FoP3B Part II Lecture 8: London equation and thermodynamics of superconductors

As seen in the previous lecture the **Meissner effect** is a unique property of superconductors and is due to *perfect* **diamagnetism**, i.e. the magnetic field is completely excluded from within the material. This is achieved by generating *surface electric currents* (also called **supercurrents**) that opposes the external field. The supercurrent is determined by the **London equation** which is described below. *The superconductor effectively has to do work to exclude the external magnetic field and this has implications for its stability*, as seen previously from the discussion on **Type I** and **Type II** superconducting behaviour. In the last part of this lecture we will calculate the thermodynamic stability of superconductors under magnetic fields.

London equation and the London penetration depth

Since the supercurrent excludes the magnetic **B**-field the two must be directly related. In the **London equation** the current density **j** is directly proportional to the *magnetic vector potential* **A** (recall $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$)[1]:

$$\mathbf{j} = -\frac{1}{\mu_o \lambda_L^2}\mathbf{A} \qquad \dots (1)$$

where μ_o is the permeability of free space and λ_L is the so-called **London penetration depth**. In defining **A** we need to specify a **gauge**, since otherwise the gradient of any scalar field can be added to **A** leaving the **B**-field unchanged (i.e. **A** is not unique without a gauge). We use the **London gauge**, where $\vec{\nabla} \cdot \mathbf{A} = 0$. This follows from the **continuity equation**, which relates the time evolution of charge carriers (i.e. superconducting electrons n_s) to the gradient in current density, i.e.

$$\frac{\partial n_s}{\partial t} = \frac{1}{e}\vec{\nabla} \cdot \mathbf{j} \qquad \dots (2)$$

where e is the electronic charge. Since we are interested in **steady state** conditions n_s must be independent of time; Equation (2) is therefore zero and the London gauge naturally follows from Equation (1).

Let us consider the implications of Equation (1). From **Ampere's law** (time independent form of Maxwell's fourth equation):

$$\vec{\nabla} \times \mathbf{B} = \mu_o \mathbf{j} \qquad \dots (3)$$

From Equation (1):

$$\vec{\nabla} \times \mathbf{j} = -\frac{1}{\mu_o \lambda_L^2}\mathbf{B}$$

[1] Derivations of the London equation are given in the Supplementary notes (non-examinable)

… (4)

Taking the curl of (3) and substituting (4) gives:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{B}\right) = -\frac{1}{\lambda_L^2}\mathbf{B}$$

… (5)

Now $\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{B}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \mathbf{B}\right) - \nabla^2\mathbf{B}$. From Maxwell's second equation $\vec{\nabla} \cdot \mathbf{B} = 0$, so that:

$$\nabla^2\mathbf{B} = \frac{1}{\lambda_\mathrm{L}^2}\mathbf{B}$$

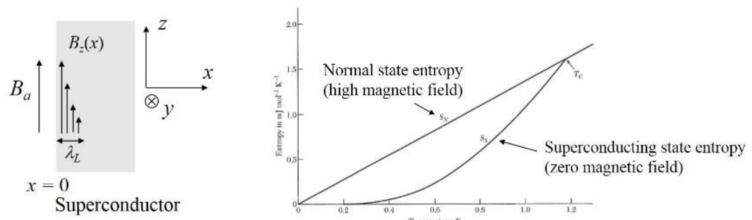
… (6)

The above equation predicts that the **B**-field decays rapidly inside a superconductor. To see this consider a relatively simple case of a **B**-field B_a applied parallel to the z-axis of a superconducting slab (Figure 1a). The **B**-field within the superconductor only varies in the x-direction and therefore Equation (6) reduces to:

$$\frac{d^2 B_z}{dx^2} = \frac{1}{\lambda_L^2}B_z$$

… (7)

where $B_z(x)$ is the z-component of the magnetic field at position x within the superconductor. It has solutions of the form $\exp(x/\lambda_L)$ and $\exp(-x/\lambda_L)$; the former can be ignored since it is not consistent with expulsion of magnetic fields from within the material. Hence using the **boundary condition** $B_z(x = 0) = B_a$, we have $B_z(x) = B_a\exp(-x/\lambda_L)$, i.e. *the magnetic field decreases exponentially within the material over a characteristic length* λ_L. Knowing the magnetic field the supercurrent **j** can be calculated using Equation (3). The supercurrent $j(x)$ flows in the y-direction and also has an exponential decrease over λ_L length scale, i.e. *the supercurrent is confined to the outer surface of the material*. <u>Note also that the direction of supercurrent flow is such that it opposes the applied **B**-field within the material</u>.



(a) **(b)**

Fig. 1: (a) Schematic of geometry used to calculate field penetration within a superconductor. (b) Entropy as a function of temperature for the superconducting and normal states. T_c is the critical phase transition temperature.

