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Simulating effective field theories

on a space-time lattice with coloured noise

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27.11.2023 .....

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"Grazie a tutti."

Matteo Zortea

Chapter 1

Introduction and outline

1.1 Quantum chromodynamics and its and phase diagram

Big picture: here we talk about QCD and the problem of the phase diagram

1.2 Effective theories

Here we first define effective theories and discuss their usefulness, then introduce RG as a technique to resolve physics at different scales.
Motivation for choices, connection to stochastic regularisation, complex langevin,

easy noise [**boo**] Motivation for choices,
The starting set up is the euclidean formulation of quantum field theory, where one typically defines a path integral Z, which, for a general scalar field $\phi(x)$ and a fermion field $\psi(x)$, assumes the form

$$Z = \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S[\phi,\psi,\bar{\psi}]} \qquad \mathcal{D}\xi = \prod_x d\xi_x, \quad \xi \in \{\phi,\psi,\bar{\psi}\} \tag{1.1}$$

One then aims at computing correlation functions via

$$\langle \xi_{x_1} \dots \xi_{x_n} \rangle = \frac{1}{Z} \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\psi\; \xi_{x_1} \dots \xi_{x_n}\, e^{-S[\phi,\psi,\bar{\psi}]} \qquad \xi_{x_i} \in \{\phi_{x_i}, \psi_{x_i}, \bar{\psi}_{x_i}\}$$

Computing physical quantities from such a direct approach is not only hard to do, but results often impossible due to the appearence of divergences in the calculations. To fix this, one often relies on expansion techniques such as perturbation theory (CITATION), in which one tries to regularise the theory order by order in an expansion on the interaction coupling, yielding finite quantities that depend on the truncation order. While this method is capable of producing incredibly precise results (g-2, fine structure, ...), it fails completely in treating non-perturbative phenomena, namely effects that cannot be captured by any order in the expansion or that are typical of strongly interacting systems. Example of such systems range from QCD, cold atoms, plasma, stuff. Moreover, such formulation is also not much suitable for numerical computations, since both the path integral and action measure are infinite dimensional objects.
Lattice field theory [1–4] is meant at first as a powerful non-perturbative regularisation tool to prevent divergences to occur and render the computation of the correlation functions finite. Moreover, it also provides a framework to study quantum field theory numerically on a computer. In order to accomplish this, one typically defines the theory on a spacetime lattice and makes use of statistical methods such as Monte

Carlo algorithms to compute observables. One may wonder how can one reconstruct the results in the continuum theory, keeping the results finite, and matching the results on the discretised theory to physically measured ones. This task, far from beeing simple, will be the focus of the next sections, in which we will first introduce relevant theoretical tools, such as the renormalisation group, and then discuss the existence of a continuum limit of a lattice theory and, if it exists, how it can be extracted. This will motivate the introduction of coloured noise in the context of continuum limits of effective theories, a technique which will be shown to be powerful also for other various reasons, which will be the main focus of the analysis carried in the remaining chapters.

Chapter 2

Theoretical background

Da qualche parte cita [5].

In this chapter we want to provide with an overview on the general theoretical framework that supports this work, and introduce the main concepts for the successive parts. Each section in this chapter is, by no means, meant as an exhaustive treatment. The description will be quite conceptual, rather than technical, and aims at recalling the main ideas and fix conventions. We ask the reader to consult appropriate references, which will be given in the corresponding sections, for a more detailed treatment of the topics.

Note that because this chapter is focused on introducing the conceptual framework, the guiding example will be, where not otherwised specified, a scalar field ϕ with the action

$$S[\phi] = \int_x \frac{1}{2} (\partial_\mu \phi_x)^2 + \frac{m_\phi^2}{2} \phi_x^2 + \frac{\lambda}{4!} \phi_x^4. \tag{2.1}$$

The framework is thus general and applies to other theories as detailed in the given references.

2.1 Block-spin renormalisation

Landau's mean field approach to study phase transition [6] gained wide popularity in the 1930's and 40's, since it was able to describe critical properties of many systems and it provided inspiration for the later Landau-Ginzburg theory of superconductivity [7]. Thus, it has soon proved to be inaccurate to predict some experimentally well proven properties of certain systems near their critical point [8]. This is because, beeing a mean field theory, it did not take into account the role of spatial fluctuations. The idea of block-spin transformation, systematically developed by Kadanoff [9], made a big step towards a deeper understanding of the scaling behaviour, and posed the basis for the later work of Wilson [10–12], which still constitutes the basis for modern approaches to renormalisation in field theory and statistical physics.

To illustrate the idea, let us consider a set of spins whose magnetisation is described by a function $\varphi(x)$. The spins are located on a discrete lattice $\mathscr{L}$, so that the function assumes values only at such sites $\varphi(x_i) = \varphi_i \neq 0 \Leftrightarrow x_i \in \mathscr{L}$. Suppose then that their interaction is described by a certain action $S[\varphi]$ and a partition function

$$Z = \sum_\varphi \mathrm{e}^{-S[\varphi]}.$$

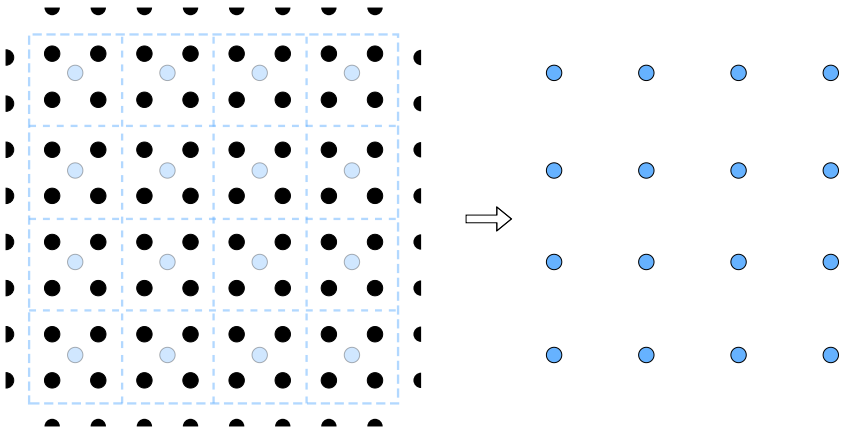


FIGURE 2.1: test, I was very bad at art

We now want to introduce a coarse-grained (or blocked) field $\bar{\varphi}$ within a spacetime cell of volume $\mathcal{V}$. Such a coarse grained field can be defined, for example, as an average over the spins within the cell $\mathcal{V}$. If the spins can only be 0 or 1 like in a Ising model, then we might opt for a majority rule [13]. We now want to find a new action S_b such that

$$Z = \sum_{\varphi} \mathrm{e}^{-S(\varphi)} = \sum_{\bar{\varphi}} \mathrm{e}^{-S_b(\bar{\varphi})}. \tag{2.2}$$

We can always, in principle, build an ad-hoc action that fulfills the above condition, but it is complicated, since S_b can be also be very different from S. For example, if the action $S[\varphi]$ contains only nearest-neighbour interactions, the new actions $S'[\phi]$ can contain higher order interactions such as nearest-to-nearest neighbour, or even more. In principle, all the terms compatible with the original symmetries are allowed. Thus, if the behaviour of the system is dominated by long-range properties, and the volume $\mathcal{V}$ is small enough, one can expect the functional form of the microscopic action S to remain approximately the same, and one has to (eventually) only adjust the couplings.
The coarse-graining procedure comes with a loss of resolution since the spacing is changed $a \to 2a$. This means that the system is not anymore capable of describing physics at scales below the new spacing. Hence one can rescale distances and momenta in the new action via $a \to a/2, p \to 2p$ connect spacing with volume. This constitutes the second step of the block-spin transformation. The whole procedure can be thought as a zoom-out with a corresponding coarse graining, in order to describe the system in terms of the relevant scales as pictured in figure 2.1. The procedure can be iterated multiple times, provided that the higher order interactions are either negligible or taken into account.
Therefore the philosophy of Kadanoff (and subsequently Wilson) was that the blocking transformation reduces the complexity of many-body systems by systematically reducing the number of degrees of freedom being taken into account [14], without changing the physical content of the theory.

2.2 Wilsonian renormalisation

The wilsonian picture of renormalisation [10, 11] is formulated in momentum space and in general is more suitable for theories in the continuum.
The idea is that a physical theory comes with an intrinsic cutoff Λ, which defines up to which scale the theory is valid. This theory might not be the one we observe directly in experiments at energy scales Λ', which we call effective theory at scale Λ'. To see how can this happen, let us consider a theory defined by the action (2.1) and a given cutoff Λ, and let us split the field as $\Phi = \phi + \varphi$ where ϕ are fields with momenta $p^2 \leq \Lambda'^2$ and φ are fields with momenta $\Lambda'^2 < p^2 \leq \Lambda^2$. This allows one to rewrite the path integral as

$$Z = \int D\Phi_\Lambda \, e^{-S_\Lambda[\Phi]} = \int D\phi_{\Lambda'} \, e^{-S_{\Lambda'}[\phi]} \int D\varphi_{\Lambda',\Lambda} \, e^{-S_{\Lambda',\Lambda}[\phi,\varphi]} = \int D\phi_{\Lambda'} \, e^{-S^{\text{eff}}_{\Lambda'}[\phi]},$$

where, in the last step, we formally performed the integral over the high-momentum field φ and defined a new effective action $S^{\text{eff}}_{\Lambda'}$ as

$$S^{\text{eff}}_{\Lambda'}[\phi] = S_{\Lambda'}[\phi] - \log\left(\int D\varphi_{\Lambda',\Lambda} \, e^{-S_{\Lambda',\Lambda}[\phi,\varphi]}\right) = S_{\Lambda'}[\phi] + \delta S_{\Lambda',\Lambda}[\phi].$$

