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DEPARTMENT OF COMPUTER SCIENCE



# Abstract

► in English . . . ◄



# Resumé

► in Danish . . . ◄



# Acknowledgments



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*Aarhus, May 2024.*





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# Chapter 1

## Introduction

►motivate and explain the problem to be addressed◄

►example of a citation: [1]◄ ►get your bibtex entries from **https://dblp.org/**◄



## Chapter 2

# Preliminaries

►Mention that the project uses heaplang and the program logic iris, and hence we need to know about them◄

### 2.1 HeapLang

►Write about heaplang◄ ►Talk about Syntactic sugar: i.e.  $e1 ;; e2 = (\text{lam } v, e2) e1$  where  $v$  is fresh, and  $\text{CAS } \dots$  as  $\text{Snd } (\text{CMPXHG } \dots)$ , and derived rules for them.◄

►Question: should formal definition of heaplang be in section, appendix, or reference to ILN?◄

### 2.2 The Iris Program Logic Framework

►Write about Iris◄ ►Seperation logic◄ ►Present some of the derivation rules◄ ►Present hoare Triples and Weakest Pre-condition◄ ►Persistent and Later◄ ►For Later, explain that it ties propositions to program-steps. Explain the löb induction rule, mention the intuition when  $P$  is hoare triple◄ ►Resource Algebra◄ ►Invariants◄ ►Fancy update modality and viewshift◄

### 2.3 Formalisation in Coq

►Mention that Iris is formalised in Coq◄ ►Stuff works in terms of weakest precondition◄ ►Mention that all the work done in this project has also been mechanised in Coq◄ ►give an overview of how the coq files relate to each chapter/section◄



## Chapter 3

# The Two-Lock Michael Scott Queue

I present here an implementation of the Two-lock MS-Queue in HeapLang. This implementation differs slightly from the original, presented in [1], but most changes simply reflect the differences in the two languages.

### 3.1 Preliminaries

The underlying data structure making up the queue is a singly-linked list. The linked-list will always contain at least one element, called the *sentinel* node, marking the beginning of the queue. Note that the sentinel node is itself not part of the queue, but all nodes following it are. The queue keeps a head pointer ( $\ell_{head}$ ) which always points to the sentinel, and a tail pointer ( $\ell_{tail}$ ) which points to some node in the linked list.

In my implementation, a node can be thought of as a triple  $(\ell_{i\_in}, v_i, \ell_{i\_out})$ . The location  $\ell_{i\_in}$  points to the pair  $(v_i, \ell_{i\_out})$ , where  $v_i$  is the value of the node, and  $\ell_{i\_out}$  either points to None which represents the null pointer, or to the next node in the linked list. When we say that a location  $\ell$  points to a node  $(\ell_{i\_in}, v_i, \ell_{i\_out})$ , we mean that  $\ell \mapsto \ell_{i\_in}$ . Hence, if we have two adjacent nodes  $(\ell_{i\_in}, v_i, \ell_{i\_out})$ ,  $(\ell_{i+1\_in}, v_{i+1}, \ell_{i+1\_out})$  in the linked list, then we have the following structure:  $\ell_{i\_in} \mapsto (v_i, \ell_{i\_out})$ ,  $\ell_{i\_out} \mapsto \ell_{i+1\_in}$ , and  $\ell_{i+1\_in} \mapsto v_{i+1}, \ell_{i+1\_out}$ .

The reader may wonder why there is an extra, intermediary "in" pointer, between the pairs of the linked list, and why the "out" pointer couldn't point directly to the next pair. In the original implementation [1], nodes are allocated on the heap. To simulate this in HeapLang, when creating a new node, we create a pointer to a pair making up the node. Now, in the C-like language used in the original specification, an assignment operator is available which is not present in HeapLang. So in order to mimic this behaviour, we model variables as pointers. In this way, we can model a variable  $x$  as a location  $\ell_x$ , and the value stored at  $\ell_x$  is the current value of  $x$ . This means that the variable  $\ell_{i\_out}$  (called "next" in the original) becomes a location  $\ell_{head}$ , and the value stored at the location is what head is currently assigned to. Since  $\ell_{i\_out}$  is supposed to be a variable

containing a pointer, then the value saved at that location will also be a pointer.

## 3.2 Implementation

The queue consists of 3 functions: initialize, enqueue, and dequeue, and as the name of the data structure suggests, the functions rely on two locks. To this end, we shall assume that we have some lock implementation given. In the accompanying coq mechanisation, a "spin-lock" is used, but the only part we really care about is its specification; this can be found in Example 8.38 in [►Cite Iris Lecture Notes◄](#).

### 3.2.1 initialize

initialize will first create a single node – the sentinel – marking the start of the linked list. It then creates two locks,  $H\_lock$  and  $T\_lock$ , protecting the head and tail pointers, respectively. Finally, it creates the head and tail pointers, both pointing to the sentinel. The queue is then a pointer to a structure containing the head, the tail, and the two locks.

Figure 3.1 illustrates the structure of the queue after initialisation. Note that one of the pointers is coloured blue. This represents a *persistent* pointer; a pointer that will never be updated again. All "in" pointers  $\ell_{i\_in}$ , are persistent, meaning that, once created, they will only ever point to  $(v_i, \ell_{i\_out})$ . We shall use the notation  $\ell \mapsto^\square v$  (introduced in [2]) to mean that  $\ell$  points persistently to  $v$ .

Note that in the original specification, a queue is a pointer to a 4-tuple  $(\ell_{head}, \ell_{tail}, H\_lock, T\_lock)$ . Since HeapLang doesn't support 4-tuples, we instead represent the queue as a pointer to a pair of pairs:  $((\ell_{head}, \ell_{tail}), (H\_lock, T\_lock))$ .

### 3.2.2 enqueue

To enqueue a value, we must create a new node, append it to the underlying linked-list, and swing the tail pointer to this new node. These three operations are depicted in figure 3.2.

enqueue takes as argument the value to be enqueued and creates a new node containing this value (corresponding to figure 3.2a). This creation doesn't interact with the underlying queue data-structure, hence why we don't acquire the  $T\_lock$  first. After creating the new node, we must make the last node in the linked list point to it. Since this operation interacts with the queue, we first acquire the  $T\_lock$ . Once we obtain the lock, we make the last node in the linked list point to our new node (figure 3.2b). Following this, we swing  $\ell_{tail}$  to the new last node in the linked list (figure 3.2c).

Figure 3.2 also illustrates when pointers become persistent; once the previous last node is updated to point to the newly inserted node, that pointer will never be updated again, hence becoming persistent.



### 3.2.3 dequeue

It is of course only possible to dequeue an element from the queue if the queue contains at least one element. Hence, the first thing dequeue does is check if the queue is empty. We can detect an empty queue by checking if the sentinel is the last node in the linked list. Being the last node in the linked list corresponds to having the "out" node be None. If this is the case, then the queue is empty and the code returns None. Otherwise, there is a node just after the sentinel, which is the first node of the queue. To dequeue it, we first read the associated value, and next we swing the head to it, making it the new sentinel. Finally, we return the value we read.

Since all of these operations interact with the queue, we shall only perform them after having acquired  $H\_lock$ .

Figure 3.3 illustrates running dequeue on a non-empty queue. Note that the only change is that the head pointer is swung to the next node in the linked list; the old sentinel is not deleted, it just become unreachable from the heap pointer. In this way, the linked list only ever grows.

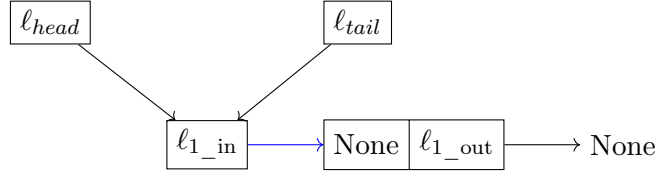


Figure 3.1: Queue after initialisation

```

1  initialize  $\triangleq$ 
2    let node = ref (None, ref (None)) in
3    let H_lock = newlock() in
4    let T_lock = newlock() in
5    ref ((ref (node), ref (node)), (H_lock, T_lock))

1  enqueue Q value  $\triangleq$ 
2    let node = ref (Some value, ref (None)) in
3    acquire(snd(snd(!Q)));
4    snd(! (snd(fst(!Q))))  $\leftarrow$  node;
5    snd(fst(!Q))  $\leftarrow$  node;
6    release(snd(snd(!Q)))

1  dequeue Q  $\triangleq$ 
2    acquire(fst(snd(!Q)));
3    let node = !(fst(fst(!Q))) in
4    let new_head = !(snd(!node)) in
5    if new_head = None then
6      release(fst(snd(!Q)));
7      None
8    else
9      let value = fst(!new_head) in

```

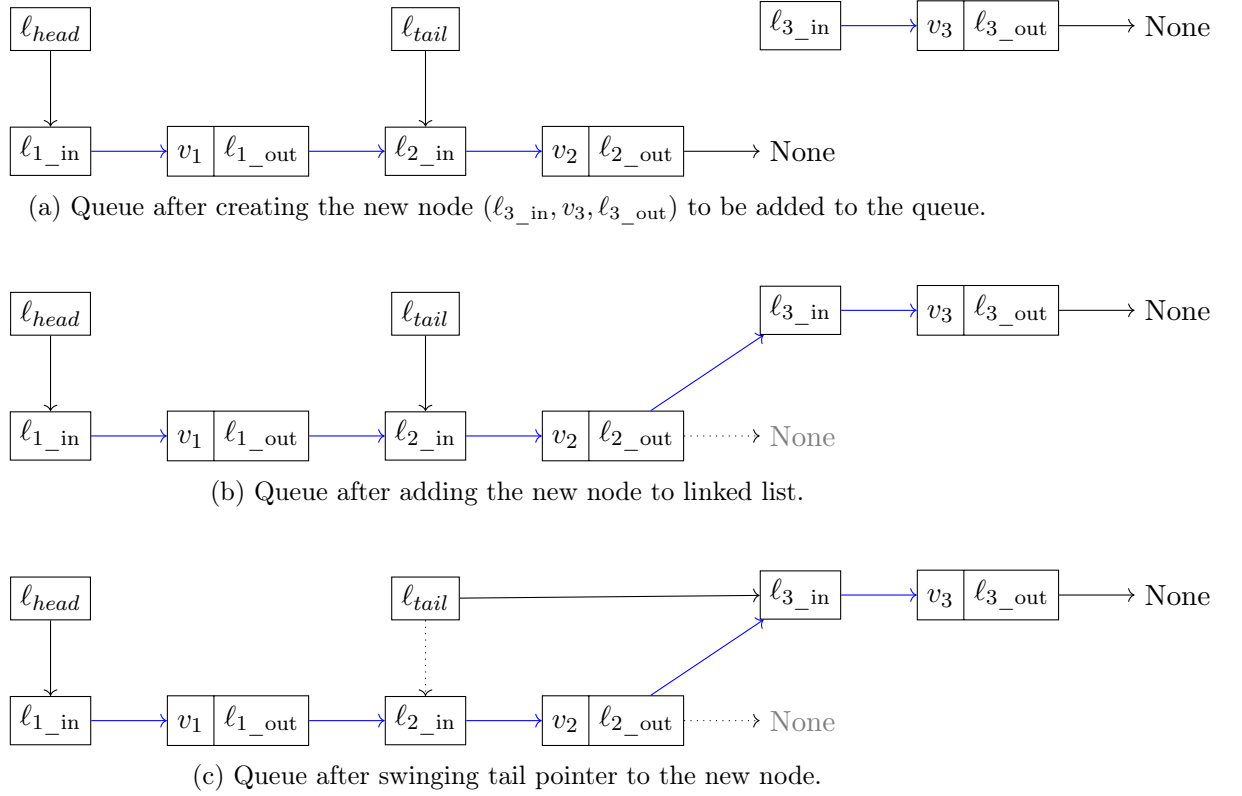


Figure 3.2: Enqueuing an element to a queue with one element.

```

10   fst(fst(!Q)) ← new_head;
11   release(fst(snd(!Q)));
12   value

```

### 3.3 Sequential Specification

Let us first prove a specification for the two-lock michael scott queue in the simple case where we don't allow for concurrency. In this case, we know that only a single thread will interact with the queue at any given point in a sequential manner. This means that we give a specification that tracks the exact contents of the queue. To this end, we shall define the abstract state of the queue, denoted  $xs_v$  as a list of HeapLang values. I.e.  $xs_v : List\ Val$ . We adopt the convention that enqueueing an element is done by adding it to the front of the list, and dequeueing removes the last element of the list (if such an element exists). The reason for this choice is purely technical.

Since the queue uses two locks, we will get two ghost names; one for each lock. For this specification, these are the only two ghost names we will need. However, for the later specifications, we will use more resource algebra, and will need more ghost names. Thus, to ease notation, we shall define the type "*SeqQghostnames*" whose purpose is to keep track of the ghost names used for a specific queue. Since we only have two ghost names for this specification, elements of *SeqQghostnames*

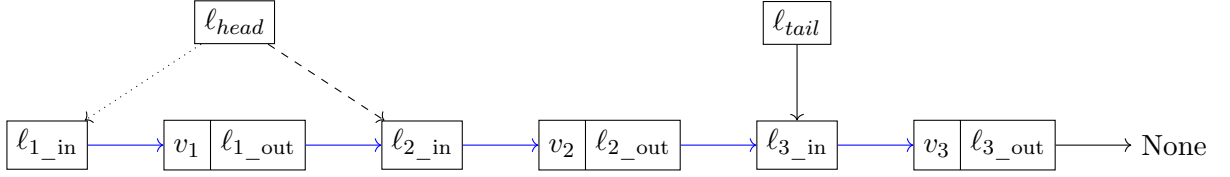


Figure 3.3: Dequeueing an element ( $v_2$ ) from a queue with two elements ( $v_2, v_3$ ). The dotted line represents the state before the dequeue, and the dashed line is the state after dequeueing.

will simply be pairs. For an element  $Q_\gamma \in SeqQnames$ , the first element of the pair, written  $Q_\gamma.\gamma_{Hlock}$ , will contain the ghost name for the head lock, and the second element,  $Q_\gamma.\gamma_{Tlock}$ , the ghost name for the tail lock.

The sequential specification we wish to prove is the following:

**Lemma 1** (Two-Lock M&S-Queue Sequential Specification).

$$\begin{aligned}
& \exists \text{is\_queue\_seq} : Val \rightarrow List\ Val \rightarrow SeqQnames \rightarrow Prop. \\
& \{True\} \text{ initialize } () \{v_q. \exists Q_\gamma. \text{is\_queue\_seq } v_q \ \square \ Q_\gamma\} \\
& \wedge \quad \forall v_q, v, xs_v, Q_\gamma. \{\text{is\_queue\_seq } v_q \ xs_v \ Q_\gamma\} \text{ enqueue } v_q \ v \{w. \text{is\_queue\_seq } v_q \ (v :: xs_v) \ Q_\gamma\} \\
& \wedge \quad \forall v_q, xs_v, Q_\gamma. \{\text{is\_queue\_seq } v_q \ xs_v \ Q_\gamma\} \\
& \quad \text{dequeue } v_q \\
& \quad \left\{ v. \begin{aligned} & (xs_v = [] * v = \text{None} * \text{is\_queue\_seq } v_q \ xs_v \ Q_\gamma) \vee \\ & (\exists x_v, xs'_v. xs_v = xs'_v ++ [x_v] * v = \text{Some } x_v * \text{is\_queue\_seq } v_q \ xs'_v \ Q_\gamma) \end{aligned} \right\}
\end{aligned}$$

The predicate  $\text{is\_queue\_seq } v_q \ xs_v \ Q_\gamma$  captures that the value  $v_q$  is a queue, whose content matches that of our abstract representation  $xs_v$ , and the queue uses the ghost names described by  $Q_\gamma$ . Note that the  $\text{is\_queue\_seq}$  predicate is not required to be persistent, hence it cannot be duplicated and given to multiple threads. This is the sense in which this specification is sequential.

