Statistical Physics: Workshop Problems 3

(1) (a) If the microstates states of a system occur with probability p_i then the entropy of the system is

$$S = -k_B \sum_i p_i \log p_i.$$

Therefore the entropy of an 8-sided dice is $k_B \log 8$ and for a 12-sided dice it's $k_B \log 12$.

(b) The phase space density of the microcanonical ensemble is constant in equilibrium.

(c) The partition functions are dimensionless.

(d) (N,V,E) in the microcanonical ensemble, (N,V,T) in the canonical ensemble and (μ,V,T) in the grand canonical ensemble (where μ is the chemical potential/Fermi level).

(2) (a) The partition function is $Z = \sum_{\text{states}} e^{-\beta \epsilon_{\text{states}}}$ where $\beta = 1/k_B T$. Therefore

$$Z = \sum_{i=1}^{3} e^{-\beta \hbar \omega_i} = e^{-\beta \hbar \omega} + e^{-\beta \hbar 2\omega} + e^{-\beta \hbar 3\omega}.$$

The probabilities are $p_i = e^{-\beta \epsilon_i}/Z$ hence

$$p_1 = \frac{e^{-\beta \hbar \omega}}{Z}, p_2 = \frac{e^{-\beta \hbar 2\omega}}{Z}, p_3 = \frac{e^{-\beta \hbar 3\omega}}{Z}.$$

(b) The limit $T \to 0$ is $\beta \to \infty$ so we have

$$(p_1) : \lim_{\beta \to \infty} \frac{e^{-\beta \hbar \omega}}{e^{-\beta \hbar \omega} + e^{-\beta \hbar 2\omega} + e^{-\beta \hbar 3\omega}} = 1,$$

$$(p_2) : \lim_{\beta \to \infty} \frac{e^{-2\beta \hbar \omega}}{e^{-\beta \hbar \omega} + e^{-\beta \hbar 2\omega} + e^{-\beta \hbar 3\omega}} = 0,$$

$$(p_3) : \lim_{\beta \to \infty} \frac{e^{-3\beta \hbar \omega}}{e^{-\beta \hbar \omega} + e^{-\beta \hbar 2\omega} + e^{-\beta \hbar 3\omega}} = 0,$$

and the limit $T \to \infty$ is $\beta \to 0$ so we have

$$(p_j) : \lim_{\beta \to 0} \frac{e^{-j\beta \hbar \omega}}{e^{-\beta \hbar \omega} + e^{-\beta \hbar 2\omega} + e^{-\beta \hbar 3\omega}} = \frac{1}{3}, j = 1, 2, 3$$

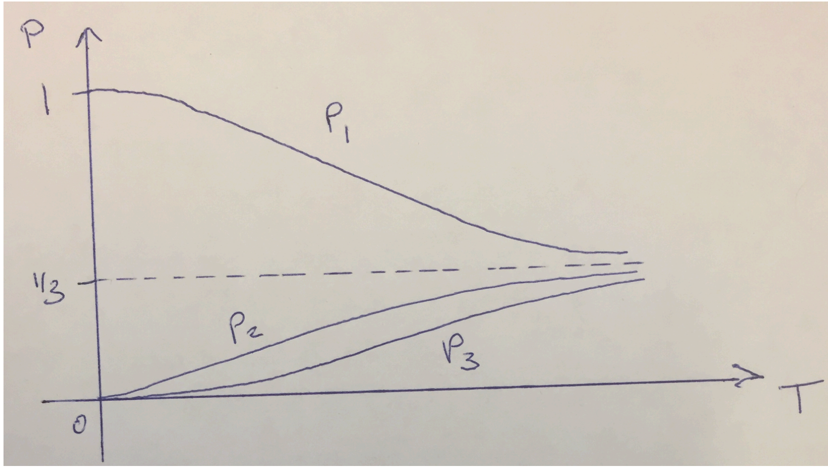
(c) Internal energy is

$$U = -\left[\frac{\partial \ln Z}{\partial \beta}\right] == \hbar \omega \frac{e^{-\beta \hbar \omega} + 2e^{-2\beta \hbar \omega} + 3e^{-3\beta \hbar \omega}}{e^{-\beta \hbar \omega} + e^{-2\beta \hbar \omega} + e^{-3\beta \hbar \omega}}.$$

Free energy is

$$F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \ln \left(e^{-\beta \hbar \omega} + e^{-\beta \hbar 2\omega} + e^{-\beta \hbar 3\omega}\right).$$

The difference between internal energy and the free energy is essentially the entropy, $S = -\beta k_B (U - F)$, and so represents the unavailability of energy to do work.



3) (a) Tabulate the possibilities with $N = 3$ and $U = 3\epsilon$

	0ϵ	1ϵ	2ϵ	$3\ \epsilon$	...	Classical Permutations
D_1	2	0	0	1		$3!/(2!\ 0!\ 0!\ 1!) = 3$
D_2	1	1	1	0		$3!/(1!\ 1!\ 1!\ 0!) = 6$
D_3	0	3	0	0		$3!/(0!\ 3!\ 0!\ 0!) = 1$
						3+6+1=10 microstates

For classical particles the distribution D_2 is most likely with a probability of 6/10.

(b) For Fermions then only one particle per state is allowed. Distributions D_1 and D_3 are not possible and so only distribution D_2 is allowed. Note also that for indistinguishable particles there is only one microstate in distribution D_2 (no permutations). We find distribution D_2 with 100% probability.

(c) For Bosons each of the classical distributions are allowed but since the particles are indistinguishable then there is only one microstate per distribution. Hence all 3 distributions are equally likely, probability 1/3.

(4) The population distribution is given by

$$\frac{n_j}{n_i} = \exp\left(\frac{\epsilon_i - \epsilon_j}{k_B T}\right)$$

therefore

$$T = \frac{\epsilon_i - \epsilon_j}{k_B \log\left(\frac{n_j}{n_i}\right)}.$$

Using the values of n_1, n_2 and n_3 given in the question, taking each possible pair we obtain the following values of T: 99.0K, 100.2K and 99.6K. Therefore, with the information given the mean value of temperature is 99.6K.