Statistical Physics: Weekly Problem 4 (SP4)

(1) They have different probabilities. The probability of each distribution increases with the number of microstates $\Omega[\{n_i\}]$ corresponding to each distribution. Among all distributions of the assembly of distinguishable particles compatible with fixed (N, U, V), the Boltzmann distribution has the greatest number of microstates and hence is the most probable. [2 marks]

(2) (a) Setting $\beta = 1/k_{\rm B}T$, the single-particle partition function is

$$Z_1 = \exp(\beta\mu_{\rm B}B) + \exp(-\beta\mu_{\rm B}B)$$

hence

$$p_\uparrow = \frac{\exp(\beta\,\mu_{\rm B}B)}{Z_1}\,, \quad p_\downarrow = \frac{\exp(-\beta\,\mu_{\rm B}B)}{Z_1}$$

[1 mark]

(b) $M/N = p_\uparrow\,\mu_{\rm B} + p_\downarrow\,(-\mu_{\rm B}) = \mu_{\rm B}\,\tanh(\beta\mu_{\rm B}B)$. [2 marks]

(c)

$$U = N[p_\uparrow\,\epsilon_\uparrow + p_\downarrow\,\epsilon_\downarrow] = N[p_\uparrow(-\mu_{\rm B}B) + p_\downarrow\,\mu_{\rm B}B] = -N\mu_{\rm B}B\,\tanh(\beta\mu_{\rm B}B)$$

The energy of N magnetic moments of magnitude M/N oriented along B is

$$E = -N(M/N)B = -N\mu_{\rm B}B\,\tanh(\beta\mu_{\rm B}B)\,,$$

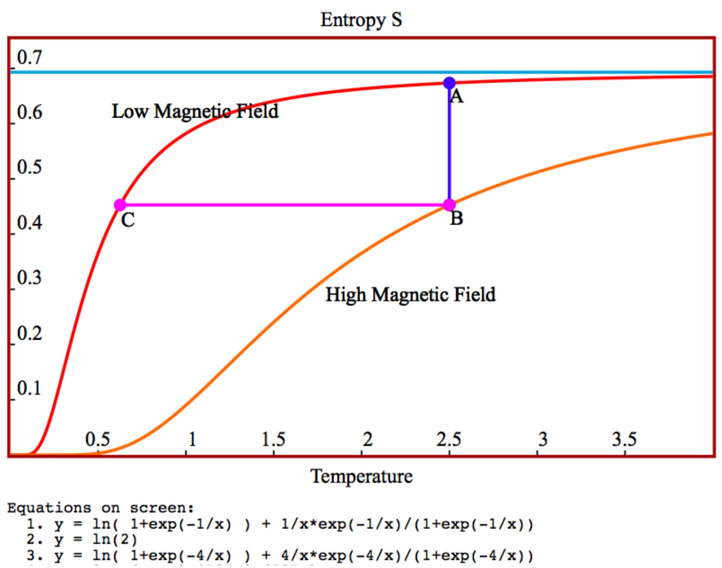
so the energy E is equal to U as expected.
[2 marks]

(d) (i) From

$$S = -Nk_{\rm B}[p_\uparrow\,\ln p_\uparrow + p_\downarrow\,\ln p_\downarrow]$$

we have that S depends on B and on T through $(\beta\mu_{\rm B}B)$ because the probabilities p_i depend on this ratio, see (a).
[1 mark]

(ii) The graph of S vs T for a high (B_h) and a low (B_l) value of B



Put the system initially at point A, at a relatively high temperature (precooled temperature) T_h, lying in a low magnetic field B_l. Then, keeping T constant

(isothermally, path $A \to B$) increase the strength of the magnetic field, so that the magnetic moments align along B. Entropy drops because there is greater order.
To cool the system down (adiabatic demagnetisation) we keep the system thermally isolated, to prevent any exchange of heat and any change of entropy $dS = 0$. Follow the path $B \to C$, by turning down the magnetic field $B_h \to B_l$ until the system reaches point C. Since S depends on the ratio B/T, it follows that during this process $B \to C$, the ratio B/T also remains fixed.

$$\frac{B_h}{T_h} = \frac{B_l}{T_l} \Rightarrow T_l = \frac{B_l}{B_h} T_h$$

Therefore, during the process $B \to C$, the temperature drops $T_h \to T_l$. [2 marks]