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Master thesis

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Investigating a Yukawa theory
with Coloured Stochastic Quantisation

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(Titel in Deutsch): (Abstract in Deutsch, max. 200 Worte)

Stochastic quantisation with coloured noise allows for a precise control over the momentum dependency of quantum fluctuations, in connection with the Wilsonian Renormalisation Group.
The technique is implemented via a Langevin Monte Carlo algorithm to investigate a fermionic field theory, namely a two flavours Yukawa model.
Applications include investigating the fluctuations content of the theory, noise-induced phase transitions and cooling the simulation by systematically removing ultraviolet degrees of freedom, encoding them in an effective description.

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“Grazie a tutti.”

Matteo Zortea

Chapter 1

Introduction and outline

1.1 Quantum chromodynamics and its phase diagram

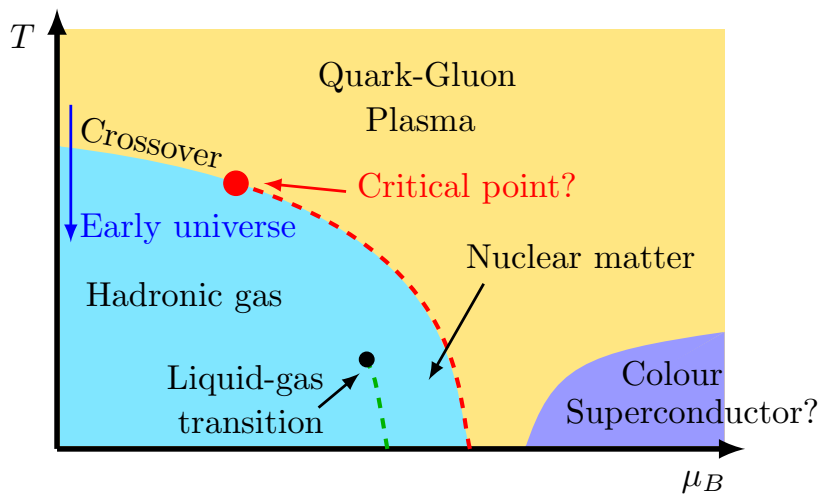


FIGURE 1.1: Conjectured phase diagram of Quantum Chromodynamics. At low temperature and chemical potential, Quantum Chromodynamics exists in a chirally broken state, where quarks and gluons are confined into hadrons and cannot be observed as single particles. As temperature or chemical potential are raised above a critical threshold, chiral symmetry is restored and particles lie in a quark-gluon plasma state. The existence and location of the critical point, as well as the colour superconducting state, is still matter of ongoing research.

Quantum Chromodynamics (QCD) plays a fundamental role in our understanding of the Standard Model. It is the theory describing the strong force among quarks, gluons and the composition of hadrons such as protons and neutrons, which constitutes the base of most of the matter around us.

The diagram reported in figure 1.1 illustrate the phases of QCD as a function of temperature (T) and baryon chemical potential (μ_B). At low temperature and chemical potential, QCD exhibits confinement [1, 2], a phenomenon for which only colour-neutral states can be observed. The confined phase is characterised by a spontaneously broken chiral symmetry, while the restoration of chiral symmetry is expected in the transition towards a quark-gluon plasma at high temperatures or chemical potential [3–6]. Moreover, also new states of matter are hypothesized, such as the existence of the colour superconducting state [7–9].

Much more has to be studied in the phase diagram, such as the precise determination of the location and nature of the critical point (if it exists at all). It's a huge challenge that needs the combination of theory, experiments, and simulations.

On the theoretical side, a wide number of results have been obtained from functional methods such as the functional renormalisation group and (fRG) and Dyson-Schwinger equations (DSE) [10–12]. The major experiments are performed at the Relativistic Heavy Ion Collider (RHIC) in New York and the Large Hadron Collider (LHC) at CERN. The Facility for Antiproton and Ion Research (FAIR) is a new collider under construction. On the simulation side, lattice field theory has become the standard approach to study QCD and particle physics and a large number of results have been obtained [13–15].

For a more detailed overview of the feature of the QCD phase diagram see [16, 17].

1.2 Lattice Field Theory

Computing physical quantities from first principle approaches is not only hard to do, but results often impossible since a naïve approach yields to diverging results. In fact, most quantum field theories need to be regularised by introducing an ultraviolet cutoff. To do this, one often relies on expansion techniques such as perturbation theory, in which one tries to regularise the theory order by order in an expansion on the interaction coupling, yielding finite quantities that depend on the truncation order. While this method is capable of producing incredibly precise results, it fails completely in treating non-perturbative phenomena, namely effects that cannot be captured by any order in the expansion or that are typical of strongly interacting systems. Phenomena such as confinement are non-perturbative.

A standard formulation of quantum field theory is also not much suitable for numerical computations, since the path integral measure and the action are infinitely dimensional objects.

Lattice field theory [18–21], which was proposed by Kenneth G. Wilson in 1974 [22], is meant at first as a powerful non-perturbative regularisation tool to prevent divergences to occur and render the computation of the correlation functions finite. Moreover, it also provides a framework to study quantum field theory numerically on a computer. In order to accomplish this, one defines the theory on a spacetime lattice and makes use of statistical methods such as Monte Carlo algorithms to compute observables. One should not view the lattice as an approximation to the continuum theory. It rather provides a definition of a theory that is undefined directly in the continuum [23]. One may then wonder how can one reconstruct a result in the continuum theory, keeping the results finite, and matching the results on the discretised theory to physically measured ones. This task, far from being simple, will be discussed in the next chapters.

1.3 The emergence of effective theories

A central role in our investigation will be played by the Wilsonian perspective of the Renormalization Group (RG). Within this paradigm, renormalization ceases to be merely a tool for taming divergences; the Wilsonian RG approach provides a rigorous way to perform a change of scale in a physical theory. The Wilson framework naturally brings to the concept of effective theory, a model where irrelevant degrees of freedom are integrated out and replaced by an effective description in terms of the relevant quantities at the given scale.

Effective theories are widely used to study QCD, aiming at tailoring to specific aspects of the strong force dynamics. Among others, we mention the Nambu-Jona-Lasinio model [24, 25], the Gross-Neveu model, the Quark-Meson model. All of these theories are Yukawa-type theories. This fact motivates the choice of model in our work.

1.4 Outline

Chapter 2 will be devoted to the introduction of the theoretical background that supports this work.

We will start with the Kadanoff and Wilson approach to renormalisation in statistical physics and quantum field theory. Lattice field theory will then be introduced as a powerful tool to simulate quantum field theory numerically and the basics of the formulation will be provided. We then turn to stochastic quantisation, the relation between noise and quantum fluctuations and how the standard formulation is modified by the presence of coloured noise. This will result in a deep connection between coloured stochastic quantisation and the renormalisation group. We then turn to the discussion of chiral symmetry and its breaking, both in the continuum and on the lattice. The chapter will end with a description of the model on which the above techniques are applied, namely a Yukawa theory.

Chapter 3 is devoted to detail techniques and methodologies of the project. We will start by explaining how the continuum model is discretised on a spacetime lattice and how it can be simulated via a Monte Carlo method based on stochastic quantisation. Some possible applications of coloured noise in lattice QFT will then be listed, and the numerical experiments that will be carried out in the last chapter will be described. The chapter will end with a description of the relevant observables that are employed to study a lattice quantum field theory.

Chapters 4 and 5 are devoted to the numerical investigation.

First, some preliminary analysis will be carried out. The procedure of extracting fermionic masses in the simulation will be illustrated, and a panoramic view over the general phase structure of the theory will be provided. The latter will guide the choice of parameter settings for all the simulations. It will then be shown how coloured noise can be used to provide a smooth interpolation between the fully classical and fully quantum picture. The introduction of noise can cause qualitative change in the behaviour of the system. In particular, it will be shown how quantum fluctuations can trigger a chiral phase transition in the model. Finally, the connection with the Kadanoff-Wilson RG will be exploited to cool a simulation by systematically encoding ultraviolet (UV) fluctuations in a redefinition of the classical action, without altering the physical content of the theory.

Chapter 2

Theoretical background

In this chapter we want to provide with an overview on the general theoretical framework that supports this work, and introduce the main concepts for the successive parts. Each section in this chapter is, by no means, meant as an exhaustive treatment. The description will be quite conceptual, rather than technical, and aims at recalling the main ideas and fix conventions. We ask the reader to consult appropriate references, which will be given in the corresponding sections, for a more detailed treatment of the topics.

2.1 The renormalisation group

Landau's mean field approach to study phase transition [26] gained wide popularity in the 1930's and 40's, since it was able to describe critical properties of many systems and it provided inspiration for the later Landau-Ginzburg theory of superconductivity [27]. Thus, it was soon proved to be inaccurate to predict experimentally well proven properties of systems near their critical point [28, 29], such as, for example, the Kondo problem [30]. This is because, being a mean field theory, it did not take into account the role of spatial fluctuations. The idea of block-spin transformation, systematically developed by Kadanoff [31], made a big step towards a deeper understanding of the scaling behaviour, and formed the basis for the later works of Wilson [29, 32, 33], which still constitutes the basis for modern approaches to renormalisation in field theory and statistical physics.

2.1.1 Block-spin RG

To illustrate the idea, let us consider a set of spins whose magnetisation is described by a function $\phi(x)$. The spins are located on a discrete lattice \mathcal{L} with spacing a , so that the function assumes values only at such sites $\phi(x_i) = \phi_i \neq 0 \Leftrightarrow x_i \in \mathcal{L}$. Suppose then that their interaction is described by a certain action $S[\phi]$ and a partition function

$$Z = \sum_{\{\phi\}} e^{-S[\phi]},$$

where the sum is performed over the possible configurations of the field ϕ .

We now want to introduce a coarse-grained (or blocked) field $\bar{\phi}$ within a spacetime cell of volume \mathcal{V} . Such a coarse grained field can be defined, for example, as an average over the spins within the cell \mathcal{V} . If the spins can only be 0 or 1 like in the Ising model, then we might opt for a majority rule [34].

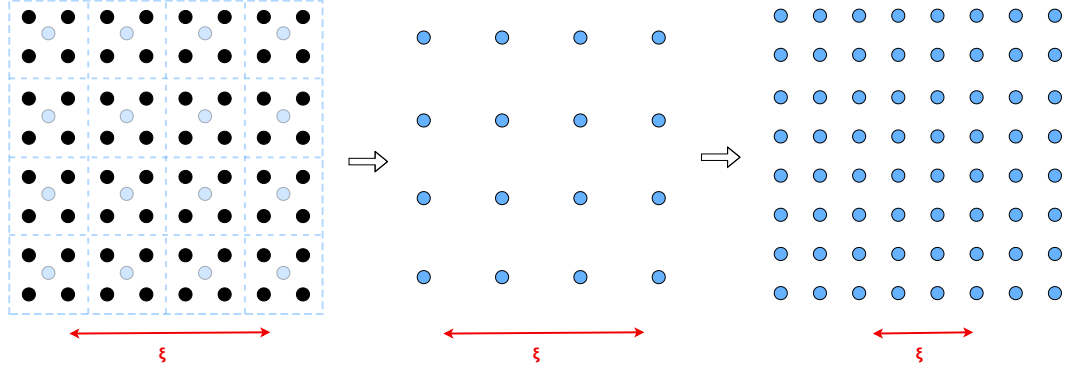


FIGURE 2.1: The two steps of the block-spin transformation. The black dots indicate the original field ϕ , while the blue dots indicate the coarse-grained field $\bar{\phi}$.

We now want to find a new action S_b such that

$$Z = \sum_{\phi} e^{-S(\phi)} = \sum_{\bar{\phi}} e^{-S_b(\bar{\phi})}. \quad (2.1)$$

Suppose for the moment that such S_b has been found. The coarse-graining procedure comes with a loss of resolution since the spacing is changed $a \rightarrow 2a$. Hence one can rescale distances and momenta in the new action via $a \rightarrow a/2$, $p \rightarrow 2p$ and then compare the result with the initial action. This constitutes the second step of the block-spin transformation.

The whole procedure can be iterated multiple times, and can be thought as a zoom-out with a corresponding coarse graining, in order to describe the system in terms of the relevant scales as pictured in figure 2.1. As made clear in the picture, the physical correlation length ξ is reduced by the block-spin step, since the description is now in terms of the coarse field $\bar{\phi}$. There are only two exceptions for this, either the correlation length is zero, or infinite. The latter case represents a fixed-point of the RG, and the system exhibits scale invariance.

Note that the condition (2.1) is non-trivial. One can, in principle, build an ad-hoc action that fulfills the condition, but it is complicated, since S_b can be also be very different from S . For example, if the action $S[\phi]$ contains only nearest-neighbour interactions, the new action $S'[\bar{\phi}]$ can contain higher order interactions such as next-to-nearest neighbour, or even more. In principle, all the terms compatible with the original symmetries are allowed, and one has often to rely on some approximations. For example, if one is interested in long range properties of the system, and the volume \mathcal{V} is sufficiently small, one can assume the functional form of the action S to

remain approximately the same, with the only change due to the dimensional rescaling of dimensionful quantities mentioned above. Thus, as the procedure is iterated and the correlation length scale is approached, one has to take account higher order interactions. For an example of the explicit construction of the RG for an Ising model, see [34].

2.1.2 Wilsonian RG

The Wilsonian picture of renormalisation [29, 32] is formulated in momentum space and in general is more suitable for theories in the continuum.

The idea is that a physical theory observed at an energy scale Λ can be seen as an effective theory of a more fundamental one, defined at scale $\Lambda_0 > \Lambda$.

To see how this can happen, let us consider a theory defined by the action $S[\phi]$ at scale Λ_0 , and let us split the field as

$$\phi = \bar{\phi} + \varphi$$

where $\bar{\phi}$ are fields with momenta $p^2 \leq \Lambda^2$ and φ are fields with $\Lambda^2 < p^2 \leq \Lambda_0^2$.

This allows to split the action as

$$S[\bar{\phi} + \varphi] = S_\Lambda[\bar{\phi}] + \delta S[\bar{\phi}, \varphi]$$

including all the dependence on φ in $\delta S[\bar{\phi}, \varphi]$.

The path integral can be rewritten as

$$\begin{aligned} Z &= \int D\phi e^{-S[\phi]} = \int D\bar{\phi}_\Lambda e^{-S_\Lambda[\bar{\phi}]} \int D\varphi e^{-\delta S[\bar{\phi}, \varphi]} \\ &= \int D\bar{\phi}_\Lambda e^{-S_\Lambda[\bar{\phi}] - S_{\text{UV}}[\bar{\phi}]} \\ &= \int D\bar{\phi}_\Lambda e^{-S_\Lambda^{\text{eff}}[\bar{\phi}]}, \end{aligned}$$

where

$$S_\Lambda^{\text{eff}} = S_\Lambda[\bar{\phi}] + S_{\text{UV}}[\bar{\phi}], \quad S_{\text{UV}}[\bar{\phi}] = -\log \left(\int D\varphi e^{-S_\Lambda[\bar{\phi}, \varphi]} \right).$$

$S_{\text{UV}}[\bar{\phi}]$ encodes all the information of UV modes with $\Lambda^2 < p^2 \leq \Lambda_0^2$.

The action $S_\Lambda[\bar{\phi}]$ has the same functional form of the initial action $S[\phi]$, but it is now defined only for fields with momenta $p^2 \leq \Lambda^2$. Instead, $S_\Lambda^{\text{eff}}[\bar{\phi}]$ constitutes an effective description of the original theory at scale Λ and depends only on degrees of freedom with $p^2 \leq \Lambda^2$.

