Statistical Physics: Weekly Problem 1 (SP1)

(a) Since $U = 3\epsilon$, the number of particles in states $k > 3$ is zero ($n_k = 0$ for $k > 3$). The number of microstates for each distribution are given by $4!/(n_0!n_1!n_2!n_3!)$. The possible distributions for the states (or levels) 0,1,2,3 are:

Distribution	(n_0, n_1, n_2, n_3)	Number of microstates
D_1	(3, 0, 0, 1)	4
D_2	(2, 1, 1, 0)	12
D_3	(1, 3, 0, 0)	4
Average [see (b)]	(2.0,1.2,0.6,0.2)	$\Omega = 20$

As the total number of microstates is $\Omega = 20$ then the total entropy is $S = k_B \ln(20) \simeq 3k_B$.

[4 mark]

(b) $\langle n_0 \rangle = (3 \times 4 + 2 \times 12 + 1 \times 4)/20 = 2,$
$\langle n_1 \rangle = (0 \times 4 + 1 \times 12 + 3 \times 4)/20 = 1.2,$
$\langle n_2 \rangle = (0 \times 4 + 1 \times 12 + 0 \times 4)/20 = 0.6,$
$\langle n_3 \rangle = (1 \times 4 + 0 \times 12 + 0 \times 4)/20 = 0.2.$

Hence (normalising back to 1):
$p_0 = 2/4 = 0.5$, $p_1 = 1.2/4 = 0.3$, $p_2 = 0.6/4 = 0.15$, $p_3 = 0.2/4 = 0.05$, and $p_k = 0$ for $k > 3$.

[3 mark]

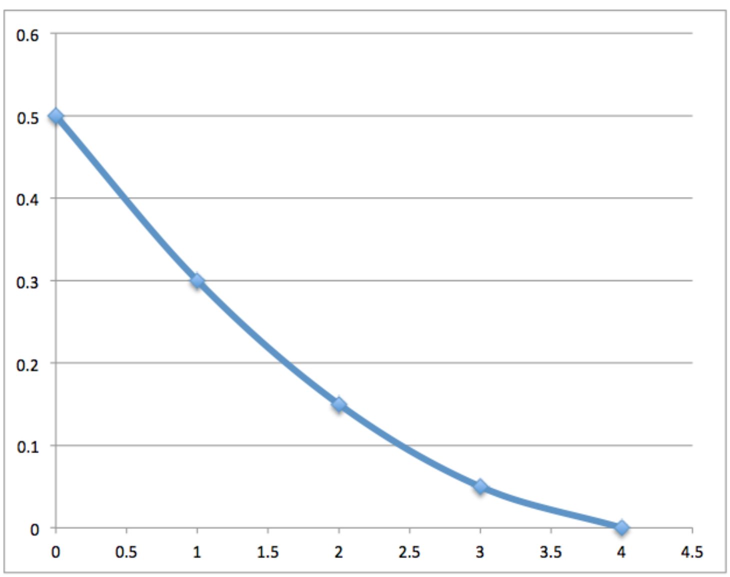


Figure 1: p_k vs k. The width, is where the curve drops to $1/e$ of the maximum (slightly below 0.2). So the width is about $\Delta = 1.5\epsilon$. Any answer between ϵ - 2ϵ is acceptable. The physical meaning of Δ is temperature.

(c) See Figure 1. [3 mark]

Statistical Physics: Weekly Problem 2 (SP2)

(a) Note that

$$a + ar + ar^2 + ar^3 + ... = \frac{a}{1-r}$$

for $|r| < 1$ and so we get

$$Z = \sum_{k=0}^{\infty} e^{-\beta\,\epsilon\,k} = \frac{1}{1 - e^{-\beta\,\epsilon}}$$

[4 marks]

(b) (i)

$$p_k = \frac{e^{-\beta\,\epsilon\,k}}{Z} = (1 - e^{-\beta\,\epsilon})\, e^{-\beta\,\epsilon\,k}$$

[1 mark]

