Statistical Physics: Weekly Problem 7 (SP7)

(1) (a) We have (from lectures, notes, books) for a free particle in a box

$$g(k)\delta k = \frac{V}{2\pi^2} k^2\,\delta k.$$

and that

$$\epsilon(k) = \frac{\hbar^2 k^2}{2M},$$

hence the single-particle partition function is

$$Z_1 = \int_0^\infty dk\, g(k)\, e^{-\beta\epsilon(k)} = \frac{V}{2\pi^2}\int_0^\infty dk\, k^2 e^{-\frac{\beta\hbar^2 k^2}{2M}} = V\left(\frac{2\pi M}{\beta h^2}\right)^{\frac{3}{2}} = \frac{V}{\lambda_{\mathrm{D}}^3},$$

[1 mark]

(b) We have the dilute gas limit when λ_{D} is much less that the average distance between gas particles. The average volume per particle is $\nu = V/N$. The average distance between particles is $\simeq \nu^{1/3}$. So, the dilute gas limit is when $\lambda_{\mathrm{D}} \ll \nu^{1/3}$ [1 mark]

(c) (i) For distinguishable particles $Z_N = (Z_1)^N$, therefore

$$\ln Z_N = N\,\ln Z_1 = N\big[\ln V - 3\ln\lambda_{\mathrm{D}}\big],$$

hence the internal energy is

$$F_{Classical} = -k_{\mathrm{B}}T\ln Z_N = -N\,k_{\mathrm{B}}T\Big[\ln V - 3\,\ln\lambda_{\mathrm{D}}\Big]$$

(ii) For indistinguishable particles $Z_N = (Z_1)^N/N!$, so

$$Z_N = \frac{Z_1^N}{N!} \quad \Rightarrow\ \ln Z_N = N\,\ln Z_1 - N\ln N + N$$

$$Z_N = N\,\ln V - 3N\,\ln\lambda_{\mathrm{D}} - N\ln N + N.$$

The internal energy is

$$F = -k_{\mathrm{B}}T\ln Z_N = -k_{\mathrm{B}}T\Big[N\,\ln V - 3N\,\ln\lambda_{\mathrm{D}} - N\ln N + N\Big]$$

$$F = -N\,k_{\mathrm{B}}T\Big[\ln(V/N) - 3\,\ln\lambda_{\mathrm{D}} + 1\Big].$$

(iii) In (ii) we divide the N-particle partition function of identical particles by $N!$ in order to compensate for over-counting the number of microstates of identical particles. For example, take a system with three distinguishable independent particles. In the state 123, the first particle is in single-particle (s.p.) state 1, the second is in s.p. state 2 and the third in s.p. state 3. We have 6 different microstates for the system of the three distinguishable particles. These are $123, 231, 312, 132, 213, 321$. However, when the particles are indistinguishable we cannot know which particle is in which state and the 6 microstates reduce to 1. So, for identical particles, we are over-counting the many-particle states by 6 (=3!).

[3 marks]

(d) The internal energy is

$$U = \frac{3}{2} N\, k_{\mathrm{B}} T$$

and the free energy is in (c), so combining these gives the classical entropy for distinguishable particles

$$S = N\, k_{\mathrm{B}} \Big[\ln \big[V/\lambda_{\mathrm{D}}^3 \big] + \frac{3}{2} \Big] = N\, k_{\mathrm{B}} \Big[\ln \big[V (2\pi M k_{\mathrm{B}} T/h^2)^{3/2} \big] + \frac{3}{2} \Big] \Rightarrow$$

$$S = N\, k_{\mathrm{B}} \ln V T^{3/2} + \frac{3}{2} N\, k_{\mathrm{B}} \Big[\ln \big[(2\pi M k_{\mathrm{B}}/h^2) \big] + 1 \Big].$$

[2 marks]

(e) The entropy depends on V, T through the product $VT^{3/2}$. Since the entropy does not change during the expansion, we have that $VT^{3/2}$ remains constant. So

$$V_1 T_1^{3/2} = V_2 T_2^{3/2} \;\Rightarrow\; T_2 = T_1 \left(\frac{V_1}{V_2} \right)^{2/3} \simeq 0.63\, T_1 = 189 \text{ K}.$$

[2 marks]

(f) Adiabatic demagnetisation was covered in lectures. Other examples exist such as Pomeranchuk cooling by adiabatic solidification of ^{3}He (look it up). [1 mark]