

Department of Physics and Astronomy
University of Heidelberg

Master thesis

in Physics

submitted by

(name and surname)

born in (place of birth)

(year of submission)

Simulating effective field theories
on a space-time lattice with coloured noise

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at the

Institute for Theoretical Physics in Heidelberg

under the supervision of

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(Titel in Deutsch): (Abstract in Deutsch, max. 200 Worte)

(Title in English): (abstract in english, at most 200 words)

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27.11.2023

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Contents

1	Introduction and outline	1
1.1	Quantum chromodynamics and its phase diagram	1
1.2	Effective theories	1
2	Theoretical background	3
2.1	The renormalisation group	3
2.1.1	Block-spin RG	3
2.1.2	Wilsonian RG	4
2.2	Lattice QFT and the continuum limit	6
2.3	Stochastic quantisation	7
2.3.1	Standard stochastic quantisation	8
2.3.2	Stochastic quantisation with coloured noise	9
2.4	Yukawa theory	11
2.4.1	Description of the model	11
2.4.2	Chiral symmetry breaking	12
2.4.3	Phase structure and order parameters	14
3	Methods and algorithms	15
3.1	Discretisation of the Yukawa theory	15
3.2	Langevin Monte Carlo	17
3.3	Applications of coloured noise in lattice QFT	17
3.3.1	Classical to quantum interpolation	17
3.3.2	Noise induced transition	17
3.3.3	Cooling and the continuum limit of effective theories	18
3.3.4	Control over temperature	19
3.4	Definition of relevant observables	19
4	Numerical investigation	21
4.1	The fermionic correlator	21
4.2	Classical to quantum interpolation	21
4.3	Cooling with coloured noise	23
4.4	Chiral fermions and a glimpse on the chiral phase transition	26
5	Conclusions and outlook	29
A	Useful relations and definitions	31
B	Wilson fermions	33
C	Algorithms and technical details	35
C.1	Conjugate Gradient algorithm and the Dirac operator	35
C.2	Bilinear noise scheme	36
	Bibliography	39

List of Figures

2.1	Block-spin transformation	4
2.2	Correlated noise	10
2.3	Classical potential and symmetry breaking	13
2.4	Yukawa phase diagram	14
4.1	Correlator as a function of the bare mass	22
4.2	CG iterations	22
4.3	Chiral symmetry breaking	27

List of Tables

“Grazie a tutti.”

Matteo Zortea

Chapter 1

Introduction and outline

1.1 Quantum chromodynamics and its phase diagram

Big picture: here we talk about QCD and the problem of the phase diagram

1.2 Effective theories

Here we first define effective theories and discuss their usefulness, then introduce RG as a technique to resolve physics at different scales.

Motivation for choices, connection to stochastic regularisation, complex langevin,

easy noise [boo] Motivation for choices,

The starting set up is the euclidean formulation of quantum field theory, where one typically defines a path integral Z , which, for a general scalar field $\phi(x)$ and a fermion field $\psi(x)$, assumes the form

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\phi, \psi, \bar{\psi}]} \quad \mathcal{D}\xi = \prod_x d\xi_x, \quad \xi \in \{\phi, \psi, \bar{\psi}\} \quad (1.1)$$

One then aims at computing correlation functions via

$$\langle \xi_{x_1} \dots \xi_{x_n} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \xi_{x_1} \dots \xi_{x_n} e^{-S[\phi, \psi, \bar{\psi}]} \quad \xi_{x_i} \in \{\phi_{x_i}, \psi_{x_i}, \bar{\psi}_{x_i}\}$$

Computing physical quantities from such a direct approach is not only hard to do, but results often impossible due to the appearance of divergences in the calculations. To fix this, one often relies on expansion techniques such as perturbation theory (**CITATION**), in which one tries to regularise the theory order by order in an expansion on the interaction coupling, yielding finite quantities that depend on the truncation order. While this method is capable of producing incredibly precise results (**g-2, fine structure, ...**), it fails completely in treating non-perturbative phenomena, namely effects that cannot be captured by any order in the expansion or that are typical of strongly interacting systems. Example of such systems range from **QCD, cold atoms, plasma, stuff**. Moreover, such formulation is also not much suitable for numerical computations, since both the path integral and action measure are infinite dimensional objects.

Lattice field theory [1–4] is meant at first as a powerful non-perturbative regularisation tool to prevent divergences to occur and render the computation of the correlation functions finite. Moreover, it also provides a framework to study quantum field theory numerically on a computer. In order to accomplish this, one typically defines the theory on a spacetime lattice and makes use of statistical methods such as Monte

Carlo algorithms to compute observables. One may wonder how can one reconstruct the results in the continuum theory, keeping the results finite, and matching the results on the discretised theory to physically measured ones. This task, far from being simple, will be the focus of the next sections, in which we will first introduce relevant theoretical tools, such as the renormalisation group, and then discuss the existence of a continuum limit of a lattice theory and, if it exists, how it can be extracted. This will motivate the introduction of coloured noise in the context of continuum limits of effective theories, a technique which will be shown to be powerful also for other various reasons, which will be the main focus of the analysis carried in the remaining chapters.

We want to mention that the Yukawa theory (2.14) includes, as a special case, the Gross-Neveu model [citation](#), which in turn is the 1+1 dimensional version of the Nambu-Jona-Lasinio model [5, 6]. More precisely, after dynamical bosonisation via a Hubbard-Stratonovich transformation [either cite or show](#), the latter theories reduce to a Yukawa model with

$$K(x, y) = m_\phi^2 \delta(x, y) \quad \lambda = 0.$$

[Da qualche parte cita \[7\].](#)

Chapter 2

Theoretical background

In this chapter we want to provide with an overview on the general theoretical framework that supports this work, and introduce the main concepts for the successive parts. Each section in this chapter is, by no means, meant as an exhaustive treatment. The description will be quite conceptual, rather than technical, and aims at recalling the main ideas and fix conventions. We ask the reader to consult appropriate references, which will be given in the corresponding sections, for a more detailed treatment of the topics.

2.1 The renormalisation group

Landau's mean field approach to study phase transition [8] gained wide popularity in the 1930's and 40's, since it was able to describe critical properties of many systems and it provided inspiration for the later Landau-Ginzburg theory of superconductivity [9]. Thus, it has soon proved to be inaccurate to predict some experimentally well proven properties of certain systems near their critical point [10]. This is because, being a mean field theory, it did not take into account the role of spatial fluctuations. The idea of block-spin transformation, systematically developed by Kadanoff [11], made a big step towards a deeper understanding of the scaling behaviour, and posed the basis for the later work of Wilson [12–14], which still constitutes the basis for modern approaches to renormalisation in field theory and statistical physics.

2.1.1 Block-spin RG

To illustrate the idea, let us consider a set of spins whose magnetisation is described by a function $\phi(x)$. The spins are located on a discrete lattice \mathcal{L} with spacing a , so that the function assumes values only at such sites $\phi(x_i) = \phi_i \neq 0 \Leftrightarrow x_i \in \mathcal{L}$. Suppose then that their interaction is described by a certain action $S[\phi]$ and a partition function

$$Z = \sum_{\phi} e^{-S[\phi]}.$$

We now want to introduce a coarse-grained (or blocked) field $\bar{\phi}$ within a spacetime cell of volume \mathcal{V} . Such a coarse grained field can be defined, for example, as an average over the spins within the cell \mathcal{V} . If the spins can only be 0 or 1 like in a Ising model, then we might opt for a majority rule [15]. We now want to find a new action S_b such that

$$Z = \sum_{\phi} e^{-S(\phi)} = \sum_{\bar{\phi}} e^{-S_b(\bar{\phi})}. \quad (2.1)$$



FIGURE 2.1: The two steps of the block-spin transformation. The black dots indicate the original field ϕ , while the blue dots indicate the coarse grained field $\bar{\phi}$.

Suppose for the moment that such S_b has been found. The coarse-graining procedure comes with a loss of resolution since the spacing is changed $a \rightarrow 2a$. Hence one can rescale distances and momenta in the new action via $a \rightarrow a/2, p \rightarrow 2p$ and then compare the result with the initial action. This constitutes the second step of the block-spin transformation. The whole procedure can be iterated multiple times, and can be thought as a zoom-out with a corresponding coarse graining, in order to describe the system in terms of the relevant scales as pictured in figure 2.1. As made clear in the picture, the physical correlation length is reduced by the block-spin step, since the description is now in terms of the coarse field $\bar{\phi}$. There are only two exceptions for this, either the correlation length is zero, or infinite. The latter case represents a fixed-point of the RG, and the system exhibits scale invariance.

Note that the condition (2.1) is non-trivial. One can, in principle, build an ad-hoc action that fulfills the above condition, but it is complicated, since S_b can be also be very different from S . For example, if the action $S[\phi]$ contains only nearest-neighbour interactions, the new action $S'[\bar{\phi}]$ can contain higher order interactions such as nearest-to-nearest neighbour, or even more. In principle, all the terms compatible with the original symmetries are allowed, and one has often to rely on some approximations.

For an example of the explicit construction of the RG for an Ising model, see [15].

2.1.2 Wilsonian RG

The wilsonian picture of renormalisation [12, 13] is formulated in momentum space and in general is more suitable for theories in the continuum.