(ii)

$$\frac{U}{N} = -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial}{\partial \beta} \ln(1 - e^{-\beta\,\epsilon}) = \frac{\epsilon\, e^{-\beta\,\epsilon}}{1 - e^{-\beta\,\epsilon}}$$

[2 marks]

(iii)

$$\frac{F}{N} = -\frac{\ln Z}{\beta} = \frac{\ln(1 - e^{-\beta\,\epsilon})}{\beta}$$

[1 mark]

(iv)

$$F = U - TS \;\Rightarrow\; \frac{S}{N} = k_B\, \beta \left[\frac{U}{N} - \frac{F}{N}\right] = k_B \left[\frac{\beta\,\epsilon\, e^{-\beta\,\epsilon}}{1 - e^{-\beta\,\epsilon}} - \ln(1 - e^{-\beta\,\epsilon})\right]$$

[2 marks]

Statistical Physics: Weekly Problem 3 (SP3)

(1) (a) For the microcanonical ensemble, all of the compatible microstates are assumed to be equally probable (this is the basic postulate of equal a priori probabilities).

[1 mark]

(b) For the canonical ensemble, the probabilities of the compatible microstates are different, depending on the energy of the microstate (which is fixed temperature, not fixed energy). The probabilities are given by the Boltzmann factor, i.e. they are proportional to $\exp(-E/(k_\mathrm{B}T))$, where E is the energy of the microstate.

[1 mark]

(2)

$$\frac{p_0}{p_1} = \exp\left(-\frac{E_0 - E_1}{k_\mathrm{B}T}\right) \;\Rightarrow\; T = \frac{E_1 - E_0}{k_\mathrm{B}\,\ln(p_0/p_1)} = \frac{(13.6 - 3.4)\mathrm{eV}}{\ln(100) \times 8.617 \times 10^{-5} eV K^{-1}} \simeq 25700K$$

[2 marks]

(3) (a) The energy parallel to B is $-\mu_B B$ and the probability that the ion will have its magnetic moment oriented parallel to B is proportional to: $p_\uparrow \propto \exp[\mu_B B/(k_\mathrm{B}T)]$.

[1 mark]

The probability that the ion will have its magnetic moment antiparallel to B is $p_\downarrow \propto \exp[-\mu_B B/(k_\mathrm{B}T)]$. These probabilities are normalised

$$p_\uparrow + p_\downarrow = 1 \;\Rightarrow\; p_\uparrow = \frac{\exp[\mu_B B/(k_\mathrm{B}T)]}{\exp[\mu_B B/(k_\mathrm{B}T)] + \exp[-\mu_B B/(k_\mathrm{B}T)]} = \frac{1}{1 + \exp[-2\mu_B B/(k_\mathrm{B}T)]}$$

[1 mark]

(b) When all the magnetic moments are parallel to B, each ion has energy $-\mu_\mathrm{B}B$. There are N ions, so the internal energy is $U = -N\mu_\mathrm{B}B$.

[1 mark]

When all the magnetic moments are oriented parallel to B, there is only way of doing this (perfect order, one microstate), $\Omega = 1$ and the entropy vanishes is $S = k_\mathrm{B}\ln\Omega = 0$.

[1 mark]

Since all the ions are in their lowest energy state, the whole system is in the ground state and the temperature must be zero. Another way: since all the ions are parallel to B, the probability must be $p_\uparrow = 1$, so $\exp[-2\mu_B B/(k_\mathrm{B}T)] = 0$. This happens when $2\mu_B B/(k_\mathrm{B}T) \to \infty$, so the temperature $T = 0$.

[1 mark]

(c) Since the internal energy is positive, we have $p_\uparrow(-\mu_\mathrm{B}B) + p_\downarrow\mu_\mathrm{B}B > 0$. As $\mu_\mathrm{B}B > 0$, we have $p_\uparrow < p_\downarrow$. From the Boltzmann distribution we have

$$\frac{p_\uparrow}{p_\downarrow} = \exp\left(\frac{2\mu_\mathrm{B}B}{k_\mathrm{B}T}\right) < 1.$$

Taking the logarithm of both sides of the inequality, we obtain

$$\frac{2\mu_\mathrm{B}B}{k_\mathrm{B}T} < 0 \;\Rightarrow\; T < 0.$$

Infinite T corresponds to equal population of up and down energy levels, $p_i = 1/2$, and hence $U = 0$. This is lower than any $U > 0$ and so, negative temperatures are "hotter" than $T = \infty$.