One can then operate a rescaling of distances and momenta to complete the Wilson RG step, according to the parameter

$$s^2 = \Lambda^2 / \Lambda_0^2.$$

If, for example, d_g is the energy dimension of a coupling g in the action, then the rescaling transformation causes

$$g \rightarrow s^{d_g} g.$$

The same applies to the fields and other dimensionful quantities. If the corrections from S_{UV} are negligible, such as for high cutoff $\Lambda \approx \Lambda_0$, then this is the only contribution to the change in the couplings and fields due to the Wilson step. More in general, for higher order iterations of the procedure, the couplings' dependence on



FIGURE 2.2: The Wilsonian perspective of the renormalisation group. Ultraviolet quantum fluctuations above the scale Λ are integrated out and effectively encoded in the coarse-grained action S_{Λ}^{eff} .

s is captured by the β -functions

$$\beta_g = \frac{d}{ds} g(s).$$

A full treatment of RG is out of scope here and we ask the reader to consult more appropriate references for more details [35].

At this point, one can clearly see the analogy with the block-spin transformation introduced in the previous section. Performing the integral over high momenta modes can be thought as performing averages (coarse graining) over neighbours. This causes a loss of resolution which can be recovered by rescaling distances and momenta. The rescaling is essential for a description of fixed points, since it can be pictured as a zoom-out.

Therefore, the philosophy of Kadanoff and Wilson approaches was that the blocking transformation reduces the complexity of many-body systems by systematically reducing the number of degrees of freedom being taken into account, without changing the physical content of the theory [36, 37].

We want to conclude this section by mentioning that the splitting of the action can be done by writing

$$S_{\Lambda}^{\text{eff}}[\bar{\phi}] = S[\phi] + \Delta S_{\Lambda}[\bar{\phi}],$$

where $\Delta S_{\Lambda}[\bar{\phi}]$ is a regulating function which typically assumes the form

$$\Delta S_{\Lambda}[\bar{\phi}] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \bar{\phi}(-p) \Lambda^2 \left(\frac{1}{r_{\Lambda}(p^2)} - 1 \right) \bar{\phi}(p). \quad (2.2)$$

For a sharp momentum cutoff one has

$$r_{\Lambda}(p^2) = \theta(p^2 - \Lambda^2).$$

This result will allow for a deep connection between coloured stochastic quantisation and the functional renormalisation group, the latter describing the functional dependence of $S_{\Lambda}^{\text{eff}}[\bar{\phi}]$ on the cutoff scale Λ [38–42].

2.2 Lattice QFT and the continuum limit

The starting set up is the Euclidean formulation of quantum field theory, where one typically defines a path integral Z , which, for a general scalar field $\phi(x)$ and a fermion field $\psi(x)$, assumes the form

$$Z = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S[\phi, \psi, \bar{\psi}]}, \quad \mathcal{D}\xi = \prod_x d\xi_x, \quad \xi \in \{\phi, \psi, \bar{\psi}\}, \quad (2.3)$$

and correlation functions are computed via

$$\langle \xi_{x_1} \dots \xi_{x_n} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \xi_{x_1} \dots \xi_{x_n} e^{-S[\phi, \psi, \bar{\psi}]}, \quad \xi_{x_i} \in \{\phi_{x_i}, \psi_{x_i}, \bar{\psi}_{x_i}\}.$$

Let us then consider a lattice \mathcal{L} , with spacing a , and N_μ points in each spacetime direction μ , hence a physical length $L_\mu = N_\mu a$. For simplicity we restricted here to a scalar field ϕ and we will recall fermionic properties only when relevant, but what follows has general validity.

The action and the path integral measure are now taken over discrete quantities

$$\begin{aligned} S = \int d^d x \mathcal{L}(\phi(x)) &\rightarrow S = a^d \sum_{n \in \mathcal{L}} \mathcal{L}(\phi(n)), \\ \prod_x d\phi(x) &\rightarrow \prod_{n \in \mathcal{L}} d\phi(n), \end{aligned}$$

where $\mathcal{L}(\phi)$ is the Lagrangian density function.

The path integral is hence

$$Z = \int \prod_n d\phi(n) e^{-S[\phi]},$$

and the probability of a field configuration ϕ is

$$p(\phi) = \frac{1}{Z} e^{-S[\phi]}. \quad (2.4)$$

Expectation values of observables are computed as

$$\langle O(\phi) \rangle = \frac{1}{Z} \int \prod_n d\phi(n) O(\phi) e^{-S[\phi]}. \quad (2.5)$$

In order to simulate a theory and perform the above sums one has to go to a finite volume and impose boundary conditions. In the space directions, we take periodic conditions

$$\begin{aligned} \phi(t, \vec{x}) &= \phi(t, \vec{x} + \vec{T}), \\ \psi(t, \vec{x}) &= \psi(t, \vec{x} + \vec{T}). \end{aligned}$$

A finite time extent is related to the temperature of the system [19, 43] via

$$\beta = 1/T = 1/L_t,$$

and boundary conditions are chosen depending on the spin-statistics of the corresponding particles, namely periodic conditions for bosons, and anti-periodic for fermions

$$\begin{aligned} \phi(t, \vec{x}) &= \phi(t + T, \vec{x}) && \text{bosons,} \\ \psi(t, \vec{x}) &= -\psi(t + T, \vec{x}) && \text{fermions.} \end{aligned}$$

Such a formulation naturally brings a momentum cutoff $\Lambda = \pi/a$ since now all the momenta are restricted to the first Brilloune zone $p_\mu \in [-\pi/a, \pi/a]$.

To compute observables one relies on Monte-Carlo methods to generate field configurations, sampling the distribution (2.4) and convergence to the statistical value given by (2.5) is expected in the limit of infinite samples $N_{\text{samp}} \rightarrow \infty$.

To recover the continuum results, one has to take $V \rightarrow \infty, a \rightarrow 0$ ¹, but this task cannot be done so straightforwardly. Continuum limits of lattice theories are intimately connected to the existence of critical points. To see why this is the case, consider the dimensionless mass gap

$$\hat{\xi} = m a$$

of a certain theory. The quantity ξ is also called correlation length and it is related to the exponential decay of correlation functions between local observables measured at different points on the lattice. When taking the continuum limit, we want $a \rightarrow 0$, while having a finite physical mass m . This implies that the correlation length $\hat{\xi}$ has to diverge: in the language of statistical physics, this is a second order phase transition. Of course, to bring the system at its critical point, where such phase transition happens, one has to tune the bare parameters g_0^i of the theory to their critical values g_0^{i*} . To do this, one should find the zeros of the lattice beta functions

$$\beta_g^{\text{latt}} = a \frac{d}{da} g(a) \stackrel{!}{=} 0, \quad a = \pi/\Lambda.$$

As this is quite hard task to do on the lattice, one typically relies on some approximation schemes such as employing perturbative continuum beta functions.

In any case, from this description it should emerge that the spacing a should not be treated as a free parameter in the continuum limit, but rather a dynamically determined quantity that depends on the couplings of the theory.

Note that in the limit $a \rightarrow 0$, one has $\Lambda \rightarrow \infty$. If one's scope is to simulate an effective theory which is expected to hold only up to a scale Λ_{phys} , one must have $\Lambda \leq \Lambda_{\text{phys}}$, with a consequent lower bound on the lattice spacing

$$a \geq a_{\text{phys}} = \pi/\Lambda_{\text{phys}}.$$

¹The order here is important, see for example [44, 45]

2.3 Stochastic quantisation

The idea of stochastic quantisation [46, 47] is that Euclidean Quantum Field theory can be thought as a system in thermal equilibrium with a heat reservoir and hence described as a stochastic process via the Langevin equation. For this, one has to introduce a fictitious time variable τ that labels the state $\phi(\tau, x)$ of the system during the evolution.

2.3.1 Standard stochastic quantisation

Let us consider, for example, a scalar field ϕ with a Euclidean action $S[\phi]$ and the following Langevin equation

$$\partial_\tau \phi(\tau, x) = -\frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta(\tau, x), \quad (2.6)$$

where $K_\phi(\tau) \equiv -\delta S[\phi]/\delta \phi(\tau, x)$ is the drift term and $\eta(\tau, x)$ is a random white noise field assumed to be normally distributed

$$P(\eta) = \frac{\exp\left(-\frac{1}{4} \int_{\tau, x} \eta^2(\tau, x)\right)}{\int D\eta \exp\left(-\frac{1}{4} \int_{\tau, x} \eta^2(\tau, x)\right)},$$

which, in particular, implies

$$\langle \eta(x, \tau) \rangle = 0, \quad \langle \eta(x, \tau) \eta(x', \tau') \rangle = 2 \delta(x - x') \delta(\tau - \tau'). \quad (2.7)$$

Stochastic average with respect to the measure $P(\eta)$ are computed via

$$\langle A(\eta) \rangle = \int D\eta P(\eta) A(\eta).$$

In momentum space, (2.7) becomes

$$\begin{aligned} \langle \eta(p, \tau) \rangle &= \left\langle \int_x e^{ipx} \eta(x, \tau) \right\rangle = \int_x \langle e^{ipx} \eta(x, \tau) \rangle = 0, \\ \langle \eta(p, \tau) \eta(q, \tau') \rangle &= \left\langle \int_{xy} e^{ipx+iqy} \eta(x, \tau) \eta(y, \tau') \right\rangle \\ &= \int_{xy} e^{ipx+iqy} \langle \eta(x, \tau) \eta(y, \tau') \rangle \\ &= 2 (2\pi)^2 \delta(p + q) \delta(\tau - \tau'). \end{aligned} \quad (2.8)$$

In absence of the noise term $\eta(\tau, x)$, equation (2.6) simply represents an evolution of the field towards the minimum of the action, and at equilibrium the field is constrained to $\partial_\tau \phi(x, \tau) = 0 = \delta S[\phi]/\delta \phi(\tau, x)$, namely to the classical equations of motion.

For any observable O , which is a function of the field, one has, for fixed time τ ,

$$\langle O(\phi(\tau)) \rangle = \int D\eta P(\eta) O(\phi(\tau)),$$

from which it follows straightforwardly using the Langevin equation and $\langle \eta \rangle = 0$, one has

$$\frac{d}{d\tau} \langle O(\phi(\tau)) \rangle = \left\langle \frac{\delta O}{\delta \phi(\tau, x)} \partial_\tau \phi(\tau, x) \right\rangle = - \left\langle \frac{\delta O}{\delta \phi(\tau, x)} \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle.$$

It follows trivially that for $O(\phi) = \phi(\tau, x)$

$$\frac{d}{d\tau} \langle \phi(\tau, x) \rangle = - \left\langle \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle \xrightarrow{\text{Equilibrium}} \left\langle \frac{\delta S}{\delta \phi(\tau, x)} \right\rangle = 0.$$

This also provides a consistency check for the correct implementation of the simulation, since the drift $K_\phi = -\delta S / \delta \phi$ is computed numerically during the evolution. One can also derive a correspondent Fokker-Planck equation [48], which can be proven to have a stationary distribution if the action is bounded from below, given by [47]

$$\mathcal{P}(\phi) = \frac{1}{Z} \exp(-S[\phi]). \quad (2.9)$$

This allows one to compute correlation functions as moments of the probability distribution (2.9). In particular, one has

$$\langle O \rangle_{P(\eta)} = \langle O \rangle_{\mathcal{P}(\phi)} \equiv \langle O \rangle. \quad (2.10)$$

Hence, the introduction of a white noise in the Langevin evolution (2.6), leads to the probability distribution (2.9), which is the same factor that one gets in the standard path integral quantisation of quantum field theory.

2.3.2 Stochastic quantisation with coloured noise

In the stochastic quantisation procedure the noise which accounts for the quantum fluctuations of the theory is assumed to be white, as defined in equations (2.7), (2.8). We now want to examine the dynamics in presence of a coloured noise, writing the Langevin equation as

$$\partial_\tau \phi(x, \tau) = - \frac{\delta S[\phi]}{\delta \phi(\tau, x)} + \eta_{\text{col}}(x, \tau),$$

with $\eta_{\text{col}}(x, \tau) = r_\Lambda(x) \eta(x, \tau)$. In particular, here we restrict to the regulating function defined as a sharp cutoff in momentum space

$$r_\Lambda(p) = \theta(\Lambda^2 - p^2). \quad (2.11)$$

The noise field in momentum space is then

$$\begin{aligned} \eta_{\text{col}}(p, \tau) &= \mathcal{F}[\eta_{\text{col}}(x, \tau)] = \mathcal{F}[r_\Lambda(x, \tau) \eta(x, \tau)] = \mathcal{F}[r_\Lambda(p, \tau)] \star \mathcal{F}[\eta(p, \tau)] \\ &= \theta(\Lambda^2 - p^2) \eta(p, \tau), \end{aligned}$$

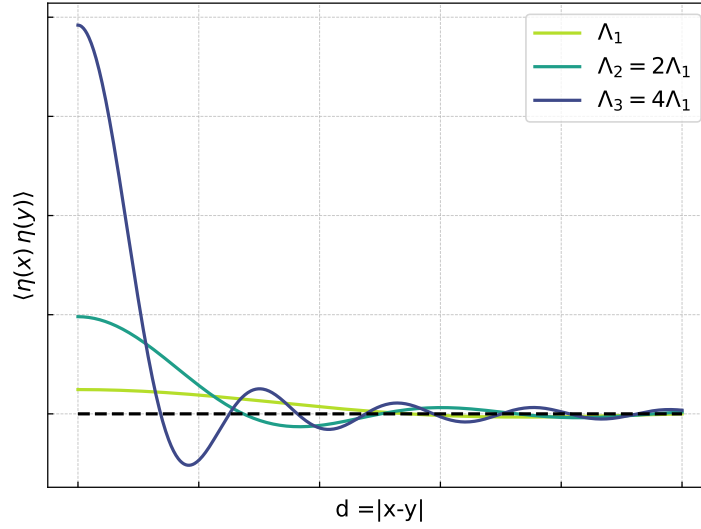


FIGURE 2.3: Noise correlation as a function of $d = |x - y|$ for three different values of the cutoff $\Lambda_1 < \Lambda_2 < \Lambda_3$, in arbitrary units. The plot is qualitative, but shows clearly that with a regulated noise, only the short-distance behaviour is affected.

where \mathcal{F} indicates the Fourier transform and \star the convolution product. An interesting quantity to look at is the position space noise correlation function

$$\begin{aligned}
 C_\eta(x, \tau, y, \tau') &= \langle \eta_{\text{col}}(x, \tau) \eta_{\text{col}}(y, \tau') \rangle \\
 &= \frac{1}{(2\pi)^4} \int D\eta P(\eta) \left[\int_{p,q} e^{-ipx - iqy} \eta_{\text{col}}(p, \tau) \eta_{\text{col}}(q, \tau') \right] \\
 &= \frac{1}{(2\pi)^4} \int_{p,q} e^{-ipx - iqy} \int D\eta [P(\eta) \eta(p, \tau) \eta(q, \tau')] \\
 &\quad \times \theta(\Lambda^2 - p^2) \theta(\Lambda^2 - q^2) \\
 &= \frac{2}{(2\pi)^2} \int_{p,q} e^{-ipx - iqy} \delta(p + q) \theta(\Lambda^2 - p^2) \theta(\Lambda^2 - q^2) \delta(\tau - \tau') \\
 &= \frac{2}{(2\pi)^2} \int_p e^{-ip(x-y)} \theta(\Lambda^2 - p^2) = \frac{1}{\pi} \int_0^\Lambda d\omega \omega J_0(\omega|x-y|),
 \end{aligned} \tag{2.12}$$

where $J_0(x)$ is a Bessel function of the first order. The integral in the last line is computed numerically as a function of $d = |x - y|$ and reported in figure 2.3 for three different values of the cutoff $\Lambda_1 < \Lambda_2 < \Lambda_3$. This shows clearly that for $|x - y| \ll 1/\Lambda$ the noise is now correlated, while for $|x - y| \gg 1/\Lambda$ the correlation function vanishes, as in the white noise case. In other words, only the short-length behaviour of the system is affected by the introduction of such a regulating term, as one could expect.

