Assignment # 2: Dynamic Aeroelasticity
Class: Structural Dynamics and Aeroelasticity,
Prof. Giuseppe Quaranta
A.Y. 2022/23

May 24, 2023

Take the last four figures of your person code ABCD, and assemble these two numbers: DA and CB[1]

$$\begin{cases} 00 \leq DA < 50 & x_1 = 1.2 \\ 50 \leq DA < 99 & x_1 = 0.7 \\ 00 \leq CB < 50 & x_2 = 1.1 \\ 50 \leq CB < 99 & x_2 = 0.6 \end{cases} \tag{1}$$

Consider the aircraft of Assignment #1. Consider the movable surface of the wing blocked. Consider the position of the centre of gravity of the wing at $x_{CG}(\bar{y}) = 0.42 \times x_2 \frac{1}{\bar{c}}$. The following inertia properties should be used

- Mass per unit span $m_w(\bar{y}) = 400$ kg/m
- Moment of Inertia per unit span $I_{\theta w}(\bar{y}) = 320$ kg m^2

For the fuselage consider the following

- Mass per unit span $m_w(\bar{y}) = x_1 \times 500$ kg/m
- Moment of Inertia per unit span $I_{\theta f}(x) = 3000$ kg m^2

For the Canard use a total lumped mass of 150 kg for each semi-surface (300 kg total) applied at the geometric centre of the surface and neglect the CG moments of inertia of each semi-surface (consider only the transport term for the semi-surface CG to the total canard CG).

1. Compute the first 10 modes of the aircraft in-vacuo (i.e. no aerodynamic forces), including the three rigid modes of plunge, pitch and roll of the aircraft.

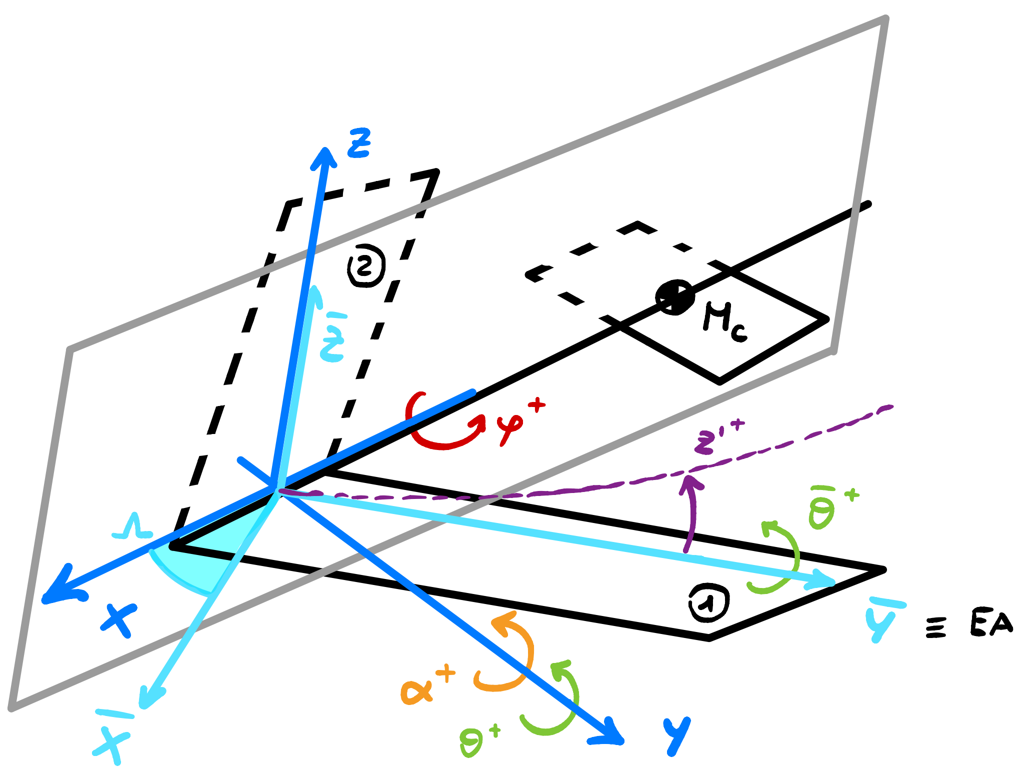
[1] Example: 10134997 $\rightarrow A = 4$, $B = 9$, $C = 9$, $D = 7$, so the two numbers will be $DA = 74$, $CB = 99$.

2. To approximate the aerodynamic behaviour use a quasi-steady strip theory approximation. Compute the flutter speed of the aircraft considering the presence of a 3% of proportional structural damping.

3. Compute the frequency response of the wing torsional moment (in terms of amplitude and phase) obtained by the rotation of the canard surface at $f = x_1 \times 15.0$ Hz.

The methods necessary to solve all steps must be written by hand on paper and a copy must be submitted together with the numeric solutions. All integrals could be solved using the symbolic integrator of MATLAB (or any other symbolic solver). The eigenvalues can be computed using the MATLAB `eig` routine (or any other numerical tool). All routines used to compute the solution must be provided too so that they can be run to verify how they work.

Clarifications on the text may be asked until May 27, 2023. To submit the responses and photos of the written solution, you must connect to a MS Form page. The answers will have to be submitted after 10 days starting from May 27, 2023 (so by 1:00 pm of June 07, 2023).



