Level 3 Condensed Matter Physics Part 1
Example Workshop 1 – Solution

1. The Fermi energy is the highest occupied electron state in a system when it is in the ground state. From the solution of the free electron Schroedinger equation it can be shown that the energy of the electron state is related to its free electron wavevector by

$$E = \frac{\hbar^2 k^2}{2m_e}$$

In three dimensions we have $k^2 = k_x^2 + k_y^2 + k_z^2$. The surface of constant energy in *k*-space, or reciprocal space, is therefore a surface of constant k^2, or a sphere. This Fermi sphere is a sphere of radius equal to the Fermi wavevector, k_F, with all of the electron states on the Fermi surface having an energy equal to the Fermi energy, E_F, where $E_F = \frac{\hbar^2 k_F^2}{2m_e}$. This is shown below.

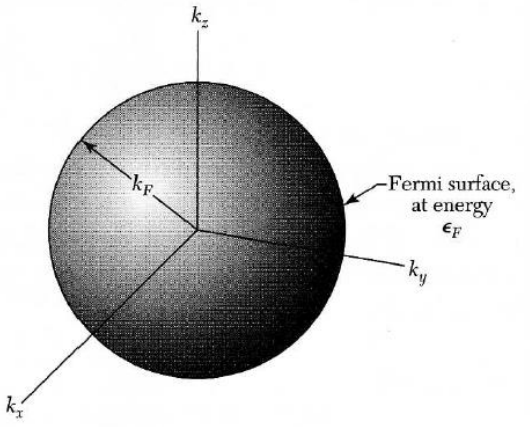
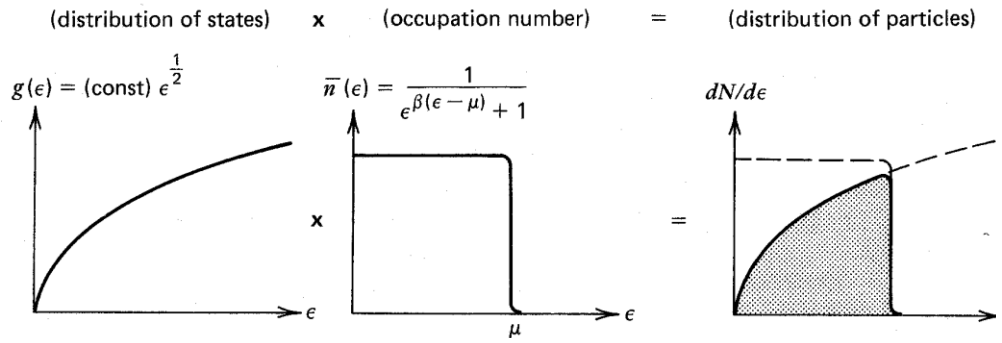


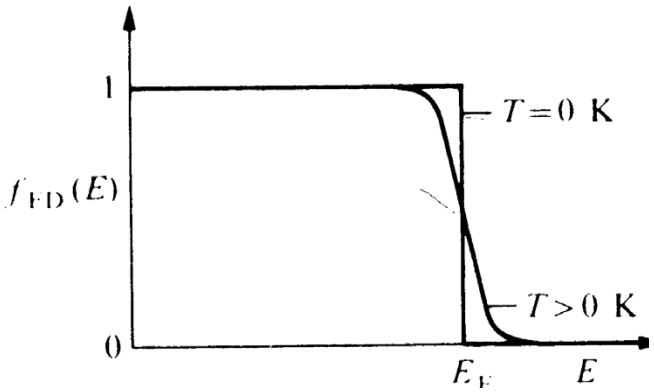
Figure 4 In the ground state of a system of N free electrons the occupied orbitals of the system fill a sphere of radius k_F, where $\epsilon_F = \hbar^2 k_F^2/2m$ is the energy of an electron having a wavevector k_F.

2. (a) The energy density of states function *g(E)* describes the number of available energy states per unit volume of material, which can be occupied by electrons, per unit energy range. The probability of occupation of those states is given by the Fermi-Dirac equation (electrons are fermions). Multiplying these two together gives us the distribution of particles as a function of energy d*N*/d*E* where

$$N = \int_0^\infty g(E) f_{\mathrm{FD}}(E) dE$$



(b) The Fermi energy is the energy of the highest occupied state in a metal (which is in its ground state). In a metal all energy states below the Fermi energy are occupied and all energy states above the Fermi energy are vacant. Note that at finite temperatures there is a thermal broadening of the transition from probability 1 to probability 0. The width of the transition is approximately $2k_BT$ around E_F. The diagram shows the behaviour of the Fermi-Dirac function at zero and finite temperatures.



(c) The Fermi energy is determined from the free electron density:

$$E_F = \frac{\hbar^2 k_F^2}{2m_e}$$

where the Fermi wavenumber is given by $k_F = (3\pi^2 n)^{1/3}$. The Fermi wavenumber can therefore be determined from the radius of the Fermi sphere. Substituting in the electron density gives $k_F = (3\pi^2 \times 6 \times 10^{28})^{1/3} = 1.21 \times 10^{10}$ m^{-1} giving an energy of $E_F = \frac{\hbar^2 (1.21\times10^{10})^2}{2m_e} = 8.96 \times 10^{-19}$ J $= 5.6$ eV. The Fermi velocity is determined by $v_F = {}^{p}/_{m_e} = {}^{\hbar k}/_{m_e} = 1.4 \times 10^6$ ms^{-1}, which is fast but not relativistic. The Fermi temperature is then $T_F = {}^{E_F}/_{k_B} = \frac{8.96 \times 10^{-19}}{1.38 \times 10^{-23}} = 6.4 \times 10^4$ K. This demonstrates that silver at room temperature is a quantum degenerate material. The thermal energy at room temperature (300 K) is just

$$E = k_B T = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21} \text{ J} = 26 \times 10^{-3} \text{ eV}$$

, which is much, much smaller.

3. The wavevector at the corner is longer than the wavevector at the midpoint of a side by the factor $\sqrt{2}$. As E $\propto$ k^2 for a free electron, the energy is higher by $(\sqrt{2})^2 = 2$. In three dimensions the energy at a corner is higher by $(\sqrt{3})^2 = 3$ than at the midpoint of a face.

4. The energy eigenvalues are

$$E_F = \frac{\hbar^2 k^2}{2m}$$

The mean value over the volume of a sphere in *k* space is

$$\langle E \rangle = \frac{\int_0^{k_F} \frac{\hbar^2 k^2}{2m} (4\pi k^2) dk}{\left(\frac{4\pi}{3} k_F^3\right)} = \frac{3}{5} \frac{\hbar^2}{2m} {k_F}^2 = \frac{3}{5} E_F$$

The total energy of *N* electrons is

$$U_0 = \frac{3}{5} N E_F$$