1 Some shorter questions

a

This result can be explained by means of the nuclear shell model. According to this model each sub-level of the nucleus can be identified by 4 quantum numbers $\{n, j, l, m_l\}$. For fixed $\{n, j, l\}$ one has that the $2l+1$ sub-levels given by $m_l = -l, -l+1, \ldots, l-1, l$ are degenerate and each of this sub-levels can contain two nucleons with different spin z-projrection ($m_s = \pm 1$).
At first one can notice that if one sub-shell is filled with two nucleons (of the same type) then the spin contibution of that sub-shell is 0. Having an even number of protons and neutrons means that they combine in couples in the sub-shell in a way such that the contribution to the spin of each sub-shell is 0. For what concerns the parity, each particle in the nucleus gives a contribute of $(-1)^l$ (the intrinsic parity of the nucleons is $+1$) and the total parity is the product of single parities. The product of the parities of two protons (neutrons) in the same sub-shell is always $+1$ since they, in particular, share the same quantum number l. Hence it is legitimate to expect an even-even nucleus to be in the state 0^+.

Answer: spin 0, parity + .

2 Nuclear binding energy

a

The mass of the atom is given by

$$M(^{48}\text{CA}) = Zm_P + (A-Z)m_N - \frac{B(A,Z)}{c^2}$$

where B indicates the binding energy, which in turns can be calculated via the semi-empirical formula (atomic units)

$$B(A,Z) = a_v A + a_s A^{2/3} + a_c \frac{Z(Z-1)}{A^{1/3}} - a_{sym}\frac{(A-2Z)^2}{A} + \delta(A,Z) \tag{1}$$

where

$$\delta(A,Z) = \begin{cases} \frac{a_P}{A^{3/4}} & \text{if Z and N are even} \\ 0 & \text{if A is odd} \\ -\frac{a_P}{A^{3/4}} & \text{if Z and N are odd} \end{cases}$$

One can simply pop in the values of A and Z into the formula using the fitted coefficients (source Wikipedia) in units of MeV/c^2

$$a_v = 15.8 \qquad a_s = 18.3 \qquad a_c = 0.714 \qquad a_{sym} = 23.2 \qquad a_P = 12$$

and obtains $B(A,Z) = 621.71\ MeV/c^2$. Hence, inserting the result into 2 one obtains

$$M(^{48}\text{Ca}) \simeq 44451.55\ MeV/c^2 \simeq 47.3\ u$$

Even though the result is closet to the experimental value there is a small difference. One can justify this difference by observing that the SEMF is not a complete theoretical model, but rather a fit to experimental data of a model with some theoretical fundations. As such, it is reasonable that the model can give an overall good description of the data but cannot perfectly describe each of them: this is especially the case of lighter atoms for which the quantum shell structures of the nucleus should be taken into account.

b

Since $B(A,Z) = \sum_i B_i(A_i, Z_i)$ one can calculate the contribution of the binding energy of the last neutron as

$$B_{neutron} = B(A,Z) - B(A-1,Z)$$

And using expression 1 to calculate the two quantities one ends up with an estimation for the energy required to extract the neutron

$$B_{neutron} \approx 15.69\ MeV/c^2$$

c

The most stable atom corresponds to the maximum point of the binding energy as a function of Z. One can find such point by imposing $\frac{dB(Z)}{dZ} = 0$. By the way one problem arises, that is that the last therm of the binding energy depends on the value of Z and we do not know it in advance. Hence the approach I chose was to set that term to 0 and find the maximum point Z : this gives a non-integer term and the correct Z is the one closer to this integer. Hence

$$0 = \frac{dB(Z)}{dZ} = \frac{a_c}{A^{1/3}}(2Z-1) + \frac{4a_{symm}}{A}(A-2Z)$$

leads to

$$Z = \frac{a_c A^{2/3} - 4a_{symm}A}{2a_c A^{2/3} - 8a_{symm}} \simeq$$

3 Higgs boson decay

a

In the resting frame of the Higgs boson the invariant mass is (atomic units)

$$W = \sqrt{(\sum_i E_i)^2 - |\sum \mathbf{p}_i|^2} = E_H = m_H \tag{2}$$

The invariant mass is a Lorent invariant, hence this quantity is the same that one observer would measure in the laboratory frame. More specifically, one can calculate W after the process using the electrons data

$$W = \sqrt{(\sum_i E_i)^2 - |\sum \mathbf{p}_i|^2} = \sqrt{(\sum_i E_i)^2} = \sqrt{\sum \sqrt{m_i^2 + p_i^2}}$$