Another intuitive and interesting aspect of the dynamics in the presence of coloured noise can be deduced by looking at the field expression in terms of the retarded Langevin Green function [47], which is here not derived, but reported from [49]

$$\phi(x, \tau) = \int_{x'} \int_{-\infty}^{\tau} d\tau' G(x - x', \tau - \tau') \left[r_\Lambda(\Delta_x) \eta(x, \tau') - \frac{\delta S}{\delta \phi} \Big|_{p=0} \phi(x', \tau) \right],$$

where

$$G(x - x', \tau - \tau') = \theta(\tau - \tau') \int_p e^{-ip \cdot (x - x')} e^{-(\tau - \tau')(p^2 + m^2)}.$$

By looking at the first term in the square bracket, one can conclude that there is no propagation of modes with momentum $p^2 \geq \Lambda^2$ due to the noise term, but one can still have contribution from modes $p^2 > \Lambda^2$ from the second term, which corresponds to the deterministic part of the equations of motion. Stated differently, UV quantum fluctuations with $p^2 > \Lambda^2$ are removed from the dynamics of ϕ , but classical degrees of freedom still contribute.

Generally speaking, the stationary distribution probability of the regulated stochastic process is given by [49]

$$\mathcal{P}_\Lambda(\phi) = \frac{1}{Z} \exp(-S_\Lambda[\phi]) = \frac{1}{Z} \exp(-(S[\phi] + \Delta S_\Lambda[\phi])), \quad (2.13)$$

where the correction term $\Delta S_\Lambda[\phi]$ reads, for a regulator $r_\Lambda(p^2)$,

$$\Delta S_\Lambda[\phi] = \frac{1}{2} \int_p \phi(p) \Lambda^2 \left(\frac{1}{r_\Lambda(p^2)} - 1 \right) \phi(-p).$$

Note that this is precisely the same regulating term that implements the cutoff on the action in the context of the Wilsonian RG, as it was shown in equation (2.2). This observation opens a deep connection between stochastic quantisation with coloured noise and the (functional) renormalisation group.

2.4 Chiral symmetry

In this section we want to introduce chiral symmetry and its breaking, both in the continuum and on the lattice.

It is useful to adopt the notation

$$\psi = (\psi^{(1)}, \dots, \psi^{(N_f)}).$$

We then introduce left-handed and right-handed spinors

$$\psi_L = (1 - \gamma_5) \psi, \quad \psi_R = (1 + \gamma_5) \psi,$$

for which

$$\psi = \frac{(1 - \gamma_5)}{2} \psi + \frac{(1 + \gamma_5)}{2} \psi = \psi_L + \psi_R.$$

2.4.1 Chiral symmetry in the continuum

The free massless Dirac Lagrangian

$$\mathcal{L}_D = \bar{\psi} \not{\partial} \psi = \bar{\psi}_L \not{\partial} \psi_L + \bar{\psi}_R \not{\partial} \psi_R \quad (2.14)$$

is symmetric under the chiral group $SU(N_f)_L \times SU(N_f)_R$, namely

$$\begin{aligned} \psi_L(x) &\rightarrow U_L \psi_L(x), & \bar{\psi}_L(x) &\rightarrow \bar{\psi}_L(x) U_L^\dagger, \\ \psi_R(x) &\rightarrow U_R \psi_R(x), & \bar{\psi}_R(x) &\rightarrow \bar{\psi}_R(x) U_R^\dagger, \end{aligned}$$

for $U_L, U_R \in SU(N_f)$.

In terms of the full spinor ψ , the chiral symmetry can be written as

$$\psi \rightarrow M \psi, \quad \bar{\psi} \rightarrow \bar{\psi} M^\dagger,$$

where

$$M = e^{i(\theta_a \tau^a + \gamma_5 \beta_a \tau^a)}$$

and $\tau^a \in su(N_f)$ are the generators of the Lie algebra of $SU(N_f)$, and for the particular case of $N_f = 2$ they are the Pauli matrices.

One can then see the chiral group as

$$SU(N_f)_L \times SU(N_f)_R \simeq SU(N_f)_V \times SU(N_f)_A,$$

where $SU(N_f)_V$ is the isospin subgroup, characterised by $\beta_a = 0$, and $SU(N_f)_A$ is the axial rotation subgroup, characterised by $\theta_a = 0$.

To be more precise, the full invariance group of the classical action is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_V,$$

where the axial and vector symmetry transformations are, respectively,

$$\psi \rightarrow e^{i\theta\gamma_5} \psi, \quad \psi \rightarrow e^{i\theta} \psi.$$

One can thus prove that axial symmetry is broken by quantum anomalies (see for example [50]), so that the symmetry in the quantum case is

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V.$$

If equal masses for each flavour are introduced, the classical action assumes the form

$$\mathcal{L}_D = \bar{\psi} \not{\partial} \psi = \bar{\psi}_L \not{\partial} \psi_L + \bar{\psi}_R \not{\partial} \psi_R + m (\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R) \quad (2.15)$$

and one can see that the mass term breaks the symmetry explicitly. This motivates the introduction of the chiral condensate

$$\langle \bar{\psi} \psi \rangle = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R$$

as an order parameter for chiral symmetry. In the presence of equal masses, the symmetry group reduces to the isospin subgroup, diagonal in flavour space

$$SU(N_f)_V \times U(1)_V.$$

Finally, if the fermions have different masses, the symmetry group reduces to

$$\underbrace{U(1) \times \cdots \times U(1)}_{N_f} \times U(1)_V.$$

2.4.2 Chiral symmetry on the lattice

The essence of chiral symmetry for fermions can be expressed as [20]

$$\{\gamma_5, D\} = 0. \quad (2.16)$$

Implementing chiral symmetry on a finite-volume spacetime lattice is a hard task. This is because, as proven by Nielsen and Ninomiya [51], one either has chiral symmetry, or solves the doubling problem. In particular, the insertion of the Wilson term in the action, as shown in Appendix B, causes explicit breaking of the symmetry. This fact makes Wilson fermions not the most appealing choice to study chiral properties on the lattice. Thus, many options have been proposed to circumvent the issue. As an example we mention the approach by Ginsparg and Wilson [52], who proposed to modify the chiral symmetry condition (2.16) to

$$\{\gamma_5, D\} = a D \gamma_5 D$$

The right hand-side of the equation vanishes in the continuum limit $a \rightarrow 0$. In this way, one can define chiral symmetry on the lattice remaining consistent in the continuum. This approach will not be pursued further and we ask the reader to consult appropriate references [19, 20] for a detailed treatment. Our approach to study chiral symmetry will be discussed and motivated in section 5.2.

2.5 Yukawa theory

2.5.1 Description of the model

Let us consider the Yukawa theory defined by the action

$$\begin{aligned} S[\phi, \psi, \bar{\psi}] &= S_\phi[\phi] + S_\psi[\psi, \bar{\psi}] + S_{\text{int}}[\phi, \psi, \bar{\psi}], \\ S_\phi[\phi] &= \int_x \phi_x \left(-\frac{\partial_x^2}{2} + \frac{m_\phi^2}{2} \right) \phi_x + \frac{\lambda}{4!} \phi_x^4, \\ S_\psi[\psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} (\not{\partial}_x + m_q) \psi_x^{(f)}, \\ S_{\text{int}}[\phi, \psi, \bar{\psi}] &= \int_x \sum_{f=1}^{N_f} g \bar{\psi}_x^{(f)} \phi_x \psi_x^{(f)}. \end{aligned} \quad (2.17)$$

One can see that the action is made of a scalar part $S_\phi[\phi]$, a fermionic part $S_\psi[\psi, \bar{\psi}]$ and a Yukawa interaction term $S_{\text{int}}[\phi, \psi, \bar{\psi}]$.

In practice we will work with fixed number of flavours $N_f = 2$, but it is useful to keep track of N_f and set it to its value when needed.

It is also convenient for later purposes to define the operators K, D represented in position space as

$$\begin{aligned} K(x, y) &= \left(-\partial_x^2 + m_\phi^2 \right) \delta(x, y), \\ D(x, y) &= (\not{\partial}_x + m_q + g\phi) \delta(x, y), \end{aligned} \quad (2.18)$$

and in momentum space as

$$\begin{aligned}\tilde{K}(p, q) &= \int_{x, y} e^{-ipx} \left(\partial_x^2 + m_\phi^2 \right) \delta(x, y) e^{iqy} = \left(\frac{p^2}{2} + \frac{m_\phi^2}{2} \right) \delta(p, q), \\ \tilde{D}(p, q) &= \int_{x, y} e^{-ipx} (\not{\partial}_x + m_q + g\phi) \delta(x, y) e^{iqy} = (\not{p}_x + m_q + g\phi) \delta(p, q).\end{aligned}\tag{2.19}$$

This allows one to rewrite the action as

$$S[\phi, \psi, \bar{\psi}] = \int_{x, y} \left[\frac{1}{2} \phi_x K_{xy} \phi_y + \frac{\lambda}{4!} \phi_x^4 \delta(x, y) + \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} D_{xy} \psi_y^{(f)} \right].$$

2.5.2 Chiral symmetry in the Yukawa model

The action written in terms of ψ_L, ψ_R , reads

$$S = S_\phi + \int_x [\bar{\psi}_L D \psi_L + \bar{\psi}_R D \psi_R + (m_q + g\phi) (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)]. \tag{2.20}$$

The action is not invariant under the full chiral group, but if $m_q = 0$ it is symmetric under the discrete chiral transformation

$$\begin{aligned}\phi &\rightarrow -\phi, \\ \psi_L &\rightarrow \gamma_5 \psi_L, & \bar{\psi}_L &\rightarrow -\bar{\psi}_L \gamma_5, \\ \psi_R &\rightarrow \gamma_5 \psi_R, & \bar{\psi}_R &\rightarrow -\bar{\psi}_R \gamma_5.\end{aligned}$$

This symmetry can be found in other toy models for QCD such as the Gross-Neveu model [53]. In order to generalise the model to the full chiral group, one has to consider an $O(4)$ scalar sector and interactions, since $O(4) \simeq SU(2) \times SU(2)$. Similar effective theories for QCD with such properties are, for example, the Nambu – Jona-Lasinio model [24, 25] and the Quark-Meson model [citationssssss](#). Thus, since this work is based on $1 + 1$ spacetime dimensions, spontaneous breaking of continuous symmetries is forbidden by the Mermin-Wagner theorem [54], hence we decided to opt for the $O(1)$ version.

We now want to discuss more in detail the phenomenom of (discrete) chiral symmetry breaking and how it can happen in the model. In fact, the latter can happen either explicitly in the classical action, if a finite quark mass is added, or spontaneously, if $\langle \phi \rangle \neq 0$.

To better see this, let us perform the fermionic path integral explicitly

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left(- \int_x \sum_{f=1}^{N_f} \bar{\psi}_x^{(f)} D \psi_x^{(f)} \right) = (\det D[\phi])^{N_f} = e^{N_f \text{Tr} \log(D[\phi])},$$

where the trace is performed over spacetime and spinor components.

The full path integral can now be expressed in terms of the resulting effective action for the scalar fields

$$Z = \int \mathcal{D}\phi e^{-S_{\text{eff}}[\phi]},$$

with

$$S_{\text{eff}}[\phi] = S_\phi[\phi] - N_f \text{Tr}_{x,s} \log D[\phi]. \tag{2.21}$$

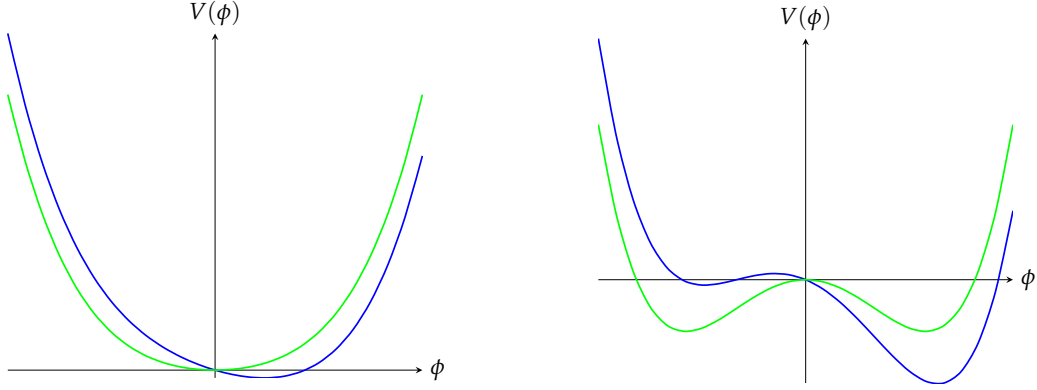


FIGURE 2.4: Symmetry breaking in the Yukawa model.
 (Left) Classical potential for $m_\phi^2 < 0$. (Right) Classical potential for $m_\phi^2 > 0$.

One can derive the classical equations of motion by imposing $\frac{\delta S}{\delta \phi} = 0$, here expressed in momentum space

$$(k^2 + m_\phi^2) \phi(x) + \frac{\lambda}{6} \phi^3(x) = N_f g \text{Tr}_s [D^{-1}(\phi(x))] = -N_f g \bar{\psi}(x) \psi(x). \quad (2.22)$$

For $\lambda = 0$, they highlight a simple proportionality relation between magnetisation and chiral condensate, which reads

$$\phi = -\frac{N_f g}{k^2 + m_\phi^2} \bar{\psi} \psi. \quad (2.23)$$

This relation, which was here derived at the classical level, is proven to hold also in mean field on the quantum level [55] and makes apparent the role of ϕ as a quark bilinear. When bosons self-interactions are added, namely when $\lambda \neq 0$, the full relation between ϕ and the chiral condensate $\bar{\psi} \psi$ is given by (2.23), but still one expects, qualitatively,

$$\langle \phi \rangle \sim \langle \bar{\psi} \psi \rangle. \quad (2.24)$$

A non-vanishing condensate is also related to a physical quark mass [55, 56], while the presence of magnetisation causes the breaking of $O(1)$ symmetry. If $\langle \phi \rangle = v$, one can write $\phi(x) = v + \varphi(x)$ and the massless lagrangian assumes the form

$$S = S_\phi + \int_x [\bar{\psi}_L D \psi_L + \bar{\psi}_R D \psi_R + g v (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)] + g \varphi(x) (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L), \quad (2.25)$$

hence showing that one can expect

$$\langle \phi \rangle \sim \langle \bar{\psi} \psi \rangle \sim m_q. \quad (2.26)$$

Figure 2.4 depicts the classical potential in the Yukawa theory. The $O(1)$ symmetry of the scalar sector is manifested in a single minimum located at $\phi = 0$ if $m_\phi^2 > 0$, or in two equivalent minima if $m_\phi^2 < 0$. As the Yukawa interaction is introduced, the symmetry can be broken either explicitly in presence of a fermionic bare mass, or spontaneously if the scalar field gains a non-zero expectation value. In this case, the classical potential assumes the shape given by the green line.

Chapter 3

Methods and algorithms

3.1 Discretisation of the Yukawa theory

In order to make the theory suitable for a numerical simulation on a computer, the continuum formulation of the Yukawa model, which has been introduced in section 2.5, has to be discretised. Here we provided a sketch of a discretisation procedure, and we refer to other resources [18–21] for further details.

