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Simulating effective field theories  
on a space-time lattice with coloured noise

This Master thesis has been carried out by Matteo Zortea

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Prof. Jan M. Pawłowski

and

Dr. Felipe Attanasio

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27.11.2023

.....

*“Grazie a tutti.”*

Matteo Zortea



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# List of Abbreviations

<b>RG</b>	<b>R</b> enormalisation <b>G</b> roup
<b>fRG</b>	<b>F</b> unctional <b>R</b> enormalisation <b>G</b> roup
<b>UV</b>	<b>U</b> ltraviolet



# Physical Constants

Speed of Light  $c_0 = 2.997\,924\,58 \times 10^8 \text{ m s}^{-1}$  (exact)





# List of Symbols

$a$	distance	m
$P$	power	W (J s <sup>-1</sup> )
$\omega$	angular frequency	rad



## Chapter 1

# Introduction

### 1.1 Quantum chromodynamics and its phase diagram

Big picture: here we talk about QCD and the problem of the phase diagram

### 1.2 The renormalisation group

### 1.3 Effective theories

Here we first define effective theories and discuss their usefulness, then introduce RG as a technique to resolve physics at different scales.

Motivation for choices, connection to stochastic regularisation, complex langevin,

easy noise [boo] Motivation for choices,



## Chapter 2

# Theoretical background

Da qualche parte cita [1].

In this chapter we want to provide with an overview on the general theoretical framework that supports this work, and introduce the main concepts for the successive parts. Each section in this chapter is, by no means, meant as an exhaustive treatment. The description will be quite conceptual, rather than technical, and aims at recalling the main ideas and fix notation. We ask the reader to consult appropriate references, which will be given in the corresponding sections, for a more detailed treatment of the topics.

The starting set up is the euclidean formulation of quantum field theory, where one typically defines a path integral  $Z$

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\phi, \psi, \bar{\psi}]} \quad (2.1)$$

and aims to compute correlation functions

$$\langle \xi_{x_1} \dots \xi_{x_n} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \xi_{x_1} \dots \xi_{x_n} e^{-S[\phi, \psi, \bar{\psi}]} \quad \xi_{x_i} \in \{\phi_{x_i}, \psi_{x_i}, \bar{\psi}_{x_i}\}$$

Computing physical quantities from such a direct approach is not only hard to do, but results often impossible due to the appearance of divergences in the calculations. To fix this, one often relies on expansion techniques such as perturbation theory (**CITATION**), in which one tries to regularise the theory order by order in an expansion on the interaction coupling, yielding finite quantities that depend on the truncation order. While this method produced incredibly precise results (**g-2, fine structure, ...**), it fails completely in treating non-perturbative phenomena, namely effects that cannot be captured by any order in the expansion or that rely on strongly interacting matter. Example of such systems range from **QCD, cold atoms, plasma, stuff**. Moreover, such formulation is also not much suitable for numerical computations, since both the path integral and action measure are infinite dimensional objects.

Lattice field theory [2–5] is meant at first as a powerful non-perturbative regularisation tool to prevent divergences to occur and render the computation of the correlation functions finite. Moreover, it also provides a framework to study quantum field theory numerically on a computer. In order to accomplish this, one typically defines the theory on a space-time lattice and makes use of statistical methods such as Monte Carlo algorithms to compute observables. One may wonder how can one reconstruct the results in the continuum theory, keeping the results finite, and matching the results on the discretised theory to physically measured ones. This task, far

from being simple, will be the focus of the next sections, in which we will first introduce relevant theoretical tools, such as the renormalisation group, and then discuss the existence of a continuum limit of a lattice theory and, if it exists, how it can be extracted. This will motivate the introduction of coloured noise in the context of continuum limits of effective theories, a technique which will be shown to be powerful also for other various reasons, which will be the main focus of the analysis carried in the remaining chapters.

## 2.1 Lattice Field Theory

As mentioned before, lattice field theory consists in formulating QFT on a space-time lattice

$$S = \int d^d x \mathcal{L}(x) t$$

## 2.2 Block-spin renormalisation

Cite Kadanoff article [PhysicsPhysiqueFizika.2.263]. Average spins and rescale stuff to keep correlation length fixed.

## 2.3 Wilsonian renormalisation

Extends block spin RG.

One splits the fields as  $\Phi = \phi + \varphi$  where  $\phi$  are fields with momenta  $p \leq b\Lambda$  and  $\varphi$  are fields with momenta  $b\Lambda < p < \Lambda$ , then one writes the path integral in terms of the Wilsonian effective action

$$Z = \int D\Phi_\Lambda e^{-S_\Lambda[\Phi]} = \int D\phi_{b\Lambda} e^{-S_{b\Lambda}[\phi]} \int D\varphi_{b\Lambda,\Lambda} e^{-S_{b\Lambda,\Lambda}[\phi,\varphi]} = \int D\phi_{b\Lambda} e^{-S_{b\Lambda}^{\text{eff}}[\phi]}$$

where

$$S_{b\Lambda}^{\text{eff}}[\phi] = S_{b\Lambda}[\phi] - \log \left( \int D\varphi_{b\Lambda,\Lambda} e^{-S_{b\Lambda,\Lambda}[\phi,\varphi]} \right) = S_{b\Lambda}[\phi] + \Delta S_{b\Lambda,\Lambda}[\phi]$$

Note that all the steps above are exact identities. In particular, performing the integral over ultraviolet modes, is the continuum version of the block spinning procedure outlined in the previous section.

Note also that  $S_{b\Lambda}[\phi]$  is the same as the initial action, but it is non-zero only for fields with  $p^2 \leq b\Lambda^2$ .

At this point one can expand  $\Delta S_{b\Lambda,\Lambda}$  in powers of the field (before, another step, see [jan pawlowski's notes](#)). Powers that are present also in  $S_{b\Lambda}$  can be absorbed into the latter by redefining the coupling

$$\begin{aligned} \phi'(x') &= \left[ b^{2-d}(1+\Delta z) \right]^{\frac{1}{2}} \phi(x), \quad m'^2 = (m^2 + \Delta m^2) \frac{1}{1+\Delta z} \frac{1}{b^2}, \quad \lambda' = (\lambda + \Delta\lambda) \frac{1}{(1+\Delta z)^2} b^{d-4}, \\ \alpha' &= (\alpha + \Delta\alpha) \frac{1}{(1+\Delta z)^2} b^d, \quad \lambda'_6 = (\lambda_6 + \Delta\lambda_6) \frac{1}{(1+\Delta z)^3} b^{2d-6}, \quad \dots \end{aligned}$$

(2.2)

higher powers are suppressed (non-renormalisable terms) are suppressed. By neglecting these higher order powers, one can bring the action in the same form as

the initial one via redefinition of the parameters. The whole procedure is non-perturbative. The result can be compared to the initial action after redefinition of all the dimensionful quantities i.e. via  $p' \equiv p/b$ .

## 2.4 Continuum limit of lattice field theories

Introduce renormalisation as a mapping as in page 40 of Montvay Munster.

Continuum limits in lattice theories are intimately connected to the existence of critical points in the theories. In fact, to take a continuum limit, one want the dimensionless correlation length  $\hat{\xi}$  to diverge: in this way one can represent an infinite number of points inside a finite volume (**explain better here**). For this to happen, the system must go under a second order phase transition, whose critical point is identified by a set of values for the bare parameters  $g_0^{i*}$ . **differentiate well between dimful and dimless.**

Consider  $O$  to be an observable which has to be matched to a physical measurable quantity, and compare it to the dimensionless quantity  $\hat{O}$  given by a lattice simulation. In general the physical observable is assumed to be a function of the spacing and the bare couplings of the theory

$$O = O(a, g_0^i)$$

while its lattice counterpart can only depend on the dimensionless coupling  $\hat{g}_0^i$ , i.e.

$$\hat{O} = \hat{O}(\hat{g}_0^i)$$

Let  $d_O$  be the physical dimension of the observable  $O$  in units of energy. Then one can relate the two quantities as

$$O(a, g_0^i) = \left(\frac{1}{a}\right)^{d_O} \hat{O}(\hat{g}_0^i) \quad (2.3)$$

We now want to address the following question: given a (small enough)  $a$ , is it possible to find a value  $\hat{g}_0^i(a)$  such that the value of  $O$  given by (2.3) does not depend on  $a$ ?

