# Matrix completion and tensor codes

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# Matrix completion

**Definition (Matrix completion matroid)**

Fix $m \leq n$ and $0 \leq d \leq m$. The matrix completion matroid $\mathcal{B}_{m,n}(d,d)$ is the matroid on $[m] \times [n]$ whose bases are the subsets $S$ of size $dm + dn - d^2$ such that, if you fill in the entries in an $m \times n$ matrix labeled by $S$ with generic complex numbers, you can fill in the remaining entries so the matrix has rank at most $d$.

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## Theorem (Bernstein)

$S$ is independent in $\mathcal{B}_{m,n}(2,2)$ if and only if the corresponding bipartite graph has an edge orientation with no alternating or directed cycles.

# Bipartite rigidity

- The bipartite rigidity matroid $\mathcal{B}_{m,n}(a, b)$ is a matroid on $[m] \times [n]$ of rank $na + mb - ab$, introduced by Kalai–Nevo–Novik.
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- Contraction of a matrix completion matroid.
- Hyperconnectivity matroid (Kalai): matroid on $\binom{[n]}{2}$, generalizing skew symmetric matrix completion.
- Symmetric matrix completion matroid: matroid on $\binom{[n]}{2} \sqcup [n]$.

# Tensor codes

- Suppose we have an $m \times n$ array of servers. The data on each server is an element of a field $k$.
- For redundancy, each column is required to lie in a fixed subspace of $k^m$, and each row is required to lie in fixed subspace of $k^n$.
- Suppose the servers labeled by $S$ fail. Can we recover all of the data?
- For simplicity, we will assume that the subspaces are *generic*.

# Tensor codes

- Let $k$ a field of characteristic $p \geq 0$. Let $v_1, \ldots, v_m$ be $m$ *generic* vectors in $k^s$ and $w_1, \ldots, w_n$ be $n$ generic vectors in $k^r$.
- We have $mn$ vectors $v_i \otimes w_j$ in $k^s \otimes k^r$.

**Definition (Tensor matroid)**

Let $\mathrm{T}_{m,n}(s, r, p)$ be the matroid on $[m] \times [n]$ whose bases are the sets $S$ of size $rs$ for which $\{v_i \otimes w_j : (i, j) \in S\}$ is a basis for $k^s \otimes k^r$.

- $S^c$ is spanning in $\mathrm{T}_{m,n}(s, r, p)$ if and only if no data is lost when the servers labeled by $S$ fail.

# Equivalence

- $\mathrm{T}_{m,n}(s, r, p)$: vectors in $k^s \otimes k^r$, where $k$ has characteristic $p$.

**Theorem (Brakensiek–Dhar–Gao–Gopi–L.)**

$\mathcal{B}_{m,n}(a, b)$ is the matroid dual of $\mathrm{T}_{m,n}(m - a, n - b, 0)$.

- No data is lost when the servers labeled by $S$ fail if and only if $S$ is independent in $\mathcal{B}_{m,n}(a, b)$.

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- No data is lost when the servers labeled by $S$ fail if and only if $S$ is independent in $\mathcal{B}_{m,n}(a, b)$.
- The hyperconnectivity matroid is dual to a $\wedge^2$ matroid, and the symmetric matrix completion matroid is dual to a $\mathrm{Sym}^2$ matroid.

# Applications to $m - a$ small

- If $m - a$ is small, then we can analyze $\mathcal{B}_{m,n}(a, b)$ using $\mathsf{T}_{m,n}(m - a, n - b, 0)$.
- We describe cocircuits in $\mathcal{B}_{m,n}(a, b)$ when $m - a \leq 3$ and give a polynomial time algorithm to check independence.

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**Theorem (Brakensiek–Dhar–Gao–Gopi–L.)**

Write $S = \bigcup_{i=1}^{m} \{i\} \times A_i \subseteq [m] \otimes [n]$. Let $S_k$ be the set of $j \in [n]$ which appear in exactly $k$ of the $A_i$. $S$ is independent in $\mathrm{T}_{m,n}(3, r, 0)$ if and only if

$$|A_i \cap A_j \cap A_k \cap A_\ell| = 0$$

$$|A_i \setminus S_3| + |S_3| \leq r$$

$$|(A_i \cap A_j) \setminus S_3| + |(A_k \cap A_\ell) \setminus S_3| + |S_3| \leq r$$

$$|A_i \setminus S_3| + |A_j \setminus S_3| + |S_2 \setminus (S_3 \cup A_i \cup A_j)| + 2|S_3| \leq 2r$$

$$|S_1| + 2|S_2| + 3|S_3| \leq 3r$$

# Applications to $m - a$ small

- We do not know a nice description of the independent sets or circuits of $\mathcal{B}_{m,n}(a,b)$ when $m - a = 3$.
- $\mathcal{B}_{m,n}(a,b)$ has a Laman-like description when $m - a \leq 2$.

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- We do not know a nice description of the independent sets or circuits of $\mathcal{B}_{m,n}(a, b)$ when $m - a = 3$.
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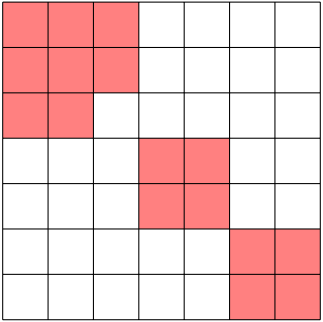

- A circuit of $\mathrm{T}_{m,n}(4, 4, p)$ for any $p$.

# Positive characteristic

- Applications use $\mathrm{T}_{m,n}(s, r, p)$ when $p > 0$, especially $p = 2$.
- We have

$$\mathrm{T}_{m,n}(m-d, n-d, p)^{\perp} \subseteq \text{matrix completion in char } p \subseteq \mathcal{B}_{m,n}(d, d).$$

**Theorem (Brakensiek–Dhar–Gao–Gopi–L.)**

If $s \leq 3, m - s \leq 1$, or $m - s = n - r = 2$, then $\mathrm{T}_{m,n}(s, r, p)$ is independent of $p$.

# Rank 2 matrix completion in characteristic p

**Theorem (Bernstein)**

$S$ is independent in $\mathcal{B}_{m,n}(2,2)$ if and only if the corresponding bipartite graph has an edge orientation with no alternating or directed cycles.

- We show that if the corresponding bipartite graph has an edge orientation with no alternating or directed cycles, then $S$ is independent in the dual of $\mathrm{T}_{m,n}(m-2, n-2, p)$ for any $p$.
- We prove a determinant is nonzero by constructing an explicit monomial with coefficient $\pm 1$. Bicoloring the edges of a bipartite graph is equivalent to orienting the edges.

# Linear algebraic matroids

- In all examples, $\mathrm{T}_{m,n}(s, r, p)$ is independent of $p \geq 0$.
- Also true for the $\wedge^2$ matroid, and for the $\mathrm{Sym}^2$ matroid except when $p = 2$.

**Theorem (Bernstein)**

$S$ is independent in the rank 2 skew-symmetric matrix completion matroid if and only if the corresponding graph has an edge orientation with no directed cycles or alternating closed trails.

- Our argument produces bases (in any characteristic) which satisfy a different-looking condition.