[1 mark]

(1) They have different probabilities. The probability of each distribution increases with the number of microstates $\Omega[\{n_i\}]$ corresponding to each distribution. Among all distributions of the assembly of distinguishable particles compatible with fixed (N, U, V), the Boltzmann distribution has the greatest number of microstates and hence is the most probable. [2 marks]

(2) (a) Setting $\beta = 1/k_\mathrm{B}T$, the single-particle partition function is

$$Z_1 = \exp(\beta\mu_\mathrm{B}B) + \exp(-\beta\mu_\mathrm{B}B)$$

hence

$$p_\uparrow = \frac{\exp(\beta\,\mu_\mathrm{B}B)}{Z_1}\,, \quad p_\downarrow = \frac{\exp(-\beta\,\mu_\mathrm{B}B)}{Z_1}$$

[1 mark]

(b) $M/N = p_\uparrow\,\mu_\mathrm{B} + p_\downarrow\,(-\mu_\mathrm{B}) = \mu_\mathrm{B}\,\tanh\,(\beta\mu_\mathrm{B}B)$. [2 marks]

(c)

$$U = N[p_\uparrow\,\epsilon_\uparrow + p_\downarrow\,\epsilon_\downarrow] = N[p_\uparrow(-\mu_\mathrm{B}B) + p_\downarrow\,\mu_\mathrm{B}B] = -N\mu_\mathrm{B}B\,\tanh\,(\beta\mu_\mathrm{B}B)$$

The energy of N magnetic moments of magnitude M/N oriented along B is

$$E = -N(M/N)B = -N\mu_\mathrm{B}B\,\tanh\,(\beta\mu_\mathrm{B}B)\,,$$

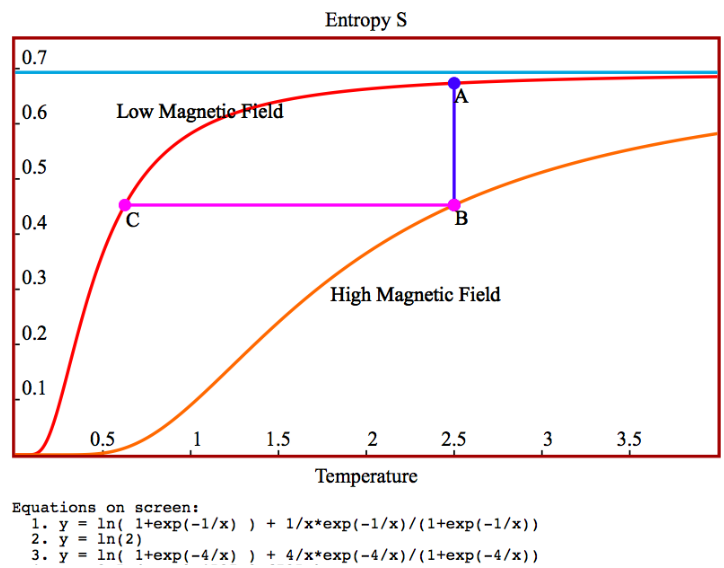
so the energy E is equal to U as expected.
[2 marks]

(d) (i) From

$$S = -Nk_\mathrm{B}[p_\uparrow\,\ln p_\uparrow + p_\downarrow\,\ln p_\downarrow]$$

we have that S depends on B and on T through $(\beta\mu_\mathrm{B}B)$ because the probabilities p_i depend on this ratio, see (a).
[1 mark]

(ii) The graph of S vs T for a high (B_h) and a low (B_l) value of B



Put the system initially at point A, at a relatively high temperature (precooled temperature) T_h, lying in a low magnetic field B_l. Then, keeping T constant

