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# NUMERICAL METHOD FOR DETERMINING A STRESS INTENSITY FACTOR IN THE CASE OF SOLIDS OF HOMOGENEOUS AND HETEROGENEOUS MATERIALS WITH A CRACK

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**Abstract:** Algorithms for determining the stress intensity factor are developed on the basis of analyzing asymptotic solutions to the problems of deformation of a homogeneous solid with a cut and a plate with a crack located at an interface between two media. The proposed algorithms are used to calculate stress intensity factors using the results of a numerical solution to the problems of loading various flat and spatial homogeneous cracked solids, as well as a plate with a crack located at the interface between two elastic media. It is shown that the calculation results are in good agreement with the data obtained by other methods.

*Keywords:* stress intensity factor, crack at the interface between two media, asymptotic solution, finite element method.

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## INTRODUCTION

Analysis of the stress-strain state of a solid with a cut is one of the main problems of linear fracture mechanics. It follows from the solution of problems for isotropic media [1] that the stress field near a crack tip has a root feature $O(r^{-1/2})$, where $r$ is the polar radius of a point lying on the normal to the crack front. The universal characteristic of this singular stress distribution is a stress intensity factor (SIF) [2]. Generally, there are three independent SIFs, which correspond to three types of displacements of the crack edges: normal detachment ($K_{\rm I}$), transverse shear ($K_{\rm II}$), and longitudinal shear ($K_{\rm III}$).

Methods for determining SIFs can be divided into three main groups: analytical, experimental, and computational. Analytical solutions are obtained for a limited class of axisymmetric and planar problems. In experiments and in a numerical solution, various quantities are determined: displacements, stresses, and energy. For example, in [3–5], experimental interference-optical methods are used to determine stresses or displacements in the vicinity of a crack, which are then approximated by analytical solutions for spatial or planar solids with cuts [6, 7].

Currently, the most developed are computational algorithms based using finite element methods, boundary element methods, or boundary integral equations. For example, when it comes to analyzing the stress-strain state of a solid with a crack by the finite element method, the singular nature of stress distribution can help one determine the displacements and the contour $J$-integral more accurately [8]. Therefore, in computer systems (ANSYS, COSMOS/M, NASTRAN, etc.), procedures for calculating SIFs from these values are used [9]. In special software units, an algorithm for approximating the surface displacements of a cut that simulates a crack with an asymptotic solution is used [2], obtained only for a plane stress state or a plane strain state is used. In [10],

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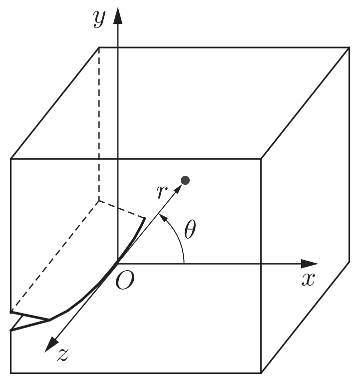


**Fig. 1.** Crack location and the coordinate system.

it is proposed to describe the displacement field in the vicinity of a crack by solving the problem for an axisymmetric body with a mathematical cut.

In its turn, stress distribution is independent of the degree of constraint of the strain along its front. Consequently, the stress field characteristics in the vicinity of the crack can be used to determine SIFs both for a cracked solid made of homogeneous material and for a solid with a crack at the interface of two solid media.

Thus study is devoted to the possibility of applying an asymptotic representation of a stress state in the vicinity of a crack for independently determining the three types of SIFs.

## 1. CRACK IN A HOMOGENEOUS SOLID

A solution to the problem of deformation of a homogeneous isotropic solid with a crack is under consideration [11]. In a plane perpendicular to the crack (Fig. 1), on a line $\theta = 0$, the dependence of stress components on the polar radius of the point $r$ has the following form:
—for a normal detachment crack,

$$\sigma_{yy} = K_{\mathrm{I}}(2\pi r)^{-1/2} + \sum_{n=1}^{\infty} a_{yyn} r^{(2n-1)/2}, \quad \sigma_{xx} = K_{\mathrm{I}}(2\pi r)^{-1/2} + \sigma_{xx}^0 + \sum_{n=1}^{\infty} a_{xxn} r^{n/2},$$
$$\sigma_{xy} = \sigma_{yz} = 0; \tag{1}$$

