

Master's Thesis Exam

Verification of the Blocking and Non-Blocking Michael-Scott Queue Algorithms

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Overview of the Project and Contributions

- Initial **goal** was to prove **safety** of the two **M&S Queues**
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 - **Sequential** specification
 - Useful for sequential clients
 - **Concurrent** specification
 - Proves **safety of concurrent queues**
 - Useful for *some* concurrent clients
 - **Doesn't track** queue contents
 - **HOCAP-style** specification
 - Stronger specification, useful for more **complex clients**
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 - Demonstrated with a PoC queue client, QueueAdd

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- **Implementations** of the M&S Queues in **HeapLang** were proven to meet the three **specifications**
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- All proofs have been **mechanised** in the **Coq proof assistant**

- 1 Queue Specifications (HOCAP)
- 2 The Two-Lock Michael-Scott Queue
- 3 Proving that the Two-Lock Michael-Scott Queue Satisfies the HOCAP-style Specification
- 4 The Lock-Free Michael-Scott Queue
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Queue Specifications (HOCAP)

Specifications for Queues

Assumptions on Queues

- Queues consists of **initialize**, **enqueue**, and **dequeue**
- **initialize** creates an **empty queue**: `[]`
- **enqueue** adds a value, v , to the **beginning of the queue** $xs_v: v :: xs_v$
- **dequeue** depends on whether queue is empty:
 - If **non-empty**, $xs_v ++ [v]$, remove value v at **end of queue** and return **Some v**
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Nature of Specifications

- Specifications written in **Iris**, a **higher order CSL**
- Expressed in terms of **Hoare triples**: $\{P\} e \{v.\Phi\ v\}$
- Hoare triples prove **partial correctness** of programs, e
- In particular: **safety**

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Authoritative view

$$\gamma \models_{\bullet} xS_v$$

Owned by queue

Fragmental view

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- Construction **ensures**:
 - **authoritative** and **fragmental** views always **agree** on abstract state of queue
 - views can only be **updated** in **unison**
- **Implemented** using the **resource algebra**: $\text{AUTH}((\text{FRAC} \times \text{AG}(\text{List Val}))^?)$
- The **desirables** are captured by the following **lemmas**

Lemmas on the Abstract State RA

$$\vdash \Vdash \exists \gamma. \gamma \Vdash_{\bullet} xs_v * \gamma \Vdash_{\circ} xs_v$$

(Abstract State Alloc)

$$\gamma \Vdash_{\bullet} xs'_v * \gamma \Vdash_{\circ} xs_v \vdash xs_v = xs'_v$$

(Abstract State Agree)

$$\gamma \Vdash_{\bullet} xs'_v * \gamma \Vdash_{\circ} xs_v \Rightarrow \gamma \Vdash_{\bullet} xs''_v * \gamma \Vdash_{\circ} xs_v$$

(Abstract State Update)

HOCAP-style Specification

- Post-condition of **initialize** specification gives **fragmental view** to **clients**
- Hoare triples for **enqueue** and **dequeue** are conditioned on **view-shifts**
- Clients must show that they can **supply** the **fragmental view**, so that the **abstract** (and concrete) **state** can be **updated**
- View-shifts and Hoare-triples **parametrised** by predicates **P** and **Q**
 - Client might have **resources** that need to be **updated** as a result of **enqueue/dequeue**
 - **P** is the clients resources **before enqueue/dequeue** and **Q** the resources **after**

Definition (HOCAP Specification)

$\exists \text{isQueue} : \text{Val} \rightarrow \text{Qgnames} \rightarrow \text{Prop}.$

$\forall v_q, G. \text{isQueue}(v_q, G) \implies \Box \text{isQueue}(v_q, G)$

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$\wedge \forall v_q, v, G, P, Q. \left(\forall xs_v. G.\gamma_{\text{Abst}} \mapsto_{\bullet} xs_v * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i \uparrow} \triangleright G.\gamma_{\text{Abst}} \mapsto_{\bullet} (v :: xs_v) * Q \right) \multimap$
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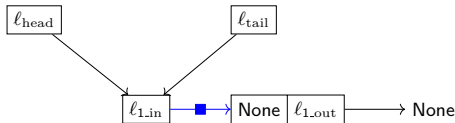
The Two-Lock Michael-Scott Queue