We then impose such condition via

$$\frac{d}{da} O(a, g_0^i) = \left( a \frac{\partial}{\partial a} - \beta(g_0^i) \frac{\partial}{\partial g_0^i} \right) O(a, g_0^i) = 0$$

with

$$\beta(g_0^i) = -a \frac{\partial g_0^i}{\partial a}$$

Integrating such  $\beta$  functions tells one how to change bare couplings as a backreaction to a change in the spacing, in order to keep observables constant. We then say that the theory admits a continuum limit if there exists some set of values  $(g_0^i)^*$  such that when  $g_0^i \rightarrow (g_0^i)^*$  one has  $\hat{\xi} \rightarrow +\infty$  and  $O \rightarrow O_{phys}$ . **Connection with fixed points and beta function. Comment also on beta functions for dimless couplings.**

Of course one does not know a priori the full lattice beta functions, but they can be computed via approximate or continuum methods. For example, in continuum perturbation theory, one can compute  $g_r^i(\Lambda) = g_r^i(\Lambda, g_0^i)$ , where  $\Lambda$  is a sharp momentum cutoff and then try to invert them to find  $g_0^i = g_0^i(\Lambda, g_r^i)$ . The connection is

then given by making the identification  $a \sim \Lambda^{-1}$ .

Generally speaking, we are interested in the set of theories in theories space that have constant renormalised couplings but different dimless couplings  $g_r^i a$  (trajectories in Kadanoff-Wilson RG).

## 2.5 Yukawa theory

Let us consider the Yukawa theory defined by the action

$$\begin{aligned}
 S[\phi, \psi, \bar{\psi}] &= S_\phi[\phi] + S_\psi[\psi, \bar{\psi}] + S_{\text{int}}[\phi, \psi, \bar{\psi}] \\
 S_\phi[\phi] &= \int_x \phi_x \left( -\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \phi_x + \frac{\lambda}{4!} \phi_x^4 \\
 S_\psi[\psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} (\not{\partial}_x + m_q) \psi_x^{(f)} \\
 S_{\text{int}}[\phi, \psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} g \bar{\psi}_x^{(f)} \phi_x \psi_x^{(f)}
 \end{aligned} \tag{2.4}$$

One can see that the action is made of a scalar part  $S_\phi[\phi]$ , a fermionic part  $S_\psi[\psi, \bar{\psi}]$  and a Yukawa interaction term  $S_{\text{int}}[\phi, \psi, \bar{\psi}]$ .

It is also convenient for later purposes to define the operators  $K, D$  represented in position space as

$$\begin{aligned}
 K(x, y) &= \left( -\partial_x^2 + m_\phi^2 \right) \delta(x, y) \\
 D(x, y) &= (\not{\partial}_x + m_q + g\phi) \delta(x, y)
 \end{aligned} \tag{2.5}$$

and in momentum space as

$$\begin{aligned}
 \tilde{K}(p, q) &= \int_{x, y} e^{-ipx} \left( \partial_x^2 + m_\phi^2 \right) \delta(x, y) e^{iqy} = \left( \frac{p^2}{2} + \frac{m_\phi^2}{2} \right) \delta(p, q) \\
 \tilde{D}(p, q) &= \int_{x, y} e^{-ipx} (\not{\partial}_x + m_q + g\phi) \delta(x, y) e^{iqy} = (\not{p}_x + m_q + g\phi) \delta(p, q)
 \end{aligned} \tag{2.6}$$

This allows one to rewrite the action as

$$S[\phi, \psi, \bar{\psi}] = \int_x \frac{1}{2} \phi_x K_{xx} \phi_x + \frac{\lambda}{4!} \phi_x^4 + \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} D_{xx} \psi_x^{(f)}$$

We introduce the left-handed and right-handed spinors

$$\psi_L = (1 - \gamma_5) \psi \quad \psi_R = (1 + \gamma_5) \psi$$

for which

$$\psi = \frac{(1 - \gamma_5)}{2} \psi + \frac{(1 + \gamma_5)}{2} \psi = \psi_L + \psi_R$$

The action written in terms of  $\psi_L, \psi_R$  reads

$$S = S_\phi + \bar{\psi}_L D \psi_L + \bar{\psi}_R D \psi_R + (m_q + g\phi) (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \tag{2.7}$$



The last equation makes clear that for  $m = 0, \langle \phi \rangle = 0$  the action is symmetric under the chiral group  $SU(2)_L \times SU(2)_R$ , namely

$$\begin{aligned}\psi_L(x) &\rightarrow U_L \psi_L(x) & \bar{\psi}_L(x) &\rightarrow \bar{\psi}_L(x) U_L^\dagger \\ \psi_R(x) &\rightarrow U_R \psi_R(x) & \bar{\psi}_R(x) &\rightarrow \bar{\psi}_R(x) U_R^\dagger\end{aligned}$$

for  $U_L, U_R \in SU(2)$ .

The main feature of the model is chiral symmetry breaking [6, 7], which can happen explicitly at the level of the classical action for a non-zero quark mass, or spontaneously when the scalar field gains a non-zero expectation value. One can in fact notice already by looking at (??), that  $\langle \phi \rangle \neq 0$  has the same effect on the action as a finite bare quark mass. This observation will be made more quantitative in section (SECCCC) where it will be shown that

$$\langle \phi \rangle \sim \langle \bar{\psi} \psi \rangle \sim m_q$$

The fermionic part of the path integral (2.1) can be performed explicitly

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( - \int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} D \psi_x^{(f)} \right) = (\det D[\phi])^{N_f} = e^{N_f \text{Tr} \log(D[\phi])}$$

where the trace is performed over space-time and spinor components.

The full path integral can now be expressed in terms of the resulting effective action for the scalar fields

$$Z = \int \mathcal{D}\phi e^{-S_{\text{eff}}[\phi]}$$

with

$$S_{\text{eff}}[\phi] = S_\phi[\phi] - \text{Tr}_{x,s,f} \log D[\phi] \quad (2.8)$$

One can derive the classical equations of motion by imposing  $\frac{\delta S}{\delta \phi} = 0$ , here expressed in momentum space

$$(k^2 + m_\phi^2) \phi(x) + \frac{\lambda}{6} \phi^3(x) = g \text{Tr}_{s,f} [D^{-1}(\phi(x))] = -g \bar{\psi}(x) \psi(x)$$

where the trace is performed over spin and flavour components. For  $\lambda = 0$ , they highlight a simple proportionality relation between magnetisation and chiral condensate, which for zero momentum reads

$$\phi(x) = -\frac{g}{m_\phi^2} \bar{\psi}(x) \psi(x) \quad (2.9)$$

The classical relation (2.9) is proven to hold also at mean field on the quantum level [8] and will be studied in the discretised theory in section 4.4.

