speyer characterized the height functions on the vertices of $P(U_{r,n})$ which induce matroid polytope subdivisions, and it is easy to check, using the Plücker relations, that the $\nu(p_B)$ satisfy his condition. It is also a consequence of a general description of projective toric varieties over a discrete valuation ring [Smi96, Section 2]. Using this second approach, Theorem 2.4 can be generalized without difficulty to torus-orbit closures in any homogeneous space.

Example 2.6. Let M be a matroid of rank r on $[n]$, and let $\mathrm{rk}_{\mathrm{M}}\colon 2^{[n]} \to \mathbb{Z}$ be the *rank function* of M, given by $\mathrm{rk}_{\mathrm{M}}(S) = \max_{B \text{ basis}} |B \cap S|$. Then the regular subdivision of $U_{r,n}$ induced by the height function $h(S) = -\mathrm{rk}_{\mathrm{M}}(S)$ on the vertices of $P(U_{r,n})$ is a matroid polytope subdivision. If M is connected, then $P(\mathrm{M})$ intersects the relative interior of $P(U_{r,n})$, and so it is a face of this subdivision.

The main purpose of these notes is to discuss the mathematics surrounding the following theorem.

Theorem 2.7. *A matroid polytope subdivision of $P(U_{r,n})$ has at most $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ interior faces of dimension $n-c$ for $c \leq \min\{r, n-r\}$, and it has no interior faces of dimension less than $\min\{r, n-r\}$.*

In fact, equality holds for the number of top-dimensional interior faces (i.e., the subdivision has $\binom{n-2}{r-1}$ top-dimensional faces) if and only if each connected matroid matroid appearing in the subdivision is a *series–parallel* matroid, in which case one has equality for the number of interior faces of each dimension.

Theorem 2.7 was conjectured in [Spe08], and it was known as Speyer's f-vector conjecture. Major progress was made in [Spe09], where Theorem 2.7 was reduced to verifying the non-negativity of a certain invariant of matroids. In [Spe09], Speyer proved this non-negativity for matroids realizable over a field of characteristic 0. After some more cases were proved in [EL23, FS24], the non-negativity was proved in general in [FSS24, BF24].

We now outline the strategy introduced in [Spe09]. For each n, define the *valuative group* Val_n to be the subgroup of functions from $\mathbb{R}^n$ to $\mathbb{R}$ generated by the indicator functions of matroid polytopes on $[n]$. There is a direct sum decomposition $\mathrm{Val}_n = \oplus_{r=0}^{n} \mathrm{Val}_{r,n}$, where $\mathrm{Val}_{r,n}$ is the subgroup generated by indicator functions of matroid polytopes of rank r. A function f from the set of matroids on $[n]$ to an abelian group A is said to be *valuative* if it factors through Val_n.

There are many easy examples of valuative invariants, such as the number of bases of a matroid, the volume of $P(\mathrm{M})$ (with respect to a volume form on the hyperplane $\sum x_i = r$), the Ehrhart polynomial of $P(\mathrm{M})$. Surprisingly, most natural invariants of a matroid turn out to be valuative, such as the Tutte polynomial [AFR10] and most "algebro-geometric" invariants such as the Bergman class [BEST23] and various K-theoretic invariants [LLPP24].

If we have a matroid polytope decomposition of $P(\mathrm{M})$, then, by a version of inclusion-exclusion [AFR10, Theorem 3.5], we have an equality

$$\mathbf{1}_{P(\mathrm{M})} = \sum_{Q \text{ interior face, } \dim Q = n-c} (-1)^{c+1} \mathbf{1}_Q. \tag{1}$$

Here $\mathbf{1}_P$ is the indicator function of P. Suppose that $f\colon \{\text{matroids on } [n]\} \to \mathbb{Z}$ is a valuative invariant such that $(-1)^{c+1} f(M) \geq 0$ for each matroid M with c connected components. Then, for any matroid polytope subdivision of $P(U_{r,n})$, we have

$$f(U_{r,n}) = \sum_{Q \text{ interior face, } \dim Q = n-c} (-1)^{c+1} f(Q).$$

Each term on the right-hand side of non-negative, and so this gives a bound on the complexity of the subdivision.

It is easy to construct valuative invariants which are non-negative on all matroids. However, constructing a valuative invariant with the sign property that is needed to use (1) to bound the complexity of a matroid polytope subdivision turns out to be quite difficult. As we will explain, such invariants arise naturally from K-theoretic constructions.

Remark 2.8. Derksen and Fink showed that that Val_n is the quotient of the free abelian group with a basis labeled by matroid on $[n]$ by the subgroup generated by relations of the form (1) [DF10, Appendix A]. See [EHL23, Appendix A] for a discussion of related questions.

3. K-groups of spaces

We now discuss the K-theory of schemes, see [Ful98, Section 15.1]. K-theory is a type of intersection theory, and, as is usual, it comes in two flavors, a homological flavor and a cohomological flavor. When the scheme is smooth, these flavors can be identified.

Let X be a scheme which is finite type over k. Let $K_\circ(X)$ be the Grothendieck group of coherent sheaves on X, i.e., $K_\circ(X)$ is the quotient of the free abelian group generated by coherent sheaves on X by the subgroup generated by relations corresponding to short exact sequences. In other words, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of coherent sheaves on X, then $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ in $K_\circ(X)$. If there is a long exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_k \to 0,$$

then we have $\sum_{i=1}^k (-1)^i [\mathcal{F}_i] = 0$ in $K_\circ(X)$ by breaking this long exact sequence into short exact sequences.

Let $K^\circ(X)$ be the Grothendieck group of vector bundles on X. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of coherent sheaves on X and $\mathcal{E}$ is a vector bundle, then $0 \to \mathcal{E} \otimes \mathcal{F}_1 \to \mathcal{E} \otimes \mathcal{F}_2 \to \mathcal{E} \otimes \mathcal{F}_3 \to 0$ is short-exact. It follows that $K^\circ(X)$ is a ring, with multiplication given by tensor product, and $K_\circ(X)$ is a module over $K^\circ(X)$.

There is a map $K^\circ(X) \to K_\circ(X)$, obtained by viewing a vector bundle as a coherent sheaf. In general, this map is neither injective nor surjective, but if X is smooth then it is an isomorphism [Ful98, Appendix B.8], essentially because every coherent sheaf has a finite resolution by vector bundles. In this case, we denote both $K^\circ(X)$ and $K_\circ(X)$ by $K(X)$.

We list some properties of K-groups. Let $\pi\colon X \to Y$ be a map of schemes which are finite type over k.

1. For any scheme X, $K_\circ(X)$ is generated (as an abelian group) by classes of structure sheaves of (integral) subvarieties of X.
2. If $X \to Y$ is proper, then there is a pushforward map $\pi_*\colon K_\circ(X) \to K_\circ(Y)$, defined by $\pi_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i \pi_* \mathcal{F}]$.
3. There is a pullback map $\pi^*\colon K^\circ(Y) \to K^\circ(X)$ defined by $\pi^*[\mathcal{E}] = [\pi^* \mathcal{E}]$.
4. If $X \to Y$ is proper, $a \in K^\circ(Y)$, and $x \in K_\circ(X)$, then

$$\pi_*(\pi^* a \cdot x) = a \cdot \pi_* x. \tag{2}$$

The property (2) is known as the *projection formula*.

Remark 3.1. Note that if X and Y are smooth, then there is a pullback map $\pi^*\colon K(Y) \to K(X)$. Given the coherent sheaf $\mathcal{F}$ on Y, $\pi^*[\mathcal{F}]$ is usually not equal to $[\pi^* \mathcal{F}]$. Rather, we resolve $\mathcal{F}$ by vector bundles and pull those back.

Particularly important is the case when X is a projective variety and Y is a point. Then $K(Y) = \mathbb{Z}$, and the pushforward map $K_\circ(X) \to \mathbb{Z}$ is equal to the Euler characteristic. I.e., the pushforward of $[\mathcal{F}]$ to a point is equal to

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}).$$

There are two practical ways to compute the Euler characteristic.

If $\mathcal{L}$ is a line bundle, then the function $a \mapsto \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$ is a polynomial. If $\mathcal{L}$ is ample, then for a sufficiently large, Serre vanishing implies that $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$ agrees with $\dim H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$. In this case, we can compute $\chi(X, \mathcal{F})$ by forming the graded module $\oplus_{a \geq 0} H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$, finding the polynomial $p(a)$ which agrees with $\dim H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$ for a sufficiently large, and then setting $a = 0$.

For example, we compute the Euler characteristic of the line bundle $\mathcal{O}(a)$ on $\mathbb{P}^N$. The corresponding graded module is $k[x_0, \ldots, x_N][a]$, i.e., the ring $k[x_0, \ldots, x_N]$, with 1 placed in degree $-a$, viewed as a module over $k[x_0, \ldots, x_N]$. For $i \geq -a$, the dimension of the ith graded piece is $\binom{N+a+i}{N}$. Setting $i = 0$, we see that

$$\chi(\mathbb{P}^N, \mathcal{O}(a)) = \binom{N+a}{N}. \tag{3}$$

If X is embedded into $\mathbb{P}^N$ for some N, then a coherent sheaf on X can be pushed forward to a coherent sheaf on $\mathbb{P}^N$. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ corresponds to a graded module $M = \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{P}^N, \mathcal{F}(i))$ over the polynomial ring $k[x_0, \ldots, x_N]$. This correspondence is a bit subtle in general, but an exact sequence of graded modules induces an exact sequence of sheaves on $\mathbb{P}^N$. By the Hilbert syzygy theorem, any graded module M over $k[x_0, \ldots, x_N]$ has a graded free resolution

$$0 \to k[x_0, \ldots, x_N] \otimes V_N \to k[x_0, \ldots, x_N] \otimes V_{N-1} \to \cdots \to k[x_0, \ldots, x_N] \otimes V_1 \to M \to 0,$$

where each V_i is a graded vector space. This induces a long exact sequence of sheaves on $\mathbb{P}^N$ involving $\mathcal{F}$, the sheafification of M, and a bunch of direct sums of lines bundles. This writes $[\mathcal{F}]$ as a linear combination of combination of classes of line bundles, and one can then use (3) to compute the Euler characteristic.

