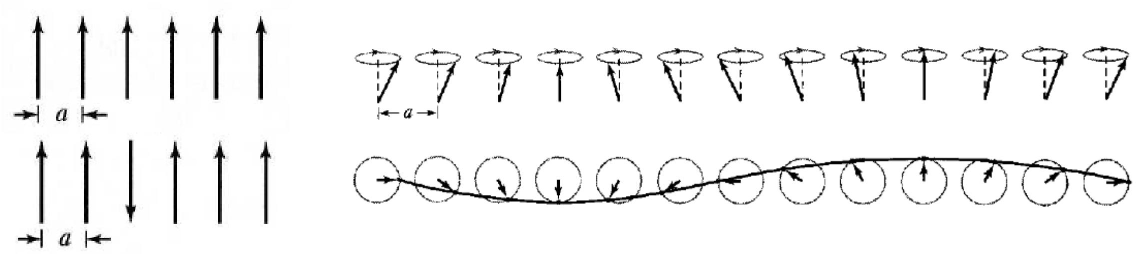
FoP3B Part I Lecture 10: Spin waves and Magnons

The magnetisation of a ferromagnet decreases monotonically with temperature due to increased disorder of the magnetic moments. The Weiss model of ferromagnetism enables us to quantify this change. For a J = ½ solid, $\frac{M}{M_s} = B_{1/2}(y) = \tanh y$, where $y = \frac{\mu_B \lambda M}{kT}$. At low temperature $y \to \infty$, so that $\tanh y = \frac{\sinh y}{\cosh y} = \frac{1-e^{-2y}}{1+e^{-2y}} \approx 1 - 2e^{-2y}$. Hence:

$$\frac{M}{M_s} = 1 - 2e^{-2y} \text{ or } \frac{M_s - M}{M_s} = 2\exp\left(-\frac{2\mu_B \lambda M}{kT}\right)$$

Experimentally however it is found that M decreases faster than the predicted variation. The Weiss model assumes that disorder is created by a full 180° spin flip on some of the atoms (Figure 1a). However, this is a high energy process, and there is an alternative mechanism for generating spin disorder, which also has a lower activation barrier. This is the formation of **spin waves** as shown in Figure 1b.



(a) **(b)**

Figure 1: (a) spin alignment at T = 0 K (top) and a spin flip at T > 0K (bottom). (b) a spin wave as viewed from the side (top) and from above (bottom).

Spin waves and Magnons

Consider a 1D solid of atoms with periodic spacing 'a'. We assume that the exchange interaction is allowed between neighbouring atoms only and has a constant exchange integral J_{ex}. Therefore, $\sum_{i,j} -J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j = \sum_j -2J_{ex}\mathbf{S}_j \cdot \mathbf{S}_{j+1}$, where the factor of '2' highlights the fact that an atom pair i,j is counted twice. The magnetic moment at position 'j' is $\boldsymbol{\mu}_j = -g_s\gamma\mathbf{S}_j$ (assuming $\mathbf{L}$ = 0). The exchange energy at 'j' with neighbours 'j-1 and 'j+1' is:

$$-2J_{ex}\mathbf{S}_j \cdot (\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) = -\boldsymbol{\mu}_j \cdot \frac{-2J_{ex}}{g_s\gamma}(\mathbf{S}_{j-1} + \mathbf{S}_{j+1})$$

The right hand side of the above expression suggests we can define an effective **B**-field ($\mathbf{B}_{eff}$) at position 'j' as $\frac{-2J_{ex}}{g_s\gamma}(\mathbf{S}_{j-1} + \mathbf{S}_{j+1})$, based on the fact that the energy of the magnetic moment $\boldsymbol{\mu}_j$ in a **B**-field is $-\boldsymbol{\mu}_j \cdot \mathbf{B}$. The torque ($\boldsymbol{\tau}$) is therefore $\boldsymbol{\tau} = \frac{d\mathbf{S}_j}{dt} = \boldsymbol{\mu}_j \times \mathbf{B}_{eff}$ or:

$$\frac{d\mathbf{S}_j}{dt} = (-g_s\gamma\mathbf{S}_j) \times \frac{-2J_{ex}}{g_s\gamma}(\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) = 2J_{ex}[\boldsymbol{S}_j \times (\boldsymbol{S}_{j-1} + \boldsymbol{S}_{j+1})]$$

A more rigorous quantum mechanical derivation shows that (note extra factor $\hbar$)*[1].