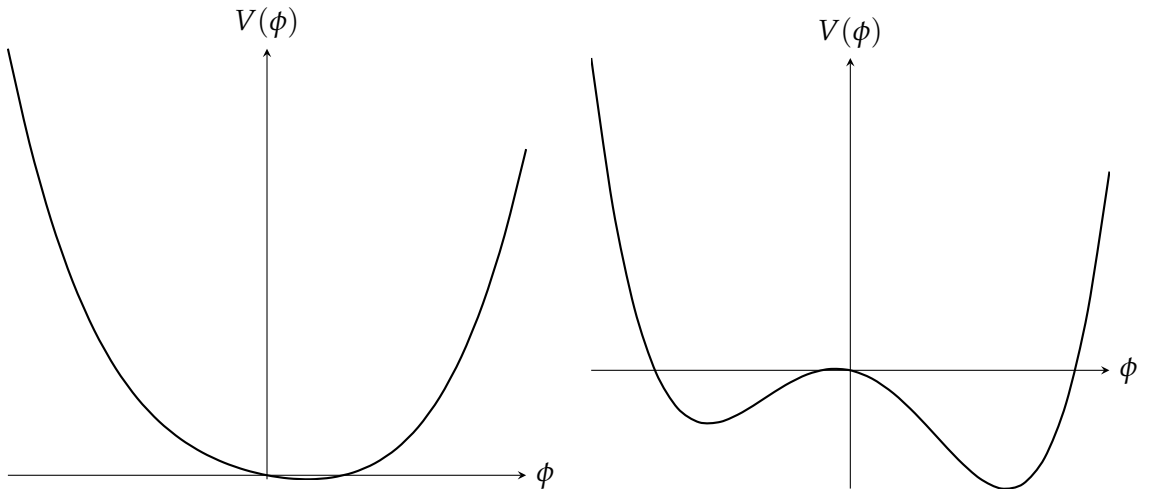


FIGURE 2.1: The introduction of the boson-fermion interaction, with a finite fermionic mass, causes the breaking of the  $O(1)$  symmetry. It shifts the equilibrium position in the symmetric phase (left) causing  $\langle \phi \rangle \neq 0$ , and tilts the potential in the broken phase (right), making the two minima not equivalent.

## Chapter 3

# Methods and algorithms

### 3.1 Discretisation of the Yukawa theory

In order to make the theory suitable for a numerical simulation on a computer, the continuum formulation of the Yukawa model, which has been introduced in section 2.5, has to be discretised. Here we provided a sketch of a discretisation procedure, and we refer to other resources [2–5] for further details.

For what concerns the bosonic part of the action, a discretisation can be done straightforwardly with the following replacements

$$\begin{aligned} \int_x &\rightarrow a^2 \sum_x \\ \partial_x^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} &\rightarrow \sum_\mu \left[ \frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2} \right] \end{aligned}$$

which yields to the lattice action

$$\begin{aligned} S_\phi[\phi] &= a^2 \left( \frac{1}{2} \sum_{m,n} \phi_m K_{mn} \phi_n + \frac{\lambda}{4!} \sum_n \phi_n^4 \right) \\ &= \frac{1}{2} \sum_{m,n} \hat{\phi}_m \hat{K}_{mn} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \sum_n \hat{\phi}_n^4 \end{aligned}$$

where we expressed everything in dimensionless quantities

$$\begin{aligned} \hat{m}_\phi^2 &= a^2 m_\phi^2 \\ \hat{\lambda} &= a^2 \lambda, \\ \hat{K}_{mn} &= a^2 K_{mn} \end{aligned} \tag{3.1}$$

The operator components  $\hat{K}_{mn}$  are the discretised version of (2.5)

$$\hat{K}_{mn} = - \sum_\mu [\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}] + \hat{m}_\phi^2 \delta_{mn} \tag{3.2}$$

and its representation in momentum space is

$$\begin{aligned}
\hat{K}_{p,q} &= \sum_{n,m} e^{ipn} \hat{K}_{nm} e^{-iqm} \\
&= \sum_{n,m} e^{ipn} \left( - \sum_{\mu} [\delta_{m,m+\mu} + \delta_{m,m-\mu} - 2\delta_{m,n}] + \hat{m}_{\phi}^2 \delta_{mn} \right) e^{-iqm} \\
&= \sum_n e^{i(p-q)n} \left[ \hat{m}_{\phi}^2 + 2 \sum_{\mu} (1 - \cos(q_{\mu})) \right] \\
&= \left[ \hat{m}_{\phi}^2 + \sum_{\mu} 4 \sin^2 \left( \frac{p_{\mu}}{2} \right) \right] \delta(p-q).
\end{aligned}$$

For what concerns the fermionic action, a naive discretisation is not sufficient, due to the well known doubling problem [2, 3] (is it fine if I cite the books? I couldn't really find a paper where the doubling problem was discussed in particular). In this work Wilson fermions [9] are employed as a way to fix such issue. Details of this formulation are explained in Appendix B. Here, only the final discretised action is reported, which reads

$$S_{\psi} [\hat{\psi}, \hat{\psi}] + S_{\text{int}} [\hat{\phi}, \hat{\psi}, \hat{\psi}] = \sum_{f=1}^{N_f} \hat{\psi}_m^{(f)} \hat{D}_{mn} \hat{\psi}_n^{(f)} \quad (3.3)$$

with  $\psi_n$  beeing a two-component spinor, and  $\hat{D}_{m,n}$  beeing the Wilson-Dirac operator (can I include  $g\phi$  in the definition of  $D$ ?) defined as

$$\begin{aligned}
\hat{D}_{m,n} &= - \left( \frac{\Gamma_{+\hat{0}}}{2} \delta_{m,m+\hat{0}} + \frac{\Gamma_{-\hat{0}}}{2} \delta_{m,m-\hat{0}} + \frac{\Gamma_{+\hat{1}}}{2} \delta_{m,m+\hat{1}} + \frac{\Gamma_{-\hat{1}}}{2} \delta_{m,m-\hat{1}} \right) \\
&\quad + (2ar + \hat{m} + \hat{g}\phi) \delta_{s,s'} \delta_{m,n}
\end{aligned} \quad (3.4)$$

Both the above and below, can I write  $ar$  like that?

The Wilson projectors  $\Gamma_{\pm\hat{\mu}}$  are defined as

$$\Gamma_{\pm\hat{\mu}} = ar \mathbb{1}_s \mp \gamma_{\mu}$$

Since  $r \in [0, 1]$  is a free parameter, in this work we set  $r = 1$ , if not otherwise specified.

In summary the discretised action for the Yukawa model is

$$S [\hat{\phi}, \hat{\psi}, \hat{\psi}] = \sum_{m,n} \hat{\phi}_m \hat{K}_{m,n} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \hat{\phi}_m^4 \delta_{m,n} + \sum_{f=1}^{N_f} \hat{\psi}_m^{(f)} \hat{D}_{mn} \hat{\psi}_n^{(f)}$$

with  $\hat{K}_{mn}, \hat{D}_{mn}$  given respectively by (3.2) and (3.4).

For later reference, we also report the discretised version of the effective action (2.8)

$$\begin{aligned}
S_{\text{eff}}[\hat{\phi}] &= S_{\phi}[\hat{\phi}] - N_f \text{Tr}_{n,s} \log \hat{D}_{nn} \\
&= \sum_{m,n} \hat{\phi}_m \hat{K}_{m,n} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \hat{\phi}_m^4 \delta_{m,n} - N_f \text{Tr}_{n,s} \log \hat{D}_{nn}
\end{aligned} \quad (3.5)$$

The full discrete path-integral reads (measure over dimless or dimful?)

$$Z = \int \prod_n d\hat{\phi}_n e^{-S_{\text{eff}}[\hat{\phi}]} \quad (3.6)$$

In the remaining of this work, both the original action  $S$  and the effective action  $S_{\text{eff}}$  will be denoted by  $S$  for simplicity. It will be clear from the context which of the two we will be referring to.

## 3.2 Stochastic quantisation and Langevin Monte Carlo

in this and next section I refer multiple times to Damgaard. Should I cite everytime or not?

In order to compute expectation values from the discretised path integral (3.6), we employ a Langevin Monte Carlo algorithm, which is based on stochastic quantisation [10, 11].

