University of Durham

PROGRESS TEST

Session:	Year:
Michaelmas Term	2021

Title: PHYS3631 Foundations of Physics 3B / PHYS4261 Foundations of Physics 4B

Statistical Physics

Attempt all questions. The marks shown in brackets for the main parts of each question are given as a guide to the weighting that the markers expect to apply.

Write your answers on A4 paper, and upload them to Gradescope.

Information

Elementary charge:	$e = 1.60 \times 10^{-19}$ C
Speed of light:	$c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Boltzmann constant:	$k_{\mathrm{B}} = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Bohr magneton:	$\mu_{\mathrm{B}} = 9.27 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
Electron mass:	$m_{\mathrm{e}} = 9.11 \times 10^{-31}$ kg
Gravitational constant:	$G = 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$
Proton mass:	$m_{\mathrm{p}} = 1.67 \times 10^{-27}$ kg
Planck constant:	$h = 6.63 \times 10^{-34}$ J s
Permittivity of free space:	$\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Magnetic constant:	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\,m^{-1}}$
Molar gas constant:	$R = 8.31\ \mathrm{J\,K^{-1}\,mol^{-1}}$
Avogadro's constant:	$N_{\mathrm{A}} = 6.02 \times 10^{23}\ \mathrm{mol^{-1}}$
Gravitational acceleration at Earth's surface:	$g = 9.81\ \mathrm{m\,s^{-2}}$
Stefan-Boltzmann constant:	$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\ m^{-2}\ K^{-4}}$
Astronomical Unit:	$\mathrm{AU} = 1.50 \times 10^{11}$ m
Parsec:	$\mathrm{pc} = 3.09 \times 10^{16}$ m
Solar Mass:	$M_{\odot} = 1.99 \times 10^{30}$ kg
Solar Luminosity:	$L_{\odot} = 3.84 \times 10^{26}$ W

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1. (a) Consider an assembly of N weakly-interacting, distinguishable particles contained in a fixed volume V, with fixed internal energy U. Are the various distributions $(n_1, n_2, \ldots)$ of the particles in single-particle states equally probable, or do they have different probabilities? State briefly what distinguishes the Boltzmann distribution from other distributions $\{n_i\}$ of the assembly of distinguishable particles. [4 marks]

   (b) In a gas of N weakly-interacting, identical Bosons in a fixed volume V, with fixed total energy U, the i-th single-particle energy level has energy ϵ_i and degeneracy g_i. Consider a distribution $(n_1, n_2, \ldots)$ of the N Bosons in single-particle energy levels, where n_i Bosons have energy ϵ_i. Show that the number of microstates, $\Omega(n_1, n_2, \ldots)$, corresponding to the distribution $(n_1, n_2, \ldots)$ is given by

$$\Omega(n_1, n_2, \ldots) = \prod_i \frac{(n_i + g_i)!}{n_i! g_i!}.$$

   What is the limit of $\Omega(n_1, n_2, \ldots)$ for a dilute gas, $g_i \gg n_i$? [4 marks]

   (c) Derive the density of states in energy, $g(\epsilon)\,\delta\epsilon$, for a particle with energy $\epsilon = \alpha\, k^{3/2}$ in three dimensions. [4 marks]

   [Hint: You may use without derivation the 3D density of states in k space

$$g(k)\,\delta k = \frac{V}{2\pi^2} k^2\,\delta k\,.$$]

   (d) The two lowest-lying energy levels of a hydrogen atom are $E_0 = -13.6$ eV and $E_1 = -3.4$ eV. Treat the hydrogen atoms as distinguishable. Ignoring degeneracies, at what temperature would we find one hundredth as many hydrogen atoms in the first excited state as in the ground state? [4 marks]

2. (a) A system of N particles is in thermal equilibrium with a heat bath at a temperature T. The single-particle energy levels ϵ_i are non-degenerate and the single-particle partition function is

$$Z_1 = \sum_i \exp(-\beta\,\epsilon_i),$$

   where $\beta = 1/k_{\mathrm{B}}T$. The average energy per particle is

$$\langle \epsilon \rangle = \sum_i p_i \epsilon_i.$$

   Show that $\langle \epsilon \rangle$ is given by

$$\langle \epsilon \rangle = -\frac{\partial \ln Z_1}{\partial \beta}$$

   where p_i is the probability that the i-th single-particle state is occupied. [3 marks]

(b) Calculate the partition function and hence average energy per particle $\langle\epsilon\rangle$ for a system of N particles, with two single-particle energy levels, $\epsilon_1 = 0$ and $\epsilon_2 = \Delta > 0$. What is the internal energy U of the system? Comment briefly on the limit of $\langle\epsilon\rangle$ for high and low temperatures. [3 marks]

(c) For the system of particles in (b), show that the entropy S is given by

$$S = Nk_B \left[\frac{\beta\Delta\, \exp(-\beta\Delta)}{1+\exp(-\beta\Delta)} + \ln[1+\exp(-\beta\Delta)]\right].$$

[4 marks]

$$\left[\begin{array}{c} \text{Hint: Use either Gibbs expression for the statistical entropy} \\ S = -Nk_{\rm B}\sum_i p_i \ln p_i, \\ \text{or alternatively use} \\ F = -Nk_{\rm B}T\ln Z_1\,, \quad S = \dfrac{U-F}{T}\,. \end{array}\right]$$