Assignment # 1: Static Aeroelasticity
Class: Structural Dynamics and Aeroelasticity, Prof. Giuseppe Quaranta
A.Y. 2022/23

April 12, 2023

Take the last four figures of your person code ABCD, and assemble these two numbers: DA and CB[1]

$$\begin{cases} 00 \leq DA < 50 & x_1 = 1.2 \\ 50 \leq DA < 99 & x_1 = 0.7 \\ 00 \leq CB < 50 & x_2 = 1.1 \\ 50 \leq CB < 99 & x_2 = 0.6 \end{cases} \tag{1}$$

Consider an aircraft with a *forward* swept wing and a *rigid* canard surface sketched in Figure 1, with the following geometric properties: wing semi-span $b = 6.10$ m, wing chord $c = 3.05$ m, sweep angle $\Lambda = -30$, fuselage length $L_f = 9.15$ m, canard chord $c_c = 3.05$ m canard semi-span $b_c = 1.525$ m. The wing is attached to the fuselage on the point with coordinate $(x_w, y_w) = (0, 0)$, while the canard is attached at $(x_c, y_c) = (-4.525, 0)$. On the wing tips, there are two ailerons with a chord equal to $c_a = \frac{1}{4}c$ and semi-span equal to $b_a = \frac{1}{2}b$. The elastic axis of the wing is positioned at 50% of the chord.

The wing is characterized by the following structural properties (with $\bar{y}$ the axis aligned with the wing beam structure)

- Bending stiffness $EI_w(\bar{y}) = x_1 \times 4.5\,10^6$ N m^2
- Torsional stiffness $GJ_w(\bar{y}) = x_2 \times 7.0\,10^6$ N m^2

The fuselage is rigid in bending and has the following torsional stiffness

- Fuselage torsional stiffness $GJ_f(x) = 12.0\,10^6$ N m^2

[1] Example: 10134997 $\rightarrow A = 4$, $B = 9$, $C = 9$, $D = 7$, so the two numbers will be $DA = 74$, $CB = 99$.

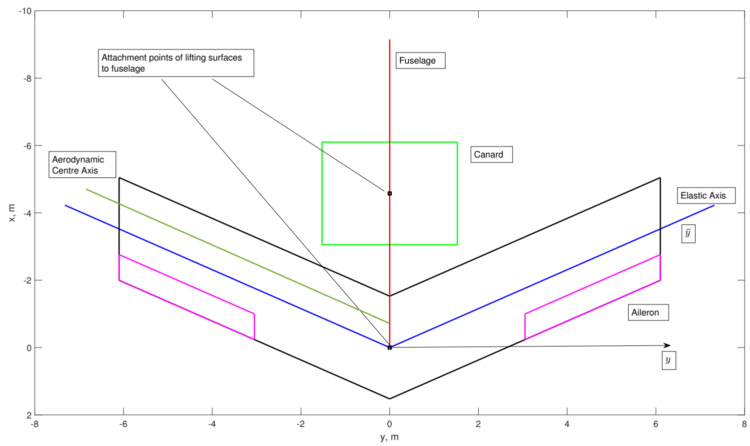

Figure 1: Aircraft planform.

Consider the aircraft subject to an incompressible airflow (i.e. $M = 0$) and use the strip theory with $C^0_{L\alpha} = 2\pi$ to model the aerodynamics for both the wing and the canard. The aerodynamic centre must be considered always placed at 1/4 of the chord. To compute the aerodynamic coefficients for the aileron, the expression provided in class (set of slides #2) must be used.

1. Compute the lowest divergence dynamic pressure of the aircraft using the Ritz-Galerkin method. Use as shape functions sinusoidal functions appropriate for the boundary conditions of the problem under analysis (remember that the aircraft is flying and not constrained in a wind tunnel). Select the number of shape functions to reach a sufficient precision.

2. Consider to rotate the ailerons by a $\beta = 1°$ (right downward, left upward). Compute the angular roll speed reached at regime by the aircraft when flying at 85% of the divergence dynamic pressure and compare it with the regime angular roll speed for the rigid aircraft. (WARNING: the canard's effect, and the effect of the flexible fuselage, must be considered. SUGGESTION: Use as shape functions to model the fuselage torsion appropriate trigonometric functions, i.e. $\sin \omega x$ and/or $\cos \omega x$ with appropriate spatial frequencies ω).

3. Plot in this condition (i.e. regime after application of $\beta = 1°$) the diagram of the torsional moment as a function of x applied to the fuselage.

4. Compute the bending moment generated by the aileron rotation applied at the wing root.

5. Make the hypothesis that it is possible to design the wing structure using composite materials so that coupling between bending and torsion exists.

Consider that the material used in the wing structure leads to the following constitutive law

$$\begin{Bmatrix} M_b \\ T \end{Bmatrix} = \begin{bmatrix} EI_w & C_w \\ C_w & GJ_w \end{bmatrix} \begin{Bmatrix} w'' \\ \theta' \end{Bmatrix} \tag{2}$$

Try to estimate a value of C_w, if exists, sufficient to have a divergences speed of the forward swept wing above that of the equivalent straight wing.

The methods necessary to solve all steps must be written by hand on paper and a copy must be submitted together with the numeric solutions. All integrals could be solved using the symbolic integrator of MATLAB (or any other symbolic solver). The eigenvalues can be computed using the MATLAB `eig` routine (or any other numerical tool). All routines used to compute the solution must be provided too so that they can be run to verify how they work.

Clarifications on the text may be asked until April 21, 2023. To submit the responses and the photos of the written solution you will have to connect to a MS Form page. The answers will have to be submitted after 10 days starting from April 22, 2023 (so by 1:00 pm of May 02, 2023). A to the MS Form page to submit the answers will be provided on April 22, 2023.