Implementation: initialize

- The queue data structure is a [linked list](#)

initialize \triangleq

```
let node = ref (None, ref (None)) in  
let H_lock = newLock() in  
let T_lock = newLock() in  
ref ((ref (node), ref (node)), (H_lock, T_lock))
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Implementation: initialize

- The queue data structure is a **linked list**
- A **node** x in the linked list is a **triple**, $x = (\ell_{\text{in}}, w, \ell_{\text{out}})$, with $\ell_{\text{in}} \mapsto (w, \ell_{\text{out}})$
- We use the following **notation** for nodes

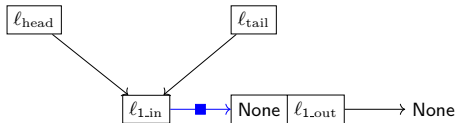
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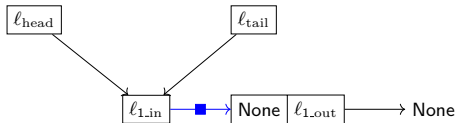
$$\text{val}(x) = w$$

$$\text{out}(x) = \ell_{\text{out}}$$

- The **initialize** function first creates an **initial head node**, x_{head}
- Then, a **lock** protecting the **head pointer**, and a **lock** protecting the **tail pointer**
- Finally, it creates the **head** and **tail pointers**, ℓ_{head} and ℓ_{tail} , both **pointing to** x_{head}

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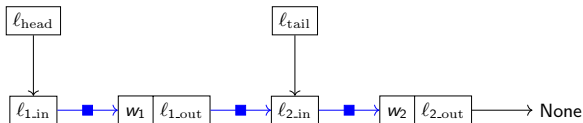


Implementation: enqueue

- The **enqueue** function consists of the following **steps**

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enqueue  $Q$  value  $\triangleq$   
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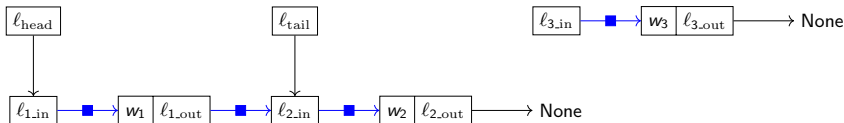


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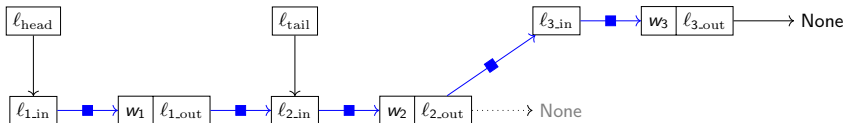


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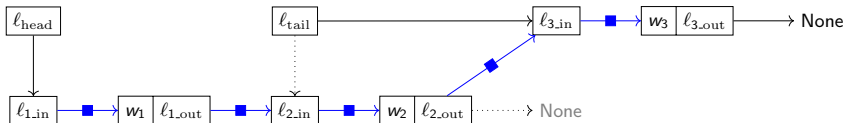


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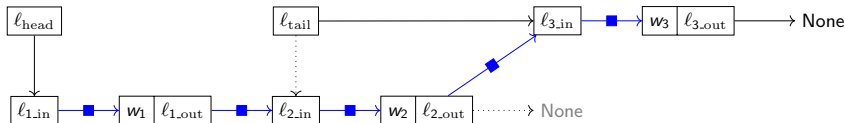
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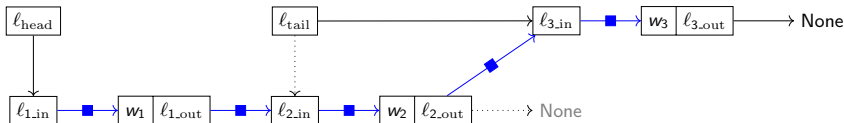
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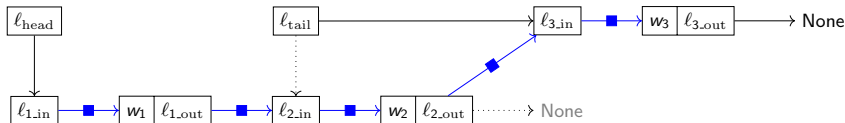
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- **dequeue ignores tail pointer** → **Tail** node can **lag** behind **head** node

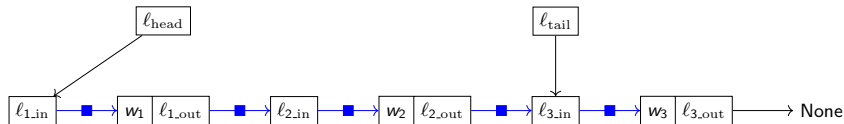
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 - If empty, **return** None
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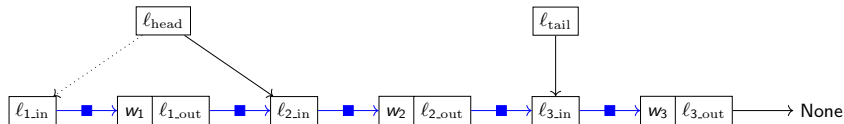
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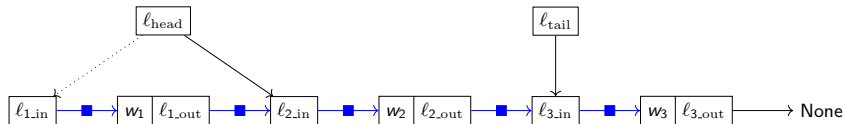
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- Dequeued node **not freed** → Linked list **only grows**

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Proving that the Two-Lock Michael-Scott Queue Satisfies the HOCAP-style Specification

The isLL Predicate

- **Idea:** express the **structure** of the linked list in terms of **points-to predicates**
- Also captures **persistent** and **non-persistent** parts of the linked list

Definition (Linked List Predicate)

$$\text{isLL_chain}([]) \triangleq \text{True}$$