The idea is that Euclidean Quantum Field theory can be thought as a system in thermal equilibrium with a heat reservoir and hence described as a stochastic process via the Langevin equation. For this, one has to introduce a fictitious time variable  $\tau$  that labels the state  $\phi(\tau, x)$  of the system during the evolution.

Let us consider, for example, a scalar field  $\phi$  with a Euclidean action  $S[\phi]$  and the following Langevin equation

$$\partial_\tau \phi(\tau, x) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta(\tau, x) \quad (3.7)$$

where  $K_\phi(\tau) \equiv -\delta S[\phi]/\delta \phi(\tau, x)$  is the drift term and  $\eta(\tau, x)$  is a random white noise field assumed to be normally distributed

$$P(\eta) = \frac{\exp\left(-\frac{1}{4} \int_{\tau, x} \eta^2(\tau, x)\right)}{\int D\eta \exp\left(-\frac{1}{4} \int_{\tau, x} \eta^2(\tau, x)\right)}$$

which, in particular, implies

$$\langle \eta(x, \tau) \rangle = 0 \quad \langle \eta(x, \tau) \eta(x', \tau') \rangle = 2 \delta(x - x') \delta(\tau - \tau') \quad (3.8)$$

Stochastic average w.r.t. the measure  $P(\eta)$  are computed via

$$\langle A(\eta) \rangle = \int D\eta P(\eta) A(\eta)$$

In momentum space, (3.8) becomes

$$\begin{aligned} \langle \eta(p, \tau) \rangle &= \left\langle \int_x e^{ipx} \eta(x, \tau) \right\rangle = \int_x \langle e^{ipx} \eta(x, \tau) \rangle = 0 \\ \langle \eta(p, \tau) \eta(q, \tau') \rangle &= \left\langle \int_{xy} e^{ipx+iqy} \eta(x, \tau) \eta(y, \tau') \right\rangle \\ &= \int_{xy} e^{ipx+iqy} \langle \eta(x, \tau) \eta(y, \tau') \rangle \\ &= 2 (2\pi)^2 \delta(p + q) \delta(\tau - \tau') \end{aligned} \quad (3.9)$$

In absence of the noise term  $\eta(\tau, x)$ , equation (3.7) simply represents an evolution of the field towards the minimum of the action, and at equilibrium the field is constrained to  $\partial_\tau \phi(x, \tau) = 0 = \delta S[\phi] / \delta \phi(\tau, x)$ , namely to the classical equations of motion.

For any observable  $O$ , which is function of the field, one has, for fixed time  $\tau$ ,

$$\langle O(\phi(\tau)) \rangle = \int D\eta P(\eta) O(\phi(\tau))$$

From which it follows straightforwardly using the Langevin equation and  $\langle \eta \rangle = 0$

$$\frac{d}{d\tau} \langle O(\phi(\tau)) \rangle = \left\langle \frac{\delta O}{\delta \phi(\tau, x)} \partial_\tau \phi(\tau, x) \right\rangle = - \left\langle \frac{\delta O}{\delta \phi}(\tau, x) \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle$$

It follows trivially that for  $O(\phi(\tau)) = \phi(\tau, x)$

$$\frac{d}{d\tau} \langle \phi(\tau, x) \rangle = - \left\langle \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle \xrightarrow{\text{Equilibrium}} \left\langle \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle = 0$$

This also provides a consistency check for the correct implementation of the simulation, since the drift  $K_\phi = -\delta S / \delta \phi$  is computed numerically during the evolution.

More generally, one can derive a correspondent Fokker-Planck equation [12], which can be proven to have a stationary distribution if the action is bounded from below, given by [11]

$$\mathcal{P}(\phi) = \frac{1}{Z} \exp(-S[\phi]) \quad (3.10)$$

This allows one to compute correlation functions as moments of the probability distribution (3.10). In particular, one has

$$\langle O \rangle_{P(\eta)} = \langle O \rangle_{P(\phi)} \equiv \langle O \rangle \quad (3.11)$$

From a more practical point of view, this idea suggests that equation (3.7) can be integrated numerically for discrete time steps  $\tau_n$  to generate field configurations distributed according to (3.10). The simplest first-order integration algorithm is [10]

$$\phi(\tau_{n+1}, x) = \phi(\tau_n, x) - \epsilon \frac{\delta S[\phi]}{\delta \phi(\tau_n, x)} + \sqrt{\epsilon} \eta(\tau_n, x) + O(\epsilon^2)$$

where  $\epsilon = \tau_{n+1} - \tau_n$ . Higher order integration schemes are possible (see e.g. [13, 14]), but not adopted in this work. Thus, an adaptive stepsize is employed as detailed in Appendix C. In this way, for any observable  $O$ , one can introduce a Monte-Carlo estimator  $\langle O \rangle_*$  which is expected to converges to the expectation value given by (3.11) in the limit of infinite samples

$$\langle O \rangle_* = \frac{1}{N_{\text{samp}}} \sum_{i=1}^{N_{\text{samp}}} O_i \xrightarrow{N_{\text{samp}} \rightarrow \infty} \langle O \rangle = \frac{1}{Z} \int D\phi O(\phi) \exp(-S[\phi]) \quad (3.12)$$

where  $O_i = O(\phi(\tau_i))$  is the sample of the observable  $O$  done at time  $\tau_i$ .

For the discretised action of the Yukawa theory (3.5) the drift reads, explicitly,

$$\begin{aligned} \frac{\partial S}{\partial \hat{\phi}_m(\tau_n)} &= \frac{\partial S_{\hat{\phi}}}{\partial \hat{\phi}_m(\tau_n)} - N_f \text{Tr}_s \left[ \sum_{j,k} \hat{D}_{jk}^{-1} \frac{\partial \hat{D}_{kj}(\hat{\phi})}{\partial \hat{\phi}_m(\tau_n)} \right] \\ &= \sum_l \hat{K}_{ml} \hat{\phi}_l + \frac{\hat{\lambda}}{6} \hat{\phi}_m^3 - \hat{g} N_f \text{Tr}_s \left[ \hat{D}_{mm}^{-1}(\hat{\phi}(\tau_n)) \right] \end{aligned} \quad (3.13)$$

While the bosonic contribution can be computed in a straightforward manner, the computation of the fermionic contribution requires the inversion of the Dirac operator. This, in general, cannot be done straightforwardly, mainly due to computational reasons. In fact, the full Dirac operator would be a  $(2 \cdot N_t \cdot N_x \cdot N_f)^2$  dimensional object and a full inversion would be very expensive. To circumvent this, we use the bilinear noise scheme [13, 15] [should I cite here or in Appendix?](#) which is illustrated in Appendix C.