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ASSIGNMENT #1, A.A. 2022/2023

Structural dynamics and Aeroelasticity

Person code : 10667431 $\longrightarrow$ $x_1 = 1.2$ e $x_2 = 1.1$

forward swept wing $\longrightarrow$ $-30° = \Lambda$

$\bar{y}$ = wing beam structure

Wing SEMISPAN = 6.10 m = b

Wing CHORD = 3.05 m = c

Wing constrain : $(x_w, y_w) = (\emptyset, \emptyset)$

Wing EA = 50% c

Wing Bending stiffness : $EI_w(\bar{y}) = x_1 \cdot 4.5 \cdot 10^6 \, Nm^2$

Wing Torsional stiffness : $GJ_w(\bar{y}) = x_2 \cdot 7.0 \cdot 10^6 \, Nm^2$

Fuselage LENGTH = 9.15 m = L

Fuselage Bending stiffness : $EI_w(\bar{y}) = \infty$

Fuselage Torsional stiffness : $GJ_w(\bar{y}) = 12.0 \cdot 10^6 \, Nm^2$

Aliron on wing

Aliron SEMISPAN : $1/2 \, b = b_a$

Aliron CHORD : $1/4 \, c = c_a$

Canard SEMISPAN = 1.525 m = b_c

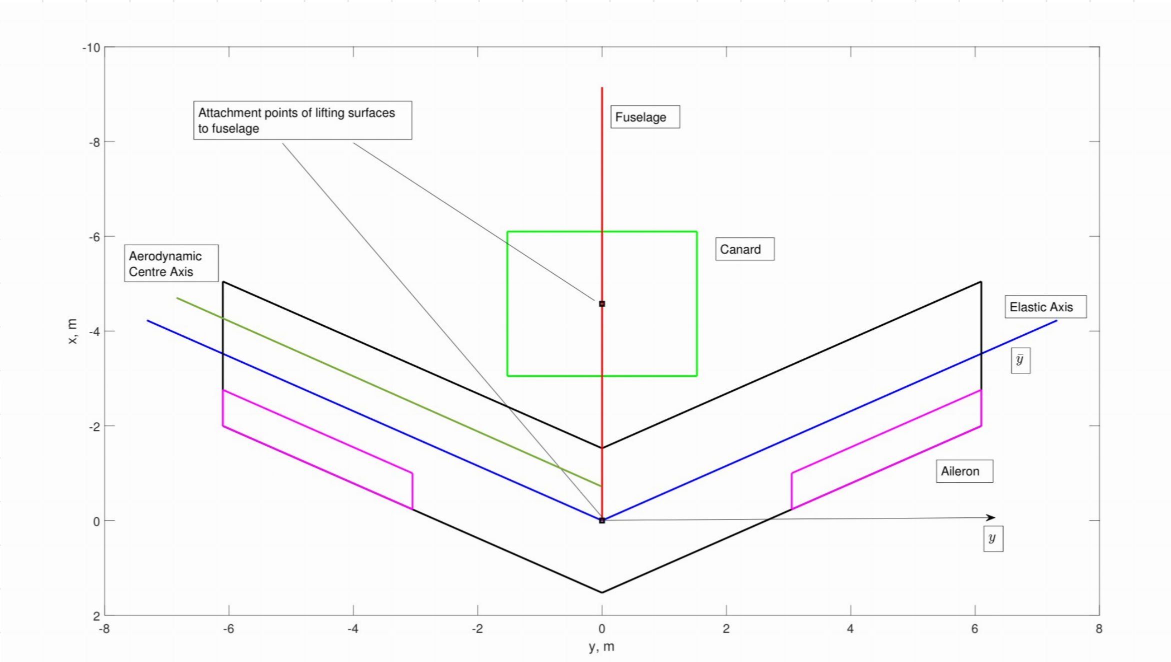
Canard CHORD = 3.05 m = c_c

Canard constrain : $(x_c, y_c) = (-4.525, 0)$

Rigid canard

Canard Bending stiffness : $EI_w(\bar{y}) = \infty$

Canard Torsional stiffness : $GJ_w(\bar{y}) = \infty$



Aircraft subjected to an incompressible airflow $(M = \emptyset)$

$e = 1/4 \, c$

Wing : $C^{\circ}_{L\alpha} = 2\pi$

$AC = 1/4 \, c$

Canard : $C^{\circ}_{L\alpha} = 2\pi$

$AC = 1/4 \, c_c$

Aliron : $C_{L\beta} = 2\left(\cos^{-1}(1-2E) + 2\sqrt{E(1-E)}\right)$

$C_{m\beta} = -2(1-E)\sqrt{E(1-E)}$

$E = c_a/c$

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(1) Compute lowest divergence dyn pressure using Ritz - GALERKIN

$\hookrightarrow$ Shape function sinusoidal (appropriate with BC)

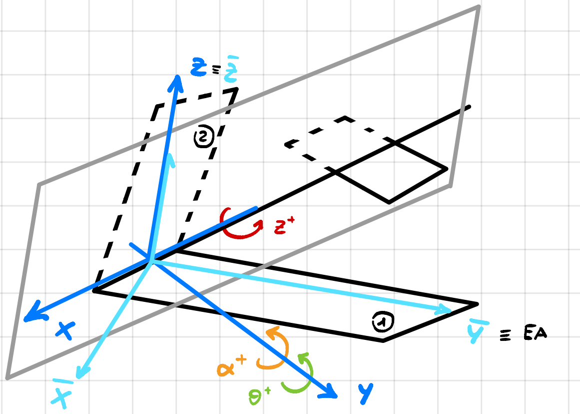
NOTE 1: Aircraft is flying, not constrained in Tunnel

NOTE 2: Select a good amount of shape function to get enought precision

We consider the aircraft in a Trim condition where every Terms in our equations is a perturbation applied on this eq. condition.

Divergence problem is a stability analysis that reguards interaction between STRUCTURAL and AERODYNAMIC stiffnesses ... since fuselage doesn't generate any aerodynamic forces could not be considered.

Since the canard is rigid we have no divergence on the canard surface.



