Department of Physics and Astronomy
University of Heidelberg

Master thesis

in Physics

submitted by

(name and surname)

born in (place of birth)

(year of submission)

Simulating effective field theories

on a space-time lattice with coloured noise

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at the

Institute for Theoretical Physics in Heidelberg

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(Titel in Deutsch): (Abstract in Deutsch, max. 200 Worte)

(Title in English): (abstract in english, at most 200 words)

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27.11.2023 ....

“Grazie a tutti.”

Matteo Zortea

Contents

List of Figures

List of Tables

List of Abbreviations

LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

Physical Constants

Speed of Light $c_0 = 2.997\,924\,58 \times 10^8\ \mathrm{m\,s^{-1}}$ (exact)

List of Symbols

a	distance	m
P	power	W (J s^{-1})
ω	angular frequency	rad

Chapter 1

Introduction

1.1 Quantum chromodynamics and its and phase diagram

Big picture: here we talk about QCD and the problem of the phase diagram

1.2 The renormalisation group

1.3 Effective theories

Here we first define effective theories and discuss their usefulness, then introduce RG as a technique to resolve physics at different scales.

Chapter 2

Theoretical background

Da qualche parte cita [1].

In this chapter we want to provide with an overview on the general theoretical framework that supports this work, and introduce the main concepts for the successive parts. Each section in this chapter is, by no means, meant as an exhaustive treatment. The description will be quite conceptual, rather than technical, and aims at recalling the main ideas and fix notation. We ask the reader to consult appropriate references, which will be given in the corresponding sections, for a more detailed treatment of the topics.

The starting set up is the euclidean formulation of quantum field theory, where one typically defines a path integral Z

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\psi \; e^{-S[\phi,\psi,\bar{\psi}]} \tag{2.1}$$

and aims to compute correlation functions

$$\langle \xi_{x_1} \dots \xi_{x_n} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\psi \; \xi_{x_1} \dots \xi_{x_n} \; e^{-S[\phi,\psi,\bar{\psi}]} \qquad \xi_{x_i} \in \{\phi_{x_i}, \psi_{x_i}, \bar{\psi}_{x_i}\}$$

Computing physical quantities from such a direct approach is not only hard to do, but results often impossible due to the appearence of divergences in the calculations. To fix this, one often relies on expansion techniques such as perturbation theory (CITATION), in which one tries to regularise the theory order by order in an expansion on the interaction coupling, yielding finite quantities that depend on the truncation order. While this method produced incredibly precise results (g-2, fine structure, ...), it fails completely in treating non-perturbative phenomena, namely effects that cannot be captured by any order in the expansion or that rely on strongly interacting matter. Example of such systems range from QCD, cold atoms, plasma, stuff. Moreover, such formulation is also not much suitable for numerical computations, since both the path integral and action measure are infinite dimensional objects.
Lattice field theory [2–5] is meant at first as a powerful non-perturbative regularisation tool to prevent divergences to occur and render the computation of the correlation functions finite. Moreover, it also provides a framework to study quantum field theory numerically on a computer. In order to accomplish this, one typically defines the theory on a space-time lattice and makes use of statistical methods such as Monte Carlo algorithms to compute observables. One may wonder how can one reconstruct the results in the continuum theory, keeping the results finite, and matching the results on the discretised theory to physically measured ones. This task, far

from beeing simple, will be the focus of the next sections, in which we will first introduce relevant theoretical tools, such as the renormalisation group, and then discuss the existence of a continuum limit of a lattice theory and, if it exists, how it can be extracted. This will motivate the introduction of coloured noise in the context of continuum limits of effective theories, a technique which will be shown to be powerful also for other various reasons, which will be the main focus of the analysis carried in the remaining chapters.

2.1 Lattice Field Theory

As mentioned before, lattice field theory consists in formulating QFT on a space-time lattice

$$S = \int d^d x\, \mathcal{L}(x)\, t$$

2.2 Block-spin renormalisation

Cite Kadanoff article [**PhysicsPhysiqueFizika.2.263**]. Average spins and rescale stuff to keep correlation length fixed.

2.3 Wilson renormalisation

Extends block spin RG.
One splits the fields as $\Phi = \phi + \varphi$ where ϕ are fields with momenta $p \leq b\Lambda$ and φ are fields with momenta $b\Lambda < p < \Lambda$, then one writes the path integral in terms of the Wilsonian effective action

$$Z = \int D\Phi_\Lambda\, e^{-S_\Lambda[\Phi]} = \int D\phi_{b\Lambda}\, e^{-S_{b\Lambda}[\phi]} \int D\varphi_{b\Lambda,\Lambda}\, e^{-S_{b\Lambda,\Lambda}[\phi,\varphi]} = \int D\phi_{b\Lambda}\, e^{-S^{\text{eff}}_{b\Lambda}[\phi]}$$

where

$$S^{\text{eff}}_{b\Lambda}[\phi] = S_{b\Lambda}[\phi] - \log\left(\int D\varphi_{b\Lambda,\Lambda}\, e^{-S_{b\Lambda,\Lambda}[\phi,\varphi]}\right) = S_{b\Lambda}[\phi] + \Delta S_{b\Lambda,\Lambda}[\phi]$$

Note that all the steps above are exact identities. In particular, performing the integral over ultraviolet modes, is the continuum version of the block spinning procedure outlined in the previous section.
Note also that $S_{b\Lambda}[\phi]$ is the same as the initial action, but it is non-zero only for fields with $p^2 \leq b\Lambda^2$.
At this point one can expand $\Delta S_{b\Lambda,\Lambda}$ in powers of the field (before, another step, see jan pawlowski's notes). Powers that are present also in $S_{b\Lambda}$ can be absorbed into the latter by redefining the coupling

$$\begin{aligned} &\phi'\left(x'\right) = \left[b^{2-d}(1+\Delta z)\right]^{\frac{1}{2}} \phi(x), \quad m'^2 = \left(m^2 + \Delta m^2\right) \frac{1}{1+\Delta z} \frac{1}{b^2}, \quad \lambda' = (\lambda + \Delta\lambda) \frac{1}{(1+\Delta z)^2} b^{d-4}, \\ &\alpha' = (\alpha + \Delta\alpha) \frac{1}{(1+\Delta z)^2} b^d, \quad \lambda_6' = (\lambda_6 + \Delta\lambda_6) \frac{1}{(1+\Delta z)^3} b^{2d-6}, \quad \ldots \end{aligned} \tag{2.2}$$

higher powers are suppressed (non-renormalisable terms) are suppressed. By neglecting these higher order powers, one can bring the action in the same form as the initial one via redefinition of the parameters. The whole procedure is non-perturbative. The result can be compared to the initial action after redefinition of all the dimensionful quantities i.e. via $p' \equiv p/b$.

