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Simulating effective field theories

on a space-time lattice with coloured noise

This Master thesis has been carried out by Matteo Zortea

at the

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(Titel in Deutsch): (Abstract in Deutsch, max. 200 Worte)

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27.11.2023 ....

“Grazie a tutti.”

Matteo Zortea

Contents

List of Figures

List of Tables

Contents

List of Figures

List of Tables

List of Abbreviations

LAH	**L**ist **A**bbreviations **H**ere
WSF	**W**hat (it) **S**tands **F**or

Physical Constants

Speed of Light $c_0 = 2.997\,924\,58 \times 10^8\ \mathrm{m\,s^{-1}}$ (exact)

List of Symbols

a	distance	m
P	power	W (J s^{-1})
ω	angular frequency	rad

Chapter 1

Introduction

1.1 QCD and phase diagram

Here we talk about QCD and the problem of the phase diagram

1.2 Effective theories (general)

Here we motivate the usefulness of effective theories and the renormalisation group to resolve physics at different scales

Chapter 2

Theoretical background

2.1 Yukawa theory

Let us consider the Yukawa theory defined by the action

$$\begin{aligned} S[\phi,\psi,\bar{\psi}] &= S_\phi[\phi] + S_\psi[\phi,\psi,\bar{\psi}] \\ S_\phi[\phi] &= \int_x \phi_x \left(-\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \phi_x + \frac{\lambda}{4!}\,\phi_x^4 \\ S_\psi[\phi,\psi,\bar{\psi}] &= \int_x \bar{\psi}_x \left(\not{\partial}_x + m_q + g\,\phi_x \right) \psi_x \end{aligned} \tag{2.1}$$

2.2 QM model

Slightly more technical: what is QM model and why it is useful for QCD

2.3 Lattice discretisation

NOTE We start from discretising the scalar theory O(N). We then introduce Wilson fermions (with details in appendix) and we write down the discretised version of the action for NJL and QM.

In this section we provide a discretised formulation of the Yukawa model introduced in section **??**.

For what concerns the bosonic part of the action, a discretisation can be done straightforwardly with the following replacements

$$\begin{aligned} \int_x \quad &\rightarrow \quad a^2 \sum_x \\ \partial_x^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} \quad &\rightarrow \quad \sum_\mu \left[\frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2} \right] \end{aligned}$$

which yields to the lattice action

$$S_\phi[\phi] = a^2 \sum_{m,n} \phi_m\, K_{mn}\, \phi_n + \frac{\lambda}{4!} \sum_n \phi_n^4$$

with

$$K_{mn} = -\sum_\mu \left[\frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2} \right] + m_\phi^2 \delta_{mn}$$

One can also express everything using dimensionless couplings

$$\begin{aligned} \hat{m}_\phi^2 &= a^2\, m_\phi^2 \\ \hat{\lambda} &= a^2\, \lambda, \\ \hat{K}_{mn} &= a^2 K_{mn} \end{aligned} \tag{2.2}$$

and the action is then described only in terms of dimensionless quantities

$$S_\phi = -\sum_{n,\mu} \hat{\phi}_n \hat{\phi}_{n+\mu} + \sum_n \left[\frac{1}{2} \left(4 + \hat{m}^2\right) \hat{\phi}_n^2 + \frac{\hat{\lambda}}{4!}\, \hat{\phi}_n^4 \right]$$

Otherwise it is customary to introduce dimensionless couplings κ, β defined via

$$\begin{aligned} \phi &\to (2k)^{\frac{1}{2}} \phi, \\ (am)^2 &\to \frac{1-2\beta}{k} - 4, \\ a^{-2}\lambda &\to \frac{6\beta}{k^2} \end{aligned} \tag{2.3}$$

and the equivalent action reads

$$S_\phi = -2k \sum_{n,\mu} \phi_n^i \phi_{n+\mu}^i + (1-2\beta) \sum_n \phi_n^i \phi_n^i + \beta \sum_n \left(\phi_n^i \phi_n^i\right)^2$$

In the following, we might use any of the two dimensionless formulations interchangeably, since they are completely equivalent given the definitions (**??**), (**??**).
For what concerns the fermionic action, a naive discretisation is not sufficient, due to the well known doubling problem. In this work Wilson fermions are employed as a way to fix such issue. Details of this formulation are explained in section **??**. Here, only the final discretised action is reported, which reads

$$S_\psi[\phi, \psi, \bar{\psi}] = \sum_{m,n} \bar{\psi}_m\, D_{m,n} \psi_n + g \sum_n \bar{\psi}_n \phi_n \psi_n$$

with ψ_n beeing a four-component spinor (2 flavour components and 2 Dirac components), and $D_{m,n}$ beeing the Wilson-Dirac operator is $g\phi$ included in the definition of D?) defined as

$$\begin{aligned} D_{m,n} = &- \left(\frac{\Gamma_{+0}}{2}\, \delta_{m,m+0} + \frac{\Gamma_{-0}}{2}\, \delta_{m,m-0} + \frac{\Gamma_{+1}}{2}\, \delta_{m,m+1} + \frac{\Gamma_{-1}}{2}\, \delta_{m,m-1} \right) \delta_{f,f'} \\ &+ (2 + m + g\phi)\ \delta_{s,s'} \delta_{m,n} \end{aligned}$$

The Wilson projectors $\Gamma_{\pm\mu}$ are defined as

$$\Gamma_{\pm\mu} = 1 \mp \gamma_\mu$$

One can then proceed by defining dimensionless fields and couplings

$$\begin{aligned} \hat{\psi} &=\to a^{\frac{1}{2}} \psi, \\ \hat{m}_q &= a m_q, \\ \hat{g} &= a g \end{aligned}$$

to describe the action only in terms of dimensionless quantities.
It is customary to perform the fermionic part of the path integral explicitly

$$\int \mathcal{D}\hat{\bar{\psi}}\mathcal{D}\hat{\psi}\ \exp\left(-\sum_{m,n}\hat{\bar{\psi}}_m\hat{D}_{mn}\,\hat{\psi}_n\right) = \det\hat{D}[\hat{\phi}] = e^{\mathrm{Tr}\log(\hat{D}[\hat{\phi}])}$$

where the trace is performed over space-time, flavour and spinor components.
The full path integral reads

$$Z = \int \mathcal{D}\hat{\phi}\ e^{-S_{\mathrm{eff}}[\hat{\phi}]}$$

with the effective action

$$S_{\mathrm{eff}}[\hat{\phi}] = S_{\phi}[\hat{\phi}] - \mathop{\mathrm{Tr}}_{x,s,f}\log D[\hat{\phi}]$$

In the remainder of this work, the dimensionless couplings and fields will be adopted, unless otherwise specified. Additionally, both the original action S and the effective action S_{eff} will be denoted by S for simplicity. It will be clear from the context to which of the two we will be referring.