The idea is that a physical theory observed at a energy scale Λ can be seen as an effective theory of a more fundamental one, defined at scale $\Lambda_0 > \Lambda$. To see how can this happen, let us consider a theory defined by the action $S[\phi]$ defined at scale Λ_0 , and let us split the field as $\phi = \bar{\phi} + \varphi$ where $\bar{\phi}$ are fields with momenta $p^2 \leq \Lambda^2$ and φ are fields with momenta $\Lambda^2 < p^2 \leq \Lambda_0^2$. This allows to split the action as $S[\bar{\phi} + \varphi] = S_\Lambda[\bar{\phi}] + \delta S[\bar{\phi}, \varphi]$, where all the dependence on φ is in $\delta S[\bar{\phi}, \varphi]$. The path integral can be rewritten as

$$\begin{aligned} Z &= \int D\phi e^{-S[\phi]} = \int D\bar{\phi}_\Lambda e^{-S_\Lambda[\bar{\phi}]} \int D\varphi e^{-\delta S[\bar{\phi}, \varphi]} \\ &= \int D\bar{\phi}_\Lambda e^{-S_\Lambda[\bar{\phi}] + S_{UV}[\bar{\phi}]} \\ &= \int D\bar{\phi}_\Lambda e^{-S_\Lambda^{\text{eff}}[\bar{\phi}]}, \end{aligned}$$

where

$$S_{UV}[\bar{\phi}] = \log \left(\int D\varphi e^{-S_\Lambda[\bar{\phi}, \varphi]} \right).$$

encodes all the information of UV modes with $\Lambda^2 < p^2 \leq \Lambda_0^2$. The action $S_\Lambda[\bar{\phi}]$ has the same functional form of the initial action $S[\phi]$, but it is now defined only for fields with momenta $p^2 \leq \Lambda^2$. Instead, $S_\Lambda^{\text{eff}}[\bar{\phi}]$ constitutes an effective action at scale Λ and depends only on degrees of freedom with $p^2 \leq \Lambda^2$. In order to set momenta back at their original scales one can operate a rescaling of distances and momenta to complete the Wilson RG step, according to the parameter

$$s^2 = \Lambda^2 / \Lambda_0^2$$

If d_g is the energy dimension of a coupling g in the action, then, the rescaling transformation causes

$$g \rightarrow s^{d_g} g.$$

The same applies to the fields and other dimensionful quantities. If the corrections from S_{UV} are negligible, such as for high cutoff $\Lambda \approx \Lambda_0$, then this is the only contribution to the change in the couplings and fields due to the Wilson step.

At this point, one can clearly see the analogy with the block-spin transformation introduced in the previous section. Performing the integral over high momenta modes can be thought as performing averages (coarse graining) over neighbours. This causes a loss of resolution which can be recovered by rescaling distances and momenta. Therefore the philosophy of Kadanoff and Wilson approaches, was that the blocking transformation reduces the complexity of many-body systems by systematically reducing the number of degrees of freedom being taken into account, without changing the physical content of the theory [16].

We want to conclude by mentioning that the splitting of the action can be done by writing

$$S_\Lambda^{\text{eff}}[\bar{\phi}] = S[\phi] + \Delta S_\Lambda^{(\text{IR})}[\bar{\phi}],$$

where $\Delta S_\Lambda^{(\text{IR})}[\hat{\phi}]$ is a regulating function which typically assumes the form

$$\Delta S_\Lambda^{(\text{IR})}[\bar{\phi}] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \bar{\phi}(-p) \Lambda^2 \left(\frac{1}{r_\Lambda(p^2)} - 1 \right) \bar{\phi}(p), \quad (2.2)$$

and for a sharp momentum cutoff

$$r_\Lambda(p^2) = \theta(p^2 - \Lambda^2).$$

This result will allow for a deep connection between coloured stochastic quantisation and the functional renormalisation group, the latter describing the functional dependence of $S_\Lambda^{\text{eff}}[\bar{\phi}]$ on the cutoff scale Λ [citationsssss](#).

2.2 Lattice QFT and the continuum limit

The starting set up is the euclidean formulation of quantum field theory, where one typically defines a path integral Z , which, for a general scalar field $\phi(x)$ and a fermion field $\psi(x)$, assumes the form

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\phi, \psi, \bar{\psi}]}, \quad \mathcal{D}\xi = \prod_x d\xi_x, \quad \xi \in \{\phi, \psi, \bar{\psi}\}, \quad (2.3)$$

and correlation functions are computed via

$$\langle \xi_{x_1} \dots \xi_{x_n} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \xi_{x_1} \dots \xi_{x_n} e^{-S[\phi, \psi, \bar{\psi}]}, \quad \xi_{x_i} \in \{\phi_{x_i}, \psi_{x_i}, \bar{\psi}_{x_i}\}.$$

Let us then consider a lattice \mathcal{L} , with spacing a , and N_μ points in each spacetime direction μ , hence a physical length L_μ . For simplicity we restricted here to a scalar field ϕ and we will recall fermionic properties only when relevant, but what follows has general validity. The action and the path integral measure are now taken over discrete quantities

$$\begin{aligned} S = \int d^d x \mathcal{L}(\phi(x)) &\rightarrow S = a^d \sum_{n \in \mathcal{L}} \mathcal{L}(\phi(n)), \\ \prod_x d\phi(x) &\rightarrow \prod_{n \in \mathcal{L}} d\phi(n), \end{aligned}$$

where $\mathcal{L}(\phi)$ is the Lagrangian density function.

The path integral is hence

$$Z = \int \prod_n d\phi(n) e^{-S[\phi]},$$

and the probability of a field configuration ϕ

$$p(\phi) = \frac{1}{Z} e^{-S[\phi]}. \quad (2.4)$$

Expectation value of observables are computes as

$$\langle O(\phi) \rangle = \frac{1}{Z} \int \prod_n d\phi(n) O(\phi) e^{-S[\phi]}. \quad (2.5)$$

In order to simulate a theory and perform the above sums one has to go to finite volumes impose boundary conditions. In the space directions, we take periodic conditions

$$\begin{aligned} \phi(t, \vec{x}) &= \phi(t, \vec{x} + T), \\ \psi(t, \vec{x}) &= \psi(t, \vec{x} + T). \end{aligned}$$

Instead, a finite time extent is related to the temperature of the system [2, 17] via

$$\beta = 1/T = 1/L_t,$$

and boundary conditions are chosen depending on the spin-statistic of the corresponding particles, namely periodic conditions for bosons, and anti-periodic for fermions

$$\begin{aligned}\phi(t, \vec{x}) &= \phi(t + T, \vec{x}) && \text{bosons,} \\ \psi(t, \vec{x}) &= -\psi(t + T, \vec{x}) && \text{fermions.}\end{aligned}$$

Such a formulation naturally brings a momentum cutoff $\Lambda = \pi/a$ since now all the momenta are restricted to the first Brillouine zone $p_\mu \in [-\pi/a, \pi/a]$.

To compute observables one relies on Monte-Carlo methods to generate field configurations, sampling the distribution (2.4) and convergence to the statistical value given by (2.5) is expected in the limit of infinite samples $N_{\text{samp}} \rightarrow \infty$. To recover the continuum results, one has to take $V \rightarrow \infty, a \rightarrow 0$ ¹, but this task cannot be done so straightforwardly. Continuum limits of lattice theories are intimately connected to the existence of critical points. To see why this is the case, consider the dimensionless mass gap

$$\hat{\xi} = m a$$

of a certain theory. The quantity ξ is also called correlation length and it is related to the exponential decay of correlation functions between local observables measured at different points on the lattice, as given by [add ref. eq.](#)

When taking the continuum limit we want $a \rightarrow 0$, while having a finite physical mass m . This implies that the correlation length $\hat{\xi}$ has to diverge: in the language of statistical physics, this is a second order phase transition. Of course to bring the system at its critical point, where such phase transition happens, one has to tune the bare parameters g_0^i of the theory to their critical values g_0^{i*} . To do this, one should find the zeros of the beta functions

$$\beta_g^{\text{latt}} = a \frac{d}{da} g(a) \stackrel{!}{=} 0, \quad a = \pi/\Lambda.$$

As this is quite hard task to do on the lattice, one typically relies on some approximation schemes such as employing perturbative continuum beta functions.

In any case, from this description it should be clear that the spacing a should not be treated as a free parameter in the continuum limit, but rather a dynamically determined quantity that depends on the couplings of the theory.

Note that in the limit $a \rightarrow 0$, one has $\Lambda \rightarrow \infty$. If one's scope is to simulate an effective theory which is expected to hold only up to a scale Λ_{phys} , one must have $\Lambda \leq \Lambda_{\text{phys}}$, with a consequent lower bound on the lattice spacing $a \geq a_{\text{phys}} = \pi/\Lambda_{\text{phys}}$.

2.3 Stochastic quantisation

The idea of stochastic quantisation [19, 20] is that Euclidean Quantum Field theory can be thought as a system in thermal equilibrium with a heat reservoir and hence described as a stochastic process via the Langevin equation. For this, one has to introduce a fictitious time variable τ that labels the state $\phi(\tau, x)$ of the system during

¹the order here is important, see for example [18]

the evolution.