2.4 Continuum limit of lattice field theories

Introduce renormalisation as a mapping as in page 40 of Montvay Munster.
Continuum limits in lattice theories are intimately connected to the existence of critical points in the theories. In fact, to take a continuum limit, one want the dimensionless correlation length $\hat{\xi}$ to diverge: in this way one can represent an infinite number of points inside a finite volume (explain better here). For this to happen, the system must go under a second order phase transition, whose critical point is identified by a set of values for the bare parameters g_0^{i*}. differentiate well between dimful and dimless.
Consider O to be an observable which has to be matched to a physical measurable quantity, and compare it to the dimensionless quantity $\hat{O}$ given by a lattice simulation. In general the physical observable is assumed to be a function of the spacing and the bare couplings of the theory

$$O = O(a, g_0^i)$$

while its lattice counterpart can only depend on the dimensionless coupling $hatg_0^i$, i.e.

$$\hat{O} = \hat{O}(g_0^i)$$

Let d_O be the physical dimension of the observable O in units of energy. Then one can relate the two quantities as

$$O(a, g_0^i) = \left(\frac{1}{a}\right)^{d_O} \hat{O}(\hat{g}_0^i) \tag{2.3}$$

We now want to address the following question: given a (small enough) a, is it possible to find a value $\hat{g}_0^i(a)$ such that the value of O given by (2.3) does not depend on a?
We then impose such condition via

$$\frac{d}{da} O(a, g_O^i) = \left(a\frac{\partial}{\partial a} - \beta(g_0^i)\frac{\partial}{\partial g_0^i} \right) O(a, g_O^i) = 0$$

with

$$\beta(g_0^i) = -a\frac{\partial g_0^i}{\partial a}$$

Integrating such β functions tells one how to change bare couplings as a backreaction to a change in the spacing, in order to keep observables constant. We then say that the theory admits a continuum limit if there exists some set of values $(g_0^i)^*$ such that when $g_0^i \to (g_0^i)^*$ one has $\hat{\xi} \to +\infty$ and $O \to O_{phys}$. Connection with fixed points and beta function. Comment also on beta functions for dimless couplings.
Of course one does not know a priori the full lattice beta functions, but they can be computed via approximate or continuum methods. For example, in continuum perturbation theory, one can compute $g_r^i(\Lambda) = g_r^i(\Lambda, g_0^j)$, where Λ is a sharp momentum cutoff and then try to invert them to find $g_0^i = g_0^i(\Lambda, g_r^j)$. The connection is

then given by making the identification $a \sim \Lambda^{-1}$.
Generally speaking, we are interested in the set of theories in theories space that have constant renormalised couplings but different dimless couplings $g_r^i\, a$ (trajectories in Kadanoff-Wilson RG).

2.5 Yukawa theory

Let us consider the Yukawa theory defined by the action

$$\begin{aligned} S[\phi, \psi, \bar{\psi}] &= S_\phi[\phi] + S_\psi[\psi, \bar{\psi}] + S_{\text{int}}[\phi, \psi, \bar{\psi}] \\ S_\phi[\phi] &= \int_x \phi_x \left(-\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \phi_x + \frac{\lambda}{4!}\, \phi_x^4 \\ S_\psi[\psi, \bar{\psi}] &= \int_x \bar{\psi}_x \left(\not{\partial}_x + m_q \right) \psi_x \\ S_{\text{int}}[\phi, \psi, \bar{\psi}] &= g \int_x \bar{\psi}_x\, \phi_x\, \psi_x \end{aligned} \tag{2.4}$$

One can see that the action is made of a scalar part $S_\phi[\phi]$, a fermionic part $S_\psi[\psi, \bar{\psi}]$ and a Yukawa interaction term $S_{\text{int}}[\phi, \psi, \bar{\psi}]$.
It is also convenient for later purposes to define the operators K, D represented in position space as

$$\begin{aligned} K(x,y) &= \left(-\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \delta(x,y) \\ D(x,y) &= \left(\not{\partial}_x + m_q + g\phi \right) \delta(x,y) \end{aligned} \tag{2.5}$$

and in momentum space as

$$\begin{aligned} \widetilde{K}(p,q) &= \int_{x,y} e^{-ipx} \left(-\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \delta(x,y)\, e^{iqy} = \left(\frac{p^2}{2} + \frac{m_\phi^2}{2} \right) \delta(p,q) \\ \widetilde{D}(p,q) &= \int_{x,y} e^{-ipx} \left(\not{\partial}_x + m_q + g\phi \right) \delta(x,y)\, e^{iqy} = \left(\not{p}_x + m_q + g\phi \right) \delta(p,q) \end{aligned} \tag{2.6}$$

This allows one to rewrite the action as

$$S[\phi, \psi, \bar{\psi}] = \int_x \phi_x K \phi_x + \frac{\lambda}{4!} \phi_x^4 + \bar{\psi}_x D \psi_x$$

We introduce the left-handed and right-handed spinors

$$\psi_L = (1 - \gamma_5)\, \psi \qquad \psi_R = (1 + \gamma_5)\, \psi$$

for which

$$\psi = \frac{(1-\gamma_5)}{2} \psi + \frac{(1+\gamma_5)}{2} \psi = \psi_L + \psi_R$$

The action written in terms of ψ_L, ψ_R reads

$$S = S_\phi + \bar{\psi}_L D \psi_L + \bar{\psi}_R D \psi_R + (m_q + g\phi)\, \left(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \right) \tag{2.7}$$