—for a transverse shear crack,

$$\sigma_{yy} = \sigma_{xx} = 0, \qquad \sigma_{xy} = K_{\mathrm{II}}(2\pi r)^{-1/2} + \sigma_{xy}^0 + \sum_{n=1}^{\infty} a_{xyn} r^{(2n-1)/2}; \tag{2}$$

—for a longitudinal shear crack,

$$\sigma_{yy} = \sigma_{xx} = \sigma_{xy} = 0, \qquad \sigma_{yz} = K_{\mathrm{III}}(2\pi r)^{-1/2} + \sigma_{yz}^0 + \sum_{n=1}^{\infty} a_{yzn} r^{(2n-1)/2}. \tag{3}$$

Expressions for the stresses $\sigma_{yy}$, $\sigma_{xy}$, and $\sigma_{yz}$ include only one of the three SIFs: $K_{\mathrm{I}}$, $K_{\mathrm{II}}$, and $K_{\mathrm{III}}$, respectively. This allows one to independently calculate these quantities in the case of arbitrary load of the solid with a crack.

If expressions (1)–(3) are restricted to the first terms of the series, then expressions for the stress components, multiplied by $r^{1/2}$, can be written as

$$p_i r^{1/2} = K_i (2\pi)^{-1/2} + a_{pi} r, \qquad i = \mathrm{I},\ \mathrm{II},\ \mathrm{III}.$$

Here $p_{\mathrm{I}} = \sigma_{yy}$, $p_{\mathrm{II}} = \sigma_{xy}$, and $p_{\mathrm{III}} = \sigma_{yz}$.

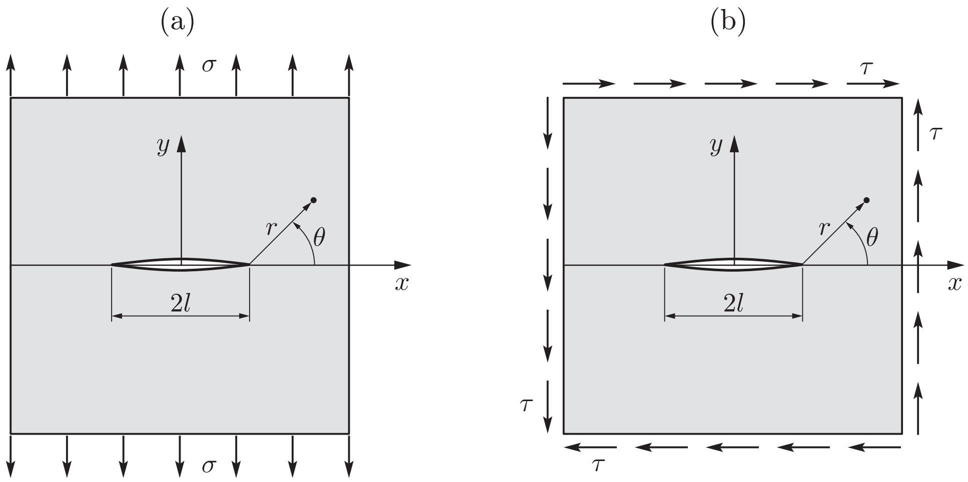


**Fig. 2.** Design diagrams of the plane problems: (a) tension; (b) pure shear.

Taking the linear approximation of the stress field in the vicinity of the crack

$$p_i r^{1/2} = A_i + B_i r, \tag{4}$$

one can determine the SIFs

$$K_i = A_i (2\pi)^{1/2}, \qquad i = \mathrm{I},\ \mathrm{II},\ \mathrm{III}. \tag{5}$$

A similar approach is used to calculate SIFs using the crack surface displacements [9].

The size of the region in the vicinity of the crack, where the approximate solution obtained with the help of linear approximation is not much different from the exact solution, is determined below. The problems of tension and pure shear of an infinite plate with a central cut of length $2l$ is described next (Fig. 2).

The same stress distribution is observed on the cut extension line ($y = 0$) both under tension and under shear [6]

$$p(r) = p\,\frac{r + l}{(r^2 + 2rl)^{1/2}}. \tag{6}$$

Here $p(r)$ denotes the normal tensile stresses $\sigma_{yy}$ of the plate and the tangential shear stresses $\sigma_{xy}$ of the plate, $p = \sigma$ refers to the normal stresses in the tension problem, $p = \tau$ stands for the tangential stresses in the shear problem, and $\sigma$ and $\tau$ denote the stresses acting at a sufficiently long distance from the cut.

