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# Assignment One

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## 1 Probability and Independence

For this question we prove or disprove the following properties of independence.

### 1.1 $(X \perp Y, W \mid Z)$ **IMPLIES** $(X \perp Y \mid Z)$ **is TRUE**

PROOF: Using the law of total probability and the conditional independence of  $X$  and the joint  $Y, W$  on  $Z$  it is clear that:

$$\begin{aligned} P(X, Y \mid Z) &= \int_W P(X, Y, W \mid Z) dW = \int_W P(X \mid Z) P(Y, W \mid Z) dW \\ &= P(X \mid Z) \int_W P(Y, W \mid Z) dW \\ &= P(X \mid Z) P(Y \mid Z) \end{aligned}$$

### 1.2 $(X \perp Y \mid Z)$ **AND** $(X, Y \perp W \mid Z)$ **IMPLIES** $(X \perp W \mid Z)$ **is TRUE**

PROOF: Notice that, by symmetry,  $(X, Y \perp W \mid Z) \Rightarrow (W \perp X, Y \mid Z)$ , and so by 1.1 we have that  $(W \perp X \mid Z)$  (and hence  $(X \perp W \mid Z)$  )

### 1.3 $(X \perp Y, W \mid Z)$ AND $(Y \perp W \mid Z)$ IMPLIES $(X, W \perp Y \mid Z)$ is TRUE

PROOF: First, notice  $(X \perp Y, W \mid Z) \implies (X \perp W \mid Z)$  by 1.1. Then, clearly:

$$\begin{aligned} P(X, Y, W \mid Z) &= P(X \mid Z)P(Y, W \mid Z) = P(X \mid Z)P(Y \mid Z)P(W \mid Z) \\ &= P(X, W \mid Z)P(Y \mid Z) \end{aligned}$$

### 1.4 $(X \perp Y \mid Z)$ AND $(X \perp Y \mid W)$ IMPLIES $(X \perp Y \mid W, Z)$ is FALSE

COUNTER EXAMPLE: Take the following Probability space:  $\Omega = \{1, 2, 3, \dots, 16\}$  as the Sample space and  $2^\Omega$  as the  $\sigma$ -algebra equipped with the Counting Measure. Consider the following random variables:

$$\begin{aligned} Z &= \mathbb{1}_{\{1,2,3,4,7,8,11,12,13\}} \\ W &= \mathbb{1}_{\{1,2,5,6,9,10,14,15,16\}} \\ X &= \mathbb{1}_{\{1,3,5,7,9\}} \\ Y &= \mathbb{1}_{\{2,7,8,9,10\}} \end{aligned}$$

Notice that

$$\begin{aligned} P(X, Y \mid Z) &= \frac{1}{9} = P(X \mid Z)P(Y \mid Z) \\ P(X, Y \mid W) &= \frac{1}{9} = P(X \mid W)P(Y \mid W) \end{aligned}$$

but

$$P(X, Y \mid Z, W) = 0 \neq P(X \mid Z, W)P(Y \mid Z, W) = \frac{1}{4}$$

## 2 Bayesian Inference and MAP

### 2.1 Given IID Data, what are the conditional independence properties for $P(\pi, x_1, \dots, x_n)$ ?

Denote  $X_A := \{X_i\}_{i \in A}$ . We can see that  $(X_A \perp X_B \mid \pi)$  for any non-intersecting collection  $A, B \subset \{1, 2, \dots, n\}$ , since, using the IID property

$$\begin{aligned} P(X_A, X_B \mid \pi) &= \prod_{i \in A \cup B} P(X_i \mid \pi) \\ &= P(X_A \mid \pi)P(X_B \mid \pi) \end{aligned}$$

i.e. the  $X_i$ 's are mutually independent, conditioned on  $\pi$ .