2.4 Continuum limit

We briefly explain how the continuum limit is, in general, taken in LFT and the problem with effective theories, so that we can motivate the use of colored noise for the next section.

We are constrained to the manifold of fixed $m^2_{\phi,0}, m_{q,0}, g_0, \lambda_0$. We use $m_{q,r}$ to fix the scales for all the other dimensionful quantities. The cutoff is hence $1/m_{q,r}a$. For perturbatively renormalisable theories, we know that we can fine tune the bare parameters (as a function of the cutoff?) to keep renormalised quantities finite as we take the continuum limit. Introduce renormalisation as a mapping as in page 40 of Montvay Munster. We are interest in the set of theories in theories space that have constant renormalised couplings (trajectories in Kadanoff-Wilson RG) but different dimless masses $m_{q,r}\,a$. Since the latter gives the cutoff, we are moving the cutoff while keeping the other renormalised couplings fixed. How does this happen? We have to compensate the change in the cutoff by a change in the bare couplings as detailed here. IN perturbation theory we can compute renormalised couplings as functions of the bare ones. We can the (HOPEFULLY) invert these relations and determine bare parameters as functions of the renormalised ones. Near the continuum limit we can expect renormalised vertex functions to depend only weakly on the spacing so that we can impose

$$a\frac{d}{da}\Gamma^n_R(p^i, g^i_R, m_{q,r}\,a) = 0$$

And since

$$\Gamma_R = Z\Gamma_0$$

one gets

$$(a\partial_a - \beta_{\mathrm{lat}}\partial g_0 + n\gamma_{\mathrm{lat}})\ \Gamma^n_0 = 0$$

Integrating these equations leaves us with functions

$$g_0^i(a)$$

that tell us how to change bare couplings as a back reaction to a change in the spacing in order to keep physics constant.
In this perspective we want renormalised fixed and we change bare ones. In the callan symanzik we keep bare fixed and we see how the renormalised change. How are the two beta functions related? See 1.271 in Montvay and Munster. In this way one gets the relation between the two beta functions. The normal one ca be computed in the continuum with standard renormalisation techniques, the lattice one deduced. They match till second order.

Chapter 3

Methods and algorithms

3.1 Langevin equation

First in general, providing path integral solutions, eq. distribution ecc. Then do the specific case for a scalar theory. Then add fermions and discuss the tracelog contribution, mentioning stochastic trace evaluation.
Scalar theory:

$$\partial_\tau \phi(\tau, x) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta(\tau, x) \tag{3.1}$$

At equilibrium the probability of a field configuration ϕ reads

$$\mathcal{P}(\phi) = \frac{1}{Z} \exp\left(-S[\phi]\right)) \tag{3.2}$$

This allow one to compute correlation functions as moments of the distribution (**??**).

3.2 Coloured noise

Mention some possible uses of colored noise beside taking continuum limits.
In the stochastic quantisation procedure the noise which accounts for the quantum fluctuations of the theory is assumed to be white noise. This means that its power spectrum is flat in momentum space, extending in all the first Brillouin zone, namely for $p_\mu\ in[-\pi/a, \pi/a]$. One could modify this spectrum, and in this case one says *colored noise*. In particular one could put a sharp cutoff on the total momentum, imposing $p^2 \leq \Lambda^2$. We refer to this particular case as *regularised noise*. An example for the one-dimensional case is reported in figure **??**. In such case the Langevin equation for the scalar field (**??**) assumes the form

$$\partial_\tau \phi(\tau, x) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + r_\Lambda(x)\, \eta(\tau, x)$$

`figures/colorednoise.png`

FIGURE 3.1: Regularised noise

where the regularising function $r_\Lambda(x)$ can be easilly expressed in momentum space as $r_\Lambda(p) = \theta(\Lambda^2 - p^2)$. One can show **Pawlowski2017CoolingNoise** that the stochastic process is now driven towards a new equilibrium distribution

$$\mathcal{P}_\Lambda(\phi) = \frac{1}{Z}\exp\left(-S_\Lambda[\phi]\right) = \frac{1}{Z}\exp\left(-(S[\phi] + \Delta S_\Lambda[\phi])\right) \tag{3.3}$$

where the correction term $S_\Lambda[\phi]$ ensures that the probability measure $\mathcal{P}_\Lambda$ vanishes for squared field's momenta greater than the cutoff Λ^2.
An explicit example of such regulator for a free scalar field can be ADD EQUATION PAWLOWSKI. This is just one example of regularisation and different ones can be chosen. See **Pawlowski2017CoolingNoise** for details.
Note that the formulation presented in this section includes also the NJL-model as a particular case, after integrating out the fermionic fields as described in section **??**.