The function $p(r)$ in dimensionless form is written as

$$s = \frac{p(r)}{p}\Big(\frac{2r}{l}\Big)^{1/2} = \Big(1 + \frac{r}{l}\Big)\Big(1 + \frac{r}{2l}\Big)^{-1/2}$$

and expanded into a series:

$$s = 1 + \sum_{n=2}^{\infty} \frac{2n-1}{2^{3n-4}}\Big(\frac{r}{l}\Big)^{n-1}.$$

If we restrict ourselves to the first term, then the following linear approximation of the theoretical stress distribution is obtained:

$$s_1 = 1 + 3r/(4l).$$

Figure 3 shows the graphs of the functions $s$ and $s_1$.

The solution of the problem of torsion of a cylinder with a ring-shaped cut is the tangential stress distribution $\sigma_{yz}$ over the diameter of the intact part of the cylinder [6]

$$\sigma_{yz} = \frac{3}{8}\,\tau_0\,\frac{l - r}{(2lr - r^2)^{1/2}} \qquad (0 \leqslant r \leqslant l).$$

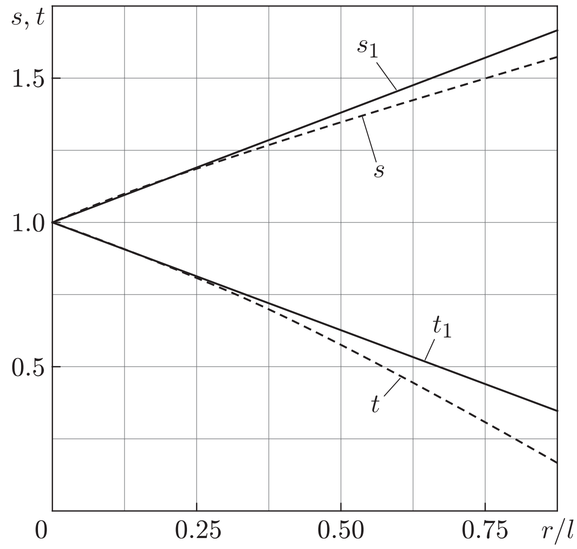


**Fig. 3.** Approximate stress distribution $(s_1, t_1)$ and exact stress distribution $(s, t)$ for the plane problem $(s, s_1)$ and spatial problem $(t, t_1)$.

For a dimensionless stress, we obtain

$$t = \frac{8\sigma_{yz}}{3\tau_0}\Big(\frac{2r}{l}\Big)^{1/2} = \Big(1 - \frac{r}{l}\Big)\Big(1 - \frac{r}{2l}\Big)^{-1/2}.$$

Here $\tau_0 = 2M/(\pi l^3)$ is the nominal tangential stress corresponding to torque $M$, and $l$ is the radius of the intact part of the cylinder.

The transformations carried out in analyzing the plane problems are used to obtain the linear representation of the function $t$:

$$t_1 = 1 - 3r/(4l).$$

Figure 3 shows the stress distributions $t$ and $t_1$ in a circular cylinder with a ring-shaped cut, which correspond to the analytical solution and the solution based on the linear approximation.

It should be noted that, for $r < 0.5l$, the difference in the approximate solution based on the linear approximation and the exact solution does not exceed 2.5%. Thus, in the case of a homogeneous load of a solid with a cut, the coefficient $A$ in Eq. (4) can be sufficiently accurately determined using the stresses acting at the points located on the crack extension line at a distance from the crack tip not exceeding $0.5l$.

The algorithm for determining SIFs using the stresses obtained with the help of the finite difference method in solving the problem of the tension of a plate with a central cut of length $2l$, located perpendicularly to the acting stresses $\sigma$ (see Fig. 2).

Figure 4 shows a finite element scheme. The size of a finite element in the vicinity of the crack is $0.05l$, and the width and height of the plate is $20l$. Figure 5 shows the stress distributions $\sigma_{yy}/\sigma$.