$$\text{isLL_chain}([x]) \triangleq \text{in}(x) \mapsto^{\square} (\text{val}(x), \text{out}(x))$$

$$\text{isLL_chain}(x :: x' :: xs) \triangleq \text{in}(x) \mapsto^{\square} (\text{val}(x), \text{out}(x)) * \text{out}(x') \mapsto^{\square} \text{in}(x') * \text{isLL_chain}(x' :: xs)$$

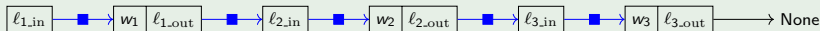
$$\text{isLL}([]) \triangleq \text{True}$$

$$\text{isLL}(x :: xs) \triangleq \text{out}(x) \mapsto \text{None} * \text{isLL_chain}(x :: xs)$$

Example

Consider the list: $xs = [(\ell_{3_in}, w_3, \ell_{3_out}); (\ell_{2_in}, w_2, \ell_{2_out}); (\ell_{1_in}, w_1, \ell_{1_out})]$.

$$\begin{aligned} \text{isLL}(xs) = & \ell_{3_out} \mapsto \text{None} * \ell_{3_in} \mapsto^{\square} (w_3, \ell_{3_out}) * \ell_{2_out} \mapsto^{\square} \ell_{3_in} * \\ & \ell_{2_in} \mapsto^{\square} (w_2, \ell_{2_out}) * \ell_{1_out} \mapsto^{\square} \ell_{2_in} * \\ & \ell_{1_in} \mapsto^{\square} (w_1, \ell_{1_out}) \end{aligned}$$



Invariant

- Queue predicate must be persistent (according to specification)
- **Problem**: the queue relies on non-persistent resources (e.g. $\ell_{\text{head}} \mapsto \text{in}(x_{\text{head}})$)
- **Solution**: identify an *invariant* (persistent), describing the resources

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 - **Solution:** identify an *invariant* (persistent), describing the resources
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- Contains abstract state of queue – existentially quantified as it can change

Definition (Two-Lock M&S Queue HOCAP Invariant)

$$I_{\text{TLH}}(\ell_{\text{head}}, \ell_{\text{tail}}, G) \triangleq \exists x s_v. G. \gamma_{\text{Abst}} \Rightarrow_{\bullet} x s_v * \quad (\text{abstract state})$$

Invariant

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Definition (Two-Lock M&S Queue HOCAP Invariant)

$$\begin{aligned} \text{I}_{\text{TLH}}(\ell_{\text{head}}, \ell_{\text{tail}}, G) &\triangleq \exists xs_v. G.\gamma_{\text{Abst}} \Rightarrow \bullet xs_v * && \text{(abstract state)} \\ &\quad \exists xs, xs_{\text{queue}}, xs_{\text{old}}, x_{\text{head}}, x_{\text{tail}}. && \text{(concrete state)} \\ &\quad xs = xs_{\text{queue}} ++ [x_{\text{head}}] ++ xs_{\text{old}} * \\ &\quad \text{isLL}(xs) * \end{aligned}$$

Invariant

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 - Asserts relation between abstract state and concrete state

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Invariant

- Queue predicate must be **persistent** (according to specification)
 - **Problem**: the queue **relies** on **non-persistent resources** (e.g. $\ell_{\text{head}} \mapsto \text{in}(x_{\text{head}})$)
 - **Solution**: identify an **invariant** (persistent), describing the resources
-
- Contains **abstract state** of queue – **existentially quantified** as it can change
 - Defines **structure** of the **concrete** linked list, xs
 - Asserts **relation** between **abstract state** and **concrete state**