The last equation makes clear that that for $m = 0, \langle\phi\rangle = 0$ the action is symmetric under the chiral group $SU(2)_L \times SU(2)_R$, namely

$$\begin{aligned} \psi_L(x) &\to U_L \psi_L(x) \qquad & \bar\psi_L(x) &\to \bar\psi_L(x) U_L^\dagger \\ \psi_R(x) &\to U_R \psi_R(x) \qquad & \bar\psi_R(x) &\to \bar\psi_R(x) U_R^\dagger \end{aligned}$$

for $U_L, U_R \in SU(2)$.
The main feature of the model is chiral symmetry breaking [6, 7], which can happen explicitly at the level of the classical action for a non-zero quark mass, or spontaneously when the scalar field gains a non-zero expectation value. One can in fact notice already by looking at (2.6), that $\langle\phi\rangle \neq 0$ has the same effect on the action as a finite bare quark mass. This observation will be made more quantitative in section (SECCCC) where it will be shown that

$$\langle\phi\rangle \sim \langle\bar\psi\psi\rangle \sim m_q$$

The fermionic part of the path integral (2.1) can be performed explicitly

$$\int \mathcal{D}\bar\psi\mathcal{D}\psi \,\exp\left(-\int_x \bar\psi_x \, D\, \psi_x\right) = \det D[\phi] = e^{\mathrm{Tr}\log(D[\phi])}$$

where the trace is performed over space-time, flavour and spinor components.
The full path integral can now be expressed in terms of the resulting effective action for the scalar fields

$$Z = \int \mathcal{D}\phi \; e^{-S_{\text{eff}}[phi]}$$

with

$$S_{\text{eff}}[\phi] = S_\phi[\phi] - \underset{x,s,f}{\mathrm{Tr}} \log D[\phi] \tag{2.8}$$

One can derive the classical equations of motion by imposing $\frac{\delta S}{\delta\phi} = 0$, here expressed in momentum space

$$(k^2 + m_\phi^2)\,\phi(x) + \frac{\lambda}{6}\,\phi^3(x) = g \underset{s,f}{\mathrm{Tr}} \left[D^{-1}(\phi(x)\right] = -g\,\bar\psi(x)\psi(x)$$

where the trace is performed over spin and flavour components. For $\lambda = 0$, they highlight a simple proportionality relation between magnetisation and chiral condensate, which for zero momentum reads

$$\phi(x) = -\frac{g}{m_\phi^2}\,\bar\psi(x)\psi(x) \tag{2.9}$$

The classical relation (2.9) is proven to hold also at mean field on the quantum level [8] and will be studied in the discretised theory in section 4.4.

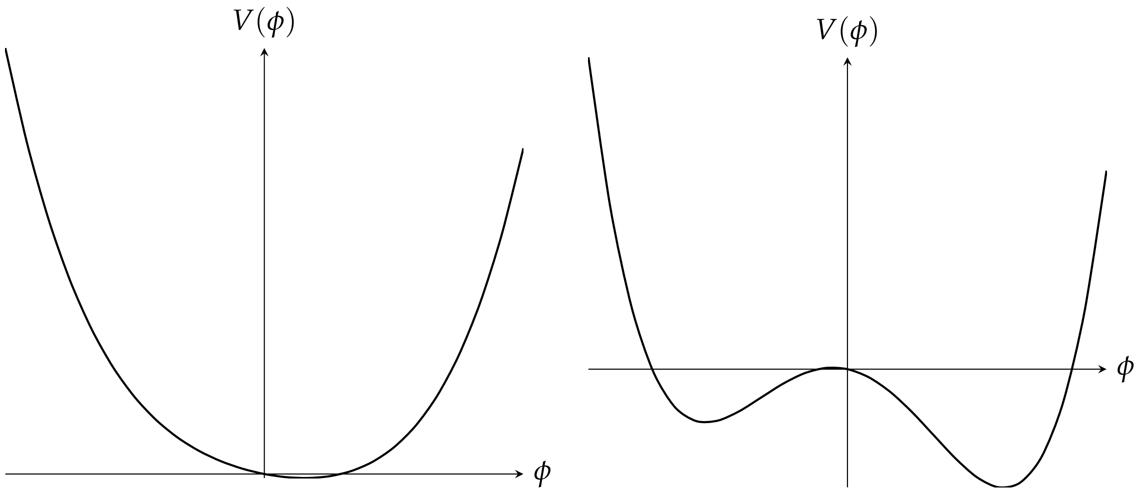

FIGURE 2.1: The introduction of the boson-fermion interaction, with a finite fermionic mass, causes the breaking of the O(1) symmetry. It shifts the equilibrium position in the symmetric phase (left) causing $\langle\phi\rangle \neq 0$, and tilts the potential in the broken phase (right), making the two minima not equivalent.

Chapter 3

Methods and algorithms

3.1 Discretisation of the Yukawa theory

In this secttt we provide a discretised formulation of the Yukawa model introduced in section **??**.

For what concerns the bosonic part of the action, a discretisation can be done straightforwardly with the following replacements

$$\begin{aligned} \int_x &\quad\rightarrow\quad a^2\sum_x \\ \partial_x^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} &\quad\rightarrow\quad \sum_\mu \left[\frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2}\right] \end{aligned}$$

which yields to the lattice action

$$S_\phi[\phi] = a^2 \sum_{m,n} \phi_m\, K_{mn}\, \phi_n + \frac{\lambda}{4!} \sum_n \phi_n^4$$

with

$$K_{mn} = -\sum_\mu \left[\frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2}\right] + m_\phi^2 \delta_{mn}$$

One can also express everything using dimensionless coupdlings

$$\begin{aligned} \hat{m}_\phi^2 &= a^2\, m_\phi^2 \\ \hat{\lambda} &= a^2\, \lambda, \\ \hat{K}_{mn} &= a^2 K_{mn} \end{aligned} \tag{3.1}$$

and the action is then described only in terms of dimensionless quantities

$$S_\phi = -\sum_{n,\mu} \hat{\phi}_n \hat{\phi}_{n+\mu} + \sum_n \left[\frac{1}{2}\left(4 + \hat{m}^2\right) \hat{\phi}_n^2 + \frac{\hat{\lambda}}{4!}\, \hat{\phi}_n^4\right]$$

Otherwise it is customary to introduce dimensionless couplings κ, β defined via

$$\begin{aligned} \phi &\rightarrow (2k)^{\frac{1}{2}} \phi, \\ (am)^2 &\rightarrow \frac{1-2\beta}{k} - 4, \\ a^{-2}\lambda &\rightarrow \frac{6\beta}{k^2} \end{aligned} \tag{3.2}$$

and the equivalent action reads

$$S_\phi = -2k \sum_{n,\mu} \phi_n^i \phi_{n+\mu}^i + (1-2\beta) \sum_n \phi_n^i \phi_n^i + \beta \sum_n \left(\phi_n^i \phi_n^i \right)^2$$