We now describe the K-theory of $\mathbb{P}^N$. This will be done more generally for Grassmannians later.

As shown above, $K(\mathbb{P}^N)$ is spanned (as an abelian group) by classes of line bundles. Repeatedly using the short exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$, where H is a hyperplane, and inducting on the dimension, we see that $K(\mathbb{P}^N)$ is generated as an abelian group by powers of $[\mathcal{O}_H]$. Using the Koszul complex, i.e., the long exact sequence

$$0 \to \mathcal{O}(-(N+1)) \to \mathcal{O}(-N)^{\oplus N+1} \to \cdots \to \mathcal{O}(-1)^{\oplus \binom{N}{2}} \to \mathcal{O}^{\oplus N} \to 0,$$

we see that $[\mathcal{O}_H]^{N+1} = ([\mathcal{O}] - [\mathcal{O}(-1)])^{N+1} = 0$, and so $K(\mathbb{P}^N)$ is generated as an abelian group by $[\mathcal{O}], [\mathcal{O}_H], [\mathcal{O}_H]^2, \ldots, [\mathcal{O}_H]^N$.

We claim that these classes form a basis for $K(\mathbb{P}^N)$. There is a pairing $K(\mathbb{P}^N) \times K(\mathbb{P}^N) \to \mathbb{Z}$ given by $(a, b) \mapsto \chi(\mathbb{P}^N, ab)$. We have

$$\chi(\mathbb{P}^N, [\mathcal{O}_H]^i [\mathcal{O}_H]^j) = \begin{cases} 1 & i + j \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix whose (i, j)th entry is $\chi(\mathbb{P}^N, [\mathcal{O}_H]^i [\mathcal{O}_H]^j)$ is nondegenerate. This means that this pairing does not descend to any quotient of the span of $[\mathcal{O}], [\mathcal{O}_H], \ldots, [\mathcal{O}_H]^N$, and so these classes must be linearly independent in $K(\mathbb{P}^N)$. We conclude that

$$K(\mathbb{P}^N) \simeq \mathbb{Z}[x]/(x^{N+1}), \quad x = [\mathcal{O}_H],$$

where H is a hyperplane. Note that, for instance by Proposition 3.2 below, $[\mathcal{O}_H]^k$ is the class of a linear subspace of $\mathbb{P}^N$ of codimension k.

The nondegeneracy of this pairing means that we can compute the class of a sheaf $[\mathcal{F}]$ in $K(\mathbb{P}^N)$ in terms of the Hilbert polynomial of $\mathcal{F}$. That is, we have $[\mathcal{F}] = \sum a_i [\mathcal{O}_H]^{N-i}$, where

$$\chi(\mathbb{P}^N, \mathcal{F} \otimes \mathcal{O}(a)) = \sum_{q=0}^{N} a_q \binom{a+q}{q}. \tag{4}$$

We see that $K_\circ(\mathbb{P}^N)$ can be identified with subspace of polynomial functions on $\mathbb{Z}$ which is spanned by the functions $a \mapsto \binom{a+q}{q}$, for $q = 0, \ldots, N$. This is exactly the space of *numerical polynomials* of degree at most N, i.e., polynomial $p \in \mathbb{Q}[t]$ such that $p(a)$ is an integer for every $a \in \mathbb{Z}$.

We now explain the sense in which K-theory is a form of intersection theory. If Y and Z are smooth subvarieties of an ambient variety X, then we say that Y and Z are *transverse* at a point $x \in Y \cap Z$ if the tangent spaces T_xY, T_xZ to Y and Z at x are transverse inside of T_xX, i.e., $\operatorname{codim} T_xY \cap T_xZ = \operatorname{codim} T_xY + \operatorname{codim} T_xZ$, where codim is the codimension inside of T_xX.

Proposition 3.2. *If Y and Z are smooth subvarieties of a smooth variety X which meet transversely, then* $[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y\cap Z}]$.

Usually one does intersection theory using singular cohomology or Chow groups. K-theory captures some refined information, as illustrated by the following examples.

Example 3.3. Let Y, Z be subvarieties of a variety X. Then there is an exact sequence

$$0 \to \mathcal{O}_{Y\cup Z} \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y\cap Z} \to 0,$$

so $[\mathcal{O}_{Y\cup Z}] = [\mathcal{O}_Y] + [\mathcal{O}_Z] - [\mathcal{O}_{Y\cap Z}]$. Note that, if Y and Z are the same dimension, then we have $[Y \cup Z] = [Y] + [Z]$ in Chow or in cohomology, so K-theory is capturing some refined information about how Y and Z meet.

Example 3.4. Let Y be the union of two skew lines in $\mathbb{P}^3$. Then $[\mathcal{O}_Y] = 2[\mathcal{O}_H]^2$ in $K(\mathbb{P}^3)$, as $[\mathcal{O}_H]^2$ is the class of a line. Let Z be the union of two lines in $\mathbb{P}^3$ that meet at a point. We have $\chi(Z, \mathcal{O}(a)) = 2a + 1$, so $[\mathcal{O}_Z] = 2[\mathcal{O}_H]^2 - [\mathcal{O}_H]^3$ by (4). Note that Y and Z have the same class in the cohomology ring $H^*(\mathbb{P}^3)$.

Furthermore, it is often the case that if Z_1 and Z_2 are subvarieties of a smooth scheme X which can be deformed to each other, we have $[\mathcal{O}_{Z_1}] = [\mathcal{O}_{Z_2}] \in K(X)$, as in the following examples.

Example 3.5. Let X be a smooth projective variety such that $K(X)$ is a finitely generated torsion-free abelian group, and the pairing $K(X) \times K(X) \to \mathbb{Z}$ given by $(a, b) \mapsto \chi(X, ab)$ is nondegenerate. For example, this holds for a Grassmannian or a smooth projective toric variety, and, more generally, for any complex variety for which the map from $K(X)$ to its topological topological K-theory is an isomorphism. Let C be a connected curve, and suppose that $\mathcal{Z}$ is a subscheme of $X \times C$ which is flat over C. Let p be the map from $X \times C$ to C. Then for any vector bundle $\mathcal{E}$ on X and points $q_1, q_2 \in C$, we have $\chi(p^{-1}(q_1) \cap \mathcal{Z}, \mathcal{E}|_{p^{-1}(q_1)}) = \chi(p^{-1}(q_2) \cap \mathcal{Z}, \mathcal{E}|_{p^{-1}(q_2)})$ because the Euler characteristic is locally constant in proper flat families [Vak25, Theorem 24.7.1]. This implies that $[\mathcal{O}_{p^{-1}(q_1)\cap\mathcal{Z}}] = [\mathcal{O}_{p^{-1}(q_2)\cap\mathcal{Z}}]$ in $K(X)$.

Example 3.6. Let X be a smooth variety, and let $\mathcal{Z}$ be a smooth subvariety of $X \times \mathbb{P}^1$ which is flat over $\mathbb{P}^1$. Let $p\colon X \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection. Then for any two points $q_1, q_2 \in \mathbb{P}^1$, we have $[\mathcal{O}_{q_1}] = [\mathcal{O}_{q_2}] \in K(\mathbb{P}^1)$. Pulling back to $X \times \mathbb{P}^1$, we see that $[\mathcal{O}_{p^{-1}(q_1)}] = [\mathcal{O}_{p^{-1}(q_2)}]$ in $K(X \times \mathbb{P}^1)$. This implies that $[\mathcal{O}_{p^{-1}(q_1)\cap\mathcal{Z}}] = [\mathcal{O}_{p^{-1}(q_2)\cap\mathcal{Z}}]$ in $K(X)$, as $[\mathcal{O}_{p^{-1}(q_1)}] \cdot [\mathcal{O}_{\mathcal{Z}}] = [\mathcal{O}_{p^{-1}(q_1)\cap\mathcal{Z}}]$ by Proposition 3.2, and similarly for q_2.

There is a direct connection between K-groups and Chow groups, which are an algebraic version of homology. If X is a scheme, $K_\circ(X)$ is equipped with a decreasing filtration $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{\dim X} \supseteq 0$, called the *coniveau filtration*. Here F_i is the subgroup generated by classes of coherent sheaves whose support has codimension at least i. There is a surjective map from the Chow groups $A_\bullet(X)$ to the associated graded $\operatorname{gr} K_\circ(X)$, obtained by sending the class of a subvariety $[Z]$ to $[\mathcal{O}_Z]$, and this map becomes an isomorphism after tensoring with $\mathbb{Q}$ [Ful98, Example 15.1.5].

4. K-theory of the Grassmannian

We will now describe the K-theory of the Grassmannian. Recall that GL_n acts on $\mathrm{Gr}(r,n)$, and so B, the Borel subgroup of upper triangular matrices in GL_n, acts on $\mathrm{Gr}(r,n)$.

Proposition 4.1. *The action of B on $\mathrm{Gr}(r,n)$ has finitely many orbits, one for each sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of $[n]$ with $n - r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$. The orbit corresponding to λ is isomorphic to $\mathbb{A}^{\lambda_1 + \cdots + \lambda_r}$.*

Let Ω_λ denote the closure of the B-orbit corresponding to λ. This is called the *Schubert variety* associated to λ. Set $|\lambda| = \lambda_1 + \cdots + \lambda_r$, so $\dim \Omega_\lambda = |\lambda|$.

