CS 170 DIS 07

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1 Chocolate Milk Factory

You have a chocolate factory that makes dark and milk chocolate. You profit \$3 per gallons of dark chocolate and \$5 per gallons of milk chocolate. You wish to maximize your profit, but you do have some constraints. You cannot make negative amounts of anything, and you can make at most 400 gallons of chocolate combined.

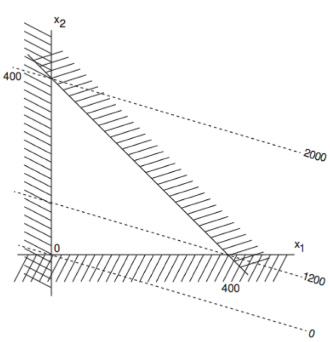
(a) Give a constrained optimization formulation of the problem.

Solution: Let x_1, x_2 respectively be the gallons of dark and milk chocolate.

$$\begin{aligned} &\max\ 3x_1 + 5x_2 \\ &\text{s.t. }\ x_1 \geq 0 \\ &\qquad x_2 \geq 0 \\ &\qquad x_1 + x_2 \leq 400 \end{aligned}$$

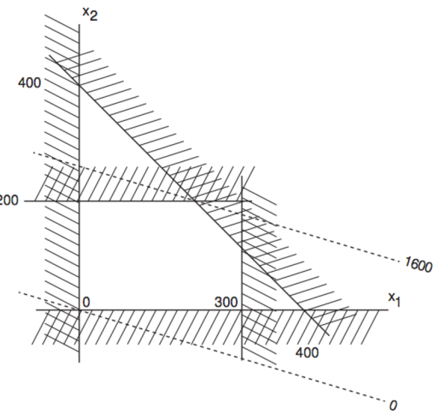
(b) Draw the feasible region. Identify the optimal solution and draw the contour lines of the objective to demonstrate the optimality of the solution.

Solution: The contours of the objective are $\mathcal{C}_c = \{(x_1, x_2) | 3x_1 + 5x_2 = c\}$. The optimal solution is $(0, 400)$.



(c) Solve again with the additional constraint that you can't make more than 300 gallons of dark and 200 gallons of milk chocolate.

Solution: $(200, 200)$.



2 Job Assignment

There are I people available to work J jobs. The value of person i working 1 day at job j is a_{ij} for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Each job is completed after it has been worked on for 1 day (summed over all workers), though partial completion still has value (ex. person i working half a day on job j is worth $a_{ij}/2$). The problem is to find an optial assignment of jobs for each person for one day.

(a) What variables should we optimize over? I.e. in the canonical linear programming definition, what is x?

Solution: An assignment x is a choice of numbers x_{ij} where x_{ij} is the portion of person i's time spent on job j.

(b) What are the constraints we need to consider? Hint: there are three major types.

Solution: First, no person i can work more than 1 day's worth of time.

$$\sum_{j=1}^{J} x_{ij} \leq 1 \qquad \text{for } i = 1, \ldots, I.$$

Second, no job j can be worked past completion:

$$\sum_{i=1}^{I} x_{ij} \leq 1 \qquad \text{for } j = 1, \ldots, J.$$

Third, we require positivity.

$$x_{ij} \geq 0 \qquad \text{for } i = 1, \ldots, I, j = 1, \ldots, J.$$

(c) What is the maximization function we are seeking?

Solution: By person i working job j for x_{ij}, they contribute value $a_{ij}x_{ij}$. Therefore, the net value is

$$\sum_{i=1,j=1}^{I,J} a_{ij}x_{ij} = A \bullet x.$$

3 Understanding convex polytopes

So far in this class we have seen linear programming defined as

$$(\mathcal{P}) = \begin{cases} \max & c^T x \\ \text{s.t.} & Ax \leq b. \end{cases}$$

Today, we explore the different properties of the region $\Omega = \{x : Ax \leq b\}$ – i.e. the region that our linear program maximizes over.

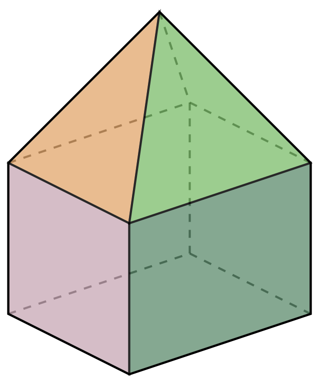


Figure 1: An example of a convex polytope. We can consider each face of the polytope as an affine inequality and then the polytope is all the points that satisfy each inequality. Notice that an affine inequality defines a half-plane and therefore is also the intersection of the half-planes.

(a) The first property that we will be interested in is *convexity*. We say that a space X is convex if for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)y \in X.$$

That is, the entire line segment $\overline{xy}$ is contained in X. Prove that Ω is indeed convex.

Solution: Let $x, y \in \Omega$. We need to show that

$$A(\lambda x + (1 - \lambda)y)) \leq b.$$

We apply the only facts we know, namely $Ax \leq b$ and $Ay \leq b$.

$$\begin{aligned} A(\lambda x + (1-\lambda)y) &= \lambda Ax + (1-\lambda)Ay \\ &\leq \lambda b + (1-\lambda)b \\ &= b. \end{aligned}$$

(b) The second property that we will be interested in is showing that linear maximations over convex polytopes achieve their maximums at the vertices. A vertex is any point $v \in \Omega$ such that v cannot be expressed as a point on the line $\overline{yz}$ for $v \neq y, v \neq z$, and $y, z \in \Omega$.

Prove the following statement: Let Ω be a convex space and f a linear function $f(x) = c^T x$. Show that the for a line $\overline{yz}$ for $y, z \in \Omega$ that $f(x)$ is maximized on the line at either y or z. I.e. show that

$$\max_{\lambda \in [0,1]} f(\lambda y + (1-\lambda)z)$$

achieves the maximum at either $\lambda = 0$ or $\lambda = 1$.

Solution: Assume without loss of generality that $f(y) \geq f(z)$. (Otherwise, swap their names). Then $c^T y \geq c^T z$. We now aim to show the maximum is achieved at $\lambda = 1$. Then,

$$\begin{aligned} f(\lambda y + (1-\lambda)z) &= c^T(\lambda y + (1-\lambda)z) \\ &= \lambda c^T y + (1-\lambda)c^T z \\ &\leq \lambda c^T y + (1-\lambda)c^T y \\ &= f(y). \end{aligned}$$

(c) Now, prove that global maximums will be achieved at vertices. For simplicity, you can assume there is a unique global maximum. Hint: Use the definition of a vertex presented above.

Solution: Assume, for contradiction, that the maximum was not achieved at a vertex and was instead achieved at a point x that was *not* a vertex. Then, there exists a line y, z containing x such that $x \neq y$ and $x \neq z$. But by the previous argument, the function achieves a maximum at either y or z. then, the maximum isn't unique. A contradiction.

This argument is the basis of the Simplex algorithm by Dantzig to solve linear programs.