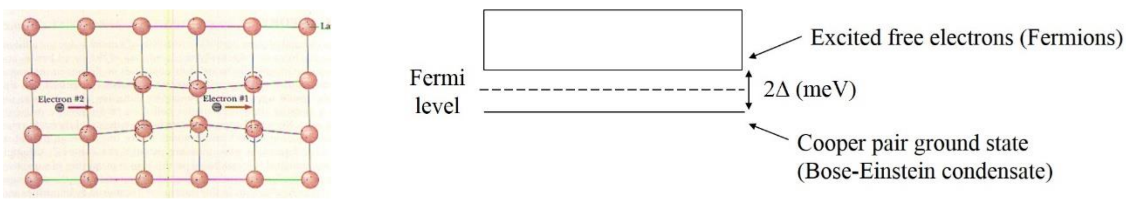
FoP3B Part II Lecture 9: Ginzburg-Landau theory of superconductivity

There are two alternative, but equivalent, theories for superconductivity: (i) BCS theory after Bardeen, Cooper and Schrieffer, and (ii) Ginzburg-Landau (GL). BCS is a *microscopic* theory, i.e. it describes superconductivity at the fundamental level of electrons in the solid. GL is a *phenomenological* theory, i.e. superconductivity is modelled at a more macroscopic level using so-called *order parameters*. In this lecture we will primarily discuss GL theory and rely on BCS to interpret the physical meaning of the order parameters.

Summary of BCS theory

Consider electrons moving in a crystal. In normal metals the direction of movement is random in the absence of an electric field. In superconductivity however a negatively charged electron will distort the ion lattice towards it (Figure 1a), so that a neighbouring electron will 'see' a lower potential region of material around the first electron and be attracted towards it. This gives rise to **Cooper pairs** of electrons that move as a single unit. At a more detailed level Cooper pairs form from **electron-phonon** interactions. *The spacing between the Cooper pair electrons can be as large as several 100 nm and the interaction is therefore weak, which is why superconductivity is a low temperature phenomenon. Furthermore, not all electrons form Cooper pairs*; in fact we have already seen (previous lecture) that the fraction of Cooper pairs increase as the material is cooled below T_c. At any given temperature the majority of electrons behave in a conventional manner, i.e. they are unbound, have finite resistivity and keep the solid from falling apart. However, some electrons close to the Fermi level are bound as Cooper pairs giving rise to superconducting behaviour.



(a) **(b)**

Fig. 1: (a) Formation of a Cooper pair and (b) electronic energy level diagram in a superconductor

The question remains why Cooper pairs have zero resistivity. Normal electrons are fermions, but with two electrons in a Cooper pair the spins take on integer values (0 or 1). *Cooper pairs are therefore bosons*. *BCS theory shows that Cooper pairs form a Bose-Einstein condensate, with a small energy gap (meV) to the next (fermion) electron level which is unoccupied (Figure 1b). The energy gap suppresses scattering processes that lead to resistivity, since scattering involves promoting an electron to a higher energy, unoccupied level.*

Ginzburg-Landau (GL) theory for homogeneous systems

We will now present GL theory for the relatively simple case of a homogeneous (i.e. uniform) material system under no applied magnetic field. <u>Close to the **critical temperature** T_c the free energy is expanded as a power series</u>:

$$G_S(T) = G_N(T) + a(T)|\psi|^2 + \frac{b(T)}{2}|\psi|^4$$

… (1)

where G_S, G_N are free energies of the superconducting and normal states, and $a(T)$, $b(T)$ are temperature dependent expansion coefficients. ψ is a complex number and is the **order parameter** in GL theory. The free energy depends on $|\psi|^2$. From BCS it can be shown that $|\psi|^2$ is the density of Cooper pairs, i.e. $|\psi|^2 = n_S/2$, where n_S is the density of superconducting electrons. For the superconducting state to be stable the G_s vs $|\psi|^2$ graph must have a minimum. Since Equation (1) is a quadratic function w.r.t $|\psi|^2$ a minimum is obtained for $b(T) > 0$.

The ground state order parameter is determined by:

$$\frac{\partial G_s}{\partial |\psi|} = 2|\psi|\{a(T) + b(T)|\psi|^2\} = 0$$
$$\Rightarrow |\psi| = 0 \text{ or } |\psi|^2 = -a(T)/b(T)$$

… (2)

For the superconducting state only the second solution for $|\psi|^2$ is valid (i.e. $|\psi| = 0$ is the solution for the normal state). For Equation (2) to give a positive $|\psi|^2$ value for the superconducting state $a(T) < 0$ for $T < T_c$ (recall $b(T) > 0$). Above T_c, $a(T) > 0$, so that the superconductor is not stable.