In the following, we might use any of the two dimensionless formulations interchangeably, since they are completely equivalent given the definitions (3.1), (3.2).
For what concerns the fermionic action, a naive discretisation is not sufficient, due to the well known doubling problem. In this work Wilson fermions are employed as a way to fix such issue. Details of this formulation are explained in section **??**. Here, only the final discretised action is reported, which reads

$$S_\psi[\phi, \psi, \bar{\psi}] = \sum_{m,n} \bar{\psi}_m \, D_{m,n} \psi_n + g \sum_n \bar{\psi}_n \phi_n \psi_n$$

with ψ_n beeing a four-component spinor (2 flavour components and 2 Dirac components), and $D_{m,n}$ beeing the Wilson-Dirac operator is $g\phi$ included in the definition of D?) defined as

$$\begin{aligned} D_{m,n} = &- \left(\frac{\Gamma_{+0}}{2} \, \delta_{m,m+0} + \frac{\Gamma_{-0}}{2} \, \delta_{m,m-0} + \frac{\Gamma_{+1}}{2} \, \delta_{m,m+1} + \frac{\Gamma_{-1}}{2} \, \delta_{m,m-1} \right) \delta_{f,f'} \\ &+ (2 + m + g\phi) \;\; \delta_{s,s'} \delta_{m,n} \end{aligned} \tag{3.3}$$

The Wilson projectors $\Gamma_{\pm\mu}$ are defined as

$$\Gamma_{\pm\mu} = 1 \mp \gamma_\mu$$

One can then proceed by defining dimensionless fields and couplings

$$\begin{aligned} \hat{\psi} &= \rightarrow a^{\frac{1}{2}} \psi, \\ \hat{m}_q &= a m_q, \\ \hat{g} &= a g \end{aligned}$$

to describe the action only in terms of dimensionless quantities.
In the remaining of this work, the dimensionless couplings and fields will be adopted, unless otherwise specified. Additionally, both the original action S and the effective action S_{eff} will be denoted by S for simplicity. It will be clear from the context to which of the two we will be referring.

3.2 Stochastic quantisation and Langevin Monte Carlo

In order to compute expectation values from the discretised path integral add ref. we employ a Langevin Monte Carlo algorithm, which is based on stochastic quantisation [9, 10].
The idea is that Euclidean Quantum Field theory can be thought as a system in thermal equilibrium with a heat reservoir and hence described as a stochastic process via the Langevin equation.
Let us consider a scalar field ϕ with a Euclidean action $S[\phi]$ and the following Langevin equation

$$\partial_\tau \phi(\tau, x) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta(\tau, x) \tag{3.4}$$

where $-\frac{\delta S[\phi]}{\delta \phi(\tau,x)}$ is the drift term and $\eta(\tau,x)$ is a random white noise field defined by

$$\langle \eta(x,\tau)\rangle = 0 \qquad \langle \eta(x,\tau)\,\eta(x',\tau')\rangle = 2\,\delta(x,x')\,\delta(\tau,\tau')$$

For $t \to +\infty$ (assuming a stationary solutions exists, but I think it is always the case if the action is bounded from below) one can prove that ADD CITATION the stationary probablity distribution is given by

$$\mathcal{P}(\phi) = \frac{1}{Z}\exp\left(-S[\phi]\right) \tag{3.5}$$

This allow one to compute correlation functions as moments of the distribution (3.5). Equation 3.4 can be integrated numerically for discrete time steps τ_n via, for example, an explicti Euler-Someone Else scheme

$$\phi(\tau_{n+1},x) = \phi(\tau_n,x) - \epsilon\,\frac{\delta S[\phi]}{\delta\phi(\tau_n,x)} + \sqrt{2\epsilon}\,\eta(\tau_n,x)$$

Higher order integration schemes are, for example, cite schemes. For the discretised action of the Yukawa theory the drift reads

$$\begin{aligned}\frac{\partial S}{\partial\phi(\tau_n,m)} &= \frac{\partial S_\phi}{\partial\phi(\tau_n,m)} - \operatorname*{Tr}_{s,f}\left[D^{-1}\frac{\partial D(\phi)}{\partial\phi(\tau_n,m)}\right] \\ &= \frac{\partial S_\phi}{\partial\phi(\tau_n,m)} - g\operatorname*{Tr}_{s,f}\left[D^{-1}(\phi(\tau_n,m))\right]\end{aligned} \tag{3.6}$$

To evaluate the trace, which is due to the fermionic contribution, we use the bilinear noise scheme add reference which is illustrated in Appendix **??**.

3.3 Langevin dynamics with coloured noise

Mention some possible uses of colored noise beside taking continuum limits.
Connection to stochastic regularisation [**boo**] In the stochastic quantisation procedure the noise which accounts for the quantum fluctuations of the theory is assumed to be white noise. This means that its power spectrum is flat in momentum space, extending in all the first Brillouin zone, namely for $p_\mu \in [-\pi/a, \pi/a]$. One could modify this spectrum, and in this case one says *colored noise*. In particular one could put a sharp cutoff on the total momentum, imposing $p^2 \leq \Lambda^2$. We refer to this particular case as *regularised noise*.
In such case the Langevin equation for the scalar field (3.4) assumes the form

$$\partial_\tau\phi(\tau,x) = -\frac{\delta S[\phi]}{\delta\phi(\tau,x)} + r_\Lambda(x)\,\eta(\tau,x)$$

where the regularising function $r_\Lambda(x)$ can be easilly expressed in momentum space as $r_\Lambda(p) = \theta(\Lambda^2 - p^2)$. One can show [11] that the stochastic process is now driven towards a new equilibrium distribution

$$\mathcal{P}_\Lambda(\phi) = \frac{1}{Z}\exp\left(-S_\Lambda[\phi]\right) = \frac{1}{Z}\exp\left(-(S[\phi] + \Delta S_\Lambda[\phi])\right) \tag{3.7}$$

where the correction term $S_\Lambda[\phi]$ ensures that the probability measure $\mathcal{P}_\Lambda$ vanishes for squared fields' momenta greater than the cutoff Λ^2.

An explicit example of such regulator for a free scalar field can be ADD EQUATION PAWLOWSKI. This is just one example of regularisation and different ones can be chosen. See [11] for details.