For what concerns the bosonic part of the action, a discretisation can be done straightforwardly with the following replacements

$$\begin{aligned} \int d^x &\rightarrow a^2 \sum_x \\ \partial_t^2 + \partial_x^2 &= \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} \rightarrow \sum_\mu \left[\frac{\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}}{a^2} \right], \end{aligned}$$

which yields to the lattice action

$$\begin{aligned} S_\phi[\phi] &= a^2 \left(\frac{1}{2} \sum_{m,n} \phi_m K_{mn} \phi_n + \frac{\lambda}{4!} \sum_n \phi_n^4 \right) \\ &= \frac{1}{2} \sum_{m,n} \hat{\phi}_m \hat{K}_{mn} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \sum_n \hat{\phi}_n^4, \end{aligned}$$

where we expressed everything in dimensionless quantities

$$\begin{aligned} \hat{m}_\phi^2 &= a^2 m_\phi^2, \\ \hat{\lambda} &= a^2 \lambda, \\ \hat{K}_{mn} &= a^2 K_{mn}. \end{aligned} \tag{3.1}$$

The operator components \hat{K}_{mn} are the discretised version of (2.18)

$$\hat{K}_{mn} = - \sum_\mu [\delta_{m,n+\mu} + \delta_{m,n-\mu} - 2\delta_{m,n}] + \hat{m}_\phi^2 \delta_{mn} \tag{3.2}$$

and its representation in momentum space is

$$\begin{aligned}
\hat{K}_{p,q} &= \sum_{n,m} e^{ipn} \hat{K}_{nm} e^{-iqm} \\
&= \sum_{n,m} e^{ipn} \left(- \sum_{\mu} [\delta_{m,m+\mu} + \delta_{m,m-\mu} - 2\delta_{m,n}] + \hat{m}_{\phi}^2 \delta_{mn} \right) e^{-iqm} \\
&= \sum_n e^{i(p-q)n} \left[\hat{m}_{\phi}^2 + 2 \sum_{\mu} (1 - \cos(q_{\mu})) \right] \\
&= \left[\hat{m}_{\phi}^2 + \sum_{\mu} 4 \sin^2 \left(\frac{p_{\mu}}{2} \right) \right] \delta(p-q).
\end{aligned}$$

For what concerns the fermionic action, a naïve discretisation is not sufficient, due to the well known doubling problem [18, 19]. In this work Wilson fermions [22] are employed as a way to fix such issue. Details of this formulation are explained in Appendix B. Here, only the final discretised action is reported, which reads

$$S_{\psi} [\hat{\psi}, \hat{\psi}] + S_{\text{int}} [\hat{\phi}, \hat{\psi}, \hat{\psi}] = \sum_{f=1}^{N_f} \hat{\psi}_m^{(f)} \hat{D}_{mn} \hat{\psi}_n^{(f)}, \quad (3.3)$$

with ψ_n being a two-component spinor, and $\hat{D}_{m,n}$ the Wilson-Dirac operator defined as

$$\begin{aligned}
\hat{D}_{m,n} &= - \left(\frac{\Gamma_{+\hat{0}}}{2} \delta_{m,m+\hat{0}} + \frac{\Gamma_{-\hat{0}}}{2} \delta_{m,m-\hat{0}} + \frac{\Gamma_{+\hat{1}}}{2} \delta_{m,m+\hat{1}} + \frac{\Gamma_{-\hat{1}}}{2} \delta_{m,m-\hat{1}} \right) \\
&\quad + (2ar + \hat{m} + \hat{g}\phi) \delta_{s,s'} \delta_{m,n}.
\end{aligned} \quad (3.4)$$

Note that the interaction term $g \bar{\psi} \phi \psi$ has been included in the definition of D . The Wilson projectors $\Gamma_{\pm\hat{\mu}}$ are defined as

$$\Gamma_{\pm\hat{\mu}} = ar \mathbb{1}_s \mp \gamma_{\mu}.$$

Since $r \in [0, 1]$ is a free parameter, in this work we set $r = 1$, if not otherwise specified.

In summary the discretised action for the Yukawa model is

$$S [\hat{\phi}, \hat{\psi}, \hat{\psi}] = \sum_{m,n} \hat{\phi}_m \hat{K}_{m,n} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \hat{\phi}_m^4 \delta_{m,n} + \sum_{f=1}^{N_f} \hat{\psi}_m^{(f)} \hat{D}_{mn} \hat{\psi}_n^{(f)}, \quad (3.5)$$

with $\hat{K}_{mn}, \hat{D}_{mn}$ given respectively by (3.2) and (3.4).

For later reference, we also report the discretised version of the effective action (2.21)

$$\begin{aligned}
S_{\text{eff}}[\hat{\phi}] &= S_{\phi}[\hat{\phi}] - N_f \text{Tr}_{n,s} \log \hat{D} \\
&= \sum_{m,n} \hat{\phi}_m \hat{K}_{m,n} \hat{\phi}_n + \frac{\hat{\lambda}}{4!} \hat{\phi}_m^4 \delta_{m,n} - N_f \text{Tr}_{n,s} \log \hat{D}_{nn}.
\end{aligned} \quad (3.6)$$

The full discrete path-integral reads

$$Z = \int \prod_n d\hat{\phi}_n e^{-S_{\text{eff}}[\hat{\phi}]}. \quad (3.7)$$

In the remaining of this work, both the original action S and the effective action S_{eff} will be denoted by S for simplicity. It will be clear from the context which of the two we will be referring to.

3.2 Langevin Monte Carlo

The relations (??) suggest that equation (2.6) can be integrated numerically for discrete time steps τ_n to generate field configurations distributed according to (2.9). The simplest first-order integration algorithm is the Euler-Majorana scheme [46]

$$\phi(\tau_{n+1}, x) = \phi(\tau_n, x) - \epsilon \frac{\delta S[\phi]}{\delta \phi(\tau_n, x)} + \sqrt{\epsilon} \eta(\tau_n, x) + O(\epsilon^2),$$

where $\epsilon = \tau_{n+1} - \tau_n$. Higher order integration schemes are possible (see e.g. [57, 58]), but not adopted in this work, and an adaptive step size is employed as detailed in Appendix C. In this way, for any observable O , one can introduce a Monte-Carlo estimator $\langle O \rangle_*$ which converges to the expectation value given by (2.10) in the limit of infinite samples

$$\langle O \rangle_* = \frac{1}{N_{\text{samp}}} \sum_{i=1}^{N_{\text{samp}}} O_i \xrightarrow{N_{\text{samp}} \rightarrow \infty} \langle O \rangle = \frac{1}{Z} \int D\phi O(\phi) \exp(-S[\phi]), \quad (3.8)$$

where $O_i = O(\phi(\tau_i))$ is the sample of the observable O done at time τ_i .

For the discretised action of the Yukawa theory (3.6) the drift reads, explicitly,

$$\begin{aligned} \frac{\partial S}{\partial \hat{\phi}_m(\tau_n)} &= \frac{\partial S_{\hat{\phi}}}{\partial \hat{\phi}_m(\tau_n)} - N_f \text{Tr}_s \left[\sum_{j,k} \hat{D}_{jk}^{-1} \frac{\partial \hat{D}_{kj}(\hat{\phi})}{\partial \hat{\phi}_m(\tau_n)} \right] \\ &= \sum_l \hat{K}_{ml} \hat{\phi}_l + \frac{\hat{\lambda}}{6} \hat{\phi}_m^3 - \hat{g} N_f \text{Tr}_s \left[(\hat{D}_{mm})^{-1}(\hat{\phi}(\tau_n)) \right]. \end{aligned} \quad (3.9)$$

While the bosonic contribution can be computed in a straightforward manner, the computation of the fermionic contribution requires the inversion of the Dirac operator. This, in general, cannot be done straightforwardly, mainly due to computational reasons. In fact, the full Dirac operator would be a $(2 \cdot N_t \cdot N_x \cdot N_f)^2$ dimensional object and a full inversion would be very expensive. In fact, D^{-1} has to be recomputed at every step and the best available algorithm as today for the matrix inversion has a computational complexity of $O(n^{2.371552})$ [59]. To circumvent this, we use the bilinear noise scheme [57, 60] which is illustrated in Appendix C.

3.3 Applications of coloured noise in lattice QFT

After the general introduction on coloured noise given in the previous paragraph, let us now look more closely on the lattice formulation and at some possible applications of the technique, some of which will be studied numerically in chapter ??.

To this end, let us consider a two-dimensional lattice with side lengths L_t, L_x and spacing $a = a_x = a_t$. This implies a maximum momentum $p_{\max} = \pi/a$ in each spacetime direction and $N_x = L_x/a, N_t = L_t/a$ points in each direction. Let us also define

$$\Lambda_0^2 \equiv (p_{\max}^x)^2 + (p_{\max}^t)^2, \quad (3.10)$$

which indicates the maximum squared momentum on the given lattice.

We then consider a simulation with a regularised noise defined by a cutoff $\Lambda \leq \Lambda_0$ and we define a dimensionless parameter

$$s^2 = \frac{\Lambda^2}{\Lambda_0^2}, \quad 0 \leq s \leq 1. \quad (3.11)$$

Note that Λ implicitly defines a length scale given by $a_{\text{eff}} = \pi/\Lambda$.

3.3.1 Classical-to-quantum interpolation

The use of coloured noise, allows for a smooth interpolation between the fully classical and fully quantum picture. In fact, one can perform various simulations changing the value of the cutoff fraction s and adding or removing quantum degrees of freedom, as pictured in figure [add figure](#). This can be used either to investigate the role of quantum fluctuations in the system, or to remove irrelevant degrees of freedom, resulting in a speed-up of the simulation.

3.3.2 Noise-induced transition

Noise-induced transition is a well known field of stochastic dynamics [\[48, 61, 62\]](#) and consists in investigating whether noise can qualitatively affect the behaviour of a system. In our case, since noise is due to quantum fluctuations, we are interested in understanding if quantum fluctuations can trigger a phase transition with respect to the classical system, with the same parameters settings.

This question will be addressed in section [5.2](#) and it will be shown that for small negative bosonic mass, the classical system lies in a state of broken symmetry, while in the quantum case, the symmetry is restored.

3.3.3 Cooling and the continuum limit of effective theories

Following the paradigm of Kadanoff-Wilson of chapter [2](#), we want to use RG properties to encode quantum fluctuations in a redefinition of the couplings of the bare action. This removes short-distance fluctuations from the simulation, without changing the physical content of the theory. Note that this would, in principle, require a detailed knowledge of the β -functions of the theory. Thus, for sufficiently high cut-off, approximate Ansatz such as dimensional rescalings are generally enough. We will pursue this way, as follows.

Let us consider a simulation with $s = 1$ and a set of bare couplings $\{g_0^i\}$, and another simulation with $s' < 1$ and new set of couplings $\{g_0^{i'}\}$.

For what concerns the scalar part of the action, a dimensional rescaling, which corresponds to a tree-level RG transformation, is rather straightforward

$$\begin{aligned} \hat{m}_\phi^2 = (a^2 m_\phi^2) &\rightarrow s^2(a^2 m_\phi^2) = s^2 \hat{m}_\phi^2, & \hat{\lambda} = (a^2 \lambda) &\rightarrow s^2(a^2 \lambda) = s^2 \hat{\lambda}, \\ \hat{\phi} = \phi &\rightarrow \phi = \hat{\phi}. \end{aligned}$$

The fermionic part needs some more careful analysis. For simplicity, let us for the moment set $N_f = 1$.

In a lattice simulation one wants to perform the integral over the fermionic fields and works with the effective action (2.21). In this case the drift is given by equation (3.9), with the fermionic contribution

$$K_\psi = g \operatorname{Tr}_s D^{-1}, \quad (3.12)$$

or, in terms of dimensionless quantities,

$$\hat{K}_\psi = (ag) \operatorname{Tr}_s (aD)^{-1}.$$

This implies that under a lattice block-spin transformation, where $a \rightarrow sa$,

$$\hat{K}_\psi \rightarrow (sag) \operatorname{Tr}_s (saD)^{-1} = \hat{K}_\psi. \quad (3.13)$$

On the other side, when computing the drift via the original action (2.17), one gets

$$\begin{aligned} K &= -\frac{\delta S}{\delta \phi} = K_\phi - g \bar{\psi} \psi = \\ &= -\hat{K}_{mn} \phi_n - \frac{\lambda}{6} \phi^3 - g \bar{\psi} \psi. \end{aligned} \quad (3.14)$$

where, in the last row, \hat{K}_{mn} indicates the bosonic operator components (3.2).

The fermionic contribution is given by

$$K'_\psi = -g \bar{\psi} \psi.$$

Note that all the terms in the equation (3.14) have dimension 2, in units of energy, which means, in particular, that after a lattice block-spin transformation where $a \rightarrow sa$, one has

$$\hat{K}'_\psi = (ag)(a\bar{\psi}\psi) \rightarrow s^2(ag)(a\bar{\psi}\psi) = s^2 \hat{K}'_\psi, \quad (3.15)$$

in contrast with (3.13). For this reason, in order to have the correct scaling, we compute the contribution to the drift without rescaling the Dirac operator (and hence the Yukawa coupling), and then rescale the whole drift via

$$\hat{K}_\psi \rightarrow s^2 \hat{K}_\psi,$$

so that the scaling dimension of the other terms in (3.14) is matched.

We want to mention that the cooling procedure has important consequences on the issue of continuum limit of low-energy effective theories. In fact, in the standard lattice regularisation procedure, one always has $a \sim \Lambda_0^{-1}$, which means that the continuum limit $a \rightarrow 0$ is always connected to the limit $\Lambda_0 \rightarrow \infty$, in accordance to what

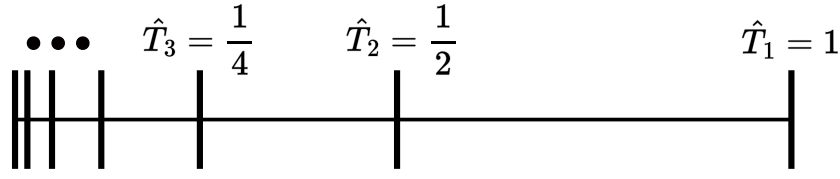


FIGURE 3.1

discussed in sections 2.1 and 2.2.

3.3.4 Control over temperature

The use of coloured noise allows one to access a wider and finer range of temperatures. In fact, as already mentioned in section 2.2, the temperature on the lattice is given by

$$T = \frac{1}{a N_t}. \quad (3.16)$$

Hence one can change temperature either by changing the spacing a , or by changing the number points in the time direction N_t . Both approaches pose serious constraints on the range of accessible temperatures. The spacing a is often determined by the energy scale at which one wants to simulate theory. For example in an effective theory, as explained in section 2.2, one has to keep $\Lambda \leq \Lambda_{\text{phys}}$, and since $a = \pi/\Lambda$, one ends with a constraint on the spacing $a \geq \pi/\Lambda_{\text{phys}}$.

The highest temperature reachable is then limited by mathematical arguments as one cannot have $N_t < 1$, even though in a simulation one often needs $N_t \geq 2$. On the other side the minimum temperature reachable is limited by computational resources as one cannot afford $N_t \rightarrow \infty$. Moreover, the resolution of high temperatures is very coarse due to the relation (3.16), in the sense that for small N_t , a change in the number of time points causes a significant variation in the temperature.

By means of coloured noise one can change the temperature by changing the effective spacing. In fact, if one follows the cooling procedure by keeping N_t fixed, one sets a new temperature which is given by

$$T_{\text{eff}} = \frac{1}{a_{\text{eff}} N_t} = \frac{1}{s a N_t} = \frac{1}{s} T,$$

where $s = \Lambda_{\text{eff}}/\Lambda \leq 1$. Note that not only the temperature has been raised, but one has also gained resolution around T_{eff} . In fact, to reach this temperature by changing N_t , one should change the number of time points to $N_t \rightarrow s N_t$, and for what explained before a lower N_t comes with a loss of resolution.

The control over temperature will not be investigated in this work. For a more detailed explanation of the procedure and its relevance in the context of the quark-meson model, we refer to [63].

3.4 Definition of relevant observables

From now on, we will indicate both the dimensionless quantities and dimensionful ones by using the notation of the latters. When beeing on discretised theory, we will be implicitly referring to the dimensionless quantities. A distinction will still be made when relevant. Moreover, a general spacetime vector will be denoted as

$$x = (t, \vec{x}),$$

where t is the temporal component and \vec{x} is the spatial component.

We define the magnetisation M of the field ϕ as,

$$M = \left\langle \frac{1}{V} \sum_n \phi_n \right\rangle.$$

Thus, in a finite volume lattice system, the absolute magnetisation

$$M = \left\langle \frac{1}{V} \left| \sum_n \phi_n \right| \right\rangle.$$

is better suited as an order parameter [20, 45].

In the continuum, phase transition is characterised by a divergence in the susceptibility, or connected two-points function

$$\chi_2 = V \left(\langle M^2 \rangle - \langle M \rangle^2 \right) = V \langle (M - \langle M \rangle)^2 \rangle.$$

On a finite volume lattice, one observes a peak that becomes sharper as the volume is increased. In practice, the susceptibility of the order parameter

$$\chi_2 = V \langle (M - \langle |M| \rangle)^2 \rangle \quad (3.17)$$

will be adopted.

