Statistical Physics: Weekly Problem 1 (SP1)

Consider four distinguishable particles where each particle can be in single-particle states k, with energy $\epsilon_k = k\,\epsilon$, where $k = 0, 1, 2, \ldots$. The system is in the microcanonical ensemble (macrostate) with total energy $U = 3\epsilon$.

(a) What is the total number of the possible microstates and what is the value of the Boltzmann entropy (in units of k_B)? [4 mark]

(b) What is the average number of particles n_k in each state k and what is the probability of occupying each state k? [3 mark]

(c) Plot the probability (p_k) versus $k\,\epsilon$. Make a rough estimate of the width Δ of the distribution in units of ϵ (the energy where the distribution reaches $1/e$ of the maximum). What is the value of Δ? What is the physical meaning of Δ? [3 mark]

Statistical Physics: Weekly Problem 2 (SP2)

A system has single-particle states k, with energy $\epsilon_k = k\,\epsilon$, where $k = 0, 1, 2, \ldots$.

(a) Derive the partition function Z for the system of particles in thermal equilibrium at temperature $k_B T = 1/\beta$. (Hint: think 'sum on geometric series' and also for simplicity, use expressions in terms of β.) [4 marks]

(b) Using the Boltzmann distribution and the partition function, Z, you derived in (a)

(i) what is the probability, p_k, that a particle will be in state k,

(ii) what is the energy per particle (U/N),

(iii) what is the free energy per particle (F/N)

(iv) and what is the entropy per particle (S/N)?

[6 marks]

Statistical Physics: Weekly Problem 3 (SP3)

(1) (a) In the *microcanonical ensemble*, or (N, U, V) macrostate, are the various microstates that are consistent with the (N, U, V) macrostate equally probable, or do they have different in general probabilities? [1 mark]

(b) Similarly, in the *canonical ensemble*, or (N, T, V) macrostate, describing a system in thermodynamic equilibrium, are the various accessible microstates equally probable, or do they have, in general, different probabilities? [1 mark]

(2) In a system of N weakly interacting particles in thermal equilibrium at temperature T, the probability that a particle will be in (single-particle) state i with energy ϵ_i is proportional to

$$p_i \propto \exp\left[-\frac{\epsilon_i}{k_\mathrm{B}T}\right],$$

i.e. the Boltzmann probability.

The two lowest-lying energy levels of a hydrogen atom have energies $\epsilon_0 = -13.6$ eV and $\epsilon_1 = -3.4$ eV. Ignoring degeneracies, at what temperature would we find one hundredth as many hydrogen atoms in the first excited state as in the ground state?
($k_\mathrm{B} = 8.617 \times 10^{-5}$ eV K^{-1}) [2 marks]

(3) A paramagnetic solid consists of N ions with spin 1/2 and magnetic moment μ_B. The system lies in a magnetic field B and each magnetic moment is oriented either parallel to the field (up), with energy $\epsilon_\uparrow = -\mu_\mathrm{B}B$, or antiparallel (down) with energy $\epsilon_\downarrow = +\mu_\mathrm{B}B$.

(a) What is the probability that an ion will have its magnetic moment oriented parallel to B? [2 marks]

(b) What is the internal energy U, entropy S and the temperature T of the system in the limit where all the magnetic moments are parallel to B? [3 marks]
Hint: do not try to think of complicated equations. Use your physical intuition.

(c) The system of ions is brought into a state where the internal energy U is positive. Show that the temperature of the system is negative. Is a negative temperature "hotter" (i.e. of higher energy) or "colder" than infinite temperature? [1 mark]

Statistical Physics: Weekly Problem 4 (SP4)

(1) Consider an assembly of N weakly-interacting, distinguishable particles contained in a fixed volume V, with fixed internal energy U. Are the various distributions $\{n_i\}$ of the particles in single-particle states equally probable, or do they have different probabilities? State briefly what distinguishes the Boltzmann distribution from other distributions $\{n_i\}$ of the assembly of distinguishable particles. [2 marks]

(2) A paramagnetic solid consists of N ions with spin 1/2 and magnetic moment μ_B. The system lies in a magnetic field B and each magnetic moment is oriented either parallel to the field (up), with energy $\epsilon_\uparrow = -\mu_\mathrm{B} B$, or antiparallel (down) with energy $\epsilon_\downarrow = +\mu_\mathrm{B} B$. The system is in contact with a heat bath at temperature T.

(a) Write down the single-particle partition function Z_1 followed by the probability $p_\uparrow$ that a magnetic moment is up and the probability $p_\downarrow$ that it is down. [1 mark]

(b) The magnetisation per ion is equal to the average magnetic moment

$$M/N = \sum_i p_i m_i,$$

where $i = \uparrow, \downarrow$ and $m_\uparrow = \mu_\mathrm{B}$, $m_\downarrow = -\mu_\mathrm{B}$. Show that the magnetisation per ion is given by

$$\frac{M}{N} = \mu_\mathrm{B} \tanh\left(\frac{\mu_\mathrm{B} B}{k_\mathrm{B} T}\right)$$

where k_B is Boltzmann's constant. [2 marks]