Proposition 4.2. *The classes of B-orbit closures $\{[\mathcal{O}_{\Omega_\lambda}]\}$ form a basis for $K(\mathrm{Gr}(r,n))$.*

Note that, when $r = 1$, this recovers the basis for $K(\mathbb{P}^{n-1})$ described previously.

4.1. **K-theoretic positivity on the Grassmannian.** As a consequence of Proposition 4.2, given any subscheme X of $\mathrm{Gr}(r,n)$, we can write $[\mathcal{O}_X] = \sum_\lambda a_\lambda [\mathcal{O}_{\Omega_\lambda}]$ for some integers a_λ. We will show, in good circumstances, these integers a_λ have certain positivity properties.

The first observation is $a_\lambda = 0$ if $|\lambda| > \dim X$. This can be seen by using the relationship between $K_\circ(\mathrm{Gr}(r,n))$ and the Chow groups $A_\bullet(\mathrm{Gr}(r,n))$. This connection also implies that if $|\lambda| = \dim X$, then a_λ is the coefficient of $[\Omega_\lambda]$ in the expansion of the fundamental class of X in $A_{\dim X}(\mathrm{Gr}(r,n))$ in terms of the classes of Schubert varieties. It follows from the Kleiman–Bertini theorem [Ful98, Lemma B.9.2] that the coefficients of the expansion of $[\Omega_\lambda]$ in terms of classes of Schubert varieties can be computed as the length of the dimensionally transverse intersection of X with a subvariety of $\mathrm{Gr}(r,n)$, and so the a_λ are non-negative in this case.

When $|\lambda| < \dim X$, then the a_λ capture more refined information about X. For example, they can be used to distinguish two skew lines in $\mathbb{P}^3$ from two lines that meet at a point.

We say that an integral scheme X over a field of characteristic 0 has *rational singularities* if a resolution of singularities $\pi \colon \tilde{X} \to X$, i.e., a proper birational map from a smooth scheme $\tilde{X}$, has $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$ for $i > 0$. If this holds for one resolution, then it holds for any resolution, because any two resolutions can be dominated by a third. In particular, if $\pi \colon \tilde{X} \to X$ is a resolution of singularities of a variety with rational singularities, then $\pi_*[\mathcal{O}_{\tilde{X}}] = [\mathcal{O}_X]$.

Example 4.3. If X is a smooth variety, a (normal) toric variety [Ful93, pg. 76], or a Schubert variety [BK05, Section 3.4], then it has rational singularities.

If X is proper and has rational singularities, then for any $a \in K^\circ(X)$ and any resolution $\pi \colon \tilde{X} \to X$, we have $\chi(X, a) = \chi(\tilde{X}, \pi^* a)$ by the projection formula. In fact, the Leray spectral sequence and the projection formula implies that, for any vector bundle $\mathcal{E}$, the natural map from $H^i(X, \mathcal{E})$ to $H^i(\tilde{X}, \pi^* \mathcal{E})$ is an isomorphism for all i.

Theorem 4.4. [Bri02] *Let X be a closed subvariety of the Grassmannian over a field of characteristic 0, and assume that X has rational singularities. Write $[\mathcal{O}_X] = \sum_\lambda a_\lambda [\mathcal{O}_{\Omega_\lambda}]$. Then $(-1)^{\dim X - |\lambda|} a_\lambda \geq 0$.*

I.e., the coefficients which are used to express $[\mathcal{O}_X]$ in the basis given by structure sheaves of Schubert varieties alternate in sign. When $\lambda = \dim X$, one has $a_\lambda \geq 0$ without any restriction on the singularities or characteristic. In the case of $\mathrm{Gr}(1,n) = \mathbb{P}^{n-1}$, Theorem 4.4 predicts the signs occurring in the expansion of the Hilbert polynomial of X using the basis $\{\binom{a+q}{q}\}_{q=0,\ldots,n-1}$ for polynomials in a of degree at most $n-1$, as in (4).

The key tool to prove K-theoretic positivity results is the *Kawamata–Viehweg* vanishing theorem. This controls the cohomology of inverses of *nef and big* line bundles on a smooth projective variety. A line bundle $\mathcal{L}$ on a projective variety X is nef if its restriction to every integral curve in X has non-negative degree. We say that $\mathcal{L}$ is big if the sections of its tensor powers have the maximal possible growth rate, i.e., if

$$\limsup_{a\to\infty} \frac{\dim H^0(X,\mathcal{L}^{\otimes a})}{(\dim X)^a} > 0.$$

If $\mathcal{L}$ is nef, then $\mathcal{L}$ is big if and only if its top self-intersection number $\int_X c_1(\mathcal{L})^{\dim X}$ is positive.

We can obtain nef and big line bundles as follow: we take a map $X \to Y$ with $\dim X = \dim Y$ and embedding of Y into projective space $\mathbb{P}^n$, and then we pull back the hyperplane bundle $\mathcal{O}(1)$ to X.

Theorem 4.5 (Kawamata–Viehweg vanishing theorem). *Let X be a (connected) smooth projective variety over a field of characteristic 0, and let $\mathcal{L}$ be a nef and big line bundle on X. Then $H^i(X,\mathcal{L}^{-1}) = 0$ for $i < \dim X$.*

By Serre duality, $H^i(X,\mathcal{L}^{-1})$ is dual to $H^{\dim X - i}(X, \omega_X \otimes \mathcal{L})$, where ω_X is the canonical bundle, so Kawamata–Viehweg vanishing is often phrased as saying that if $\mathcal{L}$ if nef and big, then $H^i(X,\omega_X\otimes\mathcal{L}) = 0$ for $i > 0$.

In the case when $\mathcal{L}$ is ample, Theorem 4.5 is called *Kodaira vanishing*, and the proof is considerably easier in this case. We will mostly use the following corollary.

Corollary 4.6. *Let X be a projective variety with rational singularities over a field of characteristic 0, and let $\mathcal{L}$ be a nef and big line bundle on X. Then $(-1)^{\dim X}\chi(X,\mathcal{L}^{-1}) \geq 0$.*

Proof. Let $\pi\colon \tilde{X} \to X$ be a projective resolution of singularities of X. Then $\pi^*\mathcal{L}$ is a nef and big line bundle on $\tilde{X}$, so, by Theorem 4.5, we have

$$\chi(\tilde{X},\pi^*\mathcal{L}^{-1}) = \sum_{i\geq 0}(-1)^i \dim H^i(X,\pi^*\mathcal{L}^{-1}) = (-1)^{\dim X}\dim H^{\dim X}(\tilde{X},\pi^*\mathcal{L}^{-1}).$$

The result follows, as $\chi(X,\mathcal{L}^{-1}) = \chi(\tilde{X},\pi^*\mathcal{L}^{-1})$ because X has rational singularities. □

Note that Theorem 4.5 and even Corollary 4.6 can fail in positive characteristic [Tot19]. Corollary 4.6 can also fail without the assumption that X has rational singularities:

Example 4.7. Choose some integers $d > n \geq 3$. By generically projecting the rational normal curve in $\mathbb{P}^d$, construct a smooth degree d rational curve in $\mathbb{P}^n$. Let Y be the cone over this curve, and let $\tilde{Y}$ be the blow-up of Y at the cone point, so the map $\pi\colon \tilde{Y} \to Y$ is a resolution of singularities. The variety $\tilde{Y}$ is isomorphic to the dth Hirzebruch surface, and, using toric geometry, one can show that $H^i(\tilde{Y},\pi^*\mathcal{O}(-1)) = 0$ for all i. But Y is not normal: there is a $(d-n)$-dimensional vector space of sections of $\mathcal{O}(1)$ on the rational curve which are not restricted from $\mathbb{P}^n$. This implies that the sheaf $\pi_*\mathcal{O}_{\tilde{Y}}/\mathcal{O}_Y$ has length at least $d-n$. Because π is an isomorphism outside of the origin, this sheaf is concentrated at the origin. Also, $R^1\pi_*\mathcal{O}_{\tilde{Y}} = R^2\pi_*\mathcal{O}_{\tilde{Y}} = 0$ because the curve is rational, so we have

$$\pi_*[\mathcal{O}_{\tilde{Y}}] = [\mathcal{O}_Y] + m[\mathcal{O}_p], \quad m \geq d-n.$$

By the projection formula (2), we have $\chi(Y,\mathcal{O}(-1)) = -m < 0$.

We now prove Theorem 4.4 in the case $r = 1$, i.e., for subschemes of projective space. The general case is similar, except that it require a more sophisticated vanishing theorem.

Proof of Theorem 4.4. We will use the observation that the pairing $K(\mathbb{P}^{n-1}) \times K(\mathbb{P}^{n-1}) \to \mathbb{Z}$ given by $(a, b) \mapsto \chi(\mathbb{P}^{n-1}, ab)$ is nondegenerate. Under this pairing, the dual basis to $\{[\mathcal{O}], [\mathcal{O}_H], \ldots, [\mathcal{O}_H]^{n-1}\}$ is $\{[\mathcal{O}_H]^{n-1}, [\mathcal{O}_H]^{n-2}[\mathcal{O}(-1)], [\mathcal{O}_H]^{n-3}[\mathcal{O}(-1)], \ldots, [\mathcal{O}(-1)]\}$. That is, we have

$$[\mathcal{O}_X] = \sum a_{n-1-i}[\mathcal{O}_H]^i, \quad \text{where } a_{n-1-i} = \chi(\mathbb{P}^{n-1}, [\mathcal{O}_X] \cdot [\mathcal{O}_H]^{n-1-i}[\mathcal{O}(-1)]).$$

By Proposition 3.2, $[\mathcal{O}_X] \cdot [\mathcal{O}_H]^{n-1-i}$ is the structure sheaf of the result of slicing X by $n-1-i$ transverse hyperplanes. Call this variety Z. By an appropriate version of Bertini's theorem, Z has dimension $\dim X - (n-1-i)$, and it has rational singularities. It follows from Corollary 4.6 that

$$(-1)^{\dim X - (n-1-i)} a_{n-1-i} = \chi(Z, \mathcal{O}(-1)) \geq 0. \qquad \square$$

Remark 4.8. If one replaces $[\mathcal{O}_X]$ with $\pi_*[\mathcal{O}_{\tilde{X}}]$, where $\pi \colon \tilde{X} \to X$ is a resolution of singularities, then Theorem 4.4 holds without any assumption on the singularities.

