1 Some shorter questions

a

This result can be explained by means of the nuclear shell model. According to this model each sub-level of the nucleus can be identified by 4 quantum numbers $\{n, j, l, m_l\}$. For fixed $\{n, j, l\}$ one has that the $2l+1$ sub-levels given by $m_l = -l, -l+1, \ldots, l-1, l$ are degenerate and each of this sub-levels can contain two nucleons with different spin z-projrection ($m_s = \pm 1$).
At first one can notice that if one sub-shell is filled with two nucleons (of the same type) then the spin contibution of that sub-shell is 0. Having an even number of protons and neutrons means that they combine in couples in the sub-shell in a way such that the contribution to the spin of each sub-shell is 0. For what concerns the parity, each particle in the nucleus gives a contribute of $(-1)^l$ (the intrinsic parity of the nucleons is $+1$) and the total parity is the product of single parities. The product of the parities of two protons (neutrons) in the same sub-shell is always $+1$ since they, in particular, share the same quantum number l. Hence it is legitimate to expect an even-even nucleus to be in the state 0^+.
Answer: spin 0, parity + .

2 Nuclear binding energy

a

The mass of the atom is given by

$$M(^{48}\text{CA}) = Zm_P + (A-Z)m_N - \frac{B(A,Z)}{c^2}$$

where B indicates the binding energy, which in turns can be calculated via the semi-empirical formula (atomic units)

$$B(A,Z) = a_v A + a_s A^{2/3} + a_c \frac{Z(Z-1)}{A^{1/3}} - a_{sym}\frac{(A-2Z)^2}{A} + \delta(A,Z) \tag{1}$$

where

$$\delta(A,Z) = \begin{cases} \frac{a_P}{A^{3/4}} & \text{if Z and N are even} \\ 0 & \text{if A is odd} \\ -\frac{a_P}{A^{3/4}} & \text{if Z and N are odd} \end{cases}$$

One can simply pop in the values of A and Z into the formula using the fitted coefficients (source Wikipedia) in units of MeV/c^2

$$a_v = 15.8 \qquad a_s = 18.3 \qquad a_c = 0.714 \qquad a_{sym} = 23.2 \qquad a_P = 12$$

and obtains $B(A,Z) = 621.71\ MeV/c^2$. Hence, inserting the result into 2 one obtains

$$M(^{48}\text{Ca}) \simeq 44451.55\ MeV/c^2 \simeq 47.3\ u$$

Even though the result is closet to the experimental value there is a small difference. One can justify this difference by observing that the SEMF is not a complete theoretical model, but rather a fit to experimental data of a model with some theoretical fundations. As such, it is reasonable that the model can give an overall good description of the data but cannot perfectly describe each of them: this is especially the case of lighter atoms for which the quantum shell structures of the nucleus should be taken into account.

b

Since $B(A,Z) = \sum_i B_i(A_i, Z_i)$ one can calculate the contribution of the binding energy of the last neutron as

$$B_{neutron} = B(A,Z) - B(A-1,Z)$$

And using expression 1 to calculate the two quantities one ends up with an estimation for the energy required to extract the neutron

$$B_{neutron} = B(A,Z) - B(A-1,Z) \approx$$

3

4

5 Quantum numbers

a

Parity

Let us consider a wavefunction $\psi(\mathbf{r})$. One can define the parity operator $\hat{P}$ as an operator such that

$$\hat{P}\,\psi(\mathbf{r}) = \psi(-\mathbf{r})$$

Since

$$\hat{P}^2\,\psi(\mathbf{r}) = \hat{P}\,\psi(-\mathbf{r}) = \psi(\mathbf{r}) \tag{2}$$

the eigenvalues of the parity operator are ± 1 and the corresponding eigenfunctions are the odd (eigenvalue -1) and even (eigenvalue $+1$) wavefunctions. If the wavefunction describes the state of particle, and the state is an eigenstate of the parity operator, the corresponding eigenvalue is also said to be the (intrinsic) parity of the particle.
Let us consider a system of two particles $A + B$ described by a wavefunction $\psi_{AB}(\mathbf{r}_A, \mathbf{r}_B)$. It can be proven that the parity of the system is given by

$$\hat{P}\psi_{AB} = \pi_A \pi_B (-1)^l \psi_{AB}$$

where l denotes the orbital angular momentum quantum number of the relative motion. Hence in a reaction of the type $A + B \to C + D$ described by a hamiltonian $\hat{H}$ that commutes with parity, parity is conserved or, in other words

$$\pi_A \pi_B (-1)^{l_{AB}} = \pi_C \pi_D (-1)^{l_{CD}}$$

This, for example, is not the case of the weak interaction where, in general, the hamiltonian operator does not commute with the parity operator.