For divergence problem we consider the wing $(1+2)$ as an Euler - Bernoulli beam free on both side

Let's start by considering the PVW applied on the wing system (as continuum):

$$\delta W_i = \int_0^{\bar{b}} \delta\bar{\theta}'^T GJ\,\bar{\theta}'\,d\bar{y} + \int_0^{\bar{b}} \delta z''^T EI\,z''\,d\bar{y} + \int_{-\bar{b}}^0 \delta\bar{\theta}'^T GJ\,\bar{\theta}'\,d\bar{y} + \int_{-\bar{b}}^0 \delta z''^T EI\,z''\,d\bar{y}$$

NOTE: $a' = \partial a / \partial \bar{y}$

$$\delta W_e = \int_0^b \delta z_{AC}^T L^w\,dy + \int_0^b \delta\alpha^T M_{AC}^w\,dy + \int_{-b}^0 \delta z_{AC}^T L^w\,dy + \int_{-b}^0 \delta\alpha^T M_{AC}^w\,dy$$

BUT $z_{AC} = z + e\theta = z + e\left(\bar{\theta}\cos\Lambda - z'\sin\Lambda\right)$ and $\alpha = \alpha_0 + \alpha_T + \alpha_B = \alpha_0 + \bar{\theta}\cos\Lambda - \bar{z}'\sin\Lambda$

Also applying Ritz-Galerkin approximation: $\bar{\theta}(\bar{y}) = N_{\bar{\theta}}(\bar{y})\cdot q_{\bar{\theta}}$ and $z(\bar{y}) = N_z(\bar{y})\cdot q_z$

NOTE 1: We neglect inertia forces

NOTE 2: The Λ angle of the two semiwings is different w.r.t. x-AXIS $\Rightarrow$ $\begin{cases} \text{semiwing } 1: \ \Lambda_1 = 330° = -30° \\ \text{semiwing } 2: \ \Lambda_2 = -\Lambda_1 \end{cases}$

This choice has been explained in APPENDIX 1

$$\delta W_i = \delta q_{\bar{\theta}}^T \int_0^{\bar{b}} N_{\bar{\theta}}'^T GJ\,N_{\bar{\theta}}'\,d\bar{y}\;q_{\bar{\theta}} + \delta q_z^T \int_0^{\bar{b}} N_z''^T EI\,N_z''\,d\bar{y}\;q_z + \delta q_{\bar{\theta}}^T \int_{-\bar{b}}^0 N_{\bar{\theta}}'^T GJ\,N_{\bar{\theta}}'\,d\bar{y}\;q_{\bar{\theta}} + \delta q_z^T \int_{-\bar{b}}^0 N_z''^T EI\,N_z''\,d\bar{y}\;q_z$$

$$\delta W_e = \int_0^b \left(\delta q_z^T N_z^T + e\cos\Lambda_1\left(\delta q_{\bar{\theta}}^T N_{\bar{\theta}}^T\right) - e\,N_z'^T\sin(\Lambda_1)\,\delta q_z^T\right) qc\,C_{L\alpha}\left(N_{\bar{\theta}} q_{\bar{\theta}}\cos\Lambda_1 - N_z'\sin(\Lambda_1)\,q_z\right)]\,dy +$$
$$+ \int_{-b}^0 \left(\delta q_z^T N_z^T + e\cos\Lambda_2\left(\delta q_{\bar{\theta}}^T N_{\bar{\theta}}^T\right) - e\,N_z'^T\sin(\Lambda_2)\,\delta q_z^T\right) qc\,C_{L\alpha}\left(N_{\bar{\theta}} q_{\bar{\theta}}\cos\Lambda_2 - N_z'\sin(\Lambda_2)\,q_z\right)]\,dy$$

SEMIWING 1

$$= \int_0^b \delta q_z^T N_z^T\,qc\,C_{L\alpha}\,N_{\bar{\theta}}\,q_{\bar{\theta}}\cos\Lambda_1\,dy - \int_0^b \delta q_z^T N_z^T\,qc\,C_{L\alpha}\,N_z'\,q_z\sin\Lambda_1\,dy +$$
$$+ \int_0^b e\,\delta q_{\bar{\theta}}^T N_{\bar{\theta}}^T\,qc\,C_{L\alpha}\,N_{\bar{\theta}}\,q_{\bar{\theta}}\cos^2\Lambda_1\,dy - \int_0^b e\,\delta q_{\bar{\theta}}^T N_{\bar{\theta}}^T\,qc\,C_{L\alpha}\,N_z'\,q_z\sin\Lambda_1\cos\Lambda_1\,dy +$$
$$- \int_0^b e\,\delta q_z^T N_z'^T qc\,C_{L\alpha}\,N_{\bar{\theta}}\,q_{\bar{\theta}}\cos\Lambda_1\sin\Lambda_1\,dy + \int_0^b e\,\delta q_z^T N_z'^T qc\,C_{L\alpha}\,q_z\,N_z'\sin^2\Lambda_1\,dy +$$

SEMIWING 2

$$+ \int_{-b}^0 \delta q_z^T N_z^T\,qc\,C_{L\alpha}\,N_{\bar{\theta}}\,q_{\bar{\theta}}\cos\Lambda_2\,dy - \int_{-b}^0 \delta q_z^T N_z^T\,qc\,C_{L\alpha}\,N_z'\,q_z\sin\Lambda_2\,dy +$$
$$+ \int_{-b}^0 e\,\delta q_{\bar{\theta}}^T N_{\bar{\theta}}^T\,qc\,C_{L\alpha}\,N_{\bar{\theta}}\,q_{\bar{\theta}}\cos^2\Lambda_2\,dy - \int_{-b}^0 e\,\delta q_{\bar{\theta}}^T N_{\bar{\theta}}^T\,qc\,C_{L\alpha}\,N_z'\,q_z\sin\Lambda_2\cos\Lambda\,dy +$$
$$- \int_{-b}^0 e\,\delta q_z^T N_z'^T qc\,C_{L\alpha}\,N_{\bar{\theta}}\,q_{\bar{\theta}}\cos\Lambda_2\sin\Lambda_2\,dy + \int_{-b}^0 e\,\delta q_z^T N_z'^T qc\,C_{L\alpha}\,q_z\,N_z'\sin^2\Lambda_2\,dy$$

Now let's rewrite the system in MATRIX FORM : $\longrightarrow$ (Virtual displacements = 1)

$$K^S_{\theta\theta} = \int_0^{\bar{b}} N_{\bar{\theta}}'^T GJ N_{\bar{\theta}}' d\bar{y} + \int_{-\bar{b}}^0 N_{\bar{\theta}}'^T GJ N_{\bar{\theta}}' d\bar{y} \qquad \Bigg\| \qquad K^S_{zz} = \int_0^{\bar{b}} N_{\bar{z}}''^T EI N_{\bar{z}}'' d\bar{y} + \int_{-\bar{b}}^0 N_{\bar{z}}''^T EI N_{\bar{z}}'' d\bar{y}$$

$$K^A_{\theta\theta} = \int_0^b e N_{\bar{\theta}}^T c C_{L\alpha} N_{\bar{\theta}} \cos^2\Lambda_1 dy + \int_{-b}^0 e N_{\bar{\theta}}^T c C_{L\alpha} N_{\bar{\theta}} \cos^2\Lambda_2 dy$$

