Statistical Physics: Weekly Problem 4 (SP4)

(1) Consider an assembly of N weakly-interacting, distinguishable particles contained in a fixed volume V, with fixed internal energy U. Are the various distributions $\{n_i\}$ of the particles in single-particle states equally probable, or do they have different probabilities? State briefly what distinguishes the Boltzmann distribution from other distributions $\{n_i\}$ of the assembly of distinguishable particles. [2 marks]

(2) A paramagnetic solid consists of N ions with spin 1/2 and magnetic moment $\mu_{\rm B}$. The system lies in a magnetic field B and each magnetic moment is oriented either parallel to the field (up), with energy $\epsilon_\uparrow = -\mu_{\rm B}B$, or antiparallel (down) with energy $\epsilon_\downarrow = +\mu_{\rm B}B$. The system is in contact with a heat bath at temperature T.

(a) Write down the single-particle partition function Z_1 followed by the probability $p_\uparrow$ that a magnetic moment is up and the probability $p_\downarrow$ that it is down. [1 mark]

(b) The magnetisation per ion is equal to the average magnetic moment

$$M/N = \sum_i p_i m_i,$$

where $i = \uparrow, \downarrow$ and $m_\uparrow = \mu_{\rm B}$, $m_\downarrow = -\mu_{\rm B}$. Show that the magnetisation per ion is given by

$$\frac{M}{N} = \mu_{\rm B} \tanh\left(\frac{\mu_{\rm B}B}{k_{\rm B}T}\right)$$

where $k_{\rm B}$ is Boltzmann's constant. [2 marks]

(c) Obtain the internal energy U of the system of ions, directly from the definition,

$$U = N \sum_i p_i \epsilon_i.$$

Compare this with the energy of N magnetic moments, each of magnitude M/N, (b) above, and oriented along B. [2 marks]

(d) (i) Using Gibbs' definition

$$S = -N k_{\rm B} \sum_i p_i \ln p_i,$$

show that the entropy of the system depends on the magnetic field B and on the temperature T through the ratio B/T. [1 mark]

(ii) Sketch the graph of the entropy versus temperature for two different applied magnetic fields and then explain how a dilute paramagnetic solid can be cooled. [2 marks]