By equating this expression to 2 and inserting the given values together with the electrons mass one obtains

$$m_h \simeq$$

b

In principle each pair of electron-positron can be generated by the Z boson, in the sense that no rules would be violated. In other words there are 4 different ways to combine the couples of electron-positron to the bosons

1. $Z \to e_1^+ + e_1^-, \quad Z^* \to e_2^+ + e_2^-$
2. $Z \to e_1^+ + e_2^-, \quad Z^* \to e_2^+ + e_1^-$
3. $Z \to e_2^+ + e_1^-, \quad Z^* \to e_1^+ + e_2^-$
4. $Z \to e_2^+ + e_2^-, \quad Z^* \to e_1^+ + e_1^-$

One can now impose the conservation of the 4−momentum in the bosons' decay and in particular, for the Z boson, this means

$$E_Z = \sqrt{m_Z^2 + p_Z^2} = E_{e^+} + E_{e^-} = \sqrt{m_{e^+}^2 + p_{e^+}^2} + \sqrt{m_{e^-}^2 + p_{e^-}^2}$$

and by rearrangin terms

$$p_Z^2 = (E_{e^+} + E_{e^-})^2 - m_Z^2$$

this equation gives the value of the momentum that the Z boson must have in order to satisfy the conservation of the energy. By explicit calculation for the 4 cases one obtains

1. $p_Z^2 \simeq -2414\ GeV/c$
2. $p_Z^2 \simeq -7350\ GeV/c$
3. $p_Z^2 \simeq 744\ GeV/c$
4. $p_Z^2 \simeq -5872\ GeV/c$

hence only in the third case the particle can be regarded as physical and not virtual.

c

d

4

5 Quantum numbers

a

Parity

Let us consider a wavefunction $\psi(\mathbf{r})$. One can define the parity operator $\hat{P}$ as an operator such that

$$\hat{P}\,\psi(\mathbf{r}) = \psi(-\mathbf{r})$$

Since

$$\hat{P}^2\,\psi(\mathbf{r}) = \hat{P}\,\psi(-\mathbf{r}) = \psi(\mathbf{r}) \tag{3}$$

the eigenvalues of the parity operator are ± 1 and the corresponding eigenfunctions are the odd (eigenvalue -1) and even (eigenvalue $+1$) wavefunctions. If the wavefunction describes the state of particle, and the state is an eigenstate of the parity operator, the corresponding eigenvalue is also said to be the (intrinsic) parity of the particle.
Let us consider a system of two particles $A + B$ described by a wavefunction $\psi_{AB}(\mathbf{r}_A, \mathbf{r}_B)$. It can be proven that the parity of the system is given by

$$\hat{P}\psi_{AB} = \pi_A \pi_B (-1)^l \psi_{AB}$$

where l denotes the orbital angular momentum quantum number of the relative motion. Hence in a reaction of the type $A + B \to C + D$ described by a hamiltonian $\hat{H}$ that commutes with parity, parity is conserved or, in other words

$$\pi_A \pi_B (-1)^{l_{AB}} = \pi_C \pi_D (-1)^{l_{CD}}$$

This, for example, is not the case of the weak interaction where, in general, the hamiltonian operator does not commute with the parity operator.

Charge conjugation

b

c

^{15}N atom has 7 protons and 8 neutrons. Since the neutrons (in the ground state) are all coupled (each subshell admits two nucleons) they do not contribute to the spin of the nucleus and the parity contribution is $+1$. Instead the protons configuration of the ^{15}N atom in the ground state is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^1$: only the last proton contributes to the interested quantities. Since the orbital p represents the quantum number $l = 1$ the parity of the the nucleus is the product of the neutrons an protons contribution, that is $(+1)\cdot(-1)^1 = -1$. The spin is then $1/2$.
There are three possible alternatives for the first excited state, all of which consists in moving a proton or a neutron to another level starting from the ground state configuration.