3.4 Lattice QFT with regularised noise

mention that here we consider for simplicity a square lattice but one can read the general version either in appendix or in Jan's paper
After the general introduction on coloured noise given in the previous paragraph, let us now look more closely on the lattice formulation and at the various applications of such techniques.
first talk about sliding the cutoff and cite Jan Philipp, Jan Pawlowski. Mention (by only citing) also the control of temperature. Then cooling: To this end, let us consider a squared two-dimensional lattice with side length $L \equiv L_x = L_t$ and spacing $a \equiv a_x = a_t$. This implies a maximum momentum $p_{\text{max}} = \pi/a$ in each space-time direction, which in turn implies $N = N_x = N_t$ points in each direction. Let us also define

$$\Lambda^2 \equiv (p^x_{\text{max}})^2 + (p^t_{\text{max}})^2 \tag{3.8}$$

which indicates the maximum squared momentum on the given lattice.
We then consider a general regularised simulation with a cutoff Λ_* and we define a dimensionless parameter

$$s^2 = \frac{\Lambda_*^2}{\Lambda^2} \qquad 0 \leq s \leq 1 \tag{3.9}$$

We then set up a simulation with $s = 1$ and a set of bare couplings $\{g_0^i\}$. As a short-hand notation for such a configuration, we introduce the following notation

$$\mathcal{C}_s = \left\{ s, a, N, \Lambda, \{g_0^i\} \right\}$$

We now want to address the following question: is it possible to compensate the change in physical observables caused by the removal of the quantum modes via regularised noise, by a rescaling of the bare parameters that enter the lattice discretised action?
In a more formal way, we want to construct a map between parameters of the simulation

$$f_{s,s'} \quad : \quad \mathcal{C}_s = \left\{ s, a, N, \Lambda, \{g_0^i\} \right\} \quad \rightarrow \quad \mathcal{C}_{s'} = \left\{ s', a', N', \Lambda', \{g_0^{i\prime}\} \right\} \tag{3.10}$$

that leaves physical observables unaltered

$$\langle \mathcal{O} \rangle_{\mathcal{C}_s} = \langle \mathcal{O} \rangle_{\mathcal{C}'_s} \tag{3.11}$$

The issue is of course related to the renormalisation transformation introduced in chapter 2.
Remembering from sections 3.1 and 2.4 that $\Lambda \sim a^{-1}$, this question would also address the problem of continuum limit of effective field theories. One then accomodates the change of the spacing (cutoff) by a change in the bare dimensionful couplings, as explained in chapter **??**. . This means that all the dimensionful quantities, couplings, momenta, fields, have to be rescaled according to their dimension, as detailed in the following lines.
For what concerns the scalr part of the action, the rescaling at tree level is rather

trivial [11, 12]

$$(a^2 m_\phi^2) \to s^2 (a^2 m_\phi^2), \quad (a^2\lambda) \to s^2(a^2\lambda), \quad \phi \to \phi$$

The fermionic part needs some more careful analysis. In a lattice simulation one wants to perform the integral over the fermionic fields and works with the effective action (2.8). In this case the drift is given by equation (3.6), with the fermionic contribution beeing

$$K_\psi = g \operatorname*{Tr}_{s,f} D^{-1} \tag{3.12}$$

or, in terms of dimensionless quantities

$$\widehat{K}_\psi = (ag) \operatorname*{Tr}_{s,f} (aD)^{-1}$$

This implies that under a lattice block-spin transformation, where $a \to sa$,

$$\widehat{K}_\psi \to (sag) \operatorname*{Tr}_{s,f} (saD)^{-1} = \widehat{K}_\psi \tag{3.13}$$

On the other side, when computing the drift via the original action (2.4), one gets

$$\begin{aligned} K(\tau,x) &= -\frac{\delta S}{\delta\phi(\tau,x)} = K_\phi(\tau,x) - g\,\bar\psi(\tau,x)\psi(\tau,x) = \\ &= -\left(-\partial_x + m_\phi^2\right)\phi - \frac{\lambda}{6}\phi^3 - g\,\bar\psi\psi \end{aligned} \tag{3.14}$$

where the fermionic contribution is given by

$$K'_\psi = -g\,\bar\psi\psi$$

Note that all the terms in the equation (3.14) have dimension 2, in units of energy, which means, in particular, that after a lattice block-spin transformation where $a \to sa$, one has

$$\widehat{K}'_\psi = (ag)(a\bar\psi\psi) \to s^2 (ag)(a\bar\psi\psi) = s^2 \widehat{K}'_\psi \tag{3.15}$$

in contrast with (3.13). For this reason, in order to have the correct scaling, we compute the contribution to the drift using (3.12) without rescaling the Dirac operator (and hence the Yukawa coupling), and then rescale the whole drift via

$$\widehat{K}_\psi \to s^2 \widehat{K}_\psi$$

so that the scaling dimension of the other terms in (3.14) is matched.

Mention that this could mean that for a higher order rescaling one might have to look at how the quark bilinear renormalises.

Chapter 4

Numerical investigation

4.1 Perfomance of Conjugate Gradient on the GPU

4.2 Test with background mesons

The Dirac operator for Wilson fermions in the yukawa model is

$$D_{nm} = \sum_{\alpha} \left[\frac{\gamma_\alpha \delta_{n+\alpha,m} - \gamma_\alpha \delta_{n-\alpha,m}}{2} + (m_q + g\phi)\, \delta_{nm} \right].$$

In momentum space it reads

$$\bar{D}_{ff'}(p) = \left(m + g\sigma \sum_{\mu} 2\sin^2\left(\frac{p_\mu}{2}\right) + i \sum_{\mu} \gamma_\mu \sin\left(p_\mu\right) \right) \delta_{ff'}$$

The inverse can be checked to be

$$\bar{D}^{-1}_{f,f'} = [m + \dots] \left(m + g\sigma \sum_{\mu} 2\sin^2\left(\frac{p_\mu}{2}\right) - i \sum_{\mu} \gamma_\mu \sin\left(p_\mu\right) \right) \delta_{ff'}$$

One can now find the pole mass by imposing $D^{-1} = 0$ and gets

$$m + \dots = 0$$

4.3 Cooling with coloured noise

In this section we report and discuss results of the Yukawa theiry using the technique of coloured noise to perform block-spin steps as outlined in the paragraph 3.3. We first set up the white noise simulation with $s = 1$ on a lattice of size 8×8, with spacing a and cutoff Λ. We then start performing complete block-spin steps as summarised in table **??**.

