Level 3 Condensed Matter Physics- Part I
Weekly problem 3 solutions

(1) Assume that the uniform **B**-field is along the z-axis. Then:

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{B}\times\mathbf{r}}{2} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & B \\ x & y & z \end{vmatrix} = \frac{1}{2}[-(By)\hat{\mathbf{i}} + (Bx)\hat{\mathbf{j}}]$$

[1 mark]

From the definition of the magnetic vector potential, the magnetic field **B** is:

$$\mathbf{B} = \vec{\nabla}\times\mathbf{A} = \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -By & Bx & 0 \end{vmatrix} = B\hat{\mathbf{k}}$$

The correct form of the magnetic field is obtained, which validates the choice of magnetic vector potential **A**.

[1 mark]

(2) From the previous question $\mathbf{A}(\mathbf{r})$ has components only in the xy-plane. We can express these in component form using polar coordinates:

$$\mathbf{A}(\mathbf{r}) = A_R\hat{\mathbf{R}} + A_\theta\hat{\boldsymbol{\theta}}$$

Making use of the relationship between Cartesian and polar coordinates we can write:

$$-\frac{By}{2} = -\frac{BR\sin\theta}{2} = A_R\cos\theta - A_\theta\sin\theta$$
$$\frac{Bx}{2} = \frac{BR\cos\theta}{2} = A_R\sin\theta + A_\theta\cos\theta$$

Solving these simultaneous equations we obtain $A_R = 0$ and $A_\theta = BR/2$, i.e. $\mathbf{A}(\mathbf{r})$ points in the $\hat{\boldsymbol{\theta}}$ direction and has magnitude that depends on R only. This shows that $\mathbf{A}(\mathbf{r})$ is rotationally invariant.

Note that it is possible to select other gauges as well, e.g. $\mathbf{A}(\mathbf{r}) = (0, Bx, 0)$. However, the symmetric gauge is used since its rotational symmetry is well-suited for studying the magnetic properties of an electron in a spherically symmetric atom.

[2 marks]

(3) Let ψ denote a wavefunction. Then using the result for $\mathbf{A}(\mathbf{r})$ from question (1):

$$\mathbf{p}\cdot\mathbf{A}\psi = \frac{-i\hbar}{2}B\left[-\frac{\partial(y\psi)}{\partial x} + \frac{\partial(x\psi)}{\partial y}\right] = \frac{i\hbar}{2}B\left[y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right]$$

[1 mark]

$$\mathbf{A} \cdot \mathbf{p}\psi = \frac{-i\hbar}{2}\left[-(By)\frac{\partial\psi}{\partial x} + (Bx)\frac{\partial\psi}{\partial y}\right] = \frac{i\hbar}{2}B\left[y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right]$$

[1 mark]

It is therefore clear that **p** and **A** commute.

(4) The current density is given by:

$$\mathbf{J}(\mathbf{r}) = -e\rho(\mathbf{r})\mathbf{v}$$

where **v** is the velocity of the electrons. This follows from the fact that $\rho(\mathbf{r})|\mathbf{v}|$ number of electrons with charge $-e$ will flow out of unit cross-sectional area within unit period of time. The change in velocity due to the magnetic field is $e\mathbf{A}(\mathbf{r})/m$, so that the induced current density is:

$$\mathbf{J}(\mathbf{r}) = \frac{e^2}{m}\rho(\mathbf{r})[\mathbf{r} \times \mathbf{B}]$$

[2 marks]

Note that the $[\mathbf{r} \times \mathbf{B}]$ term implies that there is a connection between $\mathbf{J}(\mathbf{r})$ and diamagnetism, which has energy proportional to $(\mathbf{B} \times \mathbf{r})^2$. In fact, $\mathbf{J}(\mathbf{r})$ is called the diamagnetic current density- it can be interpreted as the induced current density within the atom that opposes the applied magnetic field. Note also that for the symmetric gauge $\vec{\nabla} \cdot \mathbf{A} = 0$, and therefore $\vec{\nabla} \cdot \mathbf{J} = 0$ as well. This implies that the atom is in steady state, and there is no net flow of electrons out of the atom due to the magnetic field.

[2 marks]