3.3 Lattice QFT with regularised noise

After the general introduction on coloured noise given in the previous paragraph, let us now look more closely on the lattice formulation and at the various applications of such techniques. From a code perspective, the algorithm to regularise noise with a sharp cutoff is presented in Appendix WHICH ONE?.
Let us consider a squared two-dimensional lattice with size $L \equiv L_x = L_y$ and spacing $a \equiv a_x = a_y$. This implies a maximum momentum $p_{\max} = \pi/a$ in each space-time direction. We consider a regularised simulation with cutoff Λ'. Let us also define $\Lambda^2 \equiv (p^x_{\max})^2 + (p^y_{\max})^2$. At this point we introduce a parameter, called *cutoff fraction*, defined as $s^2 \equiv \Lambda^2/\Lambda'^2$. With this notation, $s = 1$ is the full white noise case, while for any regularised noise one has $0 < s < 1$.
We now want to address the following question: given the stationary probability distribution **??**, is it possible to compensate the change in physical observables caused by the removal of the quantum modes via regularised noise, by a rescaling of the bare parameters that enter the lattice discretised action? Remembering from sections **??** and **??** that $\Lambda \sim a^{-1}$, this question would also address the problem of continuum limit of effective field theories. In fact the question we want to address is completely equivalent to the following: can one compensate a change in the spacing (controlled by the bare parameters), by a change of the noise in the simulation, in order to keep physical observables constant?
To this end, let us consider a noise regularisation given by $s^2 = \Lambda^2/\Lambda'^2 < 1$ ($s = a'/a < 1$), which means a higher cutoff (a smaller spacing a'). We now introduce an approximate ansatz which is based on the analogy to standard block-spin transformations. Higher order corrections to this simple ansatz are discussed in WHICH SECTION?
A change in the spacing $a \to ra$ will cause the following

$$\hat{m}_0^2 = (am_0)^2 \to (ram_0)^2 = r^2(am_0)^2 = r^2\hat{m}_0^2$$
$$\Lambda \to \Lambda/r$$

Not that after rescaling, the Dirac operator transforms as $D \to sD$. This means that the full action transforms as $S = S_\phi + \mathrm{Tr}\log D \to S' = S_\phi + s^2\mathrm{Tr}\log D + s^2\mathrm{Tr}\log s$. The last term is field independent, hence one could simply drop it and the equations of motion would remain the same. Equivalently, the Langevin drift contribution of that term is zero, hence it does not contribute to the dynamics. This

s	a	Λ	N	m_ϕ^2	λ	$m_{q,0}$	g_0
1	a	Λ	8	$m_{\phi,0}^2$	λ	$m_{q,0}$	g_0
2	$a/2$	2Λ	16	$m_{\phi,0}^2/4$	$\lambda/4$	$m_{q,0}$	g_0
4	$a/4$	4Λ	32	$m_{\phi,0}^2/16$	$\lambda/16$	$m_{q,0}$	g_0
8	$a/8$	8Λ	64	$m_{\phi,0}^2/64$	$\lambda/64$	$m_{q,0}$	g_0

TABLE 3.1: Rescaling

means in particular that one does not have to rescale the couplings that enter only in the Dirac operator, such as m_q and g.

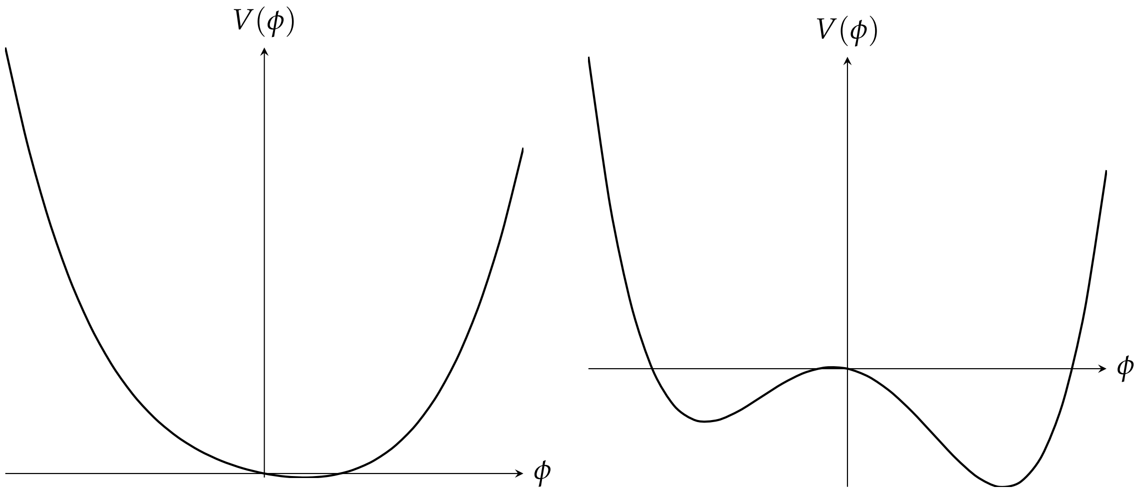

FIGURE 3.2: The introduction of the boson-fermion interaction, with a finite fermionic mass, causes the breaking of the O(1) symmetry. It shifts the equilibrium position in the symmetric phase (left) causing $\langle\phi\rangle = 0$, and tilts the potential in the broken phase (right), making the two minima not equivalent.

3.4 Langevin dynamics for NJL

This has to be rewritten for Wilson fermions, in a discretised way The effective action after solving the fermionic path integral reads

$$S[\phi] = S_\phi - \mathrm{Tr}\log\left(D(\phi)\right) = \int_x \left[\phi_x\left(-\frac{\partial^2}{2}+\frac{m_\phi^2}{2}\right)\phi_x\right] + \frac{\lambda}{4!}\phi_x^4 - \mathrm{Tr}\log\left(D(\phi_x)\right)$$

with $D(\phi) = \partial\!\!\!/ + m_q + g\phi$.
Hence the drift is

$$\frac{\delta S}{\delta\phi(x)} = \frac{\delta S_\phi}{\delta\phi(x)} - \mathrm{Tr}\left[D^{-1}\frac{\partial D}{\partial\phi(x)}\right] = \frac{\delta S_\phi}{\delta\phi(x)} - g\ \mathrm{Tr}\left[D^{-1}\right]$$

To evaluate the trace we use the bilinear noise scheme add reference

$$\mathrm{Tr}\,A = \lim_{N\to\infty}\sum_i^N \eta_i^T D_{ij}\eta_j \tag{3.4}$$

where η_i is a gaussian random field where each component is drawn from a normal distribution $\mathcal{N}(0,1)$.
More precisely each vector component η_i^α satisfies

$$\langle\eta_i^\alpha\rangle = 0 \qquad \left\langle\eta_i^\alpha\eta_j^\beta\right\rangle = \delta_{i,j}\,\delta^{\alpha\beta}$$

The series (**??**) requires in principle an infinite number of vectors to evaluate the trace exactly. In practice we truncate it and choose $N = 1$:D :D. The average over Monte Carlo samples will eventually converge nevertheless to the right result.