In the plots in figure 4.1, reported as a function of the bare mesons mass. The peek in the magnetic susceptibility deserves some comment. In general, in a finite-volume lattice theory, no phase transition can happen. To state the presence of a phase transition, one should look at the infinite volume limit, which is done by studying the volume scaling of the magnetic susceptibility. In particular we do not expect a phase transition in our model due to the spontaneous breaking of the $O(1)$ symmetry, since the presence of a finite bare quark mass already breaks the $O(1)$ symmetry explicitly. As a remark to this statement, we studied the volume scaling

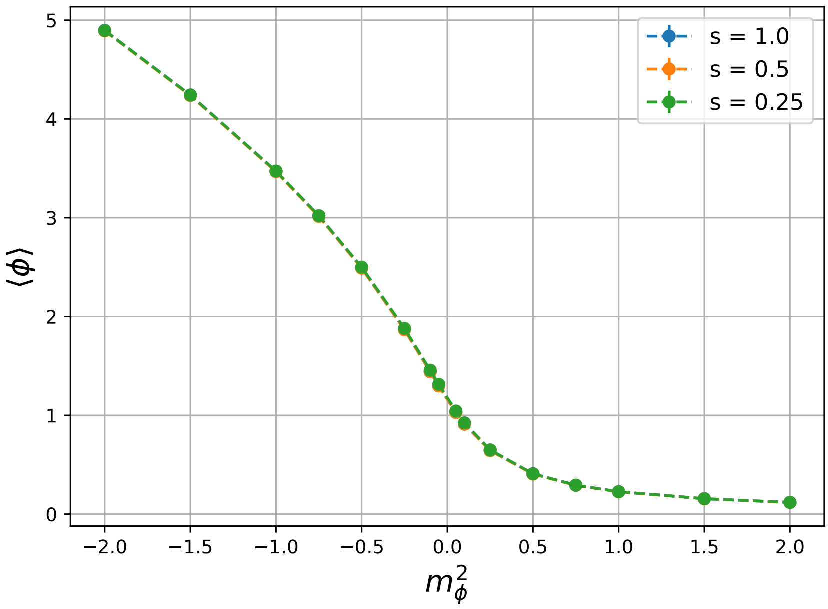

FIGURE 4.1

of χ, as reported in figure 4.2. It is clear that it converges towards XXX, implying that the system is not undergoing a phase transition.

Look at various things such as magnetization, mass, etc.
Even though is O(1) we do not observe SSB because of fermion bare quark mass.
Peak in the susceptibility does not imply P.T. $\rightarrow$ look at volume scaling.
When does L.O. rescaling ansatz breaks down?

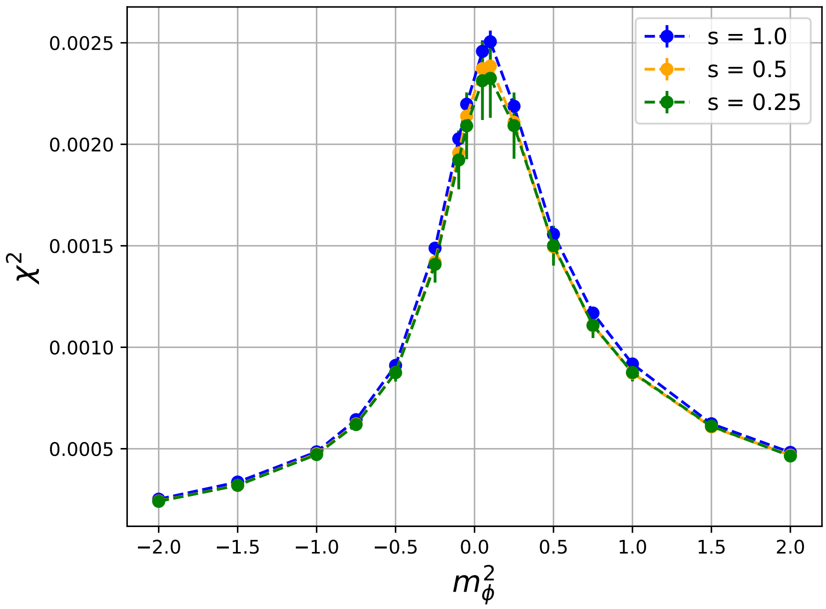

FIGURE 4.2

$$S[\phi,\bar{\psi},\psi] = \int_x \phi\left(\frac{\partial^2}{2}+\frac{m_\phi^2}{2}\right)\phi + \frac{\lambda}{4!}\phi^4 + \bar{\psi}\left(\not{\partial}+m_q+g\phi\right)\psi$$

$\lambda = 1.0$ $\quad m_\Phi^2 = 0.5$ $\quad N_t \times N_x = 8 \times 32$ $\quad m_q = 0.5$ $\quad N_{conf} = 5 \cdot 10^3$ $\quad \bar{\epsilon} = 0.01$

4.4 Classical to quantum interpolation

Let us start by analising the coloured noise field in the simulation and relevant properties that emerge from it. We consider the Yukawa model described by the continuum action **??**. In figure **??** the system is initialised in the same state for all the configurations, and then evolved with the Langevin equation with various noise fractions. The red line corresponds to the case $s = 0$, namely a classical simulation. The blue line corresponds to the case $s = 1$, namely the fully quantum case. As one can notice, the introduction of noise shifts the equilibrium expectation value of the field monotonically with the cutoff fraction: this is due to the fact that WHAT???. Note that a lower noise fraction is correlated to a faster convergence towards equilibrium. Moreover, low-distance fluctuations are suppressed due to the removal of the ultraviolet modes in the noise term.

For $\lambda = 0$ one has

$$\sigma = -\frac{g}{m_\phi^2 + k^2}\bar{\psi}\psi \tag{4.1}$$

In figure **??** one can see that equation (4.1) is verified also on the fully quantum level.

Figures **??** - **??** report a few observables as a function of the cutoff fraction s. In this case all the coupling constants are kept fixed while changing the value of s, in order to provide a smooth interpolation between the fully classical and fully quantum picture.

Each figure reports two plots corresponding to two different parameter configurations. The COLOR1 line corresponds to a system in the symmetric phase, while the COLOR2 line correspond to the broken phase. The exact parameters for the two configurations are reported under the figure.