Note that all the steps above are exact identities. At this point, one typically tries to compute $\delta S_{\Lambda',\Lambda}$ via, e.g., Feynman diagrams [15], and then expands the result in terms of local operators of the type $O_{m,n} = \partial_x^m \phi_x^n$ with respective coefficients g_{mn}. In general, this will generate many terms, some of which were already present in the original action S_Λ, while some others will be new. The former can be readily absorbed in $S_{\Lambda'}[\phi]$ by a redefitinion of the couplings. Note that the new action is not anymore capable of describing degrees of freedom above the scale Λ', and there is loss of resolution. As for the block-spin procedure, one can operate a dimensional rescaling in order to set the theory back at its original scale. To do this, let us now define a dimensionless parameter

$$r^2 = \frac{\Lambda'^2}{\Lambda^2}, \qquad 0 \leq r \leq 1,$$

and let us operating the rescaling according to

$$x \to x' = rx, \qquad p \to p' = p/r.$$

Combining the coarse-graining effects due to the high-modes integral and dimensional rescaling, the new action S^{eff}_Λ can be written as [15]

$$S^{\text{eff}}_\Lambda[\phi] = \int_x' \frac{1}{2}\left(\partial'_\mu \phi'\right)^2 + \frac{m'^2_\phi}{2}\phi'^2 + \frac{\lambda'}{4!}\phi'^4 + C'\left(\partial'_\mu \phi'\right)^4 + D'\phi'^6 + \cdots .$$

Each coefficient g' corresponding to the local operator O_{mn} is given by

$$g'(r) = r^{-d_g} \frac{(g + \delta g)}{(1 + \Delta\phi)^n}. \tag{2.3}$$

where d_g is the dimension of the coupling g, the factor s^{-d_g} appears because of the dimensional rescaling mentioned above, δg is the correction term that comes from the expansion of the high-momentum integral, and $Z_\phi = 1 + \Delta\phi$ is the wavefunction renormalisation. comment a bit on this.

At this point, one can clearly see the analogy with the block-spin transformation introduced in the previous section. Performing the integral over high momenta modes can be thought as performing averages (coarse graining) over neighbours. This causes a loss of resolution which can be recovered by rescaling dimensionful quantities, which can be pictured as a zoom out. Note that, beeing r a continuous parameter, the transformation can also be infinitesimal. In this case, the change in the couplings due to successive iterations, gives rise to a continuous flow in parameters space, which can be described by differential equations. This is captured by the beta functions

$$\beta_g = r\,\frac{dg'(r)}{dr}.$$

Clearly, if $\delta g/g \ll 1$, $\Delta\phi \ll 1$, equation (2.3) reduces, at lowest order, to a dimensional rescaling

$$\frac{g'}{g} \approx r^{-d_g} = \left(\frac{\Lambda'}{\Lambda}\right)^{-d_g},$$

and the beta function assumes the form

$$\beta_g = r\,\frac{dg(r)}{dr} = -d_g\,g(r) + \dots.$$

Fixed points are given by

$$\beta_g = 0$$

2.3 Lattice QFT and the continuum limit

punteggiatura formule.
Lattice Quantum Field Thoery is formulated on a discrete set of spacetime points. Let us then consider a lattice, with a space, with numer, with ... The action and the path integral measure are now sums over discrete quantities

$$\begin{aligned} S = \int d^d x\,\mathcal{L}(\phi(x)) \quad &\rightarrow \quad S = a^d \sum_n \mathcal{L}(\phi(n)) \\ \prod_x d\phi(x) \quad &\rightarrow \quad \prod_n d\phi(n) \end{aligned}$$

where $\mathcal{L}(\phi)$ is the Lagrangian density function add citation?.
Note that for simplicity we restricted here to a scalar field ϕ and we will recall fermionic properties only when relevant, but the treatment is valid also for the latters.
The path integral assumes the form

$$Z = \int \prod_n d\phi(n)\, e^{-S[\phi]}$$

with the probability of a field configuration ϕ beeing

$$p(\phi) = \frac{1}{Z}\, e^{-S[\phi]} \tag{2.4}$$

Expectation value of observables are computes as

$$\langle O(\phi)\rangle = \frac{1}{Z} \int \prod_n d\phi(n)\, O(\phi)\, e^{-S[\phi]} \tag{2.5}$$

In order to simulate a theory and perform the above sums one has to go to finite volumes with some boundary conditions. In the space directions, one typically takes periodic conditions

$$\phi(t,\vec{x}) = \phi(t,\vec{x}+T)$$
$$\psi(t,\vec{x}) = \psi(t,\vec{x}+T)$$

Instead, a finite time extent is related to the temperature of the system [2, 16] via

$$\beta = 1/T = 1/L_t$$

and boundary conditions are chosen depending on the spin-statistic of the corresponding particles, namely periodic conditions for bosons, and anti-periodic for fermions

$$\phi(t,\vec{x}) = \phi(t+T,\vec{x}) \qquad \text{bosons}$$
$$\psi(t,\vec{x}) = -\psi(t+T,\vec{x}) \qquad \text{fermions}$$

Such a formulation naturally brings a momentum cutoff $\Lambda = \pi/a$ since now all the momenta are restricted to the first Brilloune zone $p_\mu \in [-\pi/a, \pi/a]$.
To compute observables one relies on Monte-Carlo methods to generate field configurations sampling the distribution (2.4) and convergence to the statistical value given by (2.5) is expected for $N_{\text{samp}} \to \infty$. To recover the continuum results, one has to take $V \to 0, a \to \infty$ (this order is important, cite SOMEONE), but this task cannot be done so straightforwardly [2]. Instead, continuum limits of lattice theories are intimately connected to the existence of critical points in the theories. To see why this is the case, consider the dimensionless mass gap $\hat{\xi} = m\,a$ of a certain theory. The quantity ξ is also called correlation length and it is related to spigea cosa e di che è mostrato dopo. When taking the continuum limit we want $a \to 0$ while having a finite physical mass m. This implies that the correlation length $\hat{\xi}$ has to diverge. In the language of statistical physics, this is a second order phase transition. A divergent correlation length signifies that dire cosa significa. Of course to bring the system at its critical point, where such phase transition happens, one has to tune the bare parameters $\{g_0\}$ to their critical values $\{g_{0*}\}$. This should be done by finding zeros of the beta functions on the lattice, but often one relies on approximate solution such as employing perturbative continuum beta functions.
Note that in the limit $a \to 0$ one has $\Lambda \to \infty$. If one's scope is to simulate an effective theory which is expected to hold only up to a scale Λ_{phys}, one must have $\Lambda \leq \Lambda_{\text{phys}}$ with a consequent lower bound on the lattice spacing $a \geq a_{\text{phys}} = \pi/\Lambda_{\text{phys}}$.
Consider O to be an observable which has to be matched to a physical measurable quantity, and compare it to the dimensionless quantity $\hat{O}$ given by a lattice simulation. In general the physical observable is assumed to be a function of the spacing and the bare couplings of the theory

$$O = O(a, g_0^i)$$

while its lattice counterpart can only depend on the dimensionless coupling $hat g_0^i$, i.e.

$$\hat{O} = \hat{O}(g_0^i)$$

Let d_O be the physical dimension of the observable O in units of energy. Then one can relate the two quantities as

$$O(a, g_0^i) = \left(\frac{1}{a}\right)^{d_O} \hat{O}(\hat{g}_0^i) \tag{2.6}$$

We now want to address the following question: given a (small enough) a, is it possible to find a value $\hat{g}_0^i(a)$ such that the value of O given by (2.6) does not depend on a?
We then impose such condition via

$$\frac{d}{da} O(a, g_O^i) = \left(a\frac{\partial}{\partial a} - \beta(g_0^i)\frac{\partial}{\partial g_0^i} \right) O(a, g_O^i) = 0$$

with

$$\beta(g_0^i) = -a\frac{\partial g_0^i}{\partial a}$$

Integrating such β functions tells one how to change bare couplings as a backreaction to a change in the spacing, in order to keep observables constant. We then say that the theory admits a continuum limit if there exists some set of values $(g_0^i)^*$ such that when $g_0^i \to (g_0^i)^*$ one has $\hat{\xi} \to +\infty$ and $O \to O_{phys}$. Connection with fixed points and beta function. Comment also on beta functions for dimless couplings.
Of course one does not know a priori the full lattice beta functions, but they can be computed via approximate or continuum methods. For example, in continuum perturbation theory, one can compute $g_r^i(\Lambda) = g_r^i(\Lambda, g_0^j)$, where Λ is a sharp momentum cutoff and then try to invert them to find $g_0^i = g_0^i(\Lambda, g_r^j)$. The connection is then given by making the identification $a \sim \Lambda^{-1}$.
Generally speaking, we are interested in the set of theories in theories space that have constant renormalised couplings but different dimless couplings $g_r^i\, a$ (trajectories in Kadanoff-Wilson RG).