2.3.1 Standard stochastic quantisation

Let us consider, for example, a scalar field ϕ with a Euclidean action $S[\phi]$ and the following Langevin equation

$$\partial_\tau \phi(\tau, x) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta(\tau, x), \quad (2.6)$$

where $K_\phi(\tau) \equiv -\delta S[\phi]/\delta \phi(\tau, x)$ is the drift term and $\eta(\tau, x)$ is a random white noise field assumed to be normally distributed

$$P(\eta) = \frac{\exp\left(-\frac{1}{4} \int_{\tau, x} \eta^2(\tau, x)\right)}{\int D\eta \exp\left(-\frac{1}{4} \int_{\tau, x} \eta^2(\tau, x)\right)},$$

which, in particular, implies

$$\langle \eta(x, \tau) \rangle = 0, \quad \langle \eta(x, \tau) \eta(x', \tau') \rangle = 2 \delta(x - x') \delta(\tau - \tau'). \quad (2.7)$$

Stochastic average with respect to the measure $P(\eta)$ are computed via

$$\langle A(\eta) \rangle = \int D\eta P(\eta) A(\eta).$$

In momentum space, (2.7) becomes

$$\begin{aligned} \langle \eta(p, \tau) \rangle &= \left\langle \int_x e^{ipx} \eta(x, \tau) \right\rangle = \int_x \langle e^{ipx} \eta(x, \tau) \rangle = 0, \\ \langle \eta(p, \tau) \eta(q, \tau') \rangle &= \left\langle \int_{xy} e^{ipx+iqy} \eta(x, \tau) \eta(y, \tau') \right\rangle \\ &= \int_{xy} e^{ipx+iqy} \langle \eta(x, \tau) \eta(y, \tau') \rangle \\ &= 2 (2\pi)^2 \delta(p + q) \delta(\tau - \tau'). \end{aligned} \quad (2.8)$$

In absence of the noise term $\eta(\tau, x)$, equation (2.6) simply represents an evolution of the field towards the minimum of the action, and at equilibrium the field is constrained to $\partial_\tau \phi(x, \tau) = 0 = \delta S[\phi]/\delta \phi(\tau, x)$, namely to the classical equations of motion.

For any observable O , which is function of the field, one has, for fixed time τ ,

$$\langle O(\phi(\tau)) \rangle = \int D\eta P(\eta) O(\phi(\tau)),$$

from which it follows straightforwardly using the Langevin equation and $\langle \eta \rangle = 0$

$$\frac{d}{d\tau} \langle O(\phi(\tau)) \rangle = \left\langle \frac{\delta O}{\delta \phi(\tau, x)} \partial_\tau \phi(\tau, x) \right\rangle = - \left\langle \frac{\delta O}{\delta \phi}(\tau, x) \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle.$$

It follows trivially that for $O(\phi(\tau)) = \phi(\tau, x)$

$$\frac{d}{d\tau} \langle \phi(\tau, x) \rangle = - \left\langle \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle \xrightarrow{\text{Equilibrium}} \left\langle \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle = 0.$$

This also provides a consistency check for the correct implementation of the simulation, since the drift $K_\phi = -\delta S / \delta \phi$ is computed numerically during the evolution. More generally, one can derive a correspondent Fokker-Planck equation [21], which can be proven to have a stationary distribution if the action is bounded from below, given by [20]

$$\mathcal{P}(\phi) = \frac{1}{Z} \exp(-S[\phi]). \quad (2.9)$$

This allows one to compute correlation functions as moments of the probability distribution (2.9). In particular, one has

$$\langle O \rangle_{P(\eta)} = \langle O \rangle_{\mathcal{P}(\phi)} \equiv \langle O \rangle. \quad (2.10)$$

2.3.2 Stochastic quantisation with coloured noise

In the stochastic quantisation procedure the noise which accounts for the quantum fluctuations of the theory is assumed to be white, as defined in equations (2.7), (2.8). We now want to examine the dynamics in presence of a coloured noise, writing the Langevin equation as

$$\partial_\tau \phi(x, \tau) = - \frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta_{\text{col}}(x, \tau),$$

with $\eta_{\text{col}}(x, \tau) = r_\Lambda(x) \eta(x, \tau)$. In particular, here we restrict to the regulating function defined as a sharp cutoff in momentum space

$$r_\Lambda(p) = \theta(\Lambda^2 - p^2), \quad (2.11)$$

and we invite the reader to consult [22] for a discussion of other regulating functions. The noise field in momentum space is then

$$\begin{aligned} \eta_{\text{col}}(p, \tau) &= \mathcal{F}[\eta_{\text{col}}(x, \tau)] = \mathcal{F}[r_\Lambda(x, \tau) \eta(x, \tau)] = \mathcal{F}[r_\Lambda(x, \tau)] \star \mathcal{F}[\eta(x, \tau)] \\ &= \theta(\Lambda^2 - p^2) \eta(p, \tau). \end{aligned}$$

where \mathcal{F} indicates the Fourier transform and \star the convolution product.

An interesting quantity to look at is the position-space noise correlation function

$$\begin{aligned} C_\eta(x, \tau, y, \tau') &= \langle \eta_{\text{col}}(x, \tau) \eta_{\text{col}}(y, \tau') \rangle \\ &= \frac{1}{(2\pi)^4} \int D\eta P(\eta) \left[\int_{p,q} e^{-ipx-iqy} \eta_{\text{col}}(p, \tau) \eta_{\text{col}}(q, \tau') \right] \\ &= \frac{1}{(2\pi)^4} \int_{p,q} e^{-ipx-iqy} \int D\eta [P(\eta) \eta(p, \tau) \eta(q, \tau')] \\ &\quad \times \theta(\Lambda^2 - p^2) \theta(\Lambda^2 - q^2) \\ &= \frac{2}{(2\pi)^2} \int_{p,q} e^{-ipx-iqy} \delta(p+q) \theta(\Lambda^2 - p^2) \theta(\Lambda^2 - q^2) \delta(\tau - \tau') \\ &= \frac{2}{(2\pi)^2} \int_p e^{-ip(x-y)} \theta(\Lambda^2 - p^2) = \frac{1}{\pi} \int_0^\Lambda d\omega \omega J_0(\omega|x-y|), \end{aligned} \quad (2.12)$$

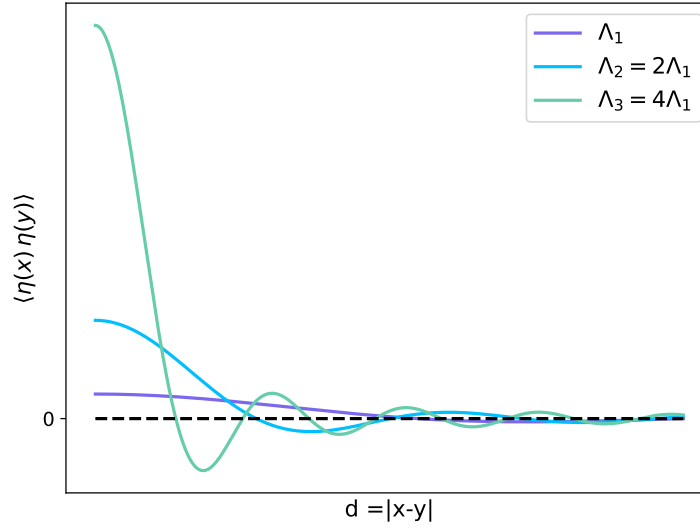


FIGURE 2.2: Noise correlation as a function of $d = |x - y|$ for three different values of the cutoff $\Lambda_1 < \Lambda_2 < \Lambda_3$, in arbitrary units. The plot is qualitative, but shows clearly that with a regulated noise, only the short-distance behaviour is affected.

where $J_0(x)$ is a Bessel function of the first order. The integral in the last line is computed numerically as a function of $d = |x - y|$ and reported in figure 2.2 for three different values of the cutoff $\Lambda_1 < \Lambda_2 < \Lambda_3$. This shows nicely that for $|x - y| \ll 1/\Lambda$ the noise is now correlated, while for $|x - y| \gg 1/\Lambda$ the correlation function vanishes, as in the white noise case. In other words, only the short-length behaviour of the system is affected by the introduction of such a regulating term, as one could expect.

Another intuitive and interesting aspect of the dynamics in the presence of coloured noise can be deduced by looking at the field expression in terms of the retarded Langevin Green function [20], which is here not derived, but reported from [22]

$$\phi(x, \tau) = \int_{x'} \int_{-\infty}^{\tau} d\tau' G(x - x', \tau - \tau') \left[r_{\Lambda}(\Delta_x) \eta(x, \tau') - \frac{\delta S}{\delta \phi} \Big|_{p=0} \phi(x', \tau) \right],$$

where

$$G(x - x', \tau - \tau') = \theta(\tau - \tau') \int_p e^{-ip \cdot (x - x')} e^{-(\tau - \tau')(p^2 + m^2)}.$$

By looking at the first term in the square bracket, one can conclude that there is no propagation of modes with momentum $p^2 \geq \Lambda^2$ due to the noise term, but one can still have contribution from modes $p^2 > \Lambda^2$ from the second term, which corresponds to the deterministic part of the equations of motion. Stated differently, UV quantum fluctuations with $p^2 > \Lambda^2$ are removed from the dynamics of ϕ , but still contribute classically.

Generally speaking, the stationary distribution probability of the regulated stochastic process is given by [22]

$$\mathcal{P}_{\Lambda}(\phi) = \frac{1}{Z} \exp(-S_{\Lambda}[\phi]) = \frac{1}{Z} \exp(-(S[\phi] + \Delta S_{\Lambda}[\phi])), \quad (2.13)$$

where the correction term $\Delta S_\Lambda[\phi]$, reads, for a regulator $r_\Lambda(p^2)$,

$$\Delta S_\Lambda[\phi] = \frac{1}{2} \int_p \phi_p \Lambda^2 \left(\frac{1}{r_\Lambda(p^2)} - 1 \right) \phi_{-p}.$$

at this point mention that this is the stationary pdf that one gets with frg for sharp cutoff, and cite some papers.