The full quantum dynamics is described by (**??**). One can find an analytical solution, in some limiting cases. For example let us consider a classical simulation with $\lambda = 0, D = m_q + g\phi$. Equilibrium will be reached when $\frac{\delta S}{\delta\phi(x)} = 0$ which means, assuming zero momentum,

$$m_\phi^2\,\phi(x) = g\,\mathrm{Tr}\left[D^{-1}(\phi(x))\right]$$

Since the fermionic kinetic term has been removed from the action by assumption and no fluctuations, except from numerical issues, are present in the simulation, the Dirac operator is diagonal and can be inverted explicitly

$$D^{-1}(\phi(x)) = \frac{1}{m_q + g\,\phi(x)} \tag{3.5}$$

yielding $\mathrm{Tr}\,D^{-1} = 4\left(m_q + g\,\phi\right)^{-1}$.
Inserting this result into (**??**) and solving for $\phi(x)$ one finds the two field configurations that minimise the action, in accordance to what pictured in figure **??**

$$\phi_{1,2}^{*} = \frac{m_q}{2\,g}\left(-1 \pm \sqrt{1 + 16\,g^2/m_q^2 m_\phi^2}\right)$$

3.5 Linear solvers in Lattice QFT ?

The full inversion of the Dirac operator is a very expensive computation, given that the Dirac operator has dimension $(2\, N_t\, N_x\, N_f)^2$, even though it is very sparse and has only few non-zero entries. One can note that for the purpose of computing the fermionic contribution to the drift force and the extraction of the physical quark mass from the correlator (details in section x and section y), only the inverse operator applied to a vector is needed. Hence it is sufficient to compute

$$\psi = D^{-1} \left|\eta\right\rangle \tag{3.6}$$

Computing ψ via equation (**??**) is equivalent to solve the linear system $D\psi = \eta$, which can be done efficiently by employing a method for sparse matrices such as Conjugate Gradient (CG) as explained in the following way.

3.5.1 inversion via CG

We want to solve the equation

$$D\,\psi = \eta$$

CG requires the matrix to be hermitian while D is only γ^5-hermitian (really? under which assumptions?). One can thus solve the linear system

$$\left(DD^\dagger\right)\xi = \eta$$

and then obtain ψ by multiplying the solution ξ by $D^\dagger$ since

$$D^\dagger \xi = D^\dagger \, \left(DD^\dagger\right)^{-1} \eta = D^{-1}\eta = \psi \tag{3.7}$$

Analogously one can calculate

$$\chi = D^\dagger \eta$$

by solving

$$\left(D^\dagger D\right)\xi = \eta$$

and then applying D to the result.

3.5.2 Pre conditioning

One can improve the solution via CG by solving a *preconditioned* equation. Suppose that we want to solve the equation

$$Mx = b$$

via CG.
Let us express the matrix A as a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.8}$$

We introduce the Schur complement of M

$$M/D = A - BD^{-1}C \tag{3.9}$$

This allows one to write M (LDU decomposition, Gaussian elimination) as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \mathbf{1}_p & -BD^{-1} \\ 0 & \mathbf{1}_q \end{bmatrix} \begin{bmatrix} M/D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbf{1}_p & 0 \\ D^{-1}C & \mathbf{1}_q \end{bmatrix} = LAR$$

Which allows for an easy block inversion

$$M^{-1} = \begin{bmatrix} I_p & 0 \\ -D^{-1}C & I_q \end{bmatrix} \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_p & -BD^{-1} \\ 0 & I_q \end{bmatrix} = L^{-1}A^{-1}R^{-1}$$

The equation to solve now reads

$$x = L^{-1}A^{-1}R^{-1}\, b \qquad \text{or} \qquad y = A^{-1}c$$

with $y = Lx$ and $c = R^{-1}b$. One can then solve the equation $y = A^{-1}c$ and get the solution x by applying L^{-1} to x.

An example of preconditioning is the even-odd preconditioning. Let us write the dirac operator in the form of equation (**??**) in the following way

$$M = \begin{pmatrix} M_{ee} & M_{eo} \\ M_{oe} & M_{oo} \end{pmatrix}$$

The Schur complement (**??**) is

$$\hat{M} \equiv M/M_{oo} =$$

3.6 Relevant quantities

Two-points function

$$\begin{aligned}
\langle \psi(x)\, \bar{\psi}(y) \rangle &= \frac{1}{Z} \int \mathcal{D}\phi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; \psi(x)\, \bar{\psi}(y)\; \exp\left(-S_\phi - \psi D \psi + \bar{\eta}\psi + \bar{\psi}\eta\right) \\
&= \frac{1}{Z} \int \mathcal{D}\phi\, \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; \frac{\delta}{\delta\bar{\eta}(x)} \frac{\delta}{\delta\eta(y)}\; \exp\left(-S_\phi - \psi D \psi + \bar{\eta}\psi + \bar{\psi}\eta\right) \\
&= \frac{1}{Z} \int \mathcal{D}\phi\; \det\left[D(\phi)\right]\; \exp\left(-S_\phi\right)\; \frac{\delta}{\delta\bar{\eta}(x)} \frac{\delta}{\delta\eta(y)}\; \exp\left(\bar{\eta} D^{-1} \eta\right) \\
&= \left\langle \left[D^{-1}(\phi)\right](x, y) \right\rangle
\end{aligned}$$

On the lattice it still holds that

$$\langle \psi_m\, \bar{\psi}_n \rangle = \left\langle \left[D^{-1}(\phi)\right]_{mn} \right\rangle$$

with D beeing the Wilson Dirac operator REFERENCE

Now consider a source at the origin at time $t = 0$, namely a state $\psi(t, x) = c\, \delta_{t,0}\, \delta_{x,0}$. After letting it propagate for a time t, me might ask how is a point on the lattice, on average, correlated to the initial source. This is quantified by the *correlator*, which, on the lattice, it is defined as

$$C(n_t,0) \equiv \frac{1}{N_x}\sum_{n_x}[\langle \psi(n_t,n_x)\,\bar{\psi}(0,0)\rangle + \langle \psi(N_t-n_t,n_x)\,\bar{\psi}(0,0)]\rangle$$