2.4 Yukawa theory

Let us consider the Yukawa theory defined by the action

$$\begin{aligned}
S[\phi, \psi, \bar{\psi}] &= S_\phi[\phi] + S_\psi[\psi, \bar{\psi}] + S_{\text{int}}[\phi, \psi, \bar{\psi}] \\
S_\phi[\phi] &= \int_x \phi_x \left(-\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \phi_x + \frac{\lambda}{4!}\phi_x^4 \\
S_\psi[\psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} \left(\not{\partial}_x + m_q \right) \psi_x^{(f)} \\
S_{\text{int}}[\phi, \psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} g\, \bar{\psi}_x^{(f)}\, \phi_x\, \psi_x^{(f)}
\end{aligned} \tag{2.7}$$

One can see that the action is made of a scalar part $S_\phi[\phi]$, a fermionic part $S_\psi[\psi, \bar{\psi}]$ and a Yukawa interaction term $S_{\text{int}}[\phi, \psi, \bar{\psi}]$.
It is also convenient for later purposes to define the operators K, D represented in position space as

$$\begin{aligned}
K(x, y) &= \left(-\partial_x^2 + m_\phi^2 \right) \delta(x, y) \\
D(x, y) &= \left(\not{\partial}_x + m_q + g\phi \right) \delta(x, y)
\end{aligned} \tag{2.8}$$

and in momentum space as

$$\begin{aligned}\widetilde{K}(p,q) &= \int_{x,y} e^{-ipx}\left(\partial_x^2 + m_\phi^2\right)\ \delta(x,y)\, e^{iqy} = \left(\frac{p^2}{2} + \frac{m_\phi^2}{2}\right)\ \delta(p,q)\\ \widetilde{D}(p,q) &= \int_{x,y} e^{-ipx}\left(\not{\partial}_x + m_q + g\phi\right)\ \delta(x,y)\, e^{iqy} = \left(\not{p}_x + m_q + g\phi\right)\ \delta(p,q)\end{aligned} \tag{2.9}$$

This allows one to rewrite the action as

$$S[\phi,\psi,\bar{\psi}] = \int_x \frac{1}{2}\,\phi_x\, K_{xx}\,\phi_x + \frac{\lambda}{4!}\,\phi_x^4 + \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)}\, D_{xx}\, \psi_x^{(f)}$$

We introduce the left-handed and right-handed spinors

$$\psi_L = (1-\gamma_5)\,\psi \qquad \psi_R = (1+\gamma_5)\,\psi$$

for which

$$\psi = \frac{(1-\gamma_5)}{2}\psi + \frac{(1+\gamma_5)}{2}\psi = \psi_L + \psi_R$$

The action written in terms of ψ_L, ψ_R reads

$$S = S_\phi + \bar{\psi}_L D\psi_L + \bar{\psi}_R D\psi_R + (m_q + g\phi)\ (\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \tag{2.10}$$

The last equation makes clear that that for $m = 0, \langle\phi\rangle = 0$ the action is symmetric under the chiral group $SU(2)_L \times SU(2)_R$, namely

$$\begin{aligned}\psi_L(x) &\to U_L\psi_L(x) \qquad & \bar{\psi}_L(x) &\to \bar{\psi}_L(x)U_L^\dagger\\ \psi_R(x) &\to U_R\psi_R(x) \qquad & \bar{\psi}_R(x) &\to \bar{\psi}_R(x)U_R^\dagger\end{aligned}$$

for $U_L, U_R \in SU(2)$.
The main feature of the model is chiral symmetry breaking [17, 18], which can happen explicitly at the level of the classical action for a non-zero quark mass, or spontaneously when the scalar field gains a non-zero expectation value. One can in fact notice already by looking at (**??**), that $\langle\phi\rangle \neq 0$ has the same effect on the action as a finite bare quark mass. This observation will be made more quantitative in section (SECCCC) where it will be shown that

$$\langle\phi\rangle \sim \langle\bar{\psi}\psi\rangle \sim m_q$$

The fermionic part of the path integral (1.1) can be performed explicitly

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\left(-\int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)}\, D\, \psi_x^{(f)}\right) = (\det D[\phi])^{N_f} = e^{N_f\,\mathrm{Tr}\,\log(D[\phi])}$$

where the trace is performed over spacetime and spinor components.
The full path integral can now be expressed in terms of the resulting effective action for the scalar fields

$$Z = \int \mathcal{D}\phi\, e^{-S_{\mathrm{eff}}[phi]}$$

with

$$S_{\mathrm{eff}}[\phi] = S_\phi[\phi] - \operatorname*{Tr}_{x,s,f} \log D[\phi] \tag{2.11}$$

One can derive the classical equations of motion by imposing $\frac{\delta S}{\delta \phi} = 0$, here expressed in momentum space

$$(k^2 + m_\phi^2)\,\phi(x) + \frac{\lambda}{6}\,\phi^3(x) = g \operatorname*{Tr}_{s,f}\left[D^{-1}(\phi(x)\right] = -g\,\bar{\psi}(x)\psi(x)$$

where the trace is performed over spin and flavour components. For $\lambda = 0$, they highlight a simple proportionality relation between magnetisation and chiral condensate, which for zero momentum reads

$$\phi(x) = -\frac{g}{m_\phi^2}\,\bar{\psi}(x)\psi(x) \tag{2.12}$$

The classical relation (2.12) is proven to hold also at mean field on the quantum level [19] and will be studied in the discretised theory in section 4.4.

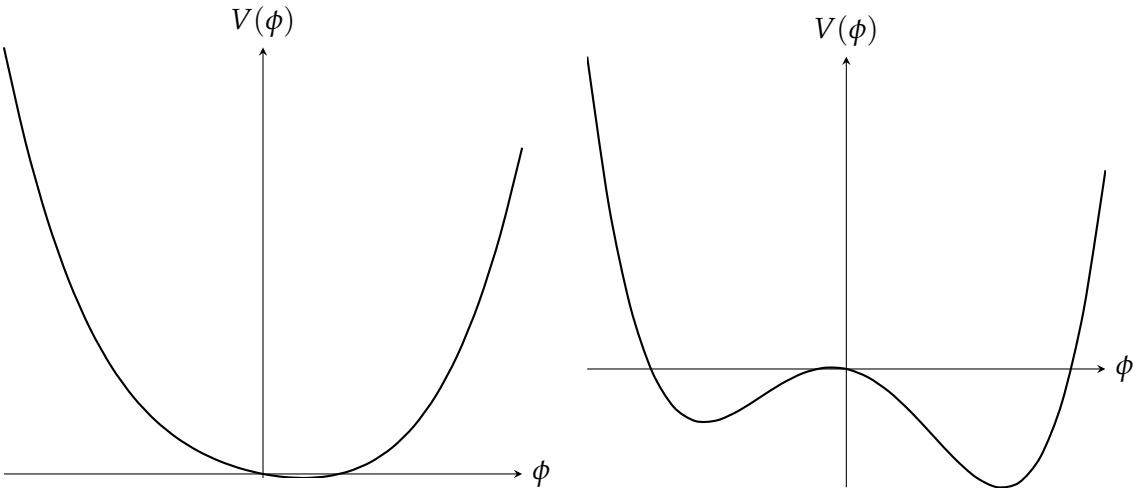


FIGURE 2.2: The introduction of the boson-fermion interaction, with a finite fermionic mass, causes the breaking of the O(1) symmetry. It shifts the equilibrium position in the symmetric phase (left) causing $\langle\phi\rangle \neq 0$, and tilts the potential in the broken phase (right), making the two minima not equivalent.

Chapter 3

Methods and algorithms

3.1 Discretisation of the Yukawa theory

In order to make the theory suitable for a numerical simulation on a computer, the continuum formulation of the Yukawa model, which has been introduced in section 2.4, has to be discretised. Here we provided a sketch of a discretisation procedure, and we refer to other resources [1–4] for further details.

For what concerns the bosonic part of the action, a discretisation can be done straightforwardly with the following replacements

$$\begin{aligned} \int d^x \quad &\to \quad a^2 \sum_x, \\ \partial_t^2 + \partial_x^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} \quad &\to \quad \sum_\mu \left[\frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2} \right], \end{aligned}$$

which yields to the lattice action

$$\begin{aligned} S_\phi[\phi] &= a^2 \left(\frac{1}{2} \sum_{m,n} \phi_m \, K_{mn} \, \phi_n + \frac{\lambda}{4!} \sum_n \phi_n^4 \right) \\ &= \frac{1}{2} \sum_{m,n} \hat{\phi}_m \, \widehat{K}_{mn} \, \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \sum_n \hat{\phi}_n^4, \end{aligned}$$

where we expressed everything in dimensionless quantities

$$\begin{aligned} \hat{m}_\phi^2 &= a^2 \, m_\phi^2, \\ \hat{\lambda} &= a^2 \, \lambda, \\ \widehat{K}_{mn} &= a^2 K_{mn}. \end{aligned} \tag{3.1}$$

The operator components $\widehat{K}_{mn}$ are the discretised version of (2.8)

$$\widehat{K}_{mn} = -\sum_\mu \left[\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\,\delta_{m,n} \right] + \hat{m}_\phi^2 \, \delta_{mn} \tag{3.2}$$

and its representation in momentum space is

$$\begin{aligned}\widehat{K}_{p,q} &= \sum_{n,m} e^{ipn}\,\widehat{K}_{nm}\,e^{-iqm} \\ &= \sum_{n,m} e^{ipn}\left(-\sum_{\mu}\left[\delta_{m,m+\mu}+\delta_{m,m-\mu}-2\delta_{m,n}\right]+\hat{m}_\phi^2\,\delta_{mn}\right)e^{-iqm} \\ &= \sum_{n} e^{i(p-q)n}\left[\hat{m}_\phi^2+2\sum_{\mu}\left(1-\cos\left(q_\mu\right)\right)\right] \\ &= \left[\hat{m}_\phi^2+\sum_{\mu}4\sin^2\left(\frac{p_\mu}{2}\right)\right]\delta(p-q).\end{aligned}$$

