Exercise 3

Let us start from equation

$$u(t) = \frac{1}{C_m} \frac{\tau_m \tau_s}{\tau_m - \tau_s} \sum_{\substack{\text{spikes}\\ t_i}}' W_i\, \Theta(t-t_i) \left[\exp\left(-\frac{t-t_i}{\tau_m}\right) - \exp\left(-\frac{t-t_i}{\tau_s}\right) \right]$$

Let us substitute $\tau_m = 2\tau_s$

$$u(t) = \frac{2\tau_s}{C_m} \sum_{\substack{\text{spikes}\\ t_i}} W_i\, \Theta(t-t_i) \left[\exp\left(-\frac{t-t_i}{2\tau_s}\right) - \exp\left(-\frac{t-t_i}{\tau_s}\right) \right] =$$

$$= \frac{2\tau_s}{C_m} \left\{ \exp\left(\frac{-t}{2\tau_s}\right) \sum_{\substack{\text{spikes}\\ t_i}} W_i\, \Theta(t-t_i) \exp\left(\frac{t_i}{2\tau_s}\right) + \right.$$

$$\left. - \exp\left(-\frac{t}{\tau_s}\right) \sum_{\substack{\text{spikes}\\ t_i}} W_i\, \Theta(t-t_i) \exp\left(\frac{t_i}{\tau_s}\right) \right\}$$

Now let us impose the condition $u(t) = \Theta$. We consider only the set of spikes preceeding the output spike-time, that is

$$C = \{ t_i \mid t_i < t_{out} \}$$

Of course $\Theta(t_{out} - t_i) = 1 \ \forall\ t_i \in C$. Hence last equation reduces to:

$$\Theta = u(t_{out}) = \frac{2\tau_s}{C_m} \left\{ \exp\left(\frac{-t}{2\tau_s}\right) \sum_{t_i \in C} W_i \exp\left(\frac{t_i}{2\tau_s}\right) - \exp\left(\frac{-t}{\tau_s}\right) \sum_{t_i \in C} W_i \exp\left(\frac{t_i}{\tau_s}\right) \right\} =$$

$$= \frac{2\tau_s}{C_m} \left\{ \exp\left(\frac{-t}{2\tau_s}\right) a_2 - \exp\left(\frac{-t}{\tau_s}\right) a_1 \right\}$$

Now let us ~~...~~ define $x \equiv \exp\left(-\frac{t}{2\tau_s}\right) \Rightarrow x^2 = \exp\left(\frac{-t}{\tau_s}\right)$ and we get

$$\frac{2\tau_s}{C_m} \left\{ a_2 x - a_1 x^2 \right\} - \Theta = 0$$

which is of the form

$$ax^2 + bx + c = 0$$

if we set

$$a = a_1 \qquad b = -a_2 \qquad c = \frac{\theta C_m}{2\tau_s}$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{a_2 \pm \sqrt{a_2^2 + \frac{2 a_2 \theta C_m}{\tau_s}}}{2a_1} =$$

Substituting the expression for x and using $\tau_s = C_m / g_L$

$$\exp(-t/2\tau_s)_{1,2} = \frac{a_2 \pm \sqrt{a_2^2 - 2a_1 g_L \theta}}{2a_1}$$

$$(t/\tau_s)_{1,2} = 2 \ln\left(\frac{2a_1}{a_2 \pm \sqrt{a_2^2 - 2a_1 g_L \theta}}\right)$$

~~Above~~ The solution with smaller time is the one of interest, since the other one corresponds to the region in which the output potential is decaying after crossing. Thus

$$\boxed{\frac{t}{\tau_s} = 2\ln\left(\frac{2a_1}{a_2 + \sqrt{a_2^2 - 2a_1 g_L \theta}}\right)}$$

Should be a 4 but I obtain 2 (!!)