Level 3 Condensed Matter Physics- Part II
Examples Class 5 Answers

(1) i) We have:

$$B_c(T) = B_c(0)\left[\frac{T_c^2 - T^2}{T_c^2}\right]$$

Since T is close to T_c we can write $(T_c^2 - T^2) = (T_c - T)(T_c + T) \approx 2T_c(T_c - T)$. Substituting this expression gives the desired result.

ii) The condensation energy according to GL theory is $[\dot{a}(T - T_c)]^2/2b$. From thermodynamics the condensation energy is $B_c(T)^2/2\mu_o$. Equating the two expressions and solving for $B_c(T)$ we get:

$$B_c(T) = \pm\sqrt{\frac{\mu_o}{b}}\,\dot{a}(T - T_c)$$

Since $\dot{a}$ and $\sqrt{(\mu_o/b)}$ are both positive only the negative solution for $B_c(T)$ is physically meaningful. We then have:

$$B_c(T) = \sqrt{\frac{\mu_o}{b}}\,\dot{a}(T_c - T)$$

This has the same form as the expression in (i).

iii) From the definition of the coherence length:

$$\xi = \left(\frac{\hbar^2}{2m|a(T)|}\right)^{1/2} = \left(\frac{\hbar^2}{2m\dot{a}(T_c - T)}\right)^{1/2}$$

For the London penetration depth we use the fact that the density of superconducting electrons is $n_s = 2|\psi|^2 = 2\dot{a}(T_c - T)/b$. Therefore:

$$\lambda_L = \left(\frac{m}{\mu_o n_s e^2}\right)^{1/2} = \left[\frac{mb}{2\mu_o \dot{a} e^2 (T_c - T)}\right]^{1/2}$$

Taking ratios:

$$\kappa = \frac{\lambda_L}{\xi} = \left(\frac{m^2 b}{\mu_o \hbar^2 e^2}\right)^{1/2}$$

κ is therefore independent of temperature and depends only on the GL parameter b.

iv) For a Type I superconductor $\kappa < 1$ and for Type II $\kappa > 1$ (strictly speaking the full GL theory actually predicts $\kappa < 1/\sqrt{2}$ for Type I and $\kappa > 1/\sqrt{2}$ for Type II). Therefore:

$$b < \frac{\mu_o \hbar^2 e^2}{m^2} \quad \text{(Type I)}$$

$$b > \frac{\mu_o \hbar^2 e^2}{m^2} \quad \text{(Type II)}$$

(2) i) The local magnetisation is $\mathbf{M}(r) = [\mathbf{B}(r) - \mathbf{B}_a]/\mu_o$. The magnetic work done per unit volume at the position r is therefore:

$$W(r) = \int_0^{B_a} \frac{B_a}{\mu_o}\left[\alpha\left(\frac{R^2 - r^2}{\lambda_L^2}\right)\right] dB_a = \frac{\alpha B_a^2}{2\mu_o \lambda_L^2}(R^2 - r^2)$$

Since $G_s(r, B_a) = G_s(0) + W(r)$ we obtain the desired result.

ii) Consider a cylinder of unit length. The average magnetic work done per unit volume $<W>$ is defined by:

$$\pi R^2 < W >= \int_0^R 2\pi r W(r) dr$$

Substituting the expression for $W(r)$ and integrating we obtain:

$$< W >= \frac{\alpha B_a^2 R^2}{4\mu_o \lambda_L^2}$$

We can therefore express the average free energy per unit volume of the superconductor phase in a magnetic field $<G_s>$ as:

$$< G_s >= G_s(0) + \frac{\alpha B_a^2 R^2}{4\mu_o \lambda_L^2}$$

At the critical field $B_a = B_c$ we have $<G_s> = G_N$, which leads to the desired result.

iii) At the critical magnetic field the work done by the magnetic field equals the free energy difference between the superconductor and normal states under zero field conditions (a constant).

For a cylinder the average work done by the magnetic field is given by $<W>$. Compare this to a bulk superconductor, where the magnetic work done is $B_a^2/2\mu_o$. We have:

$$< W >= \frac{\alpha R^2}{2\lambda_L^2}\left(\frac{B_a^2}{2\mu_o}\right)$$

Since $R << \lambda_L$ it is clear that the critical field for a cylinder will be larger than the bulk superconductor if $\alpha < 1$.

(3) i) Using the fact that the magnetic flux passing through the superconducting ring is preserved :

At $\phi = 0^o$ the entire flux Φ is provided by the external field. Therefore the supercurrent drops to zero.
At $\phi = 60^o$ the flux through the ring due to the external field is $\Phi\cos(60^o)$ or $\Phi/2$. Therefore the remaining $\Phi/2$ of flux must be provided by a reduced supercurrent of ($j/2$); note that from the Biot-Savart law the axial magnetic field due to a current carrying loop is proportional to the current.
At $\phi = 90^o$ there is no flux from the external field passing through the ring and the supercurrent is therefore j.

ii) If the external field is increased to $3B_a/2$:

At $\phi = 0^o$ a supercurrent $j/2$ will flow in the *opposite* sense to cancel the additional flux $\Phi/2$ from the external field.
At $\phi = 60^o$ the flux through the ring due to the external field is $(3\Phi/2)\cos(60^o)$ or $3\Phi/4$. The supercurrent will therefore be $j/4$; the direction of flow is unchanged.
At $\phi = 90^o$ the supercurrent in unchanged since there is no flux from the external field passing through the ring.