For what concerns the fermionic action, a naïve discretisation is not sufficient, due to the well known doubling problem [1, 2]. In this work Wilson fermions [20] are employed as a way to fix such issue. Details of this formulation are explained in Appendix B. Here, only the final discretised action is reported, which reads

$$S_\psi\left[\hat{\psi},\hat{\bar{\psi}}\right]+S_{\text{int}}\left[\hat{\phi},\hat{\psi},\hat{\bar{\psi}}\right]=\sum_{f=1}^{N_f}\hat{\bar{\psi}}_m^{(f)}\,\widehat{D}_{mn}\,\hat{\psi}_n^{(f)}, \tag{3.3}$$

with ψ_n beeing a two-component spinor, and $\widehat{D}_{m,n}$ the Wilson-Dirac operator (can I include $g\phi$ in the definition of D?) defined as

$$\begin{aligned}\widehat{D}_{m,n} = &-\left(\frac{\Gamma_{+\hat{0}}}{2}\,\delta_{m,m+\hat{0}}+\frac{\Gamma_{-\hat{0}}}{2}\,\delta_{m,m-\hat{0}}+\frac{\Gamma_{+\hat{1}}}{2}\,\delta_{m,m+\hat{1}}+\frac{\Gamma_{-\hat{1}}}{2}\,\delta_{m,m-\hat{1}}\right) \\ &+\left(2ar+\hat{m}+\hat{g}\phi\right)\,\delta_{s,s'}\,\delta_{m,n}.\end{aligned} \tag{3.4}$$

Note that the interaction term $g\,\bar{\psi}\phi\psi$ has been included in the definition of D.
The Wilson projectors $\Gamma_{\pm\hat{\mu}}$ are defined as

$$\Gamma_{\pm\hat{\mu}} = ar\,\mathbb{1}_s \mp \gamma_\mu.$$

Since $r\in[0,1]$ is a free parameter, in this work we set $r=1$, if not otherwise specified.
In summary the discretised action for the Yukawa model is

$$S\left[\hat{\phi},\hat{\psi},\hat{\bar{\psi}}\right]=\sum_{m,n}\hat{\phi}_m\,\widehat{K}_{m,n}\,\hat{\phi}_n+\frac{\hat{\lambda}}{4!}\,\hat{\phi}_m^4\,\delta_{m,n}+\sum_{f=1}^{N_f}\hat{\bar{\psi}}_m^{(f)}\,\widehat{D}_{mn}\hat{\psi}_n^{(f)},$$

with $\widehat{K}_{mn},\widehat{D}_{mn}$ given respectively by (3.2) and (3.4).
For later reference, we also report the discretised version of the effective action (2.11)

$$\begin{aligned}S_{\text{eff}}[\hat{\phi}] &= S_\phi[\hat{\phi}]-N_f\operatorname*{Tr}_{n,s}\log\widehat{D} \\ &= \sum_{m,n}\hat{\phi}_m\,\widehat{K}_{m,n}\,\hat{\phi}_n+\frac{\hat{\lambda}}{4!}\,\hat{\phi}_m^4\,\delta_{m,n}-N_f\operatorname*{Tr}_{n,s}\log\widehat{D}_{nn}.\end{aligned} \tag{3.5}$$

The full discrete path-integral reads (measure over dimless or dimful?)

$$Z=\int\prod_n d\hat{\phi}_n\,e^{-S_{\text{eff}}[\hat{\phi}]}. \tag{3.6}$$

In the remaining of this work, both the original action S and the effective action S_{eff} will be denoted by S for simplicity. It will be clear from the context which of the two we will be referring to.

3.2 Stochastic quantisation and Langevin Monte Carlo

in this and next section I refer multiple times to Damgaard. Should I cite everytime or not?

In order to compute expectation values from the discretised path integral (3.6), we employ a Langevin Monte Carlo algorithm, which is based on stochastic quantisation [21, 22].

The idea is that Euclidean Quantum Field theory can be thought as a system in thermal equilibrium with a heat reservoir and hence described as a stochastic process via the Langevin equation. For this, one has to introduce a fictitious time variable τ that labels the state $\phi(\tau, x)$ of the system during the evolution.

Let us consider, for example, a scalar field ϕ with a Euclidean action $S[\phi]$ and the following Langevin equation

$$\partial_\tau \phi(\tau, x) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta(\tau, x), \tag{3.7}$$

where $K_\phi(\tau) \equiv -\delta S[\phi]/\delta\phi(\tau, x)$ is the drift term and $\eta(\tau, x)$ is a random white noise field assumed to be normally distributed

$$P(\eta) = \frac{\exp\left(-\frac{1}{4}\int_{\tau,x} \eta^2(\tau, x)\right)}{\int D\eta \exp\left(-\frac{1}{4}\int_{\tau,x} \eta^2(\tau, x)\right)},$$

which, in particular, implies

$$\langle \eta(x, \tau)\rangle = 0, \qquad \langle \eta(x, \tau)\, \eta(x', \tau')\rangle = 2\,\delta(x - x')\,\delta(\tau - \tau'). \tag{3.8}$$

Stochastic average with respect to the measure $P(\eta)$ are computed via

$$\langle A(\eta)\rangle = \int D\eta\, P(\eta)\, A(\eta).$$

In momentum space, (3.8) becomes

$$\begin{aligned}
\langle \eta(p, \tau)\rangle &= \left\langle \int_x e^{ipx} \eta(x, \tau) \right\rangle = \int_x \left\langle e^{ipx} \eta(x, \tau) \right\rangle = 0, \\
\langle \eta(p, \tau)\eta(q, \tau')\rangle &= \left\langle \int_{xy} e^{ipx+iqy} \eta(x, \tau)\eta(y, \tau') \right\rangle \\
&= \int_{xy} e^{ipx+iqy} \langle \eta(x, \tau)\eta(y, \tau')\rangle \\
&= 2\,(2\pi)^2\,\delta(p + q)\,\delta(\tau - \tau').
\end{aligned} \tag{3.9}$$

In absence of the noise term $\eta(\tau, x)$, equation (3.7) simply represents an evolution of the field towards the minimum of the action, and at equilibrium the field is constrained to $\partial_\tau \phi(x, \tau) = 0 = \delta S[\phi]/\delta\phi(\tau, x)$, namely to the classical equations of motion.

For any observable O, which is function of the field, one has, for fixed time τ,

$$\langle O(\phi(\tau))\rangle = \int D\eta\, P(\eta)\, O(\phi(\tau)),$$

from which it follows straightforwardly using the Langevin equation and $\langle \eta \rangle = 0$

$$\frac{d}{d\tau}\langle O(\phi(\tau))\rangle = \left\langle \frac{\delta O}{\delta\phi(\tau,x)}\,\partial_\tau\phi(\tau,x)\right\rangle = -\left\langle \frac{\delta O}{\delta\phi}(\tau,x)\,\frac{\delta S}{\delta\phi(\tau,x)}\right\rangle.$$

It follows trivially that for $O(\phi(\tau)) = \phi(\tau,x)$

$$\frac{d}{d\tau}\langle\phi(\tau,x)\rangle = -\left\langle\frac{\delta S}{\delta\phi(\tau,x)}\right\rangle \quad \underset{Equilibrium}{\Longrightarrow} \quad \left\langle\frac{\delta S}{\delta\phi(\tau,x)}\right\rangle = 0.$$

This also provides a consistency check for the correct implementation of the simulation, since the drift $K_\phi = -\delta S/\delta\phi$ is computed numerically during the evolution. More generally, one can derive a correspondent Fokker-Planck equation [23], which can be proven to have a stationary distribution if the action is bounded from below, given by [22]

$$\mathcal{P}(\phi) = \frac{1}{Z}\exp\left(-S[\phi]\right). \tag{3.10}$$

This allows one to compute correlation functions as moments of the probability distribution (3.10). In particular, one has

$$\langle O\rangle_{P(\eta)} = \langle O\rangle_{\mathcal{P}(\phi)} \equiv \langle O\rangle. \tag{3.11}$$

This idea suggests that equation (3.7) can be integrated numerically for discrete time steps τ_n to generate field configurations distributed according to (3.10). The simplest first-order integration algorithm is the Euler-Majorana scheme [21]

$$\phi(\tau_{n+1},x) = \phi(\tau_n,x) - \epsilon\,\frac{\delta S[\phi]}{\delta\phi(\tau_n,x)} + \sqrt{\epsilon}\,\eta(\tau_n,x) + O(\epsilon^2),$$

where $\epsilon = \tau_{n+1} - \tau_n$. Higher order integration schemes are possible (see e.g.[24, 25]), but not adopted in this work, and an adaptive step size is employed as detailed in Appendix C. In this way, for any observable O, one can introduce a Monte-Carlo estimator $\langle O\rangle_*$ which converges to the expectiation value given by (3.11) in the limit of infinite samples

$$\langle O\rangle_* = \frac{1}{N_{\text{samp}}}\sum_{i=1}^{N_{\text{samp}}} O_i \quad \underset{N_{\text{samp}}\to\infty}{\longrightarrow} \quad \langle O\rangle = \frac{1}{Z}\int D\phi\, O(\phi)\,\exp\left(-S[\phi]\right), \tag{3.12}$$

where $O_i = O(\phi(\tau_i))$ is the sample of the observable O done at time τ_i.

For the discretised action of the Yukawa theory (3.5) the drift reads, explicitly,

$$\begin{aligned}\frac{\partial S}{\partial\hat\phi_m(\tau_n)} &= \frac{\partial S_{\hat\phi}}{\partial\hat\phi_m(\tau_n)} - N_f\,\underset{s}{\mathrm{Tr}}\left[\sum_{j,k}\widehat{D}^{-1}_{jk}\,\frac{\partial\widehat{D}_{kj}(\hat\phi)}{\partial\hat\phi_m(\tau_n)}\right]\\ &= \sum_l \widehat{K}_{ml}\,\hat\phi_l + \frac{\hat\lambda}{6}\,\hat\phi_m^3 - \hat g\,N_f\,\underset{s}{\mathrm{Tr}}\left[(\widehat{D}_{mm})^{-1}(\hat\phi(\tau_n))\right].\end{aligned} \tag{3.13}$$

While the bosonic contribution can be computed in a straightforward manner, the computation of the fermionc contribution requires the inversion of the Dirac operator. This, in general, cannot be done straightforwardly, mainly due to computational reasons. In fact, the full Dirac operator would be a $(2 \cdot N_t \cdot N_x \cdot N_f)^2$ dimensional object and a full inversion would be very expensive. In fact, D^{-1} has to be recomputed at every step and the best available algorithm as today for the matrix inversion has a computational complexity of $O(n^{2.371552})$ [26]. To circumvent this, we use the bilinear noise scheme [24, 27] should I cite here or in Appendix? which is illustrated in Appendix C.