4.2. **Bounds on matroid polytope subdivisions.** Using Theorem 4.4, we can construct valuative invariants of matroids that, at least for matroids realizable over a field of characteristic 0, have the sign property that is needed to bound the complexity of a matroid polytope subdivision.

Given a linear subspace $L \subseteq k^n$, we may expand the class of the structure sheaf of the torus orbit closure $[\mathcal{O}_{\overline{T \cdot [L]}}]$ in terms of structure sheaves of Schubert varieties:

$$[\mathcal{O}_{\overline{T \cdot [L]}}] = \sum_{\lambda} a_\lambda [\mathcal{O}_{\Omega_\lambda}].$$

Let M be the matroid of L, and let c be the number of components of M, so $\dim \overline{T \cdot [L]} = n - c$. The torus-orbit closure $\overline{T \cdot [L]}$ is a normal toric variety, and normal varieties have rational singularities by Example 4.3. Then, if k has characteristic 0, Theorem 4.4 gives that $(-1)^{n-|\lambda|-c} a_\lambda \geq 0$.

It is reasonable to expect that the class $[\mathcal{O}_{\overline{T \cdot [L]}}] \in K(\mathrm{Gr}(r, n))$ depends only on the matroid of L: often, though not always, it is possible to deform any other realization L' of M to L. Such a deformation induces a deformation of the corresponding torus-orbit closures, which implies that their K-classes are equal by Example 3.5.

In situations like this, when one has an "algebro-geometric" invariant of realizable matroids, it is almost always possible to extend the definition of this invariant to all matroids. One finds a formula for the invariant, and then the formula typically makes sense for an arbitrary matroid.

Example 2.5, together with Example 3.3, suggests that the class $[\mathcal{O}_{\overline{T \cdot [L]}}]$ behaves valuatively with respect to matroid polytope subdivisions. See [Spe09, Proposition A.3]. It is reasonable to hope that this behavior extends to non-realizable matroids.

These properties were established in [FS12], although none of them are obvious. That is, for each matroid M of rank r on $[n]$, the authors construct a class $[\mathcal{O}_{\mathrm{M}}] \in K(\mathrm{Gr}(r, n))$ that agrees with the class of the structure sheaf of the torus-orbit closure of a realization, if M is realizable. The function $\mathrm{M} \mapsto [\mathcal{O}_{\mathrm{M}}]$ is valuative.

We can construct more valuative invariants by writing

$$[\mathcal{O}_{\mathrm{M}}] = \sum a_\lambda(\mathrm{M})[\mathcal{O}_{\Omega_\lambda}] \text{ in } K(\mathrm{Gr}(r, n)).$$

As explained previously, if M is realizable over a field of characteristic 0, then $(-1)^{n-|\lambda|-c(\mathrm{M})} a_\lambda(\mathrm{M}) \geq 0$, i.e., the valuative function $\mathrm{M} \mapsto (-1)^{n-1-|\lambda|} a_\lambda(\mathrm{M})$ has the sign pattern needed to give bounds on the complexity of matroid polytope subdivisions, at least when all of the pieces are realizable over a field of characteristic 0.

It is conjectured, but not known, that $(-1)^{n-|\lambda|-c(\mathrm{M})}a_\lambda(\mathrm{M}) \geq 0$ for all matroids M [BF22, Conjecture 9.8]. However, it seems that the bounds one obtains using this strategy are not enough to prove the sharp bounds in Theorem 2.7 (except in the case of faces of dimension $n-1$, see Proposition 5.18).

5. Speyer's g-polynomial of a matroid

We now introduce the invariant which Speyer used in [Spe09] to bound the complexity of a matroid polytope subdivision. For each matroid M, he constructed a polynomial $g_{\mathrm{M}}(t) \in \mathbb{Z}[t]$ which has the property that $\mathrm{M} \mapsto (-1)^{c(\mathrm{M})+1} g_{\mathrm{M}}(t)$ is valuative. When M is realizable over a field of characteristic 0, Speyer showed that its g-polynomial has non-negative coefficients, and that the bounds it gives on the complexity of a matroid polytope subdivision are optimal, i.e., sufficient to prove Theorem 2.7 in that case. The non-negativity of the coefficients of the g-polynomial was proved in general in [FSS24, BF24].

We will explain the definition of the g-polynomial. We begin with the case of realizable matroids. We will describe the properties of the g-polynomial, and then prove the non-negativity of the coefficients when M is realizable over a field of characteristic 0.

5.1. Defining the g-polynomial for realizable matroids.

An element $i \in [n]$ is a *loop* of a matroid M if i is not contained in any basis. A *coloop* is an element which is contained in every basis. We say that M is *loopless* if it does not have any loops, and we define *coloopless* similarly.

If M is realized by $L \subseteq k^n$, then M is loopless if and only if L is not contained in any coordinate hyperplane. The loops of M are the same as the coloops of the dual matroid $\mathrm{M}^\perp$.

The following technical proposition is a consequence of Remark 2.8, because the matroid polytopes of matroids with loops or coloops are contained in the boundary of the matroid polytope of any loopless and coloopless matroid. It can also be proved directly, see [BEST23, Proof of Lemma 5.9].

Proposition 5.1. *Let Val_n^{cl} be the subgroup of Val_n generated by matroids with loops or coloops, and let Val_n^{ncl} be the subgroup generated by matroids without loops or coloops. Then $\mathrm{Val}_n = \mathrm{Val}_n^{cl} \oplus \mathrm{Val}_n^{ncl}$.*

There are several different possible definitions of the g-polynomial. However, for matroids with loops or coloops, these definitions can fail to be equivalent, so we will only define the g-polynomial for loopless and coloopless matroids.

Let $L \subseteq k^n$ be a realization of a loopless and coloopless matroid M of rank r, so $\dim L = r$, L is not contained in any coordinate hyperplane. Let $L^\perp \subseteq k^n$ be the orthogonal complement to L, so $L^\perp$ realizes $\mathrm{M}^\perp$ and is not contained in any coordinate hyperplane. There is a rational map

$$\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1},$$

given, in homogeneous coordinates, by $(x_1, \cdots, x_n) \times (y_1, \cdots, y_n) \mapsto (x_1y_1, \ldots, x_ny_n)$. Because M is loopless and coloopless, L and $L^\perp$ are both non-zero vector spaces, so we can consider their projectivizations. Using the embeddings $\mathbb{P}L \hookrightarrow \mathbb{P}^{n-1}$ and $\mathbb{P}L^\perp \hookrightarrow \mathbb{P}^{n-1}$, we obtain a rational map

$$\mathbb{P}L \times \mathbb{P}L^\perp \overset{m}{\dashrightarrow} \mathbb{P}^{n-1}.$$

Note that the image of m is contained in the hyperplane $H = \{z_1 + \cdots + z_n = 0\}$ of $\mathbb{P}^{n-1}$. As both H and $\mathbb{P}L \times \mathbb{P}L^\perp$ have dimension $n-2$, m can be viewed as a rational map between two varieties of the same dimension. The g-polynomial of M will capture certain data about this map, such as its degree.

Let Z be a smooth projective variety resolving the indeterminancy of this rational map, i.e., Z is a smooth projective variety which fits into the following triangle:

$$\begin{tikzcd} Z \arrow[d, "p"] \arrow[dr, "m"] & \\ \mathbb{P}L \times \mathbb{P}L^{\perp} \arrow[r, dashed, "m"] & H, \end{tikzcd}$$

where the map p is birational. For example, in characteristic 0, one can take the closure of the graph of m in $\mathbb{P}L \times \mathbb{P}L^{\perp} \times H$ and resolve the singularities. We will see an explicit construction of a possible choice for Z, valid in any characteristic, in Section 5.4.

We will consider the line bundle $\mathcal{O}(1)$ on H, obtained by restricting the hyperplane class from $\mathbb{P}^{n-1}$. Let $[\mathcal{O}_D]$ denote the class of a generic section of $\mathcal{O}(1)$ in $K(H)$.

Definition 5.2. *The g-polynomial $g_{\mathrm{M}}(t)$ of a realizable loopless and coloopless matroid* M *with c connected components is the polynomial whose t^i coefficient is $(-1)^{i+c}\chi(Z, m^*[\mathcal{O}(-1)] \cdot m^*[\mathcal{O}_D]^{n-i-1})$. If* M *is not loopless or not coloopless, then we set $g_{\mathrm{M}}(t) = 0$.*

The constant term of $g_{\mathrm{M}}(t)$ is 0. In order to compute the coefficient of t^i, we slice H by $n-i-1$ general hyperplanes, take the inverse image in Z, pull back $\mathcal{O}(-1)$ and compute the Euler characteristic. In particular, we see that the linear term of the g-polynomial is the degree of the map m.

