1 Some shorter questions

a

This result can be explained by means of the nuclear shell model. According to this model each sub-level of the nucleus can be identified by 4 quantum numbers $\{n, j, l, m_l\}$. For fixed $\{n, j, l\}$ one has that the $2l+1$ sub-levels given by $m_l = -l, -l+1, \ldots, l-1, l$ are degenerate and each of this sub-levels can contain two nucleons with different spin z-projrection ($m_s = \pm 1$).
At first one can notice that if one sub-shell is filled with two nucleons (of the same type) then the spin contibution of that sub-shell is 0. Having an even number of protons and neutrons means that they combine in couples in the sub-shell in a way such that the contribution to the spin of each sub-shell is 0. For what concerns the parity, each particle in the nucleus gives a contribute of $(-1)^l$ (the intrinsic parity of the nucleons is $+1$) and the total parity is the product of single parities. The product of the parities of two protons (neutrons) in the same sub-shell is always $+1$ since they, in particular, share the same quantum number l. Hence it is legitimate to expect an even-even nucleus to be in the state 0^+.

Answer: spin 0, parity + .

2 Nuclear binding energy

a

The mass of the atom is given by

$$M(^{48}_{20}\mathrm{Ca}^{28}) = Z(m_p + m_e) + (A - Z)m_n - \frac{B(A, Z)}{c^2}$$

where B indicates the binding energy, which in turns can be calculated via the semi-empirical formula (atomic units)

$$B(A, Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_{sym} \frac{(A-2Z)^2}{A} + \delta(A, Z) \tag{1}$$

where

$$\delta(A, Z) = \begin{cases} \frac{a_P}{A^{1/2}} & \text{if Z and N are even} \\ 0 & \text{if A is odd} \\ -\frac{a_P}{A^{1/2}} & \text{if Z and N are odd} \end{cases}$$

One can simply pop in the values of A and Z into the formula using the fitted coefficients (source Wikipedia) in units of MeV/c^2

$$a_v = 15.8 \qquad a_s = 18.3 \qquad a_c = 0.714 \qquad a_{sym} = 23.2 \qquad a_P = 12$$

and obtains $B(A, Z) = 412.84\ MeV/c^2$. Hence, inserting the result into 2 one obtains

$$M(^{48}\mathrm{Ca}) \simeq 44670.64\ MeV/c^2 \simeq 47.96\ u$$

Thhe result is close to the experimental value (47.952). One could justify eventual differences by observing that the SEMF is not a complete theoretical model, but rather a fit to experimental data of a model with some theoretical fundations. As such, it is reasonable that the model can give an overall good description of the data but cannot perfectly describe each of them: this is especially the case of lighter atoms for which the quantum shell structures of the nucleus should be taken into account.

b

One can calculate the energy to remove a neutron as the difference in energy between the atom's state wth the neutron and without the neutron. The energy is the sum of two terms, one that take accounts of the mass of the particles and it is simply the sum of all the masses. The second term, instead, is the binding energy. In the initial and final state the mass energy contributions are the same (we are not making any particle disappear, we are just moving one away), hence the difference is only due to the binding energy contribution. One can calculate this contribution as

$$B_{neutron} = B(A, Z) - B(A-1, Z)$$

and using expression 1 with $A = 44$ and $Z = 20$ to calculate the two quantities one ends up with an estimation for the energy required to extract the neutron

$$B_{neutron} \approx 11.14\ MeV/c^2$$

c

One can search for the most stable atom with fixed A searching for the minimum of the mass formula

$$M_A(Z) = Z(m_p + m_e) + (A - Z)m_n - B_A(Z)$$

Since one does not know if Z is even or odd in advance, one can set the pairing term in the binding energy to 0: this will lead to a non-integer Z and the true minimum value will be either the closest greater or smaller integer (one has to check which of the two has minimum energy). By direct calculation

$$0 = \frac{dM_A}{dZ} = m_p + m_e - m_n - \frac{a_c}{A^{1/3}}(A - 2Z) + 4\frac{a_{symm}}{A}(A - 2Z)$$

and rearranging terms

$$Z = \frac{m_n - (m_p + m_e) + 4a_{symm}A + a_c A^{2/3}}{2a_c A^{2/3} + 8a_{symm}} \approx 56.59 \qquad \longrightarrow \qquad Z = 56$$

