Statistical Physics: Workshop Problems 8

(1) (a) The density of states in k space is

$$g(k)dk = 2\frac{V}{2\pi^2}k^2dk$$

remembering a factor of 2 for spin and with $\epsilon = \hbar^2k^2/2m$ we have

$$g(\epsilon)d\epsilon = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\epsilon}d\epsilon.$$

At $T = 0$ the N electrons fill up the energy levels with $f_{FD}(\epsilon) = 1$ to the Fermi level so

$$\begin{aligned} N &= \int_0^{E_F} g(\epsilon)f_{FD}(\epsilon)d\epsilon \\ &= \frac{2}{3}g(E_F)E_F \end{aligned}$$

where

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}.$$

The specific heat is

$$\begin{aligned} C_V &\sim k_B^2Tg(E_F) \\ &\sim \frac{N}{E_F}k_B^2T. \end{aligned}$$

There are many ways to do this, covered in lectures and workshops. Other methods are valid.

(b) We need to obtain $g(E_F)$ which can be done in several ways, such as in (a). Using that result we get immediately that

$$\begin{aligned} \chi &= \mu_B^2g(E_F) \\ &\sim \frac{N}{E_F}\mu_B^2. \end{aligned}$$

(c) A the electrons are non-interacting then there is no potential and so the internal energy of the system is the kinetic energy. Therefore

$$\begin{aligned} \langle E_k\rangle &= \int_0^{E_F} \epsilon g(\epsilon)d\epsilon \\ &= \text{...Done many times...} = \frac{3}{5}NE_F. \end{aligned}$$

(d)

$$\begin{aligned} P &= \frac{d\langle E_k\rangle}{dV} \\ &= \frac{3}{5}\frac{NE_F}{V}. \end{aligned}$$

(2) Similar to above there are several ways to approach this, and we can simply take the results of (1) and put in the numbers. To demonstrate another method, let's do it in momentum space rather than energy space. We have the number of neutrons (or protons) on momentum range p to $p + dp$ as

$$dN = \frac{4V}{h^3} 4\pi p^2 dp$$

so the total number of neutrons is

$$N = \int dN = \frac{16\pi V}{h^3} \int_0^{p_F} p^2 dp = \frac{16\pi V}{3h^3} p_F^3$$

where p_F is the Fermi momentum. The total kinetic energy of the neutrons is

$$E_k = \int \frac{p^2}{2m} dN = \frac{16\pi V}{10h^3 m} p_F^5.$$

Therefore

$$\frac{E_k}{A} = \frac{3}{5} \frac{p_F^2}{2m},$$

noting this is the equivalent expression to (1)(c). The volume can be expressed by the usual formula for a sphere and also via the above expression for A, so

$$V = \frac{4}{3}\pi R_0^3 A = \frac{3(2\pi)^3}{16\pi} p_F^{p_F} A$$

giving

$$p_F = R_0^{-1} \left(\frac{9\pi}{8}\right)^{1/3}$$

and

$$\frac{E_k}{A} = \frac{3}{10} \left(\frac{9\pi}{8}\right)^{1/3} \frac{1}{mR_0^2} \sim 16\text{MeV}.$$

(3) We have calculated in several ways in lectures and workshops (for example, (1)(a) in this workshop) $N = \int_0^{E_F} g(\epsilon) f_{FD}(\epsilon) d\epsilon$ to get the relation between particle number and Fermi energy. This gives

$$\frac{N}{V} = \frac{8\pi}{3} \left(\frac{2mE_F}{h^2}\right)^{3/2}.$$

The condition for the reaction is $E_F \geq 0.8$ MeV so this is satisfied with a minimum value of N/V of 3.24×10^{36} m^{-3}. Multiply number density by neutron mass to get the minimum density of a neutron star to be 5.4×10^9 kg m^{-3}. A relativistic calculation gives $\sim 10^{16}$ kg m^{-3}.

(4) (a) This can be done in energy or momentum space. Let's do the momentum space calculation. Taking the usual density of states in k space we have

$$E = \frac{8\pi V}{h^3}\int_0^{p_F} \epsilon p^2 dp$$

where p_F is the Fermi momentum which is

$$p_F = \left(\frac{3Nh^3}{8\pi V}\right)^{1/3}.$$

In the extreme relativistic case we have $\epsilon = cp$ so the energy is

$$E = \frac{2\pi c V}{h^3} p_F^4$$

and the pressure $p = -(\partial E/\partial V)_{T=0}$ gives $p = E/(3V)$ and hence the equation of state is $pV = E/3$.

(b) From relativity, the extreme relativistic limit is when the rest mass is not a significant part of the total energy, i.e.

$$\epsilon = \sqrt{(mc^2)^2 + (pc)^2} \sim pc\left[1 + \frac{1}{2}\left(\frac{mc}{p}\right)^2\right].$$

Using the result in (a) we get

$$E \sim 2\pi c V\left[p_F^4 + (mcp_F)^2\right]/h^3.$$

So for (a) to be valid we require $p_F \gg mc$ hence

$$\frac{N}{V} \gg \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3$$

which can be satisfied with $N \to \infty$ for a given V or $V \to 0$ for a given N.

(5) In the stated extreme relativistic limit we have $\epsilon = pc$ and $\mu = 0$. Therefore

$$\begin{aligned}\frac{N}{V} &= \left(\frac{4\pi}{(2\pi\hbar)^3}\right)\int_0^\infty \frac{p^2}{e^{pc/k_BT}+1}dp \\ &= \frac{1}{2\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\int_0^\infty \frac{x^2}{e^x+1}dx.\end{aligned}$$

The integral is $3\zeta(3)/2 \sim 1.80309$ where ζ is the Riemann-zeta function. The average energy density is

$$\rho = \frac{N}{V}\epsilon = \frac{N}{V}k_BT,$$

and substitute for N/V to obtain the result.