A large error in calculating the stresses is observed only in the first node of the finite element grid for $r = 0.05l$. Similar results are obtained for other sizes of the finite element. Therefore, it is reasonable to determine the constant $A$ in the linear approximation (4) using the stress values in the second and third nodes of the grid at the crack extension line (see Fig. 4). As a result of the calculation (see Fig. 4), the coefficient $A = 0.71\sigma\sqrt{l}$ is obtained. Consequently, according to Eq. (5), the SIF is equal to $K_{\rm I} = 1.779\sigma\sqrt{l}$. This value is different from the theoretical value $K_{\rm I} = \sigma\sqrt{\pi l} = 1.772\sigma\sqrt{l}$ by 0.4%.

The table shows the calculation results obtained using the proposed algorithm and the algorithm for determining the SIFs with the help of the values of the cut surface displacement [9] in the numerical solution of different problems ($K/K_t$ is the ratio of the estimated value of the SIF to the theoretical value, which is determined from the data [12]). For spatial solids with a curvilinear crack, the position of the estimated point is determined by the angular coordinate $\varphi$ (Fig. 6).

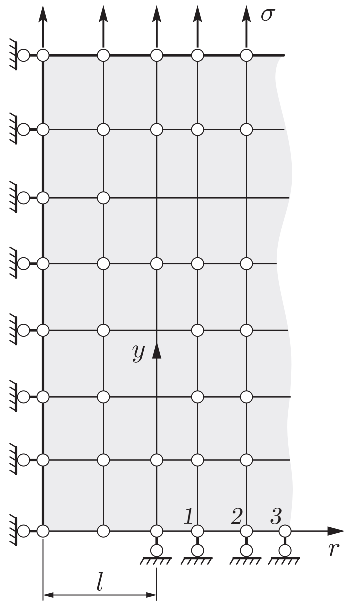


**Fig. 4.**

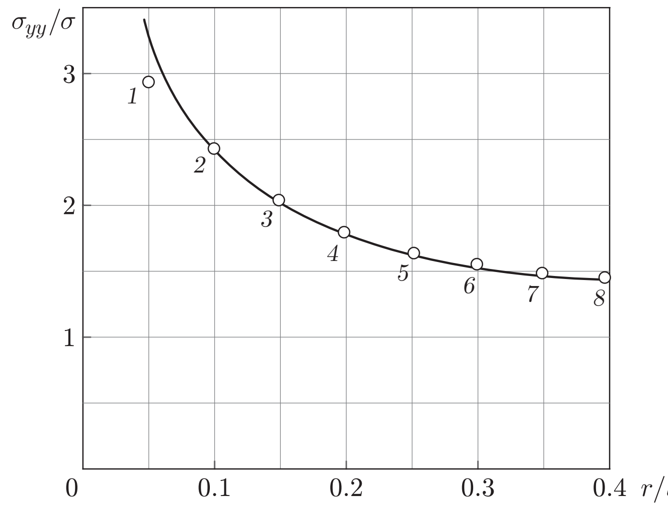


**Fig. 5.**

**Fig. 4.** Design scheme: the numbers of the nodes are denoted as *1*, *2*, and *3*.

**Fig. 5.** Stress distribution on the cut line: the curve refers to the calculation based on Eq. (6) and the points refer the calculation using the finite element method (points *1*–*8* refer to the numbers of the nodes).

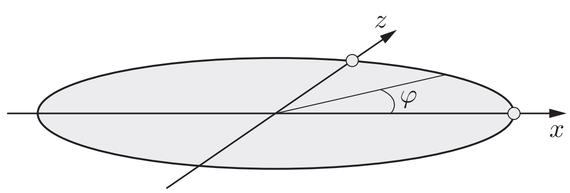


**Fig. 6.** Determination of the position of points on the edges of the curvilinear crack.

The results given in the table show that the error in determining the SIF according to the known stresses is comparable with the error in calculations based on other methods.

## 2. CRACK AT THE INTERFACE BETWEEN TWO ELASTIC MEDIA

Let there be an infinite plate with a cut located at a boundary between two infinite half-planes made of materials with different elastic characteristics: Poisson's ratios $\nu_1$ and $\nu_2$ and shear moduli $G_1$ and $G_2$. At a sufficiently large distance from the cut, the plate is loaded by normal $\sigma$ and tangential $\tau$ stresses (Fig. 7).