## 3.4 Proving the Sequential Specification

### 3.4.1 The $\text{is\_queue\_seq}$ Predicate

To prove the specification we must give a specific  $\text{is\_queue\_seq}$  predicate. To help guide us in designing this, we give the following observations about the behaviour of the implementation.

1. Head always points to the first node in the queue.
2. Tail always points to either the last or second last node in the queue.
3. All but the last pointer in the queue (the pointer to None) never change.

Observation 2 captures the fact that, while enqueueing, a new node is first added to the linked list, and then later the tail is updated to point to the newly added node. Since only one thread can enqueue a node at a time (due to the

lock), then the tail will only ever point to the last or second last due to the above. However, in a sequential setting, the tail will always appear to point to the last node, as no one can inspect the queue while the tail points to the second last.

Insight 3 means that we can mark all pointers in the queue (except the pointer to the null node) as persistent. This is technically not needed in the sequential case, but we will incorporate it now, as we will need it in the concurrent setting.

**Definition 3.4.1** (Two-Lock M&S-Queue - `is_queue_seq` Predicate).

$$\begin{aligned}
\text{is\_queue\_seq } v_q \ xs_v \ Q_\gamma &\triangleq \exists \ell_{\text{queue}}, \ell_{\text{head}}, \ell_{\text{tail}} \in \text{Loc}. \exists h_{\text{lock}}, t_{\text{lock}} \in \text{Val}. \\
&v_q = \ell_{\text{queue}} * \ell_{\text{queue}} \mapsto^\square ((\ell_{\text{head}}, \ell_{\text{tail}}), (h_{\text{lock}}, t_{\text{lock}})) * \\
&\exists xs_{\text{queue}} \in \text{List}(\text{Loc} \times \text{Val} \times \text{Loc}). \exists x_{\text{head}}, x_{\text{tail}} \in (\text{Loc} \times \text{Val} \times \text{Loc}). \\
&\text{proj\_val } xs_{\text{queue}} = \text{wrap\_some } xs_v * \\
&\text{isLL}(xs_{\text{queue}} ++ [x_{\text{head}}]) * \\
&\ell_{\text{head}} \mapsto (\text{in } x_{\text{head}}) * \\
&\ell_{\text{tail}} \mapsto (\text{in } x_{\text{tail}}) * \text{isLast } x_{\text{tail}} (xs_{\text{queue}} ++ [x_{\text{head}}]) * \\
&\text{isLock } Q_\gamma \cdot \gamma_{H\text{lock}} \ h_{\text{lock}} \ \text{True} * \\
&\text{isLock } Q_\gamma \cdot \gamma_{T\text{lock}} \ t_{\text{lock}} \ \text{True}.
\end{aligned}$$

This `is_queue_seq` predicate states that the value  $v_q$  is a location, which persistently points to the structure containing the head, the tail, and the two locks. It also connects the abstract state  $xs_v$  with the concrete state by stating that if you strip away the locations in  $xs_{\text{queue}}$  (achieved by `proj_val`) and wrap the values in the abstract state  $xs_v$  in `Some` (achieved by `wrap_some`), then the lists become equal.

Next, the predicate specifies the concrete state. There is some head node  $x_{\text{head}}$ , which the head points to. This head node and the nodes in  $xs_{\text{queue}}$  form the underlying linked list (specified using the `isLL` predicate below). There is also a tail node, which is the last node in the linked list, and the tail points to this node. The proposition `isLast x xs` simply asserts the existence of some  $xs'$ , so that  $xs = x :: xs'$ .

Finally, we have the `isLock` predicate for our two locks. Since we are in a sequential setting, then the locks are superfluous, hence they simply protect `True`.

The `isLL` predicate essentially creates the structure seen in the examples of section 3.2. It is defined in two steps. Firstly, we create all the persistent pointers in the linked list using the `isLL_chain` predicate. Note that this in effect makes `isLL_chain xs` persistent for all  $xs$ .

**Definition 3.4.2** (Linked List Chain Predicate).

$$\begin{aligned}
\text{isLL\_chain } [] &\equiv \text{True} \\
\text{isLL\_chain } [x] &\equiv \text{in } x \mapsto^\square (\text{val } x, \text{out } x) \\
\text{isLL\_chain } x :: x' :: xs &\equiv \text{in } x \mapsto^\square (\text{val } x, \text{out } x) * \text{out } x' \mapsto^\square \text{in } x * \text{isLL\_chain } x' :: xs
\end{aligned}$$

Then, to define `isLL`, we add that the last node in the linked list points to `None`.

**Definition 3.4.3** (Linked List Predicate).

$$\begin{aligned} \text{isLL } [] &\equiv \text{True} \\ \text{isLL } x :: xs &\equiv \text{out } x \mapsto \text{None} * \text{isLL\_chain } x :: xs \end{aligned}$$

For instance, if we wanted to capture the linked list in figure 3.2c, we would use the list  $xs = [(\ell_{3\_in}, v_3, \ell_{3\_out}); (\ell_{2\_in}, v_2, \ell_{2\_out}); (\ell_{1\_in}, v_1, \ell_{1\_out})]$ .  $\text{isLL } xs$  will expand to  $\ell_{3\_out} \mapsto \text{None} * \text{isLL\_chain } xs$ , and  $\text{isLL\_chain } xs$  expands to

$$\begin{aligned} \ell_{3\_in} &\mapsto^\square (x_3, \ell_{3\_out}) * \ell_{2\_out} \mapsto^\square \ell_{3\_in} * \\ \ell_{2\_in} &\mapsto^\square (x_2, \ell_{2\_out}) * \ell_{1\_out} \mapsto^\square \ell_{2\_in} * \\ \ell_{1\_in} &\mapsto^\square (x_1, \ell_{1\_out}) \end{aligned}$$

Note how this matches the structure of the linked list in figure 3.2c.

### 3.4.2 Proof outline

#### Initialise

Proving the initialise spec amounts to stepping through the code, giving us the required resources, and then using these to create an instance of  $\text{is\_queue\_seq}$  with the obtained resources. To begin with, we step through the lines creating the first node  $x_1$ , giving us locations  $\ell_{1\_in}$ ,  $\ell_{1\_out}$  with  $\ell_{1\_out} \mapsto \text{None}$  and  $\ell_{1\_in} \mapsto (\text{None}, \ell_{1\_out})$ . We can then update the latter points-to predicate to become persistent, giving us  $\ell_{1\_in} \mapsto^\square (\text{None}, \ell_{1\_out})$ . We then step to the creation of the two locks, where we shall use the newlock specification asserting that the locks should protect **True**. This gives us two ghost names,  $\gamma_{Tlock}$ ,  $\gamma_{Tlock}$ , which we will collect in a  $\text{SeqQgnames}$  pair,  $Q_\gamma$ . Next, we step through the allocations of the head, tail, and queue, which gives us locations  $\ell_{head}$ ,  $\ell_{tail}$ ,  $\ell_{queue}$ , such that both  $\ell_{head}$  and  $\ell_{tail}$  point to node  $x_1$ , and such that  $\ell_{queue}$  points to the structure containing the head, tail, and two locks. This last points to predicate we update to become persistent. With this, we now have all the resources needed to prove the post-condition:  $\exists Q_\gamma. \text{is\_queue\_seq } \ell_{queue} Q_\gamma$ . Proving this follows by a sequence of framing away the resources we obtained and instantiating existentials with the values we got above. Most noteworthy, we pick the empty list for  $xs_{queue}$ , and node  $x_1$  for  $x_{head}$  and  $x_{tail}$ .

#### Enqueue

►add line numbers to code, and refer to them in proof◄ For enqueue, we get in our pre-condition  $\text{is\_queue\_seq } v_q xs_v Q_\gamma$ , and we wish to that, if we run  $\text{enqueue } v_q v$ , then we will get  $\text{is\_queue\_seq } v_q (v :: xs_v) Q_\gamma$ . The proposition  $\text{is\_queue\_seq } v_q xs_v Q_\gamma$  gives us all the resources we will need to step through the code. Firstly, we create a new node, node  $x_{new}$ , with  $\text{val } x_{new} = v$ . We then have to acquire the lock, which will just give us **True**.

The next line adds node  $x_{new}$  to the linked list, by first finding the tail, from the queue pointer  $\ell_{queue}$ , and then finding the node that the tail points to, denoted  $x_{tail}$ , and finally writing updating the out location of  $x_{tail}$  to point to  $x_{new}$ .

The resources needed to do this are all described in  $\text{is\_queue\_seq } v_q \text{ } xs_v \text{ } Q_\gamma$ . Firstly, it tells us that  $\ell_{\text{queue}}$  points to the structure containing  $\ell_{\text{tail}}$ . Secondly, it tells us that  $\ell_{\text{tail}}$  points to  $x_{\text{tail}}$ , which is the last node in the linked list ( $xs_{\text{queue}} ++ [x_{\text{head}}]$ ). Thirdly, since we know that  $x_{\text{tail}}$  is the last node in the linked list, then by the  $\text{isLL}$  predicate, we know that  $x_{\text{tail}}$  points to  $\text{None}$  and that it has the node-like structure described by  $\text{isLL\_chain}$ . This is all we need to step through the line, adding  $x_{\text{new}}$  to the linked list. After performing the write, we then get that  $x_{\text{tail}}$  points to  $x_{\text{new}}$ , instead of  $\text{None}$ . We make this points-to predicate persistent.

The next line swings the tail to  $x_{\text{new}}$ . As describe above, we already know that  $\ell_{\text{tail}}$  points to  $x_{\text{tail}}$ , so we have the required resources to perform the write. Afterwards, we get that  $\ell_{\text{tail}}$  points to  $x_{\text{new}}$ .

Finally, we release the lock using the release specification (and we simply give back  $\text{True}$ ), and the only thing left is to prove the postcondition:  $\text{is\_queue\_seq } v_q \text{ } (v :: xs_v) \text{ } Q_\gamma$ . For the existentials, we shall pick the ones we got from the precondition, with the exception for  $xs_{\text{queue}}$  and  $x_{\text{tail}}$ . For  $xs_{\text{queue}}$ , we shall use the same  $xs_{\text{queue}}$  we got from the precondition, but with  $xs_{\text{new}}$  cons'ed to it, and for  $x_{\text{tail}}$ , we chose the new tail node:  $x_{\text{new}}$ . With these choices, proving  $\text{is\_queue\_seq } v_q \text{ } (v :: xs_v) \text{ } Q_\gamma$  is fairly straightforward.

## Dequeue

For dequeue  $v_q$ , our precondition is  $\text{is\_queue\_seq } v_q \text{ } xs_v \text{ } Q_\gamma$ , and our post condition states that either the queue is empty, or there is a tail element which is returned by the function, and removed from the queue.

Stepping through the function, we first do the superfluous acquire. Next, we get the head node  $x_{\text{head}}$  through the queue pointer  $\ell_{\text{queue}}$ . As described above for Enqueue, we get the resource to do this through  $\text{is\_queue\_seq } v_q \text{ } xs_v \text{ } Q_\gamma$ . The  $\text{is\_queue\_seq}$  predicate also tells us that  $x_{\text{head}}$  is a node in the linked list (described by the  $\text{isLL}$  predicate), hence we can step through the code in the next line, which finds the node that  $x_{\text{head}}$  is pointing to. Now, depending on whether or not the queue is empty,  $x_{\text{head}}$  either points to  $\text{None}$ , or some node  $x_{\text{head\_next}}$ . Thus, we shall perform a case analysis on  $xs_{\text{queue}}$ .

**$xs_{\text{queue}}$  is empty:** In this case, we will have that  $\text{isLL}[x_{\text{head}}]$ , which tells us that  $x_{\text{head}}$  points to  $\text{None}$ . Hence, the "then" branch of the "if" will be taken. This branch simply releases the lock and returns  $\text{None}$ . In this case, we prove the first disjunction in the post-condition. Since  $xs_v$  is reflected in  $xs_{\text{queue}}$ , then we will be able to conclude that  $xs_v$  is empty, and since we haven't modified the queue, we can create  $\text{is\_queue\_seq } v_q \text{ } xs_v \text{ } Q_\gamma$  using the same resources we got from the pre-condition.

**$xs_{\text{queue}}$  is not empty:** In this case, we can conclude that there must be some node  $x_{\text{head\_next}}$ , which is the first node in  $xs_{\text{queue}}$ . I.e.  $xs_{\text{queue}} = xs'_{\text{queue}} ++ [x_{\text{head\_next}}]$ . We can thus use the  $\text{isLL}$  predicate to conclude that  $x_{\text{head}}$  must point to  $x_{\text{head\_next}}$ . Hence the else branch will be taken. Since  $x_{\text{head\_next}}$  is part of the linked list, then  $\text{isLL}$  tells us it has the node-like structure, allowing us to extract its value in the first line of the else branch.

In the next line, we make the head pointer,  $\ell_{\text{head}}$  point to  $x_{\text{head\_next}}$ , and we

have the resource to do this through  $\text{is\_queue\_seq } v_q \ xs_v \ Q_\gamma$ .

Finally, we release the lock and return the value we got from  $x_{\text{head\_next}}$ . We must now prove the post-condition, and this time we prove the second disjunct. Since  $xs_v$  is reflected in  $xs_{\text{queue}}$ , then it must also be the case that  $xs_v$  is non-empty, and it has a first element,  $x_v$ , which is related to the first element of  $xs_{\text{queue}}$ , i.e.  $x_{\text{head\_next}}$ . This allows us to conclude that the returned value ( $\text{val } x_{\text{head\_next}}$ ) is exactly  $x_v$ , but wrapped in a `Some`, as we had to prove. Finally, we must prove  $\text{is\_queue\_seq } v_q \ xs'_v \ Q_\gamma$ , where  $xs'_v$  is  $xs_v$  but with  $x_v$  removed. For the existentials, we pick the same values we got from the precondition, with the exception of  $xs_{\text{queue}}$  and  $x_{\text{head}}$ . For  $xs_{\text{queue}}$  we pick the same  $xs_{\text{queue}}$  we got from the precondition, but with the first element,  $x_{\text{head\_next}}$  removed. By doing this,  $xs_{\text{queue}}$  will be reflexed in  $xs'_v$ . For  $x_{\text{head}}$ , we pick the new head, which we have obtained that  $\ell_{\text{head}}$  points to:  $x_{\text{head\_next}}$ . With these choices, we can prove the predicate.

### 3.5 Concurrent Specification

For the concurrent specification, we will need the predicate capturing the queue (here denoted  $\text{is\_queue\_conc}$ ) to be duplicable. To achieve this, we shall initially give up on tracking the abstract state of the queue, and instead add a predicate  $\Phi$ , which we will ensure holds for all elements of the queue. In this way, when dequeuing, we at least know that if we get some value, then  $\Phi$  holds of this value. The specification we wish to prove is as follows.