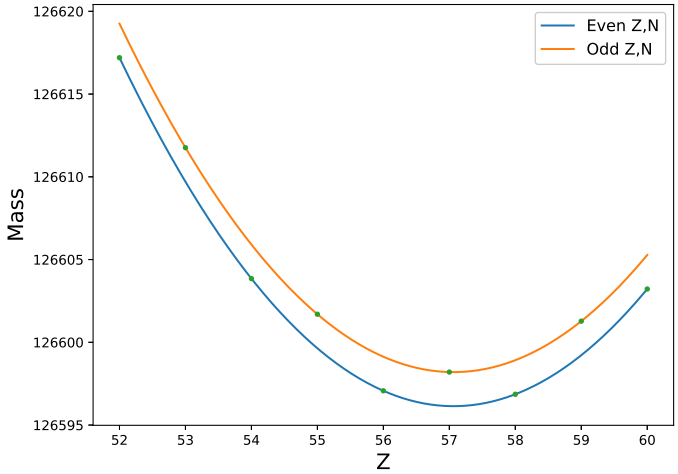

Figure 1: Even A

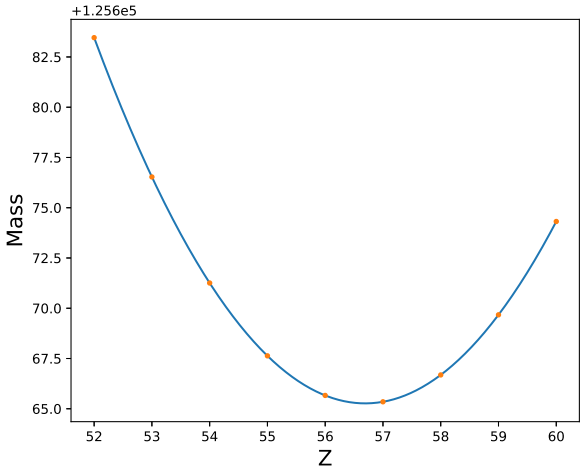

Figure 2: Odd A

3 Higgs boson decay

a

In the resting frame of the Higgs boson the invariant mass is (atomic units)

$$W = \sqrt{(\sum_i E_i)^2 - |\sum \mathbf{p}_i|^2} = E_H = m_H \tag{2}$$

The invariant mass is a Lorent invariant, hence this quantity is the same that one observer would measure in the laboratory frame. More specifically, one can calculate W after the process using the electrons data

$$W = \sqrt{(\sum_i E_i)^2 - |\sum \mathbf{p}_i|^2} = \sqrt{(\sum_i E_i)^2} = \sum \sqrt{m_i^2 + p_i^2}$$

By equating this expression to 2 and inserting the given values together with the electrons mass one obtains

$$m_h \simeq 126.2 \ GeV/c^2$$

b

In principle each pair of electron-positron can be generated by the Z boson, in the sense that no rules would be violated. In other words there are 4 different ways to combine the couples of electron-positron to the bosons

1. $Z \to e_1^+ + e_1^-, \quad Z^* \to e_2^+ + e_2^-$
2. $Z \to e_1^+ + e_2^-, \quad Z^* \to e_2^+ + e_1^-$
3. $Z \to e_2^+ + e_1^-, \quad Z^* \to e_1^+ + e_2^-$
4. $Z \to e_2^+ + e_2^-, \quad Z^* \to e_1^+ + e_1^-$