An asymptotic solution to this problem can be represented in the complex form [13]

$$(\sigma_{yy} + i\sigma_{xy})\Big|_{\theta=0} = (K_{\mathrm{I}} + iK_{\mathrm{II}})(r/l)^{ia}(2\pi r)^{-1/2}, \tag{7}$$

where $l$ is the crack half-length and $a = (2\pi)^{-1}\ln\left((G_2\varkappa_1 + G_1)/(G_1\varkappa_2 + G_2)\right)$ [$\varkappa_{1,2} = 3 - 4\nu_{1,2}$ in the case of plane deformation and $\varkappa_{1,2} = (3 - \nu_{1,2})/(1 + \nu_{1,2})$ in the case of a stress state].

The values of the SIFs obtained using numerical algorithms and reference data

| Problem | Load type (Crack type) | $K/K_t$ | |
|---|---|---|---|
| | | Stress based calculation [Eq. (4)] | Displacement based calculation [9] |
| Tension of the plate with a central cut | Tension ($K_{\rm I}$) | 0.993 | 0.959 |
| Biaxial tension of the plate with an arc-shaped cut | Biaxial tension:<br>$K_{\rm I}$<br>$K_{\rm II}$ | <br>0.889<br>0.866 | <br>0.964<br>0.843 |
| Deformation of the cylinder with a disk-shaped cut perpendicular to its axis | Tension ($K_{\rm I}$) | 0.973 | 0.989 |
| | Shear along the $x$ axis ($K_{\rm II}$) for $\varphi = 0$ | 0.914 | 0.948 |
| | Tension by accumulated forces applied to the cut edges at a distance $0.5l$ ($K_{\rm I}$):<br>$\varphi = 0$<br>$\varphi = \pi/2$<br>$\varphi = \pi$ | <br>0.982<br>0.975<br>1.005 | <br>0.955<br>0.939<br>0.976 |
| Bending of the plate with a semicircular crack arriving at the surface | Bending ($K_{\rm I}$):<br>$\varphi = 0$<br>$\varphi = \pi/2$ | <br>1.020<br>0.877 | <br>1.008<br>0.958 |
| Torsion of the cylinder with an outerradial cut | Torsion ($K_{\rm III}$) | 0.975 | 1.003 |

Expression (7) yields

$$(\sigma_{yy})\Big|_{\theta=0} = K_{\rm I}(2\pi r)^{-1/2}\cos[a\ln(r/l)] - K_{\rm II}(2\pi r)^{-1/2}\sin[a\ln(r/l)],$$

$$(\sigma_{xy})\Big|_{\theta=0} = K_{\rm II}(2\pi r)^{-1/2}\cos[a\ln(r/l)] + K_{\rm I}(2\pi r)^{-1/2}\sin[a\ln(r/l)].$$

Unlike the case with the homogeneous material, the SIFs $K_{\rm I}$ and $K_{\rm II}$ in this case cannot be related with only one stress tensor component:

$$K_{\rm I} = (2\pi r)^{1/2}\{(\sigma_{yy})\Big|_{\theta=0}\cos[a\ln(r/l)] + (\sigma_{xy})\Big|_{\theta=0}\sin[a\ln(r/l)]\},$$

$$K_{\rm II} = (2\pi r)^{1/2}\{(\sigma_{xy})\Big|_{\theta=0}\cos[a\ln(r/l)] - (\sigma_{yy})\Big|_{\theta=0}\sin[a\ln(r/l)]\}. \tag{8}$$

For the case with the plane stress state of the plate (see Fig. 7), we calculate $K_{\rm I}$ and $K_{\rm II}$ using Eqs. (8) and the numerical solution obtained by the finite element method for the following problem parameters: $\tau = 0$, $G_2/G_1 = 0.1$, $\nu_1 = \nu_2 = 0.3$, the height and width of the plate is $20l$, and the size of the finite element in the vicinity of the cut tip is $0.025l$. In this case, at the interface between the two media, the strain compatibility condition $\varepsilon_{x1} = \varepsilon_{x2}$ for $y = 0$ should be satisfied. For this purpose, the plate should be loaded by the stresses $\sigma_{x1}$ and $\sigma_{x2}$ (see Fig. 7). In accordance with the results of [13],

$$\sigma_{x2} = \frac{G_2}{G_1}\,\sigma_{x1} + \Big(\nu_2 - \frac{G_2}{G_1}\,\nu_1\Big)\sigma.$$

Consequently, $\sigma_{x2} = 0.27\sigma$ for $\sigma_{x1} = 0$.

