Lattice field theories
Exam sheet: Coding assingments

Lecturers: Jan Pawlowski j.pawlowski@thphys.uni-heidelberg.de
Felipe Attanasio pyfelipe@thphys.uni-heidelberg.de

This sheet contains 5 options for coding assignments Choose the one you feel most comfortable with. By the due date, you should send me an E-Mail with a compressed file containing your code, a makefile, and the relevant results you have obtained.

Remember to write clear code. Your future self and I will thank you.

Option 1: Electrostatic potential

Consider a two-dimensional lattice of size $L \times L$. Solve the Laplace equation,

$$\nabla^2 \phi(x, y) = 0\,, \tag{1}$$

with $0 \leq x, y \leq L$, numerically, subject to the conditions

$$\phi\left(\frac{L}{4}, \frac{L}{4}\right) = -1\,, \tag{2}$$

$$\phi(0, y) = \phi(L, y) = 1\,, \tag{3}$$

$$\phi(x, 0) = \phi(x, L) = 0\,. \tag{4}$$

Eq. (1) can be solved iteratively as

$$\begin{aligned}\phi^{(i+1)}(n_x, n_y) =& \frac{1}{4}\left[\phi^{(i)}(n_x + 1, n_y) + \phi^{(i)}(n_x - 1, n_y)\right. \\ &\left. + \phi^{(i)}(n_x, n_y + 1) + \phi^{(i)}(n_x, n_y - 1)\right]\,,\end{aligned} \tag{5}$$

where the superscript i indicates the iteration number. The initial conditions for $\phi(x, y)$ at points in $[0, L] \times [0, L]$ that are not part of the constraints (2), (3), and (4) can be chosen arbitrarily.

The number of lattice sites on each direction should be **at least** 20 and the lattice **must** be parallelised across different ranks using `MPI`, with at least 2 ranks being used. You are not required to use the lookup tables discussed in the lectures. Make a $2d$ plot of the result, such as a heatmap.

Option 2: Parallelised conjugate gradient

Consider the Dirac operator in $1 + 1$ dimensions for free fermions of mass m in the Wilson formulation,

$$D_{xy} = (2 + m)\delta_{xy} - \frac{1}{2}\sum_{\mu=0}^{1}\left(\Gamma_{+\hat{\mu}}\delta_{x+\hat{\mu},y} + \Gamma_{-\hat{\mu}}\delta_{x-\hat{\mu},y}\right)\,, \tag{6}$$

with $\Gamma_{\pm\hat{\mu}} = (1 \mp \gamma_\mu)$.

Implement *any* of the kinds of Conjugate Gradient algorith discussed in the lectures to solve the Dirac equation $D\psi = \eta$, for $\eta_x = \delta_{x,0}$, using the above Dirac operator. This computation **must** be parallelised using `MPI`, but you are not required to use the lookup tables from the lectures. You should use a lattice of size 32×32 or bigger and masses in the interval $0 \leq m \leq 10$. The inversion should be verified by computing $|\eta - D\psi|$, where $\psi = D^{-1}\eta$ is to be computed using CG. You may, *optionally*, also compute the correlator

$$C(\delta) = \frac{1}{V} \sum_{\vec{y}} \langle \psi_{\delta,\vec{y}} \overline{\psi}_0 \rangle \tag{7}$$

following exercise sheet 4, extract from it the renormalised mass and compare it with the exact formula for free Wilson fermions

$$am_R = \ln\left[1 + am\right] . \tag{8}$$

Option 3: Different implementations of conjugate gradient

Consider the Dirac operator in $1 + 1$ dimensions for free fermions of mass m in the Wilson formulation,

$$D_{xy} = (2 + m)\delta_{xy} - \frac{1}{2} \sum_{\mu=0}^{1} \left(\Gamma_{+\hat{\mu}} \delta_{x+\hat{\mu},y} + \Gamma_{-\hat{\mu}} \delta_{x-\hat{\mu},y} \right) , \tag{9}$$

with $\Gamma_{\pm\hat{\mu}} = (1 \mp \gamma_\mu)$.