3.3 Stochastic quantisation with coloured noise

In the stochastic quantisation procedure the noise which accounts for the quantum fluctuations of the theory is assumed to be white, as defined in equations (3.8), (3.9). We now want to examine the dynamics in presence of a coloured noise, writing the Langevin equation as

$$\partial_\tau \phi(x,\tau) = -\frac{\delta S[\phi]}{\delta \phi(\tau,x)} + \eta_{\text{col}}(x,\tau),$$

with $\eta_{\text{col}}(x,\tau) = r_\Lambda(x)\,\eta(x,\tau)$. In particular, here we restrict to the regulating function defined as a sharp cutoff in momentum space

$$r_\Lambda(p) = \theta(\Lambda^2 - p^2), \tag{3.14}$$

and we invite the reader to consult [28] for a discussion of other regulating functions. The noise field in momentum space is then

$$\begin{aligned}\eta_{\text{col}}(p,\tau) &= \mathcal{F}[\eta_{\text{col}}(x,\tau)] = \mathcal{F}[r_\Lambda(x,\tau)\eta(x,\tau)] = \mathcal{F}[r_\Lambda(x,\tau)] \star \mathcal{F}[\eta(x,\tau)] \\ &= \theta(\Lambda^2 - p^2)\,\eta(p,\tau).\end{aligned}$$

where $\mathcal{F}$ indicates the Fourier transform and $\star$ the convolution product.
An interesting quantity to look at is the position-space noise correlation function

$$\begin{aligned}C_\eta(x,\tau,y,\tau') &= \langle \eta_{\text{col}}(x,\tau)\,\eta_{\text{col}}(y,\tau')\rangle \\ &= \frac{1}{(2\pi)^4}\int D\eta\, P(\eta)\left[\int_{p,q} e^{-ipx-iqy}\eta_{\text{col}}(p,\tau)\,\eta_{\text{col}}(q,\tau')\right] \\ &= \frac{1}{(2\pi)^4}\int_{p,q} e^{-ipx-iqy}\int D\eta\,\left[P(\eta)\,\eta(p,\tau)\,\eta(q,\tau')\right] \\ &\quad \times \theta(\Lambda^2-p^2)\,\theta(\Lambda^2-q^2) \\ &= \frac{2}{(2\pi)^2}\int_{p,q} e^{-ipx-iqy}\,\delta(p+q)\,\theta(\Lambda^2-p^2)\,\theta(\Lambda^2-q^2)\,\delta(\tau-\tau') \\ &= \frac{2}{(2\pi)^2}\int_p e^{-ip(x-y)}\,\theta(\Lambda^2-p^2) = \frac{1}{\pi}\int_0^\Lambda d\omega\,\omega\,J_0(\omega|x-y|),\end{aligned} \tag{3.15}$$

where $J_0(x)$ is a Bessel function of the first order. The integral in the last line is computed numerically as a function of $d = |x-y|$ and reported in figure 3.1 for three different values of the cutoff $\Lambda_1 < \Lambda_2 < \Lambda_3$. This shows nicely that for $|x-y| \ll 1/\Lambda$ the noise is now correlated, while for $|x-y| \gg 1/\Lambda$ the correlation function vanishes, as in the white noise case. In other words, only the short-length behaviour

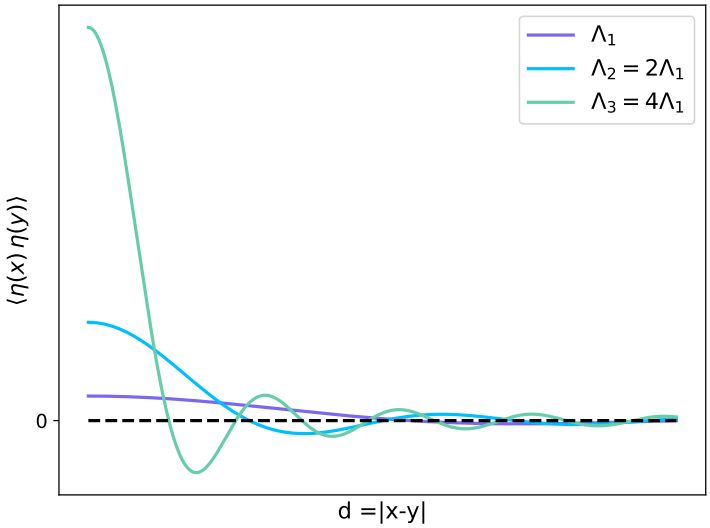


FIGURE 3.1: Noise correlation as a function of $d = |x - y|$ for three different values of the cutoff $\Lambda_1 < \Lambda_2 < \Lambda_3$, in arbitrary units. The plot is qualitative, but shows clearly that with a regulated noise, only the short-distance behaviour is affected.

of the system is affected by the introduction of such a regulating term, as one could expect.

Another intuitive and interesting aspect of the dynamics in the presence of coloured noise can be deduce by looking at the field expression in terms of the retarded Langevin Green function [22], which is here not derived, but reported from [28]

$$\phi(x,\tau) = \int_{x'} \int_{-\infty}^{\tau} \mathrm{d}\tau' G\left(x - x', \tau - \tau'\right) \left[r_\Lambda\left(\Delta_x\right) \eta\left(x, \tau'\right) - \frac{\delta S}{\delta \phi}|_{p=0}\, \phi\left(x', \tau\right) \right],$$

where

$$G\left(x - x', \tau - \tau'\right) = \theta\left(\tau - \tau'\right) \int_p \mathrm{e}^{-ip\cdot(x-x')} \mathrm{e}^{-(\tau-\tau')\left(p^2+m^2\right)}.$$

By looking at the first term in the square bracket, one can conclude that there is no propagation of modes with momentum $p^2 \geq \Lambda^2$ due to the noise term, but one can still have contribution from modes $p^2 > \Lambda^2$ from the second term, which corresponds to the deterministic part of the equations of motion. Stated differently, UV quantum fluctuations with $p^2 > \Lambda^2$ are removed from the dynamics of ϕ, but still contribute classically.

Generally speaking, the stationary distribution probability of the regulated stochastic process is given by [28]

$$\mathcal{P}_\Lambda(\phi) = \frac{1}{Z} \exp\left(-S_\Lambda[\phi]\right) = \frac{1}{Z} \exp\left(-\left(S[\phi] + \Delta S_\Lambda[\phi]\right)\right), \tag{3.16}$$

where the correction term $\Delta S_\Lambda[\phi]$, for the specific case of the regulator (3.14) reads

$$\Delta S_\Lambda[\phi] = \frac{1}{2} \int_p \phi_p\, \Lambda^2 \left(\frac{1}{r_\Lambda\left(p^2\right)} - 1 \right) \phi_{-p}.$$

at this point mention that this is the stationary pdf that one gets with frg for sharp cutoff, and cite some papers.

3.4 Applications of coloured noise in lattice QFT

Cooling and the continuum limit

After the general introduction on coloured noise given in the previous paragraph, let us now look more closely on the lattice formulation and at some possible applications of the technique.

To this end, let us consider a two-dimensional lattice with side lengths L_t, L_x and spacing $a = a_x = a_t$. This implies a maximum momentum $p_{\max} = \pi/a$ in each spacetime direction and $N_x = L_x/a, N_t = L_t/a$ points in each direction. Let us also define

$$\Lambda^2 \equiv (p^x_{\max})^2 + (p^t_{\max})^2, \tag{3.17}$$

which indicates the maximum squared momentum on the given lattice.
We then consider a simulation with a regularised noise defined by a cutoff $\Lambda_{\text{eff}} \leq \Lambda$ and we define a dimensionless parameter

$$s = \frac{\Lambda_{\text{eff}}}{\Lambda}, \qquad 0 \leq s \leq 1. \tag{3.18}$$

Note that Λ_{eff} implicitly defines a length scale given by $a_{\text{eff}} = \pi/\Lambda_{\text{eff}}$.
Let us then consider a simulation with $s = 1$ and a set of bare couplings $\{g^i_0\}$, and another simulation with $s' < 1$ and new set of couplings $\{g^{i\prime}_0\}$. We now want to address the following question: is it possible to compensate the change in physical observables caused by the removal of the UV modes via regularised noise in the second simulation, by properly fine-tuning the bare parameters that enter the lattice discretised action? In other words, we want to encode the quantum fluctuations with $p^2 > \Lambda^2_{\text{eff}}$ in a redefitinion of the classical action so that the expectation value of the observables remains unchanged.
The issue is of course related to the renormalisation transformation introduced in chapter 2. In particular we can exploit the connection between stochastic quantisation with coloured noise and the renormalisation group, as mentioned at the end of section 3.3, to accomplish the above mentioned goal.
As stated at the end of section 2.2, the Wilson RG flow of the dimensionless couplings and fields is dominated by the canonical scaling dimension of the corresponding dimensionful quantities, for high enough cutoff. We are interested in cooling the simulation by removing only the short-length fluctuations, hence we rely on this approximation as a lowest order Ansatz. This corresponds to a tree level RG rescaling.