Figure 8 shows the results of calculating the SIFs using the finite element method for $y = 0$ in a range $0.025l < r < 0.350$ [$k_{\rm I} = K_{\rm I}/K_0$ and $k_{\rm II} = K_{\rm II}/K_0$ denotes the dimensionless SIFs, and $K_0 = \sigma(\pi l)^{1/2}$]. It should be noted that, for $0.05l < r < 0.35l$, the functions $k_{\rm I}$ and $k_{\rm II}$ are virtually linear.

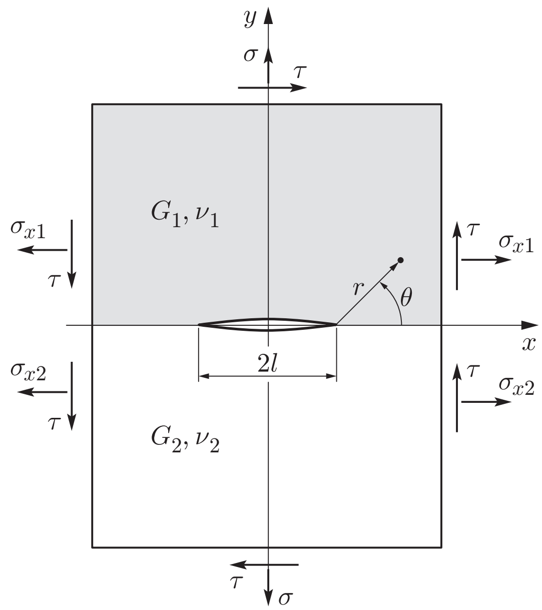


**Fig. 7.** Plate with a crack at the interface between two media.

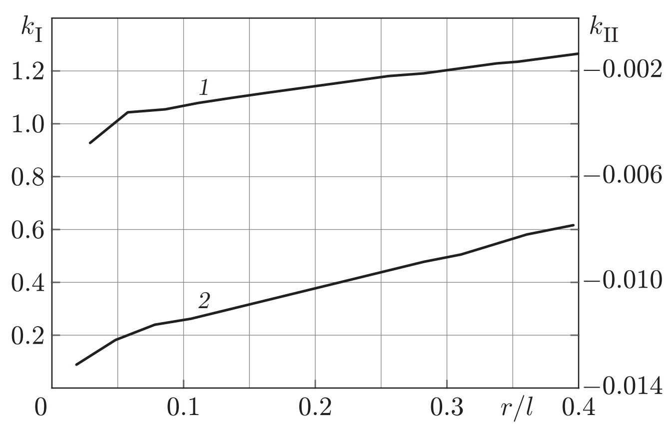


**Fig. 8.** Calculation results for the SIFs $k_{\mathrm{I}}$ (*1*) and $k_{\mathrm{II}}$ (*2*) using the finite element method.

Thus, in some vicinity of the crack tip, the right parts of Eqs. (8) can be represented as the linear functions

$$K_{\mathrm{I}}(r) = A_{\mathrm{I}} + B_{\mathrm{I}} r, \qquad K_{\mathrm{II}}(r) = A_{\mathrm{II}} + B_{\mathrm{II}} r,$$

and the values of the SIFs can be determined by extrapolating to the cut tip:

$$K_{\mathrm{I}} = A_{\mathrm{I}}, \qquad K_{\mathrm{II}} = A_{\mathrm{II}}.$$

Extrapolating the data given in Fig. 7, we obtain $K_{\mathrm{I}} = 1.0021\sigma\sqrt{\pi l}$ and $K_{\mathrm{II}} = -0.1248\sigma\sqrt{\pi l}$. These results can be compared with the analytical values of the SIFs, obtained in [14]:

$$K_{\mathrm{I}}^{\mathrm{t}} = \{\sigma[\cos(a \ln 2) + 2a \sin(a \ln 2)] + \tau[\sin(a \ln 2) - 2a \cos(a \ln 2)]\}\sqrt{\pi l},$$

$$K_{\mathrm{II}}^{\mathrm{t}} = \{\tau[\cos(a \ln 2) + 2a \sin(a \ln 2)] - \sigma[\sin(a \ln 2) - 2a \cos(a \ln 2)]\}\sqrt{\pi l}.$$

In the case under consideration, we obtain $K_{\mathrm{I}}^{\mathrm{t}} = 1.0129\sigma\sqrt{\pi l}$, and this value differs from the estimated SIF by 1%. The value $K_{\mathrm{II}}^{\mathrm{t}} = -0.1300\sigma\sqrt{\pi l}$ differs from the estimated SIF by 4%. The size of the finite element is increased in the vicinity of the crack tip up to $0.05l$, which yields $K_{\mathrm{I}} = 1.0013\sigma\sqrt{\pi l}$ (the error is 8%) and $K_{\mathrm{II}} = -0.1161\sigma\sqrt{\pi l}$ (the error is 11%).