2.4 Yukawa theory

2.4.1 Description of the model

Let us consider the Yukawa theory defined by the action

$$\begin{aligned} S[\phi, \psi, \bar{\psi}] &= S_\phi[\phi] + S_\psi[\psi, \bar{\psi}] + S_{\text{int}}[\phi, \psi, \bar{\psi}] \\ S_\phi[\phi] &= \int_x \phi_x \left(-\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \phi_x + \frac{\lambda}{4!} \phi_x^4 \\ S_\psi[\psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} (\not{\partial}_x + m_q) \psi_x^{(f)} \\ S_{\text{int}}[\phi, \psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} g \bar{\psi}_x^{(f)} \phi_x \psi_x^{(f)} \end{aligned} \quad (2.14)$$

One can see that the action is made of a scalar part $S_\phi[\phi]$, a fermionic part $S_\psi[\psi, \bar{\psi}]$ and a Yukawa interaction term $S_{\text{int}}[\phi, \psi, \bar{\psi}]$.

In practice we will work with fixed number of flavours $N_f = 2$, but it is useful to keep track of N_f and set it to its value when needed.

It is also convenient for later purposes to define the operators K, D represented in position space as

$$\begin{aligned} K(x, y) &= \left(-\partial_x^2 + m_\phi^2 \right) \delta(x, y) \\ D(x, y) &= (\not{\partial}_x + m_q + g\phi) \delta(x, y) \end{aligned} \quad (2.15)$$

and in momentum space as

$$\begin{aligned} \tilde{K}(p, q) &= \int_{x, y} e^{-ipx} \left(\partial_x^2 + m_\phi^2 \right) \delta(x, y) e^{iqy} = \left(\frac{p^2}{2} + \frac{m_\phi^2}{2} \right) \delta(p, q) \\ \tilde{D}(p, q) &= \int_{x, y} e^{-ipx} (\not{\partial}_x + m_q + g\phi) \delta(x, y) e^{iqy} = (\not{p}_x + m_q + g\phi) \delta(p, q) \end{aligned} \quad (2.16)$$

This allows one to rewrite the action as

$$S[\phi, \psi, \bar{\psi}] = \int_x \frac{1}{2} \phi_x K_{xx} \phi_x + \frac{\lambda}{4!} \phi_x^4 + \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} D_{xx} \psi_x^{(f)}$$

It is now useful to adopt the notation

$$\psi = (\psi^{(1)}, \dots, \psi^{(N_f)})$$

We then introduce left-handed and right-handed spinors

$$\psi_L = (1 - \gamma_5) \psi \quad \psi_R = (1 + \gamma_5) \psi$$

for which

$$\psi = \frac{(1 - \gamma_5)}{2} \psi + \frac{(1 + \gamma_5)}{2} \psi = \psi_L + \psi_R$$

The action written in terms of ψ_L, ψ_R , reads

$$S = S_\phi + \int_x [\bar{\psi}_L D \psi_L + \bar{\psi}_R D \psi_R + (m_q + g\phi) (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)] \quad (2.17)$$

The last equation makes clear that for $m = 0, \langle \phi \rangle = 0$ the action is symmetric under the chiral group $SU(N_f)_L \times SU(N_f)_R$, namely

$$\begin{aligned} \psi_L(x) &\rightarrow U_L \psi_L(x) & \bar{\psi}_L(x) &\rightarrow \bar{\psi}_L(x) U_L^\dagger \\ \psi_R(x) &\rightarrow U_R \psi_R(x) & \bar{\psi}_R(x) &\rightarrow \bar{\psi}_R(x) U_R^\dagger \end{aligned}$$

for $U_L, U_R \in SU(N_f)$. To be more precise, the full invariance group of the classical action is

$$SU(N_f)_A \times SU(N_f)_V \times U(1)_A \times U(1)_V$$

where the axial and vector symmetry transformations are

$$\begin{aligned} \psi &\rightarrow \dots \bar{\psi} \rightarrow \dots \\ \psi &\rightarrow \dots \bar{\psi} \rightarrow \dots \end{aligned}$$

One can thus prove that axial symmetry is broken by quantum anomalies (see for example [23]), so that the massless quantum theory has the symmetry

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V$$

If equal masses are introduced and flavours are degenerate, then the symmetry group reduces to

$$SU(N_f)_V \times U(1)_V$$

where $SU(N_f)_V$ is defined by its action on the spinors

$$\psi \rightarrow$$

Finally, if the fermions have different masses, the symmetry group reduces to

$$\left(\prod_{f=1}^{N_f} U(1)_f \right) \times U(1)_V$$

2.4.2 Chiral symmetry breaking

We now want to discuss more in detail the phenomenon of chiral symmetry breaking and how it can happen in the model. To see this, let us perform the fermionic path integral explicitly

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left(- \int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} D \psi_x^{(f)} \right) = (\det D[\phi])^{N_f} = e^{N_f \text{Tr} \log(D[\phi])},$$

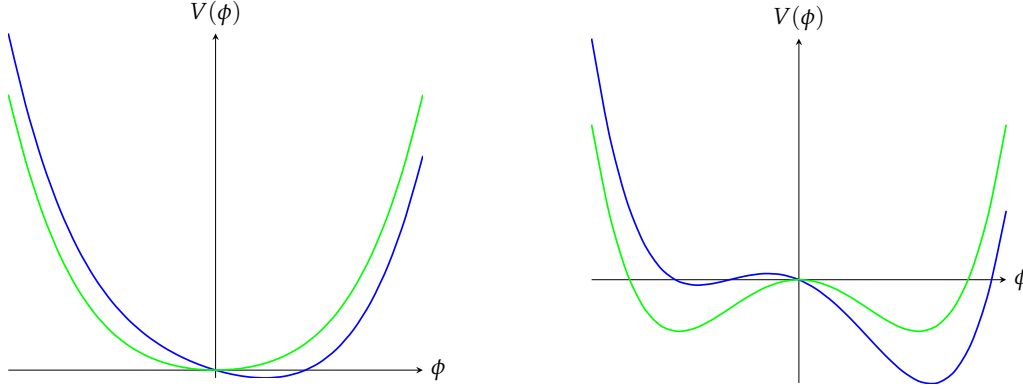


FIGURE 2.3: The introduction of the boson-fermion interaction, with a finite fermionic mass, causes explicit breaking of chiral symmetry, with consequence spontaneous breaking of the $O(1)$ symmetry according to (2.21). It shifts the equilibrium position in the symmetric phase (left) causing $\langle \phi \rangle \neq 0$, and tilts the potential in the broken phase (right), making the two minima not equivalent.

where the trace is performed over spacetime and spinor components.

The full path integral can now be expressed in terms of the resulting effective action for the scalar fields

$$Z = \int \mathcal{D}\phi e^{-S_{\text{eff}}[\phi]},$$

with

$$S_{\text{eff}}[\phi] = S_\phi[\phi] - N_f \text{Tr}_{x,s} \log D[\phi]. \quad (2.18)$$

One can derive the classical equations of motion by imposing $\frac{\delta S}{\delta \phi} = 0$, here expressed in momentum space

$$(k^2 + m_\phi^2) \phi(x) + \frac{\lambda}{6} \phi^3(x) = N_f g \text{Tr}_s [D^{-1}(\phi(x))] = -N_f g \bar{\psi}(x) \psi(x) \quad (2.19)$$

For $\lambda = 0$, they highlight a simple proportionality relation between magnetisation and chiral condensate, which reads

$$\phi = -\frac{N_f g}{k^2 + m_\phi^2} \bar{\psi} \psi \quad (2.20)$$

This relation, which was here derived at the classical level, is proven to hold also in mean field on the quantum level [24] and makes apparent the role of ϕ as a quark bilinear. When boson self-interactions are added, $\lambda \neq 0$ and the full relation between ϕ and $\bar{\psi} \psi$ is given by (2.20), but still one expects, qualitatively,

$$\langle \phi \rangle \sim \langle \bar{\psi} \psi \rangle. \quad (2.21)$$

A non-vanishing condensate is also related to a physical quark mass [24, 25], while the presence of magnetisation causes the breaking of $O(1)$ symmetry. The breaking can happen either explicitly in the action if $m_q \neq 0$, or spontaneously if $\langle \phi \rangle \neq 0$, in accordance with (2.21).

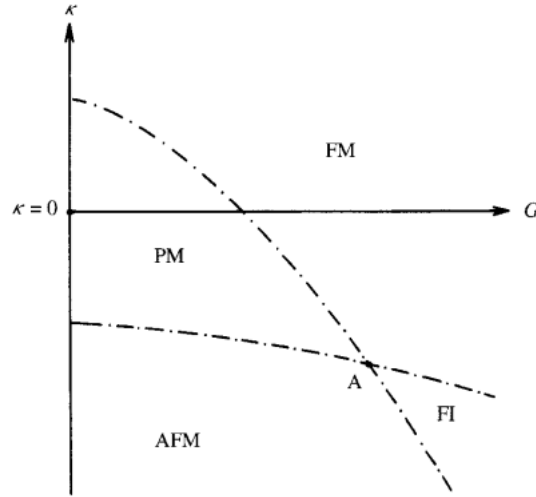


FIGURE 2.4: General phase diagram of a Yukawa theory.

I know that I wouldn't be allowed to use this picture, I will see later what to do.

2.4.3 Phase structure and order parameters

We want to conclude this chapter by commenting on the phase structure of the Yukawa theory. This will guide the choice of parameters for the investigation carried in the remaining sections. A sketch of the different phases of the model is reported in figure 2.4. The order parameters are the magnetisation $M = \langle \phi \rangle$ and the staggered magnetisation $M_s = \langle \Phi \rangle$, where

$$\Phi_x \equiv (-1)^{t+x} \phi_x = e^{i\pi(t+x)} \phi_x. \quad (2.22)$$

The four different phases are characterised by the following combinations of the order parameters

- paramagnetic phase: $M = 0, M_s = 0$,
- ferromagnetic phase: $M \neq 0, M_s = 0$,
- anti-ferromagnetic phase: $M = 0, M_s \neq 0$,
- ferrimagnetic phase: $M \neq 0, M_s \neq 0$.