Note that we sum up two waves because the source propagates both forward and backward in time due to the boundary conditions.
Since for $t \to \infty$ one has that $C(t,p) \propto e^{-E_0(p)t}$, we expect

$$C(t,p) \approx \sinh\left(E_0\left(\frac{N_t}{2} - t\right)\right)$$

Pole mass, renormalized mass, effective mass, bare mass, physical mass
In momentum space

$$\tilde{D}(p,q) = m + g\,\sigma + i\,g\,\gamma^5\,\tau^j\pi^j + \sum_\mu \sin^2\left(\frac{p_\mu}{2}\right) + \sum_\mu \gamma_\mu \sin\left(p_\mu\right)$$

Or making explicit the flavour and spinor components

$$\tilde{D}(p,q) = M\,\mathbb{1}_f \otimes \mathbb{1}_s + i\,g\,\pi^j\,\tau^j \otimes \gamma^5 + \left(\sum_\mu \sin\left(p_\mu\right)\mathbb{1}_f \otimes \gamma_\mu\right)$$

where

$$M = m + g\,\sigma + \sum_\mu \sin^2\left(\frac{p_\mu}{2}\right)$$

Effective action

$$S_{\text{eff}} = S_\phi + \text{Tr}\log D(\phi)$$

Drift force

$$K_{\phi^j} = -\frac{\delta S}{\delta \phi^j} = -\frac{\delta S_\phi}{\delta \phi^j} - \text{Tr}\left[D^{-1}\frac{\delta D}{\delta \phi^j}\right]$$

$$\frac{\delta D}{\delta \phi^j} = \begin{cases} g & \text{if } j = 0 \\ i\,g\,\gamma^5\,\tau^j & \text{if } j = 1,2,3 \end{cases}$$

Bilinear noise scheme

$$\text{Tr}\left[D^{-1}\frac{\delta D}{\delta \phi^j}\right] \approx \langle\eta|\,D^{-1}\frac{\delta D}{\delta \phi^j}\,|\eta\rangle = \langle\psi|\,\frac{\delta D}{\delta \phi^j}\,|\eta\rangle \qquad |\psi\rangle = D^{-1}\,|\eta\rangle = D^\dagger\,\underbrace{(DD^\dagger)^{-1}\,|\eta\rangle}_{\text{CG}}$$

$$\tau^1 \otimes \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\tau^2 \otimes \gamma^5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\tau^2 \otimes \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

Quark-meson model action in the continuum

$$S[\sigma, \pi^j, \psi, \bar{\psi}] = \int_x \sum_{j=1}^{3} \{ \frac{1}{2} \, (\partial_\mu \sigma)^2 + \frac{1}{2} \left(\partial_\mu \pi^j \right)^2 + \frac{m^2}{2} \left(\sigma^2 + (\pi^j)^2 \right) + \frac{\lambda}{4!} \left(\sigma^2 + (\pi^j)^2 \right)^2$$
$$+ \bar{\psi} \left(\not{\partial} + m_q \right) \psi + \bar{\psi} \left(g \, \sigma + i \, g \, \gamma^5 \pi^j \tau^j \right) \psi \}$$

3.6.1 Test with background mesons

The Dirac operator for Wilson fermions in the yukawa model is

$$D_{nm} = \sum_\alpha \left[\frac{\gamma_\alpha \delta_{n+\alpha,m} - \gamma_\alpha \delta_{n-\alpha,m}}{2} + (m_q + g\phi) \, \delta_{nm} \right].$$

In momentum space it reads

$$\bar{D}_{ff'}(p) = \delta_{ff'} \left(m + g\sigma \sum_\mu 2 \sin^2 \left(\frac{p_\mu}{2} \right) + i \sum_\mu \gamma_\mu \sin \left(p_\mu \right) \right)$$

Chapter 4

Results

4.1 Inversion of the Dirac operator - background mesons

4.2 Classical to quantum interpolation

Let us start by analising the coloured noise field in the simulation and relevant properties that emerge from it. For simplicity we hereby restrict to the NJL model analysis.

In figure **??** the system is initialised in the same state for all the configurations, and then evolved with the Langevin equation with various noise fractions. The red line corresponds to the case $s = 0$, namely a classical simulation. The blue line corresponds to the case $s = 1$, namely the fully quantum case. As one can notice, the introduction of noise shifts the equilibrium expectation value of the field monotonically with the cutoff fraction: this is due to the fact that WHAT???. Note that a lower noise fraction is correlated to a faster convergence towards equilibrium. Look at various things such as magnetization, mass, etc. As a consistency test, one could verify the classical equation of motion

$$K\sigma + \frac{\lambda}{3!}\sigma^3 = -g\,\bar{\psi}\psi \tag{4.1}$$

which is proven to hold not only classically but also on the mean-field quantum level (ADD REFERENCE TO THE PAPER CITED IN JP's THESIS).

For $\lambda = 0$ one has

$$\sigma = -\frac{g}{m_\phi^2 + k^2}\,\bar{\psi}\psi \tag{4.2}$$

In figure **??** one can see that equation (**??**) is verified also on the fully quantum level.

4.2.1 slide broken NJL

Figures **??** - **??** report a few observables as a function of the cutoff fraction s. In this case all the coupling constants are kept fixed while changing the value of s, in order to provide a smooth interpolation between the fully classical and fully quantum picture.

