(a) Note that

$$a + ar + ar^2 + ar^3 + ... = \frac{a}{1 - r}$$

for $|r| < 1$ and so we get

$$Z = \sum_{k=0}^{\infty} e^{-\beta \epsilon k} = \frac{1}{1 - e^{-\beta \epsilon}}$$

[4 marks]

(b) (i)

$$p_k = \frac{e^{-\beta \epsilon k}}{Z} = \left(1 - e^{-\beta \epsilon}\right) e^{-\beta \epsilon k}$$

[1 mark]

(ii)

$$\frac{U}{N} = -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial}{\partial \beta} \ln(1 - e^{-\beta \epsilon}) = \frac{\epsilon e^{-\beta \epsilon}}{1 - e^{-\beta \epsilon}}$$

[2 marks]

(iii)

$$\frac{F}{N} = -\frac{\ln Z}{\beta} = \frac{\ln(1 - e^{-\beta \epsilon})}{\beta}$$

[1 mark]

(iv)

$$F = U - TS \implies \frac{S}{N} = k_B \beta \left[\frac{U}{N} - \frac{F}{N}\right] = k_B \left[\frac{\beta \epsilon e^{-\beta \epsilon}}{1 - e^{-\beta \epsilon}} - \ln(1 - e^{-\beta \epsilon})\right]$$

[2 marks]