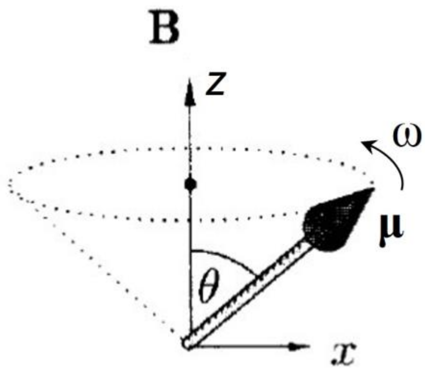
Condensed Matter Physics 3
Example Workshop 3 – Solution

1. Larmor precession

(a) The schematic of the applied field **B** and magnetic moment **μ** is shown below:



From **B** = (0,0,B) and resolving the precession formula $\frac{d\boldsymbol{\mu}}{dt} = -\gamma(\boldsymbol{\mu} \times \mathbf{B})$ into individual components:

$$\frac{d\mu_x}{dt} = -\gamma B \mu_y \qquad \text{… (1a)}$$

$$\frac{d\mu_y}{dt} = \gamma B \mu_x \qquad \text{… (1b)}$$

$$\frac{d\mu_z}{dt} = 0 \qquad \text{… (1c)}$$

Differentiating Eq. 1a w.r.t. t and substituting Eq. 1b gives:

$$\frac{d^2\mu_x}{dt^2} + (\gamma B)^2 \mu_x = 0 \qquad \text{… (2)}$$

The solution to the above equation is of the form $\mu_x = C\sin(\gamma B t) + D\cos(\gamma B t)$, where C and D are constants that must satisfy the boundary conditions, namely $\mu_x = \mu\sin\theta$ at t = 0. This gives C =0 and D = $\mu\sin\theta$, so that:

$$\mu_x = (\mu\sin\theta)\cos(\gamma B t)$$

Using similar arguments, an Equation similar to Eq. 2 can be derived for μ_y, the solution for which is $\mu_y = C\sin(\gamma B t) + D\cos(\gamma B t)$. From the boundary conditions $\mu_y = 0$ at t = 0, it follows that D =0. The value of C is obtained by noting that during precession $\sqrt{\mu_x^2 + \mu_y^2} = \mu\sin\theta$ (see figure), which implies $C = \mu\sin\theta$ and therefore:

$$\mu_y = (\mu\sin\theta)\sin(\gamma B t)$$

From Eq. 1c the μ_z component is unchanged by precession, and therefore its value is (see figure):

$$\mu_z = \mu\cos\theta$$

(b) The magnetic moment components can be written as:

$$\mu_x = (\mu\sin\theta)\cos(\omega t)$$
$$\mu_y = (\mu\sin\theta)\sin(\omega t)$$

where $\omega = \gamma B$. From the time-dependence of μ_x and μ_y it is clear that ω represents an angular frequency. Substituting $\gamma = e/2m$ for the gyromagnetic ratio gives the desired result:

$$\omega = \frac{eB}{2m}$$

(c) The energy of the magnetic moment is given by $E = -\boldsymbol{\mu} \cdot \mathbf{B}$. Since the angle θ between the vectors $\boldsymbol{\mu}$ and $\mathbf{B}$ is constant during precession, there is no change in energy and therefore no work is done by the magnetic field.

2. Diamagnetism in Germanium.

(a) The diamagnetic response of Ge will be

$$\chi_d = -\frac{\mu_0 n_{Ge} Z_{Ge} e^2 \langle r_{Ge}^2 \rangle}{6m_e}$$

The number of Ge atoms per unit volume is $n_{Ge} = \frac{N_A \rho}{M_{Ge}}$ where Avogadro's number N_A = 6.023 x 10^{23}, the molar mass, in kg is M_{Ge} = 72.63 x 10^{-3} kg and the density is ρ = 5.323 x 10^3 kg m^{-3}.

$$n_{Ge} = \frac{N_A \rho}{M_{Ge}} = \frac{6.023 \times 10^{23} \times 5.323 \times 10^3}{72.63 \times 10^{-3}} = 4.41 \times 10^{28} \text{ m}^{-3}.$$

$$Z_{Ge} = 32; \qquad \sqrt{\langle r_{Ge}^2 \rangle} = 0.12 \text{ nm}$$

$$\chi_d = -\frac{\mu_0 n_{Ge} Z_{Ge} e^2 \langle r_{Ge}^2 \rangle}{6m_e}$$
$$= -\frac{4\pi \times 10^{-7} \times 4.41 \times 10^{28} \times 32 \times (1.6 \times 10^{-19})^2 \times (0.12 \times 10^{-9})^2}{6 \times 9.11 \times 10^{-31}}$$
$$= -1.197 \times 10^{-4}$$

(b) Curie's Law, $\chi_p = \frac{C}{T} = \frac{N \times 2.356 \times 10^{-22}}{T}$. At 2 K the magnetisation, M, is zero and as $M = \chi H$, with H = 100 A m^{-1}, this implies that χ = 0.
Therefore at 2 K, the total measured magnetic susceptibility is $\chi = \chi_p + \chi_d = 0$.
We assume that we take the diamagnetic response of the arsenic atoms to be the same as germanium atoms (a reasonable assumption as they are neighbours in the 4th row of the periodic table), we have the situation where the diamagnetic contribution of the germanium cancels the paramagnetic susceptibility of the arsenic atoms.
Substituting for χ_p and χ_d and rearranging for N gives

$$N = \frac{-\chi_d T}{2.356 \times 10^{-22}} = \frac{1.197 \times 10^{-4} \times 2}{2.356 \times 10^{-22}} = 1.02 \times 10^{18}\ \mathrm{m}^{-3}.$$