figures/thermalisation.pdf

FIGURE 4.1: Thermalisation for different noise fractions

FIGURE 4.2: Caption

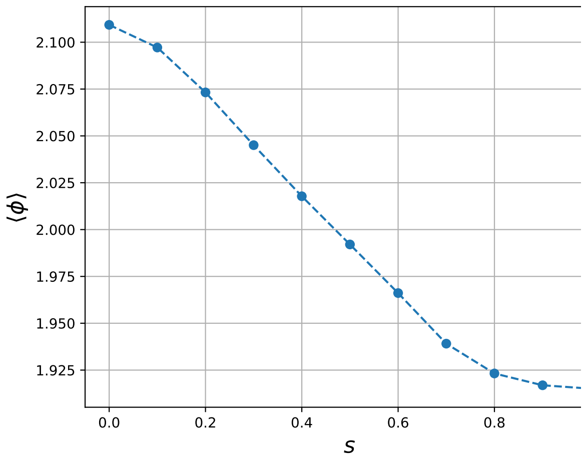

FIGURE 4.3: phi

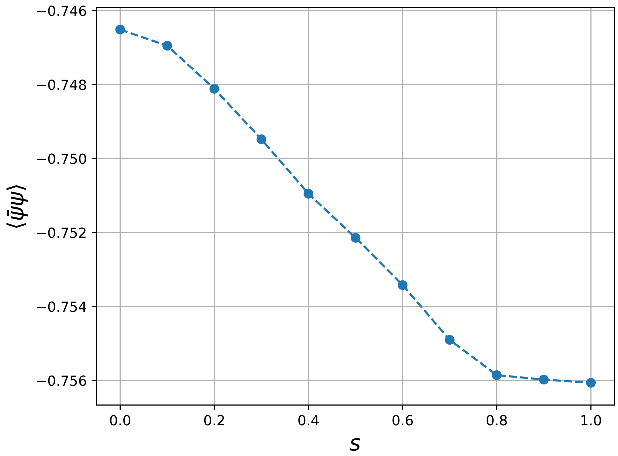

FIGURE 4.4: cond

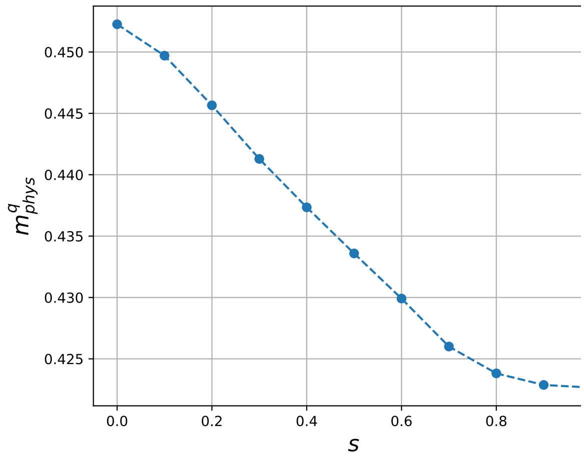

FIGURE 4.5: mq-phys

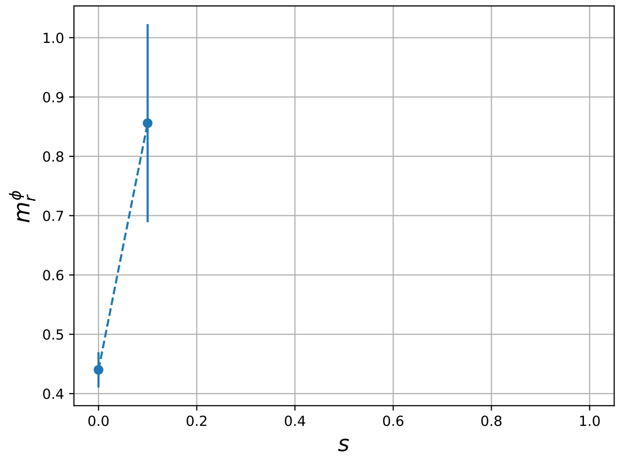

FIGURE 4.6: mphir

FIGURE 4.7: slide broken

Each figure reports two plots corresponding to two different parameter configurations. The COLOR1 line corresponds to a system in the symmetric phase, while the COLOR2 line correspond to the broken phase. The exact parameters for the two configurations are reported under the figure.

4.3 Block spin steps

In this section we report and discuss results of the NJL model and Quark-Meson model using the technique of coloured noise to perform block-spin steps as outlined in the paragraph **??**. We first set up the white noise simulation with $s = 1$ on a lattice of size 8×8, with spacing a and cutoff Λ. We then start performing complete block-spin steps as summarised in table **??**.
As in the previous section, we first focus on the NJL model, and subsequently introduce the full Quark-Meson model.

4.3.1 NJL - yukawa scan

Figure **??**, **??** report the analysis of the system for various values of the Yukawa coupling g. The classical equation of motion (**??**) is invariant under the transformation $g \to -g, \sigma \to -\sigma$. This means that one should expect, from a classical point of view, the order parameter σ to be anti-symmetric around $g = 0$. By looking at figrue **??**, one can conclude that the symmetry is still present on the quantum level, but around a point $g_0 \neq 0$ which is solution of the equation $\langle \sigma \rangle = 0$. DOES THE POINT g_0 IDENTIFY A PHASE TRANSITION?

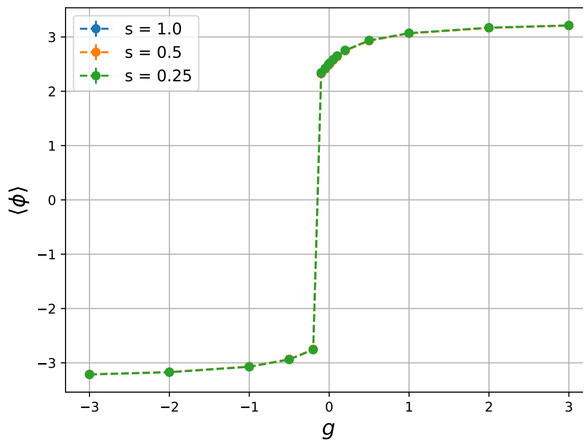

FIGURE 4.8: Magnetisation

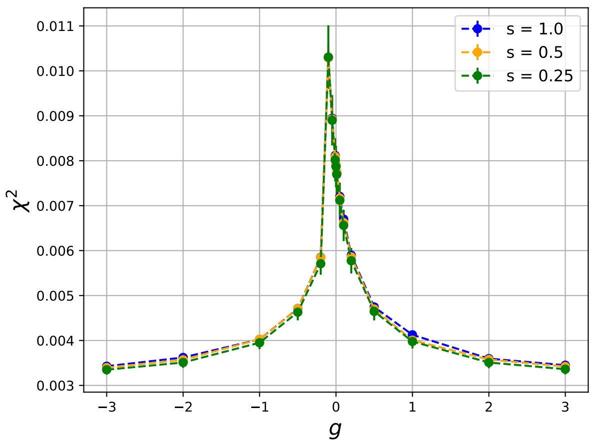

FIGURE 4.9: Susceptibility

As one can see, for the first two block spins there is perfect agreement among the various curves. From the third step, some small deviation comes in due to the momentum dependence of the coupling.