Matteo Baio, 10667431, mat. : 232805

Person code : 10667431 $\longrightarrow$ $x_1 = 1.2$ e $x_2 = 1.1$

Consider all data of ASSIGNMEN 1 and add:

Wing CG position : $X_{CG}(\bar{y}) = 0.42 \cdot {}^{x_2}/_{\bar{c}}$

Wing mass per unit span : $m_w(\bar{y}) = 400\ \mathrm{Kg/m}$

Wing moment of inertia per unit span : $I_{\theta w}(\bar{y}) = 320\ \mathrm{Kg\,m^2}$

Canard lumped mass : $M_c = 300\ \mathrm{Kg}$

Canard rotation : $f = x_1 \cdot 15\ \mathrm{Hz}$

Fuselage mass per unit span : $m_f(x) = x_1 \cdot 500\ \mathrm{Kg/m}$

Fuselage moment of inertia per unit span : $I_{\theta f}(x) = 3000\ \mathrm{Kg\,m^2}$

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1 Compute the first 10 modes of the system (including RIGID MODES)

In order to model we follow the same assumptions of ASSIGN. 1 with the differences that now INERTIA FORCES are <u>NOT</u> neglected and for this first task $q = \varnothing$

Let's start by writing PLV:

$$\delta W_i = \int_0^{b} \delta \bar{z}^{T''} EI\, \bar{z}''\, d\bar{y} + \int_{-b}^{0} \delta \bar{z}^{T''} EI\, \bar{z}''\, d\bar{y} + \int_0^{b} \delta \bar{\theta}^{T'} GJ\, \bar{\theta}'\, d\bar{y} + \int_{-b}^{0} \delta \bar{\theta}^{T'} GJ\, \bar{\theta}'\, d\bar{y} + \int_{-L_f}^{0} \delta \varphi_F^{T'} GJ\, \varphi_F'\, dx$$

$$\delta W_e = \int_0^{b} \delta \bar{z}_{CG}^{T}\, m_{\bar{w}}\, \ddot{\bar{z}}_{CG}\, d\bar{y} + \int_{-b}^{0} \delta \bar{z}_{CG}^{T}\, m_{\bar{w}}\, \ddot{\bar{z}}_{CG}\, d\bar{y} + \int_0^{b} \delta \bar{\theta}_w^{T}\, I_{\bar{\theta} w}\, \ddot{\bar{\theta}}_w\, d\bar{y} + \int_{-b}^{0} \delta \bar{\theta}_w^{T}\, I_{\bar{\theta} w}\, \ddot{\bar{\theta}}_w\, d\bar{y} + \quad \longleftarrow \text{Wing INERTIA}$$

$$+ \int_{-L_f}^{0} \delta \varphi_F^{T}\, I_{\theta F}\, \ddot{\varphi}_F\, dx + \int_{-L_f}^{0} \delta z_F^{T}\, m_F\, \ddot{z}_F\, dx + M_c\, \delta z_c^{T}\, \ddot{z}_c \quad \longleftarrow \text{Fuselage + Canard INERTIA}$$

All these variables can be expressed using KINEMATIC EQUIVALENCES as function of the three DoF of the system : $\bar{\theta}(\bar{y})$, $z(\bar{y})$, $\varphi(x)$

- WING :
  - $\bar{z}_{CG} = \bar{z} - |\bar{X}_{CG}|\,\bar{\theta}_w + \bar{\varphi}_w \bar{y}$ ← Plunge
  - $z_{AC} = z + e\theta_w + \varphi_w y$
  - $\theta_w = \bar{\theta}_w \cos\Lambda - z' \sin\Lambda$ ← Pitch
  - $\varphi_w = \varphi_F(x = \varnothing) + z'(y)$ ← Roll
  - $\alpha = \theta_w - \dot{z}_{AC}/U_\infty$

  <u>NOTE</u>: $|X_{CG}| = X_{CG}$ and $z'(\varnothing) = \varnothing$

- FUSELAGE :
  - $\theta_F = \theta_w(\varnothing)$ ← Pitch
  - φ_F ← Roll
  - $z_F = \theta_w(\varnothing)\,|x| + z(\varnothing)$ ← Plunge

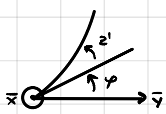
- CANARD :
  - $z_{CG}^{c} = |X_c|\,\theta_w(\varnothing) + z(\varnothing)$ ← Plunge
  - $z_{AC}^{c} = |X_c|\,\theta_w(\varnothing) + z(\varnothing) + \varphi(x_c)\, y$

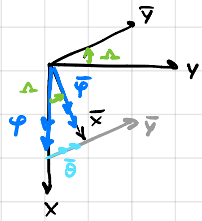
z' displacement relative to φ

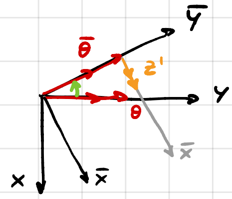
φ displacement of $\bar{y}$

$\theta_w = \bar{\theta}_w \cos\Lambda - z' \sin\Lambda$

$\varphi_w = \bar{\varphi}_w \cos\Lambda - \bar{\theta} \sin\Lambda = \varphi_F \big|_{x=\varnothing}$







Applying these equivalences the δW_e becomes:

$$= \int_0^{\bar b} \delta\left(z^T - \bar X_{CG}\bar\theta_w^T + \varphi_w^T/\cos\Lambda_1\, \bar y + \bar\theta^T \tan\Lambda_1\, \bar y + z'\bar y\right) m_{\bar w} \left(\ddot z - \bar X_{CG}\ddot{\bar\theta}_w + \ddot\varphi_w/\cos\Lambda_1\, \bar y + \ddot{\bar\theta}\tan\Lambda_1\,\bar y + z'\bar y\right) d\bar y$$

$$+ \int_0^{\bar b} \delta\bar\theta_w^T I_{\theta\bar w}\, \ddot{\bar\theta}_w\, d\bar y + \int_{-\bar b}^0 \delta\bar\theta_w^T I_{\theta\bar w}\,\ddot{\bar\theta}_w\, d\bar y + \int_{-\bar b}^0 \text{Wing } 2 + \int_{-L_f}^0 \delta\varphi_F^T I_{\theta F}\,\ddot\varphi_F\, dx$$

$$+ \int_{-L_f}^0 \delta\left(x\bar\theta_w\cos\Lambda\big|_{\bar y=0} - \cancel{x z'\sin\Lambda}\big|_{\bar y=0} + z\big|_{y=0}\right) m_F \left(x\ddot{\bar\theta}_w\cos\Lambda\big|_{\bar y=0} - \cancel{x\ddot z'\sin\Lambda}\big|_{\bar y=0} + \ddot z\big|_{y=0}\right) dx$$

