Fundamentals of Simulation Methods

University of Heidelberg, WiSe 2021/2022

Mandatory assignements - Set 2

Authors: Matteo Zortea, Anja Heim

Date: October 29, 2021

Exercise 3

a)

By defining

$$q_i \equiv \frac{\partial \mathcal{L}}{\dot{q}_i} \qquad i = 1, 2 \tag{1}$$

one can rewrite the Lagrangian equations of motion as

$$\frac{dq_1}{dt} = \frac{\partial \mathcal{L}}{\partial q_1} = -m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - (m_1 + m_2) g l_1 \sin\phi_1$$

$$\frac{dq_2}{dt} = \frac{\partial \mathcal{L}}{\partial q_2} = m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - m_2 g l_2 \sin\phi_2$$

These equations determine the evolution of the momenta q_1, q_2.

b)

If we define a state vector as $y = (\phi_1, \phi_2, q_1, q_2)T$, the evolution of our system is completely determined by the set of equations

$$\frac{dy}{dt} = f(y)$$

which, in matrix form, reads

$$\begin{pmatrix} \frac{d\phi_1}{dt} \\ \frac{d\phi_2}{dt} \\ \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \end{pmatrix} = \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ -m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - (m_1 + m_2) g l_1 \sin\phi_1 \\ m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - m_2 g l_2 \sin\phi_2 \end{pmatrix} \tag{2}$$

We now need to obtain the expression for $\dot{\phi}_i \quad i = 1, 2$. In order to do this, let us explicitly write the expression for q_1, q_2 using their definition 1.

$$q_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = (m_1 + m_2) l_1^2 \dot{\phi}_1 + m_2 l_1 l_2 \dot{\phi}_2 cos(\phi_1 - \phi_2) \equiv \alpha \dot{\phi}_1 + \beta \dot{\phi}_2$$

$$q_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = m_2 l_1 l_2 \dot{\phi}_1 \cos(\phi_1 - \phi_2) + m_2 l_2^2 \dot{\phi}_2 \equiv \beta \dot{\phi}_1 + \gamma \dot{\phi}_2$$

which can be written in matrix form as

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}$$

We can invert these expressions using the Kramer's rule

$$\dot{\phi}_1 = \frac{\begin{vmatrix} q_1 & \beta \\ q_2 & \gamma \end{vmatrix}}{\begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix}} = \frac{\gamma}{\alpha\gamma - \beta^2} q_1 - \frac{\beta}{\alpha\gamma - \beta^2} q_2 \qquad \dot{\phi}_2 = \frac{\begin{vmatrix} \alpha & q_1 \\ \beta & q_2 \end{vmatrix}}{\begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix}} = \frac{\alpha}{\alpha\gamma - \beta^2} q_2 - \frac{\beta}{\alpha\gamma - \beta^2} q_2$$

and by popping in the expressions for α, β, γ

$$\dot{\phi}_1 = \frac{1}{m_1 l_1^2(1 + m_2/m_1 \sin^2(\phi_1 - \phi_2))}\, q_1 - \frac{cos(\phi_1 - \phi_2)}{m_1 l_1 l_2(1 + m_2/m_1 \sin^2(\phi_1 - \phi_2))}\, q_2$$
$$\dot{\phi}_2 = -\frac{1}{\mu l_2^2(1 + m_2/m_1 \sin^2(\phi_1 - \phi_2))}\, q_2 - \frac{\cos(\phi_1 - \phi_2)}{m_1 l_1 l_2(1 + m_2/m_1 \sin^2(\phi_1 - \phi_2))}\, q_1$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.
We have now obtained all the expressions needed to solve equation 2 which now explicitly reads

$$\begin{pmatrix} \frac{d\phi_1}{dt} \\ \frac{d\phi_2}{dt} \\ \frac{dq_1}{dt} \\ \frac{dq_2}{dt} \end{pmatrix} = \begin{pmatrix} \frac{1}{m_1 l_1^2(1+m_2/m_1 \sin^2(\phi_1-\phi_2))}\, q_1 - \frac{cos(\phi_1-\phi_2)}{m_1 l_1 l_2(1+m_2/m_1 \sin^2(\phi_1-\phi_2))}\, q_2 \\ -\frac{1}{\mu l_2^2(1+m_2/m_1 \sin^2(\phi_1-\phi_2))}\, q_2 - \frac{\cos(\phi_1-\phi_2)}{m_1 l_1 l_2(1+m_2/m_1 \sin^2(\phi_1-\phi_2))}\, q_1 \\ -m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - (m_1 + m_2) g l_1 \sin\phi_1 \\ m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) - m_2 g l_2 \sin\phi_2 \end{pmatrix}$$

c)