### 3.3 Stochastic quantisation with coloured noise

In the stochastic quantisation procedure the noise which accounts for the quantum fluctuations of the theory is assumed to be white noise, as defined in equations (3.8), (3.9). We now want to examine the dynamics in presence of a colored noise, writing the Langevin equation as

$$\partial_\tau \phi(x, \tau) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta_{\text{col}}(x, \tau)$$

with  $\eta_{\text{col}}(x, \tau) = r_\Lambda(x) \eta(x, \tau)$ . In particular, here we restrict to the regulating function defined as a sharp cutoff in momentum space

$$r_\Lambda(p) = \theta(\Lambda^2 - p^2) \quad (3.14)$$

and we invite the reader to consult [16] for a discussion of other regulating functions. The noise field in momentum space is then

$$\begin{aligned} \eta_{\text{col}}(p, \tau) &= \mathcal{F}[\eta_{\text{col}}(x, \tau)] = \mathcal{F}[r_\Lambda(x, \tau) \eta(x, \tau)] = \mathcal{F}[r_\Lambda(x, \tau)] \star \mathcal{F}[\eta(x, \tau)] \\ &= \theta(\Lambda^2 - p^2) \eta(p, \tau) \end{aligned}$$

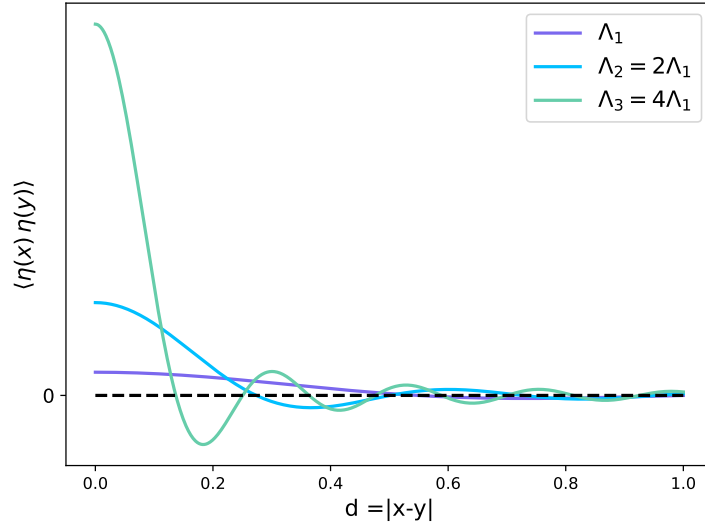


FIGURE 3.1: Noise correlation as a function of  $d = |x - y|$  for three different values of the cutoff  $\Lambda_1 < \Lambda_2 < \Lambda_3$ , in arbitrary units. The plot is qualitative, but shows clearly that with a regulated noise, small-distance ( $d \ll 1/\Lambda$ ) noise correlations are present, while the noise remains uncorrelated at large distances ( $d \gg 1/\Lambda$ ), as in the white noise case.

where  $\mathcal{F}$  indicates the Fourier transform and  $\star$  the convolution product. An interesting thing to look at is the position-space noise correlation function

$$\begin{aligned}
 C_\eta(x, \tau, y, \tau') &= \langle \eta_{\text{col}}(x, \tau) \eta_{\text{col}}(y, \tau') \rangle \\
 &= \frac{1}{(2\pi)^4} \int D\eta P(\eta) \left[ \int_{p,q} e^{-ipx-iqy} \eta_{\text{col}}(p, \tau) \eta_{\text{col}}(q, \tau') \right] \\
 &= \frac{1}{(2\pi)^4} \int_{p,q} e^{-ipx-iqy} \int D\eta [P(\eta) \eta(p, \tau) \eta(q, \tau')] \theta(\Lambda^2 - p^2) \theta(\Lambda^2 - q^2) \\
 &= \frac{2}{(2\pi)^2} \int_{p,q} e^{-ipx-iqy} \delta(p+q) \theta(\Lambda^2 - p^2) \theta(\Lambda^2 - q^2) \delta(\tau - \tau') \\
 &= \frac{2}{(2\pi)^2} \int_p e^{-ip(x-y)} \theta(\Lambda^2 - p^2) \\
 &= \frac{1}{\pi} \int_0^\Lambda d\omega \omega J_0(\omega|x-y|)
 \end{aligned} \tag{3.15}$$

where  $J_0(x)$  is a Bessel function of the first order. The integral in the last line is computed numerically as a function of  $d = |x - y|$  and reported in figure 3.1 for three different values of the cutoff  $\Lambda_1 < \Lambda_2 < \Lambda_3$ . This shows nicely that for  $|x - y| \ll 1/\Lambda$  the noise is now correlated, while for  $|x - y| \gg 1/\Lambda$  the correlation function vanishes, as in the white noise case. In other words, only the short-length behaviour of the system is affected by the introduction of such a regulating term, as one could expect.

Another intuitive and interesting aspect of the dynamics in the presence of coloured noise can be deduce by looking at the field expression in terms of the retarded Langevin Green function [11], which is here not derived, but reported



from [16]

$$\phi(x, \tau) = \int_{x'} \int_{-\infty}^{\tau} d\tau' G(x - x', \tau - \tau') \left[ r_{\Lambda}(\Delta_x) \eta(x, \tau') - \frac{\delta S}{\delta \phi} \Big|_{p=0} \phi(x', \tau) \right]$$

where

$$G(x - x', \tau - \tau') = \theta(\tau - \tau') \int_p e^{-ip \cdot (x - x')} e^{-(\tau - \tau')(p^2 + m^2)}$$

By looking at the first term in the square bracket, one can conclude that there is no propagation of modes with momentum  $p^2 \geq \Lambda^2$  due to the noise term, but one can still have contribution from modes  $p^2 > \Lambda^2$  from the second term, which corresponds to the deterministic part of the equations of motion. Stated differently, UV quantum fluctuations with  $p^2 > \Lambda^2$  are removed from the dynamics of  $\phi$ , but still contribute classically.

Generally speaking, the stationary distribution probability of the regulated stochastic process is given by [16]

$$\mathcal{P}_{\Lambda}(\phi) = \frac{1}{Z} \exp(-S_{\Lambda}[\phi]) = \frac{1}{Z} \exp(-(S[\phi] + \Delta S_{\Lambda}[\phi])) \quad (3.16)$$

where the correction term  $\Delta S_{\Lambda}[\phi]$ , for the specific case of the regulator (3.14) reads

$$\Delta S_{\Lambda}[\phi] = \frac{1}{2} \int_p \phi_p \Lambda^2 \left( \frac{1}{r_{\Lambda}(p^2)} - 1 \right) \phi_{-p}$$

at this point mention that this is the stationary pdf that one gets with frg for sharp cutoff, and cite some papers.

## 3.4 Applications of coloured noise in lattice QFT

### Cooling and the continuum limit

After the general introduction on coloured noise given in the previous paragraph, let us now look more closely on the lattice formulation and at some possible applications of the technique.

To this end, let us consider a two-dimensional lattice with side lengths  $L_t, L_x$  and spacing  $a = a_x = a_t$ . This implies a maximum momentum  $p_{\max} = \pi/a$  in each space-time direction and  $N_x = L_x/a, N_t = L_t/a$  points in each direction. Let us also define

$$\Lambda^2 \equiv (p_{\max}^x)^2 + (p_{\max}^t)^2 \quad (3.17)$$

which indicates the maximum squared momentum on the given lattice.

We then consider a simulation with a regularised noise defined by a cutoff  $\Lambda_{\text{eff}} \leq \Lambda$  and we define a dimensionless parameter

$$s = \frac{\Lambda_{\text{eff}}}{\Lambda} \quad 0 \leq s \leq 1 \quad (3.18)$$

Note that  $\Lambda_{\text{eff}}$  implicitly defines a length scale given by  $a_{\text{eff}} = \pi/\Lambda_{\text{eff}}$ .

Let us then consider a simulation with  $s = 1$  and a set of bare couplings  $\{g_0^i\}$ , and another simulation with  $s' < 1$  and new set of couplings  $\{g_0^{i'}\}$ . We now want to address the following question: is it possible to compensate the change in physical

observables caused by the removal of the UV modes via regularised noise in the second simulation, by properly fine-tuning the bare parameters that enter the lattice discretised action? In other words, we want to encode the quantum fluctuations with  $p^2 > \Lambda_{\text{eff}}^2$  in a redefinition of the classical action so that the expectation value of the observables remains unchanged.

The issue is of course related to the renormalisation transformation introduced in chapter 2. In particular we can exploit the connection between stochastic quantisation with coloured noise and the renormalisation group, as mentioned at the end of section 3.3, to accomplish the above mentioned goal.

As stated at the end of section 2.3, the Wilson RG flow of the dimensionless couplings and fields is dominated by the canonical scaling dimension of the corresponding dimensionful quantities, for high enough cutoff. We are interested in cooling the simulation by removing only the short-length fluctuations, hence we rely on this approximation as a lowest order Ansatz. This corresponds to a tree level RG rescaling.

