Lagrangian multipliers

The method of Lagrangian multipliers helps to find the extremum of a quantity, subject to constraints. For example, we want the maximum of the function

$$\max_{x_1,x_2,\ldots,x_n} f(x_1,x_2,\ldots,x_n)$$

subject to the constraint

$$g(x_1,x_2,\ldots,x_n) = \text{c}.$$

This constraint reduces the number of independent variables. In Lagrange's method, we find the unconstrained maximum of the weighted average of f and g.

$$\max_{x_1,x_2,\ldots x_n} \Big[f(x_1,x_2,\ldots,x_n) + \lambda\, g(x_1,x_2,\ldots,x_n) \Big]$$

Note that

- The weight of the main quantity $f(x_1,x_2,\ldots,x_n)$ is 1.
- The weight of $g(x_1,x_2,\ldots,x_n)$ is λ; it is initially free (undetermined).
- After the maximization, for every value of λ, we have a different maximum. f and g will have such values, so that their weighted sum is maximum.
- There will be a λ_0 for which the value of g satisfies the constraint $g(x_1,x_2,\ldots,x_n) = \text{c}$.
- For the same λ_0, f will have the maximum value consistent with the constraint.

Example: What is the maximum volume of a rectangular box of surface area $S = 24m^2$?

Let x, y, z be the length, width, and height of the box. We want to maximize the volume

$$V(x,y,z) = xyz$$

under the constraint

$$2xy + 2xz + 2yz = S \; (= 24m^2).$$

Following Lagrange's method, we shall find the unconstrained maximum of

$$F = xyz - 2\lambda\,(xy + xz + yz)$$

(Note the units of λ. What is the physical meaning of λ?) We have

$$\frac{\partial F}{\partial x} = yz - 2\lambda\,(y+z) = 0, \quad \frac{\partial F}{\partial y} = xz - 2\lambda\,(x+z) = 0, \quad \frac{\partial F}{\partial z} = xy - 2\lambda\,(x+y) = 0,$$

Equivalently

$$\begin{aligned} yz &= 2\lambda\,(y+z) \\ zx &= 2\lambda\,(z+x) \\ xy &= 2\lambda\,(x+y) \end{aligned}$$

From first two equations we get

$$\frac{y}{x} = \frac{y+z}{z+x} \;\Rightarrow\; \frac{y+z}{y} = \frac{z+x}{x} \;\Rightarrow\; \frac{z}{y} = \frac{z}{x} \;\Rightarrow\; y = x$$

From third equation we get

$$x^2 = 2\lambda 2x \;\Rightarrow\; x = 4\lambda,\; y = 4\lambda.$$

Substitution in first (or second) equation gives $z = 4\lambda$ and so,

$$V = xyz = 64\lambda^3$$

Constraint:

$$24m^2 = 96\,\lambda^2 \;\Rightarrow\; 4\lambda^2 = 1m^2 \;\Rightarrow\; \lambda = 0.5m \;\Rightarrow\; V = 8m^3$$

Main point: The method of Lagrange multipliers is a powerful method that is used to find the maximum, or the minimum, of a quantity subject to the satisfaction of a number of constraints, expressed as equalities. It works by searching for the unconstrained maximum (or minimum) of the weighted sum of the main quantity and the constraints.