It can be shown[2] that the London penetration depth λ_L is given by:

[2] See derivations of the London equations in Supplementary reading.

$$\lambda_L = \sqrt{\frac{m}{\mu_0 n_s e^2}}$$
… (8)

where m is the electron mass. It will be shown in the next lecture that *superconducting electrons form in pairs* (known as **Cooper pairs**); hence n_s is the number of superconducting electrons (= twice the number of Cooper pairs). *The fact that electrons are paired in a superconductor means that the entropy is lower compared to the normal state*. This is illustrated in Figure 1b which plots the entropy of the material in the superconducting and normal states below the critical temperature T_c. Note that the superconductor is the stable phase at temperatures below T_c, but the normal state can be produced by applying a strong enough magnetic field, thus enabling measurement of its entropy. *From Figure 1b the entropy decrease for the superconductor w.r.t the normal state becomes larger at lower temperatures, implying a greater number of Cooper pairs n_s with decreasing temperature*. There are two important implications that follow:

(i) The electrons can be divided into zero resistivity superconducting electrons (i.e. Cooper pairs) and 'normal' electrons with non-zero resistivity. The fraction of superconducting electrons increases as the material is cooled below T_c. The overall resistivity of the material is nevertheless zero at all temperatures below T_c since the electric current is carried exclusively by the zero resistivity superconducting electrons.

(ii) There is no **latent heat** or entropy change at the critical temperature T_c when the normal state transitions to the superconducting state on cooling. This is known as a **second order phase transition**. As a comparison melting of ice or boiling of water is an example of **first order transition**, which is characterised by a latent heat and entropy change due to breaking of chemical bonds.

Thermodynamics of superconductors

Consider the free energy of a superconductor under an applied magnetic field. Let $G_s[0]$ be the free energy of the superconductor in zero field conditions. *The work* (W) *done on a material by a magnetic field is given by*:

$$dW = -\mathbf{M}(\mathbf{B}) \cdot d\mathbf{B}$$
… (9)

where **M** is the magnetisation, i.e. net magnetic dipole moment per unit volume. *Equation (9) follows from the result that the potential energy of a single magnetic dipole moment* $\boldsymbol{\mu}$ *in a* **B**-*field is* $-\boldsymbol{\mu}\cdot\mathbf{B}$. The superconductor free energy $G_s[B]$ under an applied field is then:

$$G_s[B] = G_s[0] - \int_0^B \mathbf{M}(\mathbf{B}) \cdot d\mathbf{B}$$
… (10)

Since the superconductor is a perfect diamagnet $M = -H = -B/\mu_o$, so that:

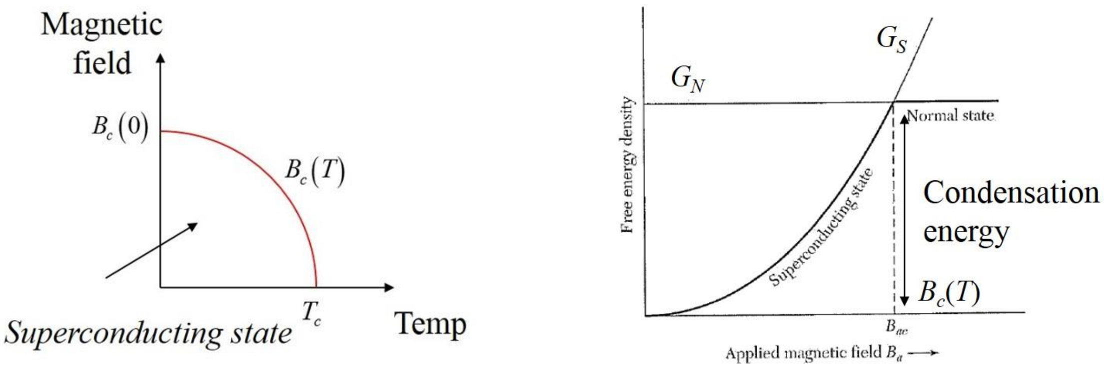
$$G_s[B] = G_s[0] + \frac{B^2}{2\mu_o} \qquad \dots (11)$$

Taking Type I superconductors as an example (Figure 2a) from Equation (11) the superconductor free energy will keep increasing with magnetic field until the critical field $B_c(T)$, beyond which the stable phase is the normal state. *At $B_c(T)$ the free energy of superconducting and normal states are equal*. Denote by $G_N[0]$ the free energy of the normal state in zero field conditions. *Since the normal state is* **paramagnetic** *its magnetisation* **M** *will be small and therefore the free energy will not change significantly with magnetic field*. Therefore $G_N[B] \approx G_N[0]$, so that:

$$G_s[B_c(T)] = G_s[0] + \frac{B_c(T)^2}{2\mu_o} = G_N[B_c(T)] = G_N[0]$$

$$\Rightarrow G_N[0] - G_s[0] = \frac{B_c(T)^2}{2\mu_o} \qquad \dots (12)$$

($G_N[0]$- $G_S[0]$) is called the **condensation energy** and represents the *free energy difference between normal and superconducting states under zero field conditions*. This is illustrated schematically in Figure 2b which plots the free energy as a function of magnetic field for the superconducting and normal states below T_c. Note the normal state energy is approximately a horizontal line, due to the paramagnetic behaviour; the superconductor energy however shows a quadratic dependence, as predicted by Equation (11). *The condensation energy of a superconductor is only μeV/atom; as a comparison the latent heat of fusion for ice melting is 63 meV/atom*.



(a) **(b)**

Fig. 2: (a) Phase diagram for a Type I superconductor and (b) Free energy vs magnetic field for the normal and superconducting states below T_c. The condensation energy is highlighted.