For what concerns the scalar part of the action, the rescaling at tree level is rather straightforward

$$\hat{m}_\phi^2 = (a^2 m_\phi^2) \rightarrow s^2(a^2 m_\phi^2) = s^2 \hat{m}_\phi^2, \quad \hat{\lambda} = (a^2 \lambda) \rightarrow s^2(a^2 \lambda) = s^2 \hat{\lambda}$$

$$\hat{\phi} = \phi \rightarrow \phi = \hat{\phi}$$

The fermionic part needs some more careful analysis. For simplicity, let us for the moment set  $N_f = 1$ .

In a lattice simulation one wants to perform the integral over the fermionic fields and works with the effective action (2.8). In this case the drift is given by equation (3.13), with the fermionic contribution beeing

$$K_\psi = g \text{Tr}_s D^{-1} \quad (3.19)$$

or, in terms of dimensionless quantities

$$\hat{K}_\psi = (ag) \text{Tr}_s (aD)^{-1}$$

This implies that under a lattice block-spin transformation, where  $a \rightarrow sa$ ,

$$\hat{K}_\psi \rightarrow (sag) \text{Tr}_s (saD)^{-1} = \hat{K}_\psi \quad (3.20)$$

On the other side, when computing the drift via the original action (2.4), one gets (omitting  $\tau$  and  $x$  dependency)

$$K(\tau, x) = -\frac{\delta S}{\delta \phi(\tau, x)} = K_\phi - g \bar{\psi} \psi =$$

$$= -\left(-\partial_x^2 + m_\phi^2\right) \phi - \frac{\lambda}{6} \phi^3 - g \bar{\psi} \psi \quad (3.21)$$

where the fermionic contribution is given by

$$K'_\psi = -g \bar{\psi} \psi$$

Note that all the terms in the equation (3.21) have dimension 2, in units of energy, which means, in particular, that after a lattice block-spin transformation where

$a \rightarrow sa$ , one has

$$\hat{K}'_\psi = (ag)(a\bar{\psi}\psi) \rightarrow s^2(ag)(a\bar{\psi}\psi) = s^2\hat{K}'_\psi \quad (3.22)$$

in contrast with (3.20). For this reason, in order to have the correct scaling, we compute the contribution to the drift without rescaling the Dirac operator (and hence the Yukawa coupling), and then rescale the whole drift via

$$\hat{K}_\psi \rightarrow s^2\hat{K}_\psi$$

so that the scaling dimension of the other terms in (3.21) is matched.

We want to mention that this has important consequences on the issue of continuum limit of low-energy effective theories. In fact, in the standard lattice regularisation procedure, one always has  $a \sim \Lambda^{-1}$ , which means that the continuum limit  $a \rightarrow 0$  is always connected to the limit  $\Lambda \rightarrow \infty$ , and this constitutes a problem for effective theories. The latters are in fact meant to be valid only up to a certain scale  $\Lambda_{\text{phys}}$ , hence one always needs  $\Lambda \leq \Lambda_{\text{phys}}$ . Cooling via coloured noise can provide a solution to this problem, since one can always keep the effective cutoff fixed at  $\Lambda_{\text{eff}} = \Lambda_{\text{phys}}$ , while taking the limits  $a \rightarrow 0, \Lambda \rightarrow \infty$ .

### Temperature control

I have to wait for some results to see whether to include this or not :D



## Chapter 4

# Numerical investigation

### 4.1 GPU implementation of the Conjugate Gradient algorithm

Idk if I have time to do this because I want to compare with CPU and the CPU code is for Quark meson.

### 4.2 Extracting quark masses

The Dirac operator for Wilson fermions in the yukawa model is

$$D_{nm} = \sum_{\alpha} \left[ \frac{\gamma_{\alpha} \delta_{n+\alpha, m} - \gamma_{\alpha} \delta_{n-\alpha, m}}{2} + (m_q + g\phi) \delta_{nm} \right].$$

In momentum space it reads

$$\bar{D}_{ff'}(p) = \left( m + g\sigma \sum_{\mu} 2 \sin^2 \left( \frac{p_{\mu}}{2} \right) + i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu}) \right) \delta_{ff'}$$

The inverse can be checked to be

$$\bar{D}_{f,f'}^{-1} = [m + \dots] \left( m + g\sigma \sum_{\mu} 2 \sin^2 \left( \frac{p_{\mu}}{2} \right) - i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu}) \right) \delta_{ff'}$$

One can now find the pole mass by imposing  $D^{-1} = 0$  and gets

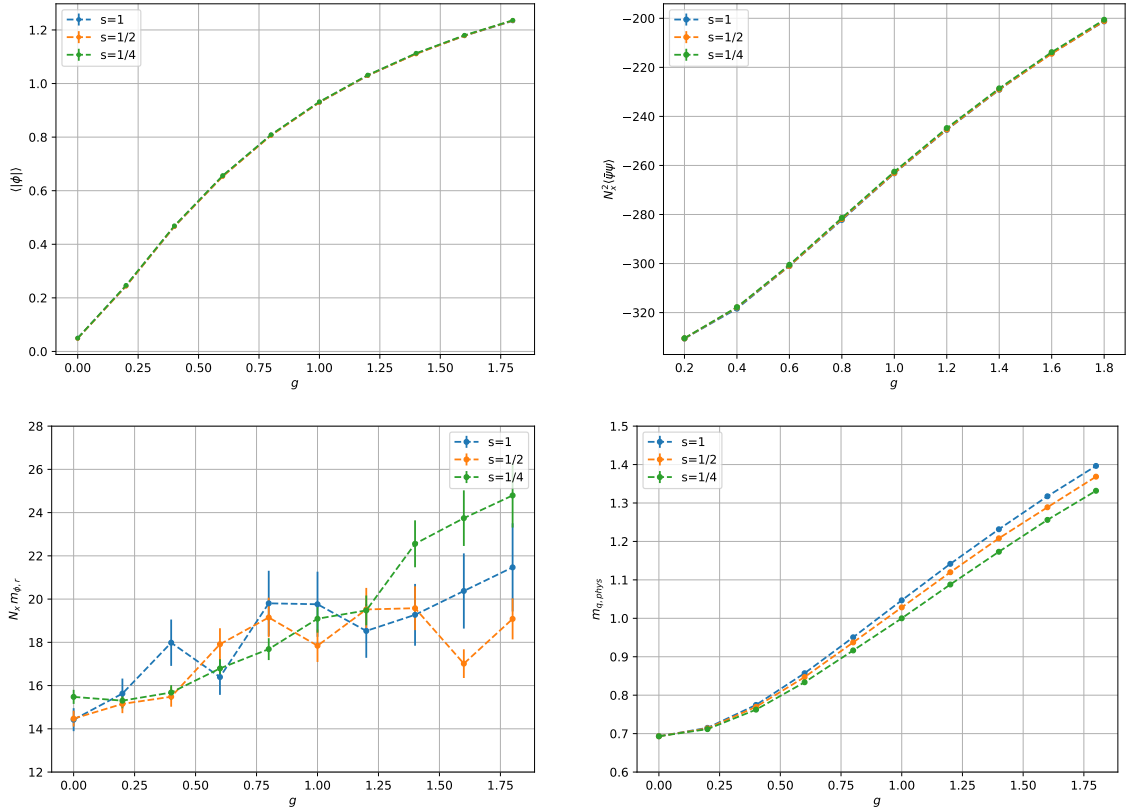
$$m + \dots = 0$$

### 4.3 Cooling with coloured noise

Look at various things such as magnetization, mass, etc.

Even though is  $O(1)$  we do not observe SSB because of fermion bare quark mass, wilson term.