The lattice observables are introduced in section 3.4.

Chapter 3

Methods and algorithms

3.1 Discretisation of the Yukawa theory

In order to make the theory suitable for a numerical simulation on a computer, the continuum formulation of the Yukawa model, which has been introduced in section 2.4, has to be discretised. Here we provided a sketch of a discretisation procedure, and we refer to other resources [1–4] for further details.

For what concerns the bosonic part of the action, a discretisation can be done straightforwardly with the following replacements

$$\begin{aligned} \int d^x &\rightarrow a^2 \sum_x \\ \partial_t^2 + \partial_x^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} &\rightarrow \sum_\mu \left[\frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2} \right], \end{aligned}$$

which yields to the lattice action

$$\begin{aligned} S_\phi[\phi] &= a^2 \left(\frac{1}{2} \sum_{m,n} \phi_m K_{mn} \phi_n + \frac{\lambda}{4!} \sum_n \phi_n^4 \right) \\ &= \frac{1}{2} \sum_{m,n} \hat{\phi}_m \hat{K}_{mn} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \sum_n \hat{\phi}_n^4, \end{aligned}$$

where we expressed everything in dimensionless quantities

$$\begin{aligned} \hat{m}_\phi^2 &= a^2 m_\phi^2, \\ \hat{\lambda} &= a^2 \lambda, \\ \hat{K}_{mn} &= a^2 K_{mn}. \end{aligned} \tag{3.1}$$

The operator components \hat{K}_{mn} are the discretised version of (2.15)

$$\hat{K}_{mn} = - \sum_\mu [\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}] + \hat{m}_\phi^2 \delta_{mn} \tag{3.2}$$

and its representation in momentum space is

$$\begin{aligned}
\hat{K}_{p,q} &= \sum_{n,m} e^{ipn} \hat{K}_{nm} e^{-iqm} \\
&= \sum_{n,m} e^{ipn} \left(- \sum_{\mu} [\delta_{m,m+\mu} + \delta_{m,m-\mu} - 2\delta_{m,n}] + \hat{m}_{\phi}^2 \delta_{mn} \right) e^{-iqm} \\
&= \sum_n e^{i(p-q)n} \left[\hat{m}_{\phi}^2 + 2 \sum_{\mu} (1 - \cos(q_{\mu})) \right] \\
&= \left[\hat{m}_{\phi}^2 + \sum_{\mu} 4 \sin^2 \left(\frac{p_{\mu}}{2} \right) \right] \delta(p-q).
\end{aligned}$$

For what concerns the fermionic action, a naïve discretisation is not sufficient, due to the well known doubling problem [1, 2]. In this work Wilson fermions [26] are employed as a way to fix such issue. Details of this formulation are explained in Appendix B. Here, only the final discretised action is reported, which reads

$$S_{\psi} [\hat{\psi}, \hat{\psi}] + S_{\text{int}} [\hat{\phi}, \hat{\psi}, \hat{\psi}] = \sum_{f=1}^{N_f} \hat{\psi}_m^{(f)} \hat{D}_{mn} \hat{\psi}_n^{(f)}, \quad (3.3)$$

with ψ_n being a two-component spinor, and $\hat{D}_{m,n}$ the Wilson-Dirac operator (can I include $g\phi$ in the definition of D ?) defined as

$$\begin{aligned}
\hat{D}_{m,n} &= - \left(\frac{\Gamma_{+\hat{0}}}{2} \delta_{m,m+\hat{0}} + \frac{\Gamma_{-\hat{0}}}{2} \delta_{m,m-\hat{0}} + \frac{\Gamma_{+\hat{1}}}{2} \delta_{m,m+\hat{1}} + \frac{\Gamma_{-\hat{1}}}{2} \delta_{m,m-\hat{1}} \right) \\
&\quad + (2ar + \hat{m} + \hat{g}\phi) \delta_{s,s'} \delta_{m,n}.
\end{aligned} \quad (3.4)$$

Note that the interaction term $g \bar{\psi} \phi \psi$ has been included in the definition of D . The Wilson projectors $\Gamma_{\pm\hat{\mu}}$ are defined as

$$\Gamma_{\pm\hat{\mu}} = ar \mathbb{1}_s \mp \gamma_{\mu}.$$

Since $r \in [0, 1]$ is a free parameter, in this work we set $r = 1$, if not otherwise specified.

In summary the discretised action for the Yukawa model is

$$S [\hat{\phi}, \hat{\psi}, \hat{\psi}] = \sum_{m,n} \hat{\phi}_m \hat{K}_{m,n} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \hat{\phi}_m^4 \delta_{m,n} + \sum_{f=1}^{N_f} \hat{\psi}_m^{(f)} \hat{D}_{mn} \hat{\psi}_n^{(f)},$$

with $\hat{K}_{mn}, \hat{D}_{mn}$ given respectively by (3.2) and (3.4).

For later reference, we also report the discretised version of the effective action (2.18)

$$\begin{aligned}
S_{\text{eff}}[\hat{\phi}] &= S_{\phi}[\hat{\phi}] - N_f \text{Tr}_{n,s} \log \hat{D} \\
&= \sum_{m,n} \hat{\phi}_m \hat{K}_{m,n} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \hat{\phi}_m^4 \delta_{m,n} - N_f \text{Tr}_{n,s} \log \hat{D}_{nn}.
\end{aligned} \quad (3.5)$$

The full discrete path-integral reads (measure over dimless or dimful?)

$$Z = \int \prod_n d\hat{\phi}_n e^{-S_{\text{eff}}[\hat{\phi}]}. \quad (3.6)$$

In the remaining of this work, both the original action S and the effective action S_{eff} will be denoted by S for simplicity. It will be clear from the context which of the two we will be referring to.

3.2 Langevin Monte Carlo

The relations (??) suggest that equation (2.6) can be integrated numerically for discrete time steps τ_n to generate field configurations distributed according to (2.9). The simplest first-order integration algorithm is the Euler-Majorana scheme [19]

$$\phi(\tau_{n+1}, x) = \phi(\tau_n, x) - \epsilon \frac{\delta S[\phi]}{\delta \phi(\tau_n, x)} + \sqrt{\epsilon} \eta(\tau_n, x) + O(\epsilon^2),$$

where $\epsilon = \tau_{n+1} - \tau_n$. Higher order integration schemes are possible (see e.g. [27, 28]), but not adopted in this work, and an adaptive step size is employed as detailed in Appendix C. In this way, for any observable O , one can introduce a Monte-Carlo estimator $\langle O \rangle_*$ which converges to the expectation value given by (2.10) in the limit of infinite samples

$$\langle O \rangle_* = \frac{1}{N_{\text{samp}}} \sum_{i=1}^{N_{\text{samp}}} O_i \xrightarrow{N_{\text{samp}} \rightarrow \infty} \langle O \rangle = \frac{1}{Z} \int D\phi O(\phi) \exp(-S[\phi]), \quad (3.7)$$

where $O_i = O(\phi(\tau_i))$ is the sample of the observable O done at time τ_i .

For the discretised action of the Yukawa theory (3.5) the drift reads, explicitly,

$$\begin{aligned} \frac{\partial S}{\partial \hat{\phi}_m(\tau_n)} &= \frac{\partial S_{\hat{\phi}}}{\partial \hat{\phi}_m(\tau_n)} - N_f \text{Tr}_s \left[\sum_{j,k} \hat{D}_{jk}^{-1} \frac{\partial \hat{D}_{kj}(\hat{\phi})}{\partial \hat{\phi}_m(\tau_n)} \right] \\ &= \sum_l \hat{K}_{ml} \hat{\phi}_l + \frac{\hat{\lambda}}{6} \hat{\phi}_m^3 - \hat{g} N_f \text{Tr}_s \left[(\hat{D}_{mm})^{-1} (\hat{\phi}(\tau_n)) \right]. \end{aligned} \quad (3.8)$$

While the bosonic contribution can be computed in a straightforward manner, the computation of the fermionic contribution requires the inversion of the Dirac operator. This, in general, cannot be done straightforwardly, mainly due to computational reasons. In fact, the full Dirac operator would be a $(2 \cdot N_t \cdot N_x \cdot N_f)^2$ dimensional object and a full inversion would be very expensive. In fact, D^{-1} has to be recomputed at every step and the best available algorithm as today for the matrix inversion has a computational complexity of $O(n^{2.371552})$ [29]. To circumvent this, we use the bilinear noise scheme [27, 30] [should I cite here or in Appendix?](#) which is illustrated in Appendix C.

3.3 Applications of coloured noise in lattice QFT

3.3.1 Classical to quantum interpolation

3.3.2 Noise induced transition

Quantum and classical system in two different phases

3.3.3 Cooling and the continuum limit of effective theories

After the general introduction on coloured noise given in the previous paragraph, let us now look more closely on the lattice formulation and at some possible applications of the technique.

To this end, let us consider a two-dimensional lattice with side lengths L_t, L_x and spacing $a = a_x = a_t$. This implies a maximum momentum $p_{\max} = \pi/a$ in each spacetime direction and $N_x = L_x/a, N_t = L_t/a$ points in each direction. Let us also define

$$\Lambda^2 \equiv (p_{\max}^x)^2 + (p_{\max}^t)^2, \quad (3.9)$$

which indicates the maximum squared momentum on the given lattice.