[1] See Supplementary Reading on DUO for a derivation.

$$\frac{d\mathbf{S}_j}{dt} = \frac{2J_{\text{ex}}}{\hbar}\left[\mathbf{S}_j \times \left(\mathbf{S}_{j-1} + \mathbf{S}_{j+1}\right)\right]$$

Let us find solutions to this equation. We assume that the spins are largely parallel to the *z*-axis (i.e. $S^z \approx S$, where S is the magnitude of the spin angular momentum), so that the *x*,*y* components, S^x and S^y, are small. This results in the following component equations:

$$\frac{dS_j^x}{dt} = \frac{2J_{\text{ex}}S}{\hbar}\left(2S_j^y - S_{j-1}^y - S_{j+1}^y\right)$$
$$\frac{dS_j^y}{dt} = -\frac{2J_{\text{ex}}S}{\hbar}\left(2S_j^x - S_{j-1}^x - S_{j+1}^x\right)$$
$$\frac{dS_j^z}{dt} \approx 0$$

Assume wave-like solutions, $S_j^x = Ae^{i(qja-\omega t)}$ and $S_j^y = Be^{i(qja-\omega t)}$ with wavenumber q and angular frequency ω. Substituting in the equations for $\frac{dS_j^x}{dt}$ and $\frac{dS_j^y}{dt}$ gives:

$$-i\omega A = \frac{4J_{\text{ex}}SB}{\hbar}[1 - \cos qa] \qquad \ldots (1)$$

$$i\omega B = \frac{4J_{\text{ex}}SA}{\hbar}[1 - \cos qa] \qquad \ldots (2)$$

(i) Dividing Equation 1 by 2 gives $A = iB$. Since $i = e^{i\pi/2}$ this implies that there is a 90° phase shift between the S^x and S^y component spin waves. In other words, there is a cyclic rotation of the spin in the *xy*-plane (Figure 1b).

(ii) Substituting $A = iB$ in either Equation 1 or 2 gives:

$$\hbar\omega = 4J_{\text{ex}}S(1 - \cos qa)$$

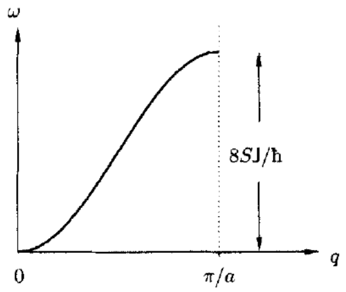


Figure 2

For a given wavenumber *q* the spin wave can only take a fixed value of energy $\hbar\omega$. The dispersion diagram is shown in Figure 2 for the first Brillouin zone. This is similar to the dispersion diagram for *acoustic phonon* waves. In fact, like phonons, spin waves have quantised energies, and can be thought of as quasi-particles called **magnons**.

Quantum Mechanical Interpretation

As mentioned earlier the activation barrier for a spin wave or magnon is smaller than a full 180° spin flip. Quantum mechanics provides some insight why this might be the case. Denote by |0> the ground state where all spins are aligned (Figure 3).

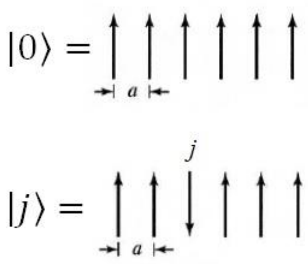


Figure 3

|0> is an eigenstate of the Hamiltonian, i.e. $\widehat{H}_{\text{mag}}|0\rangle = E_0|0\rangle$ where E_0 is the ground state energy. Now define the excited state |*j*> as the wavefunction with the spin at site '*j*' flipped (Figure 3). It can be shown that |*j*> is not an eigenstate of the Hamiltonian, i.e. $\widehat{H}_{\text{mag}}|j\rangle \neq E_j|j\rangle$ where E_j = constant.

In fact, the excited state |*q*> that is an eigenfunction of $\widehat{H}_{\text{mag}}$ is a superposition of spin flips spread out over all *N*-atoms:

$$|q\rangle = \frac{1}{\sqrt{N}} \sum_{j \in N} e^{iqja} |j\rangle$$

Note also the additional phase term e^{iqja} accompanying the linear superposition of excited state wavefunctions |*j*>. More details leading to the derivation of the above result can be found in the Supplementary Reading on DUO. A spin wave can therefore be interpreted as a *single spin flip* spread out over all atoms. Since the spin flip is delocalised the activation barrier for its formation is small. In fact, for small *q*, i.e. long wavelength spin waves, the formation energy $\hbar\omega$ is vanishingly small (Figure 3). Furthermore, the overall change in spin due to a single flip is -½ -½ = -1. The spin wave is therefore a boson (similar to a phonon).