$$+ \delta\left(x_c\bar\theta_w^T\cos\Lambda\big|_{\bar y=0} - \cancel{x_c z'\sin\Lambda}\big|_{\bar y=0} + z^T\big|_{y=0}\right) M_c \left(x_c\ddot{\bar\theta}_w\cos\Lambda\big|_{\bar y=0} - \cancel{x_c\ddot z'\sin\Lambda}\big|_{\bar y=0} + \ddot z\big|_{y=0}\right)$$

$$= \int_0^{\bar b} \delta z^T m_{\bar w}\left(z - \bar X_{CG}\bar\theta_w + \varphi_w/\cos\Lambda_1\,\bar y + \bar\theta\tan\Lambda_1\,\bar y + z'\bar y\right) d\bar y + \int_{-\bar b}^0 \text{wing } 2$$

$$- \int_0^{\bar b} \delta\bar\theta_w^T \bar X_{CG}\, m_{\bar w}\left(z - \bar X_{CG}\bar\theta_w + \varphi_w/\cos\Lambda_1\,\bar y + \bar\theta\tan\Lambda_1\,\bar y + z'\bar y\right) d\bar y + \int_{-\bar b}^0 \text{wing } 2$$

$$+ \int_0^{\bar b} \delta\varphi_{F(0)}^T/\cos\Lambda_1\,\bar y\, m_{\bar w}\left(z - \bar X_{CG}\bar\theta_w + \varphi_w/\cos\Lambda_1\,\bar y + \bar\theta\tan\Lambda_1\,\bar y + z'\bar y\right) d\bar y + \int_{-\bar b}^0 \text{wing } 2$$

$$+ \int_0^{\bar b} \delta\bar\theta^T \tan\Lambda_1\,\bar y\, m_{\bar w}\left(z - \bar X_{CG}\bar\theta_w + \varphi_w/\cos\Lambda_1\,\bar y + \bar\theta\tan\Lambda_1\,\bar y + z'\bar y\right) d\bar y + \int_{-\bar b}^0 \text{wing } 2$$

$$+ \int_0^{\bar b} \delta z'\bar y\, m_{\bar w}\left(z - \bar X_{CG}\bar\theta_w + \varphi_w/\cos\Lambda_1\,\bar y + \bar\theta\tan\Lambda_1\,\bar y + z'\bar y\right) d\bar y + \int_{-\bar b}^0 \text{wing } 2$$

$$+ \int_0^{\bar b} \delta\bar\theta_w^T I_{\theta\bar w}\,\ddot{\bar\theta}_w\, d\bar y + \int_{-\bar b}^0 \delta\bar\theta_w^T I_{\theta\bar w}\,\ddot{\bar\theta}_w\, d\bar y + \int_{-L_f}^0 \delta\varphi_f^T I_{\theta F}\,\ddot\varphi_f\, dx$$

$$+ \int_{-L_f}^0 \delta\bar\theta_w\, x\,(\bar y=0)\cos\Lambda\, m_F\left(x\ddot{\bar\theta}_w(\bar y=0)\cos\Lambda + \ddot z(\bar y=0)\right) dx + \delta x_c\bar\theta_w(\bar y=0)\cos\Lambda\, M_c\left(x_c\ddot{\bar\theta}_w(\bar y=0)\cos\Lambda + \ddot z(\bar y=0)\right)$$

$$+ \int_{-L_f}^0 \delta z(\bar y=0)\, m_F\left(x\ddot{\bar\theta}_w(\bar y=0)\cos\Lambda + \ddot z(\bar y=0)\right) dx + \delta z(\bar y=0)\, M_c\left(x_c\ddot{\bar\theta}_w(\bar y=0)\cos\Lambda + \ddot z(\bar y=0)\right)$$

From δW_i can be assembled the structural sys matrix:

$$K^s_{\bar\theta\bar\theta} = \int_0^{\bar b} \delta\bar\theta'^T GJ\,\bar\theta'\, d\bar y + \int_{-\bar b}^0 \delta\bar\theta^{T\prime} GJ\,\bar\theta'\, d\bar y$$

$$K^s_{zz} = \int_0^{\bar b} \delta\bar z''^T EI\,\bar z''\, d\bar y + \int_{-\bar b}^0 \delta\bar z''^T EI\,\bar z''\, d\bar y$$

$$K^s_{\varphi\varphi} = \int_{-L_f}^0 \delta\varphi_F'^T GJ\,\varphi_F'\, dx$$

$$\longrightarrow \quad \underline{\underline{K}} = \begin{bmatrix} K^s_{\bar\theta\bar\theta} & / & / \\ / & K^w_{zz} & / \\ / & / & K^s_{\varphi\varphi} \end{bmatrix}$$

From δW_e can be assembled the Inertia sys matrix:

$$M_{\theta\theta} = \int_0^{\bar b} \delta\bar\theta_w^T \bar X_{CG}^2\, m_w\,\ddot{\bar\theta}_w\, d\bar y - \int_0^{\bar b} 2\cdot\delta\bar\theta_w^T \bar X_{CG}\, m_w\,\ddot{\bar\theta}_w \tan\Lambda_1\,\bar y\, d\bar y + \int_0^{\bar b} \delta\bar\theta_w^T I_{\theta W}\,\ddot{\bar\theta}_w\, d\bar y \,+$$

$$+ \int_0^{\bar b} \delta\bar\theta_w^T \tan^2\Lambda_1\,\bar y^2 m_w\,\ddot{\bar\theta}_w\, d\bar y + \int \text{Wing } 2$$

$$+ \int_{-L_f}^0 \delta\bar\theta_w^T(0)\, x^2\cos^2\Lambda\, m_F\,\ddot{\bar\theta}(0)\, dx + \delta\bar\theta^T(\bar y=0)\, x_c^2\cos^2\Lambda\, M_c\,\ddot{\bar\theta}_w(y=0)$$