(isothermally, path $A \to B$) increase the strength of the magnetic field, so that the magnetic moments align along B. Entropy drops because there is greater order.
To cool the system down (adiabatic demagnetisation) we keep the system thermally isolated, to prevent any exchange of heat and any change of entropy $dS = 0$. Follow the path $B \to C$, by turning down the magnetic field $B_h \to B_l$ until the system reaches point C. Since S depends on the ratio B/T, it follows that during this process $B \to C$, the ratio B/T also remains fixed.

$$\frac{B_h}{T_h} = \frac{B_l}{T_l} \;\Rightarrow\; T_l = \frac{B_l}{B_h} T_h$$

Therefore, during the process $B \to C$, the temperature drops $T_h \to T_l$. [2 marks]

Statistical Physics: Weekly Problem 5 (SP5)

(1) (a) With $\beta = 1/(k_B T)$ the one particle partition function is

$$Z = e^{\beta\epsilon} + e^{\beta 0} + e^{-\beta\epsilon} = 1 + 2\cosh(\beta\epsilon).$$

[2 marks]

(b) The internal energy is

$$U = -N\frac{\partial \ln Z}{\partial \beta} = -N\,\epsilon\,\frac{e^{\beta\epsilon} - e^{-\beta\epsilon}}{1 + e^{\beta\epsilon} + e^{-\beta\epsilon}} = -N\epsilon\,\frac{2\sinh(\beta\epsilon)}{1 + 2\cosh(\beta\epsilon)}.$$

The heat capacity is

$$C_V = \frac{\partial U}{\partial T} = -k_B\,\beta^2\,\frac{\partial U}{\partial \beta} = Nk_B\beta^2\epsilon^2\left[\frac{e^{\beta\epsilon} + e^{-\beta\epsilon}}{1 + e^{\beta\epsilon} + e^{-\beta\epsilon}} - \frac{(e^{\beta\epsilon} - e^{-\beta\epsilon})^2}{(1 + e^{\beta\epsilon} + e^{-\beta\epsilon})^2}\right].$$

The term in square brackets doesn't simplify to a very compact expression. Any sensible simplification is fine, for example,

$$C_V = Nk_B\beta^2\epsilon^2\left[\frac{1}{1 + e^{\beta\epsilon} + e^{-\beta\epsilon}} + \frac{3}{(1 + e^{\beta\epsilon} + e^{-\beta\epsilon})^2}\right].$$

The free energy is

$$F = -Nk_B T\ln Z = -Nk_B T\ln\left[1 + e^{\beta\epsilon} + e^{-\beta\epsilon}\right].$$

The entropy is

$$S = \frac{U - F}{T} = -Nk_B\,\frac{\epsilon}{k_B T}\,\frac{e^{\beta\epsilon} - e^{-\beta\epsilon}}{1 + e^{\beta\epsilon} + e^{-\beta\epsilon}} + Nk_B\ln\left[1 + e^{\beta\epsilon} + e^{-\beta\epsilon}\right].$$

[2 marks]

(2) (a) Each single-particle energy ϵ_i can hold up to $g_i \times \eta$ particles. Therefore we have $g_i \times \eta$ "boxes" in which to distribute n_i particles, i.e. n_i boxes are full and $\eta \times g_i - n_i$ are empty. The number of ways we can do this for each energy ϵ_i is

$$\Omega_{\epsilon_i} = \frac{(\eta \times g_i)!}{n_i!\,(\eta \times g_i - n_i)!}.$$

Since the number of ways, Ω_{ϵ_i}, to distribute the n_i particles in the various levels, ϵ_i, are independent, the total number of microstates for the distribution $(n_1, n_2, \ldots)$ is the product

$$\Omega(\{n_i\}) = \prod_j \frac{(\eta \times g_j)!}{n_j!\,(\eta \times g_j - n_j)!}.$$

[2 marks]

