# Brain Inspired Computing - Sheet 7

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# Übung 7

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We define the disease set $\{P, N\}$ & the test set $\{p, n\}$ with positive ($p$ & $P$) and negative ($n$ & $N$) results.

The sensitivity is then:

$$p(p|P) = 0.95 \rightarrow p(n|P) = 1 - p(p|P) = 0.05$$

The specificity is:

$$p(n|N) = 0.9 \rightarrow p(p|N) = 1 - p(n|N) = 0.1$$

We can use the incidence to express our illness probability:

$$p(P) = \frac{50}{100\,000} \vee \frac{500}{100\,000} \vee \frac{5000}{100\,000}$$

as well as the healthy probability:

$$p(N) = 1 - p(P) = 1 - \frac{50}{100\,000} \vee 1 - \frac{500}{100\,000} \vee 1 - \frac{5000}{100\,000}$$

a) We are interested in the probability $p(N|n)$

According to the Bayes' theorem,

$$p(N|n) = \frac{p(n|N)\,p(N)}{p(n)}$$

where $p(n) = p(n|N)\,p(N) + p(n|P)\,p(P)$

we obtain:

w-

$$p(N|n) = \frac{p(n|N)\,p(N)}{p(n|N)\,p(N) + p(n|P)\,p(P)}$$

with values:

$p(N|n) \approx 0.999972 \approx 99.997\,\%$ (Incidence 50)

$\vee\ 0.99972 \approx 99.97\,\%$ (Incidence 500)

$\vee\ 0.9971 \approx 99.71\,\%$ (Incidence 5000)

b) we want: $p(P|p)$:

$$p(P|p) = \frac{p(p|P)\,p(P)}{p(p)} = \frac{p(p|P)\,p(P)}{p(p|P)\,p(P) + p(p|N)\,p(N)}$$

$\Rightarrow\ p(P|p) \approx 0.00473 \approx 0.473\,\%$ (Incidence 50)

$\vee\ 0.04556 \approx 4.556\,\%$ (Incidence 500)

$\vee\ 0.\overline{66} \approx 66{,}67\,\%$ (Incidence 5000)

We now set $p(P) = 0.6 \rightarrow p(N) = 0.4$ and see, that the results of c) & d) are no longer dependent on the actual incidence:

c) $p(P|p) = \dfrac{p(p|P)\,p(P)}{p(p|P)\,p(P) + p(p|N)\,p(N)} \approx 0.9344 \approx 93.44\,\%$

d) $p(P|n) = \dfrac{p(n|P)\,p(P)}{p(n|P)\,p(P) + p(n|N)\,p(N)} \approx 0.0769 \approx 7.69\,\%$

# Exercise 2

## (a)

$$p\left(z_k^{n+1}=1 \mid z_{\backslash k}\right)=\frac{p\left(z_k^{n+1}=1 \mid z_{\backslash k}\right)}{p\left(z_k^{n+1}=0 \mid z_{\backslash k}\right)+p\left(z_k^{n+1}=1 \mid z_{\backslash k}\right)}=\frac{1}{1+\frac{p\left(z_k^{n+1}=0 \mid z_{\backslash k}\right)}{p\left(z_k^{n+1}=1 \mid z_{\backslash k}\right)}} \tag{1}$$

Using the definition of conditional probability

$$p(y|x)=\frac{p(y,x)}{p(x)} \tag{2}$$

one gets that

$$\frac{p\left(z_k^{n+1}=0 \mid z_{\backslash k}\right)}{p\left(z_k^{n+1}=1 \mid z_{\backslash k}\right)}=\frac{p\left(z_k^{n+1}=0, z_{\backslash k}\right)}{p\left(z_k^{n+1}=1, z_{\backslash k}\right)}=\frac{p(\mathbf{z})}{p(\mathbf{z}')} \tag{3}$$

where

$$\mathbf{z}=(z_1,\ldots,z_{k-1},0,z_{k+1},\ldots,z_n) \qquad \mathbf{z}'=(z_1,\ldots,z_{k-1},1,z_{k+1},\ldots,z_n)$$

Using the probability density function

$$p(\mathbf{z})=\frac{1}{Z}\exp\left[\frac{1}{2}\mathbf{z}^t W\mathbf{z}+\boldsymbol{b}^t\mathbf{z}\right]$$

together with the fact that $\mathbf{z}$, $\mathbf{z}'$ are equal a part from the term $k$ and that $z_k=0, z'_k=1$ we have now that

$$\frac{p(\mathbf{z})}{p(\mathbf{z}')}=\exp\left[\frac{1}{2}\left(\mathbf{z}^t W\mathbf{z}+\boldsymbol{b}^t\mathbf{z}-\mathbf{z}'^t W\mathbf{z}'-\boldsymbol{b}^t\mathbf{z}'\right)\right]=\exp\left[\frac{1}{2}\left(\mathbf{z}^t W\mathbf{z}-\mathbf{z}'^t W\mathbf{z}'-b_k\right)\right]$$

The quadratic form can be worked out as

$$\mathbf{z}^t W\mathbf{z}-\mathbf{z}'^t W\mathbf{z}'=z_i W_{ij} z_j - z'_i W_{ij} z'_j = -z'_k W_{kj} z'_j - z'_i W_{ik} z'_k = -2z'_k W_{kj} z'_j$$

where $k$ is meant as a fixed index.
Inserting everyting into 1 and the fact that all the terms remain unchanged except $z_k$, which is set to 1, we get the result

$$p\left(z'_k=1 \mid z_{\backslash k}\right)=\frac{1}{1+\exp\left(-2\sum_j z'_k W_{kj} z'_j - b_k\right)}=\frac{1}{1+\exp\left(-2\sum_j W_{kj} z_j - b_k\right)}=\sigma(u_k)$$