$$M_{\theta z} = -\int_0^{\bar{b}} \delta\bar{\theta}_w^T \bar{x}_{CG}\, m_{\bar{w}}\, \ddot{z}\, d\bar{y} + \int_0^{\bar{b}} \delta\bar{\theta}_w^T \tan\Lambda_1\, \bar{y}\, m_w\, \ddot{z}\, d\bar{y} + \int_0^{\bar{b}} \delta\bar{\theta}_w^T \tan\Lambda_1\, \bar{y}^2 m_w\, \ddot{z}' + \int \text{Wing } 2$$

$$- \int_0^{\bar{b}} \delta\bar{\theta}_w^T \bar{x}_{CG}\, m_{\bar{w}}\, \bar{y}\, \ddot{z}' + \int_{-L_F}^{0} \delta\bar{\theta}_w^T(\bar{y}=\emptyset)\, x \cos\Lambda\, m_F\, \ddot{z}(\bar{y}=\emptyset)\, dx + \delta\bar{\theta}_w^T(\bar{y}=0)\, x_c\, M_c \cos\Lambda\, \ddot{z}(\bar{y}=0)$$

$$M_{\theta\varphi} = -\int_0^{\bar{b}} \delta\bar{\theta}_w^T \bar{x}_{CG}\, m_w\, \bar{y}\, {}^1/_{\cos\Lambda_1}\, \ddot{\varphi}(x=0)\, d\bar{y} + \int_0^{\bar{b}} \delta\bar{\theta}_w^T \tan\Lambda_1\, \bar{y}\, m_w\, \bar{y}\, {}^1/_{\cos\Lambda_1}\, \ddot{\varphi}(x=0)\, d\bar{y} + \int \text{Wing } 2$$

$$M_{zz} = \int_0^{\bar{b}} \delta z^T m_w\, \ddot{z}\, d\bar{y} + \int_0^{\bar{b}} \delta z^T m_w\, \bar{y}\, \ddot{z}'\, d\bar{y} + \int_0^{\bar{b}} \delta z'^T \bar{y}\, m_w\, \ddot{z}\, d\bar{y} + \int_0^{\bar{b}} \delta z'^T \bar{y}^2 m_w\, \ddot{z}'\, d\bar{y}$$

$$+ \int_{-L_F}^{0} \delta z^T(\bar{y}=0)\, m_F\, \ddot{z}(\bar{y}=0)\, dx + \delta z^T(\bar{y}=0)\, M_c\, \ddot{z}(\bar{y}=\emptyset) + \int \text{Wing } 2$$

$$M_{z\theta} = M_{\theta z}^T$$

$$M_{z\varphi} = \int_0^{\bar{b}} \delta z^T m_w\, {}^1/_{\cos\Lambda}\, \bar{y}\, \ddot{\varphi}(0)\, d\bar{y} + \int_0^{\bar{b}} \delta z'^T m_w\, \bar{y}^2\, {}^1/_{\cos\Lambda}\, \ddot{\varphi}(0) + \int \text{Wing } 2$$

$$M_{\varphi\theta} = M_{\theta\varphi}' \qquad \Big\| \qquad M_{\varphi z} = M_{z\varphi}'$$

$$M_{\varphi\varphi} = \int_0^{\bar{b}} \delta\varphi^T(x=\emptyset)\, \bar{y}^2 m_w\, {}^1/_{\cos^2\Lambda_1}\, \ddot{\varphi}(x=0)\, d\bar{y} + \int_{-\ell_f}^{0} \delta\varphi^T I_{\varphi f}\, \ddot{\varphi}\, dx + \int \text{Wing } 2$$

All the system is written and computed on MATLAB using a Ritz - Galerkin apprach. The specific shape functions used will be specified later on this paper.

Since $\delta W_i - \delta W_e = \emptyset$

$$\hookrightarrow \underline{\underline{M}} = \begin{bmatrix} M_{\bar{\theta}\bar{\theta}} & M_{\bar{\theta}z} & M_{\theta z} \\ M_{z\bar{\theta}} & M_{zz} & M_{z\varphi} \\ M_{\varphi\bar{\theta}} & M_{\varphi z} & M_{\varphi\varphi} \end{bmatrix} \Longrightarrow \underline{\underline{M}} \cdot \begin{Bmatrix} \ddot{q}_{\bar{\theta}} \\ \ddot{q}_z \\ \ddot{q}_\varphi \end{Bmatrix} + \underline{\underline{K}}^s \cdot \begin{Bmatrix} q_{\bar{\theta}} \\ q_z \\ q_\varphi \end{Bmatrix} = \emptyset$$

$$\rightsquigarrow \underline{\underline{M}}\, \ddot{\underline{q}} + \underline{\underline{K}}\, \underline{q} = \emptyset \rightsquigarrow \det\left(\lambda_i^2 \underline{\underline{M}} + \underline{\underline{K}}\right) \underline{\phi}_i = \emptyset \longrightarrow$$ General eigenvalue problem

... Solved im MATLAB using function eig:

$$[\underline{\phi}, \underline{\lambda}] = \text{eig}(\underline{\underline{K}}, -\underline{\underline{M}}) \Longrightarrow w_i = \sqrt{\lambda_i}$$

$\hookrightarrow$ $\underline{\phi}$ must made orthogonal w.r.t. $\underline{\underline{M}}$ (and $\underline{\underline{K}}$) in order to decouple the sys. $\vdash\!\to$ Ortogonalization throught GRAM - SCHMIDT

② Compute flutter speed approximating aerodynamic behavior using a QS strip theory with 3% of structural damping

