CS 170 HW 8

Due on 2018-03-18, at 11:59 pm

1 (★) Study Group

List the names and SIDs of the members in your study group.

2 (★★) Linear Programming Warm-ups

1. Find necessary and sufficient conditions on the reals a and b under which the linear program

$$\begin{aligned} \max\ & x + y \\ ax + by &\leq 1 \\ x, y &\geq 0 \end{aligned}$$

   i Is infeasible.

   ii Is unbounded.

   iii Has a unique optimal solution.

2. Write the dual to following linear program.

$$\begin{aligned} \max x + y & \\ 2x + y &\leq 3 \\ x + 3y &\leq 5 \\ x, y &\geq 0 \end{aligned}$$

   Find optimal solutions to both primal and dual LPs.

Solution:

1. The solutions to each part are as follows:

   i This LP is never infeasible as the origin will satisfy $ax + by \leq 1$ for any choice of a and b.

   ii It is sufficient that $a \leq 0$ or $b \leq 0$. If $a \leq 0$, then we can increase x (and the objective function) arbitrarily without violating any constraint. The same argument works for b and y. Conversely, suppose both a and b are positive. Let $m = \min\{a, b\}$ and notice $m > 0$. Then, $m(x + y) \leq ax + by \leq 1$, so that $x + y \leq 1/m$. Hence, the LP cannot be unbounded.

iii By a) and b), the LP has a finite optimal when a and b are positive. Suppose now $a > b$. Then, the optimal is clearly uniquely achieved at $y = \frac{1}{b}$. Similarly, if $b > a$. the unique optimum is $x = \frac{1}{a}$. However, if $a = b$, then any positive pair (x, y) such that $x + y = \frac{1}{a}$ achieves the optimum. Hence, the optimum exists and is unique if and only if a, b are positive and $a \neq b$.

2. The dual LP is:

$$\begin{aligned} \min\ 3u &\ + \ 5w \\ 2u + w &\geq 1 \\ u + 3w &\geq 1 \\ u, w &\geq 0 \end{aligned}$$

The optimal solution for the primal is $\frac{11}{5}$ given by $(x, y) = \left(\frac{4}{5}, \frac{7}{5}\right)$. The corresponding dual optimum is given by $(u, w) = \left(\frac{2}{5}, \frac{1}{5}\right)$.

3 (★★★) Provably Optimal

For the linear program

$$\begin{aligned} \max\ x_1 - 2x_3 & \\ x_1 - x_2 &\leq 1 \\ 2x_2 - x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

show that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal **using its dual**. You should not have to directly solve for the optimum of the dual.

(*Hint:* Recall that any feasible solution of the dual is an upper bound on any feasible solution of the primal)

Solution: The objective value at the claimed optimum is $3/2$. By the duality theorem, this would be optimum if and only if there is a feasible solution to the dual LP with the same objective value. The dual of the given LP is

$$\begin{aligned} \min \quad y_1 + y_2 & \\ y_1 &\geq 1 \\ -y_1 + 2y_2 &\geq 0 \\ -y_2 &\geq -2 \\ y_1, y_2 &\geq 0 \end{aligned} \qquad \equiv \qquad \begin{aligned} \min \quad y_1 + y_2 & \\ y_1 &\geq 1 \\ y_2 &\geq \frac{y_1}{2} \\ y_2 &\leq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

We then see that $y_1 = 1, y_2 = 1/2$ is a feasible dual solution, with the objective value $3/2$. Thus, the claimed primal optimal is indeed an optimal solution.

4 (★★★) Major Key

You are a locksmith tasked with producing keys $k_1, ..., k_n$ that sell for $p_1, ..., p_n$ respectively. Each key k_i takes g_i grams of gold and s_i grams of silver. You have a total of G gold and S silver to work with, and can produce as many keys of any type as you want within the time and material constraints.

1. Unfortunately, integer linear programming is an NP-complete problem. Fortunately, you have found someone to instead buy the alloys at an equivalent price! Instead of selling keys, you have decided to focus on melting the prerequisite metals together, and selling the mixture. Formulate the linear program to maximize the profit of the locksmith, and explain your decision variables, objective function, and constraints.