The susceptibility measures the Gaussian fluctuations of the order parameter, the magnetisation. We then introduce the Binder cumulant [64], which quantifies the curtos of the fluctuations

$$U_L \equiv 1 - \frac{1}{3} \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}.$$

It is particularly useful to characterise phase transitions in terms of the order parameter ϕ , since it assumes the values

$$U_L = \begin{cases} 0 & \text{symmetric phase,} \\ 2/3 & \text{broken phase.} \end{cases}$$

We define the fermionic two-points function

$$\begin{aligned}
\langle \psi(x) \bar{\psi}(y) \rangle &= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \psi(x) \bar{\psi}(y) \exp(-S_\phi - \psi D \psi + \bar{\eta} \psi + \bar{\psi} \eta) \\
&= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \frac{\delta}{\delta \bar{\eta}(x)} \frac{\delta}{\delta \eta(y)} \exp(-S_\phi - \psi D \psi + \bar{\eta} \psi + \bar{\psi} \eta) \\
&= \frac{1}{Z} \int \mathcal{D}\phi \det[D(\phi)] \exp(-S_\phi) \frac{\delta}{\delta \bar{\eta}(x)} \frac{\delta}{\delta \eta(y)} \exp(\bar{\eta} D^{-1} \eta) \\
&= \left\langle [D^{-1}(\phi)]_{mn} \right\rangle,
\end{aligned} \tag{3.18}$$

and its connected counterpart

$$\langle \psi(x) \bar{\psi}(y) \rangle_c = \langle \psi(x) \bar{\psi}(y) \rangle - \langle \psi(x) \rangle \langle \bar{\psi}(y) \rangle.$$

The chiral condensate can then be computed as

$$\langle \bar{\psi} \psi \rangle = \sum_{n,s} \langle \bar{\psi}_{n,s} \psi_{n,s} \rangle = -\text{Tr}_{n,s} (D^{-1})_{nn}.$$

On a finite volume lattice, the correlation of a field at two spacetime points is quantified by the correlator

$$C_\psi(t, 0) \equiv \frac{1}{V} \sum_x [\langle \psi(t, \vec{x}) \bar{\psi}(0, 0) \rangle_c - \langle \psi(N_t - t, \vec{x}) \bar{\psi}(0, 0) \rangle_c]. \tag{3.19}$$

Note that we sum up two waves because of the boundary conditions.

In momentum space, by using a spectral decomposition, one has

$$\begin{aligned}
\frac{1}{V} \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \langle \bar{\psi}(t, \vec{x}) \psi(0, \vec{0}) \rangle_c &= \frac{1}{V} \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \sum_k \langle 0 | \bar{\psi}(t, \vec{x}) | k \rangle \langle k | \psi(0, \vec{0}) | 0 \rangle \\
&= \frac{1}{V} \sum_{\vec{x}} e^{i\vec{p}\vec{x}} \sum_k \left\langle 0 \left| e^{-ip'x} \bar{\psi}(0, \vec{0}) e^{ip'x} \right| k \right\rangle \langle k | \psi(0, \vec{0}) | 0 \rangle \\
&= \frac{1}{V} \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \sum_k e^{ikx} \langle 0 | \bar{\psi}(0, \vec{0}) | k \rangle \langle k | \psi(0, \vec{0}) | 0 \rangle \\
&= \frac{1}{V} \sum_{\vec{x}, k} e^{-i(\vec{p}-\vec{k})\vec{x}} e^{-E_k t} |\langle 0 | \psi(0, \vec{0}) | k \rangle|^2 \\
&= \frac{1}{V} \sum_k \delta(\vec{p} - \vec{k}) e^{-i(\vec{p}-\vec{k})\vec{x}} e^{-E_k t} |\langle 0 | \psi(0, \vec{0}) | k \rangle|^2 \\
&= \frac{1}{V} \sum_n e^{-E_n(\vec{p})t} \left| \langle 0 | \psi(0, \vec{0}) | p_n \rangle \right|^2.
\end{aligned}$$

Using this, equation 3.19 can be rewritten as

$$\begin{aligned}
C(t, \vec{p}) &= \frac{1}{V} \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \left(\langle \psi(t, \vec{x}) \bar{\psi}(0, \vec{0}) \rangle_c - \langle \psi(N_t - t, \vec{x}) \bar{\psi}(0, \vec{0}) \rangle_c \right) \\
&\propto \sum_n e^{-E_n(\vec{p})t} - e^{-E_n(\vec{p})(N_t-t)} = \sum_n 2e^{-E_n(\vec{p})\frac{N_t}{2}} \sinh \left[E_n(\vec{p}) \left(\frac{N_t}{2} - t \right) \right].
\end{aligned}$$

By projecting onto $p = 0$, one can note that for large times all the contributions but the ground state are suppressed, so that the fermionic correlator assumes the form

$$C_\psi(t, 0) \approx \sinh \left(E_0^\psi \left(\frac{N_t}{2} - t \right) \right), \quad (3.20)$$

where E_0^ψ is the fermionic mass gap, which will be denoted as $m_{q,\text{phys}}$.

An important quantity to compute the correlator on the lattice are the so called time slices

$$S(t, \vec{p}) = \frac{1}{V} \sum_{\vec{x}} e^{i\vec{p}\vec{x}} \psi(t, \vec{x}), \quad \bar{S}(t, \vec{p}) = \frac{1}{V} \sum_{\vec{x}} \bar{\psi}(t, \vec{x}) e^{-i\vec{p}\vec{x}}$$

In fact, they are related to the correlator via

$$\begin{aligned} \langle \bar{S}(t_1, \vec{p}) S(t_2, -\vec{p}) \rangle_c &= \frac{1}{V^2} \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \sum_{\vec{y}} e^{i\vec{p}\vec{y}} \langle \bar{\psi}(t_1, \vec{x}) \psi(t_2, \vec{y}) \rangle_c \\ &= \frac{1}{V^2} \sum_{\vec{x}, \vec{y}} e^{-i\vec{p}(\vec{x}-\vec{y})} \langle \bar{\psi}(t_1, \vec{x}) \psi(t_2, \vec{y}) \rangle_c \\ &= \frac{1}{V} \sum_{\vec{r}} e^{-i\vec{p}\vec{r}} \langle \bar{\psi}(t_1, \vec{r}) \psi(t_2, \vec{0}) \rangle_c \\ &= C(t_1 - t_2, \vec{p}). \end{aligned}$$

Note that from the second row to the third row translation invariance has been assumed.

Using this result, together with (3.18) and (3.20), one gets to the chain of equalities

$$\begin{aligned} C_\psi(t, 0) &= \langle \bar{S}(t, 0) S(0, 0) \rangle = \frac{1}{N_x} \sum_{\vec{x}} \langle \bar{\psi}(t, \vec{x}) \psi(0, 0) \rangle \\ &= \frac{1}{N_x} \sum_{\vec{x}} [D^{-1}]((t, \vec{x}), (0, 0)) \\ &\approx \sinh \left(m_{q,\text{phys}} \left(\frac{N_t}{2} - t \right) \right) \end{aligned}$$

This allows to compute the correlator on the lattice and extract the physical quark mass via a fitting procedure. Note that we do not need the full inversion of the Dirac operator. One can in fact write

$$(D^{-1})_{n,0} = D_{nm}^{-1} \eta_m,$$

where

$$\eta_m = s \delta_{m,0}.$$

is a spinor field where only the component at the origin is non-zero, and is given by the spinor s . Hence one is reduced to solve the linear system of equations

$$\sum_n D_{mn}^{-1} \psi_m = \eta_n,$$

which can be done by means of the Conjugate Gradient algorithm, as detailed in appendix C.

By similar arguments one can introduce the bosonic two-points function and connected two-points function

$$\begin{aligned}\langle \phi(x)\phi(y) \rangle &= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} \phi(x)\phi(y) e^{-S[\phi,\psi,\bar{\psi}]} \\ \langle \phi(x)\phi(y) \rangle_c &= \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle\end{aligned}$$

and arrives at the bosonic correlator

$$\begin{aligned}C_\phi(t, 0) &\approx \cosh \left(E_0^\phi \left(\frac{N_t}{2} - t \right) \right) \\ &= \langle S(t, 0) S(0, 0) \rangle = \frac{1}{N_x} \sum_n \langle \phi(t, n) \phi(0, 0) \rangle_c,\end{aligned}\tag{3.21}$$

where the bosonic time slice is defined as

$$S(t, \vec{p}) = \frac{1}{V} \sum_{\vec{x}} e^{i\vec{p}\vec{x}} \phi(t, \vec{x}),$$

and for zero momentum is simply the space average of the field configuration. The hyperbolic cosine in (3.21) appears in place of the hyperbolic sine since one applies periodic boundary condition instead of anti-periodic in equation (3.19). The last quantity which will be used in the investigation is the renormalised boson mass $m_{\phi,r}$. A practical formula for the computation of such quantity on the lattice can be derived by looking at the expression of the scalar renormalised two-points function, which for small momentum assumes the form

$$G(p) = \frac{Z_R}{\sum_\mu p_\mu^2 + m_{\phi,r}^2 + O(p^4)}.$$

The last expression evaluated at zero momentum gives the susceptibility

$$\chi_2 = G(0) = \frac{Z_R}{m_R^2}.$$

Finally, the second momentum is, by definition,

$$\mu_2 = - \left. \frac{\partial^2}{\partial p_v \partial p_v} G(p) \right|_{p=0} = 2d \frac{Z_R}{m_R^4}.$$

Putting all together we arrive at

$$m_{\phi,r}^2 = 2d \frac{\chi_2}{\mu_2},$$

This provides a way to compute the renormalised mass, since the susceptibility can be calculated via (3.17), and the second moment as

$$\mu_2 = \sum_x x^2 \langle \phi(x) \phi(0, 0) \rangle_c = d N_x \langle S(t, 0) S(0, 0) \rangle.$$

For a more detailed derivation of these identities we refer to [49].

An important note is that translation invariance was assumed in the derivation. This does not strictly holds for every configuration, but it is assumed to hold on expectation values.

Chapter 4

Numerical investigation: preliminaries

4.1 Inversion of the Dirac operator

As mentioned in section 3.2, the inversion of the Dirac operator is a computationally expensive operation. In fact, simple algorithms to invert a matrix such as Gaussian elimination have a computational complexity of $O(n^3)$ [65], while the best algorithm available nowadays has been proved to scale as $O(n^{2.371552})$ [59].

To circumvent this, since we are often only interested in applying the inverse matrix to a vector, we make use of the Conjugate Gradient algorithm, as detailed in appendix C.

We now want to give an idea of the speed-up that one can gain by following such approach, keeping in mind that this is an operation which has to be performed at each evolution step.

To do this, we construct a Dirac operator described by equation (3.4) with bare quark mass $m_q = 0.5$ and Yukawa coupling $g = 0.5$. We then add a fluctuating scalar field

$$\phi(x) = v + \eta(x),$$

where $v = 1.0$ and η is a normally distributed number with mean 0 and unitary variance.

We then measure the time needed to compute

$$\xi = D^{-1}\psi,$$

where ψ is a vector of random numbers uniformly distributed between -1 and 1 . We study the time dependence as a function of the vector size V by averaging over 20 measures per each value of V .

Figure 4.1 reports the results of the experiment. Assuming that our algorithm has complexity $O(V^b)$, namely that when $V \rightarrow \infty$,

$$t \approx a V^b,$$

we fit our data according to the model

$$\log t = \log a + b \log V.$$

We find

$$\log a = -3.1 \pm 0.1 \quad b = 1.02 \pm 0.02.$$

Note that for a standard Gaussian elimination procedure we would have found $b \approx 3.0$. This is of particular relevance in the case of this work, since the cooling

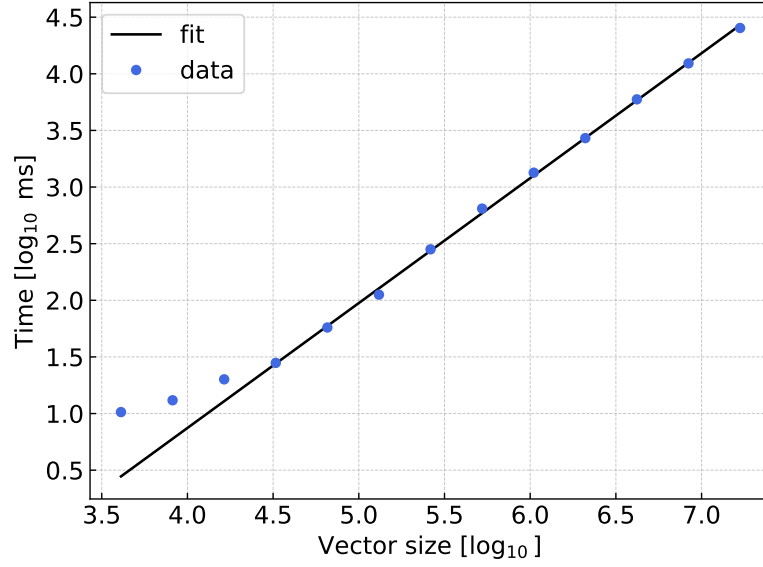


FIGURE 4.1: Computational complexity of the Conjugate Gradient algorithm in our model. The results show the operation time scales as $t \sim \mathcal{O}(V^{1.03})$ for large vector sizes, while for a standard inversion via Gaussian elimination one would have $t \sim \mathcal{O}(V^3)$.

technique via coloured noise requires doubling the number of lattice points in each direction at each block spin step; hence the volume of the lattice is increased by a factor 2^d at each iteration, where d is the spacetime dimension.

Typical lattices sizes in this work are up to $128 \times 128 \times N_f \times N_s \approx 10^5$, where N_f is the number of flavours and N_s is the number of spinor components. Therefore, a good volume scaling of the algorithm is strictly necessary.

4.2 The fermionic correlator

We now want to analyse the behaviour of the fermionic correlator and illustrate the fermionic masses extraction procedure.

Let us for the moment restrict to $g = 0$, so that the Dirac operator reduces to the one of free Wilson fermions

$$\begin{aligned} \hat{D}_{m,n} = & - \left(\frac{\Gamma_{+\hat{0}}}{2} \delta_{m,m+\hat{0}} + \frac{\Gamma_{-\hat{0}}}{2} \delta_{m,m-\hat{0}} + \frac{\Gamma_{+\hat{1}}}{2} \delta_{m,m+\hat{1}} + \frac{\Gamma_{-\hat{1}}}{2} \delta_{m,m-\hat{1}} \right) \\ & + (2ar + \hat{m}) \delta_{s,s'} \delta_{m,n}. \end{aligned} \quad (4.1)$$

We then compute the correlator (3.20) numerically via a single inversion of the Dirac operator using the Conjugate Gradient algorithm as detailed in Appendix C. The lattice volume is chosen to be 128×128 .

Figure 4.2 reports the fermionic correlator as a function of m_q . One can see that a bigger bare quark mass results in a quicker decay, according to the decay law

$$\lim_{t \rightarrow \infty} \langle \psi(t) \bar{\psi}(t) \rangle_{s,s'} \propto e^{-m_{\text{phys}} t}.$$

On the other side, a smaller mass results in a slower decay and tends to deform the characteristic shape of the correlator.