$$K^A_{\theta z} = -\int_0^b e N_{\bar{\theta}}^T c C_{L\alpha} N_z' \sin\Lambda_1 \cos\Lambda_1 dy - \int_{-b}^0 e N_{\bar{\theta}}^T c C_{L\alpha} N_z' \sin\Lambda_2 \cos\Lambda_2 dy$$

$$K^A_{z\theta} = \int_0^b N_z^T c C_{L\alpha} N_{\bar{\theta}} \cos\Lambda_1 dy - \int_0^b e N_z'^T c C_{L\alpha} N_{\bar{\theta}} \cos\Lambda_1 \sin(\Lambda_1) dy +$$
$$+ \int_{-b}^0 N_z^T c C_{L\alpha} N_{\bar{\theta}} \cos\Lambda_2 dy - \int_{-b}^0 e N_z'^T c C_{L\alpha} N_{\bar{\theta}} \cos\Lambda_2 \sin(\Lambda_2) dy$$

$$K^A_{zz} = -\int_0^b N_z^T c C_{L\alpha} N_z' \sin\Lambda_1 dy + \int_0^b N_z'^T e c C_{L\alpha} N_z' \sin^2\Lambda_1 dy +$$
$$-\int_{-b}^0 N_z^T c C_{L\alpha} N_z' \sin\Lambda_2 dy + \int_{-b}^0 N_z'^T e c C_{L\alpha} N_z' \sin^2\Lambda_2 dy$$

$$\hookrightarrow \left[\underline{\underline{K}}^S - q\,\underline{\underline{K}}^A\right] \cdot \begin{Bmatrix} q_{\bar{\theta}} \\ q_z \end{Bmatrix} = \begin{Bmatrix} \varnothing \\ \varnothing \end{Bmatrix} \Longrightarrow \text{DIVERGENCE : } \det\left([K^S - qK^A]\right) = \varnothing$$

$\hookrightarrow q_D =$ minimum real eigenvalue

NOTE : In MATLAB every integral has been computed on structural axis by changing variable dy in $d\bar{y}$ $\Longrightarrow$ $dy = d\bar{y}\cos\Lambda$

(2A) Compute Roll rate at regime when flying at 85% q_D and $\beta = 1°$

NOTE 1: Fuselage Torsion and canard effect must be considered

We consider the aircraft in a Trim condition where every Terms in our equations is a perturbation applied on this eq. condition.

Since the Fuselage doesn't generate any aerodynamic external forces doesn't enter in $\underline{\underline{K}}^A$ and It's Torsion results decoupled from the rest of the system (WING + CANARD)

$\hookrightarrow$ We will consider 3 Dof : z (wing bending), θ (wing Twist) and p (roll rate)

The roll condition is coupled with aerodynamic forces $\Longrightarrow$ must be considered and insert in $\underline{\underline{K}}^A$

$\hookrightarrow$ This can be obtained in Two ways:

- considering $\delta\varphi$ (roll angle) as virtual displacemen in PVW
- considering the rigid roll equation in parallel with PVW (without $\delta\varphi$) $\longleftarrow$ In this paper will be applied this method

It's important To consider the variation of AoT on both wing due To vertical velocity caused by roll rate

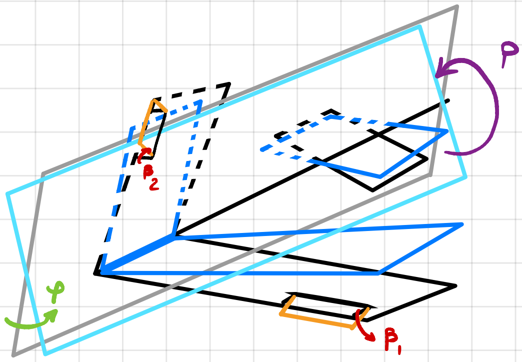
$\hookrightarrow$ on semiwing : $-\dfrac{py}{U_\infty} = -\dfrac{p\bar{y}}{U_\infty}\cos^2\Lambda$ $\longrightarrow$ NOTE : The sign of y for right and left semiwing considers the opposite movement that causes different $\delta\alpha$

In the rigid roll eq. is important to consider the damping in roll effect of the canard.

As for the sweep angle the β deflection is different too on the two semiwings:

↳ on semiwing 1: $\beta_1 = 1°$

and on semiwing 2: $\beta_2 = -1°$



Let's start by considering the PVW applied on the aircraft system (as continuum):

$$\delta W_i = \int_0^{\bar b} \delta\bar\theta'^T GJ\bar\theta'\, d\bar y + \int_0^{\bar b} \delta z''^T EI z''\, d\bar y + \int_{-\bar b}^{0} \delta\bar\theta'^T GJ\bar\theta'\, d\bar y + \int_{-\bar b}^{0} \delta z''^T EI z''\, d\bar y$$

$$\delta W_e = \int_0^b \delta z_{AC}^T L^w\, dy + \int_0^b \delta\alpha^T M_{AC}^w\, dy + \int_{-b}^0 \delta z_{AC}^T L^w\, dy + \int_{-b}^0 \delta\alpha^T M_{AC}^w\, dy$$

<u>NOTE:</u> $\bar\beta \cong \beta$

$$= \int_0^b \left(\delta z^T + e(\delta\bar\theta^T\cos\Lambda_1 - \delta z'\sin\Lambda_1)\right)\cdot\left[qc\, C_{L\alpha}\left(\bar\theta\cos\Lambda_1 - z'\sin(\Lambda_1) - \frac{yp}{U_\infty}\right)\right] dy \, +$$
$$+ \int_{b_a}^b \left(\delta\bar\theta^T\cos\Lambda_1 - \delta z'^T\sin\Lambda_1\right)\left[qc^2 C_{m\beta}\,\beta_1\right] dy + \int_{b_a}^b \left(\delta z^T + e(\delta\theta^T\cos\Lambda_1 - \delta z'^T\sin\Lambda_1)\right) qc\, C_{L\beta}\,\beta_1\, dy$$
$$+ \int_{-b}^0 \left(\delta z^T + e(\delta\bar\theta^T\cos\Lambda_2 - \delta z'\sin\Lambda_2)\right)\cdot\left[qc\, C_{L\alpha}\left(\bar\theta\cos\Lambda_2 - z'\sin(\Lambda_2) - \frac{yp}{U_\infty}\right)\right] dy \, +$$
$$+ \int_{-b}^{-b_a} \left(\delta\bar\theta^T\cos\Lambda_2 - \delta z'^T\sin\Lambda_2\right)\left[qc^2 C_{m\beta}\,\beta_2\right] dy + \int_{-b}^{-b_a} \left(\delta z^T + e(\delta\theta^T\cos\Lambda_2 - \delta z'^T\sin\Lambda_2)\right) qc\, C_{L\beta}\,\beta_2\, dy$$