In other words, by looking at the proof of Theorem 4.4 in the case of projective space, we can compute the coefficients of $g_{\mathrm{M}}(t)$ by writing the Hilbert polynomial of $\mathcal{O}(1)$ on Z as

$$\chi(Z, \mathcal{O}(a)) = \sum_{q=0}^{n-2} a_q \binom{a+q}{q}.$$

Then, for $i > 0$, the coefficient of t^i in $g_{\mathrm{M}}(t)$ is $(-1)^{i+c}a_{n-1-i}$.

The coordinate-wise multiplication of two projective varieties is sometimes called the *Hadamard product*. See [BC24] for a comprehensive survey of the literature. It often arises in rigidity theory, see, e.g., [Ber22].

At least over a field of characteristic 0, it is not hard to see that Definition 5.2 is independent of the choice of Z. For if we have another smooth projective variety Z' resolving the indeterminancy, then both Z and Z' can be dominated by another smooth variety $\tilde{Z}$ also resolving the indeterminacy, and the projection formula (2) implies that the Euler characteristic does not change when we pull back from Z or Z' to $\tilde{Z}$.

It is true, but not obvious, that the definition above does not depend on the choice of realization of M. This will be explained in Section 5.5, as well as how to extend this definition to non-realizable matroids.

Remark 5.3. In [Spe09], Speyer defines the coefficients of the g-polynomial as certain linear combinations of the invariants $a_\lambda(\mathrm{M})$. This definition agrees with Definition 5.2, at least for loopless and coloopless matroids. From the definition in [Spe09], it is tricky to see the non-negativity of the coefficients. In particular, the non-negativity does not follow from Theorem 4.4.

5.2. **Properties of g-polynomial.** We now discuss the fundamental properties of the g-polynomial. One property is clear: if M is loopless and coloopless, then the definition of $g_{\mathrm{M}}(t)$ does not change when we replace L by $L^{\perp}$.

Proposition 5.4. *We have $g_{\mathrm{M}}(t) = g_{\mathrm{M}^{\perp}}(t)$.*

Proposition 5.5. *Let L be a realization of a loopless and coloopless matroid* M *with c connected components, $S_1, \ldots, S_c$. Then the closure of the image of m is $\{(z_1, \ldots, z_n) : \sum_{j \in S_i} z_j = 0$ for $i = 1, \ldots, c\}$.*

Proof. It follows from Proposition 1.5 that the image of m is contained in $\{(z_1, \ldots, z_n) : \sum_{j \in S_i} z_j = 0$ for $i = 1, \ldots, c\}$. For the direction, it suffices to show that the dimension of the image of m is $n-c$. This reduces to the case of connected matroids. I don't know a very elementary proof this case. Later, we will compute the degree of the map $\mathbb{P}L \times \mathbb{P}L^{\perp} \dashrightarrow H$ in Proposition 5.14, and it can be proved combinatorially that the degree is positive for connected matroids, see Proposition 5.16. It can also be deduced from [Ber22, Theorem 3.1]. □

The following result will allow us to reduce computations to the case of connected matroids. It also explains the choice of indexing in the definition of the g-polynomial.

Proposition 5.6. [Spe09, Proposition 3.2] *Suppose that* $\mathrm{M} = \mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_c$. *Then*

$$g_{\mathrm{M}}(t) = g_{\mathrm{M}_1}(t) \cdots g_{\mathrm{M}_c}(t).$$

In particular, if M is connected and realizable, then Proposition 5.5 implies that the map m is dominant, and in particular it has positive degree. We see that, if M is connected and realizable matroid, the linear term of $g_{\mathrm{M}}(t)$ is positive. By Proposition 5.6, this implies the following result.

Corollary 5.7. *If* M *is a realizable loopless and coloopless matroid with* c *connected components, then the coefficient of* t^c *in* $g_{\mathrm{M}}(t)$ *is positive.*

The realizability assumption in Corollary 5.7 is not necessary. In Corollary 5.17, we explicitly compute the coefficient of t^c in terms of a well-known combinatorial invariant of matroids, and see that it is positive for any loopless matroid.

5.3. Positivity in characteristic 0.

Theorem 5.8. [Spe09, FSS24, BF24] *Let* M *be a matroid. Then the coefficients of* $g_{\mathrm{M}}(t)$ *are non-negative.*

Proof of Theorem 5.8 when M *is realizable in characteristic* 0. By Proposition 5.6, we may assume that M is connected, i.e., $c(\mathrm{M}) = 1$. Let Z_i be the inverse image under m of the result of slicing H by $n-i-1$ generic hyperplanes. By Proposition 3.2, $m^*[\mathcal{O}_D]^{n-i-1} = [\mathcal{O}_{Z_i}]$. By Bertini's theorem, Z_i is a smooth variety of dimension $i-1$. By Proposition 5.5, because M is connected, m is dominant, and so the restriction of $m^*\mathcal{O}(-1)$ to Z_i is a nef and big line bundle. Corollary 4.6 then implies that $(-1)^{i-1}\chi(Z_i, m^*\mathcal{O}(-1)) \geq 0$, implying the result. □

5.4. **Wonderful varieties and tautological bundles.** We will now relate the g-polynomial to the wonderful varieties of [DCP95] and the tautological bundles of linear subspaces that were introduce in [BEST23]. This will allow us to connect the g-polynomial to other well-studied invariants of matroids.

The *permutohedral toric variety* X_n is the toric variety obtained by blowing up $\mathbb{P}^{n-1}$ at the torus-fixed points, then the strict transform of the torus-fixed lines, and so on. It is an $(n-1)$-dimensional smooth projective toric variety whose fan is the Weyl chambers of the type A_{n-1} root system. It can be described in a number of other ways. For example, it is the torus-orbit closure of a general point in the variety of full flags in k^n, and it is the toric variety corresponding to the polytope $\sum_{r=1}^{n-1} P(U_{r,n})$, where the sum denotes Minkowski sum.

Given a subspace L of k^n which is not contained in any coordinate hyperplane, i.e., it represents a loopless matroid, the *wonderful variety* W_L is the strict transform of $\mathbb{P}L$ under the map $X_n \to \mathbb{P}^{n-1}$. I.e., it is the closure of $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$ in X_n, where $\mathbb{G}_m^n/\mathbb{G}_m$ is the torus embedded in $\mathbb{P}^{n-1}$.

The wonderful variety is a smooth and projective variety of dimension $\dim L - 1$; it can be described as an iterated blow-up of $\mathbb{P}L$ along strict transforms of intersections of $\mathbb{P}L$ with coordinate subspaces. It is also a simple normal crossings compactification of $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$.

In [BEST23], Berget, Eur, Spink, and Tseng constructed some important vector bundles $\mathcal{S}_L, \mathcal{Q}_L$ on the permutohedral variety X_n using a linear subspace L of k^n. These bundles form a short exact sequence

$$0 \to \mathcal{S}_L \to \mathcal{O}_{X_n}^{\oplus n} \to \mathcal{Q}_L \to 0. \tag{5}$$

Over a point $t \in \mathbb{G}_m^n/\mathbb{G}_m$, the torus of X_n, the fiber of $\mathcal{S}_L$ in k^n is $t^{-1}[L]$. This uniquely determines $\mathcal{S}_L$: the total space of $\mathcal{S}_L$ is the closure in $X_n \times k^n$ of the total space of the restriction of $\mathcal{S}_L$ to $\mathbb{G}_m^n/\mathbb{G}_m$. It is a theorem that this is a vector bundle: it is the pullback of a certain vector bundle from the torus-orbit closure of $[L^\perp]$ in $\mathrm{Gr}(n-r,n)$; the description of the fan of the permutohedral toric variety implies that there is a map from X_n to the toric variety of any matroid polytope. The bundle $\mathcal{Q}_L$ is defined as the quotient.

The image of the section $(1,\dots,1) \in H^0(X_n, \mathcal{O}_{X_n}^{\oplus n})$ in $\mathcal{Q}_L$ transversely cuts out W_L [BEST23, Theorem 7.10]. This identifies the restriction of $\mathcal{Q}_L$ to W_L with the normal bundle of W_L in X_n. The restriction of $\mathcal{S}_L$ to W_L is closely related to the log tangent bundle of W_L, viewed as a compactification of $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$ [BEST23, Theorem 8.8].

The geometry of Speyer's g-polynomial is closely related to the restriction of $\mathcal{Q}_L$ to W_L. The sequence (5) dualizes to give an injection $\mathcal{Q}_L^\vee \hookrightarrow \mathcal{O}_{X_n}^{\oplus n}$. Restricting to W_L and projectivizing, we obtain a subvariety $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ of $X_n \times \mathbb{P}^{n-1}$.

Identifying k^n/L with $L^\perp$, the intersection of $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ with $\mathbb{G}_m^n/\mathbb{G}_m \times \mathbb{G}_m^n/\mathbb{G}_m$ is the locus

$$\{(v, tv) : v \in \mathbb{P}L \cap \mathbb{G}^m/\mathbb{G},\, t \in \mathbb{P}L^\perp \cap \mathbb{G}_m^n/\mathbb{G}_m\}.$$

If the matroid M corresponding to L is loopless and coloopless with c connected components, then this variety is isomorphic to $(\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m) \times (\mathbb{P}L^\perp \cap \mathbb{G}_m^n/\mathbb{G}_m)$, via the map which sends (v,w) to (v, vw). In particular, if p is the projection of $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ onto $\mathbb{P}^{n-1}$, the p is a birational model for the map m considered in the definition of the g-polynomial. As a consequence, the t^i coefficient of $g_{\mathrm{M}}(t)$ is

$$(-1)^{i+c}\chi(\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee), [\mathcal{O}(-1)] \cdot ([\mathcal{O}] - [\mathcal{O}(-1)])^{n-i-1}) = \sum_{j=0}^{n-1-i} (-1)^{i+c+j} \binom{n-i-1}{j} \chi(\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee), \mathcal{O}(-j-1)).$$

This will enable us to reduce problems involving the g-polynomial to computations on X_n, where there are more tools because it is a toric variety.