4.5 Chiral fermions and a glimpse on the chiral phase transition

As explained in section **??**, in the continuum theory chiral symmetry can be broken either explicitly via a finite bare quark mass, or spontaneously if the field gains a non-zero expectation value. Moreoveor, in the discrete formulation, the introduction of the Wilson term also contributes to the explicit breaking of chiral symmetry add reference, as explained in section **??**. This, in particular, means that chiral symmetry is explicitly broken also for $m_q \to 0$. Because of this, one needs a new definition for bare mass M_q, which takes into account the Wilson term contribution, such that chiral symmetry is restored in the limit $M_q \to 0$ for vanishing expectation value of the field ϕ. A convenient way to define such M_q is the following. SSB chiral symmetry $\to$ 3 goldstone massless bosons, the pions. If the bare quark mass is zero, the physical mass of the pions has to be zero. Hence one can extract this mass on the lattice and tune m_q such that this is zero. While this is the correct way to proceed, it is very time takin since one must do mass scans and extrapolations close to singular Dirac operator. This is not the way we pursue here. Instead here we just consider naive fermions and take the limit $m_q \to 0$. This represents a physical theory with $2N_f = 4$ degenerate quarks. This is just done for the purpose of showing some interest properties of coloured noise and not (yet) to match any physical result.

In figure **??** some observables are reported as a function of the noise fraction s for different values of the bare quark mass. In the classical theory ($s = 0$) the order parameters $\langle|\phi|\rangle, \langle\bar{\psi}\psi\rangle$ are, in absolute value, bigger than in the quantum case ($s = 1$). As the chiral limit is approached $m_q \to 0$, the figure shows that the classical system lies in the broken phase, while in the quantum settings the symmetry is restored. Discuss general phase structure looking at both ϕ and $\bar{\psi}\psi$. One can see that as m_q is reduced, the systems shifts from a crossover to a second order phase transition, ahighligthed by the susceptibility χ^2 and the binder parameter U_L.

This would be better discussed as a function of bare scalar mass:
The difference gets more sharped as the base mass decreases, since the theory is closer to the chiral limit discussed above, which corresponds to the case $m_q = 0$. In this limit the systems goes under a second phase transition, highlighted by a peak in the susceptibility and and abrubt change in the Binder cumulant. In contrast, for finite mass, there is smooth crossover where the order parameters change continuously.

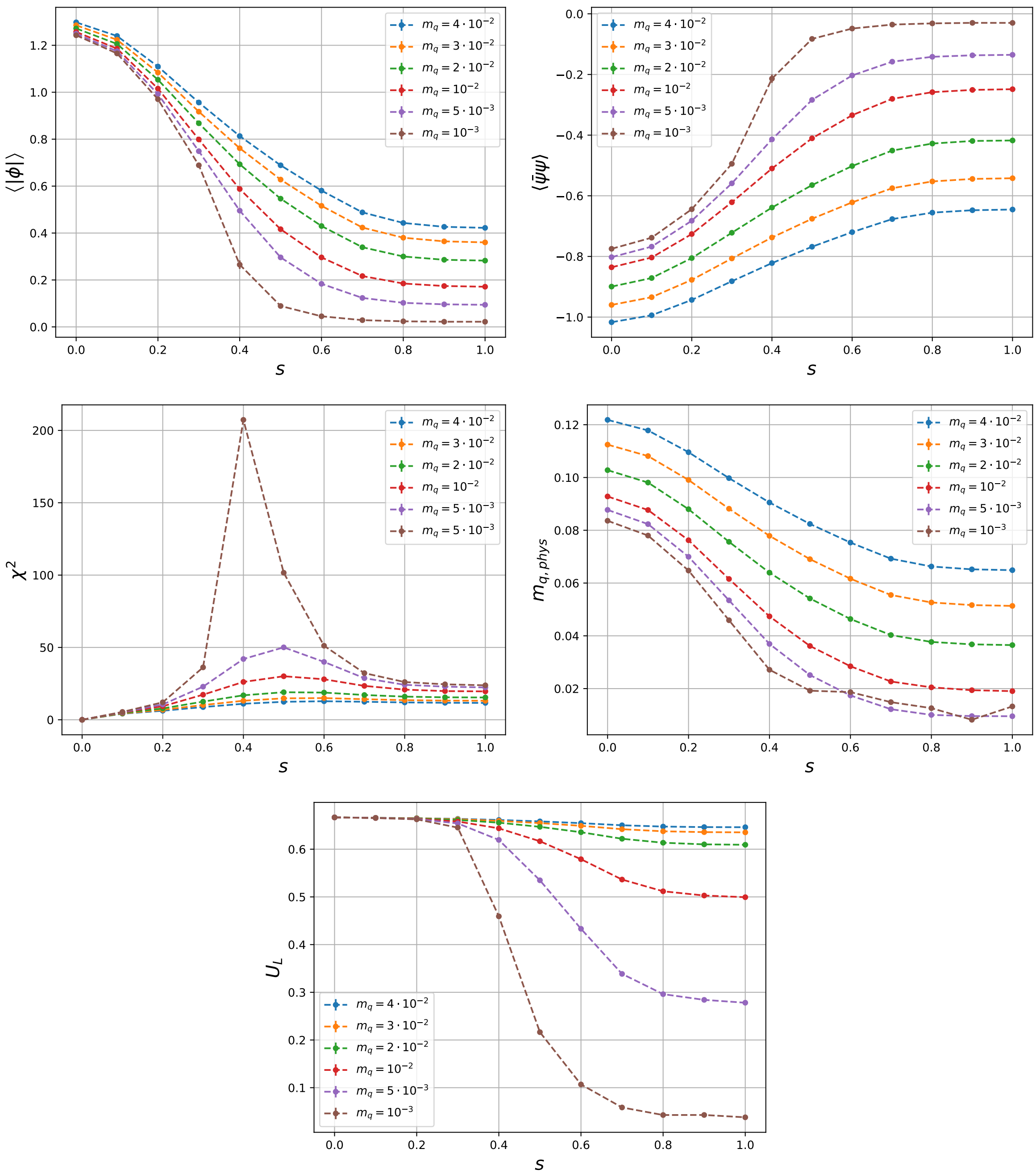

FIGURE 4.3: Chiral symmetry breaking

Chapter 5

Conclusions and outlook

Appendix A

Useful relations and definitions

In this appendix, useful relations and definitions are introduced. Fermionic wo-points function

$$\begin{aligned}
\left\langle \psi_{s,f}(x)\,\bar{\psi}_{s',f'}(y)\right\rangle &= \frac{1}{Z}\int \mathcal{D}\phi\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\;\psi(x)\,\bar{\psi}(y)\;\exp\left(-S_\phi-\psi D\psi+\bar{\eta}\psi+\bar{\psi}\eta\right)\\
&= \frac{1}{Z}\int \mathcal{D}\phi\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\;\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\;\exp\left(-S_\phi-\psi D\psi+\bar{\eta}\psi+\bar{\psi}\eta\right)\\
&= \frac{1}{Z}\int \mathcal{D}\phi\;\det\left[D(\phi)\right]\;\exp\left(-S_\phi\right)\;\frac{\delta}{\delta\bar{\eta}(x)}\frac{\delta}{\delta\eta(y)}\;\exp\left(\bar{\eta}D^{-1}\eta\right)\\
&= \left\langle \left[D^{-1}(\phi)\right]_{s,s',f,f'}(x,y)\right\rangle
\end{aligned} \tag{A.1}$$