Perhaps more interesting are the plots in figure **??**, reported as a function of the bare mesons mass. The peek in the magnetic susceptibility deserves some comment. In general, in a finite-volume lattice theory, no phase transition can happen. To state the presence of a phase transition, one should look at the infinite volume limit, which is done by studying the volume scaling of the magnetic susceptibility. In particular we do not expect a phase transition in our model due to the spontaneous breaking of the $O(1)$ symmetry, since the presence of a finite bare quark mass already breaks the $O(1)$ symmetry explicitly. As a remark to this statement, we studied the volume scaling of χ, as reported in figure **??**. It is clear that it converges towards XXX, implying that the system is not undergoing a phase transition.

Look at various things such as magnetization, mass, etc.
Even though is O(1) we do not observe SSB because of fermion bare quark mass.
Peak in the susceptibility does not imply P.T. $\rightarrow$ look at volume scaling.
When does L.O. rescaling ansatz breaks down?

4.3.2 QM model

Look at various things such as magnetization, mass, etc.
There is no P.T. as it is O(4) in 2d.
When does L.O. rescaling ansatz breaks down?

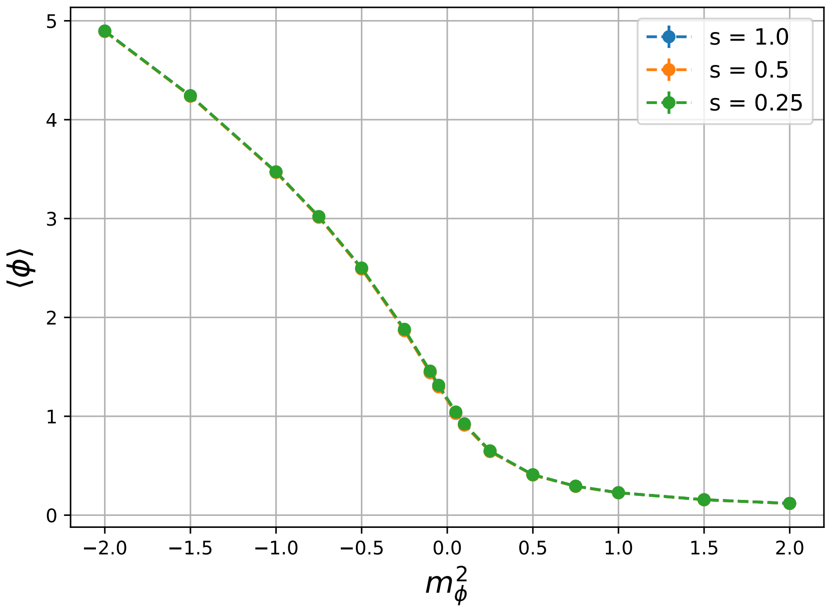

FIGURE 4.10

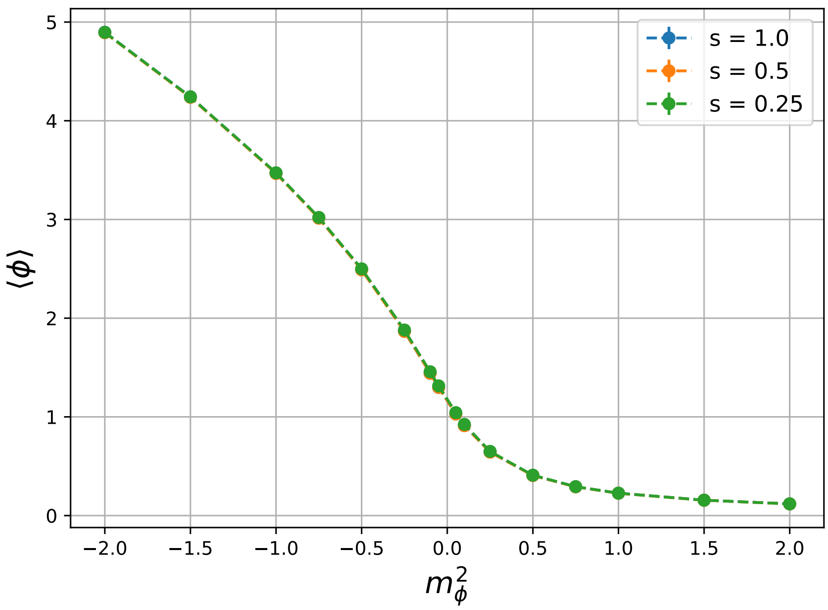

FIGURE 4.11

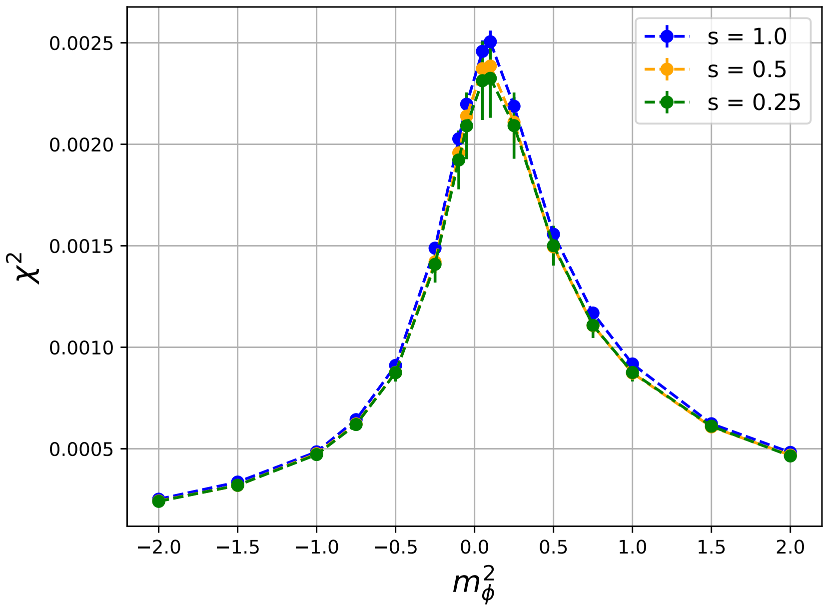

FIGURE 4.12

$$S[\phi,\bar{\psi},\psi] = \int_x \phi\left(\frac{\partial^2}{2} + \frac{m_\phi^2}{2}\right)\phi + \frac{\lambda}{4!}\phi^4 + \bar{\psi}\left(\partial\!\!\!/ + m_q + g\phi\right)\psi$$