In order for the results of calculating the SIFs to correspond to the analytical solution of the problem of deformation of the solid with a crack at the interface of two media, it is necessary to use a finite element grid in a singular region with a finite element size of less than $0.05l$.

## CONCLUSIONS

This paper described the analysis of asymptotic solutions for the cases of the homogeneous solid with a cut and the inhomogeneous plate with a crack located at the interface between two media. The stress distribution on the crack extension line in the vicinity of the crack tip was presented as the linear function of the polar radius of the point. Based on the analysis, the algorithm was developed for independently determining the stress intensity factor of three types. It followed from the results of the numerical solution of the problem of the deformation of the plane and spatial solids with a crack that algorithm was effective.

The results obtained using the stress distribution in the vicinity of the crack tip and the results obtained using other methods were compared, which confirmed the reliability of the proposed algorithms for determining the stress intensity factor.

## REFERENCES


1. M. L. Williams, “Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates in Extensions,” J. Appl. Mech. **19**, 526–528 (1952).
2. G. P. Cherepanov, *Mechanics of Brittle Fracture* (McGraw-Hill, 1979).
3. V. M. Tikhomirov and V. P. Tyrin, “Use of the Scattered Light Method in Order to Determine the Stress Intensity Factor $K_{\mathrm{III}}$ in Three-Dimensional Problems,” Prikl. Mekh. Tekh. Fiz. **31** (3), 167–170 (1990) [J. Appl. Mech. Tech. Phys. **31** (3), 500–503 (1990)].
4. V. M. Tikhomirov, “Determination of Stress Intensity Coefficients by Photoelasticity in Three-Dimensional Problems of Fracture Mechanics,” Probl. Mashinostr. Nadezh. Mashin, No. 2, 94–100 (2004).
5. V. P. Tyrin, “Application of the Holographic Interferometry Method to Determine the Stress Intensity Factor,” Prikl. Mekh. Tekh. Fiz. **31** (1), 155–158 (1990) [J. Appl. Mech. Tech. Phys. **31** (1), 142–145 (1990)].
6. H. Neuber, *Kerbspannungslehre: Grundlagen für genaue Festigkeitsberechnung mit Berücksichtigung von Konstruktionsform und Werkstoff* (Springer-Verlag, Berlin–Heidelberg, 1958).
7. S. Timoshenko and J. N. Goodier, *Theory of Elasticity* (Nauka, Moscow, 1975; McGraw–Hill, 1975).
8. E. M. Morozov and G. P. Nikishov, *Finite-Element Method in Fracture Mechanics* (Nauka, Moscow, 1980) [in Russian].
9. E. M. Morozov, A. Yu. Muizemnek, and A. S. Shadskii, *ANSYS in Engineer’s Hands: Fracture Mechanics* (Lenand, Moscow, 2008) [in Russian].
10. V. M. Tikhomirov, “Determination of Stress Intensity Factors in Three-Dimensional Problems of Fracture Mechanics,” Prikl. Mekh. Tekh. Fiz. **55** (5), 172–180 (2014) [J. Appl. Mech. Tech. Phys. **55** (5), 877–884 (2014)].
11. S. N. Atluri (ed.), *Computational Methods in the Mechanics of Fracture* (North-Holland, 1986).
12. Y. Murakami, *Stress Intensity Factors Handbook* (Pergamon Press, Oxford, 1987).
13. G. C. Sih and J. R. Rice, “The Bending of Plates of Dissimilar Materials with Cracks,” J. Appl. Mech. **31**, 477–482 (1964).
14. J. R. Rice and G. C. Sih, “Plane Problems of Cracks in Dissimilar Media,” J. Appl. Mech. **32**, 418–423 (1965).