For what concerns the scalar part of the action, the rescaling at tree level is rather straightforward

$$\hat{m}^2_\phi = (a^2 m^2_\phi) \;\to\; s^2(a^2 m^2_\phi) = s^2\,\hat{m}^2_\phi, \qquad \hat{\lambda} = (a^2\lambda) \;\to\; s^2(a^2\lambda) = s^2\hat{\lambda},$$
$$\hat{\phi} = \phi \;\to\; \phi = \hat{\phi}.$$

The fermionic part needs some more careful analysis. For simplicity, let us for the moment set $N_f = 1$.
In a lattice simulation one wants to perform the integral over the fermionic fields

and works with the effective action (2.11). In this case the drift is given by equation (3.13), with the fermionic contribution

$$K_\psi = g \operatorname*{Tr}_s D^{-1}, \tag{3.19}$$

or, in terms of dimensionless quantities,

$$\widehat{K}_\psi = (ag) \operatorname*{Tr}_s (aD)^{-1}.$$

This implies that under a lattice block-spin transformation, where $a \to sa$,

$$\widehat{K}_\psi \to (sag) \operatorname*{Tr}_s (saD)^{-1} = \widehat{K}_\psi \tag{3.20}$$

On the other side, when computing the drift via the original action (2.7), one gets (omitting τ and x dependency)

$$\begin{aligned} K(\tau, x) &= -\frac{\delta S}{\delta \phi(\tau, x)} = K_\phi - g\,\bar{\psi}\psi = \\ &= -\left(-\partial_x^2 + m_\phi^2\right)\phi - \frac{\lambda}{6}\phi^3 - g\,\bar{\psi}\psi. \end{aligned} \tag{3.21}$$

where the fermionic contribution is given by

$$K'_\psi = -g\,\bar{\psi}\psi.$$

Note that all the terms in the equation (3.21) have dimension 2, in units of energy, which means, in particular, that after a lattice block-spin transformation where $a \to sa$, one has

$$\widehat{K}'_\psi = (ag)(a\bar{\psi}\psi) \to s^2(ag)(a\bar{\psi}\psi) = s^2\widehat{K}'_\psi, \tag{3.22}$$

in contrast with (3.20). For this reason, in order to have the correct scaling, we compute the contribution to the drift without rescaling the Dirac operator (and hence the Yukawa coupling), and then rescale the whole drift via

$$\widehat{K}_\psi \to s^2\widehat{K}_\psi$$

so that the scaling dimension of the other terms in (3.21) is matched.

We want to mention that this has important consequences on the issue of continuum limit of low-energy effective theories. In fact, in the standard lattice regularisation procedure, one always has $a \sim \Lambda^{-1}$, which means that the continuum limit $a \to 0$ is always connected to the limit $\Lambda \to \infty$, and this constitutes a problem for effective theories. The latter are in fact meant to be valid only up to a certain scale Λ_{phys}, hence one always needs $\Lambda \leq \Lambda_{\text{phys}}$. Cooling via coloured noise can provide a solution to this problem, since one can always keep the effective cutoff fixed at $\Lambda_{\text{eff}} = \Lambda_{\text{phys}}$, while taking the limits $a \to 0, \Lambda \to \infty$.

Temperature control

I have to wait for some results to see whether to include this or not :D

Chapter 4

Numerical investigation

4.1 GPU implementation of the Conjugate Gradient algorithm

Idk if I have time to do this because I want to compare with CPU and the CPU code is for Quark meson.

4.2 Exracting quark masses

The Dirac operator for Wilson fermions in the yukawa model is

$$D_{nm} = \sum_{\alpha} \left[\frac{\gamma_\alpha \delta_{n+\alpha,m} - \gamma_\alpha \delta_{n-\alpha,m}}{2} + (m_q + g\phi)\, \delta_{nm} \right].$$

In momentum space it reads

$$\bar{D}_{ff'}(p) = \left(m + g\sigma \sum_{\mu} 2\sin^2\left(\frac{p_\mu}{2}\right) + i \sum_{\mu} \gamma_\mu \sin\left(p_\mu\right) \right) \delta_{ff'}$$

The inverse can be checked to be

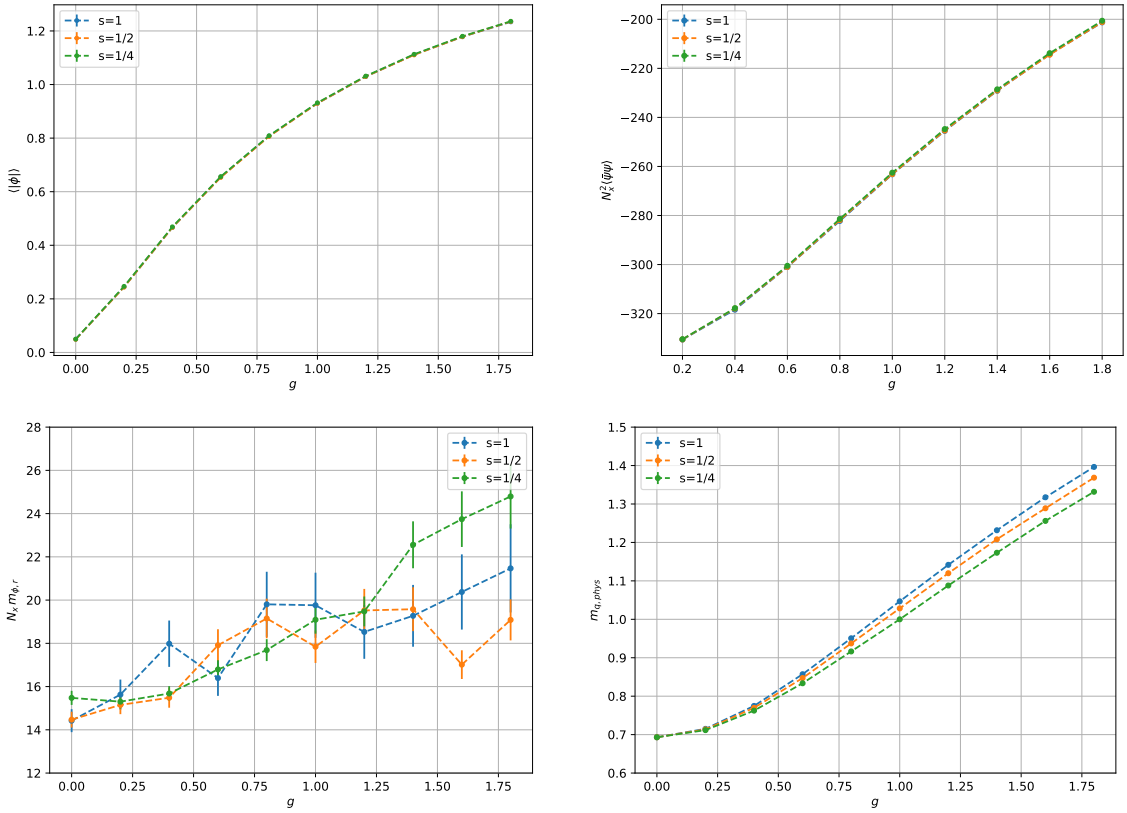
$$\bar{D}^{-1}_{f,f'} = [m + \dots] \left(m + g\sigma \sum_{\mu} 2\sin^2\left(\frac{p_\mu}{2}\right) - i \sum_{\mu} \gamma_\mu \sin\left(p_\mu\right) \right) \delta_{ff'}$$

One can now find the pole mass by imposing $D^{-1} = 0$ and gets

$$m + \dots = 0$$

4.3 Cooling with coloured noise

Look at various things such as magnetization, mass, etc.
Even though is O(1) we do not observe SSB because of fermion bare quark mass, wilson term.
When does L.O. rescaling ansatz breaks down?



$$S[\phi, \bar{\psi}, \psi] = \int_x \phi \left(\frac{\partial^2}{2} + \frac{m_\phi^2}{2} \right) \phi + \frac{\lambda}{4!}\phi^4 + \bar{\psi} \left(\not{\partial} + m_q + g\phi \right) \psi$$

$\lambda = 1.0$ $\quad m_\Phi^2 = 0.5$ $\quad N_t \times N_x = 8 \times 32$ $\quad m_q = 0.5$ $\quad N_{conf} = 5 \cdot 10^3$ $\quad \bar{\epsilon} = 0.01$

4.4 Classical to quantum interpolation

Let us start by analising the coloured noise field in the simulation and relevant properties that emerge from it. We consider the Yukawa model described by the continuum action **??**. In figure **??** the system is initialised in the same state for all the configurations, and then evolved with the Langevin equation with various noise fractions. The red line corresponds to the case $s = 0$, namely a classical simulation. The blue line corresponds to the case $s = 1$, namely the fully quantum case. As one can notice, the introduction of noise shifts the equilibrium expectation value of the field monotonically with the cutoff fraction: this is due to the fact that WHAT???. Note that a lower noise fraction is correlated to a faster convergence towards equilibrium. Moreover, low-distance fluctuations are suppressed due to the removal of the ultraviolet modes in the noise term.

For $\lambda = 0$ one has

$$\sigma = -\frac{g}{m_\phi^2 + k^2}\,\bar{\psi}\psi \tag{4.1}$$

In figure **??** one can see that equation (4.1) is verified also on the fully quantum level.

Figures **??** - **??** report a few observables as a function of the cutoff fraction s. In this case all the coupling constants are kept fixed while changing the value of s, in

order to provide a smooth interpolation between the fully classical and fully quantum picture.
Each figure reports two plots corresponding to two different parameter configurations. The COLOR1 line corresponds to a system in the symmetric phase, while the COLOR2 line correspond to the broken phase. The exact parameters for the two configurations are reported under the figure.