INERTIA and STRUCTURAL STIFFNESS matrix comes from point 1. Let's retrive AERODYNAMIC STIFFNESS and damping

$$\delta W_e^A = \int_0^b \delta z_{AC}^T L^w \, dy + \int_0^b \delta \alpha^T M_{AC}^w \, dy + \int_0^{b_c} \delta z_c^T L^c \, dy + \int_0^{b_c} \delta \alpha_c M_{AC}^c \, dy + \text{Wing } 2 + \text{Canard } 2$$

$$= \int_0^b \left(\delta z^T + e\left(\delta\bar\theta^T \cos\Lambda_1 - \delta z' \sin\Lambda_1\right) + \varphi_w y\right) \cdot \left[q c C_{L\alpha} \left(\bar\theta \cos\Lambda_1 - z' \sin(\Lambda_1) - \frac{\dot z}{U_\infty} - \frac{e \dot\theta \cos\Lambda_1}{U_\infty} + \frac{e \dot z' \sin\Lambda_1}{U_\infty} - \frac{\dot\varphi_w y}{U_\infty} \right) \right] dy \, +$$

$$+ \int_0^{b_c} \left(\delta\bar\theta_w^T\big|_0 \cos\Lambda_1 x_c - \delta z'\big|_0 \cancel{\sin\Lambda_1} x_c + z(0) + \varphi(x_c) y \right) \cdots$$

$$\cdots \left[q c_c C_{L\alpha_c} \left(\bar\theta(0) \cos\Lambda_1 - \dot z'(0) \sin\Lambda_1 - \frac{\dot{\bar\theta}(0) \cos\Lambda_1 x_c}{U_\infty} + \frac{\dot z'(0) \sin\Lambda_1 x_c}{U_\infty} - \frac{\dot z(0)}{U_\infty} - \frac{\dot\varphi(x_c) y}{U_\infty} \right) \right] dy$$

$$= \int_0^b \delta z^T \cdot \left[q c C_{L\alpha} \left(\bar\theta \cos\Lambda_1 - z' \sin(\Lambda_1) - \frac{\dot z}{U_\infty} - \frac{e \dot\theta \cos\Lambda_1}{U_\infty} + \frac{e \dot z' \sin\Lambda_1}{U_\infty} - \frac{\dot\varphi_w y}{U_\infty} \right) \right] dy + \text{Wing } 2$$

$$- \int_0^b \delta z'^T \sin\Lambda_1 \, e \left[q c C_{L\alpha} \left(\bar\theta \cos\Lambda_1 - z' \sin(\Lambda_1) - \frac{\dot z}{U_\infty} - \frac{e \dot\theta \cos\Lambda_1}{U_\infty} + \frac{e \dot z' \sin\Lambda_1}{U_\infty} - \frac{\dot\varphi_w y}{U_\infty} \right) \right] dy + \text{Wing } 2$$

$$+ \int_0^b \delta\bar\theta^T \cos\Lambda_1 \, e \left[q c C_{L\alpha} \left(\bar\theta \cos\Lambda_1 - z' \sin(\Lambda_1) - \frac{\dot z}{U_\infty} - \frac{e \dot\theta \cos\Lambda_1}{U_\infty} + \frac{e \dot z' \sin\Lambda_1}{U_\infty} - \frac{\dot\varphi_w y}{U_\infty} \right) \right] dy + \text{Wing } 2$$

$$+ \int_0^b \delta\varphi_w \, y \left[q c C_{L\alpha} \left(\bar\theta \cos\Lambda_1 - z' \sin(\Lambda_1) - \frac{\dot z}{U_\infty} - \frac{e \dot\theta \cos\Lambda_1}{U_\infty} + \frac{e \dot z' \sin\Lambda_1}{U_\infty} - \frac{\dot\varphi_w y}{U_\infty} \right) \right] dy + \text{Wing } 2$$

$$+ \int_0^{b_c} \delta\bar\theta_w^T(0) \cos\Lambda_1 x_c \cdot \left[q c_c C_{L\alpha_c} \left(\bar\theta_w(0) \cos\Lambda_1 - \cancel{z'(0) \sin\Lambda_1} - \frac{\dot{\bar\theta}(0) \cos\Lambda_1 x_c}{U_\infty} + \cancel{\frac{\dot z'(0) \sin\Lambda_1 x_c}{U_\infty}} - \frac{\dot z(0)}{U_\infty} - \frac{\dot\varphi(x_c) y}{U_\infty} \right) \right] dy + \text{Canard } 2$$

$$- \int_0^{b_c} \cancel{\delta z'^T(0) \sin\Lambda_1 x_c} \cdot \left[q c_c C_{L\alpha_c} \left(\bar\theta_w(0) \cos\Lambda_1 - \cancel{z'(0) \sin\Lambda_1} - \frac{\dot{\bar\theta}(0) \cos\Lambda_1 x_c}{U_\infty} + \cancel{\frac{\dot z'(0) \sin\Lambda_1 x_c}{U_\infty}} - \frac{\dot z(0)}{U_\infty} - \frac{\dot\varphi(x_c) y}{U_\infty} \right) \right] dy + \text{Canard } 2$$

$$+ \int_0^{b_c} \delta z(y{=}0)^T \cdot \left[q c_c C_{L\alpha_c} \left(\bar\theta_w(0) \cos\Lambda_1 - \cancel{z'(0) \sin\Lambda_1} - \frac{\dot{\bar\theta}(0) \cos\Lambda_1 x_c}{U_\infty} + \cancel{\frac{\dot z'(0) \sin\Lambda_1 x_c}{U_\infty}} - \frac{\dot z(0)}{U_\infty} - \frac{\dot\varphi(x_c) y}{U_\infty} \right) \right] dy + \text{Canard } 2$$

$$+ \int_0^{b_c} \delta\varphi(x_c)^T y \cdot \left[q c_c C_{L\alpha_c} \left(\bar\theta_w(0) \cos\Lambda_1 - \cancel{z'(0) \sin\Lambda_1} - \frac{\dot{\bar\theta}(0) \cos\Lambda_1 x_c}{U_\infty} + \cancel{\frac{\dot z'(0) \sin\Lambda_1 x_c}{U_\infty}} - \frac{\dot z(0)}{U_\infty} - \frac{\dot\varphi(x_c) y}{U_\infty} \right) \right] dy + \text{Canard } 2$$

From PLV is possible to write all Terms of $\underline{\underline{K}}^A$ and $\underline{\underline{C}}^A$:

SEMPLIFICATION : Since we are considering QS flutter condition $\alpha = \theta - \dot z / U_\infty \Rightarrow$ Only Terms will be considered for damping $\underline{\underline{C}}^A$

$$C_{zz}^A = -\int_0^b \delta z^T q c C_{L\alpha} \, \dot z / U_\infty \, dy + \int_0^b \delta z^T q c C_{L\alpha} \frac{e \dot z' \sin\Lambda_1}{U_\infty} dy - \int_0^{b_c} \delta z(0)^T q c_c C_{L\alpha_c} \, \dot z(0) / U_\infty \, dy$$

$$+ \int_0^b \delta z'^T \sin\Lambda_1 \, e \, q c C_{L\alpha} \, \dot z / U_\infty \, dy - \int_0^b \delta z'^T \sin\Lambda_1 \, e \, q c C_{L\alpha} \frac{e \dot z' \sin\Lambda_1}{U_\infty} dy$$

$$C_{z\theta}^A = -\int_0^b \delta z^T q c C_{L\alpha} \frac{e \dot{\bar\theta} \cos\Lambda_1}{U_\infty} dy + \int_0^b \delta z'^T \sin\Lambda_1 \, e \, q c C_{L\alpha} \frac{e \dot{\bar\theta} \cos\Lambda_1}{U_\infty} dy - \int_0^{b_c} \delta z(0)^T q c_c C_{L\alpha_c} \frac{\dot{\bar\theta}(0) \cos\Lambda_1 x_c}{U_\infty} dy$$

$$C_{z\varphi}^A = -\int_0^b \delta z^T q c C_{L\alpha} \frac{\dot\varphi_w y}{U_\infty} dy + \int_0^b \delta z'^T \sin\Lambda_1 \, e \, q c C_{L\alpha} \frac{\dot\varphi_w y}{U_\infty} dy - \int_0^{b_c} \delta z(0)^T q c_c C_{L\alpha_c} \frac{\dot\varphi(x_c) y}{U_\infty} dy$$

$$C^A_{\theta z} = -\int_0^b \delta\bar{\theta}^T \cos\Lambda_1\, e\, q\, c\, C_{L\alpha}\, \dot{z}/U_\infty\, dy + \int_0^b \delta\bar{\theta}^T \cos\Lambda_1\, e\, q\, c\, C_{L\alpha} \frac{e\,\dot{z}'\sin\Lambda_1}{U_\infty}\, dy - \int_0^{b_c} \delta\bar{\theta}^T(0)\cos\Lambda_1\, x_c\, q\, c_c\, C_{L\alpha_c} \frac{\dot{z}(0)}{U_\infty}\, dy$$

$$C^A_{\theta\theta} = -\int_0^b \delta\bar{\theta}^T \cos\Lambda_1\, e\, q\, c\, C_{L\alpha} \frac{e\,\dot{\theta}\cos\Lambda_1}{U_\infty}\, dy - \int_0^{b_c} \delta\bar{\theta}^T(0)\cos\Lambda_1\, x_c\, q\, c_c\, C_{L\alpha_c} \frac{\dot{\theta}(0)\cos\Lambda_1\, x_c}{U_\infty}\, dy$$

$$C^A_{\theta\varphi} = -\int_0^b \delta\bar{\theta}^T \cos\Lambda_1\, e\, q\, c\, C_{L\alpha} \frac{\dot{\varphi}_w\, y}{U_\infty}\, dy - \int_0^{b_c} \delta\bar{\theta}^T(0)\cos\Lambda_1\, x_c\, q\, c_c\, C_{L\alpha_c} \frac{\dot{\varphi}(x_c)\, y}{U_\infty}\, dy$$

$$C^A_{\varphi z} = -\int_0^b \delta\varphi_w^T\, y\, q\, c\, C_{L\alpha}\, \dot{z}/U_\infty\, dy + \int_0^b \delta\varphi_w^T\, y\, q\, c\, C_{L\alpha} \frac{e\,\dot{z}'\sin\Lambda_1}{U_\infty}\, dy - \int_0^b \delta\varphi^T(x_c)\, y \cdot q \cdot c_c \cdot C^c_{L\alpha} \cdot \dot{z}(0)/U_\infty\, dy$$

$$C^A_{\varphi\theta} = -\int_0^b \delta\varphi_w^T\, y\, q\, c\, C_{L\alpha} \frac{e\,\dot{\theta}\cos\Lambda_1}{U_\infty}\, dy - \int_0^{b_c} \delta\varphi(x_c)\, y \cdot q\, c_c\, C^c_{L\alpha} \cdot \frac{\dot{\theta}(0)\cos\Lambda_1\, x_c}{U_\infty}\, dy$$

$$C^A_{\varphi\varphi} = -\int_0^b \delta\varphi_w^T\, y\, q\, c\, C_{L\alpha} \frac{\dot{\varphi}_w\, y}{U_\infty}\, dy - \int_0^{b_c} \delta\varphi(x_c)\, y \cdot q\, c_c\, C^c_{L\alpha} \cdot \frac{\dot{\varphi}(x_c)\, y}{U_\infty}\, dy$$

$$K^A_{zz} = -\int_0^b \delta z^T q\, c\, C_{L\alpha}\, z'\sin\Lambda_1\, dy + \int_0^b \delta z'^T \sin\Lambda_1\, e\, q\, c\, C_{L\alpha}\, z'\sin\Lambda_1 \qquad \Big\| \qquad K^A_{z\varphi} = \emptyset$$

$$K^A_{z\theta} = \int_0^b \delta z^T q\, c\, C_{L\alpha}\, \bar{\theta}\cos\Lambda_1\, dy - \int_0^b \delta z'^T \sin\Lambda_1\, e\, q\, c\, C_{L\alpha}\, \theta\cos\Lambda_1 + \int_0^{b_c} \delta z(y=0)^T q\, c_c\, C^c_{L\alpha}\, \bar{\theta}_w(0)\cos\Lambda_1\, dy + \dots$$

$$K^A_{\theta z} = -\int_0^b \delta\bar{\theta}^T \cos\Lambda_1\, e\, q\, c\, C_{L\alpha}\, z'\sin\Lambda_1\, dy \qquad \Big\| \qquad K^A_{\theta\varphi} = \emptyset$$