Also applying Ritz-Galerkin approximation: $\bar\theta(\bar y) = N_\theta(\bar y)\cdot q_{\bar\theta}$ and $z(\bar y) = N_z(\bar y)\cdot q_z$

$$\delta W_e = \int_0^b \left(\delta q_z^T N_z^T + e(\delta q_{\bar\theta}^T N_{\bar\theta}^T\cos\Lambda_1 - \delta q_z^T N_z'^T\sin\Lambda_1)\right)\cdot\left[qc\, C_{L\alpha}\left(N_{\bar\theta} q_{\bar\theta}\cos\Lambda_1 - N_z'\sin(\Lambda_1)\, q_z - \frac{yp}{U_\infty}\right)\right] dy$$
$$+ \int_{b_a}^b \left(\delta q_\theta^T N_\theta^T\cos\Lambda_1 - \delta q_z^T N_z'^T\sin\Lambda_1\right)\left[qc^2 C_{m\beta}\,\beta_1\right] dy + \int_{b_a}^b \left(\delta q_z^T N_z^T + e(\delta q_{\bar\theta}^T N_{\bar\theta}^T\cos\Lambda_1 - \delta q_z^T N_z'^T\sin\Lambda_1)\right) qc\, C_{L\beta}\,\beta_1\, dy$$
$$+ \int_{-b}^0 \left(\delta q_z^T N_z^T + e(\delta q_{\bar\theta}^T N_{\bar\theta}^T\cos\Lambda_2 - \delta q_z^T N_z'^T\sin\Lambda_2)\right)\cdot\left[qc\, C_{L\alpha}\left(N_{\bar\theta} q_{\bar\theta}\cos\Lambda_2 - N_z'\sin(\Lambda_2)\, q_z - \frac{yp}{U_\infty}\right)\right] dy$$
$$+ \int_{-b}^{-b_a} \left(\delta q_\theta^T N_\theta^T\cos\Lambda_2 - \delta q_z^T N_z'^T\sin\Lambda_2\right)\left[qc^2 C_{m\beta}\,\beta_2\right] dy + \int_{-b}^{-b_a} \left(\delta q_z^T N_z^T + e(\delta q_{\bar\theta}^T N_{\bar\theta}^T\cos\Lambda_2 - \delta q_z^T N_z'^T\sin\Lambda_2)\right) qc\, C_{L\beta}\,\beta_2\, dy$$

Let's also consider the rigid roll equation

$$0 = \int_0^b qc\left[C_{L\alpha}\left(\bar\theta\cos\Lambda_1 - \bar z'\sin\Lambda_1 - \frac{py}{U_\infty}\right)\cdot y\right] dy + \int_0^{b_c} qc_c\, C_{L\alpha}\left(-\frac{py}{U_\infty}\right)\cdot y\, dy + \int_{b_a}^b qc\, C_{L\beta}\,\beta_1\cdot y\, dy \, +$$
$$\int_{-b}^0 qc\left[C_{L\alpha}\left(\bar\theta\cos\Lambda_2 - \bar z'\sin\Lambda_2 - \frac{py}{U_\infty}\right)\cdot y\right] dy + \int_{-b_c}^0 qc_c\, C_{L\alpha}\left(-\frac{py}{U_\infty}\right)\cdot y\, dy + \int_{-b}^{-b_a} qc\, C_{L\beta}\,\beta_2\cdot y\, dy$$

Now let's explicit all terms of δW_e and compute the solutions of the linear system:

$$\delta W_{e_1} = \int_0^b \delta q_z^T N_z^T \, qc \, C_{L\alpha} \, N_{\bar\theta} \, q_{\bar\theta} \cos\Lambda_1 \, dy - \int_0^b \delta q_z^T N_z^T qc \, C_{L\alpha} \, N_z' q_z \sin\Lambda_1 \, dy \, +$$
$$+ \int_0^b e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T \, qc \, C_{L\alpha} \, N_{\bar\theta} \, q_{\bar\theta} \cos^2\Lambda_1 \, dy - \int_0^b e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T qc \, C_{L\alpha} \, N_z' q_z \sin\Lambda_1 \cos\Lambda_1 \, dy \, +$$
$$- \int_0^b e \, \delta q_z^T N_z'^T qc \, C_{L\alpha} N_{\bar\theta} q_{\bar\theta} \sin\Lambda_1 \cos\Lambda_1 dy + \int_0^b e \, \delta q_z^T N_z'^T qc \, C_{L\alpha} \, \delta q_z N_z' \sin^2\Lambda_1 dy$$
$$- \int_0^b \delta q_z^T N_z^T qc \, C_{L\alpha} \frac{yP}{U_\infty} dy - \int_0^b e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T c q \, C_{L\alpha} \frac{yP}{U_\infty} \cos\Lambda_1 dy + \int_0^b e \delta q_z N_z'^T qc C_{L\alpha} \frac{yP}{U_\infty} \sin\Lambda_1 dy$$
$$+ \int_{b_a}^b \delta q_z^T N_z^T qc \, C_{L\beta} \, \beta_1 \, dy + \int_{b_a}^b e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T cq \, C_{L\beta} \, \beta \cos\Lambda_1 dy - \int_{b_a}^b e \, \delta q_z^T N_z'^T qc \, C_{L\beta} \, \beta_1 \sin\Lambda_1 dy$$
$$+ \int_{b_a}^b \delta q_{\bar\theta}^T N_{\bar\theta}^T \cos\Lambda_1 \, q \left(c^2 C_{m\beta} \, \beta_1\right) dy - \int_{b_a}^b \delta q_z^T N_z'^T \sin\Lambda_1 \, q \left(c^2 C_{m\beta} \, \beta_1\right) d\bar y$$