One could do the same calculation for $z_k^{n+1}=0$ and would of course find that

$$p\left(z'_k=0 \mid z_{\backslash k}\right)=1-\sigma(u_k)$$

## (b)

The equality is trivially verified if $z_k=z'_k$, hence we assume $z_k \neq z'_k$. We have then to prove that

$$p_{\mathrm{T}}\left(z_k^{n+1}=0 \mid z_k^n=1,\mathbf{z}_{\backslash k}\right) p\left(z_k^n=1 \mid \mathbf{z}_{\backslash k}\right)=p_{\mathrm{T}}\left(z_k^{n+1}=1 \mid z_k^n=0,\mathbf{z}_{\backslash k}\right) p\left(z_k^n=0 \mid \mathbf{z}_{\backslash k}\right)$$

or

$$\frac{p_{\mathrm{T}}\left(z_k^{n+1}=0 \mid z_k^n=1,\mathbf{z}_{\backslash k}\right)}{p_{\mathrm{T}}\left(z_k^{n+1}=1 \mid z_k^n=0,\mathbf{z}_{\backslash k}\right)}=\frac{p\left(z_k^n=0 \mid \mathbf{z}_{\backslash k}\right)}{p\left(z_k^n=1 \mid \mathbf{z}_{\backslash k}\right)} \tag{4}$$

Since the updating probability of the term $k$ does not depend on the value of the latter at the previous step, but just relies on the value of all the other components and the new value for $z_k$ we may write

$$p_{\mathrm{T}}\left(z_k^{n+1} \mid z_k^n, \mathbf{z}_{\setminus k}\right) = p_{\mathrm{T}}\left(z_k^{n+1} \mid \mathbf{z}_{\setminus k}\right) = \begin{cases} \sigma(u_k) & \text{if } z_k^{n+1} = 1 \\ 1 - \sigma(u_k) & \text{if } z_k^{n+1} = 0 \end{cases}$$

The right hand-side member of equation 4 can be rewritten using 2 as

$$\frac{p\left(z_k^n = 0 \mid \mathbf{z}_{\setminus k}\right)}{p\left(z_k^n = 1 \mid \mathbf{z}_{\setminus k}\right)} = \frac{p\left(z_k^n = 0, \mathbf{z}_{\setminus k}\right)}{p\left(z_k^n = 1, \mathbf{z}_{\setminus k}\right)} = \frac{p(\mathbf{z})}{p(\mathbf{z}')}$$

and equality 4 follows from 3.

# BIC #7

## EX 03

$$E(z) = \frac{1}{2} z^t W z + b^t z$$

ISING : $z_k \in \{-1, 1\}$

Boltzmann : $z_k \in \{0, 1\}$

a) $W_{ks} = 4 \hat{W}_{ks}$

$b_k = 2\hat{b}_k - 2\sum_{s=1}^{k} \hat{W}_{ks}$

Maps $\Omega = \{0,1\}^k \to \hat{\Omega} = \{-1, 1\}^k$

$E(z) = E(\hat{z}) + C \quad \forall \hat{z} = 2z - 1$ and $C = \frac{1}{2}\sum_{k,s=1}^{k} \hat{W}_{ks} - \sum_{k=1}^{k} \hat{b}_k$

$E(z) = 2 z^t \hat{W} z + 2\hat{b}^T z - 2\sum_{s=1}^{k} \hat{W}_{sk} z_k =$ → here implicit $\sum_k$

$E(\hat{z} = 2z-1) = 2z^t \hat{W} z + \frac{1}{2} 1^t \hat{W} 1 + 2\hat{b}^t z - b^t 1$ where $1_k = 1$

$\underbrace{- 1^t \hat{W} z - z^t \hat{W} 1}_{= -2\sum_{s=1}^{k} \hat{W}_{sk} z_k}$ (sum over k)

then, $-b^t 1 = -\sum_k b_k$ and $\frac{1}{2} 1^t \hat{W} 1 = \frac{1}{2}\sum_{k,s=1}^{k} \hat{W}_{ks}$

b) $p(z) = \frac{1}{Z} e^{-E(z)} \to p'(z) = \frac{1}{Z'} e^{-E(z) + c}$

$p'(z) = \frac{1}{Z'} e^c e^{-E(z)} \Rightarrow \frac{1}{Z} = \frac{e^c}{Z'}$ → the normaliz. factor reestablish the correct probability

→ the probability does not change

[Also can be seen as $Z' = \sum_k e^c e^{-E(z)} = e^c \sum_k E^{-E(z)} = e^c Z$]

c) $b_1 = b_2 = 0 \quad W_{12} = W_{21} \geq 0$

$\Omega$: $\begin{array}{cc} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} & \begin{array}{c} z^1 \\ z^2 \\ z^3 \\ z^4 \end{array} \end{array}$ $\hat{\Omega}$: $\begin{array}{cc} \begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} & \begin{array}{c} \hat{z}^1 \\ \hat{z}^2 \\ \hat{z}^3 \\ \hat{z}^4 \end{array} \end{array}$

$W_{12} = 0$ : $E(z) = 0 \ \forall z \in \Omega$, $E(\hat{z}) = 0 \ \forall \hat{z} \in \hat{\Omega}$

If we increase $W_{12}$, we obtain always a positive (or null) contribution from $z^t W z$ in $\Omega$.

In $\hat{\Omega}$, $2 z^t W z$ count an extra contribution, that is balanced from the term $-2\sum_{j=1}^{k} W_{kj} z_k$. Part of $W$ act then as a bias term in $\hat{\Omega}$.