## 2.2 Derive $P(\pi \mid x^{(1)}, \dots, x^{(n)})$

Note that, ignoring normalization constants we can see that:

$$\begin{aligned} P(\pi \mid x^{(1)}, \dots, x^{(n)}) &\propto P(x^{(1)}, \dots, x^{(n)} \mid \pi) P(\pi \mid \alpha) \\ &\propto \prod_{j=1}^k \pi_j^{\sum_{i=1}^n x_j^{(i)}} \prod_{j=1}^k \pi_j^{\alpha_j - 1} = \prod_{j=1}^k \pi_j^{N_j + \alpha_j - 1} \end{aligned}$$

Where  $N_j = \sum_{i=1}^n x_j^{(i)}$ . We recognize the above as the unnormalized density of a Dirichlet Random Variable, and so  $\pi \mid x^{(1)}, \dots, x^{(n)} \sim \text{Dirichlet}(\{N_j + \alpha_j\}_{j=1}^k)$

## 2.3 Derive $P(x^{(1)}, \dots, x^{(n)})$

To compute the data density we integrate out  $\pi$

$$\begin{aligned} P(x^{(1)}, \dots, x^{(n)}) &= \int_{\pi \in \Delta_k} P(\pi, x^{(1)}, \dots, x^{(n)}) d\pi \\ &= \int_{\pi \in \Delta_k} P(x^{(1)}, \dots, x^{(n)} \mid \pi) P(\pi \mid \alpha) d\pi \\ &= \int_{\pi \in \Delta_k} \prod_{j=1}^k \pi_j^{N_j} \frac{1}{C(\alpha)} \prod_{j=1}^k \pi_j^{\alpha_j - 1} d\pi \end{aligned}$$

Where  $C(\alpha) = C(\alpha_1, \dots, \alpha_k) = \frac{\prod_{j=1}^k \Gamma(\alpha_j)}{\Gamma(\sum_{j=1}^k \alpha_j)}$

$$\begin{aligned} &= \frac{1}{C(\alpha)} \int_{\pi \in \Delta_k} \prod_{j=1}^k \pi_j^{N_j + \alpha_j - 1} d\pi \\ &= \frac{C(N + \alpha)}{C(\alpha)} \end{aligned}$$

Since  $\frac{1}{C(N + \alpha)} \prod_{j=1}^k \pi_j^{N_j + \alpha_j - 1} \sim \text{Dirichlet}(N + \alpha)$

## 2.4 Derive MAP estimate $\hat{\pi}$ for $\pi$ and compare it to the MLE estimator

We assume that  $\alpha_j > 1 \forall j$ .

Notice that

$$\begin{aligned} \max_{\pi \in \Delta_k} P(\pi \mid x^{(1)}, \dots, x^{(n)}) &= \max_{\pi \in \Delta_k} P(x^{(1)}, \dots, x^{(n)} \mid \pi) P(\pi \mid \alpha) \\ &= \max_{\pi \in \Delta_k} \prod_{j=1}^k \pi_j^{N_j + \alpha_j - 1} \end{aligned}$$

And using the monotonicity of  $\log$  along with the definition of  $\Delta_k$  we can write this as the equivalent constrained optimization:

$$\begin{aligned} & \text{Maximize } l(\pi) = \sum_{j=1}^k (N_j + \alpha_j - 1) \log(\pi_j) \\ & \text{subject to the constraint } g(\pi) = 1 - \sum_{i=1}^k \pi_i = 0 \end{aligned}$$

We solve this using the method of Lagrange Multipliers. We look for any stationary points i.e.

$$\begin{aligned} \nabla_{\pi}(l(\pi) + \lambda g(\pi)) &= 0 \\ \nabla_{\lambda}(l(\pi) + \lambda g(\pi)) &= 0 \end{aligned}$$

And solving for each  $\pi_j$  yields

$$\begin{aligned} \nabla_{\pi_j}(l(\pi_j) + \lambda g(\pi_j)) &= \frac{N_j + \alpha_j - 1}{\pi_j} - \lambda = 0 \\ \Rightarrow \pi_j &= \frac{N_j + \alpha_j - 1}{\lambda} \end{aligned}$$

And solving for  $\lambda$  yields

$$\begin{aligned} \nabla_{\lambda}(l(\pi) + \lambda g(\pi)) &= 1 - \sum_{i=1}^k \pi_i = 0 \\ \Rightarrow 1 - \sum_{i=1}^k \frac{N_i + \alpha_i - 1}{\lambda} &= 0 \\ \Rightarrow \lambda &= \sum_{i=1}^k N_i + \alpha_i - 1 = N + \sum_{i=1}^k (\alpha_i - 1) \end{aligned}$$

And putting the above together, we get that each  $\pi_j = \frac{N_j + \alpha_j - 1}{N + \sum_{i=1}^k (\alpha_i - 1)}$ .