$$K^A_{\theta\theta} = \int_0^b \delta\bar{\theta}^T \cos\Lambda_1\, e\, q\, c\, C_{L\alpha}\, \bar{\theta}\cos\Lambda_1\, dy + \int_0^{b_c} \delta\bar{\theta}_w^T(0)\cos\Lambda_1\, x_c\, q\, c_c\, C_{L\alpha_c}\, \bar{\theta}_w(0)\cos\Lambda_1\, dy$$

$$K^A_{\varphi z} = -\int_0^b \delta\varphi_w^T\, y\, q\, c\, C_{L\alpha}\, z'\sin\Lambda_1\, dy \qquad \Big\| \qquad K^A_{\varphi\varphi} = \emptyset$$

$$K^A_{\varphi\theta} = \int_0^b \delta\varphi_w^T\, y\, q\, c\, C_{L\alpha}\, \bar{\theta}\cos\Lambda_1\, dy + \int_0^{b_c} \delta\varphi(x_c)^T\, y\, q\, c_c\, C_{L\alpha_c}\, \bar{\theta}_w(0)\cos\Lambda_1\, dy$$

$$\underline{\underline{C}}^a = \begin{bmatrix} C^a_{\bar\theta\bar\theta} & C^a_{\bar\theta z} & C^a_{\bar\theta\varphi} \\ C^a_{z\bar\theta} & C^a_{zz} & C^a_{z\varphi} \\ C^a_{\varphi\bar\theta} & C^a_{\varphi z} & C^a_{\varphi\varphi} \end{bmatrix} \quad \text{and} \quad \underline{\underline{K}}^a = \begin{bmatrix} K^a_{\bar\theta\bar\theta} & K^a_{\bar\theta z} & K^a_{\bar\theta\varphi} \\ K^a_{z\bar\theta} & K^a_{zz} & K^a_{z\varphi} \\ K^a_{\varphi\bar\theta} & K^a_{\varphi z} & K^a_{\varphi\varphi} \end{bmatrix}$$

$\Longrightarrow$ Because ... $\delta W_i - \delta W_e = \emptyset$

The full dynamic system becomes: $\underline{\underline{M}}\,\ddot{\underline{q}} + (-\underline{\underline{C}}^a + \underline{\underline{C}}^s)\,\dot{\underline{q}} + (\underline{\underline{K}}^s - \underline{\underline{K}}^a)\,\underline{q} = \emptyset$

Using modal scomposition can be simplified:

$$\underline{\underline{\phi}}^T \underline{\underline{M}}\, \underline{\underline{\phi}}\, \ddot{\underline{z}} + \underline{\underline{\phi}}^T \underline{\underline{C}}\, \underline{\underline{\phi}}\, \dot{\underline{z}} + \underline{\underline{\phi}}^T \underline{\underline{K}}\, \underline{\underline{\phi}}\, \underline{z} = \emptyset$$

The stability of the system can be easly analyzed as a general eigenvalue problem by converting it in STATE SPACE FORM

$\Longrightarrow$ The modal system expressed in state space modal form becomes:

$$\begin{bmatrix} \ddot{\underline{z}} \\ \dot{\underline{z}} \end{bmatrix} = \begin{bmatrix} -\underline{\underline{M}}_m^{-1}\underline{\underline{C}}_m & -\underline{\underline{M}}_m^{-1}\underline{\underline{K}}_m \\ \underline{\underline{I}} & \emptyset \end{bmatrix} \begin{bmatrix} \dot{\underline{z}} \\ \underline{z} \end{bmatrix} \iff \dot{\underline{x}} = \underline{\underline{A}} \cdot \underline{x}$$

The eigenvalues of the system can be easly computed in MATLAB

↳ By computing eigenvalues for different speed is possible to compute the locus root and the function $g(v)$

... Where $g(v)$ is the interpolation of g-damping values at different speed

NOTE: $g\text{-damping} = \dfrac{Re(\text{eigenvalue}_i)}{|Im(\text{eigenvalue}_i)|}$

NOTE: for the analysis has been considered complex conjugate eigenvalues with $Im > \emptyset$

The flutter condition occours when an eigenvalue with $Im \neq \emptyset$ crosses the $Re = \emptyset$ axiz

↳ And so when g-damping value becomes $> \emptyset$

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③ Compute the frequency response of wing torsional moment obtained by rotation of canard surface at $f = x_1 \cdot 15\ Hz$

The DYNAMIC SYSTEM is the same seen before with the add of a forcing term given by the rotation of the canard

⟹ From external work the forcing term can be seen as: $\int_0^{b_c} \delta z_{AC}^{c^T} \cdot q c_c C_{L\alpha}^c \beta_{canard}(t)\, dy$

↳ $\int_0^{b_c} \left(\delta\bar{\theta}_w^T\big|_0 \cos\Lambda_1 x_c + z(\emptyset) + \varphi(x_c) y \right) q c_c C_{L\alpha_c} \beta_{canard}(t)\, dy$

$$F_{\bar{\theta}} = \int_0^{b_c} \delta\bar{\theta}_w^T\big|_0 \cos\Lambda_1 x_c \, q c_c C_{L\alpha_c} \beta_{canard}(t)\, dy$$

$$F_z = \int_0^{b_c} \delta z^T(\emptyset)\, q c_c C_{L\alpha_c} \beta_{canard}(t)\, dy$$

$$F_\varphi = \int_0^{b_c} \delta\varphi^T(x_c) y \; q c_c C_{L\alpha_c} \beta_{canard}(t)\, dy$$

$$\rightarrow \underline{F} = \begin{Bmatrix} F_\theta \\ F_z \\ F_\varphi \end{Bmatrix} \xrightarrow[\text{DECOMPOSITION}]{\text{MODAL}} \underline{F}_m = \underline{\underline{\phi}}^T \cdot \underline{F}$$

For exemple let's consider a fly condition with $U_{eas} = 30\ m/s$. In this condition our system is stable and linear. Since the input $\beta(t)$ is $\beta_0 e^{j\omega t}$ with $\omega = x_1 \cdot 15\ Hz$, $\beta_0 = 5°$