Consider now the *Taylor expansion* of a function $f(x)$ about the point $x = x_o$:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \cdots$$

… (3)

Instead of x, x_o we can substitute T, T_c and instead of $f(x)$ we can write either $a(T)$ or $b(T)$. For $a(T)$ we set the first term on the RHS equal to zero and retain only the second term. Therefore $a(T) \approx \dot{a}(T - T_c)$, where $\dot{a}$ is a positive constant. This satisfies the desired properties for positive and negative $a(T)$ values above and below T_c. Similarly for $b(T)$ only the first term on the RHS of Equation (3) is retained, so that $b(T) \approx b$, where b is a positive constant. Therefore:

$$|\psi|^2 = \begin{cases} \left[\frac{\dot{a}(T_c - T)}{b}\right] & T < T_c \\ 0 & T > T_c \end{cases}$$

… (4)

This is illustrated in Figure 2. The minimum in the graph is the Cooper pair density for the superconducting state. From Equation (4) the Cooper pair density increases with cooling below T_c, as required. The **condensation energy**, $G_N - G_S$, is found by substituting the $|\psi|^2$ value for $T < T_c$ from (4) into Equation (1). The result is $[\dot{a}(T - T_c)]^2/2b$. From thermodynamics the condensation energy is $[B_c(T)]^2/2\mu_o$, where $B_c(T)$ is the critical magnetic field (see previous lecture). Equating the GL and thermodynamic expressions for the condensation energy gives a value for $\dot{a}^2/b$ in terms of measurable parameters such as $B_c(T)$ and T_c. This value for $\dot{a}^2/b$ can then be used to predict further properties, such as entropy using GL theory. The results are consistent with the experimental observation of superconductivity being a second order phase transition, with no latent heat or entropy change at the critical temperature T_c.

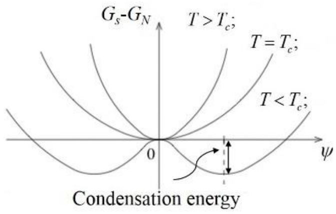


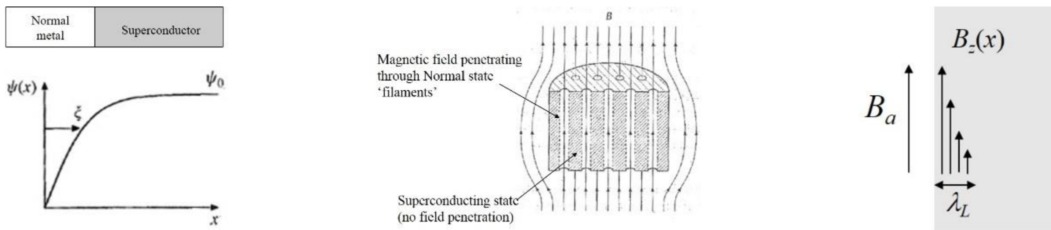
Fig. 2: Free energy ($G_S - G_N$) vs $|\psi|$ graphs as a function of temperature. The minimum in the curve gives the order parameter $|\psi|^2$ for the material. At $T>T_c$, $|\psi|^2 = 0$ and ($G_S - G_N$) = 0, meaning that only the normal state is allowed. For $T<T_c$ however $|\psi|^2 >0$ and ($G_S - G_N$) < 0, so that the superconductor is the more stable phase. The condensation energy is also indicated.

Coherence length and Type I vs Type II behaviour

GL theory can be extended to inhomogeneous systems as well, such as the interface between a normal metal and superconductor (Figure 3a). *The order parameter ψ for this system increases from zero in the normal metal to the equilibrium value ψ_0 for the superconductor over a characteristic distance ξ, which is known as the* **coherence length**. $|\psi|^2$ represents the density of Cooper pairs, so the fact that the order parameter is lower near the normal metal-superconductor interface means that some of the Cooper pairs are destroyed. This is due to the fact that the two electrons forming the Cooper pair have a relatively large separation (despite being bound), so that in order not to be broken up by the normal metal they have to be sufficiently far away from the interface. *Thus the coherence length ξ is of the order of the Cooper pair separation.* From GL theory the coherence length is given by:

$$\xi(T) = \left(\frac{\hbar^2}{2m|a(T)|}\right)^{1/2} = \left[\frac{\hbar^2}{2m\dot{a}(T_c - T)}\right]^{1/2} \quad \dots (5)$$

The coherence length therefore decreases as the material is cooled below T_c. The fact that the order parameter and Cooper pair density is lower near the interface also means that the *local* condensation energy for that region is smaller than the bulk superconductor. This has important implications for Type I vs Type II superconducting behaviour. As shown in Figure 3b the characteristic feature of Type II superconductors is the **mixed or vortex state**, where normal and superconducting regions exist side by side, similar to Figure 3a.