To check that our stationary point is a maximum, we compute the determinant of the Hessian and check if it is negative.

If we compared  $\hat{\pi}^{MLE}$  with  $\hat{\pi}^{MAP}$ , as  $k$  becomes very large, for any  $k$ ,  $\hat{\pi}_k^{MAP}$  will shrink since the probabilities of each  $\hat{\pi}_k^{MAP}$  will be necessarily non-zero, whereas  $\hat{\pi}_j^{MLE} = 0$  whenever  $N_k = 0$  (which would be very frequent when  $k$  gets larger than  $N$ ).

### 3 Properties of estimators

#### 3.1 Find MLE of $X_1, X_2, \dots, X_n \stackrel{iid}{\sim} \text{Poisson}(\lambda)$ and determine its Bias, Variance and Consistency

We want to find a  $\lambda$  to maximize the following:

$$L(\lambda \mid x_1, x_2, \dots, x_n) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

to simplify the above, we take the log:

$$l(\lambda) = -n\lambda + \log(\lambda) \sum_{i=1}^n x_i - \log(x_i!)$$

We differentiate with respect to  $\lambda$  and find the stationary point  $\lambda^*$ .

$$\begin{aligned} \partial l(\lambda) &= -n + \frac{\sum_{i=1}^n x_i}{\lambda} = 0 \\ \Rightarrow \lambda^* &= \frac{\sum_{i=1}^n x_i}{n} \end{aligned}$$

The second derivative is negative for all  $\lambda \neq 0$ , and so  $\lambda^*$  is a maximum.

$$\partial^2 l(\lambda) = -\frac{\sum_{i=1}^n x_i}{\lambda^2} < 0$$

We compute the mean and variance of  $\lambda^{MLE}$

$$E(\lambda^{MLE}) = \frac{\sum_{i=1}^n E(X_i)}{n} = \frac{n\lambda}{n} = \lambda \quad (3.1)$$

$$Var(\lambda^{MLE}) = \frac{Var(\sum_{i=1}^n X_i)}{n^2} = \frac{\sum_{i=1}^n Var(X_i)}{n^2} = \frac{\lambda}{n} \quad (3.2)$$

From (3.1) we see that  $\lambda^{MLE}$  is unbiased, and using (3.2) and  $Var(\lambda^{MLE}) \rightarrow 0$ , we have that  $\lambda^{MLE}$  is consistent.

### 3.2 Given $X_1, X_2, \dots, X_n \stackrel{iid}{\sim} Bern(p)$ and estimator $\hat{p} := \frac{1}{10} \sum_{i=1}^{10} X_i$ determine its Bias, Variance and Consistency

We compute the mean and variance of  $\hat{p}$

$$E(\hat{p}) = \frac{1}{10} \sum_{i=1}^{10} E(X_i) = \frac{10p}{10} = p \quad (3.3)$$

$$Var(\hat{p}) = \frac{1}{100} Var\left(\sum_{i=1}^{10} X_i\right) = \frac{p(1-p)}{10} \quad (3.4)$$

And we use  $\sum_{i=1}^{10} X_i \sim Bin(10p, 10p(1-p))$  for (3.4). We can see that  $\hat{p}$  is unbiased, but is not consistent in  $l^2$ , since  $E(\|\hat{p} - p\|^2) = Bias(\hat{p})^2 + Var(\hat{p}) = \frac{p(1-p)}{10} \not\rightarrow 0$

### 3.3 Given $X_1, X_2, \dots, X_n \stackrel{iid}{\sim} Unif(\theta)$ find MLE and determine its Bias, Variance and Consistency

We want to find a  $\theta$  to maximize the following:

$$\begin{aligned} L(\theta \mid x_1, x_2, \dots, x_n) &= \prod_{i=1}^n \frac{1}{\theta} \mathbb{1}_{\{x_i \leq \theta\}} \\ &= \left(\frac{1}{\theta}\right)^n \prod_{i=1}^n \mathbb{1}_{\{x_i \leq \theta\}} \\ &= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{x_1 \leq \theta, \dots, x_n \leq \theta\}} \\ &= \left(\frac{1}{\theta}\right)^n \mathbb{1}_{\{x_{(n)} \leq \theta\}} \end{aligned}$$

where  $x_{(n)}$  is the  $n$ th order statistic.