The lattice version becomes

$$\langle\psi_m\,\bar{\psi}_n\rangle = \left\langle\left[D^{-1}(\phi)\right]_{mn}\right\rangle$$

with D beeing the Wilson Dirac operator.
From this, it follows straightforwardly

$$\langle\bar{\psi}\psi\rangle = \operatorname*{Tr}_{x,s,f} D^{-1}$$

where $\langle\bar{\psi}\psi\rangle = \sum\limits_{x,s,f}\left\langle\bar{\psi}_{s,f}(x)\,\psi_{s,f}(x)\right\rangle$.

The correlator is defined as

$$C(n_t,0) \equiv \frac{1}{N_x}\sum_{n_x}\left[\langle\psi(n_t,n_x)\,\bar{\psi}(0,0)\rangle + \langle\psi(N_t-n_t,n_x)\,\bar{\psi}(0,0)\right]\rangle$$

Note that we sum up two waves because the source propagates both forward and backward in time due to the boundary conditions.
Since for $t\to\infty$ one has that $C(t,p)\propto e^{-E_0(p)t}$, we expect

$$C(t,p)\approx\sinh\left(E_0\left(\frac{N_t}{2}-t\right)\right)$$

Pole mass, renormalized mass, effective mass, bare mass, physical mass
Effective action

$$S_{\text{eff}} = S_\phi + \operatorname{Tr}\log D(\phi)$$

Drift force

$$K_{\phi^j} = -\frac{\delta S}{\delta\phi^j} = -\frac{\delta S_\phi}{\delta\phi^j} - \operatorname{Tr}\left[D^{-1}\frac{\delta D}{\delta\phi^j}\right]$$

Appendix B

Wilson fermions

I am not really sure on whether to put this appendix or not, maybe I will just cite some papers that talk about Wilson fermions

choice of the basis

Appendix C

Algorithms and technical details

C.1 Conjugate Gradient algorithm and the Dirac operator

The full inversion of the Dirac operator is a very expensive computation, given that the Dirac operator has dimension $(2\, N_t\, N_x\, N_f)^2$, even though it is very sparse and has only few non-zero entries. One can note that for the purpose of computing the fermionic contribution to the drift force and the extraction of the physical quark mass from the correlator (details in section x and section y), only the inverse operator applied to a vector is needed. Hence it is sufficient to compute

$$\psi = D^{-1} \left|\eta\right\rangle \tag{C.1}$$

Computing ψ via equation (C.1) is equivalent to solve the linear system $D\psi = \eta$, which can be done efficiently by employing a method for sparse matrices such as Conjugate Gradient (CG) as explained in the following way.

We want to solve the equation

$$D\,\psi = \eta$$

CG requires the matrix to be hermitian while D is only γ^5-hermitian (really? under which assumptions?). One can thus solve the linear system

$$\left(DD^\dagger\right)\xi = \eta$$

and then obtain ψ by multiplying the solution ξ by $D^\dagger$ since

$$D^\dagger\xi = D^\dagger\ \left(DD^\dagger\right)^{-1}\eta = D^{-1}\eta = \psi \tag{C.2}$$

Analogously one can calculate

$$\chi = D^\dagger\eta$$

by solving

$$\left(D^\dagger D\right)\xi = \eta$$

and then applying D to the result.

One can improve the solution via CG by solving a *preconditioned* equation. Suppose that we want to solve the equation

$$Mx = b$$

via CG.
Let us express the matrix A as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{C.3}$$

We introduce the Schur complement of M

$$M/D = A - BD^{-1}C \tag{C.4}$$

This allows one to write M (LDU decomposition, Gaussian elimination) as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbf{1}_p & -BD^{-1} \\ 0 & \mathbf{1}_q \end{bmatrix} \begin{bmatrix} M/D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbf{1}_p & 0 \\ D^{-1}C & \mathbf{1}_q \end{bmatrix} = LAR$$

Which allows for an easy block inversion

$$M^{-1} = \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix} \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & -BD^{-1} \\ 0 & I_q \end{bmatrix} = L^{-1}A^{-1}R^{-1}$$

The equation to solve now reads

$$x = L^{-1}A^{-1}R^{-1}\,b \qquad \text{or} \qquad y = A^{-1}c$$

with $y = Lx$ and $c = R^{-1}b$. One can then solve the equation $y = A^{-1}c$ and get the solution x by applying L^{-1} to x.

An example of preconditioning is the even-odd preconditioning. Let us write the dirac operator in the form of equation (C.3) in the following way

$$M = \begin{pmatrix} M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{pmatrix}$$

The Schur complement (C.4) is

$$\hat{M} \equiv M/M_{oo} =$$

C.2 Bilinear noise scheme

$$\mathrm{Tr}\left[D^{-1}\frac{\delta D}{\delta\phi^j}\right] \approx \langle\eta|\,D^{-1}\frac{\delta D}{\delta\phi^j}\,|\eta\rangle = \langle\psi|\,\frac{\delta D}{\delta\phi^j}\,|\eta\rangle \qquad |\psi\rangle = D^{-1}\,|\eta\rangle = D^{\dagger}\,\underbrace{(DD^{\dagger})^{-1}\,|\eta\rangle}_{\mathrm{CG}}$$

$$\mathrm{Tr}\,A = \frac{1}{N}\lim_{N\to\infty}\sum_{i}^{N}\eta_i^T D_{ij}\eta_j \tag{C.5}$$

where η_i is a gaussian random field where each component is drawn from a normal distribution $\mathcal{N}(0,1)$.
More precisely each vector component η_i^α satisfies

$$\langle\eta_i^\alpha\rangle = 0 \qquad \left\langle\eta_i^\alpha\eta_j^\beta\right\rangle = \delta_{i,j}\,\delta^{\alpha\beta}$$

The series (C.5) requires in principle an infinite number of vectors to evaluate the trace exactly. In practice we truncate it and choose $N = 1$:D :D. The average over Monte Carlo samples will eventually converge nevertheless to the right result.

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