CS 170 HW 4

Due on 2017-02-19, at 11:59 pm

1 (★) Study Group

List the names and SIDs of the members in your study group.

2 (★) Short Answer Questions

(a) Let G be a dag and suppose the vertices $v_1, \ldots, v_n$ are in topologically sorted order. If there is a path from v_i to v_j in G, are we guaranteed that $i < j$?

(b) Let $G = (V, E)$ be any strongly connected directed graph. Is it always possible to remove one vertex (as well as edges to and from it) from the graph so that the remaining directed graph is strongly connected?

(c) Give an example where the greedy set cover algorithm does not produce an optimal solution.

(d) Let G be a connected undirected graph with positive lengths on all the edges. Let s be a fixed vertex. Let $d(s, v)$ denote the distance from vertex s to vertex v, i.e., the length of the shortest path from s to v. If we choose the vertex v that makes $d(s, v)$ as small as possible, subject to the requirement that $v \neq s$, then does every edge on the path from s to v have to be part of every minimum spanning tree of G?

(e) The same question as above, except now no two edges can have the same length.

3 (★★★) Arbitrage

Shortest-path algorithms can also be applied to currency trading. Suppose we have n currencies $C = \{c_1, c_2, \ldots, c_n\}$: e.g., dollars, Euros, bitcoins, dogecoins, etc. For any pair i, j of currencies, there is an exchange rate $r_{i,j}$: you can buy $r_{i,j}$ units of currency c_j at the price of one unit of currency c_i. Assume that $r_{i,i} = 1$ and $r_{i,j} \geq 0$ for all i, j.

(a) The Foreign Exchange Market Organization (FEMO) has hired Oski, a CS170 alumnus, to make sure that it is not possible to generate a profit through a cycle of exchanges; that is, for any currency $i \in C$, it is not possible to start with one unit of currency i, perform a series of exchanges, and end with more than one unit of currency i. (That is called *arbitrage*.) Give an efficient algorithm for the following problem: given a set of exchange rates $r_{i,j}$ and two specific currencies s, t, find the most advantageous sequence of currency exchanges for converting currency s into currency t. We recommend that you represent the currencies and rates by a graph whose edge lengths are real numbers.

[Provide 4 part solution.]

(b) In the economic downturn of 2016, the FEMO had to downsize and let Oski go, and the currencies are changing rapidly, unfettered and unregulated. As a responsible citizen and in light of what you saw in lecture this week, this makes you very concerned: it may now be possible to find currencies $c_{i_1}, \ldots, c_{i_k}$ such that $r_{i_1,i_2} \times r_{i_2,i_3} \times \cdots \times r_{i_{k-1},i_k} \times r_{i_k,i_1} > 1$. This means that by starting with one unit of currency c_{i_1} and then successively converting it to currencies $c_{i_2}, c_{i_3}, \ldots, c_{i_k}$ and finally back to c_{i_1}, you would end up with more than one unit of currency c_{i_1}. Such anomalies last only a fraction of a minute on the currency exchange, but they provide an opportunity for profit.

You decide to step up to help out the World Bank. Given an efficient algorithm for detecting the presence of such an anomaly. You may use the same graph representation as for part (a).

[Provide 4 part solution.]

4 (★★) Proof of Correctness for Greedy Algorithms

This question guides you through writing a proof of correctness for a greedy algorithm. A doctor's office has n customers, labeled $1, 2, \ldots, n$, waiting to be seen. They are all present right now and will wait until the doctor can see them. The doctor can see one customer at a time, and we can predict exactly how much time each customer will need with the doctor: customer i will take $t(i)$ minutes.

(a) We want to minimize the average waiting time (the average of the amount of time each customer waits before they are seen, not counting the time they spend with the doctor). What order should we use? You do not need to justify your answer for this part. (Hint: sort the customers by ____)

(b) Let $x_1, x_2, \ldots, x_n$ denote an ordering of the customers (so we see customer x_1 first, then customer x_2, and so on). Prove that the following modification, if applied to any order, will never increase the average waiting time:

- If $i < j$ and $t(x_i) \geq t(x_j)$, swap customer i with customer j.

(For example, if the order of customers is $3, 1, 4, 2$ and $t(3) \geq t(4)$, then applying this rule with $i = 1$ and $j = 3$ gives us the new order $4, 1, 3, 2$.)

(c) Let u be the ordering of customers you selected in part (a), and x be any other ordering. Prove that the average waiting time of u is no larger than the average waiting time of x—and therefore your answer in part (a) is optimal.

Hint: Let i be the smallest index such that $u_i \neq x_i$. Use what you learned in part (b). Then, use proof by induction (maybe backwards, in the order $i = n, n-1, n-2, \ldots, 1$, or in some other way).

5 (★★★) Preventing Conflict

A group of n guests shows up to a house for a party, and any two guests are either friends or enemies. There are two rooms in the house, and the host wants to distribute guests among

the rooms, breaking up as many pairs of enemies as possible. The guests are all waiting outside the house and are impatient to get in, so the host needs to assign them to the two rooms quickly, even if this means that it's not the best possible solution. Come up with a linear-time algorithm that breaks up at least half the number of pairs of enemies as the best possible solution, and prove your answer.

You can treat the input as a graph $G = (V, E)$ with edge relations denoting enemies.

[Provide 4 Part Solution]

6 (★★) Updating a MST

You are given a graph $G = (V, E)$ with positive edge weights, and a minimum spanning tree $T = (V, E')$ with respect to these weights; you may assume G and T are given as adjacency lists. Now suppose the weight of a particular edge $e \in E$ is modified from $w(e)$ to a new value $\hat{w}(e)$. You wish to quickly update the minimum spanning tree T to reflect this change, without recomputing the entire tree from scratch. There are four cases. In each, give a linear time algorithm for updating the tree. For each, use the four part algorithm format. However, for some of the cases the main idea, proof and justification of running time may be quite brief. Also, you may either write pseudocode for each case separately, or write a single algorithm which includes all cases.

(a) $e \notin E'$ and $\hat{w}(e) > w(e)$

(b) $e \notin E'$ and $\hat{w}(e) < w(e)$

(c) $e \in E'$ and $\hat{w}(e) < w(e)$

(d) $e \in E'$ and $\hat{w}(e) > w(e)$