**Lemma 2** (Two-Lock M&S-Queue Concurrent Specification).

$\exists \text{is\_queue\_conc} : (Val \rightarrow Prop) \rightarrow Val \rightarrow ConcQnames \rightarrow Prop.$

$\forall \Phi : Val \rightarrow Prop.$

$$\begin{aligned} & \forall v_q, Q_\gamma. \text{is\_queue\_conc } \Phi \ v_q \ Q_\gamma \implies \Box \text{is\_queue\_conc } \Phi \ v_q \ Q_\gamma \\ & \wedge \quad \{True\} \text{ initialize } () \{v_q. \exists Q_\gamma. \text{is\_queue\_conc } \Phi \ v_q \ Q_\gamma\} \\ & \wedge \quad \forall v_q, v, Q_\gamma. \{\text{is\_queue\_conc } \Phi \ v_q \ Q_\gamma * \Phi \ v\} \text{ enqueue } v_q \ v \{v. True\} \\ & \wedge \quad \forall v_q, Q_\gamma. \{\text{is\_queue\_conc } \Phi \ v_q \ Q_\gamma\} \text{ dequeue } v_q \{v.v = \text{None} \vee (\exists x_v. v = \text{Some } x_v * \Phi \ x_v)\} \end{aligned}$$

Note that the type of the collection of ghost names here is  $ConcQnames$ , as we will require more ghost names than before. The new ghost names are used for "tokens" which are introduced in the following section.

### 3.6 Proving the Concurrent Specification

#### 3.6.1 The $\text{is\_queue\_conc}$ Predicate

As we did for the sequential specification, we note here some useful observations about the implementation.

1. Nodes in the linked list are never deleted. Hence, the linked list only ever grows.

2. The tail can lag one node behind Head.
3. At any given time, the queue is in one of four states:
  - (a) No threads are interacting with the queue (**Static**)
  - (b) A thread is enqueueing (**Enqueue**)
  - (c) A thread is dequeuing (**Dequeue**)
  - (d) A thread is enqueueing and a thread is dequeuing (**Both**)

Observation 2 might seem a little surprising, and indeed it stands in contrast to property 5 in [1], which states that the tail never lags behind head. I also didn't realise this possibility until a proof attempt using a model that "forgot" old nodes lead to an unprovable case (see section 3.6.2). The situation can occur when the queue is empty, and a thread performs an incomplete enqueue; it attaches the new node to the end, but before it can swing the tail to this new node, another thread performs a dequeue, which dequeues this new node, swinging the head to it. Now the tail is lagging a node behind the head.

It is not possible for the tail to point more than one node behind the head, as in order for this to happen, more nodes must be enqueued, but this can't happen before the current enqueue finishes, which will update the tail and bring it up to speed with the head.

Fortunately, this isn't an issue for safety, but a consequence of this possibility is that when modelling the queue, we must remember at least one "old" node (i.e. a dequeued node), as the tail might be pointing to this node. For the sake of simplicity in the model, the choice is made to remember an arbitrary amount of old nodes, which is represented by the list  $xs_{old}$ .

Observation 3 is a simple consequence of the implementation using two locks.

Since we want `is_queue_conc` to be persistent, then we cannot directly state the points-to predicates as we did in the sequential case. However, we will still need all the same resources to be able to prove the specification. The solution is to have an invariant which describes the concrete state of the queue. In the proofs, when we need access to some resource, we shall then access it by opening the invariant. We now present the invariant and explain it afterwards.



**Definition 3.6.1** (Two-Lock M&S-Queue Concurrent Invariant).

$$\begin{aligned}
& \text{queue\_invariant } \Phi \ell_{\text{head}} \ell_{\text{tail}} Q_\gamma \triangleq \\
& \exists x_{s_v}. \text{All } x_{s_v} \Phi * \tag{abstract state} \\
& \exists x_s, x_{s_{\text{queue}}}, x_{s_{\text{old}}}, x_{\text{head}}, x_{\text{tail}}. \tag{concrete state} \\
& x_s = x_{s_{\text{queue}}} ++ [x_{\text{head}}] ++ x_{s_{\text{old}}} * \\
& \text{isLL } x_s * \\
& \text{proj\_val } x_{s_{\text{queue}}} = \text{wrap\_some } x_{s_v} * \\
& ( \\
& \quad \ell_{\text{head}} \mapsto (\text{in } x_{\text{head}}) * \ell_{\text{tail}} \mapsto (\text{in } x_{\text{tail}}) * \text{isLast } x_{\text{tail}} x_s * \tag{Static} \\
& \quad \text{ToknE } Q_\gamma * \text{ToknD } Q_\gamma * \text{TokUpdated } Q_\gamma \\
& \vee \\
& \quad \ell_{\text{head}} \mapsto (\text{in } x_{\text{head}}) * \ell_{\text{tail}} \mapsto \frac{1}{2} (\text{in } x_{\text{tail}}) * \tag{Enqueue} \\
& \quad (\text{isLast } x_{\text{tail}} x_s * \text{TokBefore } Q_\gamma \vee \text{isSndLast } x_{\text{tail}} x_s * \text{TokBefore } Q_\gamma) * \\
& \quad \text{TokE } Q_\gamma * \text{ToknD } Q_\gamma \\
& \vee \\
& \quad \ell_{\text{head}} \mapsto \frac{1}{2} (\text{in } x_{\text{head}}) * \ell_{\text{tail}} \mapsto (\text{in } x_{\text{tail}}) * \text{isLast } x_{\text{tail}} x_s * \tag{Dequeue} \\
& \quad \text{ToknE } Q_\gamma * \text{TokD } Q_\gamma * \text{TokUpdated } Q_\gamma \\
& \vee \\
& \quad \ell_{\text{head}} \mapsto \frac{1}{2} (\text{in } x_{\text{head}}) * \ell_{\text{tail}} \mapsto \frac{1}{2} (\text{in } x_{\text{tail}}) * \tag{Both} \\
& \quad (\text{isLast } x_{\text{tail}} x_s * \text{TokBefore } Q_\gamma \vee \text{isSndLast } x_{\text{tail}} x_s * \text{TokBefore } Q_\gamma) * \\
& \quad \text{TokE } Q_\gamma * \text{TokD } Q_\gamma \\
& )
\end{aligned}$$

In contrast to the sequential specification, the abstract state is now existentially quantified, hence the exact contents of the queue are not tracked. Instead, we have added the proposition *All*  $x_{s_v} \Phi$ , which states that all values in  $x_{s_v}$  (i.e. the values currently in the queue) satisfy the predicate  $\Phi$ . This will allow us to conclude that dequeued values satisfy  $\Phi$ .

The concrete state of the queue is still reflected in the abstract state through projecting out the values of the nodes (*proj\_val*), and wrapping the values in the queue in *Some* (*wrap\_some*). Another difference is that we now also keep track of an arbitrary number of "old" nodes; nodes that are behind the head node,  $x_{\text{head}}$ . As discussed above, this inclusion is due to observation 2.

As before, we also assert that the concrete state forms a linked list, as described by the *isLL* predicate.

The final part of the invariant describes the four possible states of the queue, as described in 3. Since the resources used by the queue are inside an invariant, and enqueueing/dequeueing threads need to access the resources of the queue multiple times, then we will have to open and close the invariant multiple times. Each time we open the invariant, the existentially quantified variables will not be the same as those from early accesses of the invariant (as they are existentially

quantified). Thus, the threads must be able to "match up" variables from previous accesses to later accesses. The way we shall achieve this is by allowing threads to keep a *fraction* of the points-to predicate that it is using. For instance, an enqueueing thread will have to access the points-to predicate concerning  $\ell_{tail}$  multiple times, and in between accesses of the invariant, it can get to keep half of the points-to predicate. Thus, when it opens the invariant later, it will have  $\ell_{tail} \mapsto^{\frac{1}{2}}$  in  $x_{tail}$  from an earlier access, and it will obtain the existence of some new  $x'_{tail}$ , such that  $\ell_{tail} \mapsto^{\frac{1}{2}}$  in  $x'_{tail}$ . Combining the two points-to predicates allows us to conclude that in  $x_{tail} =$  in  $x'_{tail}$ . In this way, we can match up variables from earlier accesses to variables in later accesses.

In the **Static** state where no thread is interacting with the queue, the queue owns all of the points-to predicates concerning the head and tail.

In the **Enqueue** state, the enqueueing thread owns half of the tail pointer, and we distinguish between two cases, as discussed in 2: either the enqueueing thread has yet to add the new node to the linked list and  $x_{tail}$  is still the last node, or the new node has been added, but the tail pointer hasn't been updated, meaning that  $x_{tail}$  is the second last node (*isSndLast* is defined similarly to *isLast*).

In the **Dequeue** state, the dequeueing thread owns half of the head pointer, and the tail is as in the **Static** state.

Finally, the **Both** state is essentially a combination of the **Enqueue** and **Dequeue** states.

To track which state the queue is in, we use *tokens*. Tokens are defined using the exclusive resource algebra on the singleton set:  $\text{EX}()$ . This resource algebra only has one valid element, and combining two elements will give the non-valid element  $\perp$ . Thus, if we own a particular token, then, upon opening the invariant, we can rule out certain states simply because they mention the token we own. We will use several tokens, each of which is the valid element of their own instance of  $\text{EX}()$ . Different instances are distinguish between using ghost names. Hence, each token will be represented by a ghost name. As we did for the sequential specification, we group these ghost names into a tuple  $Q_\gamma$ , and write, for instance  $\text{TokE } Q_\gamma$  to refer to the valid element of a particular instance. We proceed to explain the meaning of each of the tokens used in the invariant.

- $\text{ToknE } Q_\gamma$  represents that no threads are enqueueing.
- $\text{TokE } Q_\gamma$  represents that a thread is enqueueing.
- $\text{ToknD } Q_\gamma$  represents that no threads are dequeueing.
- $\text{TokD } Q_\gamma$  represents that a thread is dequeueing.
- $\text{TokBefore } Q_\gamma$  represents that an enqueueing thread has not yet added the new node to the linked list.
- $\text{TokBefore } Q_\gamma$  represents that an enqueueing thread has added the new node to the linked list, but not yet swung the tail.
- $\text{TokUpdated } Q_\gamma$  is defined as  $\text{TokBefore } Q_\gamma * \text{TokBefore } Q_\gamma$ , and represents that the queue is up to date.

**Note:** The concurrent specification for the two-lock Michael Scott Queue *can* be proven using the queue invariant 3.6.1, and the proof outline below will also be using this. However, a simpler (but arguably less intuitive) queue invariant was discovered. This simpler invariant is equivalent to 3.6.1 and has the benefit of being easier to work with in the mechanised proofs. Thus, in the mechanised proofs, the simpler variant is used. The simpler variant can be found in the appendix **►add appendix◄**.

With this, we can now give our definition of `is_queue_conc`. In the below, we let  $\mathcal{N}$  be some namespace.

**Definition 3.6.2** (Two-Lock M&S-Queue - `is_queue_conc` Predicate).

$$\begin{aligned} \text{is\_queue\_conc } \Psi \ v_q \ Q_\gamma &\triangleq \exists \ell_{\text{queue}}, \ell_{\text{head}}, \ell_{\text{tail}} \in \text{Loc}. \exists h_{\text{lock}}, t_{\text{lock}} \in \text{Val}. \\ &v_q = \ell_{\text{queue}} * \ell_{\text{queue}} \mapsto^\square ((\ell_{\text{head}}, \ell_{\text{tail}}), (h_{\text{lock}}, t_{\text{lock}})) * \\ &\boxed{\text{queue\_invariant} \Phi \ell_{\text{head}} \ell_{\text{tail}} Q_\gamma}^{\mathcal{N}.\text{queue}} * \\ &\text{isLock } Q_\gamma.\gamma_{H\text{lock}} \ h_{\text{lock}} \ (\text{TokD } Q_\gamma) * \\ &\text{isLock } Q_\gamma.\gamma_{T\text{lock}} \ t_{\text{lock}} \ (\text{TokE } Q_\gamma). \end{aligned}$$

In contrast to the sequential specification, the locks now protect `TokE`  $Q_\gamma$  and `TokD`  $Q_\gamma$ . The idea is that, when an enqueueing thread obtains  $t_{\text{lock}}$ , they will obtain the `TokE`  $Q_\gamma$  token, which allows them to conclude that the queue state is either **Static** or **Dequeue**. Similarly for a dequeueing thread. We now proceed to prove the specification using the above `is_queue_conc` predicate.

### 3.6.2 Proof outline

Firstly, we must show that `is_queue_conc` is persistent. This however follows from the fact that invariants are persistent, the `isLock` predicates are persistent, persistent points-to predicates are persistent, and persistency is preserved by `*` and quantifications (rules: `persistently-sep`, `persistently- $\wedge$` , `persistently- $\exists$` ).

The proofs structure for the specifications are largely similar to the sequential counterparts. The major difference is that we don't have access to the resources all the time; we must get them from the invariant. Further we also have to keep track of which state we are in. For the proof outlines below, these points will be the main focus.

#### Initialise

We first step through the first line which gives us the sentinel node of the linked list. Next, we must create the two locks. To create the two tokens that the locks must protect, we use the `ghost-alloc` rule twice, which gives us two ghost names, one for each of the tokens. We put the ghost names into a tuple  $Q_\gamma$ , and write `TokE`  $Q_\gamma$  and `TokD`  $Q_\gamma$  for the two ghost resources created by the `ghost-alloc` rule. We then create the locks, giving up the two tokens. Following this, we create the  $\ell_{\text{queue}}$ ,  $\ell_{\text{head}}$ , and  $\ell_{\text{tail}}$  pointers. All that remains then is to prove the postcondition; the `is_queue_conc` predicate. The persistent points-to predicate we got when we stepped through the code, and

the `isLock` predicates we got when we created the locks. So all that remains is the invariant. We create the *queue\_invariant* in the **Static** state, most of which is analogous to the sequential specification. However, we will also need to supply the tokens required by the **Static** state. Thus, we allocate the four tokens `ToknE`  $Q_\gamma$ , `ToknD`  $Q_\gamma$ , `TokBefore`  $Q_\gamma$ , and `TokBefore`  $Q_\gamma$  in the same way we allocated `TokE`  $Q_\gamma$  and `TokD`  $Q_\gamma$ . We combine `TokBefore`  $Q_\gamma$  and `TokBefore`  $Q_\gamma$  to get `TokUpdated`  $Q_\gamma$ , and we now have all the tokens we need to create the *queue\_invariant* in the **Static** state. To create the invariant from *queue\_invariant*, we use the `Inv-alloc` rule (FUP).