(b) To find the most probable distribution $(n_i, n_2, \ldots)$, maximize the entropy under the usual constraints of fixed N, U, i.e. maximize

$$\frac{S}{k_B} - \alpha N - \beta U =$$

$$= \sum_i \left[[(\eta\times g_i)\ln(\eta\times g_i) - \eta\times g_i] - (n_i \ln n_i - n_i) - [(\eta\times g_i - n_i)\ln(\eta\times g_i - n_i) - (\eta\times g_i - n_i)] - \alpha n_i - \beta n_i \epsilon_i \right]$$

$$= \sum_i \left[(\eta \times g_i)\ln(\eta \times g_i) - n_i \ln n_i - (\eta \times g_i - n_i)\ln(\eta \times g_i - n_i) - \alpha n_i - \beta n_i \epsilon_i \right].$$

The derivative with respect to any of the n_j is zero, so

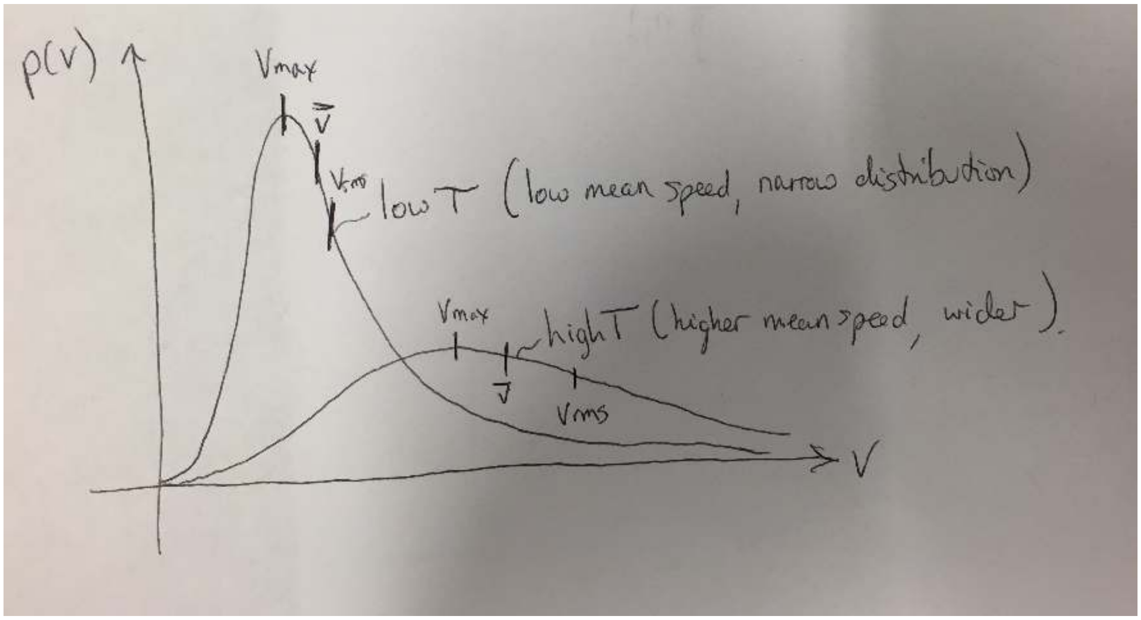
$$\begin{aligned}
\frac{\partial}{\partial n_j}\left\{ \frac{S}{k_B} - \alpha N - \beta U \right\} &= 0 \\
-\ln n_j - 1 + \ln(\eta \times g_j - n_j) + 1 - \alpha - \beta\epsilon_j &= 0 \\
\ln(\eta \times g_j/n_j - 1) &= \alpha + \beta\epsilon_j \\
\eta \times g_j/n_j &= e^{\alpha}\, e^{\beta\epsilon_j} + 1 \\
\frac{n_j}{g_j} &= \frac{1}{(e^{\alpha}/\eta)\, e^{\beta\epsilon_j} + (1/\eta)} \\
&= \frac{1}{A\, e^{\beta\epsilon_j} + (1/\eta)}
\end{aligned}$$

where $A = e^{\alpha}/\eta$. [2 marks]

(c) This distribution function reduces to the Fermi Dirac distribution for $\eta = 1$. It reduces to the Maxwell-Boltzmann distribution for large η $(\eta \to \infty)$. [2 marks]

Statistical Physics: Weekly Problem 6 (SP6)

(1) (a) Sketch of distribution and note that $p(0) = 0$ and $p(\infty) \to 0$.



