Statistical Physics: Workshop Problems 7

(1) (a) The density of states in k-space for weakly interacting particles moving at relativistic speeds is the same as the non-relativistic case,

$$g(k)dk = \frac{V}{2\pi^2}k^2 dk,$$

however the energy-wavevector (momentum) relation differs; it is $\epsilon = c\hbar k$. Calculate the density of states with respect to energy, $g(\epsilon)d\epsilon$.

(b) What is the partition function for the relativistic particles? Compare the temperature dependence of the partition function to that of the non-relativistic case.

(2) (a) The density of states of non-relativistic particles is

$$g(\epsilon)d\epsilon = \frac{2\pi V}{h^3}(2M)^{3/2}\sqrt{\epsilon}d\epsilon.$$

If a system contains N particles the Fermi energy, E_F, can be defined via

$$N = \int_0^{E_F} g(\epsilon)f(\epsilon)d\epsilon$$

where $f(\epsilon)$ is a statistical distribution function. Find an expression for the Fermi energy and also the Fermi temperature of Fermions at $T = 0$.

(b) Similarly, calculate the Fermi energy for a system of N relativistic Fermions.

(c) Show that the internal energy, U, of the relativistic Fermion gas is $3NE_F/4$. Similarly show that the internal energy in the non-relativistic case is $3NE_F/5$.

(d) Calculate the Fermi energy (in eV) and Fermi temperature of

(i) liquid ^{3}He with density 0.0823 g cm^{-3},

(ii) electrons in aluminium which has valence 3 and a density of 2.7 g cm^{-3},

(iii) neutrons in the nucleus of ^{16}O given that the radius r of an atomic nucleus is $r \approx 1.2A^{1/3}$ fm where A is the atomic mass number.

(3) (a) This question uses results from (2) and (3). The electrons, protons and neutrons in a white dwarf star obey quantum statistics as the system is very dense, in fact so dense that $T = 0$ is a reasonable approximation (compare Fermi temperature of an atomic nucleus). Assume the star has radius R, mass M and contains equal numbers of protons, neutrons and electrons, and that the electrons can be treated non-relativistically. Show that the internal energy of the electrons is given by

$$U_{\text{elec}} = 0.0088\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}$$

where m_e is the electron mass and m_p is the proton or neutron mass.

(b) The gravitational potential energy of a star is

$$U_{\text{grav}} = -\frac{3}{5}\frac{GM^2}{R}.$$

Let $U_{\text{total}} = U_{\text{elec}} + U_{\text{grav}}$, then the white dwarf's radius, R, will be the value that minimises U_{total}. Derive an expression for the equilibrium radius, $R(M)$, of a white dwarf star as a function of the star's mass, M.

(c) Show that the radius of a white dwarf star that is the mass of our sun is approximately the radius of the Earth.

(d) Calculate the Fermi energy of the electrons in a white dwarf star of one solar mass. Do you think the non-relativistic treatment of the electrons was reasonable?

(e) If the electrons were treated relativistically show that $U_{\text{elec}} \sim R^{-1}$ rather than R^{-2}.

(f) Use (e) to explain why a white dwarf star will not be stable over a certain mass and will collapse further.