MXB103 Exam Notes

Ordinary Differential Equations

Euler's Method

what's given: function, where you want to approximate, step value (h), and initial condition (when n = 0)
follow the formula: $y_{n+1} = y_n + hf'(x_n)$

n	x_n	y_n
0	initial x	initial y
1	initial x + h	follow fomula
2	$x_1 + h$	follow fomula
3	$x_2 + h$	follow fomula
4	$x_3 + h$	follow fomula
5	$x_4 + h$	follow fomula

Second Order Taylor Method

1. Identify $f(x)$ and the point a.
2. Compute derivatives $f'(x)$ and $f''(x)$.
3. Evaluate $f'(a)$ and $f''(a)$.
4. Construct the Taylor polynomial:

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

5. Use $P_2(x)$ to approximate $f(x)$ near a.

Modified Euler's Method

1. Given y_n and x_n, compute the initial slope $k_1 = f(x_n, y_n)$.
2. Predict the next value $\tilde{y}_{n+1} = y_n + k_1 h$, where h is the step size.
3. Compute the corrected slope $k_2 = f(x_n + h, \tilde{y}_{n+1})$.
4. Calculate the next value $y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$.
5. Update $x_{n+1} = x_n + h$ and repeat from step 1 for further iterations.

Runge-Kutta 4th Order (RK4)

1. Start with initial conditions y_n and x_n.
2. Calculate the first slope $k_1 = hf(x_n, y_n)$.
3. Compute the second slope $k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$.
4. Calculate the third slope $k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$.
5. Compute the fourth slope $k_4 = hf(x_n + h, y_n + k_3)$.
6. Estimate the next value $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.
7. Increment $x_{n+1} = x_n + h$ and use the new x_{n+1} and y_{n+1} for the next iteration.

Polynomial Interpolation

Lagrange Interpolating Polynomial

1. sub abscissas into given function to get y values.
2. find the Polynomial for the amount of abscissas. Ex: $(x_0, y_0), (x_1, y_1), (x_2, y_2)$
$\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0$ + same thing for y_1 and y_2

Newton's Divided Difference

$a_0 = y_0$, $a_1 = \frac{y_1-y_0}{x_1-x_0}$, $a_2 = \frac{\frac{y_2-y_1}{x_2-x_0} - \frac{y_1-y_0}{x_1-x_0}}{x_2-x_0}$

x_i	zeroth	first	second	third
x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	$f[x_1]$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
x_2	$f[x_2]$	$f[x_2, x_3]$		
x_3	$f[x_3]$			

$P_3(x) = f[x_0] = f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$

1. Sub given x and y values into first and second columns of the table.
2. Compute first divided difference using above a_1 formula.
3. Compute second divided difference using above a_2 formula
4. Sub into above formula.

Newton's Forward Difference

ONLY use when the abscissas are equally spaced.

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	
x_2	y_2	Δy_2		
x_3	y_3			

1. sub given x and y values into first and second columns of table.
2. Compute remaining columns. Ex: $\Delta y_0 = y_1 - y_0$
3. Sub into given polynomial equation.

Helpful Additional Info

0.1 Differential equation rules:

Constant Rule: $c' = 0$ where c is a constant.
Power Rule: $(x^n)' = nx^{n-1}$ for any real number n.
Sum Rule: $(u+v)' = u' + v'$
Difference Rule: $(u-v)' = u' - v'$
Product Rule: $(uv)' = u'v + uv'$
Quotient Rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
Exponential Rule: $(a^x)' = a^x \ln(a)$ where $a > 0$ and $a \neq 1$.
Logarithmic Rule: $(\ln(x))' = \frac{1}{x}$ for $x > 0$.
Trigonometric Rules:
$(\sin(x))' = \cos(x)$
$(\cos(x))' = -\sin(x)$
$(\tan(x))' = \sec^2(x)$
Inverse Trigonometric Rules:
$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$
$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$
$(\arctan(x))' = \frac{1}{1+x^2}$

0.2 Initial Value Problem Examples:

$Given : \frac{dy}{dx} = (6x-3), y(0) = 4$
$dx \cdot \frac{dy}{dx} = (6x-3)dx$
$dy = (6x-3)dx$
$\int dy\ = \int (6x-3)\, dx$
$y = \frac{6x^2}{2} - 3x + C$
Sub in initial value
$4 = 3(0)^2 - 3(0) + C$
$C = 4$
Plug constant C back into function:
$y = \frac{6x^2}{2} - 3x + 4$

$Given : \frac{dy}{dx} = 2xy$, with initial value: $y(0) = 3$
$dy = 2xy \cdot dx$
$\frac{1}{y} \cdot dy = \frac{2xy}{y} dx$
$\int \frac{1}{y} dy\ = \int 2x \cdot dx$
$lny = \frac{2x^2}{2} + C$
$lny = x^2 + C$
$e^{lny} = e^{x^2+C}$
$y = e^C \cdot e^{x^2}$
$y = Ce^{x^2}$
Sub in initial value
$3 = Ce^0$
$C = 3$