## Enqueue

We first step through the first line which gives us the new node  $x_{new}$ . We then acquire the tail lock  $t_{lock}$ , giving us `TokE`  $Q_\gamma$ . In the next line we must dereference the tail pointer, in order to get the tail node  $x_{tail}$ . This information, however, is inside the invariant. Invariant can only be opened if the expression being considered is atomic, but we can always make it atomic using the `bind` rule. Thus, we open the invariant, and since we have `TokE`  $Q_\gamma$ , we know that the queue is in state **Static** or **Dequeue**. In any case, we get that  $\ell_{tail} \mapsto \text{in } x_{tail}$ , and that  $x_{tail}$  is the last node in the linked list. We can then dereference  $\ell_{tail}$ , and must then close the invariant. We split up the points-to predicate  $\ell_{tail} \mapsto \text{in } x_{tail}$  in two, which leaves us with two of  $\ell_{tail} \mapsto^{\frac{1}{2}} \text{in } x_{tail}$ . We keep one of them, and use the other to close the invariant in the before case of state **Enqueue** or **Both**, depending on which state we opened the invariant into. By doing this, we give up `TokE`  $Q_\gamma$ , but we gain `ToknE`  $Q_\gamma$  and `TokBefore`  $Q_\gamma$ . We can now step to the point where  $x_{tail}$ 's out is updated to point to  $x_{new}$ . However, the points-to predicate concerning out  $x_{tail}$  isn't persistent, and is hence inside the invariant. We thus have to open the invariant again. Since we have `ToknE`  $Q_\gamma$  and `TokBefore`  $Q_\gamma$ , we know that we are in the before case of either state **Enqueue** or **Both**. We now get a different tail node,  $x'_{tail}$ , with  $\ell_{tail} \mapsto^{\frac{1}{2}} \text{in } x'_{tail}$ . However, since we kept  $\ell_{tail} \mapsto^{\frac{1}{2}} \text{in } x_{tail}$ , we can combine these, allowing us to conclude that  $\text{in } x_{tail} = \text{in } x'_{tail}$ . Due to the structure of nodes (as described by `isLL`), we can further conclude that  $x_{tail} = x'_{tail}$ . This now gives us that  $\text{out } x_{tail} \mapsto \text{None}$ , and we can perform the store, adding  $x_{new}$  to the linked list. We now wish to close the invariant in the after case of either state **Enqueue** or **Both**, giving up `TokBefore`  $Q_\gamma$ , and obtaining `TokBefore`  $Q_\gamma$ . When closing the invariant we shall pick as the abstract state  $v :: xs_v$ , where  $v$  is the enqueued value, and  $xs_v$  the abstract state we got when we opened the invariant. Note that in the pre-condition of the hoare-triple, we have  $\Phi v$ , hence we will be able to conclude  $All(v :: xs_v) \Phi$ . For the concrete state, we pick  $x_{new} :: xs$ , where  $xs$  is the concrete state we got when we opened the invariant. With these choices, we can close the invariant.

The next line swings the tail pointer to  $x_{new}$ . But to perform this store, we must first know that  $\ell_{tail}$  points to something. This resource is inside the invariant, so we must open the invariant one last time. Due to our tokens, we know that we are in the after case of state **Enqueue** or **Both**. This time, we get some  $x''_{tail}$ , with  $\ell_{tail} \mapsto^{\frac{1}{2}} x''_{tail}$ , but we also get that  $x''_{tail}$  is only the second last node

in the linked list. Hence there is some other node  $x'_{new}$ , which is the last node, with  $x''_{tail}$  pointing to it. As before, we use the points to predicate of  $\ell_{tail}$  to get that  $x''_{tail} = x_{tail}$ . Since  $x_{tail}$  points to  $x_{new}$ , and  $x''_{tail}$  points to  $x'_{new}$ , we can further conclude that  $x_{new} = x'_{new}$ . Thus, we can perform the store, which now gives us that  $\ell_{tail}$  points to  $x'_{new}$ ; the last node in the linked list. With this, we can close the invariant in state **Static Dequeue**, giving up ToknE  $Q_\gamma$  and TokUpdated  $Q_\gamma$ , but getting TokE  $Q_\gamma$ . Finally, the code releases the lock, which we can do since we have TokE  $Q_\gamma$ . The postcondition only says **True**, so there is nothing left to prove.

## Dequeue

We first acquire the lock, which gives us TokD  $Q_\gamma$ . Next, we must get the head node, by dereferencing  $\ell_{head}$ . To do this, we must open the invariant. We open it in state **Static** or **Enqueue**, and conclude that there is some head node,  $x_{head}$ , with  $\ell_{head} \mapsto x_{head}$ . We perform the read, and take half of the points-to predicate. We then close the invariant in state **Dequeue** or **Both**, giving up TokD  $Q_\gamma$ , but gaining ToknD  $Q_\gamma$ . Next, we must find out what  $x_{head}$  points to by dereferencing out  $x_{head}$ . To perform this dereference, we must open the invariant. Using the token, we conclude that we open it in state **Dequeue** or **Both**. In any case, we get that there is some  $x'_{head}$  with  $\ell_{head} \mapsto^{\frac{1}{2}} x'_{head}$ . Using the fractional points-to predicate we kept from earlier, we can conclude that  $x'_{head} = x_{head}$ . We now perform a case analysis on the contents of the queue:  $xs_{queue}$ .

*xs<sub>queue</sub> is empty:* In this case, we conclude that  $x_{head}$  points to None. We then perform the dereference of out  $x_{head}$ , giving us None. We close the invariant in state **Static** or **Enqueue**, giving up ToknD  $Q_\gamma$  and obtaining TokD  $Q_\gamma$ . We then step through the code, and since out  $x_{head}$  dereferenced to None, we take the if branch. We release the lock, giving up TokD  $Q_\gamma$ . The return value is None, so to finish the proof, we change the post-condition to prove the left disjunct.

*xs<sub>queue</sub> is not empty:* We can now conclude that  $x_{head}$  points to some node  $x_{head\_next}$ , which is the first node in  $xs_{queue}$ . We perform the dereference, which gives us in  $x_{head\_next}$ . We close the invariant in **Dequeue** or **Both**. We step through the code, taking the else branch. We extract the value from  $x_{head\_next}$  (which we have access to since it is persistent). Next, we must swing  $\ell_{head}$  to  $x_{head\_next}$ , which requires that we know that  $\ell_{head}$  points to something. Hence, we open the invariant in state **Dequeue** or **Both**, which gives us  $\ell_{head} \mapsto^{\frac{1}{2}} x''_{head}$ . We combine this with our half of the points-to predicate to conclude that  $x''_{head} = x_{head}$ . We then perform the store, giving us  $\ell_{head} \mapsto$  in  $x_{head\_next}$ . Closing the invariant now consists of removing the head element  $x_v$  from the abstract state  $xs_v$ , putting  $x_{head}$  into  $xs_{old}$ , removing  $x_{head\_next}$  from  $xs_{queue}$  (which means that  $xs_{queue}$  is still reflected in  $xs_v$ ) and letting  $x_{head\_next}$  become the new  $x_{head}$ . In removing  $x_v$  from  $xs_v$  we may also extract  $\Phi x_v$  from  $All x_v. \Phi$ . With these changes, we can close the invariant in state **Static** or **Enqueue**, giving up ToknD  $Q_\gamma$ , and obtaining TokD  $Q_\gamma$ .

All that is left now is releasing the lock, which we do by giving up TokD  $Q_\gamma$ , and

we are left with the return value  $\text{val } x_{\text{head\_next}}$ . We change the post-condition to prove the second disjunct. Since  $xs_{\text{queue}}$  was reflected in  $xs_v$ , and  $x_{\text{head\_next}}$  was the head of  $xs_{\text{queue}}$ , and  $x_v$  the head of  $xs_v$ , then we can conclude that  $\text{val } x_{\text{head\_next}} = \text{Some } x_v$ . And since we had  $\Phi x_v$ , we can then finish the proof by choosing  $x_v$  as the witness in the post-condition and frame away  $\Phi x_v$ .

### Discussing the need for $xs_{\text{old}}$

As mentioned in the observations, it is possible for the tail to lag one node behind the head. This insight lead to including the old nodes of the queue in the queue invariant. This addition manifests in the end of the proof of dequeue. When we open the invariant to swing  $\ell_{\text{head}}$  to the  $x_{\text{head\_next}}$ , we get that the entire linked list is  $xs$ . After performing the store, we can then close the invariant with the same  $xs$  that we opened the queue to, just written differently to signify that  $x_{\text{head}}$  is now "old", and  $x_{\text{head\_next}}$  is the new head node. Because of this, we can supply the same predicate concerning the *tail* that we got when we opened the invariant, since this only mentions  $xs$ , which remains the same.

Had we not used an  $xs_{\text{old}}$  and essentially just "forgotten" old nodes, we couldn't have done this. Say that we defined  $xs$  as  $xs = xs_{\text{queue}} ++ [x_{\text{head}}]$  instead. Then, once we have to close the invariant, we cannot supply  $xs$ , which we got when we opened the invariant. Our only choice (due to the fact that  $\text{loc}_{\text{head}}$  must point to  $x_{\text{head\_next}}$ ) is to close the invariant with  $xs' = xs_{\text{queue}} = xs'_{\text{queue}} ++ [x_{\text{head\_next}}]$ . However, clearly  $xs' \neq xs$ , so we cannot supply the same predicate concerning the *tail* that we got when opening the invariant, since this predicate talks about  $xs$ , not  $xs'$ . Now, if we opened the invariant in the state **Dequeue**, then we could conclude  $\text{isLast}x_{\text{tail}}xs'$  from  $\text{isLast}x_{\text{tail}}xs$ , due to the relationship between  $xs$  and  $xs'$ , and still be able close the invariant. However, if we opened the invariant in state **Both**, then we would need to assert  $\text{isSndLast}x_{\text{tail}}xs'$  from  $\text{isSndLast}x_{\text{tail}}xs$ . This is however not provable, since  $\text{isSndLast}x_{\text{tail}}xs$  allows for the case where  $xs'_{\text{queue}}$  is empty, which makes  $xs' = [x_{\text{head\_next}}]$ , disallowing us to prove  $\text{isSndLast}x_{\text{tail}}xs'$ .

## 3.7 Hocap-style Specification

When proving the concurrent specification, we were quite careful with tracking the state of the queue, and to some extent, even it's contents. The contents may have been existentially quantified, but through saving half a pointer, we could match up the contents of the queue between invariant openings. Given this precision in the proof, the reader may wonder if it is possible to give a more precise spec: one which is both concurrent and allows tracking of the contents of the queue. Indeed, this is possible, and we will explore such a spec in this section. We shall refer to this spec as a hocap-style spec (higher order concurrent abstract predicate) since the spec will be concurrent and parametrised by abstract predicates. This spec is more general than both the sequential and concurrent specs in the sense that those specs can be derived from the hocap-style spec. We prove this in section 3.8.

As before, we cannot simply parametrise the `is_queue` predicate with the abstract state of the queue, as we wish for it to be concurrent. So to allow clients to keep track of the contents of the queue, we will "split" the abstract state up in two parts, the authoritative view and the fragmental view. The client will then own the fragmental view, allowing them to keep track of the contents of the queue, whereas the `is_queue` predicate will own the authoritative view. We will in particular make sure that, if one has both the fragmental and authoritative views, then these agree on the abstract state of the queue. Further, it is only possible to update the abstract state of the queue (through the `fup`) if one possess both the authoritative and fragmental views.

We shall use the resource algebra  $\text{AUTH}((\text{FRAC} \times \text{AG}(\text{List Val}))^?)$  to achieve the above. *List Val* is the abstract state. It is wrapped in the agreement RA, AG, which ensures that if one owns two elements, then they agree on the abstract state. The fractional RA FRAC, denotes how much of the fragmental view is owned; the fragmental view can be split up, which is handled by the clients. We collect FRAC and  $\text{AG}(\text{List Val})$  in the product RA, whose elements are then pairs of fractions and abstract states. The option RA  $?$ , makes the product RA unital which is required by the AUTH construction. AUTH is the authoritative resource algebra and gives us the authoritative and fragmental views, and governs that they can only be updated in unison.

As before, we collect the ghost names we will need in a tuple, this time of type *Qghostnames*. It is similar to *ConcQghostnames*, but with one additional ghost name:  $\gamma_{\text{Abst}}$  which is used for elements in the resource algebra we constructed above.

For an abstract state  $xs_v$  and a tuple  $Q_\gamma$  of type *Qghostnames*, we shall use the notation  $Q_\gamma \models_\bullet xs_v$  for the ownership assertion  $[\bullet(1, \text{ag } xs_v)]^{Q_\gamma \cdot \gamma_{\text{Abst}}}$ , meaning that the authoritative view of the abstract state associated with  $Q_\gamma \cdot \gamma_{\text{Abst}}$  is  $xs_v$ . Similarly we write  $Q_\gamma \models_\circ xs_v$  for the assertion  $[\circ(1, \text{ag } xs_v)]^{Q_\gamma \cdot \gamma_{\text{Abst}}}$ .

With this, we now give the hocap-style specification, and explain it afterwards.

**Lemma 3** (Two-Lock M&S-Queue Hocap Specification).

$\exists \text{is\_queue} : \text{Val} \rightarrow \text{Qghostnames} \rightarrow \text{Prop}.$

$$\begin{aligned}
& \forall v_q, Q_\gamma. \text{is\_queue } v_q Q_\gamma \implies \Box \text{is\_queue } v_q Q_\gamma \\
& \wedge \{ \text{True} \} \text{initialize}() \{ v_q. \exists Q_\gamma. \text{is\_queue } v_q Q_\gamma * Q_\gamma \models_\circ [] \} \\
& \wedge \forall v_q, v, Q_\gamma, P, Q. \left( \forall xs_v. Q_\gamma \models_\bullet xs_v * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i \uparrow} \triangleright Q_\gamma \models_\bullet v :: xs_v * Q \right) \multimap \\
& \quad \{ \text{is\_queue } v_q Q_\gamma * P \} \text{enqueue } v_q v \{ w.Q \} \\
& \wedge \forall v_q, Q_\gamma, P, Q. \left( \forall xs_v. Q_\gamma \models_\bullet xs_v * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i \uparrow} \triangleright \left( \begin{aligned} & (xs_v = [] * Q_\gamma \models_\bullet xs_v * Q \text{ None}) \\ & \vee \left( \begin{aligned} & \exists x_v, xs'_v. xs_v = xs'_v ++ [x_v] * \\ & Q_\gamma \models_\bullet xs'_v * Q \text{ (Some } x_v) \end{aligned} \right) \end{aligned} \right) \right) \multimap \\
& \quad \{ \text{is\_queue } v_q Q_\gamma * P \} \text{dequeue } v_q \{ v.Q \}
\end{aligned}$$

Firstly, we require `is_queue` to be persistent, giving us support for concurrent clients.

Next, the initialise spec gives clients an additional resource in the postcondition: the ownership of the fragmental view of the empty list,  $Q_\gamma \models_\circ []$ . As discussed above, this allows them to keep track of the contents of the queue.

Finally, the specs for enqueue and dequeue have been parametrised by two predicates  $P$  and  $Q$ . The clients get to pick  $P$  and  $Q$ , and the choice depends on what the client wishes to prove;  $P$  describes those resources that the client has before enqueue or dequeue, and  $Q$  the resources it will have after. Hence  $P$  is in the precondition and  $Q$  in the postcondition of the hoare triple. However, before the client gets access to the hoare triple for enqueue or dequeue they must prove a viewshift. This viewshift states how the abstract state of the queue will change as a result of running enqueue or dequeue, and further shows that  $P$  can be updated to  $Q$ . Note that the consequent **►or righthand-side?◄** of the viewshift contains a  $\triangleright$ . This signifies that the update in the abstract state is tied to a step in the code. The mask on the viewshift further disallows opening of invariants in the namespace  $\mathcal{N}.i$ . This is because, when proving the specs, we will use an invariant within this namespace. Thus, to be able to use the viewshift while our invariant is open, we must make sure the viewshift doesn't use our invariant (since invariants can only be opened once, before being closed).