We then consider a simulation with a regularised noise defined by a cutoff $\Lambda_{\text{eff}} \leq \Lambda$ and we define a dimensionless parameter

$$s = \frac{\Lambda_{\text{eff}}}{\Lambda}, \quad 0 \leq s \leq 1. \quad (3.10)$$

Note that Λ_{eff} implicitly defines a length scale given by $a_{\text{eff}} = \pi/\Lambda_{\text{eff}}$.

Let us then consider a simulation with $s = 1$ and a set of bare couplings $\{g_0^i\}$, and another simulation with $s' < 1$ and new set of couplings $\{g_0^{i'}\}$. We now want to address the following question: is it possible to compensate the change in physical observables caused by the removal of the UV modes via regularised noise in the second simulation, by properly fine-tuning the bare parameters that enter the lattice discretised action? In other words, we want to encode the quantum fluctuations with $p^2 > \Lambda_{\text{eff}}^2$ in a redefinition of the classical action so that the expectation value of the observables remains unchanged.

The issue is of course related to the renormalisation transformation introduced in chapter 2. In particular we can exploit the connection between stochastic quantisation with coloured noise and the renormalisation group, as mentioned at the end of section 2.3.2, to accomplish the above mentioned goal.

As stated at the end of section 2.1.2, the Wilson RG flow of the dimensionless couplings and fields is dominated by the canonical scaling dimension of the corresponding dimensionful quantities, for high enough cutoff. We are interested in cooling the simulation by removing only the short-length fluctuations, hence we rely on this approximation as a lowest order Ansatz. This corresponds to a tree level RG rescaling.

For what concerns the scalar part of the action, the rescaling at tree level is rather straightforward

$$\begin{aligned} \hat{m}_\phi^2 = (a^2 m_\phi^2) &\rightarrow s^2 (a^2 m_\phi^2) = s^2 \hat{m}_\phi^2, & \hat{\lambda} = (a^2 \lambda) &\rightarrow s^2 (a^2 \lambda) = s^2 \hat{\lambda}, \\ \hat{\phi} = \phi &\rightarrow \phi = \hat{\phi}. \end{aligned}$$

The fermionic part needs some more careful analysis. For simplicity, let us for the moment set $N_f = 1$.

In a lattice simulation one wants to perform the integral over the fermionic fields and works with the effective action (2.18). In this case the drift is given by equation (3.8), with the fermionic contribution

$$K_\psi = g \text{Tr}_s D^{-1}, \quad (3.11)$$

or, in terms of dimensionless quantities,

$$\hat{K}_\psi = (ag) \operatorname{Tr}_s (aD)^{-1}.$$

This implies that under a lattice block-spin transformation, where $a \rightarrow sa$,

$$\hat{K}_\psi \rightarrow (sag) \operatorname{Tr}_s (saD)^{-1} = \hat{K}_\psi \quad (3.12)$$

On the other side, when computing the drift via the original action (2.14), one gets (omitting τ and x dependency)

$$\begin{aligned} K(\tau, x) &= -\frac{\delta S}{\delta \phi(\tau, x)} = K_\phi - g \bar{\psi} \psi = \\ &= -\left(-\partial_x^2 + m_\phi^2\right) \phi - \frac{\lambda}{6} \phi^3 - g \bar{\psi} \psi. \end{aligned} \quad (3.13)$$

where the fermionic contribution is given by

$$K'_\psi = -g \bar{\psi} \psi.$$

Note that all the terms in the equation (3.13) have dimension 2, in units of energy, which means, in particular, that after a lattice block-spin transformation where $a \rightarrow sa$, one has

$$\hat{K}'_\psi = (ag)(a\bar{\psi}\psi) \rightarrow s^2(ag)(a\bar{\psi}\psi) = s^2\hat{K}'_\psi, \quad (3.14)$$

in contrast with (3.12). For this reason, in order to have the correct scaling, we compute the contribution to the drift without rescaling the Dirac operator (and hence the Yukawa coupling), and then rescale the whole drift via

$$\hat{K}_\psi \rightarrow s^2 \hat{K}_\psi$$

so that the scaling dimension of the other terms in (3.13) is matched.

We want to mention that this has important consequences on the issue of continuum limit of low-energy effective theories. In fact, in the standard lattice regularisation procedure, one always has $a \sim \Lambda^{-1}$, which means that the continuum limit $a \rightarrow 0$ is always connected to the limit $\Lambda \rightarrow \infty$, and this constitutes a problem for effective theories. The latter are in fact meant to be valid only up to a certain scale Λ_{phys} , hence one always needs $\Lambda \leq \Lambda_{\text{phys}}$. Cooling via coloured noise can provide a solution to this problem, since one can always keep the effective cutoff fixed at $\Lambda_{\text{eff}} = \Lambda_{\text{phys}}$, while taking the limits $a \rightarrow 0, \Lambda \rightarrow \infty$.

3.3.4 Control over temperature

I have to wait for some results to see whether to include this or not :D

3.4 Definition of relevant observables

magnetisation

$$M = \left\langle \frac{1}{V} \sum_n \phi_n \right\rangle$$

absolute magnetisation

$$M = \left\langle \frac{1}{V} \left| \sum_n \phi_n \right| \right\rangle$$

Susceptibility of the field

$$\chi_2 = V \left(\langle M^2 \rangle - \langle M \rangle^2 \right) = V \langle (M - \langle M \rangle)^2 \rangle$$

Fermionic two-points function

$$\begin{aligned} \langle \psi_m \bar{\psi}_n \rangle &= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi(x) \bar{\psi}(y) \exp(-S_\phi - \psi D \psi + \bar{\eta} \psi + \bar{\psi} \eta) \\ &= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \frac{\delta}{\delta \bar{\eta}_m} \frac{\delta}{\delta \eta_n} \exp(-S_\phi - \psi D \psi + \bar{\eta} \psi + \bar{\psi} \eta) \\ &= \frac{1}{Z} \int \mathcal{D}\phi \det[D(\phi)] \exp(-S_\phi) \frac{\delta}{\delta \bar{\eta}_m} \frac{\delta}{\delta \eta_n} \exp(\bar{\eta} D^{-1} \eta) \\ &= \langle [D^{-1}(\phi)]_{mn} \rangle \end{aligned} \quad (3.15)$$

with D being the Wilson Dirac operator.

From this, the condensate follows straightforwardly

$$\langle \bar{\psi} \psi \rangle = \sum_{n,s} \langle \bar{\psi}_{n,s} \psi_{n,s} \rangle = -\text{Tr}_{n,s} (D^{-1})_{nn}$$

The fermionic correlator is defined as

$$C_\psi(t, 0) \equiv \frac{1}{N_x} \sum_{n_x} [\langle \psi(t, n_x) \bar{\psi}(0, 0) \rangle - \langle \psi(N_t - t, n_x) \bar{\psi}(0, 0) \rangle]$$

Note that we sum up two waves because the source propagates both forward and backward in time due to the boundary conditions.

Since for $t \rightarrow \infty$ one has that $C(t, p) \propto e^{-E_0(p)t}$, we expect

$$C_\psi(t, p) \approx \sinh \left(E_0 \left(\frac{N_t}{2} - t \right) \right)$$

Bosonic correlator

$$C_\phi(t, 0) \equiv \frac{1}{N_x} \sum_{n_x} [\langle \phi(t, n_x) \phi(0, 0) \rangle + \langle \phi(N_t - t, n_x) \bar{\phi}(0, 0) \rangle] \approx \cosh \left(E_0 \left(\frac{N_t}{2} - t \right) \right)$$

Chapter 4

Numerical investigation

4.1 The fermionic correlator

The Dirac operator for Wilson fermions in the yukawa model is

$$D_{nm} = \sum_{\alpha} \left[\frac{\gamma_{\alpha} \delta_{n+\alpha, m} - \gamma_{\alpha} \delta_{n-\alpha, m}}{2} + (m_q + g\phi) \delta_{nm} \right].$$

In momentum space it reads

$$\bar{D}_{ff'}(p) = \left(m + g\sigma \sum_{\mu} 2 \sin^2 \left(\frac{p_{\mu}}{2} \right) + i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu}) \right) \delta_{ff'}$$

The inverse can be checked to be

$$\bar{D}_{ff'}^{-1} = [m + \dots] \left(m + g\sigma \sum_{\mu} 2 \sin^2 \left(\frac{p_{\mu}}{2} \right) - i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu}) \right) \delta_{ff'}$$

One can now find the pole mass by imposing $D^{-1} = 0$ and gets

$$m + \dots = 0$$

4.2 Classical to quantum interpolation

Let us start by analysing the coloured noise field in the simulation and relevant properties that emerge from it. We consider the Yukawa model described by the continuum action ???. In figure ?? the system is initialised in the same state for all the configurations, and then evolved with the Langevin equation with various noise fractions. The red line corresponds to the case $s = 0$, namely a classical simulation. The blue line corresponds to the case $s = 1$, namely the fully quantum case. As one can notice, the introduction of noise shifts the equilibrium expectation value of the field monotonically with the cutoff fraction: this is due to the fact that WHAT???. Note that a lower noise fraction is correlated to a faster convergence towards equilibrium. Moreover, low-distance fluctuations are suppressed due to the removal of the ultraviolet modes in the noise term.

For $\lambda = 0$ one has

$$\sigma = -\frac{g}{m_{\phi}^2 + k^2} \bar{\psi} \psi \quad (4.1)$$

In figure ?? one can see that equation (4.1) is verified also on the fully quantum level.

