Statistical Physics: Workshop Problem 1 Answers

(1) (a) The first person will belong to some DNA profile anyway. The probability that the second person will belong in the same profile as the first person is $p = 199/(50 \times 10^6 - 1)$. Because the numbers are large, this is almost equal to:

$$\frac{200}{50 \times 10^6} = \frac{1}{250\,000}$$

(b)

$$\frac{1}{200}$$

(2) The first person will have birthday on some day. The probability the second person does not have the same birthday is 364/365 and so on. The result is:

$$\frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \frac{361}{365} = 0.973$$

(3) The distributions are:

Distribution	$(n_0, n_1, n_2, n_3, n_4, n_5, n_6)$	Number of microstates
(1)	(3, 0, 0, 0, 0, 0, 1)	4
(2)	(2, 1, 0, 0, 0, 1, 0)	12
(3)	(2, 0, 1, 0, 1, 0, 0)	12
(4)	(1, 2, 0, 0, 1, 0, 0)	12
(5)	(2, 0, 0, 2, 0, 0, 0)	6
(6)	(1, 1, 1, 1, 0, 0, 0)	24
(7)	(0, 3, 0, 1, 0, 0, 0)	4
(8)	(1, 0, 3, 0, 0, 0, 0)	4
(9)	(0, 2, 2, 0, 0, 0, 0)	6
Average	(1.333,1,0.714,0.476,0.286,0.143,0.048)	$\Omega = 84$
Probabilities	(0.33,0.25,0.18,0.12,0.07,0.04,0.01)	$S = k_B \ln 84 = 4.43\, k_B$

Table 1:
$\langle n_0 \rangle = 112/84 = 1.33$,
$\langle n_1 \rangle = 84/84 = 1$,
$\langle n_2 \rangle = 60/84 = 0.714$,
$\langle n_3 \rangle = 40/84 = 0.476$,
$\langle n_4 \rangle = 24/84 = 0.286$,
$\langle n_5 \rangle = 12/84 = 0.143$,
$\langle n_6 \rangle = 4/84 = 0.048$.

(4) Since the standard deviation $\sigma = 5\Omega$ and the mean $\mu = 100\Omega$, the first part is the probability that the variable takes a value within a standard deviation from the mean. The probability is given by

$$P(\mu - \sigma \leq x \leq \mu + \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma}^{\mu+\sigma} dx \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

With the change of variable $x \to (x-\mu)/\sigma$, the probability becomes

$$P(\mu - \sigma \leq x \leq \mu + \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} dx \, \exp\left(-\frac{x^2}{2}\right)$$

From the table

$$P(\mu - \sigma \leq x \leq \mu + \sigma) = 0.68269$$

The probability that the resistor has resistance between 80 Ω and 120 Ω is:

$$P(\mu - 4\sigma \leq x \leq \mu + 4\sigma) = 0.99994$$

The probability that the resistance is outside these values is 0.00006.

Hence, the probability that the resistance is less than 80 Ω is half that value (0.00003):

$$P(x < \mu + \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu - 4\sigma} dx \, \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\Rightarrow \; P(x < \mu - 4\sigma) = \frac{1 - P(\mu - 4\sigma \leq x \leq \mu + 4\sigma)}{2} = 0.00003$$

Note that strictly, the lower limit of the integral should be 0 rather than $-\infty$, but because $\mu/\sigma = 20 \gg 1$, and the Gaussian decays extremely rapidly with distance so the integrand in over negative values of x is negligible.