One can now impose the conservation of the 4−momentum in the bosons' decay and in particular, for the Z boson, this means

$$E_Z = \sqrt{m_Z^2 + p_Z^2} = E_{e^+} + E_{e^-} = \sqrt{m_{e^+}^2 + p_{e^+}^2} + \sqrt{m_{e^-}^2 + p_{e^-}^2}$$

and by rearrangin terms

$$p_Z^2 = (E_{e^+} + E_{e^-})^2 - m_Z^2$$

this equation gives the value of the momentum that the Z boson must have in order to satisfy the conservation of the energy. By explicit calculation for the 4 cases one obtains

1. $p_Z^2 \simeq -2414 \ GeV/c$
2. $p_Z^2 \simeq -7350 \ GeV/c$
3. $p_Z^2 \simeq 744 \ GeV/c$
4. $p_Z^2 \simeq -5872 \ GeV/c$

hence only in the third case the particle can be regarded as physical and not virtual.

c

d

4

5 Quantum numbers

a

Parity

Let us consider a wavefunction $\psi(\mathbf{r})$. One can define the parity operator $\hat{P}$ as an operator such that

$$\hat{P}\,\psi(\mathbf{r}) = \psi(-\mathbf{r})$$

Since

$$\hat{P}^2\,\psi(\mathbf{r}) = \hat{P}\,\psi(-\mathbf{r}) = \psi(\mathbf{r}) \tag{3}$$

the eigenvalues of the parity operator are ± 1 and the corresponding eigenfunctions are the odd (eigenvalue -1) and even (eigenvalue $+1$) wavefunctions. If the wavefunction describes the state of particle, and the state is an eigenstate of the parity operator, the corresponding eigenvalue is also said to be the (intrinsic) parity of the particle.
Let us consider a system of two particles $A + B$ described by a wavefunction $\psi_{AB}(\mathbf{r}_A, \mathbf{r}_B)$. It can be proven that the parity of the system is given by

$$\hat{P}\psi_{AB} = \pi_A \pi_B (-1)^l \psi_{AB}$$

where l denotes the orbital angular momentum quantum number of the relative motion. Hence in a reaction of the type $A + B \to C + D$ described by a hamiltonian $\hat{H}$ that commutes with parity, parity is conserved or, in other words

$$\pi_A \pi_B (-1)^{l_{AB}} = \pi_C \pi_D (-1)^{l_{CD}}$$

This, for example, is not the case of the weak interaction where, in general, the hamiltonian operator does not commute with the parity operator.

Charge conjugation

The charge conjugation operator is an operator that changes the sign of all the forces' quantum charges, specifically electric charge, baryon number, lepton number, flavor charges, isosping, ... In particular, if applied to a particle, the C-parity operator transforms the particle into its antiparticle. If a particle is in a state $|\psi\rangle$ then the action of the conjugation operator reads

$$\hat{C}\,|\psi\rangle = |\bar{\psi}\rangle$$

where $|\bar{\psi}\rangle$ represents the antiparticle's state.
The eigenstates of the C-parity operator are the systems neutral to all force charges like the photon or particle-antparticle bound states. In this case

$$\hat{C}\,|\psi\rangle = C_\psi\,|\psi\rangle$$

and C_ψ is said to be the C-parity of the particle. As for the the parity operator, the eigenvalues can be only ± 1.
Both P-parity and C-parity are conserved in the processes governed by all the fundamental forces but the weak force.

b

c

^{15}N atom has 7 protons and 8 neutrons. Since the neutrons (in the ground state) are all coupled (each sub-shell admits two nucleons) they do not contribute to the spin of the nucleus and the parity contribution is $+1$. Instead the protons configuration of the ^{15}N atom in the ground state is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^1$: only the last proton contributes to the interested quantities. Since the orbital p represents the quantum number $l = 1$ the parity of the the nucleus is the product of the neutrons an protons contribution, that is $(+1)\cdot(-1)^1 = -1$. The spin is then $1/2$.
There are three possible alternatives for the first excited state, all of which consists in moving a proton or a neutron to another level starting from the ground state configuration.