 - Identifies possible **queue states**: **Static**, **Enqueue**, **Dequeue**, and **Both**
 - Two locks \rightarrow Four queue states
 - Invariant describes the queue resources in each state
 - See next slide

Definition (Two-Lock M&S Queue HOCAP Invariant)

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Invariant (Queue States)

- **Idea**: the enqueueing thread keeps half of tail pointer between invariant openings
- **Guarantees** that the pointer is **not updated** (full pointer needed for update)
- **Similarly** for the dequeueing thread
- **Enqueue** and **Both** also captures “gap” between adding x_{new} and swinging ℓ_{tail}
- **Tokens** used to reason about which **state** queue is in

Definition (Two-Lock M&S Queue HOCAP Invariant – continued)

...

$\ell_{\text{head}} \mapsto \text{in}(x_{\text{head}}) * \ell_{\text{tail}} \mapsto \text{in}(x_{\text{tail}}) * \text{isLast}(x_{\text{tail}}, xs) *$ (Static)

$\text{TokNE } G * \text{TokND } G * \text{TokUpdated } G$

✓ $\ell_{\text{head}} \mapsto \text{in}(x_{\text{head}}) * \ell_{\text{tail}} \mapsto \frac{1}{2} \text{in}(x_{\text{tail}}) *$ (Enqueue)

$(\text{isLast}(x_{\text{tail}}, xs) * \text{TokBefore } G \vee \text{isSndLast}(x_{\text{tail}}, xs) * \text{TokAfter } G) *$

$\text{TokE } G * \text{TokND } G$

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Queue Predicate

- We now define the **queue predicate** in terms of our **invariant**

Definition (Two-Lock M&S Queue - isQueue Predicate)

$$\begin{aligned} \text{isQueue}(v_q, G) &\triangleq \exists \ell_{\text{queue}}, \ell_{\text{head}}, \ell_{\text{tail}} \in \text{Loc}. \exists h_{\text{lock}}, t_{\text{lock}} \in \text{Val}. \\ v_q &= \ell_{\text{queue}} * \ell_{\text{queue}} \mapsto^{\square} ((\ell_{\text{head}}, \ell_{\text{tail}}), (h_{\text{lock}}, t_{\text{lock}})) * \\ &\quad \boxed{\text{TLH}(\ell_{\text{head}}, \ell_{\text{tail}}, G)}^{\mathcal{N}.\text{queue}} * \\ &\quad \text{isLock}(G.\gamma_{\text{Hlock}}, h_{\text{lock}}, \text{TokD } G) * \\ &\quad \text{isLock}(G.\gamma_{\text{Tlock}}, t_{\text{lock}}, \text{TokE } G) \end{aligned}$$

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- The queue predicate is **persistent**, as all its constituents are
- **Proving** that Two-Lock M&S Queue satisfies the **HOCAP-style specification** then consists of proving the **Hoare triples** for **initialize**, **enqueue**, and **dequeue**
- We here **focus** on **enqueue**

Proof Sketch of the Hoare Triple for enqueue

$$\forall v_q, v, G, P, Q. \quad (\forall xs_v. G.\gamma_{\text{Abst}} \models \bullet xs_v * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i \uparrow} \triangleright G.\gamma_{\text{Abst}} \models \bullet (v :: xs_v) * Q) \multimap \\ \{\text{isQueue}(v_q, G) * P\} \text{ enqueue } v_q \ v \ \{w.Q\}$$

Proof Sketch of the Hoare Triple for enqueue