1. Move the $1p_{1/2}$ proton to the $1d_{5/2}$ level
   $\longrightarrow$ the new proton configuration is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^{-2}\,(1d_{5/2})^1$ and the spin-parity is $5/2^+$.
2. Move the $1p_{3/2}$ proton to the $1p_{1/2}$ level
   $\longrightarrow$ the new proton configuration is $(1s)^2\,(1p_{3/2})^{-1}\,(1p_{1/2})^2$ and the spin-parity is $3/2^-$.
3. Move the $1p_{1/2}$ neutron to the $1d_{5/2}$ level
   $\longrightarrow$ the new neutorn configuration is $(1s)^2\,(1p_{3/2})^2\,(1p_{1/2})^{-1}\,(1d_{5/2})^1$. The parity here is non-trivial because of the angular momemnta addition rules.

By looking at the suggested table, one can notice that the first excited state is described by configuration 1, the second excited state by configuration 3 and the third excited state is described by configuration 2.

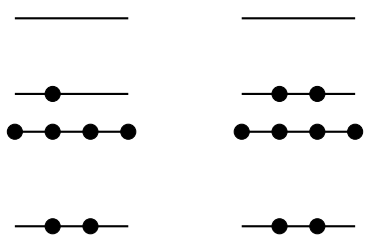

Figure 1: nothing

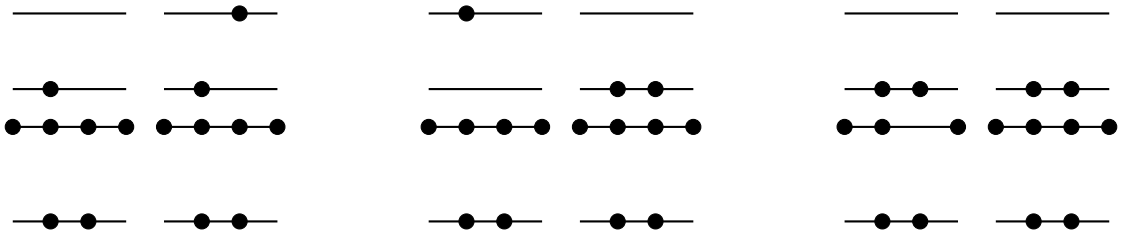

Figure 2: nothing

6

7 Radioactive decay

a

Said $N(t)$ the number of atoms present at time t, the radioactive decay law states that

$$\mathcal{A} = -\frac{dN}{dt} = \lambda N \tag{4}$$

where λ is called the decay constant.
By integrating it in time one obtains

$$N(t) = N_0\, e^{-\lambda t} \tag{5}$$

Now if we consider a chain decay of the type $A \to B \to C$ with initial conditions

$$N_A(t=0) = N_0 \qquad N_B(t=0) = N_C(t=0) = 0$$

one can immediately obtain the number of A atoms as a function of time by applying 4

$$N_A(t) = N_0 e^{-\lambda_A t}$$

To study the number of atoms of type B one has to take account of two factors: the number of "created" atoms by means of A's decay, and the number of atoms decayed into C. This means that the radioactive law reads

$$N_B(t) = -\frac{1}{\lambda_B}\frac{dN_B(t)}{dt} + N_{A\to B}(t)$$

where the first term takes account of the fact on the right side member represents the B's decay and the the last term represents the number of B obtained by A's decaying.
One can now substitute 5 obtaining

$$\frac{dN_B(t)}{dt} + \lambda_B N_B(t) = N_0\, e^{-\lambda_A t}$$

The general solution of this differential equation is

$$N_B(t) = e^{-A(t)}\left(N_B(0) + \int_0^t N_0 e^{-\lambda_A s} e^{A(s)}\, ds\right)$$

where $A(s) = \lambda_B \int dt = \lambda_B t$. Hence

$$N_B(t) = e^{-\lambda_B t}\int_0^t N_0 e^{-(\lambda_A-\lambda_B)s}\, ds = \frac{N_0\, e^{-\lambda_B t}}{\lambda_A - \lambda_B}\left(1 - e^{-(\lambda_A-\lambda_B)t}\right) =$$
$$= N_0\,\frac{e^{-\lambda_A t} - e^{-\lambda_B t}}{\lambda_B - \lambda_A}$$

b

Let us consider the single-decay law

$$N(t) = N_0 e^{-\lambda t}$$

One can notice here that λ's physical units are inverse of time. The time $t_{1/2}$ is defined as the time necessary to halve the number of atoms fro N_0. This can be found by imposing $N(t) = N_0/2$ and solving for t one obtains