1. Move the $1p_{1/2}$ proton to the $1d_{5/2}$ level
   $\longrightarrow$ the new proton configuration is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^{-2}\,(1d_{5/2})^1$ and the spin-parity is $5/2^+$.

2. Move the $1p_{3/2}$ proton to the $1p_{1/2}$ level
   $\longrightarrow$ the new proton configuration is $(1s)^2\,(1p_{3/2})^{-1}\,(1p_{1/2})^2$ and the spin-parity is $3/2^-$.

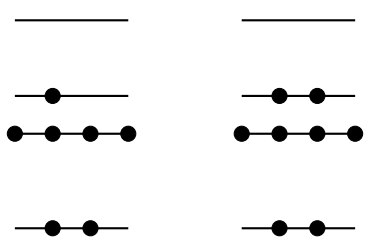

Figure 3: nothing

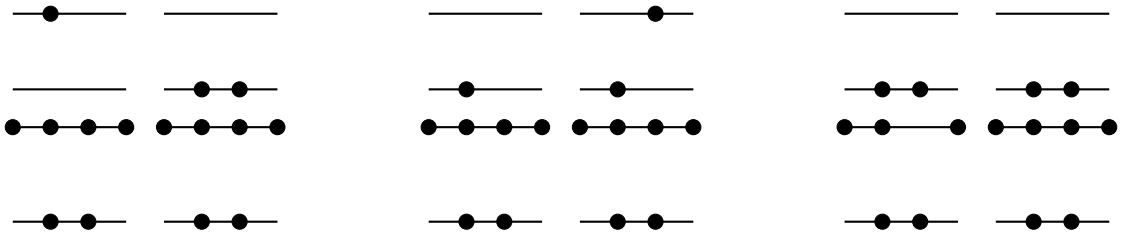

Figure 4: nothing

3. Move the $1p_{1/2}$ neutron to the $1d_{5/2}$ level
   $\longrightarrow$ the new neutorn configuration is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^{-1}\,(1d_{5/2})^1$. The parity here is non-trivial because of the angular momemnta addition rules.

By looking at the suggested table, one can notice that the first excited state is described by configuration 1, the second excited state by configuration 3 and the third excited state is described by configuration 2.

d

The third excited state is the one that has $J^p = \frac{3}{2}^-$. From this state the nucleus can decay into the second excited state, the first excited state and the ground state. Let us first consider the decay to the ground state. When the excited proton returns to the $1p_{3/2}$ level there is no change in parity and one has that $\Delta J = J_{exc} - J_{ground} = 1$. This means that one must have odd-L magnetic fields and even-L electric fields.
Because of the general selection rules in the gamma decays one has that, said L the photon's angular momentum

$$|J_{exc} - J_{ground}| \leq L \leq J_{exc} + J_{ground} \qquad \longrightarrow \qquad 1 \leq L \leq 2$$

The allowed states are

$$M_1, E_2$$

For $A = 15$ one has that (ADD REFERENCE TO KRANE)

$$\frac{\lambda(E_2)}{\lambda(M_1)} \approx 10^{-3} \tag{4}$$

hence we expect the M_1 decay to be the most probable one, but with a non-neglectable contribution of E_2.
In an analogous way one can find that for the decays to the second and first excited states the relations reported in table 1. For the transition $2 \to 1$ one has that the dominant field is E_2 ($2 \leq L \leq 3$ and no

Transition	ΔE	$\Delta\pi$	L	Fields	Dominants
$3 \to 0$	$\approx 6.3\ MeV$	no	$1 \leq L \leq 2$	M_1, E_2	M_1, E_2
$3 \to 1$	$\approx 1\ MeV$	yes	$1 \leq L \leq 4$	E_1, M_2, E_3, M_4	E_1
$3 \to 2$	$\approx 1\ MeV$	yes	$1 \leq L \leq 2$	E_1, M_2	E_1

