Statistical Physics: Workshop Problems 4

(1) Consider a particle in a box. Derive the density of states in k space, $g(k)$, and in energy space, $g(\epsilon)$, in (a) two and in (b) three dimensions.

(2) The single-particle partition function for a particle in a 3D box is

$$Z_1 = \int_0^\infty d\epsilon\, g(\epsilon)\, e^{-\beta\epsilon} = V \left(\frac{2\pi M}{\beta h^2}\right)^{3/2}.$$

(a) What is the partition function Z_N for a gas of N such particles in volume V, classically?

(b) Obtain the internal energy U, the free energy F and entropy S of the gas of particles.

(c) Are the internal energy, the free energy and the entropy of the gas extensive quantities? [Hint: in order to check, double the number of particles and the volume and check if U, F also double.]

(d) When the particles in the gas are indistinguishable, how can one correct the N particle partition function in order to obtain extensive quantities? What is the extensive expression for the entropy?

(e) A container with a gas of identical particles is divided by a partition in two compartments of equal volume ($V/2$) and equal number ($N/2$) of particles.

(i) The two compartments contain the same kind of indistinguishable particles and hence the removal of the partition is a reversible process that does not involve the exchange of heat or work. What is the change in entropy when the partition is removed? Is this result expected? (Use the expression for the entropy where the indistinguishability of particles is taken into account).

(ii) Each of the two compartments contains a different kind of indistinguishable particles. What is the change in entropy when the partition is removed? (Entropy of mixing.)

(3) Consider a gas of N free fermions of spin $1/2$ enclosed either in a box of side a and volume V (in 3D), or enclosed in a square of side a and area A (in 2D). At $T = 0$, the N fermions occupy the single-particle states in k-space, from $k = 0$ up to the Fermi wave vector $k = k_F$. The Fermi energy (ϵ_F) is

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2M}$$

where M is the mass of the free fermions.

(a) What is the density of states (in energy) of the fermion gas in the two cases?

(b) Calculate the Fermi wave vector k_F and the Fermi energy ϵ_F at $T = 0$ for the two cases. Note that the Fermi energy is when $n(k_F) = N$.