Problem Set 10

Exercises for course Fundamentals of Simulation Methods, WS 2021

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Hand in until Wednesday, 19.01.2022, 23:59
Tutorials times: 20-21.01.2022
Group 1: Brooke | Thursday 11:00 - 13:00
Group 2: Glen | Thursday 14:00 - 16:00
Group 3: Jan | Friday 11:00 - 13:00

1) Entropy evolution of an ideal gas

Consider the *entropic function*

$$A \equiv \frac{P}{\rho^\gamma} = (\gamma - 1)\frac{u}{\rho^{\gamma-1}} \tag{1}$$

for an ideal gas with adiabatic index γ (while A is not identical to the specific thermodynamic entropy, it is a direct function of it and thus can be used in lieu of it).

(a) Show from the differential form of the Euler equations that A remains constant along the path of a fluid element, or in other words, that

$$\frac{\mathrm{D}A}{\mathrm{D}t} = 0. \tag{2}$$

(b) You may have been told that entropy increases at shock waves. How can we reconcile this statement with the above result?

2) Simple advection problem

Consider the most simple 1D advection problem. It is define by a partial differential equation (PDE) of the form

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, \tag{3}$$

where $u = u(x, t)$ is a function of x and t, and v is a constant parameter. Think of u being the density and v being the sound speed.

Show that if we are given any function $q(x)$, then

$$u(x, t) = q(x - vt) \tag{4}$$

is a solution of the advection equation.

3) Derivation of the wave equation in 1D

Now we turn to the Euler equations as defined in the lecture, yet we take the 1D case:

$$\frac{\partial \rho}{\partial t} + \nabla (\rho v) = 0 \tag{5}$$

$$\frac{\partial}{\partial t} (\rho v) + \nabla \left(\rho v^2 + P\right) = 0 \tag{6}$$

$$\frac{\partial}{\partial t} (\rho \varepsilon_{\mathrm{tot}}) + \nabla \left[(\rho \varepsilon_{\mathrm{tot}} + P)\, v\right] = 0 \quad , \tag{7}$$

with $\varepsilon_{\mathrm{tot}} = u + v^2/2$ the total energy per unit mass and u the thermal energy per unit mass.
Take a simple isothermal equation of state, $P = c_{\mathrm{s}}^2 \rho$, with c_{s} being the isothermal sound speed. Show that they lead to a simple wave equation for density perturbations.

To do so, look at the behaviour of small perturbations around the equilibrium solution in all relevant quantities. For example, the density can be written as $\rho(x,t) = \rho_0 + \delta\rho(x,t)$, and so forth. Recall that the equilibrium state is homogeneous and stationary. Only consider terms up to linear order in the perturbation.