$$t_{1/2} = \frac{1}{\lambda} \log 2 \equiv \tau \log 2$$

where I introduced the mean lifetime $\tau \equiv 1/\lambda$. This last term, which has units of time, has a particular meaning. To understand this, one can first think of

$$p(t) = \frac{N(t)}{\int N(t)\, dt}$$

as a probability distribution function related to the probability of a particle to decay. One can then calculate the expected lifetime of such particle

$$\langle t \rangle = \frac{\int tN(t)\, dt}{\int N(t)\, dt} = \frac{\int_0^\infty te^{-\lambda t}\, dt}{\int_0^\infty e^{-\lambda t}} = \frac{1}{\lambda}$$

and it is exactly τ.
Another interesting quantity is the natural decay width Γ. One can make plot the probability of a decay to occur as function of the decaying particle's energy: this is given by the Breit Wigner formula. The half width at half maximum of this distribution is the factor Γ which has units of energy and is related to the average lifetime of the particle trough the energy-time uncertainty relation

$$\Delta E\, \Delta t >= \frac{\hbar}{2}$$

EXPAND AND EXPLAIN BETTER HERE

c

$1mCi = 37MBq$. Taking the solution

$$N_1(t) = N_0\, e^{-\lambda t}$$

and imposing $\mathcal{A}(t=0) = -\frac{dN_1(t)}{dt}|_{t=0} = 37MBq$ one obtains

$$N_0 = \frac{37MBq}{\lambda} \simeq 29689798324$$

d

COMMENTS HERE

e

The maximum activity of ^{139}Ba is registered after 1.099 hours (see script below).

f

The activities of ^{139}Cs and ^{139}Ba becomes equal after 0.709 hours (see script below)

```
from matplotlib import pyplot as plt
import numpy as np
import matplotlib

matplotlib.rcParams['mathtext.fontset'] = 'stix'
matplotlib.rcParams['font.family'] = 'STIXGeneral'

T_halve_Cs = 9.27/60 # Half life time of 139Cs in hours
T_halve_Ba = 82.93/60 # Half life time of 139Ba int hours
lam_Cs = np.log(2)/T_halve_Cs # Decay width of 139Cs
lam_Ba = np.log(2)/T_halve_Ba # Decay width of 139Ba
```