It might seem a bit strange that the client has to prove that the abstract state can be updated, but remember that the client owns the fragmental view, and that both this and the authoritative view, which is owned by the queue, is needed to update the abstract state. When proving the viewshift, clients aren't updating the abstract state of the queue, they are merely showing that they can supply the fragmental view, allowing the abstract state to be updated. This then enables the queue to update the authoritative view of the abstract state (using the proved viewshift) in conjunction with updating the concrete view.

Exactly how the client supplies the fragmental view depends on what the client wants to achieve. We will see two options, when we derive the sequential and concurrent specs from this hocap spec.

### 3.7.1 The `is_queue` Predicate

Our definition of `is_queue` is almost the same as `is_queue_conc`, so we only mention the difference here. The full definition can be found in the appendix **►appendix◄**. The difference is that we no longer take the predicate  $\Psi$ , and the collection of ghost names is now of type  $Qnames$ . Similarly, the queue invariant, *queue\_invariant*, doesn't require the  $\Psi$  any more. Further, the assertion  $All\ xs_v\ \Psi$  is changed to  $Q_\gamma \Rightarrow_\bullet xs_v$ .

### 3.7.2 Proof outline

The proofs are largely similar to the concurrent spec, but now, instead of having to handle the  $\Psi$  predicate, we must work with the authoritative and fragmental views of the abstract state.

For initialise, we must additionally get ownership of the authoritative and fragmental view of the abstract state, both of which should state that it is empty. I.e. we must get  $Q_\gamma \Rightarrow_\bullet [] * Q_\gamma \Rightarrow_\circ []$ . We achieve this by own-op and own-allocate, which requires us to show that  $\bullet(1, \mathbf{ag}\ []) \cdot \circ(1, \mathbf{ag}\ []) \in \mathcal{V}$ . This follows by the definitions of the resource algebras. We use  $Q_\gamma \Rightarrow_\bullet []$  to establish the queue invariant, and  $Q_\gamma \Rightarrow_\circ []$  to prove the post-condition.



For enqueue and dequeue, the only real changes are the points in the proof, where the concrete state of the queue is updated.

**Enqueue** We start by assuming the viewshift which allows us to update  $P$  to  $Q$  and  $Q_\gamma \Rightarrow_\bullet xs_v$  to  $Q_\gamma \Rightarrow_\bullet v :: xs_v$ , for any  $xs_v$ . We must then prove the hoare triple for the expression enqueue  $v_q v$ . The only real change from the previous proof happens the second time we open the invariant; the first and third times, the abstract state doesn't change, hence we can simply frame away the newly added authoritative fragment concerning the abstract state, and continue as we did before. The second time we open the invariant, it is around the expression that adds the newly created node to the linked list (**►add line number◄**). When opening it, we get  $Q_\gamma \Rightarrow_\bullet xs_v$ . As before, we also get all the resources to match up variables and step through the code, updating the concrete state. To close the invariant, we must make the same choice of abstract state as we did previously:  $v :: xs_v$ . This, however, requires us to obtain  $Q_\gamma \Rightarrow_\bullet v :: xs_v$ . However, since we have  $Q_\gamma \Rightarrow_\bullet xs_v$  and  $P$  (from the precondition), we can apply the viewshift to obtain it, along with  $Q$ . This then allows us to close the invariant, and the proof proceeds as previously. At the end, we must also prove the postcondition  $Q$ , but this is no issue as we obtained that from the viewshift.

**Dequeue** We assume the viewshift and proceed as in the concurrent proof until we get to the second time we open the invariant, which is around the expression that reads head's next node. It is here that we figure out whether or not the queue is empty by doing case analysis on  $xs_{queue}$ .

In the case that the queue is empty, then the abstract state of the queue will not change. We thus apply the viewshift (we have  $P$  from the precondition and  $Q_\gamma \Rightarrow_\bullet xs_v$  from the invariant), which gives us the disjunct. The right disjunct states that the abstract state  $xs_v$  is non-empty, but since the abstract state is reflected in  $xs_{queue}$ , which *is* empty, then we know that the right disjunct is impossible. Hence we may assume the left disjunct. I.e.  $xs_v = [] * Q_\gamma \Rightarrow_\bullet xs_v * Q$  None. We now proceed as before, this time giving up  $Q_\gamma \Rightarrow_\bullet xs_v$  to close the invariant. After stepping through the code, we are left with proving the postcondition:  $Q$  None, which we got from the viewshift.

If the queue is not empty, then we do not apply the viewshift (as the abstract state doesn't change within this invariant opening), and simply continue as we did previously. The next time we open the invariant is around the expression that writes the new head to  $\ell_{head}$ . It is this store that updates the abstract state of the queue, so it is within this invariant opening that we apply the viewshift (again, we have  $P$  from the precondition and  $Q_\gamma \Rightarrow_\bullet xs_v$  from the invariant). This time, we know that  $xs_{queue}$  is non-empty, and since  $xs_v$  is reflected in  $xs_{queue}$ , then we can conclude that the first disjunct is impossible, so the viewshift gives us  $\exists x_v, xs'_v. xs_v = xs'_v ++ [x_v] * Q_\gamma \Rightarrow_\bullet xs'_v * Q$  (Some  $x_v$ ). As before, we conclude that Some  $x_v$  is the return value (through the reflection between  $xs_{queue}$  and  $xs_v$ ), and proceed to close the invariant, this time giving up  $Q_\gamma \Rightarrow_\bullet xs'_v$ . Stepping through the code, we end up having to prove the post-condition  $Q$  (Some  $x_v$ ), which we got from the viewshift.

## 3.8 Deriving Sequential and Concurrent specs from Hocap

In this section we show that we can derive the sequential and concurrent specifications from sections 3.3 and 3.5 from the Hocap-style specification. The derivations will need to show how to update the abstract state of the queue. To help with this, we use the following lemmas, both of which follow from the definitions of the involved resource algebras. The first shows the authoritative and fragmental views of the abstract state agree.

**Lemma 4** (Abstract state agree).  $\forall xs_v, xs'_v.$   
 $Q_{\gamma H} \Rightarrow_{\bullet} xs_v * Q_{\gamma H} \Rightarrow_{\circ} xs'_v \vdash xs_v = xs'_v$

►consider showing proof◄ The second shows that, if we own both the authoritative and fragmental views, we are allowed to update the abstract state to whatever we like.

**Lemma 5** (Abstract state update).  $\forall xs_v, xs'_v, xs''_v.$   
 $Q_{\gamma H} \Rightarrow_{\bullet} xs_v * Q_{\gamma H} \Rightarrow_{\circ} xs'_v \vdash Q_{\gamma H} \Rightarrow_{\bullet} xs'_v * Q_{\gamma H} \Rightarrow_{\circ} xs''_v$

►consider showing proof◄

### 3.8.1 Deriving Sequential spec

We define the `is_queue_seq` predicate as follows.

**Definition 3.8.1** (Two-Lock M&S-Queue - `is_queue_seq` Predicate (Derive)).

$$\begin{aligned} \text{is\_queue\_seq } v_q \ xs_v \ Q_{\gamma S} &\triangleq \exists Q_{\gamma H} \in Qnames. \\ &\quad \text{proj\_Qnames\_seq } Q_{\gamma H} = Q_{\gamma S} * \\ &\quad \text{is\_queue } v_q \ Q_{\gamma H} * \\ &\quad Q_{\gamma H} \Rightarrow_{\circ} xs_v \end{aligned}$$

Here, `proj_Qnames_seq`  $Q_{\gamma H}$  simply creates an element of `SeqQnames`, with ghost names matching those of  $Q_{\gamma H}$ . `is_queue` is the predicate from the hocap-style spec, hence we know that it is duplicable.

The **sequential initialise spec** follows almost directly from hocap-style initialise spec. They only differ in the post-condition. The post-condition in the hocap-style spec states  $\exists Q_{\gamma H}. \text{is\_queue } v_q \ Q_{\gamma H} * Q_{\gamma H} \Rightarrow_{\circ} []$ , whereas we have to prove  $\exists Q_{\gamma S}. \text{is\_queue\_seq } v_q \ [] \ Q_{\gamma S}$ . If we choose `proj_Qnames_seq`  $Q_{\gamma H}$  for  $Q_{\gamma S}$ , then the equality in `is_queue_seq` becomes trivially true, and the postcondition we must prove follows from the hocap-style postcondition.

To prove the **sequential enqueue spec**, assume some  $v_q, v, xs_v$ , and  $Q_{\gamma S}$ . We must then show the hoare-triple concerning the expression: `enqueue`  $v_q \ v$ . To do this, we shall use the hocap-style spec for `enqueue`. This requires us to pick  $P$  and  $Q$ , and prove the resulting viewshift. We choose  $P \triangleq Q_{\gamma H} \Rightarrow_{\circ} xs_v$  and  $Q \triangleq Q_{\gamma H} \Rightarrow_{\circ} (v :: xs_v)$ . Note that with this choice, the hoare triple we get after proving the viewshift almost matches the hoare triple we have to prove. The main thing we need is `is_queue`  $v_q \ Q_{\gamma H}$  in the postcondition. However, since

is\_queue is persistent, and it is present in the precondition, we may assume it in the postcondition. Hence, all we have to prove is the viewshift:

$$\forall xs'_v. Q_\gamma \models_\bullet xs'_v * Q_{\gamma H} \models_\circ xs_v \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i\uparrow} \triangleright Q_\gamma \models_\bullet v :: xs'_v * Q_{\gamma H} \models_\circ (v :: xs_v)$$

So assume some  $xs'_v$ ,  $Q_\gamma \models_\bullet xs'_v$  and  $Q_{\gamma H} \models_\circ xs_v$ . We must then prove  $\models_{\mathcal{E} \setminus \mathcal{N}.i\uparrow} \triangleright Q_\gamma \models_\bullet (v :: xs'_v) * Q_{\gamma H} \models_\circ (v :: xs_v)$  By property 4,  $xs_v = xs'_v$ , hence, we can apply property 5 to update the authoritative and fragmental views to  $(v :: xs_v)$ , which is what we wanted.

We use a similar approach to above to prove the **sequential dequeue spec**. So we assume some  $v_q$ ,  $xs_v$ , and  $Q_{\gamma S}$ , and must then prove the hoare-triple concerning the expression: dequeue  $v_q$ . This time, we use the hocap-style dequeue spec, with the following choices:  $P \triangleq Q_{\gamma H} \models_\circ xs_v$ , and  $Q \ v \triangleq (xs_v = [] * v = \text{None} * Q_{\gamma H} \models_\circ xs_v) \vee (\exists x_v, xs'_v. xs_v = xs'_v ++ [x_v] * v = \text{Some } x_v * Q_{\gamma H} \models_\circ xs'_v)$ . In the same way as above, the hoare triple we get matches the one we have to prove (after a bit of manipulation). So we only have to prove the viewshift. First we conclude that the abstract states in the authoritative and fragmental views are equal. Then we do a case analysis on the abstract state,  $xs_v$ . If  $xs_v$  is empty, then we prove the left disjunct in the consequent of the viewshift, *without* updating the authoritative and fragmental views. If  $xs_v$  is non-empty, i.e.  $xs_v = xs'_v ++ [x_v]$  for some  $xs'_v$  and  $x_v$ , then we prove the right-side of the consequent in the viewshift, using property 5 to update the authoritative and fragmental views to the new abstract state  $(xs'_v)$ .

### 3.8.2 Deriving Concurrent spec

Remember that we need the is\_queue\_conc predicate to be persistent, hence we cannot simply assert  $Q_{\gamma H} \models_\circ xs_v$  as we did for is\_queue\_seq. Instead we will put it into an invariant. The predicate we will use looks as follows.

**Definition 3.8.2** (Two-Lock M&S-Queue - is\_queue\_conc Predicate (Derive)).

$$\begin{aligned} \text{is\_queue\_conc } v_q \ xs_v \ Q_{\gamma C} &\triangleq \exists Q_{\gamma H} \in Q_{\text{gnames}}. \\ &\text{proj\_}Q_{\text{gnames\_conc}} \ Q_{\gamma H} = Q_{\gamma C} * \\ &\text{is\_queue } v_q \ Q_{\gamma H} * \\ &\boxed{\exists xs_v. Q_{\gamma H} \models_\circ xs_v * \text{All } xs_v \ \Psi}^{\mathcal{N}.c} \end{aligned}$$

Persistency of is\_queue\_conc follows by the persistency of is\_queue and the fact that invariants and equalities are persistent.

The concurrent initialise spec follows from the hocap-style initialise spec, after allocating the invariant in the post-condition. We achieve this by applying the rules Ht-csq-vs and inv-alloc, to put the assertions  $Q_{\gamma H} \models_\circ xs_v$  and  $\text{All } [] \Psi$  (which is trivially true) in the postcondition of the hocap style spec into an invariant.

Next, to prove the enqueue spec, we assume some  $v_q$ ,  $v$ , and  $Q_{\gamma C}$ , and must then prove the hoare triple concerning the expression: enqueue  $v_q \ v$ . We specialise the hocap-style enqueue spec with  $P \triangleq \Psi \ v$  and  $Q \triangleq \text{True}$ . The hoare

triple we get after proving the viewshift matches the hoare triple we must prove, except that its precondition is weaker: it doesn't mention the invariant or the equality. Hence, the hoare triple we have to prove simply follows by the rule of consequence. To prove the viewshift, we must supply the full fragmental view. When deriving the sequential spec, we had this available through  $P$ . But this time we shall get it by opening the invariant in `is_queue_conc`. Proving the viewshift is then similar to what we did for the sequential spec.

To derive the dequeue spec, we pick  $P \triangleq \text{True}$  and  $Q\ v \triangleq v = \text{None} \vee (\exists x_v. v = \text{Some } x_v * \Psi\ x_v)$ . Again, the hoare triple we get after proving the viewshift is exactly the hoare triple we must prove, except that its precondition is weaker. Hence, we only have to prove the viewshift. This is done analogously to the sequential case (i.e. case distinction on  $xs_v$ ), except this time we get  $Q_{\gamma H} \Rightarrow_{\circ} xs_v$  through the invariant, and in the case where  $xs_v$  is not empty, i.e.  $xs_v = xs'_v ++ [x_v]$  for some  $xs'_v$  and  $x_v$ , we extract  $\Psi\ x_v$  from  $All\ xs_v\ \Psi$ , and use this to prove the right disjunct in  $Q$ .

## Chapter 4

# The Lock-Free Michael Scott Queue

### 4.1 Introduction

In this chapter we will study the Lock-Free Michael Scott Queue. As with the two-lock version, the original implementation can be found in [1]. As the name "Lock-Free" suggests, the implementation doesn't rely on locks to achieve correct behaviour. Instead, it uses the atomic operation: **CAS** which we discussed in section 2.1 **►Make sure you have discussed it◄**.