(c) Obtain the internal energy U of the system of ions, directly from the definition,

$$U = N \sum_i p_i \epsilon_i.$$

Compare this with the energy of N magnetic moments, each of magnitude M/N, (b) above, and oriented along B. [2 marks]

(d) (i) Using Gibbs' definition

$$S = -N k_\mathrm{B} \sum_i p_i \ln p_i,$$

show that the entropy of the system depends on the magnetic field B and on the temperature T through the ratio B/T. [1 mark]

(ii) Sketch the graph of the entropy versus temperature for two different applied magnetic fields and then explain how a dilute paramagnetic solid can be cooled. [2 marks]

Statistical Physics: Weekly Problem 5 (SP5)

(1) (a) Consider a system of spin one particles localised on a lattice (i.e. distinguishable) in a magnetic field. The energy levels associated with the three spins states $(-1, 0, 1)$ have energies $(-\epsilon, 0, \epsilon)$. Calculate the single-particle partition function Z. [2 mark]

(b) The same system is at a temperature T. Calculate the internal energy U, heat capacity C_V, free energy F and entropy S. [2 marks]

(2) (a) In a hypothetical system of identical particles, restricted to move in two dimensions, each single particle state may hold up to η particles, where η is a fixed positive integer. The degeneracy of each single-particle energy ϵ_i is g_i. (This hypothetical system is a model for 2D particles known as anyons, or the fractional statistics gas.)

Justify that the number of microstates $\Omega(n_1, n_2, \ldots)$ for the distribution $(n_1, n_2, \ldots)$ of N such identical particles in the single-particle energies ϵ_i of the system is

$$\Omega(\{n_i\}) = \prod_j \frac{(\eta \times g_j)!}{n_j!\,(\eta \times g_j - n_j)!}$$

[2 marks]

(b) For the same hypothetical system of identical particles show that the distribution function, or fractional occupancy, $f_i = n_i/g_i$, of each single particle energy is

$$\frac{n_i}{g_i} = \frac{1}{A\,e^{\beta\epsilon_i} + (1/\eta)}$$

where A and β are Lagrange multipliers. [2 marks]

(c) What are the limits of the distribution function for small and large η? [2 marks]

Statistical Physics: Weekly Problem 6 (SP6)

(1) In this question you will need to evaluate integrals of the form

$$I_n(\alpha) \;=\; \int_0^\infty x^n e^{-\alpha x^2} dx.$$

Simply look up the expressions for these standard integrals and use them (they are also given in the lectures).

The probability distribution of speeds of particles in a gas is given by the Maxwell-Boltzmann (MB) distribution

$$p(v)dv = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) dv.$$

(a) Sketch $p(v)$ as a function of v for (i) a low temperature and (ii) a high temperature. Indicate the v dependence for small v, and the trend at large v. [2 marks]

(b) The expressions for the most probable speed v_{max}, the mean speed $\bar{v}$, and the r.m.s. speed v_{rms} were evaluated in lectures. State them here and mark them on the graph. [2 mark]

(c) Calculate v_{max}, $\bar{v}$, and v_{rms} for a system composed of gaseous neon atoms at room temperature. The mass of a neon atom is 3.37×10^{-26} kg. [2 marks]

(d) In two dimensions the probability distribution of speeds of particles in a gas is

$$p(v)\, dv = C\, v\, \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

where C normalises the probability.

(i) Calculate C. [2 marks]

(ii) Find the expressions for the most probable speed v_{max}, the mean speed $\bar{v}$, and the r.m.s. speed v_{rms} of particles in this two dimensional gas. [2 marks]

Statistical Physics: Weekly Problem 7 (SP7)

(1) (a) Derive the single-particle partition function, Z_1, for a free particle of mass M in three dimensions, constrained to a box of volume V, in thermal equilibrium at temperature T. You will need common integrals of the form $\int_0^\infty x^n e^{-bx^2} dx$ which you can look up and state without proof. [1 mark]

(b) Give a criterion for the dilute gas limit in terms of the thermal de Broglie wavelength $\lambda_{\mathrm{D}} = \sqrt{h^2/2\pi M k_{\mathrm{B}} T}$. [1 mark]

(c) The free energy for a gas of N weakly interacting particles in a volume V at temperature T is given by $F = -k_{\mathrm{B}} T \ln Z_N$, where Z_N is the N-particle partition function. Find the free energy of

(i) A gas of distinguishable particles. [1 mark]

(ii) A gas of indistinguishable particles. [1 mark]

(iii) Explain why there is a difference between the answers to parts (i) and (ii). [1 mark]

(d) Show that the entropy for a classical gas of monoatomic distinguishable particles is

$$S = N k_{\mathrm{B}} \ln V T^{3/2} + \frac{3}{2} N k_{\mathrm{B}} \left[\ln \frac{2\pi M k_{\mathrm{B}}}{h^2} + 1 \right].$$

[2 marks]

(e) A classical gas of distinguishable monoatomic particles is in thermal equilibrium at temperature $T = 300$ K in a container of volume V. The gas is allowed to expand adiabatically (without change in its entropy) until it occupies twice the initial volume. What is the temperature of the gas after the expansion? [2 marks]

(f) Mention two examples of cooling under constant entropy. One example example was given in lectures, you'll have to research for another. [1 mark]