$$\forall v_q, v, G, P, Q. \left(\forall x_{s_v}. G.\gamma_{\text{Abst}} \models \bullet x_{s_v} * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i \uparrow} \triangleright G.\gamma_{\text{Abst}} \models \bullet (v :: x_{s_v}) * Q \right) \multimap \{ \text{isQueue}(v_q, G) * P \} \text{enqueue } v_q \ v \{ w.Q \}$$

(Proof)

Assume the **view-shift**, and the persistent information in **isQueue**(v_q , $Qgnames$):

- $v_q = \ell_{\text{queue}} * \ell_{\text{queue}} \mapsto^{\square} ((\ell_{\text{head}}, \ell_{\text{tail}}), (h_{\text{lock}}, t_{\text{lock}}))$
- $\boxed{\text{ITLH}(\ell_{\text{head}}, \ell_{\text{tail}}, G)}^{\mathcal{N}.queue}$
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$\{P\}$

`let node = ref (Some v, ref (None)) in`

`acquire(snd(snd(! v_q)));`

`e_t = !(snd(fst(! v_q)))`

`snd(! (e_t)) ← node;`

`snd(fst(! v_q)) ← node;`

`release(snd(snd(! v_q)))`

$\{Q\}$

Proof Sketch of the Hoare Triple for enqueue

$\forall v_q, v, G, P, Q. (\forall x_{sv}. G.\gamma_{\text{Abst}} \mapsto \bullet x_{sv} * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i \uparrow} \triangleright G.\gamma_{\text{Abst}} \mapsto \bullet (v :: x_{sv}) * Q) \multimap$
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$\{P\}$

let $node = \text{ref}(\text{Some } v, \text{ref}(\text{None}))$ in (create node x_{new})

$\{P * \text{out}(x_{\text{new}}) \mapsto \text{None}\}$

acquire($\text{snd}(\text{snd}(!v_q))$);

$e_t = !(\text{snd}(\text{fst}(!v_q)))$

$\text{snd}(!e_t) \leftarrow node$;

$\text{snd}(\text{fst}(!v_q)) \leftarrow node$;

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{P}
let node = ref(Some v, ref(None)) in (create node xnew)
{P * out(xnew) ↦ None}
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{P * out(xnew) ↦ None * TokE G}
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$$\forall v_q, v, G, P, Q. \quad (\forall x_{S_v}. G. \gamma_{\text{Abst}} \mapsto \bullet. x_{S_v} * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}. i \uparrow} \triangleright G. \gamma_{\text{Abst}} \mapsto \bullet. (v :: x_{S_v}) * Q) \multimap \{ \text{isQueue}(v_q, G) * P \} \text{ enqueue } v_q \ v \{ w. Q \}$$

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{P * out(xnew) ↦ None * ℓtail ↦ ½ in(xtail) * TokNE G * TokAfter G}
  snd(! (et)) ← node;   (make xtail point to xnew. VS. ITLH: Enqueue/Both (before) → Enqueue/Both (after))
{Q * ℓtail ↦ ½ in(xtail) * TokNE G * TokBefore G}
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{Q * ℓtail ↦ 1/2 in(xtail) * TokNE G * TokBefore G}
  snd(fst(! vq)) ← node; (swing tail pointer to xnew. ITLH: Enqueue/Both (after) → Static/Dequeue)
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{Q * ℓtail ↦ 1/2 in(xtail) * TokNE G * TokBefore G}
  snd(fst(! vq)) ← node; (swing tail pointer to xnew. ITLH: Enqueue/Both (after) → Static/Dequeue)
{Q * TokE G}
  release(snd(snd(! vq))) (release tail lock)
{Q}
```