**►Introduce what we will go through in the chapter◄**

### 4.2 Implementation

```
1  initialize  $\triangleq$ 
2    let node = ref (None, ref (None)) in
3    ref (ref (node), ref (node))

1  enqueue Q value  $\triangleq$ 
2    let node = ref (Some value, ref (None)) in
3    (rec loop_ =
4      let tail = !(snd (!Q)) in
5      let next = !(snd (!tail)) in
6      if tail = !(snd (!Q)) then
7        if next = None then
8          if CAS (snd (!tail)) next node then
9            CAS (snd (!Q)) tail node
10           else loop ()
11         else CAS (snd (!Q)) tail next; loop ()
12       else loop ()
13    ) ()

1  dequeue Q  $\triangleq$ 
2    (rec loop_ =
```

```

3      let head = !(fst(!Q)) in
4      let tail = !(snd(!Q)) in
5      let p = newproph in
6      let next = !(snd(!head)) in
7      if head = Resolve(!(fst(!Q)), p, ()) then
8          if head = tail then
9              if next = None then
10                 None
11             else
12                 CAS(snd(!Q)) tail next; loop ()
13         else
14             let value = fst(!next) in
15             if CAS (Fst(!Q)) head next then
16                 value
17             else loop ()
18     else loop ()
19 )()

```

### 4.3 Reachability

An important aspect in the correctness of the lock-free M&S-queue is which nodes a particular node is able to *reach* through the linked list (i.e. by following the chain of pointers), and how the head and tail pointers change during the lifetime of the queue.

Firstly, the underlying linked list still only ever grows, and it does so only at the end. Hence, the set of nodes that a given node can reach only ever grows. Further, all nodes can always reach the last node in the linked list.

Secondly, similarly to the two-lock variant, the correctness of the queue relies on the fact that the head and tail pointers are only ever swung towards the end of the linked list. That is, if a node can reach, say, the tail node at one point during the program, then it can reach any future tail nodes.

Thirdly, whereas it was possible for the tail node to lag behind the head node in the two-lock version, it is not possible in the lock-free version. Indeed, if such a scenario could happen, dequeue could crash! Consider the scenario where the head node is the last node in the linked list (hence the queue is empty), and the tail is lagging behind the head. If someone invokes dequeue, the check on line 8, which is supposed to detect an empty queue or a lagging tail will result to false, and hence, incorrectly, take the ‘else’ branch, which assumes that there is something to dequeue. But since the queue is empty, then *next* – the node after head – is None, and when we try to dereference *next* on line 14, we will crash. Therefore, our invariant must ensure that the tail never lags behind the head.

To capture these properties, we introduce two notions of reachability: concrete reachability and abstract reachability, which we introduce in the following sections. This way of modelling the queue was originally introduced in [2]. The presentation here borrows the same ideas, but the presentation differs in the sense that it is node-oriented instead of location-oriented. Moreover, we prove some additional

properties of reachability which allows us to simplify the queue invariant slightly.

### 4.3.1 Concrete Reachability

We say that a node  $x_n$  in a linked list can concretely reach a node  $x_m$  when, if we start traversing succeeding nodes (by following the out and in pointers starting from  $x_n$ ), we will eventually get to  $x_m$ . If this is the case, we write  $x_n \rightsquigarrow x_m$ . We allow for traversing zero nodes to reach  $x_m$ , which essentially means that all nodes can reach themselves. Formally, we define  $x_n \rightsquigarrow x_m$  inductively as follows.

**Definition 4.3.1** (Concrete Reachability).  $x_n \rightsquigarrow x_m \triangleq \text{in } x_n \mapsto^\square (\text{val } x_n, \text{out } x_n) * (x_n = x_m \vee \exists x_p. \text{out } x_n \mapsto^\square \text{in } x_p * x_p \rightsquigarrow x_m)$

This definition firstly states that  $x_n$  is a node:  $\text{in } x_n \mapsto^\square (\text{val } x_n, \text{out } x_n)$ . Secondly,  $x_n$  is either the node to be reached  $x_m$ , or it has a succeeding node  $x_p$ , which can reach  $x_m$ . Note that the points-to propositions are all persistent, which mimics the fact that the linked list is only ever changed by appending new nodes to the end. **►decide which of the following two sentences makes more sense◄** This in turn makes concrete reachability a persistent predicate. This in turn makes  $x_n \rightsquigarrow x_m$  persistent for all  $x_n$  and  $x_m$ .

We proceed to prove some useful lemmas about concrete reachability.

**Lemma 6** (reach-reflexive).  $x_n \rightsquigarrow x_n ** \text{in } x_n \mapsto^\square (\text{val } x_n, \text{out } x_n)$

*Proof.* The  $\rightarrow$  direction follows directly by the definition. To prove the  $*$ -direction, it suffices to show  $(x_n = x_n \vee \exists x_p. \text{out } x_n \mapsto^\square \text{in } x_p * x_p \rightsquigarrow x_n)$ . Clearly, this follows as the left disjunction holds.  $\square$

**Lemma 7** (reach-transitive).  $x_n \rightsquigarrow x_m * x_m \rightsquigarrow x_o * x_n \rightsquigarrow x_o$

*Proof.* We proceed by induction in  $x_n \rightsquigarrow x_m$ .

B.C. In the base case,  $x_n = x_m$ . We get to assume that  $x_m \rightsquigarrow x_o$ , and must prove  $x_n \rightsquigarrow x_o$ . Since  $x_n = x_m$ , we are done.

I.C. In the inductive case, we assume that  $x_n$  is a node that points to some  $x_p$ , which satisfies  $x_m \rightsquigarrow x_o * x_p \rightsquigarrow x_o$ . Assuming  $x_m \rightsquigarrow x_o$ , we must prove  $x_n \rightsquigarrow x_o$ .

To prove  $x_n \rightsquigarrow x_o$  we must first show that  $x_n$  is a node, which we have already established. Next, we must show that either  $x_n = x_o$ , or  $x_n$  steps to some  $x'_p$  which can reach  $x_o$ . We prove the second case by choosing our  $x_p$  for  $x'_p$ . Thus, we have to show  $x_p \rightsquigarrow x_o$ . This then follow by the induction hypothesis together with our assumption that  $x_m \rightsquigarrow x_o$ .

$\square$

**Lemma 8** (reach-from-is-node).  $x_n \rightsquigarrow x_m * \text{in } x_n \mapsto^\square (\text{val } x_n, \text{out } x_n)$

*Proof.* This follow immediately from the definition of concrete reachability.  $\square$

**Lemma 9** (reach-to-is-node).  $x_n \rightsquigarrow x_m * \text{in } x_m \mapsto^\square (\text{val } x_m, \text{out } x_m)$

*Proof.* We proceed by induction in  $x_n \rightsquigarrow x_m$ . The base case follows by lemma 8 above. In the inductive case, we assume that  $x_n$  points to some  $x_p$ , which reaches  $x_m$ . Our induction hypothesis is in  $x_m \mapsto^\square (\text{val } x_m, \text{out } x_m)$ , which is also our proof obligation, so we are done.  $\square$

**Lemma 10** (reach-last).  $x_n \rightsquigarrow x_m \multimap \text{out } x_n \mapsto \text{None} \multimap x_n = x_m \multimap \text{out } x_n \mapsto \text{None}$

*Proof.* Assuming  $x_n \rightsquigarrow x_m$  and  $\text{out } x_n \mapsto \text{None}$ , we must prove that  $x_n = x_m$  and  $\text{out } x_n \mapsto \text{None}$ . By  $x_n \rightsquigarrow x_m$ , we know that either  $x_n = x_m$ , or  $x_n$  points to some  $x_p$  and  $x_p \rightsquigarrow x_m$ . The first case immediately gives us everything we need to prove both goals. If we are in the second case, then we know that  $\text{out } x_n \mapsto^\square$  in  $x_p$ . But by our initial assumption,  $\text{out } x_n \mapsto \text{None}$ . Since in  $x_p$  is a location, then this is clearly a contradiction.  $\square$

### 4.3.2 Abstract Reachability

As discussed, we wish to capture that if a node can reach either the head or tail node at one point during the program, then it can reach any future head or tail nodes. To do this we introduce the notion of abstract reachability. The idea is to introduce ghost variables that can "point" to nodes in the linked list, just as the head and tail pointers do. We shall write  $\gamma \rightsquigarrow x$  to mean that the ghost variable  $\gamma$  *abstractly points* to the node  $x$ . We shall construct the abstract points-to predicate so that we can update  $\gamma \rightsquigarrow x$  to  $\gamma \rightsquigarrow y$  only if  $x$  can concretely reach  $y$ , i.e.  $x \rightsquigarrow y$ . This additional restriction compared to the normal points-to predicate is what allows us to capture the property described above. We write  $x \dashrightarrow \gamma$  to mean that the node  $x$  can *abstractly reach* the ghost variable  $\gamma$ . The idea is that, if we have established  $x \dashrightarrow \gamma$ , then no matter what node  $\gamma$  abstractly points to, for instance  $\gamma \rightsquigarrow y$ , we can conclude  $x \rightsquigarrow y$ . This means that if we update  $\gamma \rightsquigarrow y$  in the future to, say  $\gamma \rightsquigarrow z$ , then we can conclude that  $x \rightsquigarrow z$ .

To define the abstract points-to predicate and the abstract reach predicate, we create the following resource algebra:  $\text{AUTH}(\mathcal{P}(\text{Node}))$ , where  $\text{Node} = (\text{Loc} \times \text{Val}) \times \text{Loc}$ . Here, the resource algebra  $\mathcal{P}(\text{Node})$  denotes the set of subsets of  $\text{Node}$ , with union as the operation. The empty set is the unit element, meaning that  $\mathcal{P}(\text{Node})$  is unital. We may now define abstract reach and abstract points-to as follows.

**Definition 4.3.2** (Abstract Reach).  $x \dashrightarrow \gamma \triangleq \llbracket \circ \{x\} \rrbracket^\gamma$

**Definition 4.3.3** (Abstract Points to).  $\gamma \rightsquigarrow x \triangleq \exists s. \llbracket \bullet s \rrbracket^\gamma \multimap \bigstar_{x_m \in s} x_m \rightsquigarrow x$

One should think about sets  $s \in \mathcal{P}(\text{Node})$  as specifying which nodes can abstractly reach a certain ghost variable. Due to how the authoritative resource algebra work, the assertion  $\llbracket \circ \{x\} \rrbracket^\gamma$  essentially states that  $x$  is *one* of the nodes that can reach the node that  $\gamma$  points to. This is because, when combining fragmental and authoritative elements, we get that the fragmental element is "smaller" than the authoritative. In this case, "smaller" amounts to "subset".



Hence,  $\llbracket \circ \{x\} \rrbracket^\gamma$  means that, whatever the authoritative set is, it will contain the node  $x$ .

The authoritative set is existentially quantified as it can change over time, but whatever it is, we assert that all the nodes it contains *can* concretely reach the node that the ghost name is currently pointing to. This choice of definitions enables us to prove the properties we desired of abstract reachability above. The four lemmas below is all we need for abstract reachability when we prove the specification for the lock-free M&S-queue later.

Firstly, if we have a node, we may allocate some ghost variable  $\gamma$  which points to it and assert that the node can reach  $\gamma$ .

**Lemma 11** (Abs-Reach-Alloc).  $\blacktriangleright$ *is it a viewshift?* $\blacktriangleleft x \rightsquigarrow x \Rightarrow (\exists \gamma. \gamma \mapsto x * x \dashv\vdash \gamma)$

*Proof.* We assume  $x \rightsquigarrow x$  and must show  $\models \exists \gamma'. \gamma' \mapsto x * x \dashv\vdash \gamma'$ . By definition or the authoritative RA, the element  $\bullet \{x\} \cdot \circ \{x\}$  is valid. Hence by the Ghost-alloc rule, we may get  $\models \exists \gamma. \llbracket \bullet \{x\} \cdot \circ \{x\} \rrbracket^\gamma$ . By Upd-mono, we may strip away the update modality on the goal and the previous assertion. Thus, we must prove  $\exists \gamma'. \gamma' \mapsto x * x \dashv\vdash \gamma'$ , and we have that  $\exists \gamma. \llbracket \bullet \{x\} \cdot \circ \{x\} \rrbracket^\gamma$ . We use  $\gamma$  as the witness in the goal meaning we must prove  $\gamma \mapsto x * x \dashv\vdash \gamma$ . We can split the ownership of the authoritative and fragmental parts up using Own-op, giving us  $\llbracket \bullet \{x\} \rrbracket^\gamma$  and  $\llbracket \circ \{x\} \rrbracket^\gamma$ . The latter assertion is equivalent to  $x \dashv\vdash \gamma$ , which matches the second obligation in the goal. To prove the first obligation, we must give some set as witness, and show that all nodes in the set can reach  $x$ . We of course choose  $\{x\}$  as the witness, and must then prove that  $x \rightsquigarrow x$ , which we assumed in the beginning.  $\square$

The second lemma allows us to get a *concrete* reachability predicate out of an abstract one. If a ghost name  $\gamma_m$  currently points abstractly to some node  $x_m$ , then any node that can abstractly reach  $\gamma$ , can also *concretely* reach  $x_m$ .

**Lemma 12** (Abs-Reach-Concr).  $x_n \dashv\vdash \gamma_m * \gamma_m \mapsto x_m * x_n \rightsquigarrow x_m * \gamma_m \mapsto x_m$

*Proof.* Assuming  $x_n \dashv\vdash \gamma_m$  and  $\gamma_m \mapsto x_m$ , we must show  $x_n \rightsquigarrow x_m$  without "consuming"  $\gamma_m \mapsto x_m$ . From  $\gamma_m \mapsto x_m$  we can deduce that there is some set  $s$  so that  $\llbracket \bullet s \rrbracket^{\gamma_m}$  and  $\bigstar_{x' \in s} x' \rightsquigarrow x_m$ . Since we own both  $\llbracket \bullet s \rrbracket^{\gamma_m}$  and  $\llbracket \circ \{x_n\} \rrbracket^{\gamma_m}$  (from  $x_n \dashv\vdash \gamma_m$ ), we may conclude that their product is valid, which in our instantiation of the authoritative RA equates to  $x_n \in s$ . We may now frame away the second part of the goal,  $\gamma_m \mapsto x_m$ , using  $\llbracket \bullet s \rrbracket^{\gamma_m}$  and  $\bigstar_{x' \in s} x' \rightsquigarrow x_m$ . Note that we get to keep the latter assertion as reach is persistent. Thus, from that assertion and by  $x_n \in s$ , we can deduce that  $x_n \rightsquigarrow x_m$ , which is what we had to prove.  $\square$

We can also go the other way, and get an abstract reachability predicate out of a concrete one. If a ghost variable  $\gamma_m$  points abstractly to some node  $x_m$ , and a node  $x_n$  can *concretely* reach  $x_m$ , then we may deduce that  $x_n$  can *abstractly* reach  $\gamma_m$ , meaning that  $x_n$  can reach any node that  $\gamma_m$  will ever point to in the future.

**Lemma 13** (Abs-Reach-Abs).  $x_n \rightsquigarrow x_m * \gamma_m \rightsquigarrow x_m \Rightarrow x_n \dashrightarrow \gamma_m * \gamma_m \rightsquigarrow x_m$

*Proof.* Assuming  $x_n \rightsquigarrow x_m$  and  $\gamma_m \rightsquigarrow x_m$  we must conclude  $\models x_n \dashrightarrow \gamma_m$ . From  $\gamma_m \rightsquigarrow x_m$  we know that there is some set  $s$  so that  $\boxed{\bullet s}^{\gamma_m}$  and  $\bigstar_{x' \in s} x' \rightsquigarrow x_m$ . There are now two cases to consider: either  $x_n \in s$  or  $x_n \notin s$ .

$x_n \in s$  By the definition of our authoritative RA, if a set  $y$  is a subset of  $s$ , then we may update our ghost resources to obtain ownership of the fragment  $y$ . In our case, since  $x_n \in s$ , we may update our resources to additionally get  $\boxed{\circ \{x_n\}}^{\gamma_m}$ , which is exactly what we wanted. Since we still have  $\boxed{\bullet s}^{\gamma_m}$ , we can also prove  $\gamma_m \rightsquigarrow x_m$ .

