# CS 170 DIS 04

### Released on 2018-02-14

## 1 Worst-case Instances for Greedy Set-Cover

Recall the set cover problem:

Input: A set of elements $B$ and sets $S_1, \ldots, S_m \subseteq B$.

Output: A selection of the $S_i$ whose union is $B$ (i.e. that contain every element of $B$).

Cost: Number of sets picked.

The natural strategy to solve this problem is a greedy approach: At every step, pick the set that covers the most uncovered elements of $B$. In the book, we proved that this greedy strategy over-estimates the optimal number of sets by a factor of at most $O(\log n)$, where $n = |B|$. In this problem we will prove that this bound is tight.

Show that for any integer $n$ that is a power of 2, there is an instance of the set cover problem (i.e. a collection of sets $S_1, \ldots, S_m$) with the following properties:

i. There are $n$ elements in the base set $B$.

ii. The optimal cover uses just two sets.

iii. The greedy algorithm picks at least $O(\log n)$ sets.

## 2 Money Changing.

Fix a set of positive integers called *denominations* $x_1, x_2, \ldots, x_n$ (think of them as the integers 1, 5, 10, and 25). The problem you want to solve for these denominations is the following: Given an integer $A$, express it as

$$A = \sum_{i=1}^{n} a_i x_i$$

for some nonnegative integers $a_1, \ldots, a_n \geq 0$.

1. Under which conditions on the denominations $x_i$ are you able to do this for all integers $A > 0$?

2. Given any set of denominations $x_i$, not necessarily satisfying the conditions in the first part, describe the set of *sufficiently large* integers $A$ that you can express as $A = \sum_{i=1}^{n} a_i x_i$ with nonnegative $a_i$. In other words you should prove a statement like "If $A$ exceeds $X$, then we can write $A = \sum_{i=1}^{n} a_i x_i$ for $a_i \geq 0$ if and only if $A$ has the following simple property...." You do not have to give $X$ explicitly, but prove it exists.

3. Suppose that you want, given $A$, to find the nonnegative $a_i$'s that satisfy $A = \sum_{i=1}^{n} a_i x_i$, and such that the sum of all $a_i$'s is minimal —that is, you use the smallest possible number of coins. Define a *greedy algorithm* for this problem.

4. Show that the greedy algorithm finds the optimum $a_i$'s in the case of the denominations 1, 5, 10, and 25, and for any amount $A$.
5. Give an example of a denomination where the greedy algorithm fails to find the optimum $a_i$'s for some $A$. Do you know of an actual country where such a set of denominations exists?
6. How far from the optimum number of coins can the output of the greedy algorithm be, as a function of the denominations?

## 3 Midterm Discussion