Notice that as  $\theta$  decreases  $L(\theta \mid x_1, x_2, \dots, x_n)$  increases, but we cannot take  $\theta^{MLE}$  to be arbitrarily small, since  $L(\theta \mid x_1, x_2, \dots, x_n) = 0$  whenever  $\theta < x_{(n)}$ . Hence  $\theta^{MLE} = x_{(n)}$ .

Note that  $F_{x_{(n)}}(x) = P(x_{(n)} < x) = P(x_1 < x) \cdots P(x_n < x) = \left(\frac{x}{\theta}\right)^n \mathbb{1}_{\{x \leq \theta\}}$

And so

$$f_{x_{(n)}}(x) = \partial F_{x_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1} & 0 \leq x \leq \theta \\ 0 & o.w. \end{cases}$$

We compute the mean and variance of  $\theta^{MLE}$

$$\begin{aligned} E(\theta^{MLE}) &= \int_0^\theta \frac{n}{\theta^n} x^n dx = \frac{n}{n+1} \frac{x^{n+1}}{\theta^n} \Big|_0^\theta = \frac{n}{n+1} \theta \\ E((\theta^{MLE})^2) &= \int_0^\theta \frac{n}{\theta^n} x^{n+1} dx = \frac{n}{n+2} \frac{x^{n+2}}{\theta^n} \Big|_0^\theta = \frac{n}{n+2} \theta^2 \\ Var(\theta^{MLE}) &= E((\theta^{MLE})^2) - E(\theta^{MLE})^2 \\ &= \frac{n}{n+2} \theta^2 - \left(\frac{n}{n+1} \theta\right)^2 = \frac{n\theta^2}{(n+2)(n+1)^2} \end{aligned}$$

So  $\theta^{MLE}$  is biased. It's consistent, though, since  $E(\theta^{MLE}) \rightarrow \theta$  and  $Var(\theta^{MLE}) \rightarrow 0$

### 3.4 Given $X_1, X_2, \dots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$ show that the MLE $(\hat{\mu}, \hat{\sigma}^2)$ is $\left(\bar{X}, \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}\right)$ and determine the Bias, Variance and Consistency for $\hat{\sigma}^2$

It is enough to show that  $(\hat{\mu}, \hat{\sigma}^2)$  satisfy  $\nabla l(\hat{\mu}, \hat{\sigma}^2) = 0$  and  $\nabla^2 l(\hat{\mu}, \hat{\sigma}^2) = 0$  is a negative definite matrix.

This can be seen by computing the partial derivatives of  $l(\mu, \sigma^2) = \frac{n}{2} \log(2\pi) - \frac{n}{2} \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2$

$$\frac{\partial l(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} \left( \sum_{i=1}^n x_i - n\mu \right) \quad (3.5)$$

$$\frac{\partial l(\mu, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n (x_i - \mu)^2 \quad (3.6)$$

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu^2} = -\frac{n}{\sigma^2} \quad (3.7)$$

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial (\sigma^2)^2} = \frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3} \sum_{i=1}^n (x_i - \mu)^2 \quad (3.8)$$

$$\frac{\partial^2 l(\mu, \sigma^2)}{\partial \mu \partial \sigma^2} = -\frac{1}{\sigma^2} \left( \sum_{i=1}^n x_i - n\mu \right) \quad (3.9)$$

Assuming  $\sigma^2 \neq 0$ . After setting (3.5) to 0 we get

$$\sum_{i=1}^n x_i = n\mu \Rightarrow \mu = \bar{X} \quad (3.10)$$

Setting (3.6) to 0 yields

$$-\frac{n}{2(\sigma^2)^2} \left( \sigma^2 - \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 \right) = 0 \Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 \quad (3.11)$$