To find behaviour at small v we can simplify the structure of $p(v)$ by setting all the constants to 1 giving $p(v) \sim v^2 \exp(-v^2)$. For small v we have $\exp(-v^2) \sim 1$ hence we get

$$\lim_{v \to 0} p(v) \sim v^2$$

[2 marks]

(b) From lectures we evaluated the integrals to find the average values. The results are

(i) most probable speed

$$v_{max} = \sqrt{\frac{2}{\beta m}} \quad \text{or} \quad \sqrt{2}\sqrt{\frac{k_B T}{m}},$$

(ii) mean speed

$$\bar{v} = \sqrt{\frac{8}{\pi}} \frac{1}{\sqrt{\beta m}} \quad \text{or} \quad \sqrt{\frac{8}{\pi}}\sqrt{\frac{k_B T}{m}},$$

(iii) r.m.s speed

$$v_{rms} = \sqrt{\frac{3}{\beta m}} \quad \text{or} \quad \sqrt{3}\sqrt{\frac{k_B T}{m}}.$$

[1 mark]

If we define $v_T = 1/\sqrt{\beta m} = \sqrt{k_B T/m}$ and then in units of v_T we have

$$v_{max} = 1.414\, v_T, \quad \bar{v} = 1.596\, v_T, \quad v_{rms} = 1.732\, v_T$$

making it straightforward to mark on a graph, shown in (a). [1 mark]

(c) Some calculator time. We have $m = 3.36 \times 10^{-26}$ kg and $T = 300$ K so $v_T = 351$ ms^{-1}. This gives $v_{max} = 496$ ms^{-1}, $\bar{v} = 560$ ms^{-1} and $v_{rms} = 607$ ms^{-1}. [2 marks]

(d) (i) For normalisation

$$
\begin{aligned}
1 &= \int_0^\infty p(v)\, dv \\
&= C \int_0^\infty dv\, v \exp\left(-\frac{mv^2}{2k_BT}\right) \\
&= C\, I_1(\alpha) = \frac{C}{2\alpha} \\
&= \frac{C\, k_B\, T}{m} \\
\Rightarrow C &= \frac{m}{k_BT}
\end{aligned}
$$

[2 marks]

The probability distribution in both 2D and 3D $p(v = 0) = 0$. In 3D for small v we had $p(v) \sim v^2$ and in 2D, we have $p(v) \sim v$. For large v in both cases the probability distribution goes to zero.

(ii) The most probable speed is

$$\frac{dp(v)}{dv} = Ce^{-\frac{\beta mv^2}{2}} \left[1 - \beta mv^2\right] = 0 \quad \Rightarrow v_{max} = \frac{1}{\sqrt{\beta m}} = v_T,$$

the mean speed is

$$\bar{v} = \frac{\int_0^\infty dv\, v^2 e^{-\frac{\beta mv^2}{2}}}{\int_0^\infty dv\, v e^{-\frac{\beta mv^2}{2}}} = \frac{I_2(\lambda)}{I_1(\lambda)} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} = \sqrt{\frac{\pi}{2\beta m}} = \sqrt{\frac{\pi}{2}}\, v_T = 1.253\, v_T,$$

and the RMS speed is

$$v_{rms}^2 = \frac{\int_0^\infty dv\, v^3 e^{-\frac{\beta mv^2}{2}}}{\int_0^\infty dv\, v e^{-\frac{\beta mv^2}{2}}} = \frac{I_3(\lambda)}{I_1(\lambda)} = \frac{1}{\lambda} = \frac{2}{\beta m} = 2\, v_T^2 \;\Rightarrow\; v_{rms} = \sqrt{2}\, v_T = 1.414\, v_T.$$

[2 marks]

Statistical Physics: Weekly Problem 7 (SP7)

