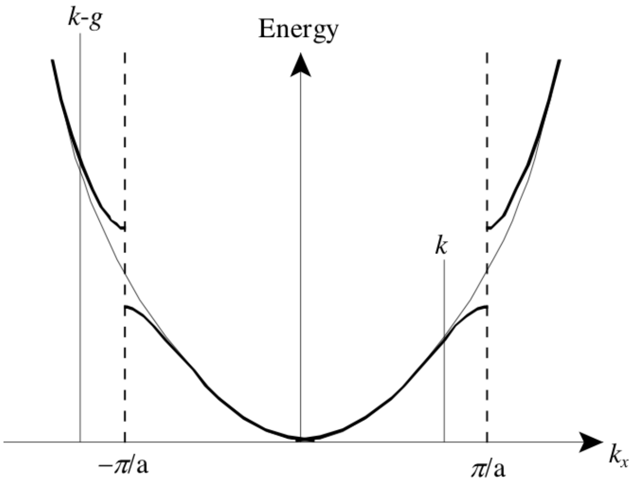
Level 3 Condensed Matter Physics- Part I
Weekly problem 2 solutions

(1) The group velocity is the velocity of electrons in Bloch states in energy bands. For ordinary traveling waves we have $v_g = \frac{\partial \omega}{\partial k}$ and in Bloch states where $E = \hbar\omega$ we have $v_g = \frac{1}{\hbar}\frac{dE}{dk}$ [1 mark].

Thus, the velocity depends on the gradient of the $E(k)$ relationship and this in the nearly-free electron approximation is not constant across the Brillouin zone.
[1 mark]



The diagram, right, shows the typical $E(k)$ behaviour for nearly free electrons in a metal across the first Brillouin zone from $-\frac{\pi}{a}$ to $+\frac{\pi}{a}$.
[1 marks]
Unlike the free electron parabola, the electron group velocity can vary, from zero at the zone centre and zone boundaries up to a maximum close to the middle of the zone at $\pm\frac{\pi}{2a}$. [1 mark]

(2) We know that

$$v_g = \frac{1}{\hbar}\frac{dE}{dk} = \frac{1}{\hbar}\frac{d}{dk}(Ck^2 - Dk^4) = \frac{1}{\hbar}(2Ck - 4Dk^3) = \frac{1}{\hbar}2Ck(1 - 2Dk^2).$$

[1 mark]
The group velocity is zero at $k = 0$ and at $k = \pm\pi/a$. [1 mark]
Setting $k = \pi/a$ and equating v_g to zero implies that $D = \frac{1}{2}\frac{a^2}{\pi^2}$. [1 mark]

The group velocity starts off as zero at the start of the Brillouin zone centre and increases to a maximum value close to the middle of the zone before decreasing again to zero at the Brillouin zone boundary.

(3) Evaluating and substituting for D gives $\frac{d^2E}{dk^2} = 2C - 6C\frac{a^2k^2}{\pi^2}$. Since the system is isotropic (i.e. the energy E depends only on the magnitude of k) this gives an effective mass of $\hbar^2 \Big/ \left[2C - 6C\frac{a^2k^2}{\pi^2}\right]$. Evaluating at $k = 0$ gives $m_{eff} = \frac{\hbar^2}{2C}$ and at $k = \pm\frac{\pi}{a}$ gives $m_{eff} = -\frac{\hbar^2}{4C}$.

[3 marks]