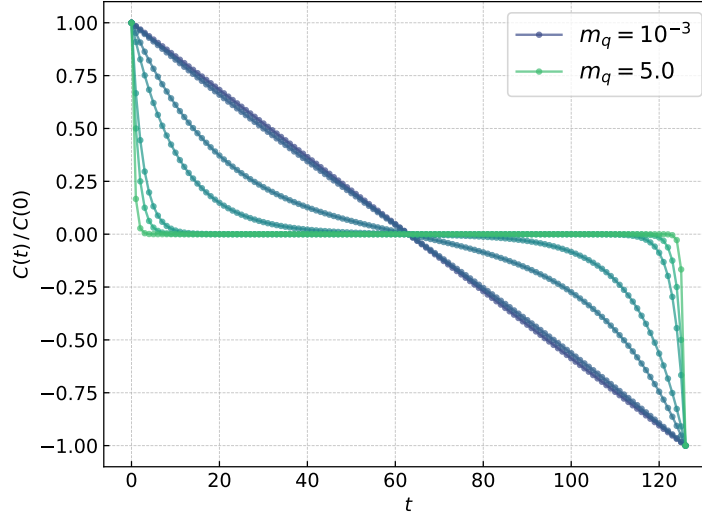


FIGURE 4.2: Normalised fermionic correlator for different values of the bare quark mass. A bigger mass results in a quicker decay of the correlator.

Figure 4.3 shows the number of iterations needed for convergence of the Conjugate Gradient algorithm. While the exact number depends on the desired tolerance, one can clearly see that the number of iterations grows as $m_q \rightarrow 0$, due to an increase in the condition number [65].

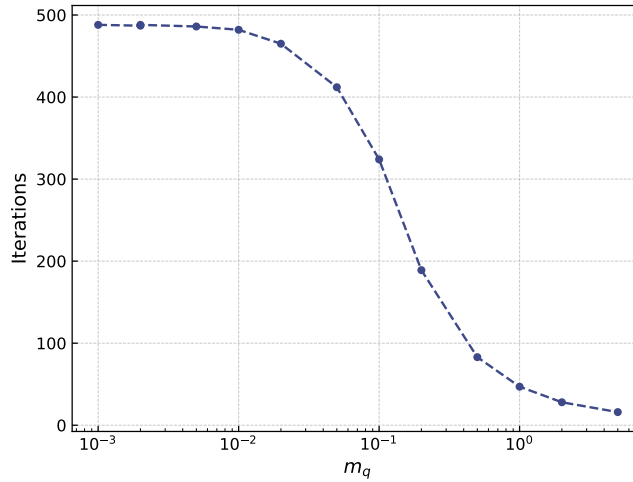


FIGURE 4.3: Conjugate Gradient algorithm iterations as a function of the bare quark mass. As $m_q \rightarrow 0$ the number of iterations grows due to an increase in the condition number.

For the Dirac operator 4.1, one can derive an analytical expression for the physical mass, the pole of the propagator.

To do this, let us consider the momentum space expression of the Dirac operator,

which is derived in appendix [B](#)

$$\bar{D}(p) = \hat{m}_q + \sum_{\mu} 2 \sin^2 \left(\frac{p_{\mu} a}{2} \right) + i \sum_{\mu} \gamma_{\mu} \sin (p_{\mu} a) .$$

This can be straightforwardly inverted to

$$\bar{D}^{-1}(p) = \frac{\hat{m}_q + \sum_{\mu} 2 \sin^2 \left(\frac{p_{\mu} a}{2} \right) - i \sum_{\mu} \gamma_{\mu} \sin (p_{\mu} a)}{\hat{m}_q + \sum_{\mu} 2 \sin^2 \left(\frac{p_{\mu} a}{2} \right) + \sum_{\mu} \sin^2 (p_{\mu} a)} .$$

One can now find the pole by imposing the denominator evaluated at $p^{\mu} = (im_{\text{phys}}, 0)$ to zero:

$$\left[\hat{m}_q + \sum_{\mu} 2 \sin^2 \left(\frac{p_{\mu} a}{2} \right) \right]_{p_{\mu}=(im_{\text{phys}},0)}^2 + \left[\sum_{\mu} \gamma_{\mu} \sin (p_{\mu} a) \right]_{p_{\mu}=(im_{\text{phys}},0)}^2 = 0 .$$

This results in a transcendental equation

$$\left[\hat{m}_q - 2 \sinh^2 \left(\frac{\hat{m}_{\text{phys}}}{2} \right) \right]^2 - \sinh^2 (\hat{m}_{\text{phys}}) = 0 ,$$

which has the solution

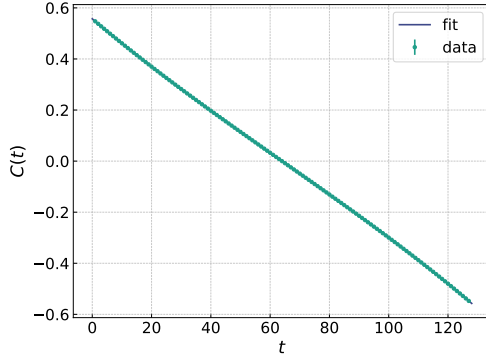
$$\hat{m}_{\text{phys}} = \log (1 + \hat{m}_q) . \quad (4.2)$$

We then choose three values of the bare quark mass, compute the correlator numerically and perform a fit according to [\(3.20\)](#), in order to extract the physical mass. We then compare it to the theoretical value given by [\(4.2\)](#). The results are reported in figures [4.4](#) and table [4.1](#).

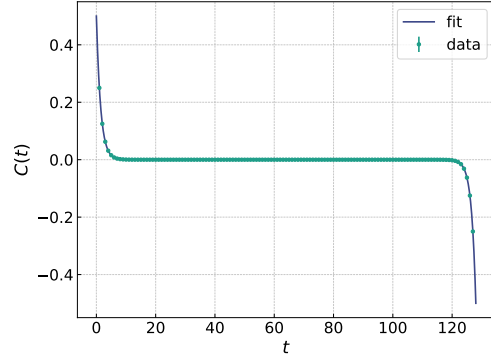
Note that in an interacting theory, excited states are, strictly speaking, negligible only for $t \rightarrow \infty$. This means that they may affect the characteristic shape of the correlator, and one has to either consider only a smaller window centered around $t = N/2$, or to make a fit to a portion of the exponential decay, in order to extract the mass [I might try to bring an example](#).

m_q	teo	fit
1.0	0.6931471805599453	0.6931537171644739
0.1	0.09531017980432493	0.09531020915059212
0.01	0.009950330853168092	0.009950277657505842

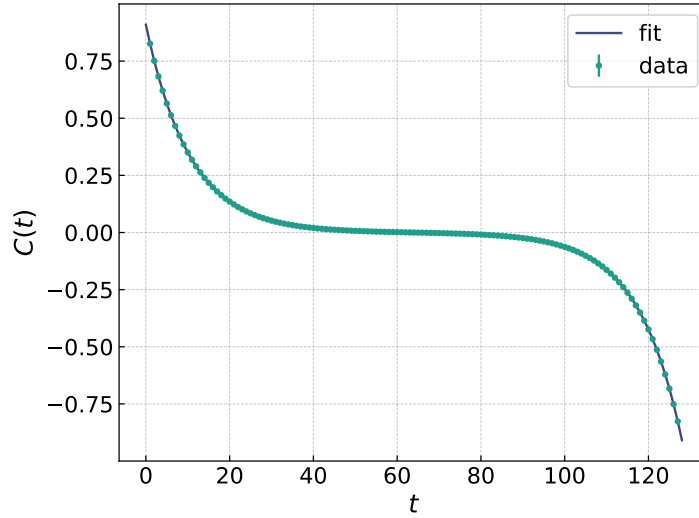
TABLE 4.1: The correlator for free Wilson fermions is fitted to equation (3.20) and compared to its analytical value given by (4.2). The precision for the Conjugate Gradient algorithm was set to $r^2 \leq 10^{-10}$.



(A) $m_q = 0.01$



(B) $m_q = 1.0$



(C) $m_q = 0.1$

FIGURE 4.4: Fit of the fermionic correlator according to equation (3.20), for three different values of the bare quark mass. The results are reported in table 4.1.

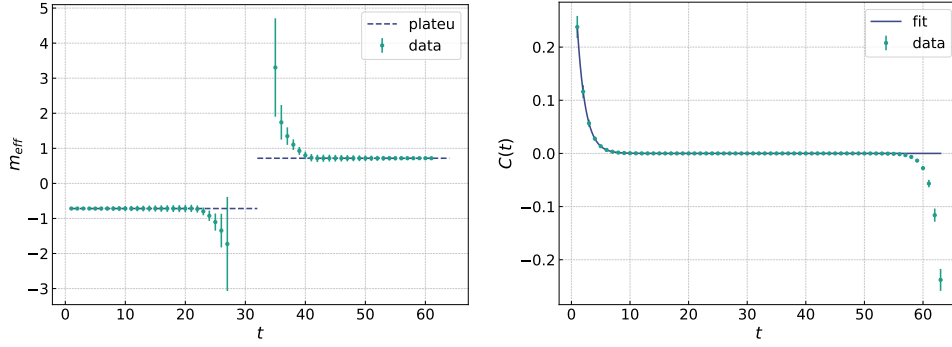


FIGURE 4.5: Effective mass

4.3 Phase structure

We want to start the analysis of the Yukawa theory by doing a parameters scan, in order to have a global picture of the phase diagram in the presence of Wilson fermions. For the remaining of this section, we set the value of the fermionic bare mass to $m_q = 1.0$.

Figure 4.6 reports a slice of the phase diagram in the $g - m_\phi^2$ plane.

For $g = 0$, the fermionic and bosonic theories are independent. In particular the scalar theory reduces to a $O(1)$ interacting theory: for $m_\phi^2 > 0$ the system lies in a symmetric state identified by $\langle \phi \rangle = 0$. As the mass goes to negative values, the scalar field gains a non-zero expectation value, signaling the spontaneous breaking of the $O(1)$ symmetry.

When a finite Yukawa coupling is added, the finite bare quark mass and the Wilson term, that break chiral symmetry explicitly, are fed to the scalar field, which gains a vacuum expectation value due to the relation (2.26).

The plot of the chiral condensate shows that chiral symmetry is always broken in the model, even for $g = 0$. Because of this, for $g \neq 0$, one can never speak of a proper phase transition. Thus, one still have a smooth crossover between the two phases. This fact is reflected in the plot of the magnetic susceptibility, which peaks around $m_\phi^2 = 0$ only for small values of g .

As explained in section 2.5, the presence of a background scalar field can be interpreted as a bare quark mass. This is reflected in the plot of $m_{q,\text{phys}}$ which resembles the one of $\langle \phi \rangle$, except for $g = 0$, since the two theories are disconnected.

Figure 4.7 reports the phase diagram in the $\lambda - g$ plane. The behavior can be understood qualitatively by means of classical arguments. For $g = 0$, one expects a field magnetisation of order

$$v = \sqrt{-\frac{6m_\phi^2}{\lambda}}.$$

Hence, as λ is increased, the expectation value of the field is reduced. If $g \neq 0$, the fermion mass and the Wilson term contributions increase the expectation value of the field, which in turn works as an additional bare quark mass, hence increasing $m_{q,\text{phys}}$. The behavior of the susceptibility is analogous to the previous case: only if $g = 0$ one has a proper phase transition, reflected in the single peak. The precise value of λ at which this happens depends, mainly, on the choice of m_ϕ^2 . maybe do a

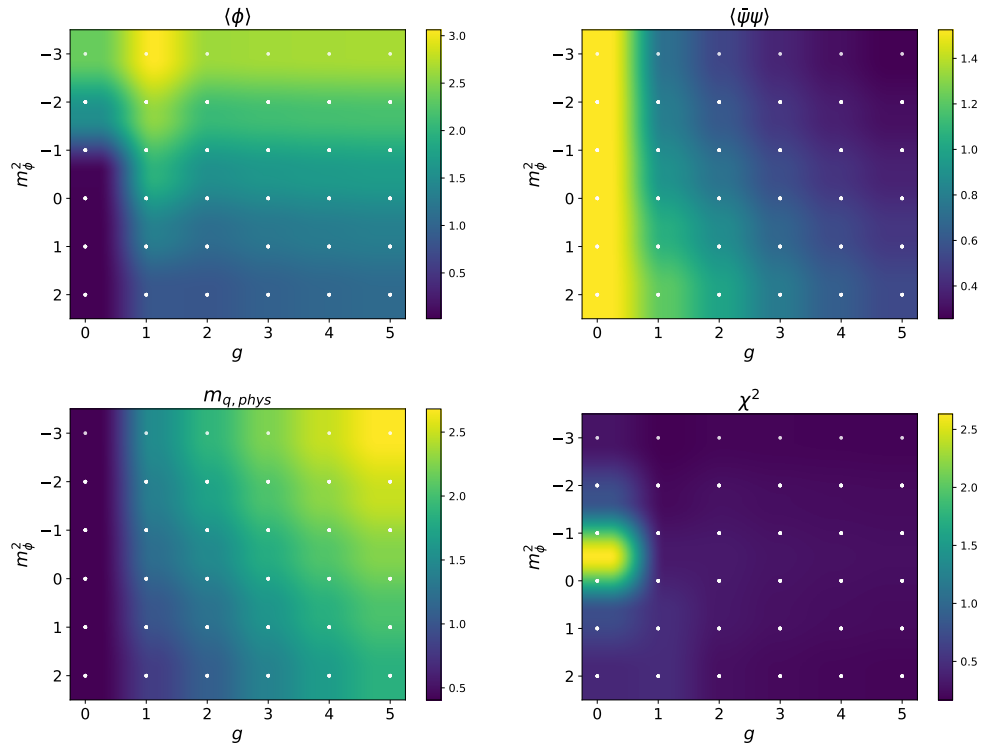


FIGURE 4.6: Slice of the phase diagram at fixed $\lambda = 2.0$.
Lattice size 32×32 , $N_{\text{conf}} \approx \mathcal{O}(5 \cdot 10^4)$.

scan also for $m_\phi^2 > 0$

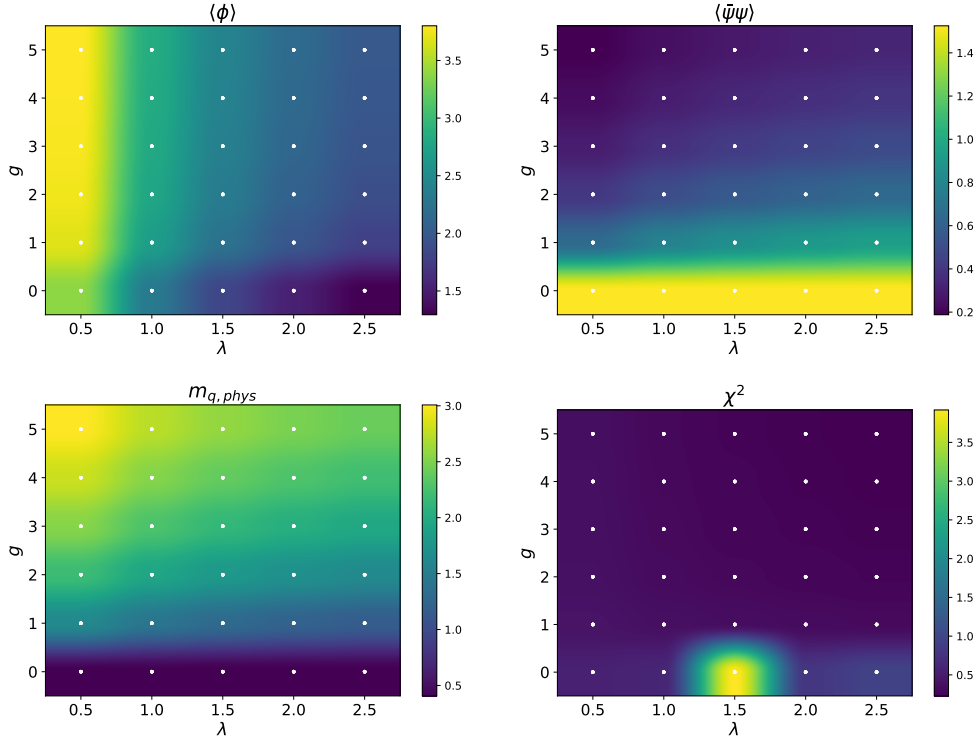


FIGURE 4.7: Slice of the phase diagram at fixed $m_\phi^2 = -1.0$.
Lattice size 32×32 , $N_{\text{conf}} \approx \mathcal{O}(10^4)$.

