MSAS – Assignment #1: Simulation

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1 Implicit equations

Exercise 1

Let $\mathbf{f}$ be a two-dimensional vector-valued function $\mathbf{f}(\mathbf{x}) = (x_2^2 - x_1 - 2,\ -x_1^2 + x_2 + 10)^\top$, where $\mathbf{x} = (x_1, x_2)^\top$. Find the zero(s) of $\mathbf{f}$ by using Newton's method with $\partial \mathbf{f}/\partial \mathbf{x}$ 1) computed analytically, and 2) estimated through finite differences. Which version is more accurate?

(3 points)

Write your answer here

- Develop the exercises in one Matlab script; name the file `lastname123456_Assign1.m`
- Organize the script in sections, one for each exercise; use local functions if needed.
- Download the PDF from the Main menu.
- Create a single .zip file containing both the report in PDF and the MATLAB file. The name shall be `lastname123456_Assign1.zip`.
- Red text indicates where answers are needed; be sure there is no red stuff in your report.
- In your answers, be concise: to the point.
- **Deadline for the submission: Nov 20 2023, 23:59**.
- **Load the compressed file to the Homework folder on Webeep**.

2 Numerical solution of ODE

Exercise 2

The Initial Value Problem $\dot{x} = x - 2t^2 + 2$, $x(0) = 1$, has analytic solution $x(t) = 2t^2 + 4t - e^t + 2$. 1) Implement a general-purpose, fixed-step Heun's method (RK2); 2) Solve the IVP in $t \in [0, 2]$ for $h_1 = 0.5$, $h_2 = 0.2$, $h_3 = 0.05$, $h_4 = 0.01$ and compare the numerical vs the analytical solution; 3) Repeat points 1)–2) with RK4; 4) Trade off between CPU time & integration error.

(4 points)

Write your answer here

Exercise 3

Let $\dot{\mathbf{x}} = A(\alpha)\mathbf{x}$ be a two-dimensional system with $A(\alpha) = [0, 1; -1, 2\cos\alpha]$. Notice that $A(\alpha)$ has a pair of complex conjugate eigenvalues on the unit circle; α denotes the angle from the $\mathrm{Re}\{\lambda\}$-axis. 1) Write the operator $F_{\mathrm{RK2}}(h, \alpha)$ that maps $\mathbf{x}_k$ into $\mathbf{x}_{k+1}$, namely $\mathbf{x}_{k+1} = F_{\mathrm{RK2}}(h, \alpha)\,\mathbf{x}_k$. 2) With $\alpha = \pi$, solve the problem "Find $h \geq 0$ s.t. $\max\left(|\mathrm{eig}(F(h, \alpha))|\right) = 1$". 3) Repeat point 2) for $\alpha \in [0, \pi]$ and draw the solutions in the $(h\lambda)$-plane. 4) Repeat points 1)–3) with RK4.

(5 points)

Write your answer here

Exercise 4

Consider the IVP $\dot{\mathbf{x}} = A(\alpha)\mathbf{x}$, $\mathbf{x}(0) = [1, 1]^T$, to be integrated in $t \in [0, 1]$. 1) Take $\alpha \in [0, \pi]$ and solve the problem "Find $h \geq 0$ s.t. $\|\mathbf{x}_{\mathrm{an}}(1) - \mathbf{x}_{\mathrm{RK1}}(1)\|_\infty = \mathrm{tol}$", where $\mathbf{x}_{\mathrm{an}}(1)$ and $\mathbf{x}_{\mathrm{RK1}}(1)$ are the analytical and the numerical solution (with RK1) at the final time, respectively, and $\mathrm{tol} = \{10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$. 2) Plot the four locus of solutions in the $(h\lambda)$-plane; plot also the function evaluations vs tol for $\alpha = \pi$. 3) Repeat points 1)–2) for RK2 and RK4.

(4 points)

Write your answer here

Exercise 5

Consider the backinterpolation method $\mathrm{BI2}_{0.4}$. 1) Derive the expression of the linear operator $B_{\mathrm{BI2}_{0.4}}(h, \alpha)$ such that $\mathbf{x}_{k+1} = B_{\mathrm{BI2}_{0.4}}(h, \alpha)\mathbf{x}_k$. 2) Following the approach of point 3) in Exercise 3, draw the stability domain of $\mathrm{BI2}_{0.4}$ in the $(h\lambda)$-plane. 3) Derive the domain of numerical stability of $\mathrm{BI2}_\theta$ for the values of $\theta = [0.1, 0.3, 0.7, 0.9]$.

(5 points)

Write your answer here

Exercise 6

Consider the IVP $\dot{\mathbf{x}} = B\mathbf{x}$ with $B = [-180.5, 219.5; 179.5, -220.5]$ and $\mathbf{x}(0) = [1, 1]^T$ to be integrated in $t \in [0, 5]$. Notice that $\mathbf{x}(t) = e^{Bt}\mathbf{x}(0)$. 1) Solve the IVP using RK4 with $h = 0.1$; 2) Repeat point 1) using implicit extrapolation technique IEX4; 3) Compare the numerical results in points 1) and 2) against the analytic solution; 4) Compute the eigenvalues associated to the IVP and represent them on the $(h\lambda)$-plane both for RK4 and IEX4; 5) Discuss the results.

(4 points)

Write your answer here

Exercise 7

Consider the two-dimensional IVP

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2}\left[1 + 8\sin(t)\right] x_1 \\ (1 - x_1)x_2 + x_1 \end{bmatrix}, \qquad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1) Solve the IVP using AB3 in $t \in [0, 3]$ for $h = 0.1$; 2) Repeat point 1) using AM3, ABM3, and BDF3; 3) Discuss the results.

(5 points)

Write your answer here