Let $\pi \colon \mathbb{P}_{W_L}(\mathcal{Q}_L^\vee) \to W_L$ be the projection. We can compute the Euler characteristic on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ by pushing forward along π. It follows from the fact that $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee) = \operatorname{Proj} \operatorname{Sym} \mathcal{Q}_L$ that, for $a \geq 0$, we have $\pi_*\mathcal{O}(a) = \operatorname{Sym}^a \mathcal{Q}_L$. Furthermore, we have $R^i\pi_*\mathcal{O}(a) = 0$ for $i > 0$, $a \geq 0$, for example by the theorem on cohomology and base change. Grothendieck duality implies that $R^i\pi_*\mathcal{O}(a) = 0$ for $i < n-r-1, a < 0$ or $i = r, -(n-r) < a < 0$, and $R^{n-r-1}\pi_*\mathcal{O}(-a) = \det \mathcal{Q}_L^\vee \otimes \operatorname{Sym}^{a-n+r} \mathcal{Q}_L^\vee$ for $a \leq -(n-r)$. We deduce the following formula for the g-polynomial.

Proposition 5.9. *Let L be a realization of a matroid M. For $i > 0$, the t^i coefficient of $g_{\mathrm{M}}(t)$ is*

$$(-1)^{i+c(\mathrm{M})+n-r-1} \sum_{j=n-r+1}^{n-1-i} (-1)^j \binom{n-i-1}{j} \chi(W_L, \det \mathcal{Q}_L^\vee \otimes \operatorname{Sym}^{j+1-n+r} \mathcal{Q}_L^\vee).$$

For example, the coefficient of t^i is 0 if $i > r$ because the sum is empty. Because $g_{\mathrm{M}}(t) = g_{\mathrm{M}^\perp}(t)$, we see that the t^i coefficient of $g_{\mathrm{M}}(t)$ is 0 if $i > n-r$ as well, so the degree of g is at most $\min\{r, n-r\}$. If $i = r$, there is only one term in the sum. We see that the coefficient of t^r is $(-1)^{r+c(\mathrm{M})}\chi(W_L, \det \mathcal{Q}_L^\vee)$.

Remark 5.10. The description of the g-polynomial in terms of $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ does not show the symmetry of the g-polynomial under matroid duality. There is a more symmetric model for Z, related to the conormal fan of a matroid of [ADH23].

5.5. **Defining the g-polynomial for non-realizable matroids.** We can use the formula in Proposition 1.5 to see that the g-polynomial does not depend on the choice of realization, and to define the g-polynomial for non-realizable matroids.

For a matroid M on $[n]$, the authors of [BEST23] constructed a class $[\mathcal{Q}_{\mathrm{M}}]$ in $K(X_n)$. When M is realized by L, then $[\mathcal{Q}_{\mathrm{M}}] = [\mathcal{Q}_L]$. In particular, the class of $[\mathcal{Q}_L]$ does not depend on the choice of realization. Define $[\mathcal{S}_{\mathrm{M}}]$ as $[\mathcal{O}_{X_n}^{\oplus n}] - [\mathcal{Q}_{\mathrm{M}}]$; similarly, $[\mathcal{S}_{\mathrm{M}}] = [\mathcal{S}_L]$ for any realization L of M. We have the following key property.

Proposition 5.11. [BEST23, Proposition 5.6] *The function which assigns a matroid to any fixed polynomial function in the symmetric or exterior powers of $[\mathcal{S}_{\mathrm{M}}]$, $[\mathcal{Q}_{\mathrm{M}}]$, and their duals is valuative.*

In [LLPP24], the authors defined a ring $K(\mathrm{M})$ for every loopless matroid M. There is a surjection map from $K(X_n)$ to $K(\mathrm{M})$. If M is realized by L, then there is an identification of $K(W_L)$ with $K(\mathrm{M})$. For instance, the inclusion of W_L into X_n induces a restriction map $K(X_n) \to K(W_L)$, which is surjective. The kernel of the map from $K(X_n)$ to $K(\mathrm{M})$ is the same as the map from $K(X_n)$ to $K(W_L)$.

The authors of [LLPP24] constructed an "Euler characteristic" map $\chi\colon K(\mathrm{M}) \to \mathbb{Z}$ which satisfies an analogue of projection formula. When M is realized by L, this Euler characteristic map coincides with the usual one. Suppose that L is a realization of M. Recall that there is a section of $\mathcal{Q}_L$ which transversely cuts out W_L inside of X_n. This induces a Koszul resolution of $\mathcal{O}_{W_L}$: there is an exact sequence

$$0 \to \wedge^{n-r} \mathcal{Q}_L^{\vee} \to \wedge^{n-r-1} \mathcal{Q}_L^{\vee} \to \cdots \to \mathcal{Q}_L^{\vee} \to \mathcal{O}_{X_n} \to \mathcal{O}_{W_L} \to 0.$$

This implies that $[\mathcal{O}_{W_L}] = \sum_{i \geq 0} (-1)^i \wedge^i [\mathcal{Q}_L^{\vee}]$ in $K(X_n)$. The projection formula (2) then means that if M is realization and a is a class in $K(X_n)$, then

$$\chi(W_L, a) = \chi(X_n, a \cdot \sum_{i \geq 0} (-1)^i \wedge^i [\mathcal{Q}_L^{\vee}]).$$

By [LLPP24, Proposition 5.6], for any class $a \in K(X_n)$ and any loopless matroid M, we have

$$\chi(\mathrm{M}, a) = \chi(X_n, a \cdot \sum_{i \geq 0} (-1)^i \wedge^i [\mathcal{Q}_{\mathrm{M}}^{\vee}]). \tag{6}$$

Definition 5.12. *Let* M *be matroid. If* M *has a loop or coloop, then define $g_{\mathrm{M}}(t)$ to be* 0*. Otherwise, define $g_{\mathrm{M}}(t)$ to be the polynomial whose constant term is* 0*, and whose t^i coefficient for $i > 0$ is*

$$(-1)^{i+c(\mathrm{M})+n-r-1} \sum_{j=n-r+1}^{n-1-i} (-1)^j \binom{n-i-1}{j} \chi(\mathrm{M}, \det[\mathcal{Q}_{\mathrm{M}}]^{\vee} \otimes \mathrm{Sym}^{j+1-n+r}[\mathcal{Q}_{\mathrm{M}}]^{\vee}).$$

Then the following result is immediate by (6), Proposition 5.1, and Proposition 5.11.

Proposition 5.13. *The assignment $\mathrm{M} \mapsto (-1)^{c(\mathrm{M})} g_{\mathrm{M}}(t)$ is a valuative invariant of matroids.*

5.6. **The beta invariant of a matroid.** Now that we have defined the g-polynomial for all matroids, we will compute its linear term. This computation will show that the linear term of the g-polynomial is strictly positive for connected matroids, which will be crucial in the proof of Theorem 2.7.

The *beta invariant* $\beta(\mathrm{M})$ is an important invariants of matroids. It can be defined using a deletion-contraction recursion. We have $\beta(U_{0,1}) = 0, \beta(U_{1,1}) = 1$, and if $i \in [n]$ is an element which is not a loop or coloops, then $\beta(\mathrm{M}) = \beta(\mathrm{M}/i) + \beta(\mathrm{M} \setminus i)$. It can also be defined as the coefficient of x in the Tutte polynomial of M.

Proposition 5.14. *Let* M *be a loopless and coloopless matroid. The coefficient of t in $g_{\mathrm{M}}(t)$ is $\beta(\mathrm{M})$.*

This can be proved in several ways. In [Spe09], Speyer directly verified that the linear term of the g-polynomial satisfied the recursion defining the beta invariant.

Proof of Proposition 5.14. One can massage Definition 5.2 to see that the linear term of the g-polynomial is degree of the top Segre class of $[\mathcal{Q}_{\mathrm{M}}]^{\vee}$. In the realizable case, this is because the coefficient of t in $g_{\mathrm{M}}(t)$ is the top self-intersection of $\mathcal{O}(1)$ on $\mathbb{P}_{W_L}(\mathcal{Q}_L^{\vee})$. The Segre classes of $[\mathcal{Q}_{\mathrm{M}}]^{\vee}$ are the Chern classes of $[\mathcal{S}_{\mathrm{M}}]^{\vee}$. The result then follows from [BEST23, Theorem 6.2]. $\square$

Remark 5.15. When M is realized by L, one can deduce that $\int_{W_L} c_{r-1}(\mathcal{S}_L^{\vee}) = \beta(\mathrm{M})$ by using the logarithmic Poincaré-Hopf theorem. In [BEST23, Theorem 8.8], it is shown that $\mathcal{S}_L$ is an extension of the log tangent bundle of W_L by a trivial bundle, so $c_{r-1}(\mathcal{S}_L)$ is the top Chern class of the log tangent bundle. The logarithmic Poincaré-Hopf theorem states that the degree of the top Chern class of the log tangent bundle of a simple normal crossings compactification of a variety U is the topological Euler characteristic of U. It is known that the Euler characteristic of $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$ is $(-1)^{r-1}\beta(\mathrm{M})$.

Proposition 5.16. [Cra67] *A matroid* M *on at least* 2 *elements is connected if and only if* $\beta(\mathrm{M})$ *is positive.*

The difficult part of Proposition 5.16 is to show that if M is a connected matroid on $[n]$ and $i \in [n]$, then either M/i or $\mathrm{M} \setminus i$ is connected. The following result follows from Proposition 5.14 and Proposition 5.6.