(c) Relative permeability is related to susceptibility by $\mu_r = 1 + \chi$. To achieve a value of 10^5 we first substitute in for χ, $\mu_r = 1 + \chi = 1 + {}^{C}/_{T} + \chi_d$, and rearranging for T gives

$$T = {}^{C}/_{(\mu_r - 1 - \chi_d)} = \frac{1.02 \times 10^{18} \times 2.356 \times 10^{-22}}{10^5 - 1 + 1.197 \times 10^{-4}} = 2.40 \times 10^{-9}\ \mathrm{K}$$

To calculate the flux density at this value for the relative permeability we use $B = \mu_r \mu_0 H$ and for the infinite solenoid, $H = nI$ where n is the number of turns per unit length and I is the current through the solenoid. Substituting in for H and rearranging gives an expression for the current

$$I = {}^{B}/_{(n\mu_r\mu_0)} = {}^{1}/_{(10^3 \times 10^5 \times 4 \times \pi \times 10^7)} = 7.96 \times 10^{-3}\ \mathrm{A}$$

3. Paramagnetism in a solid with angular momentum *J*.

(a) We have $J_z = m_J \hbar$, where $m_J = -J, -J+1, \dots, J-1, J$.

The z-component of the magnetic dipole moment is $\mu_z = -\gamma g_J (m_J \hbar) = -\mu_B g_J m_J$ and the energy in a magnetic field is $-\mu_z B = \mu_B g_J m_J B$. The partition function Z is therefore:

$$Z = \sum_{m_J=-J}^{J} \exp\left(\frac{\mu_B g_J m_J B}{kT}\right) = \sum_{m_J=-J}^{J} \exp(m_J x)$$

Using the formula for a geometric series to solve for Z:

$$\sum_{m_J=-J}^{J} \exp(m_J x) = e^{-Jx}[1 + e^x + \cdots + e^{2Jx}] = e^{-Jx}\left[\frac{e^{(2J+1)x} - 1}{e^x - 1}\right] = \frac{e^{(J+1)x} - e^{-Jx}}{e^x - 1}$$

Dividing numerator and denominator by $e^{x/2}$:

$$Z = \frac{e^{(2J+1)x/2} - e^{-(2J+1)x/2}}{e^{x/2} - e^{-x/2}} = \frac{\sinh[(2J+1)x/2]}{\sinh[x/2]}$$

(b) The average magnetic moment is given by:

$$\langle \mu_z \rangle = \frac{-\mu_B g_J \sum_{m_J=-J}^{J} m_J \exp(m_J x)}{\sum_{m_J=-J}^{J} \exp(m_J x)}$$

From the definition of the partition function $Z = \sum_{m_J=-J}^{J} \exp(m_J x)$ it is clear that:

$$\frac{\partial Z}{\partial x} = \sum_{m_J=-J}^{J} m_J \exp(m_J x)$$

So that:

$$\langle \mu_z \rangle = \frac{-\mu_B g_J}{Z} \frac{\partial Z}{\partial x}$$

(c) The saturation magnetic moment $(\mu_z)_s = -\mu_B g_J J$ and consequently:

$$\frac{M}{M_s} = \frac{\langle \mu_z \rangle}{(\mu_z)_s} = \frac{1}{JZ} \frac{\partial Z}{\partial x}$$

Using the fact that $Z = \frac{e^{(2J+1)x/2} - e^{-(2J+1)x/2}}{e^{x/2} - e^{-x/2}}$ it follows that:

$$\frac{\partial Z}{\partial x} = \left(\frac{2J+1}{2}\right) \frac{\cosh[(2J+1)x/2]}{\sinh[x/2]} - \frac{1}{2} \frac{\cosh[x/2] \sinh[(2J+1)x/2]}{\sinh^2[x/2]}$$

Hence:

$$\frac{M}{M_s} = \frac{1}{JZ} \frac{\partial Z}{\partial x} = \left(\frac{2J+1}{2J}\right) \coth[(2J+1)x/2] - \frac{1}{2J} \coth[x/2]$$

Making the substitution $y = xJ$ we obtain the desired result.

(d) The Brillouin function:

$$B_J(y) = \frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} y\right) - \frac{1}{2J} \coth\left(\frac{y}{2J}\right)$$

For $J = \infty$ the first term is simply coth(y). For the second term:

$$\lim_{J \to \infty} \coth\left(\frac{y}{2J}\right) = \lim_{J \to \infty} \frac{e^{\frac{y}{2J}} + e^{-\frac{y}{2J}}}{e^{\frac{y}{2J}} - e^{-\frac{y}{2J}}}$$
$$= \frac{\left(1 + \frac{y}{2J}\right) + \left(1 - \frac{y}{2J}\right)}{\left(1 + \frac{y}{2J}\right) - \left(1 - \frac{y}{2J}\right)} = \frac{2J}{y}$$

Substituting we obtain:

$$B_\infty(y) = \coth y - \frac{1}{y}$$