$x_n \notin s$  In this case we may update  $\boxed{\bullet s}^{\gamma_m}$  so that the set also includes  $x_n$ . The reason we may do this, is because, according to the  $\mathcal{P}(\text{Node})$  RA, we may update a set  $X$  to  $Y$ , as long as  $X \subseteq Y$ . Thus, we can update our resource to get  $\boxed{\bullet \{x_n\} \cup s}^{\gamma_m}$ . As in the previous case, we can further get  $\boxed{\circ \{x_n\}}^{\gamma_m}$  out of this, which we use to frame away the goal  $x_n \dashrightarrow \gamma_m$ . To prove  $\gamma_m \rightsquigarrow x_m$ , we use the set  $\{x_n\} \cup s$ , and immediately frame away the authoritative part, which we owned. We are left with having to prove  $\bigstar_{x' \in \{x_n\} \cup s} x' \rightsquigarrow x_m$ . However, by  $\bigstar_{x' \in s} x' \rightsquigarrow x_m$  and our assumption that  $x_n \rightsquigarrow x_m$ , we can easily conclude this.

□

The final lemma allows us update abstract pointers. As discussed above, we will require that whatever node we update the pointer to is a successor of the current node. That is, if a ghost variable  $\gamma_m$  currently points to  $x_m$ , then we must show that  $x_m$  can reach  $x_o$ , before we can update  $\gamma$  to point abstractly to  $x_o$ . After the update we additionally get that  $x_o$  can abstractly reach  $\gamma$ .

**Lemma 14** (Abs-Reach-Advance).  $\gamma_m \rightsquigarrow x_m * x_m \rightsquigarrow x_o \Rightarrow \gamma_m \rightsquigarrow x_o * x_o \dashrightarrow \gamma_m$

*Proof.* Assuming  $\gamma_m \rightsquigarrow x_m$  and  $x_m \rightsquigarrow x_o$  we must prove  $\models \gamma_m \rightsquigarrow x_o * x_o \dashrightarrow \gamma_m$ . From  $\gamma_m \rightsquigarrow x_m$ , we get some set  $s$  so that  $\boxed{\bullet s}^{\gamma_m}$  and  $\bigstar_{x' \in s} x' \rightsquigarrow x_m$ . As we did in previous proof (lemma 13), we update  $\boxed{\bullet s}^{\gamma_m}$  so that the set additionally contains  $x_o$ . Thus, we get  $\boxed{\bullet \{x_o\} \cup s}^{\gamma_m}$ . From this, we may extract ownership of the fragmental part:  $\boxed{\bullet \{x_o\}}^{\gamma_m}$ , which we use to prove the second part of the goal.

We are thus left with proving  $\gamma_m \rightsquigarrow x_o$ . We use  $\{x_o\} \cup s$  as witness for the authoritative set, and immediately frame away the ownership assertion of the authoritative part. We are left with proving  $\bigstar_{x' \in \{x_o\} \cup s} x' \rightsquigarrow x_o$ . We already know that  $\bigstar_{x' \in s} x' \rightsquigarrow x_m$  and  $x_m \rightsquigarrow x_o$ . Thus, by transitivity of reach (lemma 7), we may conclude  $\bigstar_{x' \in \{x_o\} \cup s} x' \rightsquigarrow x_o$ . Thus, we are done if we can prove that  $x_o \rightsquigarrow x_o$ , which by 6 amount to showing that  $x_o$  is a node. However, since  $x_m \rightsquigarrow x_o$ , then, by 9, we know that this is the case. □

## 4.4 Specification for Lock-Free M&S-Queue

From the perspective of a client, the two-lock M&S-Queue and the lock-free M&S-Queue should behave similarly – they should both behave as a concurrent queue. Hence, in this section, we will prove specifications that are almost identical to those we proved for the two-lock M&S-queue; a sequential, a concurrent, and a hcap-style spec.

As we showed in the previous chapter, the sequential and concurrent specifications can be derived from the hcap-style spec *without* referring to the actual implementation. Thus, in this chapter we will only focus on proving the hcap-style spec – the derivations of the sequential and concurrent specs will be practically identical to that of the last chapter.

The only two differences between the hcap spec we prove for the lock-free version compared to the two-lock version (lemma 3) is the collection of ghost names, and the fact that the expressions in our hoare-triples – initialize, enqueue, and dequeue – refer to the lock-free versions from section 4.2.

The collection  $Qghostnames$  will contain  $\gamma_{Abst}$  whose purpose is the same as before; to keep track of the abstract state of the queue. Additionally, we will have  $\gamma_{Head}$ ,  $\gamma_{Tail}$ , and  $\gamma_{Last}$ , which will abstractly point to the head, tail, and last node, respectively.

### 4.4.1 The `is_queue` predicate

We will again be needing an invariant to make the predicate persistent. The invariant we define has some commonalities with the invariant we used for the two-lock variant, but it incorporates the differences we discussed earlier in the chapter. In particular, it is important for the correctness of the queue that the tail doesn't lag behind the head. As such, our invariant will not allow for this behaviour. This has the extra implication that the head node is always the oldest node, meaning that we do not need to keep track of older nodes,  $xs_{old}$ .

Unlike the two-lock variant, we assert the existence of an additional node  $x_{last}$ , which invariantly is the last (newest added) node in the linked list. This helps us reason about where the head and tail nodes are located; enqueue distinguishes between the cases where the tail is last and not last, and similarly for dequeue and head.

In this way,  $x_{head}$  is the first node,  $x_{last}$  is the last node, and  $x_{tail}$  either lies somewhere in between, is one of them, or, in the case where the queue is empty, is both of them. To force this structure, we use our abstract reachability predicate from the previous section.

We proceed to define the invariant.

**Definition 4.4.1** (Lock-Free M&S-Queue Invariant).

$$\begin{aligned}
& \text{queue\_invariant } \ell_{\text{head}} \ell_{\text{tail}} Q_\gamma \triangleq \\
& \exists xs_v. Q_\gamma \models_\bullet xs_v^* \quad (\text{abstract state}) \\
& \exists xs, xs_{\text{queue}}, x_{\text{head}}, x_{\text{tail}} x_{\text{last}}. \quad (\text{concrete state}) \\
& xs = xs_{\text{queue}} ++ [x_{\text{head}}]^* \\
& \text{isLL } xs^* \\
& \text{isLast } x_{\text{last}} xs \\
& \text{proj\_val } xs_{\text{queue}} = \text{wrap\_some } xs_v^* \\
& \ell_{\text{head}} \mapsto \text{in } x_{\text{head}}^* \\
& \ell_{\text{tail}} \mapsto \text{in } x_{\text{tail}}^* \\
& Q_\gamma \cdot \gamma_{\text{Head}} \mapsto x_{\text{head}} * x_{\text{head}} \dashrightarrow Q_\gamma \cdot \gamma_{\text{Tail}}^* \\
& Q_\gamma \cdot \gamma_{\text{Tail}} \mapsto x_{\text{tail}} * x_{\text{tail}} \dashrightarrow Q_\gamma \cdot \gamma_{\text{Last}}^* \\
& Q_\gamma \cdot \gamma_{\text{Last}} \mapsto x_{\text{last}}
\end{aligned}$$

The `is_queue` predicate is now quite simple: it states that the value representing the queue is a location which points persistently to a pair of locations, the head and tail pointers, which satisfy the invariant we defined above.

**Definition 4.4.2** (Lock-Free M&S-Queue - `is_queue` Predicate).

$$\begin{aligned}
\text{is\_queue } v_q Q_\gamma & \triangleq \exists \ell_{\text{queue}}, \ell_{\text{head}}, \ell_{\text{tail}} \in \text{Loc}. \\
v_q & = \ell_{\text{queue}} * \ell_{\text{queue}} \mapsto^\square (\ell_{\text{head}}, \ell_{\text{tail}})^* \\
& \boxed{\text{queue\_invariant } \ell_{\text{head}} \ell_{\text{tail}} Q_\gamma}^{\mathcal{N}.\text{queue}}
\end{aligned}$$

#### 4.4.2 Proof outline

We instantiate the specification with our definition of `is_queue` (4.4.2). By the definition of `is_queue` we easily show that `is_queue` is persistent. What remains to be shown is the specifications for `initialize`, `enqueue`, and `dequeue`. Both `enqueue` and `dequeue` has code that attempt to swing the tail pointer forward (for `enqueue`, lines 9 and 11, and for `dequeue`, line 12). These all behave similarly, so we additionally prove a specification for swinging the tail.

##### Initialise

**Lemma 15** (Lock-Free M&S-Queue Specification - Initialise).

$$\{ \text{True} \} \text{initialize}() \{ v_q. \exists Q_\gamma. \text{is\_queue } v_q Q_\gamma * Q_\gamma \models_\circ [] \}$$

*Proof.* We first step through line 2 which creates a new node:  $x_1 = (\ell_{1\_in}, \text{None}, \ell_{1\_out})$ , with  $\ell_{1\_out} \mapsto \text{None}$  and  $\ell_{1\_in} \mapsto (\text{None}, \ell_{1\_out})$ , the latter of which we make persistent. Next, we step through line 3 which gives us some locations  $\ell_{\text{head}}$ ,  $\ell_{\text{tail}}$ , and  $\ell_{\text{queue}}$ , with  $\ell_{\text{head}} \mapsto x_1$  and  $\ell_{\text{tail}} \mapsto x_1$ , and finally  $\ell_{\text{queue}} \mapsto (\ell_{\text{head}}, \ell_{\text{tail}})$ . As we did for the two-lock version, we allocate an empty abstract queue, giving us some ghost name  $\gamma_{\text{Abst}}$  that we put into  $Q_\gamma$ , and the resources  $Q_\gamma \models_\bullet [] * Q_\gamma \models_\circ []$ .

To allocate the invariant, we must additionally obtain abstract reach propositions. Since  $x_1$  is a node, we may use lemma 6 to conclude  $x_1 \rightsquigarrow x_1$ . We can now use lemma 11 three times, giving us ghost names  $\gamma_{Head}$ ,  $\gamma_{Tail}$ ,  $\gamma_{Last}$  which we again put into  $Q_\gamma$ , and the resources

$$Q_\gamma.\gamma_{Head} \rightsquigarrow x_1 * Q_\gamma.\gamma_{Tail} \rightsquigarrow x_1 * x_1 \dashrightarrow Q_\gamma.\gamma_{Tail} * Q_\gamma.\gamma_{Last} \rightsquigarrow x_1 * x_1 \dashrightarrow Q_\gamma.\gamma_{Last}$$

We now have all the resources we need to allocate the invariant with the head, tail, and last node being  $x_1$ .

With the invariant allocated, proving the post-condition becomes straightforward.  $\square$

## Swing Tail

The specification we wish to prove is the following.

**Lemma 16** (Swing Tail).  $\forall \ell_{head}, \ell_{tail}, x_{tail}, x_{newtail}, Q_\gamma.$   
 $\{ \boxed{\text{queue\_invariant } \ell_{head} \ell_{tail} Q_\gamma}^{\mathcal{N}.queue} * x_{tail} \rightsquigarrow x_{newtail} * x_{newtail} \dashrightarrow Q_\gamma.\gamma_{Last} \}$   
**CAS**  $\ell_{tail}$  in  $x_{tail}$  in  $x_{newtail}$   
 $\{v.v = \text{true} \vee v = \text{false}\}$

*Proof.* The rule for **CAS** demands that we have a points-to predicate for  $\ell_{tail}$ . This is available inside the invariant, so we proceed to open it. This tells us that there is some  $x'_{tail}$  so that  $\ell_{tail} \mapsto x'_{tail}$ . We consider both cases of the CAS:

**Case CAS** succeeds. It must then have been the case that in  $x'_{tail} =$  in  $x_{tail}$ . Since we have  $x_{tail} \rightsquigarrow x_{newtail}$ , we know that  $x_{tail}$  is a node (lemma 8). From the invariant, we additionally got that  $Q_\gamma.\gamma_{Tail} \rightsquigarrow x'_{tail}$  and  $x_{head} \dashrightarrow Q_\gamma.\gamma_{Tail}$ , which by lemma 12 means that  $x_{head} \rightsquigarrow x'_{tail}$ . We can thus also conclude that  $x'_{tail}$  is a node (lemma 9). So since both  $x_{tail}$  and  $x'_{tail}$  are nodes, and in  $x'_{tail} =$  in  $x_{tail}$ , then it must be that  $x_{tail} = x'_{tail}$ . In other words, we have now know that  $Q_\gamma.\gamma_{Tail} \rightsquigarrow x_{tail}$ . Since the **CAS** succeeded, we now have that  $\ell_{tail} \mapsto$  in  $x_{newtail}$ . Since the invariant demands that  $Q_\gamma.\gamma_{Tail}$  and  $\ell_{tail}$  agree on the node they point to, we must update  $Q_\gamma.\gamma_{Tail} \rightsquigarrow x_{tail}$  to  $Q_\gamma.\gamma_{Tail} \rightsquigarrow x_{newtail}$ . We can do this using lemma 14 as we assumed  $x_{tail} \rightsquigarrow x_{newtail}$ . With this, we can close the invariant again, using  $x_{newtail}$  as the tail node. The **CAS** evaluates to **true** which we use to prove the first disjunct of the post-condition.

**Case CAS** Fails. Since the **CAS** failed, nothing was updated, and we can close the invariant again with the same resources we got out of it. The **CAS** evaluates to **false**, hence we can prove the second disjunct in the post-condition.  $\square$

## Enqueue

**Lemma 17** (Lock-Free M&S-Queue Specification - Enqueue).

$$\forall v_q, v, Q_\gamma, P, Q. \left( \forall x_{sv}. Q_\gamma \Rightarrow_\bullet x_{sv} * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i^\dagger} \triangleright Q_\gamma \Rightarrow_\bullet v :: x_{sv} * Q \right) \multimap \{ \text{is\_queue } v_q \ Q_\gamma * P \} \text{ enqueue } v_q \ v \{ w.Q \}$$

*Proof.* We assume the viewshift and proceed to prove the hoare triple. By definition of `is_queue`, we know that the queue  $v_q$  is a location  $\ell_{queue}$  and there are locations  $\ell_{head}$  and  $\ell_{tail}$  so that

$$\ell_{queue} \mapsto^\square (\ell_{head}, \ell_{tail}) * \boxed{\text{queue\_invariant} \ell_{head} \ell_{tail} Q_\gamma}^{\mathcal{N}.queue} \quad (4.1)$$

We first step through line 2 which creates a new node  $x_{new}$ , so that

$$\text{in } x_{new} \mapsto^\square (\text{Some } v, \text{out } x_{new}) \quad (4.2)$$

$$\text{out } x_{new} \mapsto \text{None} \quad (4.3)$$

The next line is the beginning of the looping function. We proceed by löb induction, which allows us to assume the hoare triple we wish to prove *later*. This means that, if we reach a recursive call, we will have the hoare triple that we must prove – the later will be immediately stripped when we apply  $()$  and “step into” the recursive function.