Charge conjugation

c

^{15}N atom has 7 protons and 8 neutrons. Since the neutrons (in the ground state) are all coupled (each subshell admits two nucleons) they do not contribute to the spin of the nucleus and the parity contribution is $+1$. Instead the protons configuration of the ^{15}N atom in the ground state is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^1$: only the last proton contributes to the interested quantities. Since the orbital p represents the quantum number $l = 1$ the parity of the the nucleus is the product of the neutrons an protons contribution, that is $(+1)\cdot(-1)^1 = -1$. The spin is then $1/2$.
There are three possible alternatives for the first excited state, all of which consists in moving a proton or a neutron to another level starting from the ground state configuration.

1. Move the $1p_{1/2}$ proton to the $1d_{5/2}$ level
   $\longrightarrow$ the new proton configuration is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^{-2}\,(1d_{5/2})^1$ and the spin-parity is $1/2^+$.

2. Move the $1p_{3/2}$ proton to the $1d_{5/2}$ level
   $\longrightarrow$ the new proton configuration is $(1s)^2\,(1p_{3/2})^{-1}\,(1p_{1/2})^2$ and the spin-parity is $3/2^-$.

3. Move the $1p_{1/2}$ neutron to the $1d_{5/2}$ level
   $\longrightarrow$ the new neutorn configuration is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^{-1}\,(1d_{5/2})^1$. The parity here is non-trivial because of the angular momemnta addition rules.

By looking at the suggested table, one can notice that the first excited state is described by configuration 3, the second by configuration 1 and the third excited state is described by configuration 2.

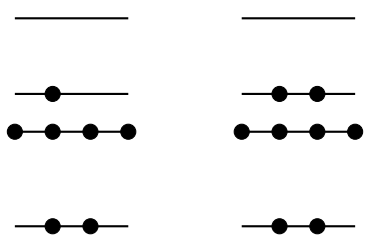

Figure 1: nothing

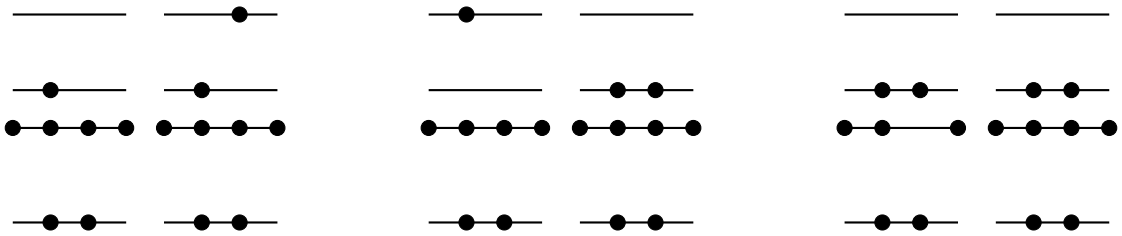

Figure 2: nothing

6

7 Radioactive decay

a

Said $N(t)$ the number of atoms present at time t, the radioactive decay law states that

$$\mathcal{A} = -\frac{dN}{dt} = \lambda N \tag{3}$$

where λ is called the decay constant.
By integrating it in time one obtains

$$N(t) = N_0\, e^{-\lambda t} \tag{4}$$

Now if we consider a chain decay of the type $A \to B \to C$ with initial conditions

$$N_A(t=0) = N_0 \qquad N_B(t=0) = N_C(t=0) = 0$$

one can immediately obtain the number of A atoms as a function of time by applying 3

$$N_A(t) = N_0 e^{-\lambda_A t}$$

To study the number of atoms of type B one has to take account of two factors: the number of "created" atoms by means of A's decay, and the number of atoms decayed into C. This means that the radioactive law reads

$$N_B(t) = -\frac{1}{\lambda_B}\frac{dN_B(t)}{dt} + N_{A\to B}(t)$$

where the first term takes account of the fact on the right side member represents the B's decay and the the last term represents the number of B obtained by A's decaying.
One can now substitute 4 obtaining

$$\frac{dN_B(t)}{dt} + \lambda_B N_B(t) = N_0\, e^{-\lambda_A t}$$

The general solution of this differential equation is

$$N_B(t) = e^{-A(t)}\ \left(N_B(0) + \int_0^t N_0 e^{-\lambda_A s} e^{A(s)}\, ds\right)$$

where $A(s) = \lambda_B \int dt = \lambda_B t$. Hence

$$N_B(t) = e^{-\lambda_B t}\ \int_0^t N_0 e^{-(\lambda_A-\lambda_B)s}\, ds = \frac{N_0\, e^{-\lambda_B t}}{\lambda_A - \lambda_B}\ \left(1 - e^{-(\lambda_A-\lambda_B)t}\right) =$$
$$= N_0\, \frac{e^{-\lambda_A t} - e^{-\lambda_B t}}{\lambda_B - \lambda_A}$$