4.5 Chiral fermions and a glimpse on the chiral phase transition

As explained in section **??**, in the continuum theory chiral symmetry can be broken either explicitly via a finite bare quark mass, or spontaneously if the field gains a non-zero expectation value. Moreoveor, in the discrete formulation, the introduction of the Wilson term also contributes to the explicit breaking of chiral symmetry add reference, as explained in section **??**. This, in particular, means that chiral symmetry is explicitly broken also for $m_q \to 0$. Because of this, one needs a new definition for bare mass M_q, which takes into account the Wilson term contribution, such that chiral symmetry is restored in the limit $M_q \to 0$ for vanishing expectation value of the field ϕ. A convenient way to define such M_q is the following. SSB chiral symmetry $\to$ 3 goldstone massless bosons, the pions. If the bare quark mass is zero, the physical mass of the pions has to be zero. Hence one can extract this mass on the lattice and tune m_q such that this is zero. While this is the correct way to proceed, it is very time takin since one must do mass scans and extrapolations close to singular Dirac operator. This is not the way we pursue here. Instead here we just consider naive fermions and take the limit $m_q \to 0$. This represents a physical theory with $2N_f = 4$ degenerate quarks. This is just done for the purpose of showing some interest properties of coloured noise and not (yet) to match any physical result.

In figure **??** some observables are reported as a function of the noise fraction s for different values of the bare quark mass. In the classical theory ($s = 0$) the order parameters $\langle|\phi|\rangle$, $\langle\bar{\psi}\psi\rangle$ are, in absolute value, bigger than in the quantum case ($s = 1$). As the chiral limit is approached $m_q \to 0$, the figure shows that the classical system lies in the broken phase, while in the quantum settings the symmetry is restored. Discuss general phase structure looking at both ϕ and $\bar{\psi}\psi$. One can see that as m_q is reduced, the systems shifts from a crossover to a second order phase transition, ahighligthed by the susceptibility χ^2 and the binder parameter U_L.

This would be better discussed as a function of bare scalar mass:

The difference gets more sharped as the base mass decreases, since the theory is closer to the chiral limit discussed above, which corresponds to the case $m_q = 0$. In this limit the systems goes under a second phase transition, highlighted by a peak in the susceptibility and and abrubt change in the Binder cumulant. In contrast, for finite mass, there is smooth crossover where the order parameters change continuously.

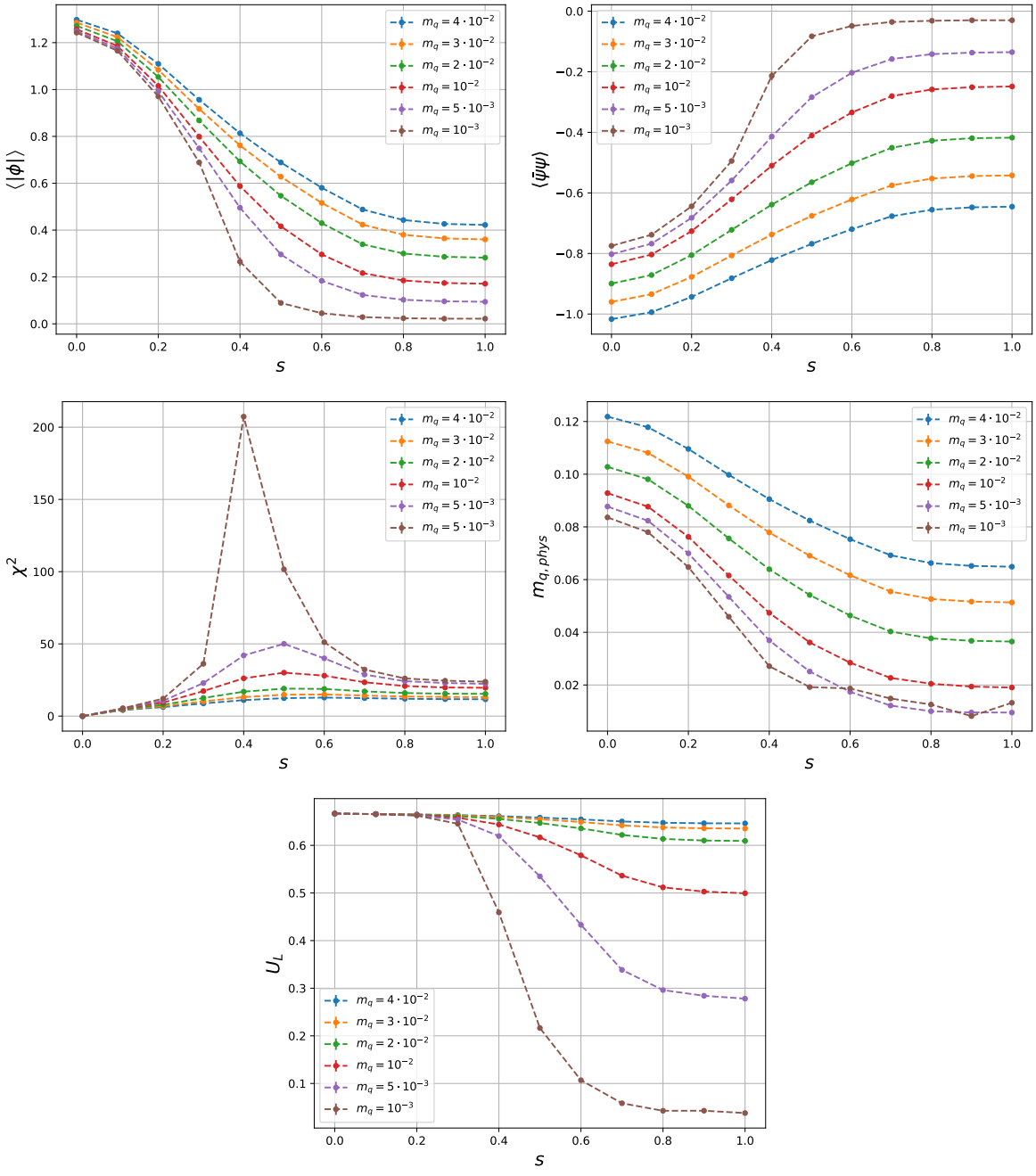


FIGURE 4.1: Chiral symmetry breaking

Chapter 5

Conclusions and outlook

Appendix A

Useful relations and definitions

In this appendix, useful relations and definitions are introduced. Fermionic wo-points function

$$\begin{aligned}\left\langle \psi_{s,f}(x)\,\bar{\psi}_{s',f'}(y)\right\rangle &= \frac{1}{Z}\int \mathcal{D}\phi\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\;\psi(x)\,\bar{\psi}(y)\;\exp\left(-S_\phi-\psi D\psi+\bar{\eta}\psi+\bar{\psi}\eta\right)\\ &= \frac{1}{Z}\int \mathcal{D}\phi\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\;\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\;\exp\left(-S_\phi-\psi D\psi+\bar{\eta}\psi+\bar{\psi}\eta\right)\\ &= \frac{1}{Z}\int \mathcal{D}\phi\,\det\left[D(\phi)\right]\;\exp\left(-S_\phi\right)\;\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\;\exp\left(\bar{\eta}D^{-1}\eta\right)\\ &= \left\langle\left[D^{-1}(\phi)\right]_{s,s',f,f'}(x,y)\right\rangle\end{aligned} \tag{A.1}$$

The lattice version becomes

$$\langle\psi_m\,\bar{\psi}_n\rangle = \left\langle\left[D^{-1}(\phi)\right]_{mn}\right\rangle$$

with D beeing the Wilson Dirac operator.
From this, it follows straightforwardly

$$\langle\bar{\psi}\psi\rangle = \operatorname*{Tr}_{x,s,f} D^{-1}$$

where $\langle\bar{\psi}\psi\rangle = \sum\limits_{x,s,f}\left\langle\bar{\psi}_{s,f}(x)\,\psi_{s,f}(x)\right\rangle$.

The correlator is defined as

$$C(n_t,0) \equiv \frac{1}{N_x}\sum_{n_x}\left[\langle\psi(n_t,n_x)\,\bar{\psi}(0,0)\rangle + \langle\psi(N_t-n_t,n_x)\,\bar{\psi}(0,0)]\right\rangle$$

Note that we sum up two waves because the source propagates both forward and backward in time due to the boundary conditions.
Since for $t\to\infty$ one has that $C(t,p)\propto e^{-E_0(p)t}$, we expect

$$C(t,p) \approx \sinh\left(E_0\left(\frac{N_t}{2}-t\right)\right)$$

Pole mass, renormalized mass, effective mass, bare mass, physical mass
Effective action

$$S_{\text{eff}} = S_\phi + \operatorname{Tr}\log D(\phi)$$

Drift force

$$K_{\phi^j} = -\frac{\delta S}{\delta\phi^j} = -\frac{\delta S_\phi}{\delta\phi^j} - \operatorname{Tr}\left[D^{-1}\,\frac{\delta D}{\delta\phi^j}\right]$$

Appendix B

Wilson fermions

I am not really sure on whether to put this appendix or not, maybe I will just cite some papers that talk about Wilson fermions

choice of the basis

Appendix C

Algorithms and technical details

cite also felipe for bilinear

C.1 Conjugate Gradient algorithm and the Dirac operator

The full inversion of the Dirac operator is a very expensive computation, given that the Dirac operator has dimension $(2\,N_t\,N_x\,N_f)^2$, even though it is very sparse and has only few non-zero entries. One can note that for the purpose of computing the fermionic contribution to the drift force and the extraction of the physical quark mass from the correlator (details in section x and section y), only the inverse operator applied to a vector is needed. Hence it is sufficient to compute

$$\psi = D^{-1}\,|\eta\rangle \tag{C.1}$$

Computing ψ via equation (C.1) is equivalent to solve the linear system $D\psi = \eta$, which can be done efficiently by employing a method for sparse matrices such as Conjugate Gradient (CG) as explained in the following way.

We want to solve the equation

$$D\,\psi = \eta$$

CG requires the matrix to be hermitian while D is only γ^5-hermitian (really? under which assumptions?). One can thus solve the linear system

$$\left(DD^\dagger\right)\xi = \eta$$

and then obtain ψ by multiplying the solution ξ by $D^\dagger$ since

$$D^\dagger\xi = D^\dagger\,\left(DD^\dagger\right)^{-1}\eta = D^{-1}\eta = \psi \tag{C.2}$$

Analogously one can calculate

$$\chi = D^\dagger\eta$$

by solving

$$\left(D^\dagger D\right)\xi = \eta$$

and then applying D to the result.