Table 1

parity change) while for the decay $2 \to 0$ the dominant field is E_1 ($0 \leq L \leq 1$ and parity change). Hence

from the second excited state the nucleus cand ecay to the ground state via E_1 radiation, or it can decay to the first excited state via E_2 radiation. By comparing probabilities (REFERENCE TO KRANE)

$$\frac{P_{2\to 0}}{P_{2\to 1}} \approx \frac{\lambda(E_1)}{\lambda(E_2)} = \frac{1}{7.3}\cdot 10^7 \cdot A^{-2/3} > 1$$

6

7 Radioactive decay

a

Said $N(t)$ the number of atoms present at time t, the radioactive decay law states that

$$\mathcal{A} = -\frac{dN}{dt} = \lambda N \tag{5}$$

where λ is called the decay constant.
By integrating it in time one obtains

$$N(t) = N_0\, e^{-\lambda t} \tag{6}$$

Now if we consider a chain decay of the type $A \to B \to C$ with initial conditions

$$N_A(t=0) = N_0 \qquad N_B(t=0) = N_C(t=0) = 0$$

one can immediately obtain the number of A atoms as a function of time by applying 5

$$N_A(t) = N_0 e^{-\lambda_A t}$$

To study the number of atoms of type B one has to take account of two factors: the number of "created" atoms by means of A's decay, and the number of atoms decayed into C. This means that the radioactive law reads

$$N_B(t) = -\frac{1}{\lambda_B}\frac{dN_B(t)}{dt} + N_{A\to B}(t)$$

where the first term takes account of the fact on the right side member represents the B's decay and the the last term represents the number of B obtained by A's decaying.
One can now substitute 6 obtaining

$$\frac{dN_B(t)}{dt} + \lambda_B N_B(t) = N_0\, e^{-\lambda_A t}$$

The general solution of this differential equation is

$$N_B(t) = e^{-A(t)}\left(N_B(0) + \int_0^t N_0 e^{-\lambda_A s} e^{A(s)}\, ds\right)$$

where $A(s) = \lambda_B \int dt = \lambda_B t$. Hence

$$N_B(t) = e^{-\lambda_B t}\int_0^t N_0 e^{-(\lambda_A-\lambda_B)s}\, ds = \frac{N_0\, e^{-\lambda_B t}}{\lambda_A-\lambda_B}\left(1 - e^{-(\lambda_A-\lambda_B)t}\right) =$$
$$= N_0\,\frac{e^{-\lambda_A t} - e^{-\lambda_B t}}{\lambda_B - \lambda_A}$$

b

Let us consider the single-decay law

$$N(t) = N_0 e^{-\lambda t}$$

One can notice here that λ's physical units are inverse of time. The time $t_{1/2}$ is defined as the time necessary to halve the number of atoms fro N_0. This can be found by imposing $N(t) = N_0/2$ and solving for t one obtains

$$t_{1/2} = \frac{1}{\lambda}\log 2 \equiv \tau \log 2$$

where I introduced the mean lifetime $\tau \equiv 1/\lambda$. This last term, which has units of time, has a particular meaning. To understand this, one can first think of

$$p(t) = \frac{N(t)}{\int N(t)\, dt}$$

as a probability distribution function related to the probability of a particle to decay. One can then calculate the expected lifetime of such particle

$$\langle t \rangle = \frac{\int tN(t)\, dt}{\int N(t)\, dt} = \frac{\int_0^\infty te^{-\lambda t}\, dt}{\int_0^\infty e^{-\lambda t}} = \frac{1}{\lambda}$$

and it is exactly τ.
Another interesting quantity is the natural decay width Γ. One can make plot the probability of a decay to occur as function of the decaying particle's energy: this is given by the Breit Wigner formula. The half width at half maximum of this distribution is the factor Γ which has units of energy and is related to the average lifetime of the particle trough the energy-time uncertainty relation