Corollary 5.17. *If* M *is a loopless and coloopless matroid with connected components* $\mathrm{M}_1, \dots, \mathrm{M}_c$*, then the coefficient of* t^c *in* $g_{\mathrm{M}}(t)$ *is* $\beta(\mathrm{M}_1) \cdots \beta(\mathrm{M}_c)$*. In particular, the coefficient of* t^c *is strictly positive.*

Finally, we note the following interpretation of the beta invariant.

Proposition 5.18. [Spe09, Theorem 5.1] *Let* $\lambda = (n-r, 1^{r-1})$ *be the hook shape. Then* $\beta(\mathrm{M}) = a_{\lambda}(\mathrm{M})$*.*

5.7. **Bounding the complexity of a matroid polytope subdivision.**

Proposition 5.19. [Spe09, Proposition 3.1] *We have*

$$g_{U_{r,n}}(t) = \sum_{i=1}^{\min\{r,n-r\}} \frac{(n-i-1)!}{(r-i)!(n-r-i)!(i-1)!} t^i.$$

Theorem 5.20. *A matroid polytope subdivision of* $P(U_{r,n})$ *where all matroids* M *appearing have* $g_{\mathrm{M}}(t) \geq 0$ *has at most* $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ *interior faces of dimension* $n-c$ *for* $c \leq \min\{r, n-r\}$*, and it has no interior faces of dimension less than* $\min\{r, n-r\}$*.*

Combined with Theorem 5.8, Theorem 5.20 proves Theorem 2.7.

Proof of Theorem 5.20. The matroid polytope of a matroid with a loop or coloop is contained in the boundary of $P(U_{r,n})$, so all interior faces of a matroid polytope subdivision of $P(U_{r,n})$ correspond to loopless and coloopless matroids. Using (1) and Proposition 5.13, we deduce that

$$-g_{U_{r,n}}(t) = \sum_{\mathrm{M},\, P(\mathrm{M}) \text{ interior face}} (-1)^{c(\mathrm{M})+1}(-1)^{c(\mathrm{M})} g_{\mathrm{M}}(t).$$

Looking at the coefficient of t^c for $c \leq \min\{r, n-r\}$, we see that $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ is a sum of the coefficient of t^c in $g_{\mathrm{M}}(t)$ over all matroids where $P(\mathrm{M})$ is an interior face. By Corollary 5.17 (or Corollary 5.7 for realizable matroids), each interior face of dimension $n-c$ contributes at least 1 to this sum, and each other face contributes non-negatively, and so there are at most $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ faces of dimension $n-c$. If $c > \min\{r, n-r\}$, the coefficient of t^c on the left-hand side is 0, so there can be no faces of dimension $n-c$. $\square$

Remark 5.21. In [Spe09], Speyer shows that if M is a *series-parallel matroid*, i.e., the graphic matroid of a graph which is obtained from a parallel edge by repeatedly either subdividing an edge or doubling an edge, then $g_{\mathrm{M}}(t) = t$. It is known that if M is a matroid, then $\beta(\mathrm{M}) = 1$ if and only if M is a series-parallel matroid [Bry71, Theorem 7.6]. This implies that a matroid M has $g_{\mathrm{M}}(t) = t^{c(\mathrm{M})}$ if and only if it is a direct sum of series-parallel matroids (such matroids are called (loopless and coloopless) *quasi series-parallel matroids*, see [FL24]). It is known that a face of the matroid polytope of a quasi series-parallel matroids is a quasi series-parallel matroids, and so if all $(n-1)$-dimensional faces in a matroid polytope subdivision of $P(U_{r,n})$ are series-parallel, then all interior faces are quasi series-parallel, and the bounds in Theorem 2.7 are sharp. Furthermore, if any $(n-1)$-dimensional face is not a series-parallel matroid, then there are fewer than $\binom{n-2}{r-1}$ $(n-1)$-dimensional faces.

6. Positivity via Cohen–Macaulayness

The driving force behind all of the K-theoretic positivity phenomena that we have seen has been Corollary 4.6. In situations where the hypotheses of Corollary 4.6 are not satisfied, it has been difficult to prove directly that the conclusion holds. For instance, one could try to prove Theorem 5.8 by finding a combinatorial interpretation of the coefficients of the g-polynomial, or by finding a non-negative recursive formula. It is possible to prove that the β invariant of a matroid, i.e., the linear of the g-polynomial, is non-negative in either of these ways. But no one has been able to do this for the other coefficients.

Corollary 4.6 concerns the value of the Hilbert polynomial of $\mathcal{L}$ at -1. This is difficult to get at directly, because it involves understanding the higher cohomology of $\mathcal{L}^{-1}$. It is usually much easier to analyze the sections and cohomology of positive twists of $\mathcal{L}$. It is sometimes possible to use this to prove the conclusion of Corollary 4.6 by proving a stronger result, known as *arithmetical Cohen–Macaulayness*. The significance of Cohen–Macaulayness in combinatorial problems was first realized by Stanley [Sta75, Sta80].

Let $A^\bullet = A^0 \oplus A^1 \oplus \cdots$ be a finitely generated graded algebra of Krull dimension d, with $A^0 = k$. Then $A^\bullet$ is *Cohen–Macaulay* if there are homogeneous elements $\theta_1, \ldots, \theta_d$ such that, for $i = 1, \ldots, d-1$,

$$\theta_i \text{ is not a zero-divisor in } A^\bullet/(\theta_1, \ldots, \theta_{i-1}).$$

The elements $\theta_1, \ldots, \theta_d$ are called a *homogeneous system of parameters*. In most cases that relevant to combinatorics, the ideal generated by A^1 is primary to the homogeneous maximal ideal. For example, this is true if $A^\bullet$ is generated as a ring by A^1. As k is infinite, this is equivalent to the existence of a homogeneous system of parameters consisting of elements of A^1, which we call a *linear system of parameters* (l.s.o.p.). Any d general elements of A^1 will do.

For example, $A^\bullet$ is automatically Cohen–Macaulay if $d = 0$. If $d = 1$, then $A^\bullet$ is Cohen–Macaulay if and only if A^1 contains an element which is not a zero-divisor. This holds if and only if $\operatorname{Spec} A^\bullet$ is generically reduced.

A graded ring being Cohen–Macaulay gives very strong restrictions on the Hilbert function of $A^\bullet$. For a graded module $M^\bullet$, let $\operatorname{Hilb}_M(z) = \sum_{i=0}^{\infty} \dim M^i \cdot z^i$ denote the Hilbert series of $M^\bullet$.

Proposition 6.1. *Assume that $A^\bullet$ admits an l.s.o.p. If $A^\bullet$ is Cohen–Macaulay of Krull dimension $d+1$, then we can write*

$$\operatorname{Hilb}_A(z) = \frac{h_0 + h_1 + \cdots + h_e}{(1-z)^d},$$

where $h_i \geq 0$ for each i. If $A^\bullet$ is generated in degree 1, then $(h_0, \ldots, h_e)$ is the Hilbert function of an algebra which is generated in degree 1.

Proof. Choose an l.s.o.p. $\theta_1, \ldots, \theta_d$. There is an exact sequence

$$0 \to (\theta_1) \to A^\bullet \to A^\bullet/(\theta_1) \to 0.$$

Because θ_1 is not a zero-divisor, we see that the Hilbert series of (θ_1) is $z\operatorname{Hilb}_A(z)$. Using the additivity of Hilbert series in short exact sequences, we see that

$$(1-z)\operatorname{Hilb}_A(z) = \operatorname{Hilb}_{A/(\theta_1)}(z).$$

Repeating this argument, and using that θ_i is not a zero-divisor on $A/(\theta_1, \ldots, \theta_{i-1})$, we see that

$$(1-z)^d \operatorname{Hilb}_A(z) = \operatorname{Hilb}_{A/(\theta_1,\ldots,\theta_d)}(z).$$

Note that $A^\bullet/(\theta_1, \ldots, \theta_d)$ is a graded algebra of Krull dimension 0, and so its Hilbert series is a polynomial with non-negative coefficients, proving the claim about the form on the Hilbert series and the non-negativity of the h_i. If $A^\bullet$ is generated in degree 1, then so is $A^\bullet/(\theta_1, \ldots, \theta_d)$, proving the second part. □

Often, the Hilbert function will be a polynomial. This holds in most cases which are relevant to combinatorics. If p is a polynomial of degree d, then we can write

$$\sum_{a\geq 0} p(a)t^a = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}}, \tag{7}$$

where the coefficients $(h_0^*, \ldots, h_d^*)$ are related to p by the equation

$$p(a) = \sum_{q=0}^{d} h_q^* \binom{a+d-q}{d}. \tag{8}$$

See [Sta12, Section 4.3]. Note that (8) implies that $(-1)^d p(-1) = h_d^*$, and that the lead term of p is $(h_0^* + \cdots + h_d^*)/d!$.

Note that the polynomial $a \mapsto \binom{a+d-q}{d}$ is the Hilbert polynomial of $\mathcal{O}(-q)$ on $\mathbb{P}^d$. The polynomials $\{\binom{a+d-q}{d}\}_{q=0,\ldots,d}$ form a basis for the space of numerical polynomials of degree at most d, and computing the numerator of the series (7) for a polynomial of degree d is the same thing as expanding it in this basis.

If X is a d-dimension subscheme of $\mathbb{P}^N$, then $[\mathcal{O}_X]$ lies in the piece F_{N-d} of the coniveau filtration on $K(\mathbb{P}^N)$. The classes $\{[\mathcal{O}_H]^{N-d}[\mathcal{O}(-q)]\}_{q=0,\ldots,d}$ form a basis for F_{N-d}, and computing the numerator of the series $\sum_{a\geq 0}\chi(X, \mathcal{O}(a))t^a$ is the same as computing the expansion of $[\mathcal{O}_X]$ in this basis.