$$\delta W_{e_2} = \int_{-b}^0 \delta q_z^T N_z^T \, qc \, C_{L\alpha} \, N_{\bar\theta} \, q_{\bar\theta} \cos\Lambda_2 \, dy - \int_{-b}^0 \delta q_z^T N_z^T qc \, C_{L\alpha} \, N_z' q_z \sin\Lambda_2 \, dy \, +$$
$$+ \int_{-b}^0 e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T \, qc \, C_{L\alpha} \, N_{\bar\theta} \, q_{\bar\theta} \cos^2\Lambda_2 \, dy - \int_{-b}^0 e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T qc \, C_{L\alpha} \, N_z' q_z \sin\Lambda_2 \cos\Lambda_2 \, dy \, +$$
$$- \int_{-b}^0 e \, \delta q_z^T N_z'^T qc \, C_{L\alpha} N_{\bar\theta} q_{\bar\theta} \sin\Lambda_2 \cos\Lambda_2 dy + \int_{-b}^0 e \, \delta q_z^T N_z'^T qc \, C_{L\alpha} \, \delta q_z N_z' \sin^2\Lambda_2 dy$$
$$- \int_{-b}^0 \delta q_z^T N_z^T qc \, C_{L\alpha} \frac{yP}{U_\infty} dy - \int_{-b}^0 e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T c q \, C_{L\alpha} \frac{yP}{U_\infty} \cos\Lambda_2 dy + \int_{-b}^0 e \delta q_z N_z'^T qc C_{L\alpha} \frac{yP}{U_\infty} \sin\Lambda_2 dy$$
$$+ \int_{-b}^{-b_a} \delta q_z^T N_z^T qc \, C_{L\beta} \, \beta_2 \, dy + \int_{-b}^{-b_a} e \, \delta q_{\bar\theta}^T N_{\bar\theta}^T cq \, C_{L\beta} \, \beta_2 \cos\Lambda_2 dy - \int_{-b}^{-b_a} e \, \delta q_z^T N_z'^T qc \, C_{L\beta} \, \beta_2 \sin\Lambda_2 dy$$
$$+ \int_{-b}^{-b_a} \delta q_{\bar\theta}^T N_{\bar\theta}^T \cos\Lambda_2 \, q \left(c^2 C_{m\beta} \, \beta_2\right) dy - \int_{-b}^{-b_a} \delta q_z^T N_z'^T \sin\Lambda_2 \, q \left(c^2 C_{m\beta} \, \beta_2\right) d\bar y$$

$\Longrightarrow$ $\delta W_e = \delta W_{e_1} + \delta W_{e_2}$ $\rightsquigarrow$ Now let's build up the matrixes of the system:

$$\begin{bmatrix} K^S_{\theta\theta} & \varnothing & \varnothing \\ \varnothing & K^S_{zz} & \varnothing \\ \varnothing & \varnothing & \varnothing \end{bmatrix} - \begin{bmatrix} K^A_{\theta\theta} & K^A_{\theta z} & K^A_{\theta P} \\ K^A_{z\theta} & K^A_{zz} & K^A_{zP} \\ K^A_{P\theta} & K^A_{Pz} & K^A_{PP} \end{bmatrix} = \begin{Bmatrix} f_\theta \\ f_z \\ f_P \end{Bmatrix} \Longrightarrow$$

We can solve using "\" MATLAB command

$\Longrightarrow$ The last element of the solution array is going to be the rollrate

2B Compare the elastic rollrate with the one of the equivalent RIGID AIRCRAFT

To compute the rigid roll rate we must use only the rigid roll equation without consider $\bar\theta$ and z displacement:

$$0 = \int_0^b qc\left[C_{L\alpha}\left(\cancel{\bar\theta \cos\Lambda_1} - \cancel{z' \sin\Lambda_1} - \frac{PY}{U_\infty}\right) \cdot Y\right] dy + \int_0^{b_c} qc_c \, C_{L\alpha}\left(-\frac{PY}{U_\infty}\right) \cdot y \, dy + \int_{b_a}^b qc \, C_{L\beta} \, \beta_1 \cdot y \, dy \, +$$
$$+ \int_{-b}^0 qc\left[C_{L\alpha}\left(\cancel{\bar\theta \cos\Lambda_2} - \cancel{z' \sin\Lambda_2} - \frac{PY}{U_\infty}\right) \cdot Y\right] dy + \int_{-b_c}^0 qc_c \, C_{L\alpha}\left(-\frac{PY}{U_\infty}\right) \cdot y \, dy + \int_{-b}^{-b_a} qc \, C_{L\beta} \, \beta_2 \cdot y \, dy$$

$\hookrightarrow$ $[K^A_{PP}]\{P\} = \{f_P\}$ $\rightsquigarrow$ We solve the system using "\" MATLAB command

(3) Plot in the condition "2" the diagram of the Torsional moment as a function of x applied to the fuselage

Computation of TORSIONAL MOMENT acting on the fuselage:

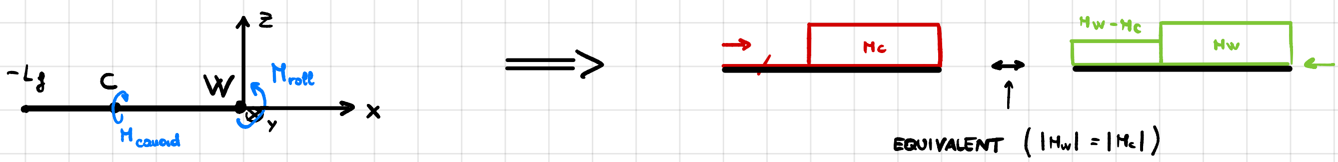
- WING MOMENT $= \int_0^b qcC_{L\alpha}\left(\bar{\theta}\cos\Lambda_1 - z'\sin\Lambda_1 - \frac{pY}{U_\infty}\right)\cdot Y\,dy + \int_{b_a}^{b} qcC_{L\beta}\,\beta_1 Y\,dy \; +$

$$+ \int_{-b}^{0} qcC_{L\alpha}\left(\bar{\theta}\cos\Lambda_2 - z'\sin\Lambda_2 - \frac{pY}{U_\infty}\right)\cdot Y\,dy + \int_{-b}^{-b_a} qcC_{L\beta}\,\beta_2 Y\,dy$$

- CANARD MOMENT $= \int_0^{b_c} qcC_{L\alpha}\left(-\frac{pY}{U_\infty}\right)Y\,dy - \int_{-b_c}^{0} qcC_{L\alpha}\frac{pY}{U_\infty}Y\,dy$