$\lambda = 1.0 \qquad m_\Phi^2 = 0.5 \qquad N_t \times N_x = 8\times 32 \qquad m_q = 0.5 \qquad N_{conf} = 5\cdot 10^3 \qquad \bar{\epsilon} = 0.01$

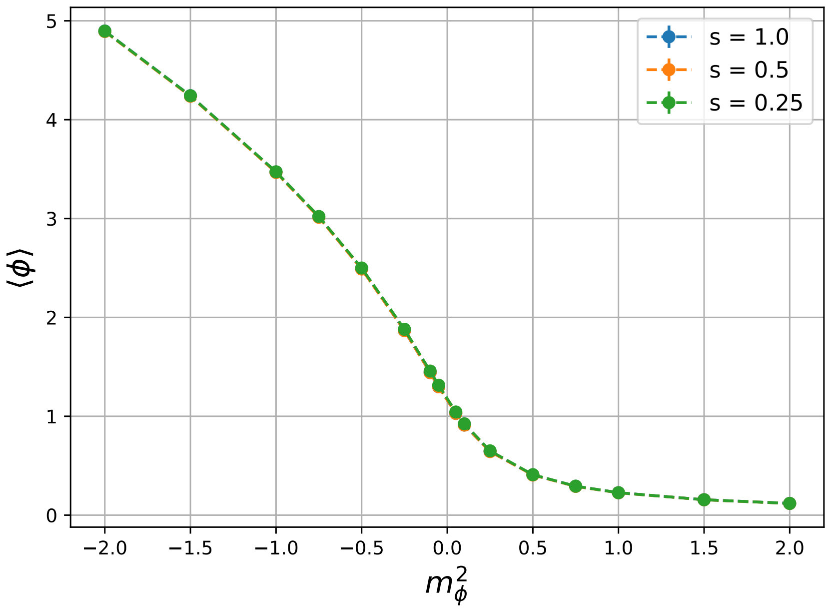

FIGURE 4.13: Magnetization

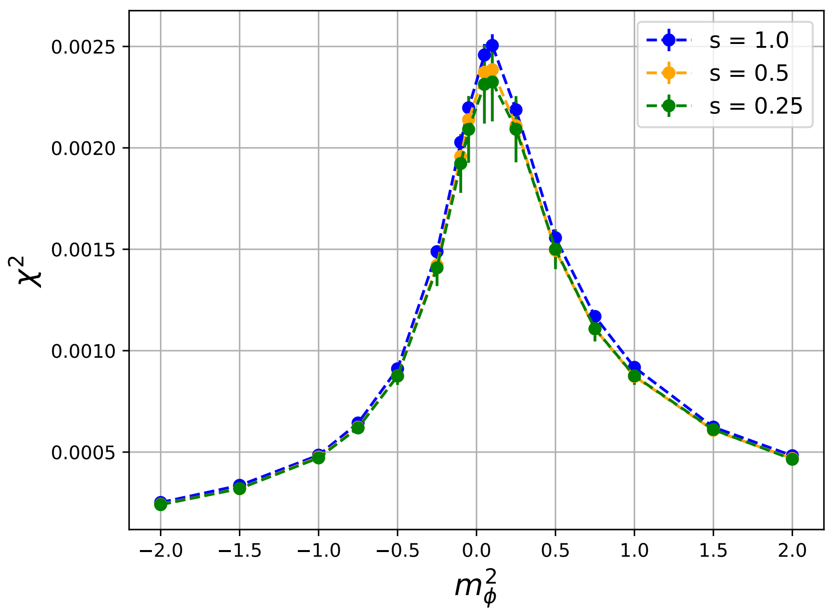

FIGURE 4.14: Magnetic susceptibility

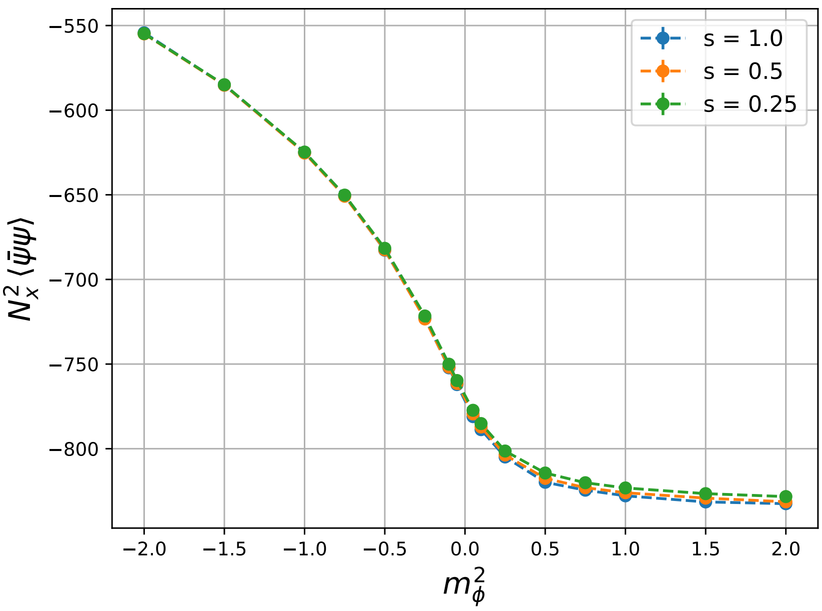

FIGURE 4.15: Condensate

Chapter 5

Conclusions

Appendix A

Frequently Asked Questions

A.1 How do I change the colors of links?

The color of links can be changed to your liking using:
`\hypersetup{urlcolor=red}`, or
`\hypersetup{citecolor=green}`, or
`\hypersetup{allcolor=blue}`.
If you want to completely hide the links, you can use:
`\hypersetup{allcolors=.}`, or even better:
`\hypersetup{hidelinks}`.
If you want to have obvious links in the PDF but not the printed text, use:
`\hypersetup{colorlinks=false}`.