$$\Delta E\, \Delta t >= \frac{\hbar}{2}$$

EXPAND AND EXPLAIN BETTER HERE

c

$1mCi = 37MBq$. Taking the solution

$$N_1(t) = N_0\, e^{-\lambda t}$$

and imposing $\mathcal{A}(t=0) = -\frac{dN_1(t)}{dt}|_{t=0} = 37MBq$ one obtains

$$N_0 = \frac{37MBq}{\lambda} \simeq 29689798324$$

d

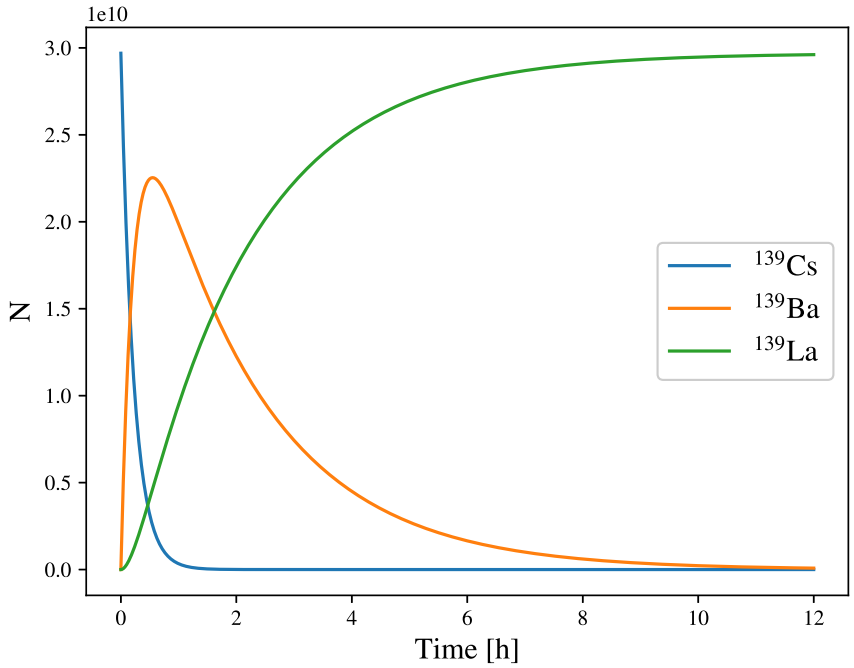

Figure 5: Radioactive decay chain

COMMENTS HERE

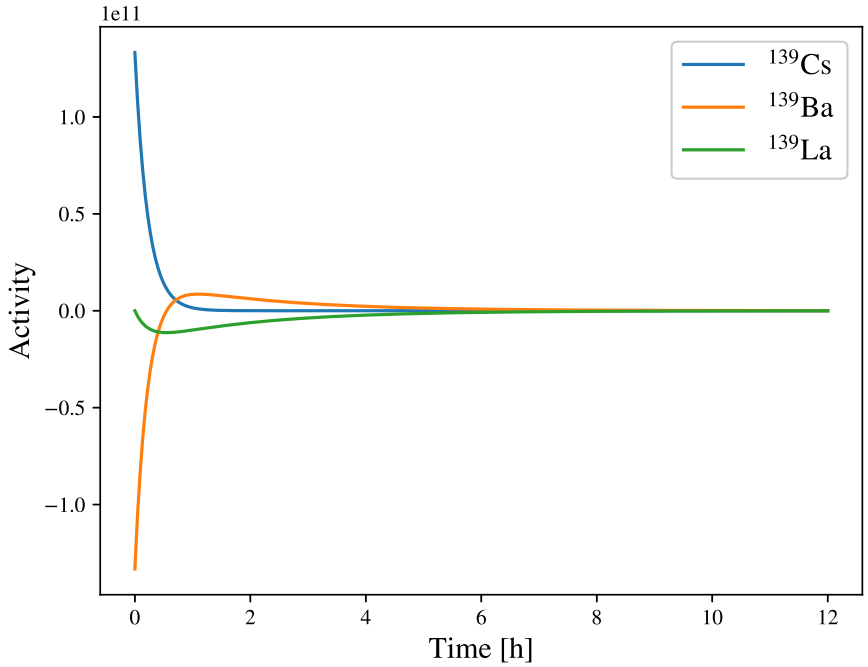

Figure 6: Radioactive decay chain

e

The maximum activity of ^{139}Ba is registered after 1.099 hours (see script below).