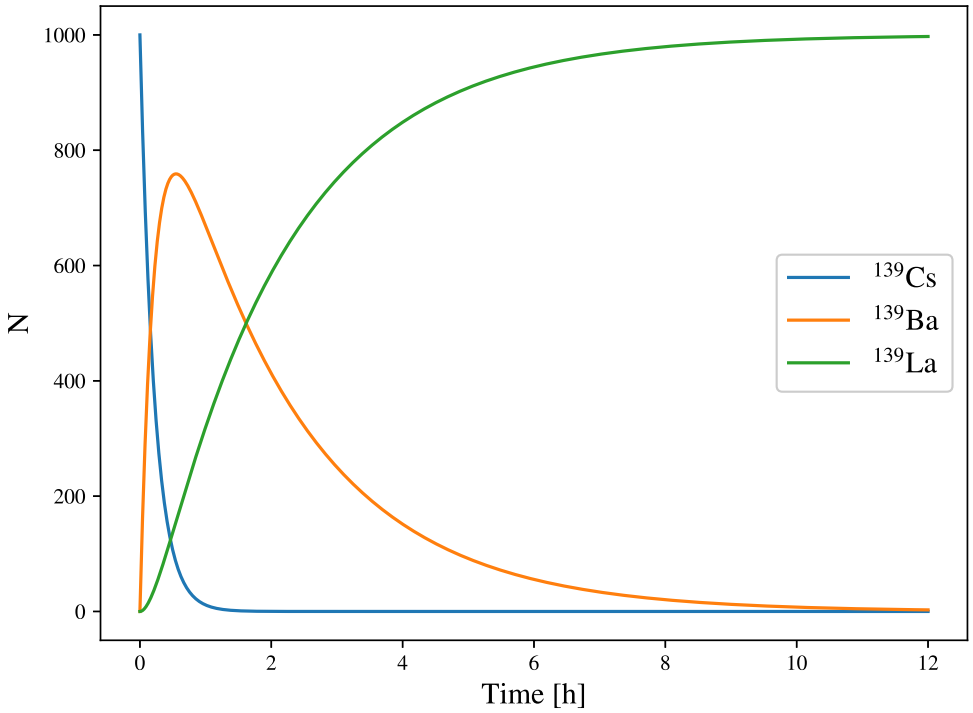

Figure 3: Radioactive decay chain

```
from matplotlib import pyplot as plt
import numpy as np
import matplotlib

matplotlib.rcParams['mathtext.fontset'] = 'stix'
matplotlib.rcParams['font.family'] = 'STIXGeneral'

T_halve_Cs = 9.27/60 # Half life time of 139Cs in hours
T_halve_Ba = 82.93/60 # Half life time of 139Ba int hours
lambda_Cs = np.log(2)/T_halve_Cs # Decay width of 139Cs
lambda_Ba = np.log(2)/T_halve_Ba # Decay width of 139Ba

print(np.log(2), T_halve_Cs, lambda_Cs, lambda_Ba, lambda_Cs-lambda_Ba)
N0 = 1000

# First decay
def N1(N0, t, lambda1):
    return N0*np.exp(-lambda1*t)

# Second decay
def N2(N0, t, lambda1, lambda2):
    return N0 * lambda1 * (np.exp(-lambda1*t) - np.exp(-lambda2*t)) / (lambda2 - lambda1

# Third decay
def N3(N0, t, lambda1, lambda2):
    print((lambda1 - lambda2))
    return N0 * (lambda1*(1-np.exp(-lambda2*t)) - lambda2*(1-np.exp(-lambda1*t))) / (lam

# Time values to evaluate the numbers of species
time_ticks = np.linspace(0, 12, 10000)
```

```
N_Cs = N1(N0, time_ticks, lambda_Cs)
N_Ba = N2(N0, time_ticks, lambda_Cs, lambda_Ba)
N_La = N3(N0, time_ticks, lambda_Cs, lambda_Ba)

# Plotting
plt.plot(time_ticks, N_Cs, label='$^{139}$Cs')
plt.plot(time_ticks, N_Ba, label='$^{139}$Ba')
plt.plot(time_ticks, N_La, label='$^{139}$La')

plt.xlabel('Time [h]', fontsize=14)
plt.ylabel('N', fontsize=14)
plt.legend(fontsize=14)

plt.show()
```