(1) (a) We have (from lectures, notes, books) for a free particle in a box

$$g(k)\delta k = \frac{V}{2\pi^2} k^2\, \delta k.$$

and that

$$\epsilon(k) = \frac{\hbar^2 k^2}{2M},$$

hence the single-particle partition function is

$$Z_1 = \int_0^\infty dk\, g(k)\, e^{-\beta\epsilon(k)} = \frac{V}{2\pi^2} \int_0^\infty dk\, k^2 e^{-\frac{\beta\hbar^2 k^2}{2M}} = V\left(\frac{2\pi M}{\beta h^2}\right)^{\frac{3}{2}} = \frac{V}{\lambda_{\mathrm{D}}^3},$$

[1 mark]

(b) We have the dilute gas limit when λ_{D} is much less that the average distance between gas particles. The average volume per particle is $\nu = V/N$. The average distance between particles is $\simeq \nu^{1/3}$. So, the dilute gas limit is when $\lambda_{\mathrm{D}} \ll \nu^{1/3}$ [1 mark]

(c) (i) For distinguishable particles $Z_N = (Z_1)^N$, therefore

$$\ln Z_N = N\,\ln Z_1 = N\big[\ln V - 3\ln\lambda_{\mathrm{D}}\big],$$

hence the internal energy is

$$F_{Classical} = -k_{\mathrm{B}}T\ln Z_N = -N\,k_{\mathrm{B}}T\Big[\ln V - 3\,\ln\lambda_{\mathrm{D}}\Big]$$

(ii) For indistinguishable particles $Z_N = (Z_1)^N/N!$, so

$$Z_N = \frac{Z_1^N}{N!} \quad \Rightarrow \ln Z_N = N\,\ln Z_1 - N\ln N + N$$

$$Z_N = N\,\ln V - 3N\,\ln\lambda_{\mathrm{D}} - N\ln N + N.$$

The internal energy is

$$F = -k_{\mathrm{B}}T\ln Z_N = -k_{\mathrm{B}}T\Big[N\,\ln V - 3N\,\ln\lambda_{\mathrm{D}} - N\ln N + N\Big]$$

$$F = -N\,k_{\mathrm{B}}T\Big[\ln(V/N) - 3\,\ln\lambda_{\mathrm{D}} + 1\Big].$$

(iii) In (ii) we divide the N-particle partition function of identical particles by $N!$ in order to compensate for over-counting the number of microstates of identical particles. For example, take a system with three distinguishable independent particles. In the state 123, the first particle is in single-particle (s.p.) state 1, the second is in s.p. state 2 and the third in s.p. state 3. We have 6 different microstates for the system of the three distinguishable particles. These are $123, 231, 312, 132, 213, 321$. However, when the particles are indistinguishable we cannot know which particle is in which state and the 6 microstates reduce to 1. So, for identical particles, we are over-counting the many-particle states by 6 (=3!).

[3 marks]

(d) The internal energy is

$$U = \frac{3}{2} N\, k_{\mathrm{B}} T$$

and the free energy is in (c), so combining these gives the classical entropy for distinguishable particles

$$S = N\, k_{\mathrm{B}} \Big[\ln \big[V/\lambda_{\mathrm{D}}^3 \big] + \frac{3}{2} \Big] = N\, k_{\mathrm{B}} \Big[\ln \big[V (2\pi M k_{\mathrm{B}} T/h^2)^{3/2} \big] + \frac{3}{2} \Big] \Rightarrow$$

$$S = N\, k_{\mathrm{B}} \ln V T^{3/2} + \frac{3}{2} N\, k_{\mathrm{B}} \Big[\ln \big[(2\pi M k_{\mathrm{B}}/h^2) \big] + 1 \Big].$$

[2 marks]

(e) The entropy depends on V, T through the product $VT^{3/2}$. Since the entropy does not change during the expansion, we have that $VT^{3/2}$ remains constant. So

$$V_1 T_1^{3/2} = V_2 T_2^{3/2} \;\Rightarrow\; T_2 = T_1 \left(\frac{V_1}{V_2} \right)^{2/3} \simeq 0.63\, T_1 = 189 \text{ K}.$$

[2 marks]

(f) Adiabatic demagnetisation was covered in lectures. Other examples exist such as Pomeranchuk cooling by adiabatic solidification of ^{3}He (look it up). [1 mark]