Figures ?? - ?? report a few observables as a function of the cutoff fraction s . In this case all the coupling constants are kept fixed while changing the value of s , in

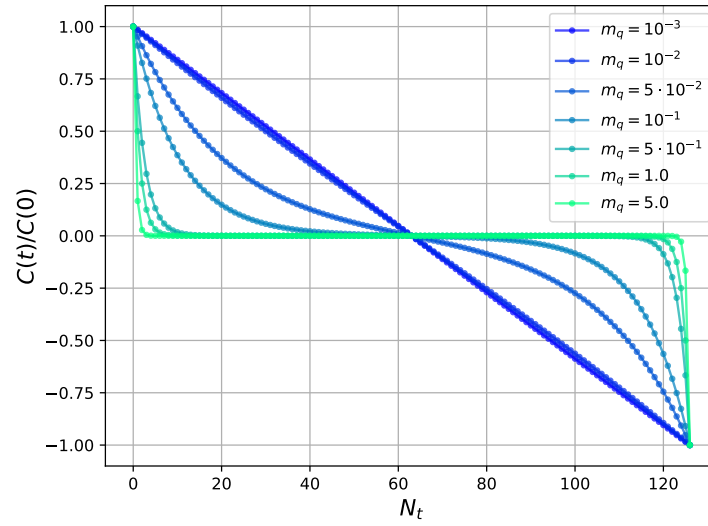


FIGURE 4.1: Correlator as a function of the bare mass

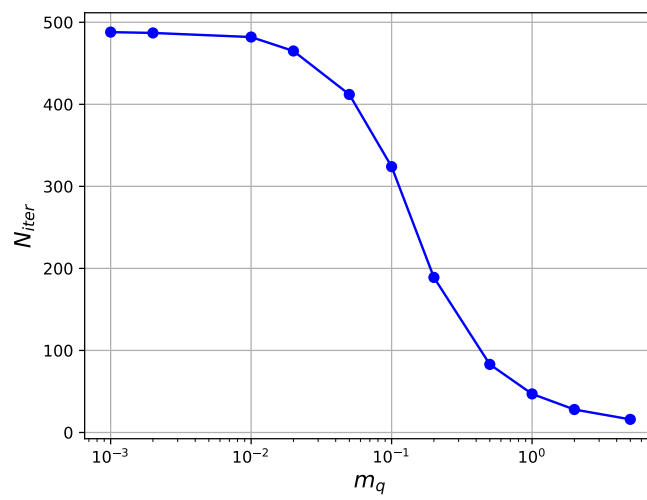
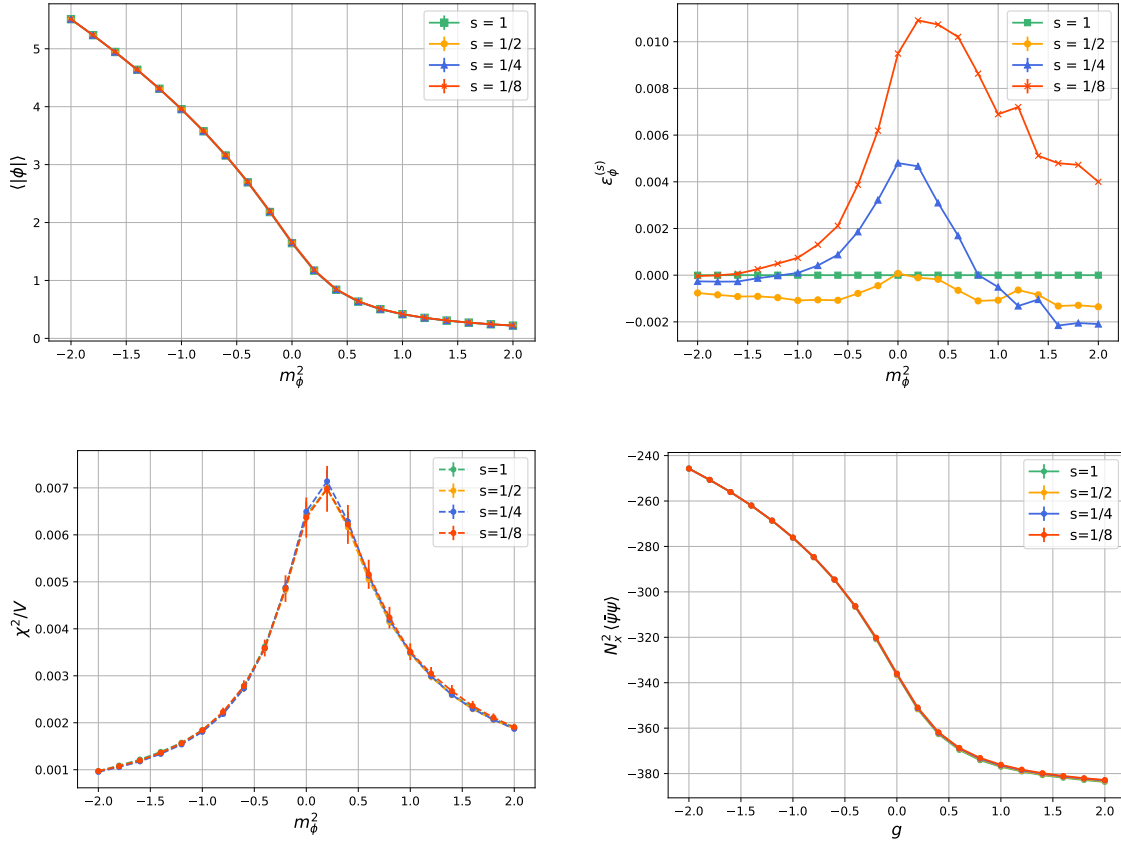


FIGURE 4.2: CG iterations



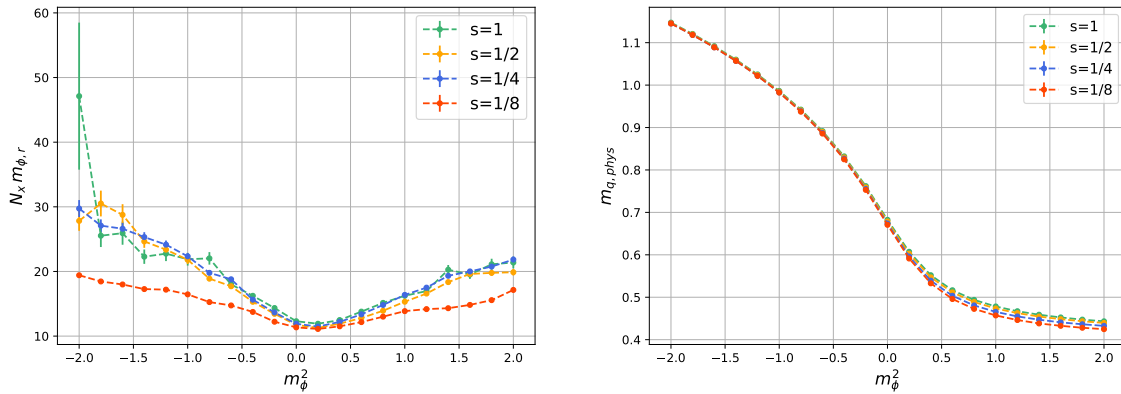
order to provide a smooth interpolation between the fully classical and fully quantum picture.

Each figure reports two plots corresponding to two different parameter configurations. The COLOR1 line corresponds to a system in the symmetric phase, while the COLOR2 line correspond to the broken phase. The exact parameters for the two configurations are reported under the figure.

4.3 Cooling with coloured noise

Look at various things such as magnetization, mass, etc.

Even though is O(1) we do not observe SSB because of fermion bare quark mass,



wilson term.

When does L.O. rescaling ansatz breaks down?

$$S[\phi, \bar{\psi}, \psi] = \int_x \phi \left(\frac{\partial^2}{2} + \frac{m_\phi^2}{2} \right) \phi + \frac{\lambda}{4!} \phi^4 + \bar{\psi} (\not{\partial} + m_q + g\phi) \psi$$

$$\lambda = 1.0 \quad m_\Phi^2 = 0.5 \quad N_t \times N_x = 8 \times 32 \quad m_q = 0.5 \quad N_{conf} = 5 \cdot 10^3 \quad \bar{\epsilon} = 0.01$$

4.4 Chiral fermions and a glimpse on the chiral phase transition

As explained in section ??, in the continuum theory chiral symmetry can be broken either explicitly via a finite bare quark mass, or spontaneously if the field gains a non-zero expectation value. Moreover, in the discrete formulation, the introduction of the Wilson term also contributes to the explicit breaking of chiral symmetry [add reference](#), as explained in section ?. This, in particular, means that chiral symmetry is explicitly broken also for $m_q \rightarrow 0$. Because of this, one needs a new definition for bare mass M_q , which takes into account the Wilson term contribution, such that chiral symmetry is restored in the limit $M_q \rightarrow 0$ for vanishing expectation value of the field ϕ . A convenient way to define such M_q is the following. SSB chiral symmetry $\rightarrow 3$ goldstone massless bosons, the pions. If the bare quark mass is zero, the physical mass of the pions has to be zero. Hence one can extract this mass on the lattice and tune m_q such that this is zero. While this is the correct way to proceed, it is very time taking since one must do mass scans and extrapolations close to singular Dirac operator. This is not the way we pursue here. Instead here we just consider naive fermions and take the limit $m_q \rightarrow 0$. This represents a physical theory with $2N_f = 4$ degenerate quarks. This is just done for the purpose of showing some interesting properties of coloured noise and not (yet) to match any physical result.

In figure ?? some observables are reported as a function of the noise fraction s for different values of the bare quark mass. In the classical theory ($s = 0$) the order parameters $\langle |\phi| \rangle$, $\langle \bar{\psi}\psi \rangle$ are, in absolute value, bigger than in the quantum case ($s = 1$). As the chiral limit is approached $m_q \rightarrow 0$, the figure shows that the classical system lies in the broken phase, while in the quantum settings the symmetry is restored. [Discuss general phase structure looking at both \$\phi\$ and \$\bar{\psi}\psi\$](#) . One can see that as m_q is reduced, the system shifts from a crossover to a second order phase transition, highlighted by the susceptibility χ^2 and the binder parameter U_L .

