Statistical Physics: Weekly Problem 3 (SP3)

(1) (a) For the microcanonical ensemble, all of the compatible microstates are assumed to be equally probable (this is the basic postulate of equal a priori probabilities).

[1 mark]

(b) For the canonical ensemble, the probabilities of the compatible microstates are different, depending on the energy of the microstate (which is fixed temperature, not fixed energy). The probabilities are given by the Boltzmann factor, i.e. they are proportional to $\exp(-E/(k_{\mathrm{B}}T))$, where E is the energy of the microstate.

[1 mark]

(2)

$$\frac{p_0}{p_1} = \exp\left(-\frac{E_0 - E_1}{k_{\mathrm{B}}T}\right) \Rightarrow T = \frac{E_1 - E_0}{k_{\mathrm{B}} \ln(p_0/p_1)} = \frac{(13.6 - 3.4)\mathrm{eV}}{\ln(100) \times 8.617 \times 10^{-5} eV K^{-1}} \simeq 25700K$$

[2 marks]

(3) (a) The energy parallel to B is $-\mu_B B$ and the probability that the ion will have its magnetic moment oriented parallel to B is proportional to: $p_\uparrow \propto \exp[\mu_B B/(k_{\mathrm{B}}T)]$.

[1 mark]

The probability that the ion will have its magnetic moment antiparallel to B is $p_\downarrow \propto \exp[-\mu_B B/(k_{\mathrm{B}}T)]$. These probabilities are normalised

$$p_\uparrow + p_\downarrow = 1 \Rightarrow p_\uparrow = \frac{\exp[\mu_B B/(k_{\mathrm{B}}T)]}{\exp[\mu_B B/(k_{\mathrm{B}}T)] + \exp[-\mu_B B/(k_{\mathrm{B}}T)]} = \frac{1}{1 + \exp[-2\mu_B B/(k_{\mathrm{B}}T)]}$$

[1 mark]

(b) When all the magnetic moments are parallel to B, each ion has energy $-\mu_{\mathrm{B}} B$. There are N ions, so the internal energy is $U = -N\mu_{\mathrm{B}} B$.

[1 mark]

When all the magnetic moments are oriented parallel to B, there is only way of doing this (perfect order, one microstate), $\Omega = 1$ and the entropy vanishes is $S = k_{\mathrm{B}} \ln \Omega = 0$.

[1 mark]

Since all the ions are in their lowest energy state, the whole system is in the ground state and the temperature must be zero. Another way: since all the ions are parallel to B, the probability must be $p_\uparrow = 1$, so $\exp[-2\mu_B B/(k_{\mathrm{B}}T)] = 0$. This happens when $2\mu_B B/(k_{\mathrm{B}}T) \to \infty$, so the temperature $T = 0$.

[1 mark]

(c) Since the internal energy is positive, we have $p_\uparrow(-\mu_B B) + p_\downarrow \mu_B B > 0$. As $\mu_B B > 0$, we have $p_\uparrow < p_\downarrow$. From the Boltzmann distribution we have

$$\frac{p_\uparrow}{p_\downarrow} = \exp\left(\frac{2\mu_B B}{k_B T}\right) < 1.$$

Taking the logarithm of both sides of the inequality, we obtain

$$\frac{2\mu_B B}{k_B T} < 0 \;\Rightarrow\; T < 0.$$

Infinite T corresponds to equal population of up and down energy levels, $p_i = 1/2$, and hence $U = 0$. This is lower than any $U > 0$ and so, negative temperatures are "hotter" than $T = \infty$.

[1 mark]