2. Formulate the dual of the linear program from part (a), and explain your decision variables, objective function, and constraints. The explanations provide economic intuition behind the dual. We will only be grading the dual formulation.
   Hint: Formulate the dual first, then think about it from the perspective of the locksmith when negotiating prices for buying G gold and S silver if they had already signed a contract for the prices for the output alloys p_i. Think about the breakeven point, from which the locksmith's operations begin to become profitable for at least one alloy.

Solution:

(a) The decision variables x_i correspond to the amount of mixture created for each key.

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} x_i p_i && \text{(Maximize profit)} \\ & \sum_{i=1}^{n} x_i g_i \leq G && \text{(Use at most } G \text{ grams of gold)} \\ & \sum_{i=1}^{n} x_i s_i \leq S && \text{(Use at most } S \text{ grams of silver)} \\ x_i \geq 0 \quad & \forall i \in [1...n] && \text{(Cannot produce negative amounts of metal)} \end{aligned}$$

(b) The decision variables y_G, y_S correspond to the prices of the means of production: gold, and silver. The dual poses the following question: if the prices of gold and silver were originally too high for the locksmith's operations to be profitable, how low can they drop before breaking even? The solution returns the breakeven point; lower prices would finally allow the locksmith to become profitable.

$$\begin{aligned} \min \quad & G y_G + S y_S && \text{(Minimize total cost of materials)} \\ g_i y_G + s_i y_S \geq p_i \quad & \forall i \in [1...n] && \text{(Cost for producing mixture } i \text{ is at least } p_i) \\ & y_G, y_S \geq 0 && \text{(Cannot set negative prices)} \end{aligned}$$

5 (★★) Zero-Sum Battle

Two Pokemon trainers are about to engage in battle! Each trainer has 3 Pokemon, each of a single, unique type. They each must choose which Pokemon to send out first. Of course each trainer's advantage in battle depends not only on their own Pokemon, but on which Pokemon their opponent sends out.

The table below indicates the competitive advantage (payoff) Trainer A would gain (and Trainer B would lose). For example, if Trainer B chooses the fire Pokemon and Trainer A chooses the rock Pokemon, Trainer A would have payoff 2.

		Trainer B:		
		ice	grass	fire
Trainer A:	dragon	-10	3	3
	steel	4	-1	-3
	rock	6	-9	2

Feel free to use an online LP solver to solve your LPs in this problem.
Here is an example of a Python LP Solver and its Tutorial.

1. Write an LP to find the optimal strategy for Trainer A. What is the optimal strategy and expected payoff?

2. Now do the same for Trainer B. What is the optimal strategy and expected payoff?

Solution:

1. d = probability that A picks the dragon type
   s = probability that A picks the steel type
   r = probability that A picks the rock type

$$
\begin{aligned}
\max \quad & z \\
-10d + 4s + 6r &\geq z && \text{(B chooses ice)} \\
3d - s - 9r &\geq z && \text{(B chooses grass)} \\
3d - 3s + 2r &\geq z && \text{(B chooses fire)} \\
d + s + r &= 1 \\
d, s, r &\geq 0
\end{aligned}
$$

   The optimal strategy is $d = 0.3346,\ s = 0.5630,\ r = 0.1024$ for an optimal payoff of -0.48.

2. i = probability that B picks the ice type
   g = probability that B picks the grass type
   f = probability that B picks the fire type

$$
\begin{aligned}
\min \quad & z \\
-10i + 3g + 3f &\leq z && \text{(A chooses dragon)} \\
4i - g - 3f &\leq z && \text{(A chooses steel)} \\
6i - 9g + 2f &\leq z && \text{(A chooses rock)} \\
i + g + f &= 1 \\
i, g, f &\geq 0
\end{aligned}
$$

B's optimal strategy is $i = 0.2677,\ g = 0.3228,\ f = 0.4094$. The value for this is -0.48, which is the payoff for A. The payoff for B is 0.48, since the game is zero-sum.

(Note for grading: Equivalent LPs are of course fine. It is fine for part (b) to maximize B's payoff instead of minimizing A's. For the strategies, fractions or decimals close to the solutions are fine, as long as the LP is correct.)