Finally, let us give a look at the phase structure in the $\lambda - m_\phi^2$ plane. Since here one always has $g \neq 0$, the expectation value of the scalar field and the chiral condensate are never zero, since the symmetry is broken independently of the choice of m_ϕ^2 and λ . This fact is also reflected in the mass plot, where one can clearly see that $m_{q,phys}$ is always proportional to the field. Moreover, the susceptibility in this case does not show a single peak as the phase transition can only happen for $g = 0$. Note in fact the difference in magnitude with respect to figures 4.6 and 4.7.

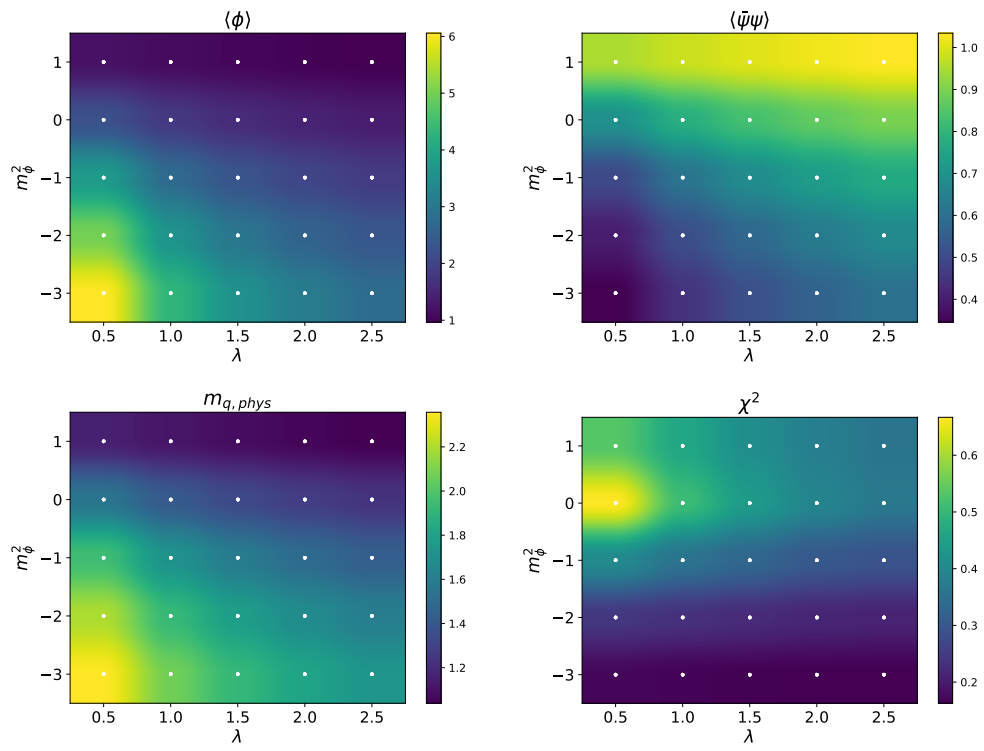


FIGURE 4.8: Slice of the phase diagram at fixed $g = 1.5$.
Lattice size 64×64 , $N_{\text{conf}} \approx \mathcal{O}(10^4)$.

Chapter 5

Numerical investigation: coloured noise

5.1 Classical-to-quantum interpolation

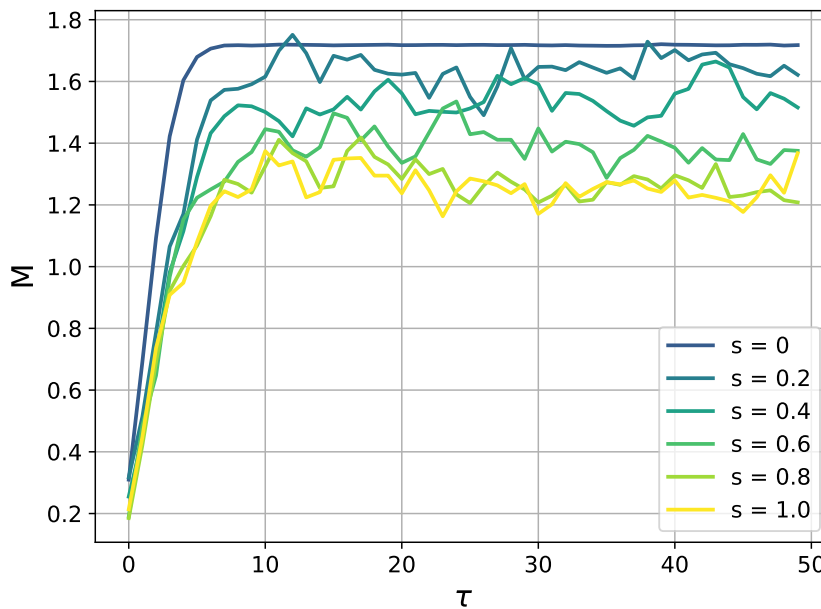


FIGURE 5.1: Thermalisation of the system for different values of the noise fraction s . As noise is added, the equilibrium state of the system shifts accordingly. Coloured noise allows for a smooth interpolation between the fully classical and fully quantum picture.

Lattice size 64×64 , $m_\phi^2 = -1.0$, $\lambda = 0.5$, $g = 0.08$, $m_q = 0.5$.

Let us start by analysing the coloured noise field in the simulation and relevant properties that emerge from it. We consider the Yukawa model described by the continuum action (2.17) and its discrete version (3.5), (3.6).

The system is initialised in the same state for all the configurations on a 64×64 lattice. We consider a simulation with $s = 1$ and then progressively lower the cutoff fraction s , keeping fixed all the quantities in the classical action.

Figure 5.1 shows the system thermalisation for different values of s , namely the Langevin evolution from the initial state to equilibrium. The blue line corresponds to the case $s = 0$, a classical simulation, while the yellow line corresponds to the $s = 1$, the fully quantum case. All the parameters settings are reported under the figure. As quantum modes are removed via coloured noise, the system shifts its

equilibrium point.

We now want to make use of coloured noise to understand the relation between scalar field, quark condensate and fermionic mass, which was qualitatively discussed in section 2.5 by means of classical arguments.

Keeping all the parameters fixed, we perform simulations changing the value of the cutoff fraction s , and we measure the three quantities above mentioned: the results are reported in figure 5.2. One can clearly see that as quantum modes are added, one has

$$\langle \phi \rangle \sim \langle \bar{\psi} \psi \rangle \sim m_q.$$

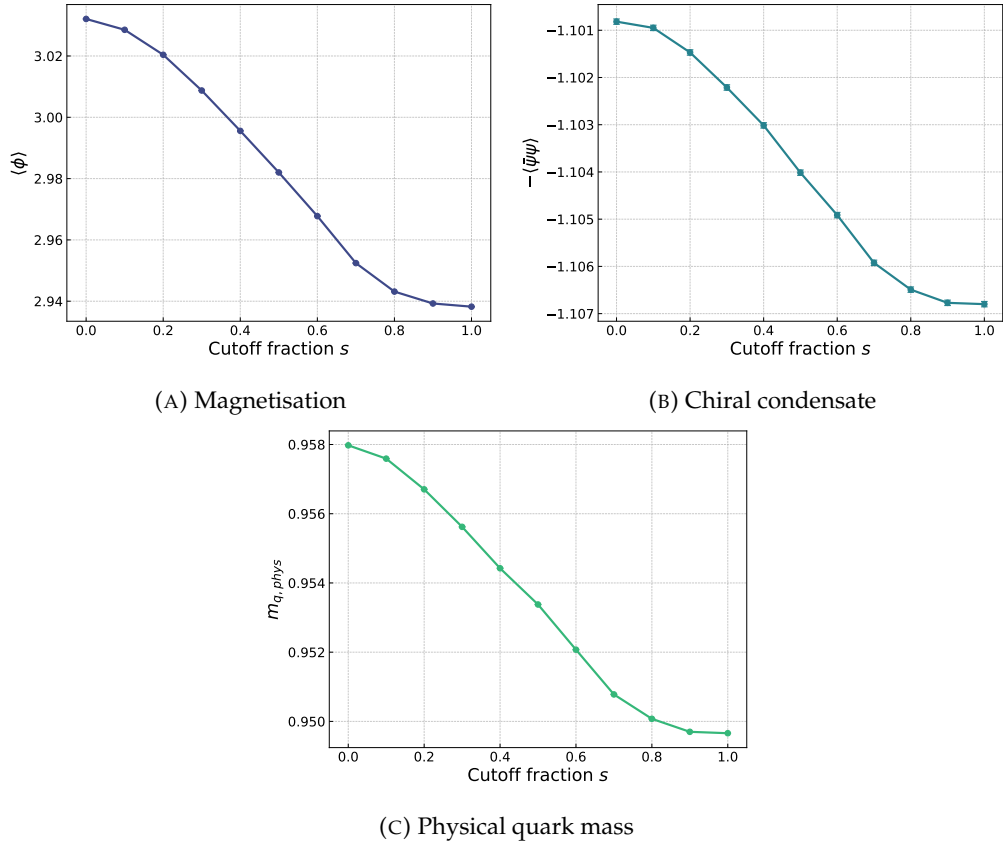


FIGURE 5.2: Relation between magnetisation, chiral condensate and mass, which was motivated in section 2.5.
 $m_\phi^2 = -1.0$, $\lambda = 0.7$, $g = 0.2$, $m_q = 1.0$
 Lattice size 64×64 , $N_{\text{conf}} \approx \mathcal{O}(5 \cdot 10^4)$.

This makes clear at first that the scalar field ϕ effectively plays the role of quark bilinear in the dynamics. Moreover, the physical mass of the quark grows proportionally to the scalar field magnetisation at all levels in the quantum theory.

The interconnection of these phenomena is an important result if one interprets it in the context of effective theories. In fact, often in QCD, the dynamical chiral symmetry breaking mechanism is described by low energy Yukawa-type effective models such as Gross-Neveu, Nambu - Jona-Lasinio or Quark Meson, to name some examples, where a scalar field can emerge after bosonisation via a Hubbard-Stratonovich transformation [cite](#).

Independently of this, studying the behavior of observables as a function of the cut-off fraction s , allows to understand better the role of fluctuations in the system and which degrees of freedom give more contribution to the dynamics.

5.2 Chiral fermions and noise-induced transition

As explained in section 2.5, chiral symmetry can be broken, in the continuum theory, either explicitly via the introduction of a finite bare quark mass, or spontaneously if the field gains a non-zero expectation value.

Moreover, in the discrete formulation, the introduction of the Wilson term contributes to the explicit breaking of chiral symmetry, as shown in appendix B. This, in particular, means that chiral symmetry is explicitly broken also for $m_q \rightarrow 0$. Because of this, one needs a new definition of the bare mass M_q , which takes into account the Wilson term contribution, such that chiral symmetry is restored in the limit $M_q \rightarrow 0$. In a lattice study of a theory such as two-flavours QCD, what one typically does [19, 20] is the following: the spontaneous breaking of chiral symmetry generates three goldstone massless bosons, the pions. If the bare quark mass is zero, the physical mass of the pions has to be zero, as a consequence of Goldstone's theorem [66]. Hence one can measure the pions mass and find a value m_q^* such that when $m_q \rightarrow m_q^*$, one has $m_\pi \rightarrow 0$.

This approach is not feasible in our theory, since it is described by a discrete chiral symmetry, and no Goldstone modes appear as a consequence. A possible way of proceeding [67] could be to tune the bare quark mass as a function of the interaction couplings in order to set the renormalised mass to zero, causing the divergence of a correlation length. In this case, the physical quantities of interest become independent of the underlying lattice, and one expects to recover the desired continuum QFT. A discussion on the issue, together with other proposals for the definition of the bare quark mass for similar theories can be found in [68, 69]. We do not follow such approaches here, mostly because of time limits, but also because it is not our purpose to match any precise model, but rather investigate properties of the coloured noise technique. In this section we just consider naïve fermions and the chiral limit is reached for $m_q \rightarrow 0$. Thus, one has to keep in mind that due to the fermion doubling this represents a theory with $2N_f = 4$ degenerate quarks.

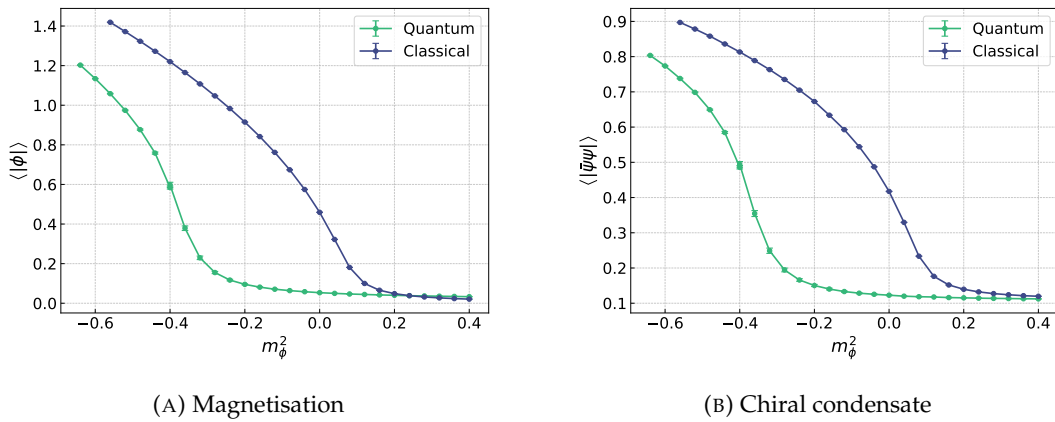


FIGURE 5.3: Scalar mass scan of the quantum and classical theories. One can see that for values of scalar squared mass $-0.2 \leq m_\phi^2 \leq 0$, the classical and the quantum systems lie in two different phases. $\lambda = 1.951, g = 0.08$, Lattice size 32×32 , $N_{\text{conf}} \approx \mathcal{O}(10^4)$

In figure 5.3, the (absolute) magnetisation and the chiral condensate are studied as a function of the bosonic mass squared, both in the fully quantum and fully classical theory. One can notice that the classical system undergoes a phase transition at values of m_ϕ^2 bigger than the quantum counterpart.

Note that as the bare quark mass is small but finite, this is not a proper phase transition and the latter will be, eventually, reached in the limit $m_q \rightarrow 0$.

We then pick a value $m_\phi^{2*} = -0.123$. Using coloured noise, we then interpolate between the classical and quantum picture for different values of the quark mass, namely $m_q = 10^{-2}, 5 \cdot 10^{-3}, 10^{-3}$.

