Statistical Physics: Workshop Problems 7

(1) (a) Simply substitute for $k = \epsilon/c\hbar$ and $dk = d\epsilon/c\hbar$ to get

$$g(\epsilon)d\epsilon \;=\; \frac{V}{2\pi^2 c^3 \hbar^3}\epsilon^2 d\epsilon.$$

(b) The partition function for a system with density of states $g(\epsilon)d\epsilon$ is

$$Z \;=\; \int_0^\infty g(\epsilon)e^{-\beta\epsilon}d\epsilon,$$

therefore we have

$$\begin{aligned} Z &= \int_0^\infty \frac{V}{2\pi^2 c^3 \hbar^3}\epsilon^2 e^{-\epsilon/(k_B T)}d\epsilon \\ &= \frac{V}{\hbar^3 c^3 \pi^2}\left(k_B T\right)^3. \end{aligned}$$

The non-relativistic case is calculated similarly with $\epsilon = \hbar^2 k^2/2m$. Its temperature dependence goes as $T^{3/2}$ rather than T^3. Of course with the partition function as a function of T (or β) it's straightforward to calculate thermodynamic quantities such as internal energy, free energy, specific heat, etc.

(2) (a) Given the density of states $g(\epsilon)$ in the question, $f_{FD}(\epsilon) = 1$ for $\epsilon < E_F$ (zero otherwise) at $T = 0$ and the definition of Fermi energy, we get

$$\begin{aligned} N &= \frac{2\pi V}{h^3}(2M)^{3/2}\int_0^{E_F}\sqrt{\epsilon}d\epsilon \\ \Rightarrow E_F &= \frac{\hbar^2}{2M}\left(\frac{3\pi^2 N}{V}\right)^{2/3}. \end{aligned}$$

(Note change from h to $\hbar$ absorbing factors of 2π).

(b) We have the relativistic density of states from (1) and the same simple expression for $f_{FD}(\epsilon)$ at $T = 0$, therefore

$$\begin{aligned} N &= \int_0^{E_F}\frac{V}{2\pi^2 c^3\hbar^3}\epsilon^2 d\epsilon \\ \Rightarrow E_F &= hc\left(\frac{3N}{8\pi V}\right)^{1/3}. \end{aligned}$$

(c) We have the density of states for relativistic Fermions $g(\epsilon)d\epsilon = A\epsilon^2 d\epsilon$ from question (1) (constants collected together into A) and $f_{FD}(\epsilon)$ at $T = 0$. Therefore the average energy per particle is

$$\begin{aligned} \langle\epsilon\rangle &= \frac{\int_0^{E_F}\epsilon g(\epsilon)f_{FD}(\epsilon)d\epsilon}{\int_0^{E_F} g(\epsilon)f_{FD}(\epsilon)d\epsilon} \\ &= \frac{A\int_0^{E_F}\epsilon\epsilon^2 d\epsilon}{A\int_0^{E_F}\epsilon^2 d\epsilon} \\ &= 3E_F/4, \end{aligned}$$

therefore the total energy for N particles is $3NE_F/4$. In the non-relativistic case (collecting constants into B) $g(\epsilon)d\epsilon = B\sqrt{\epsilon}$ giving

$$\begin{aligned}\langle\epsilon\rangle &= \frac{B\int_0^{E_F}\epsilon\sqrt{\epsilon}d\epsilon}{B\int_0^{E_F}\sqrt{\epsilon}d\epsilon}\\ &= 3E_F/5\end{aligned}$$

so the energy for N particles is $3NE_F/5$.

(d) Calculator time. The point of this question is that you think about the order of magnitude of the the values of these quantities in a wide range of different systems. The Fermi energies and temperatures ($T_F = E_F/k_B$) are

(i) 0.425 meV and 4.9 K.

(ii) 11.5 eV and 134,000 K.

(iii) 33 MeV and 3.8×10^{11} K.

(3) (a) For non-relativistic electrons we have

$$\begin{aligned}U_{\text{elec}} &= \frac{3}{5}NE_F\\ &= \frac{3}{5}N\frac{\hbar^2}{2m}\left(\frac{3\pi^2N}{V}\right)^{2/3}.\end{aligned}$$

Now we have to obtain N and V from the information given in the question. There are $m_p + m_p$ protons and neutrons in the star which make up (near enough) all of the mass, so the number of electrons will be $N = M/2m_p$. The star has radius R and so $V = 4\pi R^3/3$. Substitute this into the above equation for energy gives

$$U_{\text{elec}} = \frac{3}{5}\frac{M}{2m_p}\frac{h^2}{8\pi^2m_e}\left(3\pi^2\frac{M}{2m_p}\frac{3}{4\pi R^3}\right)^{2/3}.$$

Collecting all of the numbers and π's together into a single value we can simplify to

$$U_{\text{elec}} = 0.0088\frac{h^2M^{5/3}}{m_e m_p^{5/3}R^2}.$$

(b) The total energy is of the form

$$U_{\text{total}} = -\frac{A}{R} + \frac{B}{R^2},$$

where A and B are the constants gathered together in the expressions for U_{grav} and U_{elec}. The energy will be minimised when $dU_{\text{total}}/dR = 0$ at $R = 2B/A$. Therefore we obtain

$$\begin{aligned}R &= 2\times0.0088\frac{h^2M^{5/3}}{m_e m_p^{5/3}R^2}\frac{5}{3GM^2}\\ &= \frac{0.028h^2}{m_e m_p^{5/3}G}M^{-1/3}.\end{aligned}$$

(c) Looking up the constants and putting them into the equation for the radius of a white dwarf gives $R \approx 6000$ km.

(d) For one solar mass and the radius in (c), the Fermi energy is ~ 0.5 MeV. This is about the rest mass energy of an electron hence a relativistic treatment is required.

(e) The above set of calculations can be performed using the relativistic expressions, however it's easier just to note that the expression for the Fermi energy (and hence average electron energy) of a relativistic Fermion gas scales as $(N/V)^{1/3} \sim R^{-1}$.

(f) The white dwarf is now unstable to gravitational collapse as, for large enough M, there is no minimum in energy. This is called the Chandrasekhar limit.