# CS 170 Dis 9

**Released on 2017-03-21**

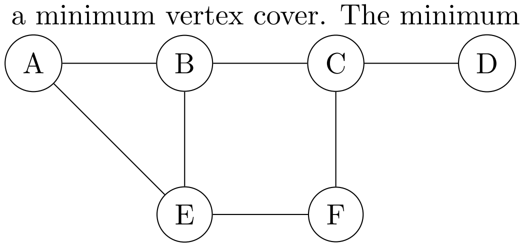
## 1 Maximal Matching

Let $G = (V, E)$ be a (not necessarily bipartite) undirected graph. A *maximal matching*, $M$, is a matching in which no edge can be added while keeping it a matching. Show that the size of any maximal matching is at least half the size of a maximum matching $M^*$.

**Solution:** Assume for contradiction that there is a maximal matching $M$ whose size is less than half the size of a maximal matching $M^*$. We will show that we can improve the matching on $M$ by adding an edge from $M^*$, contradicting the claim that $M$ is maximal. To see this note that for each $(u, v) \in M$, there are at most two edges in $M^*$ incident on $u$ or $v$ (one for each vertex). Therefore, at most $2|M|$ edges in $M^*$ are incident on some vertex appearing in $M$. Since $2|M| < |M^*|$, there is an edge $e \in M^*$ not incident on any vertex in $M$. So $M \cup \{e\}$ is a matching, and $M$ is not maximal.

## 2 Reducing Vertex Cover to Set Cover

In the minimum vertex cover problem, we are given an undirected graph $G(V, E)$ and asked to find the smallest set $U \subseteq V$ that "covers" the set of edges $E$. In other words, we want to find the smallest set $U$ such that for each $(u, v) \in E$, either $u$ or $v$ is in $U$ ($U$ is not necessarily unique). For example, in the following graph, $\{A, E, C, D\}$ is a vertex cover, but not a minimum vertex cover. The minimum vertex covers are $\{B, E, C\}$ and $\{A, E, C\}$.




Recall the following definition of the minimum Set Cover problem: Given a set $U$ of elements and a collection $S_1, \ldots, S_m$ of subsets of $U$, what is the smallest collection of these sets whose union equals $U$? So, for example, given $U := \{a, b, c, d\}$, $S_1 := \{a, b, c\}$, $S_2 := \{b, c\}$, and $S_3 := \{c, d\}$, a solution to the problem is the collection of $S_1$ and $S_3$.

Give an efficient reduction from the Minimum Vertex Cover Problem to the Minimum Set Cover Problem.

**Solution:** Let $G = (V, E)$ be an instance of the Minimum Vertex Cover Problem. Create an instance of the Minimum Set Cover Problem where $U = E$ and for each $u \in V$, the set $S_u$ contains all edges adjacent to $u$. Let $C = \{S_{u_1}, S_{u_2}, \ldots, S_{u_k}\}$ be a set cover. Then our corresponding vertex cover will be $u_1, u_2, \ldots, u_k$. To see this is a vertex cover, take any $(u, v) \in E$. Since $(u, v) \in U$, there is some set $S_{u_i}$ containing $(u, v)$, so $u_i$ equals $u$ or $v$ and $(u, v)$ is covered in the vertex cover.

Now take any vertex cover $u_1, \ldots, u_k$. To see that $S_{u_1}, \ldots, S_{u_k}$ is a set cover, take any $(u, v) \in E$. By the definition of vertex cover, there is an $i$ such that either $u = u_i$ or $v = u_i$. So $(u, v) \in S_{u_i}$, so $S_{u_1}, \ldots, S_{u_k}$ is a set cover.

Since every vertex cover has a corresponding set cover (and vice-versa) and minimizing set cover minimizes the corresponding vertex cover, the reduction holds.

## 3 Midterm Discussion