[This would be better discussed as a function of bare scalar mass:](#)

The difference gets more sharpened as the bare mass decreases, since the theory is closer to the chiral limit discussed above, which corresponds to the case $m_q = 0$. In this limit the system goes under a second phase transition, highlighted by a peak in the susceptibility and an abrupt change in the Binder cumulant. In contrast, for finite mass, there is a smooth crossover where the order parameters change continuously.

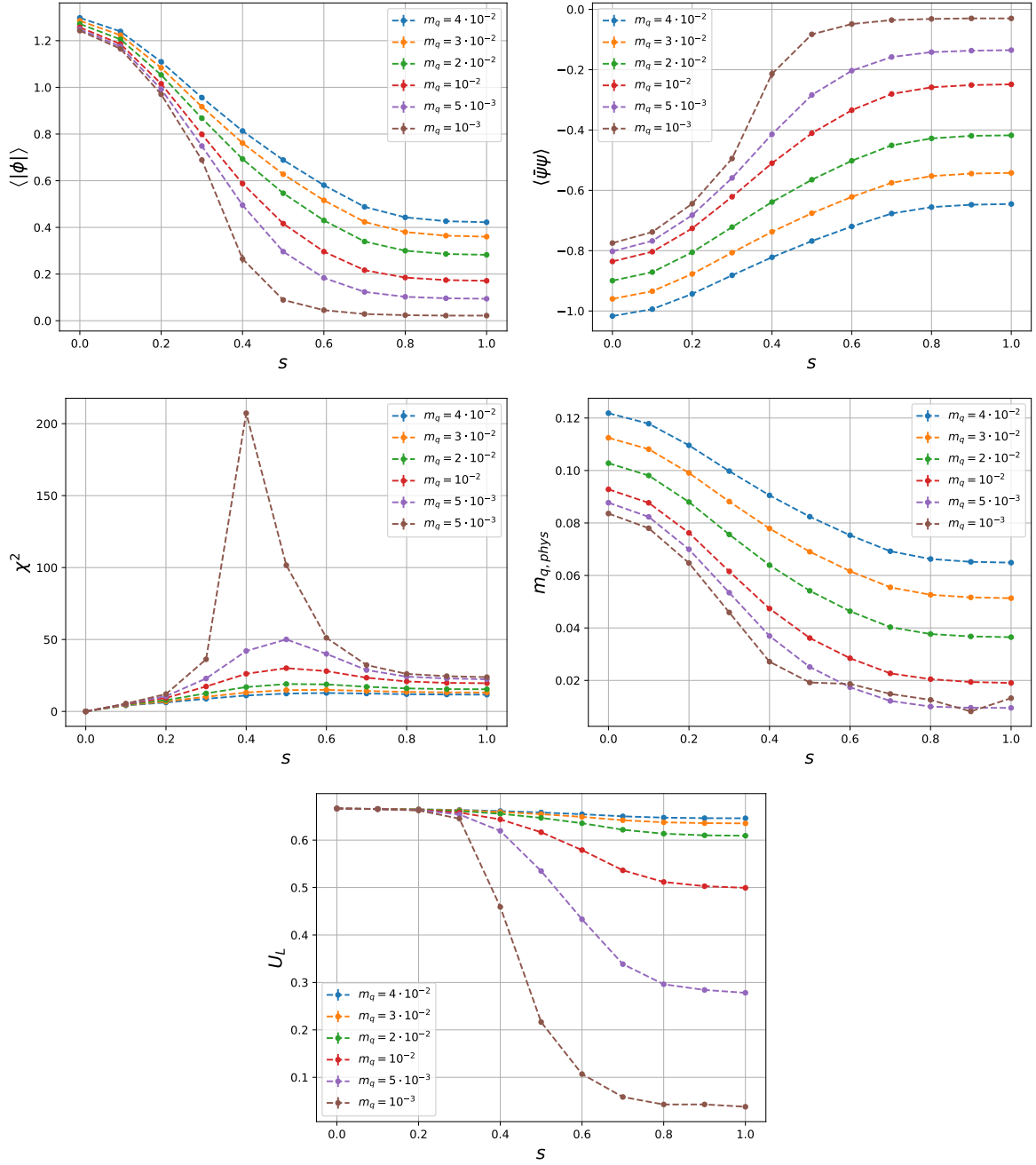


FIGURE 4.3: Chiral symmetry breaking

Chapter 5

Conclusions and outlook

Appendix A

Useful relations and definitions

In this appendix, useful relations and definitions are introduced.

Effective action

$$S_{\text{eff}} = S_{\phi} + \text{Tr} \log D(\phi)$$

Drift force

$$K_{\phi^j} = -\frac{\delta S}{\delta \phi^j} = -\frac{\delta S_{\phi}}{\delta \phi^j} - \text{Tr} \left[D^{-1} \frac{\delta D}{\delta \phi^j} \right]$$

Appendix B

Wilson fermions

I am not really sure on whether to put this appendix or not, maybe I will just cite some papers that talk about Wilson fermions

choice of the basis

Appendix C

Algorithms and technical details

cite also felipe for bilinear

C.1 Conjugate Gradient algorithm and the Dirac operator

The full inversion of the Dirac operator is a very expensive computation, given that the Dirac operator has dimension $(2 N_t N_x N_f)^2$, even though it is very sparse and has only few non-zero entries. One can note that for the purpose of computing the fermionic contribution to the drift force and the extraction of the physical quark mass from the correlator (details in section x and section y), only the inverse operator applied to a vector is needed. Hence it is sufficient to compute

$$\psi = D^{-1} |\eta\rangle \quad (\text{C.1})$$

Computing ψ via equation (C.1) is equivalent to solve the linear system $D\psi = \eta$, which can be done efficiently by employing a method for sparse matrices such as Conjugate Gradient (CG) as explained in the following way.

We want to solve the equation

$$D \psi = \eta$$

CG requires the matrix to be hermitian while D is only γ^5 -hermitian (really? under which assumptions?). One can thus solve the linear system

$$(DD^\dagger) \xi = \eta$$

and then obtain ψ by multiplying the solution ξ by D^\dagger since

$$D^\dagger \xi = D^\dagger (DD^\dagger)^{-1} \eta = D^{-1} \eta = \psi \quad (\text{C.2})$$

Analogously one can calculate

$$\chi = D^\dagger \eta$$

by solving

$$(D^\dagger D) \xi = \eta$$

and then applying D to the result.

One can improve the solution via CG by solving a *preconditioned* equation. Suppose that we want to solve the equation

$$Mx = b$$

via CG.

Let us express the matrix A as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (\text{C.3})$$

We introduce the Schur complement of M

$$M/D = A - BD^{-1}C \quad (\text{C.4})$$

This allows one to write M (LDU decomposition, Gaussian elimination) as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbf{1}_p & -BD^{-1} \\ 0 & \mathbf{1}_q \end{bmatrix} \begin{bmatrix} M/D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbf{1}_p & 0 \\ D^{-1}C & \mathbf{1}_q \end{bmatrix} = LAR$$

Which allows for an easy block inversion

$$M^{-1} = \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & -BD^{-1} \\ 0 & I_q \end{bmatrix} = L^{-1}A^{-1}R^{-1}$$

The equation to solve now reads

$$x = L^{-1}A^{-1}R^{-1}b \quad \text{or} \quad y = A^{-1}c$$

with $y = Lx$ and $c = R^{-1}b$. One can then solve the equation $y = A^{-1}c$ and get the solution x by applying L^{-1} to x .

An example of preconditioning is the even-odd preconditioning. Let us write the dirac operator in the form of equation (C.3) in the following way

$$M = \begin{pmatrix} M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{pmatrix}$$

The Schur complement (C.4) is

$$\hat{M} \equiv M/M_{oo} =$$

C.2 Bilinear noise scheme

$$\text{Tr} \left[D^{-1} \frac{\delta D}{\delta \phi^j} \right] \approx \langle \eta | D^{-1} \frac{\delta D}{\delta \phi^j} | \eta \rangle = \langle \psi | \frac{\delta D}{\delta \phi^j} | \eta \rangle \quad |\psi\rangle = D^{-1} |\eta\rangle = D^\dagger \underbrace{(DD^\dagger)^{-1} |\eta\rangle}_{\text{CG}}$$

$$\text{Tr} A = \frac{1}{N} \lim_{N \rightarrow \infty} \sum_i^N \eta_i^T D_{ij} \eta_j \quad (\text{C.5})$$

where η_i is a gaussian random field where each component is drawn from a normal distribution $\mathcal{N}(0, 1)$.

More precisely each vector component η_i^α satisfies

$$\langle \eta_i^\alpha \rangle = 0 \quad \langle \eta_i^\alpha \eta_j^\beta \rangle = \delta_{i,j} \delta^{\alpha\beta}$$

The series (C.5) requires in principle an infinite number of vectors to evaluate the trace exactly. In practice we truncate it and choose $N = 1 : \mathbf{D} : \mathbf{D}$. The average over Monte Carlo samples will eventually converge nevertheless to the right result.

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List of Abbreviations

RG	R enormalisation G roup
fRG	F unctional R enormalisation G roup
UV	U ltraviolet
QFT	Q uantum F ield T heory

Physical Constants

Speed of Light $c_0 = 2.997\,924\,58 \times 10^8 \text{ m s}^{-1}$ (exact)

List of Symbols