When does L.O. rescaling ansatz breaks down?



$$S[\phi, \bar{\psi}, \psi] = \int_x \phi \left( \frac{\partial^2}{2} + \frac{m_\phi^2}{2} \right) \phi + \frac{\lambda}{4!} \phi^4 + \bar{\psi} (\not{\partial} + m_q + g\phi) \psi$$

$$\lambda = 1.0 \quad m_\phi^2 = 0.5 \quad N_t \times N_x = 8 \times 32 \quad m_q = 0.5 \quad N_{conf} = 5 \cdot 10^3 \quad \bar{\epsilon} = 0.01$$

#### 4.4 Classical to quantum interpolation

Let us start by analysing the coloured noise field in the simulation and relevant properties that emerge from it. We consider the Yukawa model described by the continuum action ???. In figure ?? the system is initialised in the same state for all the configurations, and then evolved with the Langevin equation with various noise fractions. The red line corresponds to the case  $s = 0$ , namely a classical simulation. The blue line corresponds to the case  $s = 1$ , namely the fully quantum case. As one can notice, the introduction of noise shifts the equilibrium expectation value of the field monotonically with the cutoff fraction: this is due to the fact that WHAT???. Note that a lower noise fraction is correlated to a faster convergence towards equilibrium. Moreover, low-distance fluctuations are suppressed due to the removal of the ultraviolet modes in the noise term.

For  $\lambda = 0$  one has

$$\sigma = -\frac{g}{m_\phi^2 + k^2} \bar{\psi} \psi \quad (4.1)$$

In figure ?? one can see that equation (4.1) is verified also on the fully quantum level.

Figures ?? - ?? report a few observables as a function of the cutoff fraction  $s$ . In this case all the coupling constants are kept fixed while changing the value of  $s$ , in order to provide a smooth interpolation between the fully classical and fully quantum picture.

Each figure reports two plots corresponding to two different parameter configurations. The COLOR1 line corresponds to a system in the symmetric phase, while the COLOR2 line correspond to the broken phase. The exact parameters for the two configurations are reported under the figure.

## 4.5 Chiral fermions and a glimpse on the chiral phase transition

As explained in section ??, in the continuum theory chiral symmetry can be broken either explicitly via a finite bare quark mass, or spontaneously if the field gains a non-zero expectation value. Moreover, in the discrete formulation, the introduction of the Wilson term also contributes to the explicit breaking of chiral symmetry [add reference](#), as explained in section ?. This, in particular, means that chiral symmetry is explicitly broken also for  $m_q \rightarrow 0$ . Because of this, one needs a new definition for bare mass  $M_q$ , which takes into account the Wilson term contribution, such that chiral symmetry is restored in the limit  $M_q \rightarrow 0$  for vanishing expectation value of the field  $\phi$ . A convenient way to define such  $M_q$  is the following. SSB chiral symmetry  $\rightarrow$  3 goldstone massless bosons, the pions. If the bare quark mass is zero, the physical mass of the pions has to be zero. Hence one can extract this mass on the lattice and tune  $m_q$  such that this is zero. While this is the correct way to proceed, it is very time taking since one must do mass scans and extrapolations close to singular Dirac operator. This is not the way we pursue here. Instead here we just consider naive fermions and take the limit  $m_q \rightarrow 0$ . This represents a physical theory with  $2N_f = 4$  degenerate quarks. This is just done for the purpose of showing some interesting properties of coloured noise and not (yet) to match any physical result.

In figure ?? some observables are reported as a function of the noise fraction  $s$  for different values of the bare quark mass. In the classical theory ( $s = 0$ ) the order parameters  $\langle |\phi| \rangle$ ,  $\langle \bar{\psi}\psi \rangle$  are, in absolute value, bigger than in the quantum case ( $s = 1$ ). As the chiral limit is approached  $m_q \rightarrow 0$ , the figure shows that the classical system lies in the broken phase, while in the quantum settings the symmetry is restored. [Discuss general phase structure looking at both  \$\phi\$  and  \$\bar{\psi}\psi\$](#) . One can see that as  $m_q$  is reduced, the system shifts from a crossover to a second order phase transition, highlighted by the susceptibility  $\chi^2$  and the binder parameter  $U_L$ .

[This would be better discussed as a function of bare scalar mass:](#)

The difference gets more sharpened as the bare mass decreases, since the theory is closer to the chiral limit discussed above, which corresponds to the case  $m_q = 0$ . In this limit the system goes under a second phase transition, highlighted by a peak in the susceptibility and an abrupt change in the Binder cumulant. In contrast, for finite mass, there is a smooth crossover where the order parameters change continuously.



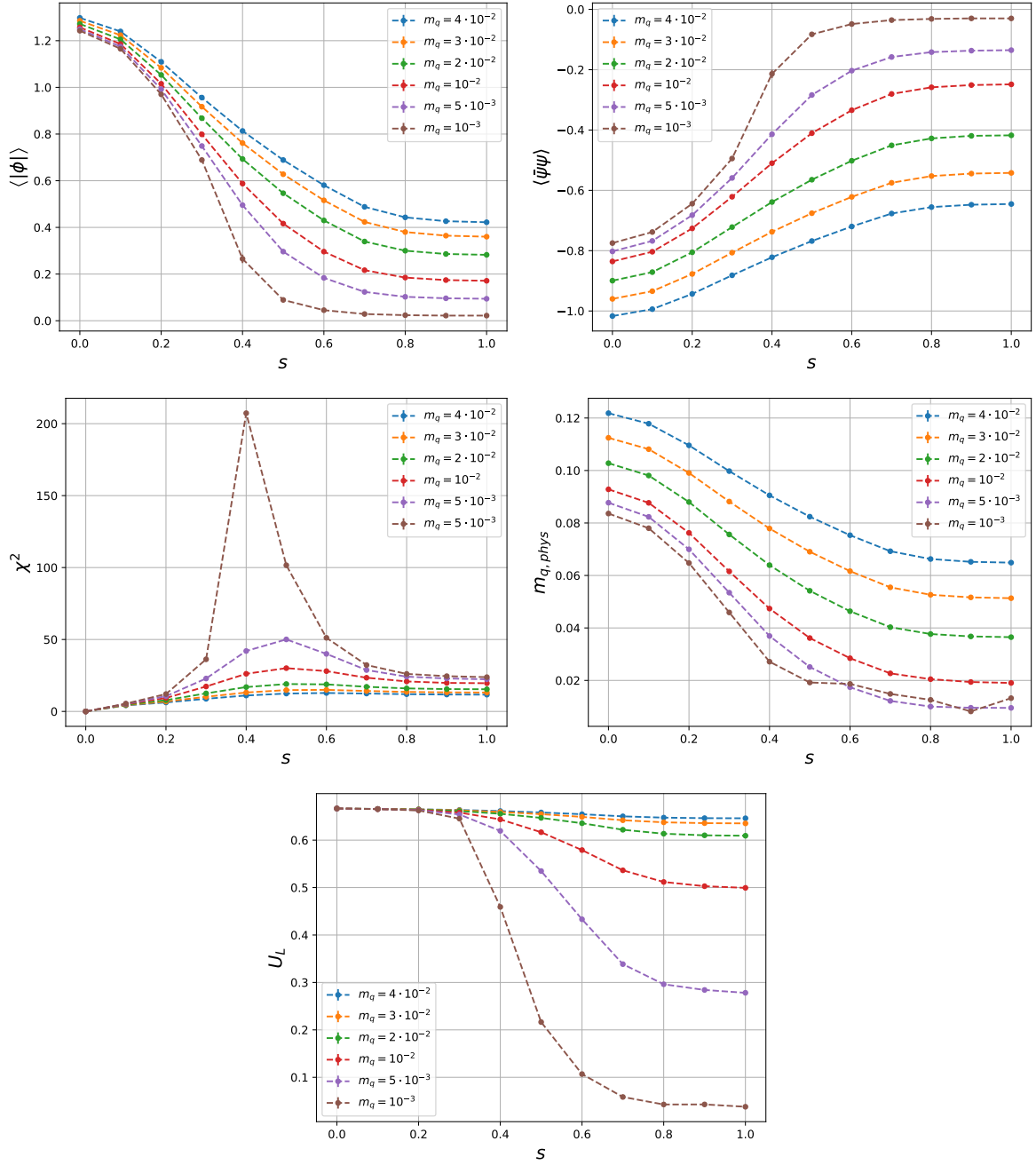


FIGURE 4.1: Chiral symmetry breaking



## **Chapter 5**

# **Conclusions and outlook**



## Appendix A

# Useful relations and definitions

In this appendix, useful relations and definitions are introduced. Fermionic two-points function

