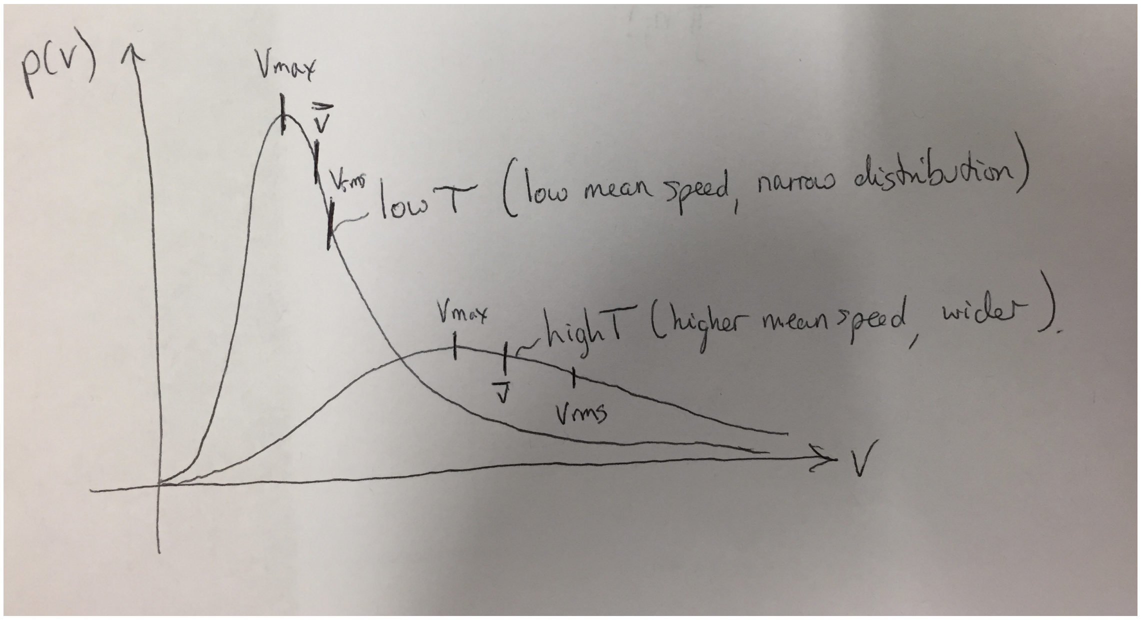
Statistical Physics: Weekly Problem 6 (SP6)

(1) (a) Sketch of distribution and note that $p(0) = 0$ and $p(\infty) \to 0$.



To find behaviour at small v we can simplify the structure of $p(v)$ by setting all the constants to 1 giving $p(v) \sim v^2 \exp(-v^2)$. For small v we have $\exp(-v^2) \sim 1$ hence we get

$$\lim_{v \to 0} p(v) \sim v^2$$

[2 marks]

(b) From lectures we evaluated the integrals to find the average values. The results are

(i) most probable speed

$$v_{max} = \sqrt{\frac{2}{\beta m}} \quad \text{or} \quad \sqrt{2}\sqrt{\frac{k_B T}{m}},$$

(ii) mean speed

$$\bar{v} = \sqrt{\frac{8}{\pi}} \frac{1}{\sqrt{\beta m}} \quad \text{or} \quad \sqrt{\frac{8}{\pi}}\sqrt{\frac{k_B T}{m}},$$

(iii) r.m.s speed

$$v_{rms} = \sqrt{\frac{3}{\beta m}} \quad \text{or} \quad \sqrt{3}\sqrt{\frac{k_B T}{m}}.$$

[1 mark]

If we define $v_T = 1/\sqrt{\beta m} = \sqrt{k_B T/m}$ and then in units of v_T we have

$$v_{max} = 1.414\, v_T, \quad \bar{v} = 1.596\, v_T, \quad v_{rms} = 1.732\, v_T$$

making it straightforward to mark on a graph, shown in (a). [1 mark]

(c) Some calculator time. We have $m = 3.36 \times 10^{-26}$ kg and $T = 300$ K so $v_T = 351$ ms^{-1}. This gives $v_{max} = 496$ ms^{-1}, $\bar{v} = 560$ ms^{-1} and $v_{rms} = 607$ ms^{-1}. [2 marks]

(d) (i) For normalisation

$$\begin{aligned}
1 &= \int_0^\infty p(v)\, dv \\
&= C \int_0^\infty dv\, v \exp\left(-\frac{mv^2}{2k_BT}\right) \\
&= C\, I_1(\alpha) = \frac{C}{2\alpha} \\
&= \frac{C\, k_B\, T}{m} \\
\Rightarrow C &= \frac{m}{k_BT}
\end{aligned}$$

[2 marks]
The probability distribution in both 2D and 3D $p(v = 0) = 0$. In 3D for small v we had $p(v) \sim v^2$ and in 2D, we have $p(v) \sim v$. For large v in both cases the probability distribution goes to zero.

(ii) The most probable speed is

$$\frac{dp(v)}{dv} = Ce^{-\frac{\beta m v^2}{2}}\left[1 - \beta m v^2\right] = 0 \quad \Rightarrow v_{max} = \frac{1}{\sqrt{\beta m}} = v_T,$$

the mean speed is

$$\bar{v} = \frac{\int_0^\infty dv\, v^2 e^{-\frac{\beta m v^2}{2}}}{\int_0^\infty dv\, v e^{-\frac{\beta m v^2}{2}}} = \frac{I_2(\lambda)}{I_1(\lambda)} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} = \sqrt{\frac{\pi}{2\beta m}} = \sqrt{\frac{\pi}{2}}\, v_T = 1.253\, v_T,$$

and the RMS speed is

$$v_{rms}^2 = \frac{\int_0^\infty dv\, v^3 e^{-\frac{\beta m v^2}{2}}}{\int_0^\infty dv\, v e^{-\frac{\beta m v^2}{2}}} = \frac{I_3(\lambda)}{I_1(\lambda)} = \frac{1}{\lambda} = \frac{2}{\beta m} = 2\, v_T^2 \;\Rightarrow\; v_{rms} = \sqrt{2}\, v_T = 1.414\, v_T.$$

[2 marks]