↳ Is possible to compute the forced response using FREQUENCY DOMAIN

Anyway the system can't be decoupled with eigenmode because of K^A and C^A
$\Rightarrow$ is necessary to turn it in STATE SPACE FORM and compute the transfer matrix

$$\begin{cases} \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \\ \underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u} \end{cases} \quad \text{where...} \quad \underline{x} = \begin{Bmatrix} \underline{z} \\ \dot{\underline{z}} \end{Bmatrix} \quad \text{and} \quad \underline{z} = \begin{Bmatrix} z_{\bar{\theta}} \\ z_{\bar{z}} \\ z_{\varphi} \end{Bmatrix} \longrightarrow \text{modal general coordinates}$$

The matrices are :

Since we are interested in generalized states

$$\underline{\underline{A}} = \begin{bmatrix} \emptyset & \underline{\underline{I}} \\ -\underline{\underline{M}}_m^{-1}\underline{\underline{K}}_m & -\underline{\underline{M}}_m^{-1}\underline{\underline{C}}_m \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} \emptyset \\ \underline{\underline{M}}_m^{-1}\underline{\underline{F}}_m \end{bmatrix}, \quad \underline{\underline{C}} = [\,1\,,\,\emptyset\,], \quad \underline{\underline{D}} = \emptyset$$

Transform in frequency domain : $z(t) = z e^{j\omega t}$, $\dot{z}(t) = j\omega z e^{j\omega t}$, $F(t) = F e^{j\omega t}$

$$\Rightarrow \quad \underline{y} = \underline{\underline{G}}(j\omega) \cdot \underline{u} \quad \text{where} \quad \underline{\underline{G}}(j\omega) = \underline{\underline{C}} \cdot (j\omega\,\underline{\underline{I}} - \underline{\underline{A}})^{-1}\,\underline{\underline{B}} + \underline{\underline{D}}$$

$\hookrightarrow$ In this way we can compute the response to input $\omega = 18$ Hz

The process can be easly implemented in MATLAB with functions :

- 'ss' $\longrightarrow$ defines state space system from matrices A, B, C, D
- 'frd' $\longrightarrow$ convert dynamic state space system to frequency - response data model form
- 'bode' $\longrightarrow$ returns the response data at the specified frequency

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SHAPE FUNCTION DEFINITION

The DoF of the system are expressed using Ritz - Galerkin method as :

$$\bar{\theta} = \underline{N}_{\bar{\theta}}(\bar{y}) \cdot \underline{q}_{\bar{\theta}} \,, \quad \bar{z} = \underline{N}_{\bar{z}}(\bar{y}) \cdot \underline{q}_{\bar{z}} \,, \quad \varphi = \underline{N}_{\varphi}(x) \cdot \underline{q}_{\varphi}$$

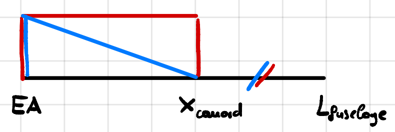
The shapes functions of $\bar{\theta}$ and $\bar{z}$ are the same of the one chosen in assignment 1 with the add of a rigid one

$\hookrightarrow$ This guarantee to consider even the rigid body modes

$$\Rightarrow \quad N_{\bar{\theta}} = [\,1\,,\, N_{\bar{\theta},\text{assign 1}}\,] \,, \quad N_{\bar{z}} = [\,1\,,\, N_{\bar{z},\text{assign 1}}\,]$$

The shapes functions of φ instead has been chosen considering the fuselage torsional moment computed in assignment 1

It results constant between $[EA_{wing}\,,\, X_{canard}]$ and null between $[X_{canard}\,,\, L_{fuselage}]$



$\Longrightarrow$ φ = twist angle of the fuselage will results :

$H_t = \varphi_{/x}(x)$

$\varphi(x)$

$$\Longrightarrow \quad \varphi(x) = \begin{cases} 1 - x/x_{canard} \,, & [0, x_{canard}] \\ \emptyset \,, & [x_{canard} \,, L_{fuselage}] \end{cases}$$

In addition we have to add the rigid mode shape function (for roll)

↳ So the complete shape function vector will be : $N_\varphi = [1 \,, \varphi(x)]$

.... In MATLAB

Each time we perform an integration on dx w.r.t. $\varphi(x)$ the integral is executed between $-x_{canard}$ and $\emptyset$.

↳ This because $\varphi(x)$ for $x < x_{canard}$ is $= \emptyset$

STRUCTURAL DAMPING

The structural damping matrix has been computed as follows :

$$\underline{\underline{C}}_s = \text{diag}\left\{ 2 \cdot \mu_i \cdot \xi_{\%} \cdot \omega_i \right\} \quad \text{where :} \quad \begin{array}{l} - \mu_i \text{ are modal masses} \\ - \omega_i \text{ are eigenfrequencies} \end{array}$$

LIST of RESULTS

(Computed using 12 shapes : 5 $\bar{\theta}$, 5 z and 2 φ)

1. ω_i = ∅ , ∅ , ∅ , 5.233103 , 7.168191 , 16.240622 , 19.098612 , 27.836466 , 41.200274 , 59.549901 Hz

2. $U_{flutter}$ = 4.2 m/s NOTE : U_D computed at $\rho = 1.225\ kg/m^3$

3. $|\cdot|$ = 0.084943 , 0.000004 , 0.00000 , 0.000007 , 0.000001
   $\measuredangle$ = 179.9907 , -13.9514 , 166.3903 , 20.1675 , 176.8695

   ← On MATLAB code are computed even magnitude and phase associated to z and φ

NOTE :

The script has been given with shape functions number set to : num_Ntheta = num_Nz = 2 and num_Phi = 1

This guarantee a good precision and reasonable computation time.

All the code has been written and run on MATLAB R2023a with SYMBOLIC MATH TOOLBOX (ver 9.3) and CONTROL SYSTEM TOOLBOX (ver 10.13)