Using that $[\mathcal{O}(-1)] = [\mathcal{O}] - [\mathcal{O}_H]$, we see that $[\mathcal{O}(-q)] = ([\mathcal{O}] - [\mathcal{O}_H])^q$. This implies that

$$\binom{a+d-q}{d} = \sum_{i=0}^{q} (-1)^i \binom{q}{i}\binom{a+d-i}{d-i}. \tag{9}$$

The sign-definiteness of this transformation implies that if the numerator in (7) is non-negative, then the coefficients in the expansion of p into the polynomials $\{\binom{a+q}{q}\}$ has alternating signs. For example, we have the following corollary.

Corollary 6.2. *Let X be a subscheme of $\mathbb{P}^N$ of dimension d such that the ring $\bigoplus_{a\geq 0} H^0(X, \mathcal{O}(a))$ is a Cohen–Macaulay ring. Assume that $\chi(X, \mathcal{O}(a)) = \dim H^0(X, \mathcal{O}(a))$ for $a \geq 0$. Then in the expansion $[\mathcal{O}_X] = \sum a_i [\mathcal{O}_H]^{N-d+i}$, we have $(-1)^i a_i \geq 0$.*

Proof. It follows from the fact that restriction of $\mathcal{O}(1)$ to X is globally generated by the ring $\bigoplus_{a\geq 0} H^0(X, \mathcal{O}(a))$ is normalization of the subring which is generated in degree 1, and in particular it is finitely generated as a module over the subring which is generated in degree 1. This implies that $\bigoplus_{a\geq 0} H^0(X, \mathcal{O}(a))$ admits an l.s.o.p., and so Proposition 6.1 and the assumption that $\chi(X, \mathcal{O}(a)) = \dim H^0(X, \mathcal{O}(a))$ for $a \geq 0$ implies that, if we write

$$\sum_{a\geq 0} \chi(X, \mathcal{O}(a))t^a = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}},$$

then $h_i^* \geq 0$ for all i. From (9), we deduce that

$$a_i = (-1)^i \sum_{q=i}^{d} \binom{q}{i} h_q^*. \qquad \square$$

Note that Corollary 6.2 does not require the characteristic to be 0 or for X to have rational singularities (or even be integral). This opens the door to verifying the hypotheses of Corollary 6.2 by degenerating X to a reducible scheme. Additionally, while it is difficult to define varieties whose geometry is related to the combinatorics of a non-realizable matroid, it is not hard to construct reducible schemes related to non-realizable matroids. One can hope to prove that these schemes satisfy the hypothesis of Corollary 6.2.

Even though the inequalities given by the Cohen–Macaulayness of the section ring of a variety are much stronger than those given by Theorem 4.4, they can be easier to prove. Unfortunately, these inequalities are so strong that they fail in the case of $m^*\mathcal{O}(1)$ on Z.

Example 6.3. Let M be the Fano matroid F_7. Then $g_{\mathrm{M}}(t) = 3t^3 + 5t^2 + 3t$, see [Spe09, Section 10], so

$$\begin{aligned}\chi(Z, \mathcal{O}(a)) &= 3\binom{a+5}{5} - 5\binom{a+4}{4} + 3\binom{a+3}{3} \\ &= \frac{a^5}{40} + \frac{a^4}{6} + \frac{13a^3}{24} + \frac{4a^2}{3} + \frac{29a}{15} + 1 \\ &= \binom{a+5}{5} - \binom{a+4}{5} + 3\binom{a+3}{5},\end{aligned}$$

i.e., the numerator of the series (7) is $1 - t + 3t^2$. There are also counterexamples to the inequalities in Proposition 6.1 which are representable in characteristic 0, see [Fer23, Table 1].

In the case when $A^\bullet$ is generated in degree 1, one obtains bounds on how fast the numerator of the Hilbert series can grow. These bounds are given explicitly by the following result.

Definition 6.4. *A sequence $(h_0, h_1, \dots, h_d)$ of integers is a* Macaulay vector *if $(h_0, h_1, \dots, h_d, 0, 0, \dots)$ is the Hilbert function of a graded artinian k-algebra $A^\bullet$ which is generated in degree 1 and has $A^0 = k$.*

Macaulay vectors are also called M-vectors and O-sequences. Macaulay gave an explicit description of these vectors as follows [BH93, Theorem 4.2.10]. Given positive integers n and d, there is a unique expression

$$n = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}, \quad k_d > k_{d-1} > \cdots > k_\delta \geq 1.$$

Set $n^{\langle d \rangle} = \binom{k_d+1}{d+1} + \cdots + \binom{k_\delta+1}{\delta+1}$, and define $0^{\langle i \rangle} = 1$ for all i. Then $(1, a_1, \dots, a_d)$ is a Macaulay vector if and only if $0 \leq a_{t+1} \leq a_t^{\langle t \rangle}$ for all $t \geq 1$.

For example, let P be a lattice polytope in $\mathbb{R}^n$, i.e., the convex hull of a finite number of points in $\mathbb{Z}^n$. For a natural number a, let $\mathrm{ehr}_P(a)$ be the number of lattice points in the ath dilate of P. It is known that ehr_P is a polynomial in a, called the *Ehrhart polynomial*. Let R_P denote the semigroup algebra of P, see Definition 1.16. The following result is due to Hochster.

Proposition 6.5. [BH93, Theorem 6.3.5] *Let P be a lattice polytope. Then the semigroup algebra R_P is Cohen–Macaulay.*

Furthermore, R_P is a finitely generated module over the subring of R_P which is generated in degree 1. This implies the existence of an l.s.o.p. Note that the Krull dimension of R_P is $\dim P + 1$.

Corollary 6.6. *Let P be a lattice polytope of dimension d. Then we can write*

$$\sum_{a\geq 0}\mathrm{ehr}_P(a)t^a = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}},$$

with $h_i^* \geq 0$ for each i. If, for each a, every lattice point in aP is a sum of a lattice points in P, then $(h_0^*, \dots, h_d^*)$ is a Macaulay vector.

Proof. The Hilbert function of R_P is given by the Ehrhart polynomial, so the result follows from Proposition 6.1 and Proposition 6.5. □

7. Exercises

First exercise session:

(1) Check that the uniform matroid $U_{r,n}$ is realizable.
(2) Prove Proposition 1.5, either directly or by using moment polytopes. Hint for the direct proof: consider the linear forms vanishing on L of minimal support.
(3) Let B_1, B_2 be distinct bases of a matroid M. Show that for each $i \in B_2 \setminus B_1$, there is $j \in B_1 \setminus B_2$ such that $B_1 \cup i \setminus j$ is a basis of M.
(4) Prove that the dimension of $P(\mathrm{M})$ is $n - c$, where c is the number of connected components of M.
(5) (For those familiar with matroids). Given a matroid M on $[n]$ and a subset S of $[n]$ of positive rank, the *S-principal truncation* $T_S\mathrm{M}$ is the matroid whose bases are the sets of the form $B \setminus f$, where B is a basis of M and f is in the closure of S. For example, $T_i\mathrm{M}$ is $M/i \oplus U_{0,1}$ if i is a flat of rank 1. Let M be any matroid, and let S_1, S_2 be subsets of $[n]$. Assuming that all of the relevant principal truncations are defined, show that either there is a matroid polytope subdivision of $P(T_{S_1\cup S_2}T_{S_1\cup S_2}\mathrm{M})$ with interior faces $P(T_{S_1}T_{S_1\cup S_2}\mathrm{M})$, $P(T_{S_2}T_{S_1\cup S_2}\mathrm{M})$, and $P(T_{S_1}T_{S_2}\mathrm{M})$, or all four matroids are equal. Show that the subdivision of $P(U_{2,4})$ in Example 2.3 arises in this way.
(6) Let $S_1, \dots, S_k$ be a collection of subsets of $\binom{[n]}{r}$ such that the size of the symmetric difference of any pair is at least 3. Let M be the matroid with bases $\binom{[n]}{r} \setminus \{S_1, \dots, S_k\}$. Show that M is a connected matroid. Show that

$$\mathrm{N}_i = \{T : T \text{ has symmetric difference of size at most 2 with } S_i\}.$$

is the set of bases of a connected matroid. Show that there is a subdivision of $P(U_{r,n})$ whose $(n-1)$-dimensional interior faces are $P(\mathrm{M}), P(\mathrm{N}_1), \dots, P(\mathrm{N}_k)$.

Second exercise session:

(1) Show that $K(\mathbb{A}^n) \simeq \mathbb{Z}$.
(2) For $a > 0$, using that $[\mathcal{O}(-1)] = [\mathcal{O}] - [\mathcal{O}_H]$, compute the K-class of $[\mathcal{O}_{V(x^a)}]$ in $K(\mathbb{P}^2)$ in the basis given by $[\mathcal{O}], [\mathcal{O}_H], [\mathcal{O}_H]^2$. Use this to compute $\chi(\mathbb{P}^2, \mathcal{O}_{V(x^a)}(-1))$.
(3) Compute $\chi(\mathbb{P}^2, \mathcal{O}_{V(x^a)}(-1))$ by taking a graded free resolution of the module over $k[x_0, x_1, x_2]$ corresponding to $\mathcal{O}_{V(x^a)}$.
(4) Compute $\chi(\mathbb{P}^2, \mathcal{O}_{V(x^a)}(-1))$ by computing the Hilbert polynomial of $V(x^a)$.

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Email address: `mattlarson@princeton.edu`