Line 4 first dereferences to the tail pointer  $\ell_{tail}$  using the resources in 4.1. We open the invariant to obtain the points-to predicate concerning  $\ell_{tail}$ . We get that  $\ell_{tail}$  points to some  $x_{tail}$ . Using the resources from the invariant, we may conclude the following persistent information:

$$x_{tail} \dashrightarrow Q_\gamma.\gamma_{Last} \quad (4.4)$$

$$\text{in } x_{tail} \mapsto^\square (\text{val } x_{tail}, \text{out } x_{tail}) \quad (4.5)$$

The first part is directly from the invariant, and the second we may derive using lemmas 12 and 9. We perform the load, and close the invariant.

The next line (line 5) finds out what  $x_{tail}$  points to. Using 4.5 we step to  $!(\text{out } x_{tail})$ . The points-to predicate required to perform this dereference is owned by the invariant (as it might be non-persistent), so we open the invariant again. We get that there is some  $x_{last}$ , with  $Q_\gamma.\gamma_{Last} \mapsto x_{last}$ . From this, 4.4, and lemma 12 we conclude  $x_{tail} \rightsquigarrow x_{last}$ . This gives us two cases to consider: either  $x_{tail}$  *is*  $x_{last}$  (meaning that  $x_{tail}$  is not lagging behind), or it points to some node  $x_{tail\_next}$  which can reach  $x_{last}$  (meaning that  $x_{tail}$  is lagging behind).

**Case**  $x_{tail} = x_{last}$ . Since we had *isLast* $x_{last}xs$ , we know that  $x_{tail}$  is the last node in the linked list, hence it points to None. We perform the load which sets *next* to None, and close the invariant.

We proceed to the consistency check on line 6. As before, the points-to predicate for  $\ell_{tail}$  is in the invariant, so we open it. We get  $\ell_{tail} \mapsto x'_{tail}$ , for some  $x'_{tail}$ . Using this, we perform the dereference and close the invariant. The branch taken now depends on whether or not  $x'_{tail}$  is consistent with  $x_{tail}$ . In case they aren't, we take the ‘else’ branch on line 12, which simply consists of a recursive call to the looping function. We are done by the induction hypothesis (from the löb induction).

If they are consistent, we take the ‘then’ branch and step to line 7. Here we check whether or not *next* is None. We already know this is the case, so we proceed to line 8. This consists of a **CAS** instruction which attempts to add  $x_{new}$  to the linked list. The **CAS** will succeed if and only if  $\ell_{tail}$

still points to None. We open the invariant to gain access to the relevant points-to predicate. Similarly to what we did earlier, we apply lemma 12 to conclude  $x_{tail} \rightsquigarrow x'_{last}$ , where  $x'_{last}$  is the current last node of the linked list (according to the invariant). As before, we perform case analysis on  $x_{tail} \rightsquigarrow x'_{last}$ .

**Case**  $x_{tail} = x'_{last}$ . We now know that  $x_{tail}$  is still the last node in the linked list, hence out  $x_{tail} \mapsto \text{None}$ , and the CAS will succeed. This instruction makes  $x_{tail}$  point to  $x_{new}$ , which essentially adds it to the linked list. Thus, the value in  $x_{new}$  becomes enqueued. In other words, this is a linearisation point, so we must apply the viewshift. We instantiate the viewshift with the abstract state of the queue  $xs_v$  from the invariant opening, and supply  $Q_\gamma \models_\bullet xs_v$  from the invariant and the  $P$  from the pre-condition. We hence get  $Q_\gamma \models_\bullet v :: xs_v$  and  $Q$ .

When closing the invariant, we use  $(v :: xs_v)$  for the abstract state,  $(x_{new} :: xs)$  for the concrete state,  $x_{new} :: xs_{queue}$  for the queue, and we take  $x_{new}$  to be the last node. The head and tail nodes remain the same. This means we give up  $Q_\gamma \models_\bullet v :: xs_v$  that we got from the viewshift, and the points-to predicate 4.3 (used to assert  $\text{isLL}(x_{new} :: xs)$ ). The only thing left to prove is  $Q_\gamma \cdot \gamma_{Last} \rightsquigarrow x_{new}$ . From the invariant opening, we have  $Q_\gamma \cdot \gamma_{Last} \rightsquigarrow x_{tail}$ . Since  $x_{tail} \rightsquigarrow x_{new}$ , we may apply lemma 14 to update the abstract points-to resource to  $Q_\gamma \cdot \gamma_{Last} \rightsquigarrow x_{new}$  and additionally obtain  $x_{new} \dashrightarrow Q_\gamma \cdot \gamma_{Last}$ . With this, we can close the invariant, and step to line 9. This line attempts to swing the tail, so we apply our swing-tail lemma (lemma 16) by supplying our invariant,  $x_{tail} \rightsquigarrow x_{new}$ , and  $x_{new} \dashrightarrow Q_\gamma \cdot \gamma_{Last}$ . This tells us that the CAS is safe, and it either succeeds or fails. The resulting value is the returned value of the enqueue function, but since the post-condition is simply  $Q$ , which we own, we are done.

**Case** out  $x_{tail} \mapsto^\square$  in  $x_{tail\_next} * x_{tail\_next} \rightsquigarrow x_{last}$ . Since  $x_{tail}$  doesn't point to None, the CAS will fail. We close the invariant and step to line 10. We finish by applying the induction hypothesis.

**Case** out  $x_{tail} \mapsto^\square$  in  $x_{tail\_next} * x_{tail\_next} \rightsquigarrow x_{last}$ . Using this we perform the load, which sets  $next$  to in  $x_{tail\_next}$ . Before closing the invariant, we apply lemma 13 with  $x_{tail\_next} \rightsquigarrow x_{last}$  and  $Q_\gamma \cdot \gamma_{Last} \rightsquigarrow x_{last}$  to obtain  $x_{tail\_next} \dashrightarrow Q_\gamma \cdot \gamma_{Last}$ . We now proceed to close the invariant. Next, we reach the consistency check. We handle it similarly to the previous case: open the invariant, get some  $x'_{tail}$ , close the invariant, and in case of inconsistency, apply induction hypothesis. If the nodes are consistent, we step to line 7. This time, the check will fail as  $next$  is in  $x_{tail\_next}$  which is not None. Hence we step to line 11 which attempts to swing the tail pointer. We here apply lemma 16 which we can do as we own the invariant,  $x_{tail} \rightsquigarrow x_{tail\_next}$ , and  $x_{tail\_next} \dashrightarrow Q_\gamma \cdot \gamma_{Last}$ . We step through to the recursive call and finish by applying the induction hypothesis.

□

## Deque

**Lemma 18** (Lock-Free M&S-Queue Specification - Enqueue).

$$\forall v_q, Q_\gamma, P, Q. \left( \forall xs_v. Q_\gamma \Rightarrow_\bullet xs_v * P \Rightarrow_{\mathcal{E} \setminus \mathcal{N}.i^\dagger} \triangleright \left( \begin{array}{l} (xs_v = [] * Q_\gamma \Rightarrow_\bullet xs_v * Q \text{ None}) \\ \vee \left( \begin{array}{l} \exists x_v, xs'_v. xs_v = xs'_v ++ [x_v] * \\ Q_\gamma \Rightarrow_\bullet xs'_v * Q \text{ (Some } x_v) \end{array} \right) \end{array} \right) \right) \right) \multimap$$

$$\{\text{is\_queue } v_q \ Q_\gamma * P\} \text{ dequeue } v_q \{v.Q \ v\}$$

*Proof.* We assume the viewshift and must prove the hoare triple. As before, we know from `is_queue` that the queue  $v_q$  is a location  $\ell_{queue}$  and there are locations  $\ell_{head}$  and  $\ell_{tail}$  so that

$$\ell_{queue} \mapsto^\square (\ell_{head}, \ell_{tail}) * \boxed{\text{queue\_invariant} \ell_{head} \ell_{tail} Q_\gamma}^{\mathcal{N}.queue} \quad (4.6)$$

The body of `dequeue` is the loop, so we immediately apply *l b* induction. We step through the function application, and into the looping function to line 3. This line dereferences  $\ell_{head}$ , so we open the invariant to access the associated points-to predicate. We obtain that  $\ell_{head}$  points to some  $x_{head}$ , meaning the load results to in  $x_{head}$ . We also derive the following information

$$\text{in } x_{head} \mapsto^\square (\text{val } x_{head}, \text{out } x_{head}) \quad (4.7)$$

$$x_{head} \dashrightarrow Q_\gamma.\gamma_{Head} \quad (4.8)$$

$$x_{head} \dashrightarrow Q_\gamma.\gamma_{Tail} \quad (4.9)$$

$$x_{head} \dashrightarrow Q_\gamma.\gamma_{Last} \quad (4.10)$$

From the abstract points-to predicates from the invariant and lemma 12 we get that  $x_{head} \rightsquigarrow x_{tail}$ , so by lemma 8, we know that  $x_{head}$  is a node. This shows 4.7. By reflexivity of *reach* (lemma 6) we additionally know that  $x_{head} \rightsquigarrow x_{head}$ . Lemma 13 then gives us 4.8 and 4.9.

Lastly, we use lemma 12 to deduce that  $x_{tail} \rightsquigarrow x_{last}$ . By transitivity of *reach* (lemma 7) we have that  $x_{head} \rightsquigarrow x_{last}$ , which we use with lemma 13 to get 4.10. We now close the invariant and step to line 4 which attempts to read  $\ell_{tail}$ . We open the invariant, which tells us that  $\ell_{tail}$  points to some  $x_{tail}$  (not necessarily the same as the previous invariant opening), and we perform the load. From the abstract points-to and *reach* predicates from the invariant together with 4.9 and lemmas 12, 9, and 13 we get the following

$$\text{in } x_{tail} \mapsto^\square (\text{val } x_{tail}, \text{out } x_{tail}) \quad (4.11)$$

$$x_{head} \rightsquigarrow x_{tail} \quad (4.12)$$

$$x_{tail} \dashrightarrow Q_\gamma.\gamma_{Tail} \quad (4.13)$$

$$(4.14)$$

We close the invariant and step to line 5. This line creates our *prophecy variable*  $p$ , which will be resolved on line 7. This allows us to reason about the result of the consistency check: we will later show that the expression associated with  $p$  (i.e.  $!(\text{fst}(!Q))$ ) evaluates to some value  $v_p$ , but we can already now case on whether  $v_p$  will be equal to in  $x_{head}$  – the left hand side of the equality check on line 7.



**Case** in  $x_{head} = v_p$ . We continue to line 6, which finds out what  $x_{head}$  points to. As  $x_{head}$  could be the last node in the linked list, we don't have the relevant points-to predicate. We therefore open the invariant. We get the three nodes  $x'_{head}$ ,  $x'_{tail}$ , and  $x_{last}$ . Specifically,  $x_{last}$  is the last node in the linked list, and  $Q_{\gamma \cdot \gamma_{Last}} \mapsto x_{last}$ . Combining this with 4.10 and lemma 12 we conclude  $x_{head} \rightsquigarrow x_{last}$ . This gives us two cases to consider: either  $x_{head} = x_{last}$  or  $x_{head}$  points to some  $x_{head\_next}$ , which reaches  $x_{last}$ .

**Case**  $x_{head} = x_{last}$ . This corresponds to the queue being empty, which we derive below.

As  $x_{last}$  is the last node, we have that out  $x_{head} \mapsto \text{None}$ , hence the load resolves to None. Using the abstract points-to predicates from the invariant together with 4.8, 4.9, and lemma 12 we get  $x_{head} \rightsquigarrow x'_{head}$  and  $x_{head} \rightsquigarrow x_{tail'}$ . We can now apply lemma 10 three times to conclude  $x_{head} = x'_{head} = x_{tail'} = x_{tail}$ . Since  $x_{head'}$  points to None, then  $xs_{queue}$  has to be empty (if it wasn't we could deduce that  $x_{head'}$  pointed to a node). This also implies that abstract state of the queue  $xs_v$ , is empty,  $xs_v = []$ .

Because the load resolved to None, then the variable  $next$  in the code will be None, and since we are in the case where the consistency check passes, and since we have derived that  $x_{head} = x_{tail}$ , we already know now that dequeue will return None. In other words, this is a linearisation point.

Since  $xs_v = []$ , then our abstract state predicate from the invariant states  $Q_{\gamma} \models_{\bullet} []$ . We thus instantiate the viewshift with  $[]$ , and supply the  $P$  from the pre-condition. As  $xs_v = []$  we can conclude that the first disjunct must be true (the second contains a contradiction), so we get  $Q \text{ None}$  and  $Q_{\gamma} \models_{\bullet} []$ . As we haven't changed any resources, we can close the invariant again.

We reach the consistency check on line 7. By 4.6, we know that  $!(fst(!Q))$  steps to  $!(\ell_{head})$ , but to resolve the prophecy, we must first show what  $!(\ell_{head})$  evaluates to. This resource is inside the invariant so we open it. We get that  $\ell_{head} \mapsto$  in  $x''_{head}$  for some node  $x''_{head}$ . We close the invariant and resolve the prophecy:  $!(fst(!Q))$  evaluated to in  $x''_{head}$ . In other words,  $v_p =$  in  $x''_{head}$ , and therefore in  $x_{head} =$  in  $x''_{head}$ . Since the remaining if statement on line 7 compares in  $x_{head}$  to in  $x''_{head}$ , we know that we will take the 'then' branch, so we step to line 8. Since  $x_{head} = x_{tail}$  and  $next$  was set to None, we step to line 10 which returns None. The post-condition thus requires us to prove  $Q \text{ None}$ , which we already have.

**Case** out  $x_{tail} \mapsto^{\square}$  in  $x_{tail\_next} * x_{tail\_next} \rightsquigarrow x_{last}$ . **►finish case◄**

**Case** in  $x_{head} \neq v_p$ . We step to line 6 which finds out what  $x_{head}$  points to. To access the relevant points-to predicate, open the invariant. Importantly, we get that there is some last node of the linked list,  $x_{last}$ , with  $Q_{\gamma \cdot \gamma_{Last}} \mapsto x_{last}$ . By combining this with 4.10 and lemma 12 we deduce that  $x_{head} \rightsquigarrow x_{last}$  which means that either  $x_{head}$  is the last node, and hence out  $x_{head} \mapsto$

None, or there is some other node  $x_{head\_next}$  and out  $x_{head} \mapsto^\square x_{head\_next}$ . This shows that the load is safe. The actual value that the load resolves to is unimportant, as it won't be used in this case. We thus perform the load, close the invariant and step to line 7.

By 4.6, we know that  $!(fst(!Q))$  steps to  $!(\ell_{head})$ , so to resolve the prophecy, we show what  $!(\ell_{head})$  evaluates to. We open the invariant, which gives us that  $\ell_{head} \mapsto \text{in } x'_{head}$  for some node  $x'_{head}$ . We close the invariant again, and resolve the prophecy:  $!(fst(!Q))$  evaluated to  $\text{in } x'_{head}$ . That is,  $v_p = \text{in } x'_{head}$ , and therefore  $\text{in } x_{head} \neq \text{in } x'_{head}$ . We hence take the 'else' branch and step to line 18. This consists of a recursive call to the loop function, hence we are done by the induction hypothesis.

□

## 4.5 Discussion

►Mention that one can remove the consistency check and hence also the prophecy◄

## Chapter 5

# Conclusion and Future work

►conclude on the problem statement from the introduction◄

►Mention the possibility of simplifying queue invariant for lock-free by removing isLL (and adding x\_last -> None)◄



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## Appendix A

# The Technical Details