The Lock-Free Michael-Scott Queue

Implementation (Consistency-Check Free)

- Queue data structure is **still** a **linked list**
- The **lock-free versions** of **initialize**, **enqueue**, and **dequeue** perform the **same manipulations** of the linked list as **two-lock versions**
- **Difference** is how the manipulations take place

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- Queue data structure is **still** a **linked list**
- The **lock-free versions** of **initialize**, **enqueue**, and **dequeue** perform the **same manipulations** of the linked list as **two-lock versions**
- **Difference** is how the manipulations take place – now with **CAS** instructions
- **Ensures** no **overwrites** of enqueued nodes, and no node **dequeued twice**

```
initialize  $\triangleq$   
  let node = ref (None, ref (None)) in  
  ref (ref (node), ref (node))  
  
enqueue Q value  $\triangleq$   
  let node = ref (Some value, ref (None)) in  
  (rec loop_ =  
    let tail = !(snd(! Q)) in  
    let next = !(snd(! tail)) in  
    if next = None then  
      if CAS (snd(! tail)) next node then  
        CAS (snd(! Q)) tail node  
      else loop ()  
    else CAS (snd(! Q)) tail next; loop ()  
  ) ()
```

```
dequeue Q  $\triangleq$   
  (rec loop_ =  
    let head = !(fst(! Q)) in  
    let tail = !(snd(! Q)) in  
    let next = !(snd(! head)) in  
    if head = tail then  
      if next = None then  
        None  
      else  
        CAS (snd(! Q)) tail next; loop ()  
    else  
      let value = fst(! next) in  
      if CAS (fst(! Q)) head next then  
        value  
      else loop ()  
  )()
```

Consistency Checks and Prophecies

- Original implementation has **consistency checks** to deal with **ABA problem**
- Creates **complications** when proving adherence to **HOCAP-style specification**
- In **dequeue**, the point at which to apply the **view-shift** (to update P to Q) depends on whether **queue is empty** and result of **future consistency check**
- **Prophecies** allow us to reason about future computations (e.g. **consistency check**)

```
initialize  $\triangleq$   
  let node = ref (None, ref (None)) in  
  ref (ref (node), ref (node))  
  
enqueue Q value  $\triangleq$   
  let node = ref (Some value, ref (None)) in  
  (rec loop_ =  
    let tail = !(snd(! Q)) in  
    let next = !(snd(! tail)) in  
    if tail = !(snd(! Q)) then  
      if next = None then  
        if CAS (snd(! tail)) next node then  
          CAS (snd(! Q)) tail node  
        else loop ()  
      else CAS (snd(! Q)) tail next; loop ()  
    else loop ()  
  ) ()
```

```
dequeue Q  $\triangleq$   
  (rec loop_ =  
    let head = !(fst(! Q)) in  
    let tail = !(snd(! Q)) in  
    let p = newproph in  
    let next = !(snd(! head)) in  
    if head = Resolve(!(fst(! Q)), p, ()) then  
      if head = tail then  
        if next = None then  
          None  
        else  
          CAS (snd(! Q)) tail next; loop ()  
      else  
        let value = fst(! next) in  
        if CAS (fst(! Q)) head next then  
          value  
        else loop ()  
    else loop ()  
  )()
```