Since the fuselage must be in eq we will have two moment equal and opposite

↳ By analyzing the fuselage as a beam with two concentrate moment at wingroot and canard's root we can write the internal force diagram:



(4) Compute the bending moment generated by the aileron rotation applied at the wing root:

If we consider the damping in roll effect the moment will be:

WING MOMENT $= \int_0^b qcC_{L\alpha}\left(\bar{\theta}\cos\Lambda_1 - z'\sin\Lambda_1 - \frac{pY}{U_\infty}\right)\cdot Y\,dy + \int_{b_a}^{b} qcC_{L\beta}\,\beta_1 Y\,dy \; +$

$$+ \int_{-b}^{0} qcC_{L\alpha}\left(\bar{\theta}\cos\Lambda_2 - z'\sin\Lambda_2 - \frac{pY}{U_\infty}\right)\cdot Y\,dy + \int_{-b}^{-b_a} qcC_{L\beta}\,\beta_2 Y\,dy$$

If we consider only the moment generated by the movable surfaces

ALEIRON MOMENT $= \int_{b_a}^{b} qcC_{L\beta}\,\beta_1 Y\,dy + \int_{-b}^{-b_a} qcC_{L\beta}\,\beta_2 Y\,dy$

(5) Compute the value C_w of the structural matrix that makes the divergence speed of the forward swept wing higher than the equivalent straight wing one

First of all we must find the divergence pressure for the equivalent straight wing

↳ This can be done by repeating point 1 calculation with: $\begin{cases} \Lambda_1 = \emptyset \\ \Lambda_2 = \emptyset \end{cases}$

Once we have the $q_{D\_straight}$ we can modify the system written at point one

↳ We must add extradiagonal Terms at $\underline{\underline{K}}^S$ with C_w as a paremetre :

$$K^S_{\theta\theta} = \int_0^{\bar{b}} N_{\bar{\theta}}^{'T} GJ N_{\bar{\theta}}' d\bar{y} + \int_{-\bar{b}}^{0} N_{\bar{\theta}}^{'T} GJ N_{\bar{\theta}}' d\bar{y} \qquad K^S_{zz} = \int_0^{\bar{b}} N_{\bar{z}}^{''T} EI N_{\bar{z}}'' d\bar{y} + \int_{-\bar{b}}^{0} N_{\bar{z}}^{''T} EI N_{\bar{z}}'' d\bar{y}$$

$$K^S_{\theta z} = \int_0^{\bar{b}} N_{\bar{\theta}}^{'T} C_w N_z'' d\bar{y} + \int_{-\bar{b}}^{0} N_{\bar{\theta}}^{'T} C_w N_z'' d\bar{y} \qquad K^S_{z\theta} = \int_0^{\bar{b}} N_{\bar{z}}^{''T} C_w N_{\bar{\theta}}' d\bar{y} + \int_{-\bar{b}}^{0} N_{\bar{z}}^{''T} C_w N_{\bar{\theta}}' d\bar{y}$$

... While $\underline{\underline{K}}^A$ stays the same

AT this point we can iterate the computation of q_D for different values of C_w while we find that $q_D(C_w) > q_{D\_straight}$

SHAPE FUNCTION DECISION

One last important aspect to analyze is the build of shape functions for $\bar{\theta}$ (wing twist angle) and $\bar{z}$ (wing bending as displacement on z axis) variables

Since the aircraft is free to fly we don't have cinematic boundary conditions

↳ Anyway since the wing has been seen as a continuos EULER BERNOULLI beam we must consider the presence of the fuselage at $y = \varnothing$

$\Rightarrow$ Some compatibility conditions in $Y = \varnothing$ must be considered : $\begin{cases} \bar{\theta}(\varnothing) = \varnothing \\ \bar{z}(\varnothing) = \varnothing \\ \bar{z}'(\varnothing) = \varnothing \end{cases}$

In addition we have some naturel condition that may be usefull To increase convergence

THETA SHAPE FUNCTION

The shapes functions chosen are both 'sin' and 'cos'
In order To guarantee compatibility we pick for 'cos' shapes : $\cos(\omega_c \bar{y}) - 1$

The frequency of the shapes functions has been found with application of The only naturel condition for theta : $GJ\,\theta'(-\bar{b}) = GJ\,\theta'(\bar{b}) = \varnothing$

- $\cos(\omega_c \bar{y}) - 1 \xrightarrow{d\bar{y}} -\omega_c \sin(\omega_c \bar{y}) = \varnothing \Rightarrow \omega_c = \varnothing + m\frac{\pi}{\bar{b}}, \quad m \in \mathbb{N}$
- $\sin(\omega_s \bar{y}) \xrightarrow{d\bar{y}} \omega_s \cos(\omega_s \bar{y}) = \varnothing \Rightarrow \omega_s = \frac{\pi}{2\bar{b}} + m\frac{\pi}{\bar{b}}, \quad m \in \mathbb{N}$

So in The end θ shapes function vector will be :

$$N_{\bar{\theta}} = \left[\cos\left(m_c \frac{\pi}{\bar{b}} \bar{y}\right) - 1 ,\ \sin\left(\left(\frac{\pi}{2\bar{b}} + m_s \frac{\pi}{\bar{b}}\right)\bar{y}\right) \right], \quad \text{with : } \begin{array}{l} m_c = 1 : \#Shape\_Theta \in \mathbb{N} \\ m_s = 0 : (\#Shape\_Theta - 1) \in \mathbb{N} \end{array}$$

z SHAPE FUNCTION

The shapes functions chosen are both 'sin' and 'cos'
In order to guarantee compatibility we pick for 'cos' shapes: $\cos(\omega_c \bar{y}) - 1$ and for 'sin' shapes: $\sin(\omega_s \bar{y}) - \bar{y}\omega_s$

The frequency of the shapes functions has been found with application of one of the two natural conditions: $EIz''(-\bar{b}) = EIz''(\bar{b}) = \emptyset$ and $EIz'''(-\bar{b}) = EIz'''(\bar{b}) = \emptyset$

For semplicity we apply the first of the two:

- $\cos(\omega_c \bar{y}) - 1 \xrightarrow{d\bar{y}^2} -\omega_c^2 \cos(\omega_c \bar{y}) = \emptyset \implies \omega_c = \frac{\pi}{2b} + m\frac{\pi}{b}, \quad m \in \mathbb{N}$
- $\sin(\omega_s \bar{y}) - \omega_s \bar{y} \xrightarrow{d\bar{y}^2} -\omega_s^2 \sin(\omega_s \bar{y}) = \emptyset \implies \omega_s = \emptyset + m\frac{\pi}{b}, \quad m \in \mathbb{N}$

So in the end z shapes function vector will be:

$$N_z = \left[\cos\left(\left(\frac{\pi}{2b} + m\frac{\pi}{b}\right)\bar{y}\right) - 1, \ \sin\left(m\frac{\pi}{b}\bar{y}\right) - \left(m\frac{\pi}{b}\bar{y}\right) \right] \text{ with: } \begin{array}{l} m_s = 1 : \#\text{Shape\_Theta} \in \mathbb{N} \\ m_c = 0 : (\#\text{Shape\_Theta} - 1) \in \mathbb{N} \end{array}$$

NOTE: Both groups of shapes function will be as "big" necessary to reach convergence of the results

↳ This has been done throught an iterative process

LIST of RESULTS

(Computed using 8 shapes, 4 cos and 4 sin)

1. $q_D = 8319.857578$ Pa $\quad U_D = 116.548020$ m/s

   NOTE: U_D computed at $\rho = 1.225$ kg/m³

2. $P_{rigid} = 0.207394$ rad/s $\quad P_{elastic} = 0.179724$ rad/s

3. $M_{canard\,Root} = -M_{wing\,Root} = -535.950367$ Nm

4. $M_{Aileron} = 40200.223886$ Nm

5. $U_{d\,straight} = 238.848534$ m/s $\quad C_w = 29058500.00$ Nm²

   NOTE: $U_{D\,straight}$ computed at $\rho = 1.225$ kg/m³

NOTE:

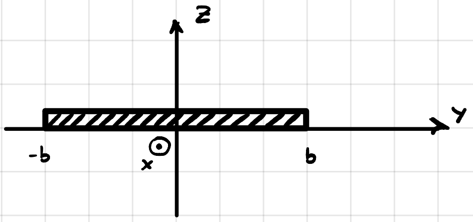
The script has been given with shape functions number set to: num_Ntheta = num_Nz = 2.
This guarantee a good precision and reasonable computation time.

In order to get better results (same as reported on this paper) num_Ntheta and num_Nz must be set to 4.

All the code has been written and run on MATLAB R2023a with SYMBOLIC MATH TOOLBOX (ver 9.3) and CONTROL SYSTEM TOOLBOX (ver 10.13)

APPENDIX 1

The beam considered To model the wing follows EULER-BERNOULLI MODEL



Since The sweep angle is differen from zero is particular important To consider The bending and Twist displacement of the structure.

These displacements must be Taken into account To write δz_{AC} and $\delta\alpha$.

In the next figure is shown how have been retrived $\delta\alpha$ and $\delta\theta$ (which is usefull To compute δz_{AC})

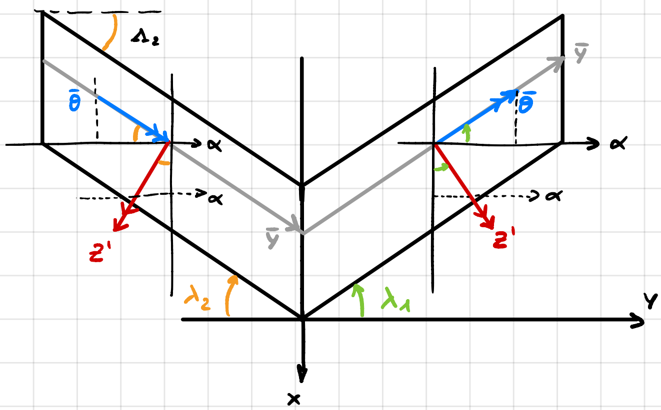
$\theta_2 = \bar{\theta}\cos\lambda_2 - z'\sin\lambda_2$

but $\lambda_2 = 30°$ for construction

$\Rightarrow$ Rewriting in function of Λ_2

$\theta_2 = \bar{\theta}\cos\Lambda_2 - z'\sin\Lambda_2$

$\hookrightarrow \alpha = \theta_2 - \frac{PY}{U_\infty}$



$\theta_1 = \bar{\theta}\cos\lambda_1 + z'\sin\lambda_1$

but $\lambda_1 = 30°$ for construction

$\Rightarrow$ Rewriting in function of Λ_1

$\theta_1 = \bar{\theta}\cos\Lambda_1 - z'\sin\Lambda_1$

$\hookrightarrow \alpha = \theta_1 - \frac{PY}{U_\infty}$

NOTE 1: z' is flipped between The Two semiwings because is obtained from $\partial z/\partial \bar{y}$ and since $\bar{y}$ changes $\Rightarrow$ z' changes Too

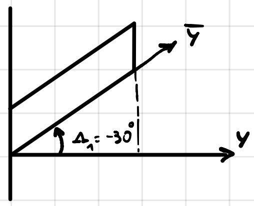
$\hookrightarrow$ In order To Keep always The same sign convention the Term in z' for The left semwing must have The opposite sign of the right one.

NOTE 2: The Term "$-PY/U_\infty$" which considers The variation of α due To a different wind velocity direction induced by the roll movement mustn't be change

$\hookrightarrow$ The sign of 'Y' infact guarantee the consistency of the variation for both semiwings

Every integral of aerodynamic stiffness has been written initially in aerodynamic system. Then, through a change of variable, has been computed on The local structural system

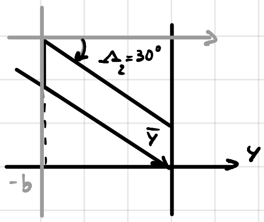
This has be done in a way that guarantee same sign convention for both semiwings in aerodynamic frame



$\bar{y}\cos(\Lambda_1) = y$

$\Rightarrow$ INTEGRATION:

$\int_0^b (\dots)\,dy = \int_0^{\bar{b}} (\dots)\cos(\Lambda_1)\,d\bar{y}$



$\bar{y}\cos(\Lambda_2) = y$

$\Rightarrow$ INTEGRATION:

$\int_{-b}^0 (\dots)\,dy = \int_{-\bar{b}}^0 (\dots)\cos(\Lambda_2)\,d\bar{y}$