And substituting  $\mu$  with  $\bar{X}$  from (3.11) gives us

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \quad (3.12)$$

Hence  $\nabla l(\hat{\mu}, \hat{\sigma}^2) = 0$ . Finally to show that  $\nabla^2 l(\hat{\mu}, \hat{\sigma}^2)$  is negative definite, we substitute  $\hat{\mu}, \hat{\sigma}^2$  from (3.10) and (3.12) into (3.7), (3.8), and (3.9) to obtain the following:

$$\nabla^2 l(\hat{\mu}, \hat{\sigma}^2) = \begin{bmatrix} -\frac{n}{\hat{\sigma}^2} & 0 \\ 0 & -\frac{n}{2\hat{\sigma}^2} \end{bmatrix}$$

It is clear that this is negative definite, since its eigenvalues are both negative. Next we find the mean and variance of the MLE  $\hat{\sigma}^2$ .

$$E(\hat{\sigma}^2) = \frac{1}{n} E \left( \sum_{i=1}^n (X_i - \bar{X})^2 \right) = \frac{1}{n} E \left( \sum_{i=1}^n X_i^2 - n\bar{X}^2 \right) \quad (3.13)$$

$$= \frac{1}{n} \left( \sum_{i=1}^n E(X_i^2) - nE(\bar{X}^2) \right) = E(X_i^2) - E(\bar{X}^2) \quad (3.14)$$

$$= \mu^2 + \sigma^2 - \left( \mu^2 + \frac{\sigma^2}{n} \right) = \frac{n-1}{n} \sigma^2 \quad (3.15)$$

$$Var(\hat{\sigma}^2) = \frac{1}{n^2} Var \left( \sum_{i=1}^n (X_i - \bar{X})^2 \right) = \frac{(\sigma^2)^2}{n^2} Var \left( \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \right) \quad (3.16)$$

$$= \frac{(\sigma^2)^2}{n^2} 2(n-1) \quad (3.17)$$

Where (3.15) comes from the Central Limit Theorem, i.e. that  $\bar{X} \sim N(\mu, \frac{\sigma}{n})$  and (3.17) comes from  $\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi_{n-1}^2$ .

We can see from (3.15) that  $\hat{\sigma}^2$  is biased. It's consistent, however, since  $E(\hat{\sigma}^2) \rightarrow \sigma^2$  and  $Var(\hat{\sigma}^2) \rightarrow 0$ .

## 4 Empirical Experimentation

We simulated 10,000 observations from a standard Gaussian distribution and to assess the theoretical results for  $\hat{\sigma}^2$  as computed in section 3.4.

First, we note that the empirical distribution plotted in figure 4.1 looks  $\chi_4^2$  distributed, as it should, given the claim used in (3.17).

Secondly we compute the bias and variance of  $\hat{\sigma}^2$ .

Using (3.15) we compute the Bias as

$$E(\hat{\sigma}^2 - \sigma^2) = \frac{n-1}{n} \sigma^2 - \sigma^2 = -\frac{\sigma^2}{n} = -\frac{1}{5}$$

Our estimate of the bias was -0.20137 – which is quite close.

Using (3.17) we compute the variance as

$$Var(\hat{\sigma}^2) = \frac{(\sigma^2)^2}{n^2} 2(n-1) = \frac{8}{25} = 0.32$$

Our estimate of the variance was 0.3188. Again, this is quite close.



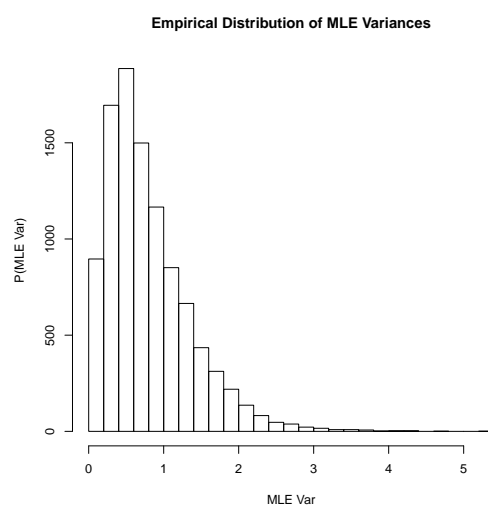


Figure 4.1: Empirical Distribution of  $\hat{\sigma}^2$