Probabilistic Graphical Models Notes

Section 1: Background

Conditional Independence: Given RVs X, Y, Z we say that $X \perp Y \mid Z$ iff $P(X, Y|Z) = P(X \mid Z)P(Y \mid Z)$
Useful properties:

a) Decomposition: $X \perp Y, Z \mid W \Rightarrow \begin{cases} X \perp Y \mid Z \\ X \perp Y \mid W \end{cases}$

b)

An ordering $I : v \mapsto \{1, \cdots, n\}$ is **Topological** iff $j \in \pi_i \Rightarrow I(j) < I(i)$ *i.e. parents always come before their children*
If G is a DAG, then $\exists$ a Topological Ordering on G

Section 2: Directed Graphical Models

Given DAG $G = (V, E)$

$$\mathcal{L}(G) = \left\{ p \text{ is a dist over } x_v : \exists \text{ factors } f_i \text{ s.t. } p(x_v) = \textstyle\prod_{i=1}^n f_i(x_i; x_{\pi_i}) \text{ and } f_i \text{ satisfies: } \begin{array}{l} f_i > 0 \\ \sum_{x_i} f_i(x_i; x_{\pi_i}) = 1 \\ f_i : Dom(x_i)^2 \mapsto [0,1] \end{array} \right\}$$

Proposition 1 (Leaf Plucking Property). *if n is a leaf of G, $p(x_v) \in \mathcal{L}(G - \{n\})$*

Proof. $P(x_v) = p(x_{1:n-1}, x_n) = f_n(x_n; x_{\pi_n}) \prod_{i \neq n} f_i(x_i, x_{\pi_i})$. Next marginalizing out x_n and using that it is a leaf, we have:

$$p(x_{1:n-1}) = \underbrace{\sum_{x_n} f_n(x_n; x_{\pi_n})}_{\text{sums to 1}} \underbrace{\prod_{i=1}^{n-1} f_i(x_i; x_{\pi_i})}_{\text{Does not contain } x_n} = \prod_{i=1}^{n-1} f_i(x_i; x_{\pi_i})$$

So $p(x_v) \in \mathcal{L}(G - \{n\})$ as needed. ☐

Proposition 2 (Factors are Conditional PMFs). *Let $p \in \mathcal{L}(G)$ and $\{f_j\}$ be a factorization, then $\forall i$, $P(x_i|x_{\pi_i}) = f_i(x_i; x_{\pi_i})$*

Proof. WLOG let $\{1, \cdots, n\}$ be a Topological Ordering and use Theorem 1 to get that $p(x_{1:i}) \in \mathcal{L}(G - \{i+1, \cdots, n\})$, and so $p(x_{1:i}) = \underbrace{\prod_{j=1}^{i-1} f_j(x_j; x_{\pi_j})}_{\text{Call this } g(x_{1:i-1})} f_i(x_i; x_{\pi_i})$. We partition $\{1:i\}$ as $\{i\} \cup \pi_i \cup A$ and get that:

$$p(x_i \mid x_{\pi_i}) = \frac{\sum_{x_A} f_i(x_i; x_{\pi_i}) g(x_{1:i-1})}{\sum_{x_A} \sum_{x_i'} f_i(x_i'; x_{\pi_i}') g(x_{1:i-1})} = \frac{f_i(x_i; x_{\pi_i}) \sum_{x_A} g(x_{1:i-1})}{\sum_{x_i'} f_i(x_i'; x_{\pi_i}') \sum_{x_A} g(x_{1:i-1})} = f_i(x_i; x_{\pi_i})$$

☐

Note: adding edges adds more distributions i.e. $E \subseteq E'$ and $G' = (V, E')$ then $\mathcal{L}(G) \subseteq \mathcal{L}(G')$

Proposition 3. $p \in \mathcal{L}(G) \iff x_i \perp x_{nd(i)} \mid \pi_i$

Proof. $(\Rightarrow)$ $(\Leftarrow)$ ☐

References