$$\begin{aligned}
 \langle \psi_{s,f}(x) \bar{\psi}_{s',f'}(y) \rangle &= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi(x) \bar{\psi}(y) \exp(-S_\phi - \psi D\psi + \bar{\eta}\psi + \bar{\psi}\eta) \\
 &= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \frac{\delta}{\delta \bar{\eta}(x)} \frac{\delta}{\delta \eta(y)} \exp(-S_\phi - \psi D\psi + \bar{\eta}\psi + \bar{\psi}\eta) \\
 &= \frac{1}{Z} \int \mathcal{D}\phi \det[D(\phi)] \exp(-S_\phi) \frac{\delta}{\delta \bar{\eta}(x)} \frac{\delta}{\delta \eta(y)} \exp(\bar{\eta} D^{-1} \eta) \\
 &= \left\langle [D^{-1}(\phi)]_{s,s',f,f'}(x,y) \right\rangle
 \end{aligned} \tag{A.1}$$

The lattice version becomes

$$\langle \psi_m \bar{\psi}_n \rangle = \left\langle [D^{-1}(\phi)]_{mn} \right\rangle$$

with  $D$  beeing the Wilson Dirac operator.

From this, it follows straightforwardly

$$\langle \bar{\psi}\psi \rangle = \text{Tr}_{x,s,f} D^{-1}$$

where  $\langle \bar{\psi}\psi \rangle = \sum_{x,s,f} \langle \bar{\psi}_{s,f}(x) \psi_{s,f}(x) \rangle$ .

The correlator is defined as

$$C(n_t, 0) \equiv \frac{1}{N_x} \sum_{n_x} [\langle \psi(n_t, n_x) \bar{\psi}(0, 0) \rangle + \langle \psi(N_t - n_t, n_x) \bar{\psi}(0, 0) \rangle]$$

Note that we sum up two waves because the source propagates both forward and backward in time due to the boundary conditions.

Since for  $t \rightarrow \infty$  one has that  $C(t, p) \propto e^{-E_0(p)t}$ , we expect

$$C(t, p) \approx \sinh \left( E_0 \left( \frac{N_t}{2} - t \right) \right)$$

Pole mass, renormalized mass, effective mass, bare mass, physical mass

Effective action

$$S_{\text{eff}} = S_\phi + \text{Tr} \log D(\phi)$$

Drift force

$$K_{\phi^j} = -\frac{\delta S}{\delta \phi^j} = -\frac{\delta S_\phi}{\delta \phi^j} - \text{Tr} \left[ D^{-1} \frac{\delta D}{\delta \phi^j} \right]$$



## Appendix B

# Wilson fermions

I am not really sure on whether to put this appendix or not, maybe I will just cite some papers that talk about Wilson fermions

### choice of the basis





## Appendix C

# Algorithms and technical details

### C.1 Conjugate Gradient algorithm and the Dirac operator

The full inversion of the Dirac operator is a very expensive computation, given that the Dirac operator has dimension  $(2 N_t N_x N_f)^2$ , even though it is very sparse and has only few non-zero entries. One can note that for the purpose of computing the fermionic contribution to the drift force and the extraction of the physical quark mass from the correlator (details in section x and section y), only the inverse operator applied to a vector is needed. Hence it is sufficient to compute

$$\psi = D^{-1} |\eta\rangle \quad (\text{C.1})$$

Computing  $\psi$  via equation (C.1) is equivalent to solve the linear system  $D\psi = \eta$ , which can be done efficiently by employing a method for sparse matrices such as Conjugate Gradient (CG) as explained in the following way.

We want to solve the equation

$$D\psi = \eta$$

CG requires the matrix to be hermitian while  $D$  is only  $\gamma^5$ -hermitian (really? under which assumptions?). One can thus solve the linear system

$$(DD^\dagger)\xi = \eta$$

and then obtain  $\psi$  by multiplying the solution  $\xi$  by  $D^\dagger$  since

$$D^\dagger\xi = D^\dagger (DD^\dagger)^{-1} \eta = D^{-1}\eta = \psi \quad (\text{C.2})$$

Analogously one can calculate

$$\chi = D^\dagger\eta$$

by solving

$$(D^\dagger D)\xi = \eta$$

and then applying  $D$  to the result.

One can improve the solution via CG by solving a *preconditioned* equation. Suppose that we want to solve the equation

$$Mx = b$$

via CG.

Let us express the matrix  $A$  as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (\text{C.3})$$

We introduce the Schur complement of  $M$

$$M/D = A - BD^{-1}C \quad (\text{C.4})$$

This allows one to write  $M$  (LDU decomposition, Gaussian elimination) as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbf{1}_p & -BD^{-1} \\ 0 & \mathbf{1}_q \end{bmatrix} \begin{bmatrix} M/D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbf{1}_p & 0 \\ D^{-1}C & \mathbf{1}_q \end{bmatrix} = LAR$$

Which allows for an easy block inversion

$$M^{-1} = \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & -BD^{-1} \\ 0 & I_q \end{bmatrix} = L^{-1}A^{-1}R^{-1}$$

The equation to solve now reads

$$x = L^{-1}A^{-1}R^{-1}b \quad \text{or} \quad y = A^{-1}c$$

with  $y = Lx$  and  $c = R^{-1}b$ . One can then solve the equation  $y = A^{-1}c$  and get the solution  $x$  by applying  $L^{-1}$  to  $x$ .

An example of preconditioning is the even-odd preconditioning. Let us write the dirac operator in the form of equation (C.3) in the following way

$$M = \begin{pmatrix} M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{pmatrix}$$

The Schur complement (C.4) is

$$\hat{M} \equiv M/M_{oo} =$$

## C.2 Bilinear noise scheme

$$\text{Tr} \left[ D^{-1} \frac{\delta D}{\delta \phi^j} \right] \approx \langle \eta | D^{-1} \frac{\delta D}{\delta \phi^j} | \eta \rangle = \langle \psi | \frac{\delta D}{\delta \phi^j} | \eta \rangle \quad |\psi\rangle = D^{-1} |\eta\rangle = D^\dagger \underbrace{(DD^\dagger)^{-1}}_{\text{CG}} |\eta\rangle$$

$$\text{Tr} A = \frac{1}{N} \lim_{N \rightarrow \infty} \sum_i^N \eta_i^T D_{ij} \eta_j \quad (\text{C.5})$$

where  $\eta_i$  is a gaussian random field where each component is drawn from a normal distribution  $\mathcal{N}(0, 1)$ .

More precisely each vector component  $\eta_i^\alpha$  satisfies

$$\langle \eta_i^\alpha \rangle = 0 \quad \langle \eta_i^\alpha \eta_j^\beta \rangle = \delta_{ij} \delta^{\alpha\beta}$$

The series (C.5) requires in principle an infinite number of vectors to evaluate the trace exactly. In practice we truncate it and choose  $N = 1 : \mathbf{D} : \mathbf{D}$ . The average over

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Monte Carlo samples will eventually converge nevertheless to the right result.



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