f

The activities of ^{139}Cs and ^{139}Ba becomes equal after 0.709 hours (see script below)

```
from matplotlib import pyplot as plt
import numpy as np
import matplotlib

matplotlib.rcParams['mathtext.fontset'] = 'stix'
matplotlib.rcParams['font.family'] = 'STIXGeneral'

T_halve_Cs = 9.27/60 # Half life time of 139Cs in hours
T_halve_Ba = 82.93/60 # Half life time of 139Ba int hours
lam_Cs = np.log(2)/T_halve_Cs # Decay width of 139Cs
lam_Ba = np.log(2)/T_halve_Ba # Decay width of 139Ba

N0 = 37e6/(lam_Cs/60/60)

print("N0:", int(N0))

# First decay
def N1(N0, t, lam1):
    return N0*np.exp(-lam1*t)

# Second decay
def N2(N0, t, lam1, lam2):
    return N0 * lam1 * (np.exp(-lam1*t) - np.exp(-lam2*t)) / (lam2 - lam1)

# Third decay
def N3(N0, t, lam1, lam2):
    return N0 * (lam1*(1-np.exp(-lam2*t)) - lam2*(1-np.exp(-lam1*t))) / (lam1 - lam2)
```

```
# Activity of the first reaction
def A1(N0, t, lam1):
    return lam1*N0*np.exp(-lam1*t)

def A2(N0, t, lam1, lam2):
    return -N0 * lam1/(lam2-lam1) * (lam2*np.exp(-lam2*t) - lam1*np.exp(-lam1*t))

def A3(N0, t, lam1, lam2):
    return -N0 * lam1*lam2/(lam1-lam2) * (np.exp(-lam2*t) - np.exp(-lam1*t))

# Time values to evaluate the numbers of species
time_ticks = np.linspace(0, 12, 10000)

# N(t)
N_Cs = N1(N0, time_ticks, lam_Cs)
N_Ba = N2(N0, time_ticks, lam_Cs, lam_Ba)
N_La = N3(N0, time_ticks, lam_Cs, lam_Ba)

# Activities
A_Cs = A1(N0, time_ticks, lam_Cs)
A_Ba = A2(N0, time_ticks, lam_Cs, lam_Ba)
A_La = A3(N0, time_ticks, lam_Cs, lam_Ba)

# Finding requested quantities
max_A_Ba = print("Maximum activity: %.3f at time %.3f hours" % (max(A_Ba), time_ticks[np
diff = np.abs(A_Ba - A_Cs)
print("Activities of Cs and Ba are equal at time %.3f hours" % (time_ticks[np.where(diff

# Plotting N(t)
plt.plot(time_ticks, N_Cs, label='$^{139}$Cs')
plt.plot(time_ticks, N_Ba, label='$^{139}$Ba')
plt.plot(time_ticks, N_La, label='$^{139}$La')
plt.xlabel('Time [h]', fontsize=14)
plt.ylabel('N', fontsize=14)
plt.legend(fontsize=14)
plt.show()

# Plotting activities
plt.plot(time_ticks, A_Cs, label='$^{139}$Cs')
plt.plot(time_ticks, A_Ba, label='$^{139}$Ba')
plt.plot(time_ticks, A_La, label='$^{139}$La')
plt.xlabel('Time [h]', fontsize=14)
plt.ylabel('Activity', fontsize=14)
plt.legend(fontsize=14)
plt.show()
```