Implement the *even-odd preconditioned, mixed precision* version of CG to solve the Dirac equation $D\psi = \eta$, for $\eta_x = \delta_{x,0}$, using the above Dirac operator. You may *optionally* parallelise your code, but it is not required. You should use a lattice of size 32×32 or bigger and masses in the interval $0 \leq m \leq 10$. The inversion should be verified by computing $|\eta - D\psi|$, where $\psi = D^{-1}\eta$ is to be computed using CG. You may, *optionally*, also compute the correlator

$$C(\delta) = \frac{1}{V} \sum_{\vec{y}} \langle \psi_{\delta,\vec{y}} \overline{\psi}_0 \rangle \tag{10}$$

following exercise sheet 4, extract from it the renormalised mass and compare it with the exact formula for free Wilson fermions

$$am_R = \ln\left[1 + am\right] . \tag{11}$$

Option 4: SU(3) Yang-Mills with HMC and MPI

Use the HMC algorithm to simulate the SU(3) pure Yang-Mills theory in 1+1 dimensions.

$$S = \beta \sum_n \sum_{\mu<\nu} \left\{ 1 - \frac{1}{2N_c} \mathrm{Tr}\left[U_{\mu\nu}(n) + U^\dagger_{\mu\nu}(n) \right] \right\} , \tag{12}$$

where $\beta = 2N_c/g_0$, g_0 is the bare coupling, N_c is the number of colours, and

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n) \tag{13}$$

is the plaquette at site n in the $\mu - \nu$ plane. Take $N_c = 3$.

The lattice volume should be **at least** 32×32 and the lattice **must** be divided amongst different ranks, with communication handled by `MPI`. You are not required to use the lookup tables discussed in the lectures. You should compute the (ensemble) average of the Polyakov loop,

$$\mathcal{P} = \frac{1}{3}\frac{1}{V}\sum_{\vec{n}} \mathrm{Tr}\left\{\prod_{n_t=0}^{N_t-1}\left[U_0(n_t,\vec{n})\right]\right\}, \tag{14}$$

for inverse couplings in the range $1 \leq \beta \leq 7$ and make a plot.

Option 5: Gross-Neveu model in $1+1$ dimensions

The Gross-Neveu model with mass-degenerate fermions in $1+1$ dimensions is given by the continuum action

$$S = \int d^2x \left[\sum_{f=1}^{N_f}\overline{\psi}^{(f)}\left(\gamma_\mu\partial_\mu + m\right)\psi^{(f)} - \frac{g}{2N_f}\left(\sum_{f=1}^{N_f}\overline{\psi}^{(f)}\psi^{(f)}\right)^2\right], \tag{15}$$

or its lattice counterpart, using Wilson fermions,

$$\begin{aligned} S_{\mathrm{lat}} = &\sum_x\sum_{f=1}^{N_f}\left[(2+m)\overline{\psi}_x^{(f)}\psi_x^{(f)} - \frac{1}{2}\overline{\psi}_x^{(f)}\sum_\mu\left(\Gamma_{+\hat{\mu}}\psi_{x+\hat{\mu}}^{(f)} + \Gamma_{-\hat{\mu}}\psi_{x-\hat{\mu}}^{(f)}\right)\right] \\ &- \frac{g}{2N_f}\sum_x\left(\sum_{f=1}^{N_f}\overline{\psi}_x^{(f)}\psi_x^{(f)}\right)^2, \end{aligned} \tag{16}$$

In order to ensure positivity of the integration measure, we take $N_f = 2$.

After performing a Hubbard-Stratonovich transformation, use the HMC algorithm and pseudofermions to simulate this model. The lattice size must be **at least** 16×16, with Yukawa coupling $g = 0.1$ and fermion mass in the range $-0.3 \leq m \leq 0.3$. You **must** compute the chiral condensate, $\langle\overline{\psi}\psi\rangle$, for different masses and plot it. Note that since both fermion flavours are identical their chiral condensates will be identical.

You may use any version of conjugate gradient to compute the pseudofermion contribution, and also may use `MPI` for paralellisation – but it is not required. You are free to choose the Hubbard-Stratonovich transformation that you consider most convenient.