Proving that the Lock-Free Michael-Scott Queue Satisfies the HOCAP-style Specification

- The queue relies on some **important properties** to function **correctly**:
 - The head and tail are **only moved forward** in the linked list
 - The **tail cannot lag** behind the **head** (unlike in the two-lock version)

Reachability

- The queue relies on some **important properties** to function **correctly**:
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- We capture these properties with a notion of **reachability**
- Consists of a **concrete** and **abstract** version of reachability

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- We capture these properties with a notion of **reachability**
- Consists of a **concrete** and **abstract** version of reachability

Concrete Reachability

- Concrete reachability essentially captures a **section** of the **linked list** (à la isLL)

Reachability

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Lemmas for Reachability (simplified)

$$x \leadsto x \Rightarrow \exists \gamma. \gamma \mapsto x \quad (\text{Abs Reach Alloc})$$

$$x_n \dashrightarrow \gamma_m * \gamma_m \mapsto x_m \multimap x_n \leadsto x_m \quad (\text{Abs Reach Concr})$$

$$x_n \leadsto x_m * \gamma_m \mapsto x_m \Rightarrow x_n \dashrightarrow \gamma_m \quad (\text{Abs Reach Abs})$$

$$\gamma_m \mapsto x_m * x_m \leadsto x_o \Rightarrow \gamma_m \mapsto x_o \quad (\text{Abs Reach Advance})$$

Invariant and Queue Predicate

- Invariant is very **similar** to **Two-Lock variant**
- Main **difference**: instead of **the four state**, we now capture the **afforementioned properties** using abstract reachability
- **Queue predicate** is as before, but we now reflects that the **absence of locks**
- Proving rest of **HOCAP-style specification** is now **fairly straightforward**...

Definition (Invariant)

$$\begin{aligned}
 I_{LFH}(\ell_{head}, \ell_{tail}, G) &\triangleq \\
 \exists xs_v. G.\gamma_{Abst} &\mapsto_{\bullet} xs_v * \\
 \exists xs, xs_{queue}, x_{head}, x_{tail}, x_{last}. \\
 xs = xs_{queue} ++ [x_{head}] * \\
 isLL(xs) * \\
 isLast(x_{last}, xs) * \\
 projVal(xs_{queue}) = wrapSome(xs_v) * \\
 \ell_{head} &\mapsto in(x_{head}) * \\
 \ell_{tail} &\mapsto in(x_{tail}) * \\
 G.\gamma_{Head} &\mapsto x_{head} * x_{head} \dashrightarrow G.\gamma_{Tail} * \\
 G.\gamma_{Tail} &\mapsto x_{tail} * x_{tail} \dashrightarrow G.\gamma_{Last} * \\
 G.\gamma_{Last} &\mapsto x_{last}
 \end{aligned}$$

Definition (isQueue Predicate)

$$\begin{aligned}
 isQueue(v_q, G) &\triangleq \exists \ell_{queue}, \ell_{head}, \ell_{tail} \in Loc. \\
 v_q &= \ell_{queue} * \\
 \ell_{queue} &\mapsto^{\square} (\ell_{head}, \ell_{tail}) * \\
 \boxed{I_{LFH}(\ell_{head}, \ell_{tail}, G)}^{\mathcal{N}.queue}
 \end{aligned}$$