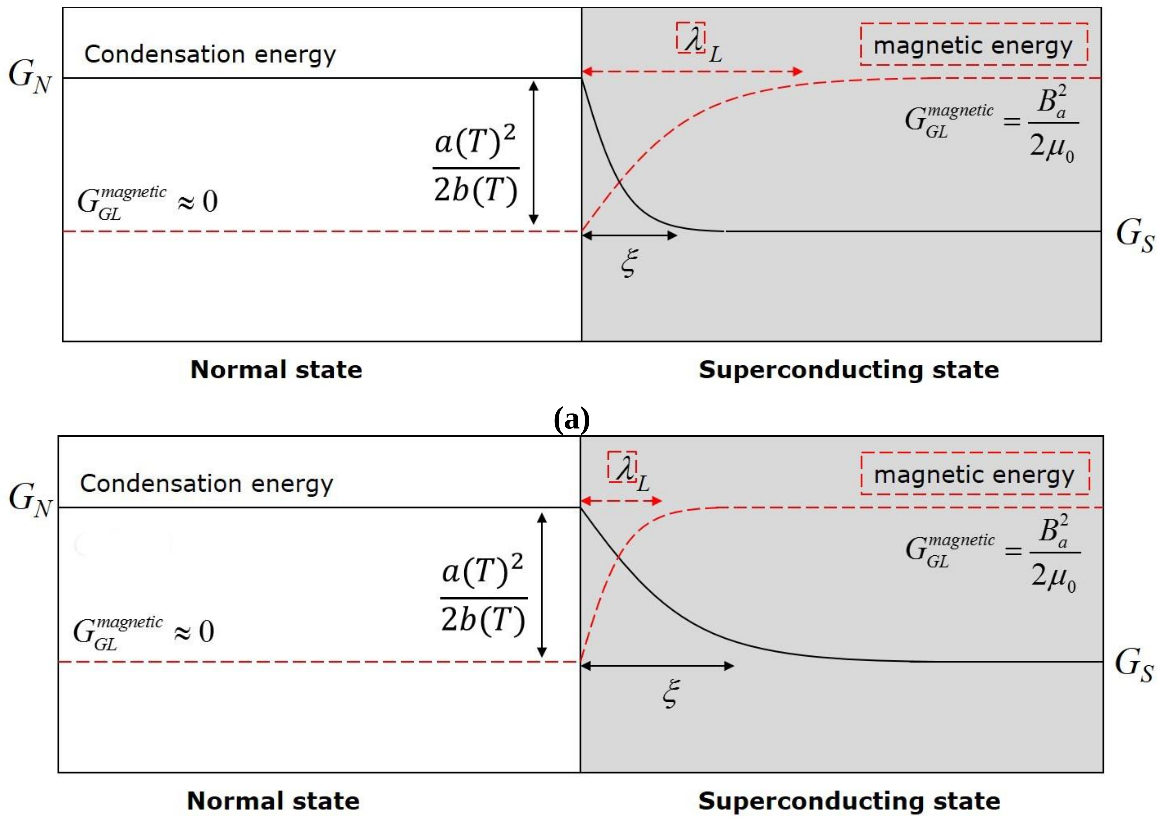


(a) **(b)** **(c)**

Fig. 3: (a) Order parameter as a function of distance from a normal metal-superconductor interface. (b) Microstructure of the mixed or vortex state in a Type II superconductor. (c) Magnetic field penetration into a superconductor.

Consider now a normal metal-superconductor interface under an applied magnetic field. *With no magnetic field the free energy of the superconductor will decrease to its bulk value over the coherence length ξ, due to condensation of Cooper pairs*. If a magnetic field B_a is then applied, the field will penetrate into the normal metal, but will be repelled from within the **diamagnetic** superconductor over a characteristic length λ_L, the **London penetration depth** (Figure 3c). *Repelling the magnetic field will add an extra energy term to the superconductor, which varies as $B^2/2\mu_0$, where B is the magnitude of the field repelled*. These two contributions, condensation energy and magnetic energy are shown schematically in Figure 4a as a function of distance from the normal metal-superconductor interface. The diagram for Figure 4a has $\xi < \lambda_L$; the condensation energy decreases rapidly at the interface, while the magnetic field energy increases relatively slowly. The net result is a *relatively low* interfacial energy. Therefore a $\xi < \lambda_L$ interface is energetically allowed and leads to Type II behaviour (Figure 3b).

On the other hand Figure 4b shows the opposite case where $\xi > \lambda_L$. Here the condensation energy decreases relatively slowly at the interface, while the magnetic energy increases rapidly. The net result is a *relatively high* interfacial energy. The interface is therefore energetically unfavourable and cannot exist. This leads to Type I behaviour with no mixed phase. Type I vs Type II behaviour is therefore governed by the relative magnitudes of ξ and λ_L.



(b)

Fig. 4: Condensation and magnetic free energy contributions for (a) $\xi < \lambda_L$ Type II superconductor and (b) $\xi > \lambda_L$ Type I superconductor.