One can improve the solution via CG by solving a *preconditioned* equation. Suppose that we want to solve the equation

$$Mx = b$$

via CG.
Let us express the matrix A as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{C.3}$$

We introduce the Schur complement of M

$$M/D = A - BD^{-1}C \tag{C.4}$$

This allows one to write M (LDU decomposition, Gaussian elimination) as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbf{1}_p & -BD^{-1} \\ 0 & \mathbf{1}_q \end{bmatrix} \begin{bmatrix} M/D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbf{1}_p & 0 \\ D^{-1}C & \mathbf{1}_q \end{bmatrix} = LAR$$

Which allows for an easy block inversion

$$M^{-1} = \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & -BD^{-1} \\ 0 & I_q \end{bmatrix} = L^{-1}A^{-1}R^{-1}$$

The equation to solve now reads

$$x = L^{-1}A^{-1}R^{-1}\, b \qquad \text{or} \qquad y = A^{-1}c$$

with $y = Lx$ and $c = R^{-1}b$. One can then solve the equation $y = A^{-1}c$ and get the solution x by applying L^{-1} to x.

An example of preconditioning is the even-odd preconditioning. Let us write the dirac operator in the form of equation (C.3) in the following way

$$M = \begin{pmatrix} M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{pmatrix}$$

The Schur complement (C.4) is

$$\hat{M} \equiv M/M_{oo} =$$

C.2 Bilinear noise scheme

$$\mathrm{Tr}\left[D^{-1}\frac{\delta D}{\delta \phi^j}\right] \approx \langle \eta | D^{-1} \frac{\delta D}{\delta \phi^j} | \eta \rangle = \langle \psi | \frac{\delta D}{\delta \phi^j} | \eta \rangle \qquad |\psi\rangle = D^{-1}|\eta\rangle = D^\dagger \underbrace{(DD^\dagger)^{-1}|\eta\rangle}_{\text{CG}}$$

$$\mathrm{Tr}\ A = \frac{1}{N} \lim_{N\to\infty} \sum_i^N \eta_i^T D_{ij} \eta_j \tag{C.5}$$

where η_i is a gaussian random field where each component is drawn from a normal distribution $\mathcal{N}(0,1)$.
More precisely each vector component η_i^α satisfies

$$\langle \eta_i^\alpha \rangle = 0 \qquad \left\langle \eta_i^\alpha \eta_j^\beta \right\rangle = \delta_{i,j}\, \delta^{\alpha\beta}$$

The series (C.5) requires in principle an infinite number of vectors to evaluate the trace exactly. In practice we truncate it and choose $N = 1$:D :D. The average over Monte Carlo samples will eventually converge nevertheless to the right result.

Bibliography

[1] Istvan Montvay and Gernot Münster. *Quantum Fields on a Lattice*. 1994. DOI: 10.1017/cbo9780511470783.

[2] Heinz J Rothe. *Lattice Gauge Theories*. 4th. WORLD SCIENTIFIC, 2012. DOI: 10.1142/8229. URL: https://www.worldscientific.com/doi/abs/10.1142/8229.

[3] Christof Gattringer and Christian B. Lang. *Lattice quantum chromodynamics*. DOI: 10.1036/1097-8542.yb100080. URL: https://doi.org/10.1036%2F1097-8542.yb100080.

[4] Michael Creutz. *Quarks, Gluons and Lattices*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2023. DOI: 10.1017/9781009290395.

[5] Andrea Carosso. *Novel Approaches to Renormalization Group Transformations in the Continuum and on the Lattice*. 2020. arXiv: 2006.07481 [hep-lat].

[6] L. D. Landau. "On the theory of phase transitions". In: *Zh. Eksp. Teor. Fiz.* 7 (1937). Ed. by D. ter Haar, pp. 19–32. DOI: 10.1016/B978-0-08-010586-4.50034-1.

[7] Vitaly L. Ginzburg. "On Superconductivity and Superfluidity (What I Have and Have Not Managed to Do), as well as on the 'Physical Minimum' at the Beginning of the 21st Century". In: *ChemPhysChem* 5.7 (2004), pp. 930–945. DOI: https://doi.org/10.1002/cphc.200400182. eprint: https://chemistry-europe.onlinelibrary.wiley.com/doi/pdf/10.1002/cphc.200400182. URL: https://chemistry-europe.onlinelibrary.wiley.com/doi/abs/10.1002/cphc.200400182.

[8] Tian Yu Cao, ed. *Conceptual foundations of quantum field theory. Proceedings, Symposium and Workshop, Boston, USA, March 1-3, 1996*. Cambridge, UK: CUP, 1999.

[9] Leo P. Kadanoff. "Scaling laws for ising models near T_c". In: *Physics Physique Fizika* 2 (6 June 1966), pp. 263–272. DOI: 10.1103/PhysicsPhysiqueFizika.2.263. URL: https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.2.263.

[10] Kenneth G. Wilson. "Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture". In: *Phys. Rev. B* 4 (9 Nov. 1971), pp. 3174–3183. DOI: 10.1103/PhysRevB.4.3174. URL: https://link.aps.org/doi/10.1103/PhysRevB.4.3174.

[11] Kenneth G. Wilson. "Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior". In: *Phys. Rev. B* 4 (9 Nov. 1971), pp. 3184–3205. DOI: 10.1103/PhysRevB.4.3184. URL: https://link.aps.org/doi/10.1103/PhysRevB.4.3184.

[12] Kenneth G. Wilson and Michael E. Fisher. "Critical Exponents in 3.99 Dimensions". In: *Phys. Rev. Lett.* 28 (4 Jan. 1972), pp. 240–243. DOI: 10.1103/PhysRevLett.28.240. URL: https://link.aps.org/doi/10.1103/PhysRevLett.28.240.

[13] John Cardy. "The renormalization group idea". In: *Scaling and Renormalization in Statistical Physics*. Cambridge Lecture Notes in Physics. Cambridge University Press, 1996, pp. 28–60. DOI: 10.1017/CBO9781316036440.004.

[14] Kenneth G. Wilson and J. Kogut. "The renormalization group and the ϵ expansion". In: *Physics Reports* 12.2 (1974), pp. 75–199. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(74)90023-4. URL: https://www.sciencedirect.com/science/article/pii/0370157374900234.

[15] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Reading, USA: Addison-Wesley, 1995. ISBN: 978-0-201-50397-5.

[16] Michel Le Bellac. *Thermal Field Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1996. DOI: 10.1017/CBO9780511721700.

[17] Y. Nambu and G. Jona-Lasinio. "Dynamical model of elementary particles based on an analogy with superconductivity. i". In: *Physical Review* 122.1 (1961). ISSN: 0031899X. DOI: 10.1103/PhysRev.122.345.

[18] Y. Nambu and G. Jona-Lasinio. "Dynamical model of elementary particles based on an analogy with superconductivity. II". In: *Physical Review* 124.1 (1961). ISSN: 0031899X. DOI: 10.1103/PhysRev.124.246.

[19] Michael Buballa. *NJL-model analysis of dense quark matter*. 2005. DOI: 10.1016/j.physrep.2004.11.004.

[20] Kenneth G. Wilson. "Confinement of quarks". In: *Phys. Rev. D* 10 (8 Oct. 1974), pp. 2445–2459. DOI: 10.1103/PhysRevD.10.2445. URL: https://link.aps.org/doi/10.1103/PhysRevD.10.2445.

[21] G. Parisi and Y.-S. Wu. "Perturbation Theory without Gauge Fixing". In: *Scientia Sinica* (24 1981), p. 483.

[22] Poul H. Damgaard and Helmuth Hüffel. "Stochastic quantization". In: *Physics Reports* 152.5-6 (Aug. 1987), pp. 227–398. ISSN: 0370-1573. DOI: 10.1016/0370-1573(87)90144-X.

[23] Crispin Gardiner. *Stochastic Methods: A Handbook for the Natural and Social Sciences*. Springer Berlin, Heidelberg, 2009, pp. XVIII, 447.

[24] G. G. Batrouni et al. "Langevin simulations of lattice field theories". In: *Phys. Rev. D* 32 (10 Nov. 1985), pp. 2736–2747. DOI: 10.1103/PhysRevD.32.2736. URL: https://link.aps.org/doi/10.1103/PhysRevD.32.2736.

[25] Andreas S. Kronfeld. "Dynamics of Langevin simulations". In: *Prog. Theor. Phys. Suppl.* 111 (1993), pp. 293–312. DOI: 10.1143/PTPS.111.293. arXiv: hep-lat/9205008.

[26] Virginia Vassilevska Williams et al. *New Bounds for Matrix Multiplication: from Alpha to Omega*. 2023. arXiv: 2307.07970 [cs.DS].

[27] Haim Avron and Sivan Toledo. "Randomized Algorithms for Estimating the Trace of an Implicit Symmetric Positive Semi-Definite Matrix". In: *J. ACM* 58.2 (Apr. 2011). ISSN: 0004-5411. DOI: 10.1145/1944345.1944349. URL: https://doi.org/10.1145/1944345.1944349.

[28] Jan M. Pawlowski, Ion Olimpiu Stamatescu, and Felix P.G. Ziegler. "Cooling stochastic quantization with colored noise". In: *Physical Review D* 96.11 (2017). ISSN: 24700029. DOI: 10.1103/PhysRevD.96.114505.

List of Abbreviations

RG	**R**enormalisation **G**group
fRG	**F**unctional **R**enormalisation **G**group
UV	**U**ltra**v**iolet
QFT	**Q**uantum **F**ield **Theory**

Physical Constants

Speed of Light $c_0 = 2.997\,924\,58 \times 10^8\,\mathrm{m\,s^{-1}}$ (exact)

List of Symbols