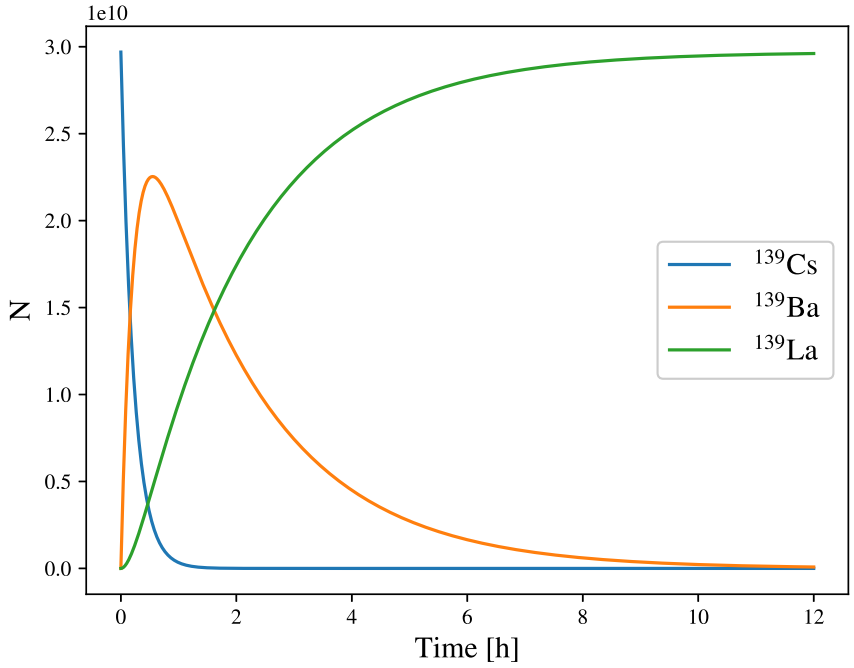

Figure 3: Radioactive decay chain

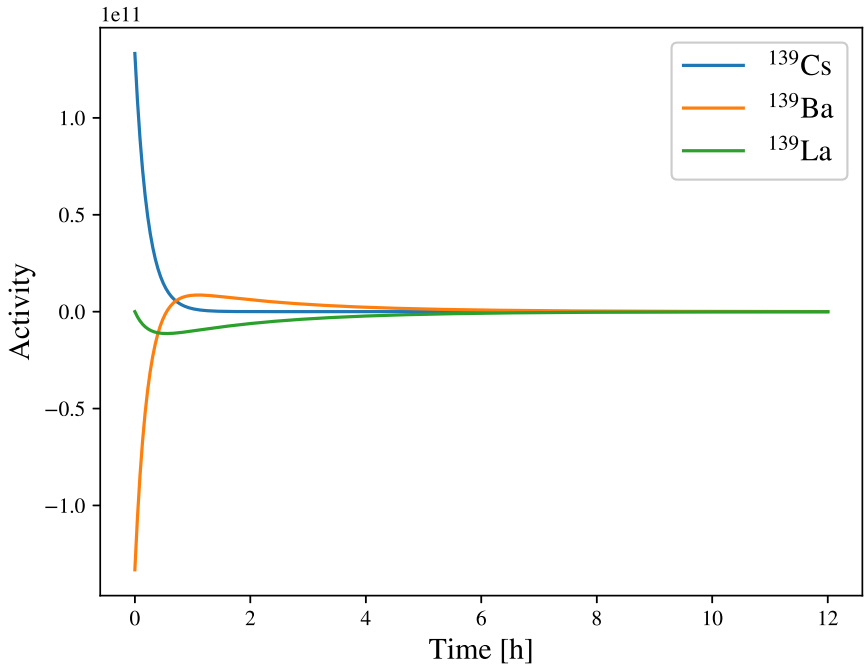

Figure 4: Radioactive decay chain

```
N0 = 37e6/(lam_Cs/60/60)

print("N0:", int(N0))

# First decay
def N1(N0, t, lam1):
    return N0*np.exp(-lam1*t)

# Second decay
def N2(N0, t, lam1, lam2):
    return N0 * lam1 * (np.exp(-lam1*t) - np.exp(-lam2*t)) / (lam2 - lam1)

# Third decay
def N3(N0, t, lam1, lam2):
    return N0 * (lam1*(1-np.exp(-lam2*t)) - lam2*(1-np.exp(-lam1*t))) / (lam1 - lam2)

# Activity of the first reaction
def A1(N0, t, lam1):
    return lam1*N0*np.exp(-lam1*t)

def A2(N0, t, lam1, lam2):
    return -N0 * lam1/(lam2-lam1) * (lam2*np.exp(-lam2*t) - lam1*np.exp(-lam1*t))

def A3(N0, t, lam1, lam2):
    return -N0 * lam1*lam2/(lam1-lam2) * (np.exp(-lam2*t) - np.exp(-lam1*t))

# Time values to evaluate the numbers of species
time_ticks = np.linspace(0, 12, 10000)

# N(t)
N_Cs = N1(N0, time_ticks, lam_Cs)
N_Ba = N2(N0, time_ticks, lam_Cs, lam_Ba)
N_La = N3(N0, time_ticks, lam_Cs, lam_Ba)

# Activities
A_Cs = A1(N0, time_ticks, lam_Cs)
A_Ba = A2(N0, time_ticks, lam_Cs, lam_Ba)
A_La = A3(N0, time_ticks, lam_Cs, lam_Ba)

# Finding requested quantities
max_A_Ba = print("Maximum activity: %.3f at time %.3f hours" % (max(A_Ba), time_ticks[np
diff = np.abs(A_Ba - A_Cs)
print("Activities of Cs and Ba are equal at time %.3f hours" % (time_ticks[np.where(diff

# Plotting N(t)
plt.plot(time_ticks, N_Cs, label='$^{139}$Cs')
plt.plot(time_ticks, N_Ba, label='$^{139}$Ba')
plt.plot(time_ticks, N_La, label='$^{139}$La')
plt.xlabel('Time [h]', fontsize=14)
plt.ylabel('N', fontsize=14)
plt.legend(fontsize=14)
plt.show()

# Plotting activities
plt.plot(time_ticks, A_Cs, label='$^{139}$Cs')
plt.plot(time_ticks, A_Ba, label='$^{139}$Ba')
plt.plot(time_ticks, A_La, label='$^{139}$La')
plt.xlabel('Time [h]', fontsize=14)
plt.ylabel('Activity', fontsize=14)
plt.legend(fontsize=14)
```

```
plt.show()
```