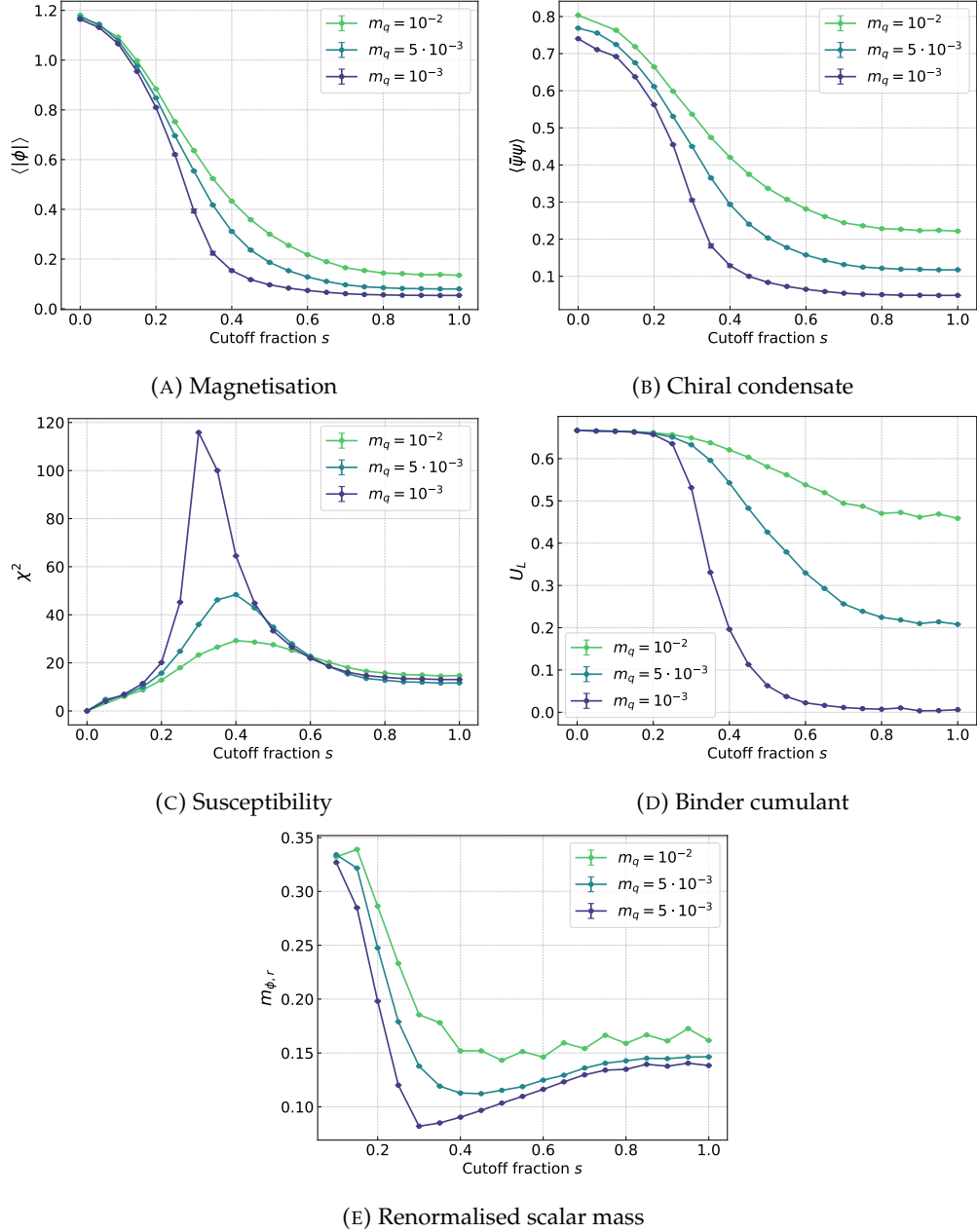


FIGURE 5.4: $m_\phi^2 = -0.123$, $\lambda = 1.951$, $g = 0.08$,
Lattice size 64×64 , $N_{\text{conf}} \approx \mathcal{O}(5 \cdot 10^4)$

One can clearly see in figure 5.4 that as the quark mass is lowered the difference between the two phases gets more pronounced, indicating a phase transition.

Both the magnetisation and the chiral condensate are finite in the classical theory and get washed out as quantum fluctuations are added. The peak in the susceptibility shows the transition is happening for $0.2 \leq s \leq 0.4$, but the resolution is not enough to determine the precise value of s .

The Binder parameters assumes the value $U_L = 0$ in the symmetric phase, and $U_L = 2/3$ in the broken phase, as expected. Finally, the scalar renormalised mass tends to zero as the transition point is approached, signaling the divergence of the correlatin length.

Hence all the five observables are in agreement and signal the presence of a phase transition.

Intuitively speaking one can interpret the situation by looking at figure 2.4. For $m_\phi^2 < 0$, the classical potential has two minima that are slightly inequivalent due to the presence of a small yukawa coupling. In the classical theory, the system is constrained at the global minimum, since there is no stochasticity and the kinetic term is not strong enough to cause the hopping between the two minima. At the quantum level, fluctuations allow such transition bringing $\langle \phi \rangle \rightarrow 0$ and restoring the symmetry. Note that in order to confirm that a phase transition took place, one should first consider $V \rightarrow \infty$ and then $m_q \rightarrow 0$. If the order parameters of the classical and quantum theories still indicates that the system is in two different phases, one can draw the conclusion.

In this context, it would be interesting to look at the physical quark mass. In fact, neglecting the bare quark mass $m_q \approx 0$, one would expect the fermions to be massive in the classical theory due to the presence of the background field ϕ , which work as an additional mass. In the classical-to-quantum transition, as the field expectation vanishes, the fermions become massless. Thus, due to the choice of discretisation, namely employing naive fermions, it is not possible to extract the quark mass in a straightforward way due to the proprties of the doublers. The latter exhibits in fact an incorrect momentum dispersion relation [70, 71]. [we thought it was the choice of the source but it is not. I might try to add an appendix in which I show the plot of what happens there.](#)

5.3 Cooling with coloured noise

Let us now consider one of the main applications of coloured noise, namely the cooling technique.

We first set up a white noise simulation $s = 1$, and then progressively lower s . The lowering of the cutoff is compensated by a change in the couplings, as explained in section 3.3. We study the behavior of the system both as a function of the bosonic mass squared m_ϕ^2 and the Yukawa coupling g . A summary of the parameters choice for both the experiments is reported in tables 5.1 and 5.2.

s	N_t	N_x	m_ϕ^2	λ	g	m_q	K_ψ
1	16	16	m_ϕ^2	0.4	0.3	0.5	K_ψ
1/2	32	32	$m_\phi^2/4$	0.1	0.3	0.5	$K_\psi/4$
1/4	64	64	$m_\phi^2/16$	0.025	0.3	0.5	$K_\psi/16$
1/8	128	128	$m_\phi^2/64$	0.00625	0.3	0.5	$K_\psi/64$

TABLE 5.1: Parameter settings in the cooling procedure for the bosonic mass squared scan. Each coupling in the bosonic action is rescaled according to its canonical dimension, while the fermionic sector rescaling is implemented directly at the drift level, as detailed in section 3.3

s	N_t	N_x	m_ϕ^2	λ	g	m_q	K_ψ
1	16	16	0.5	0.7	g	1.0	K_ψ
1/2	32	32	0.125	0.175	g	1.0	$K_\psi/4$
1/4	64	64	0.0625	0.04375	g	1.0	$K_\psi/16$
1/8	128	128	0.03125	0.001094	g	1.0	$K_\psi/64$

TABLE 5.2: Parameters setting in the cooling procedure for the Yukawa coupling scan. Each coupling in the bosonic action is rescaled according to its canonical dimension, while the fermionic sector rescaling is implemented directly at the drift level, as detailed in section 3.3

Figure 5.5 reports the magnetisation, its susceptibility and the chiral condensate as a function of the bare scalar mass squared m_ϕ^2 , while figure 5.6 reports the investigation as a function of the Yukawa coupling. One can clearly see that there is a general good agreement for the first three block-spin transformations, even if based on tree level arguments.

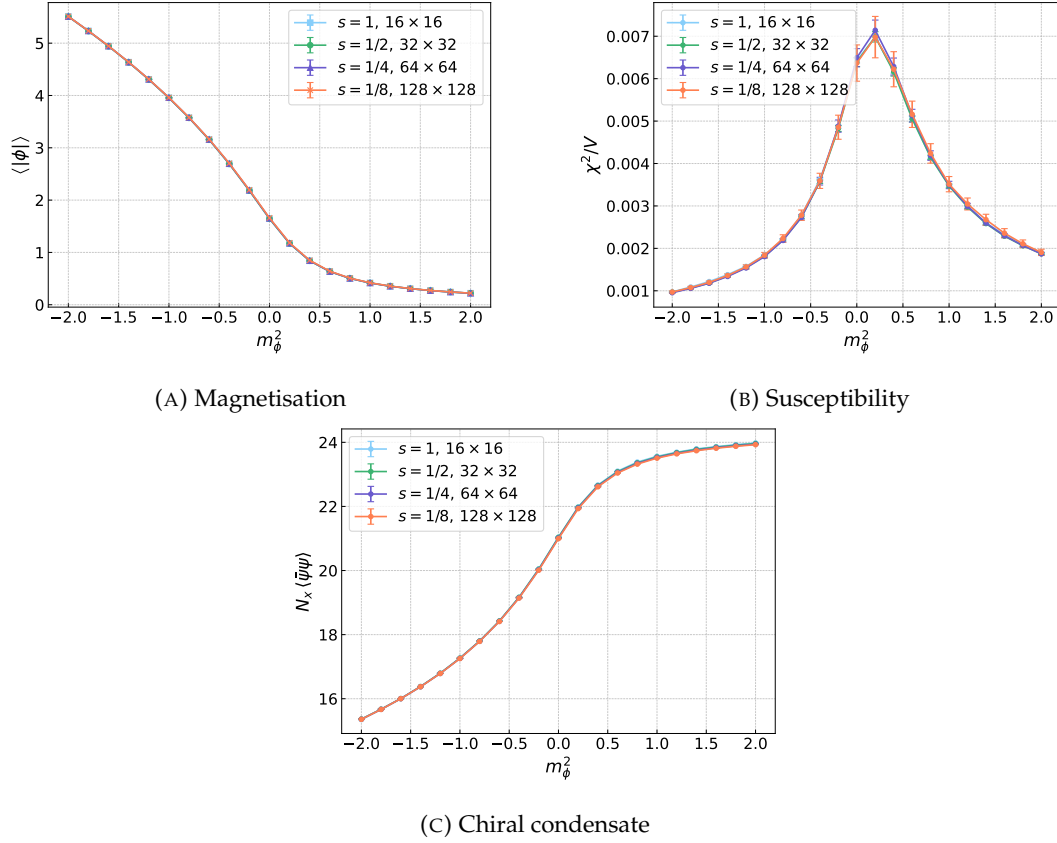


FIGURE 5.5: Cooling via coloured noise. The absolute magnetisation, its susceptibility and the chiral condensate are compared after performing block-spins transformations as a function of the bosonic mass squared. All the parameters are reported in table 5.1.

$$N_{\text{conf}} \approx \mathcal{O}(5 \cdot 10^4).$$

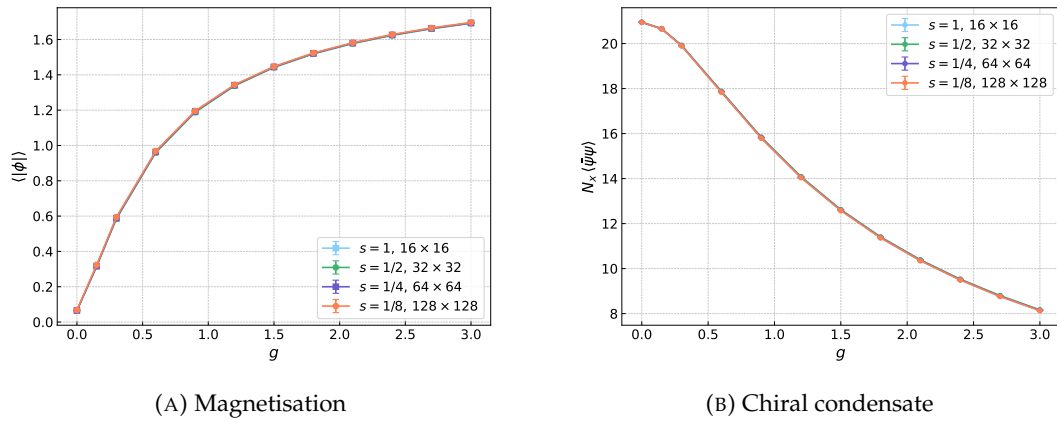


FIGURE 5.6: Cooling via coloured noise. The absolute magnetisation and the chiral condensate are compared after performing block-spins transformations as a function of the Yukawa coupling.

All the parameters are reported in table 5.2.

$$N_{\text{conf}} \approx \mathcal{O}(5 \cdot 10^4).$$

The most relevant observables are the magnetisation and the chiral condensate, since they are the order parameters of the theory. We therefore want to provide a more detailed comparison of the performance of the procedure. To this end we define the relative errors

$$\begin{aligned}\epsilon_\phi(s) &= \frac{\langle |M| \rangle_s - \langle |M| \rangle}{\langle |M| \rangle}, \\ \epsilon_\psi(s) &= \frac{N_x^{(s)} \langle |\tilde{\psi} \hat{\psi}| \rangle_s - N_x \langle |\tilde{\psi} \hat{\psi}| \rangle}{N_x \langle |\tilde{\psi} \hat{\psi}| \rangle}.\end{aligned}\quad (5.1)$$

The deviation quantified by such parameters is reported in figure 5.7 and figure 5.8.

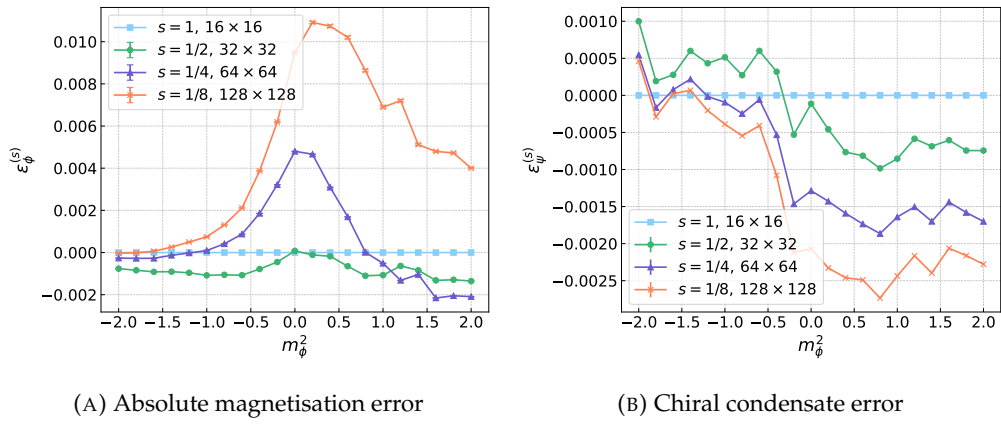


FIGURE 5.7: Relative error of the absolute magnetisation and chiral condensate in the cooling procedure for various values of the noise fraction s .

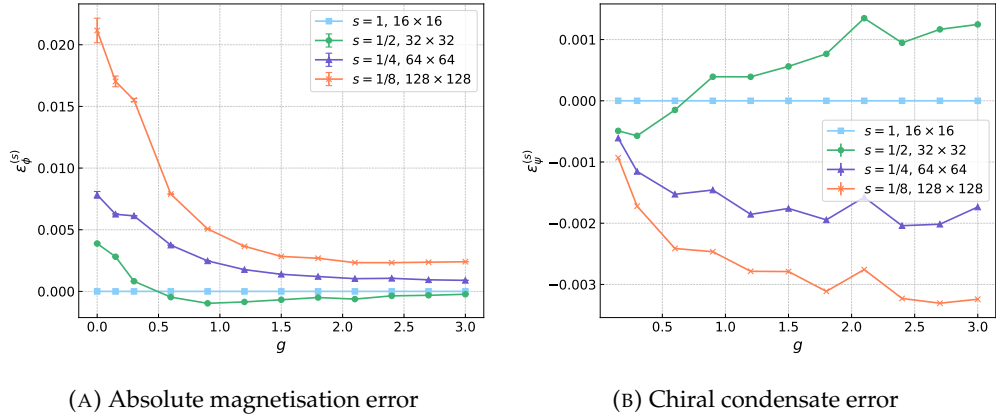


FIGURE 5.8: Relative error of the absolute magnetisation and chiral condensate in the cooling procedure for various values of the noise fraction s .

Errors obviously increase with successive iterations, since the approximations are multiple.

First of all, a tree level rescaling neglects coupling momentum dependencies which do not stem from spacetime rescaling; the assumption is thus justified for high enough cutoffs.

Moreover, the assumption that any change due to a modification of the cutoff can be absorbed by a redefinition of the couplings, is itself non-trivial. In fact, following Wilsonian's RG perspective, as momentum shells are integrated out, all the relevant operators, including those not present in the classical action, grow as the cutoff is lowered. As before, the assumption is justified for high enough cutoffs.

A quantitative analysis on this aspect was done in [49], where it was proposed a procedure to systematically test until when it is possible to accommodate the cutoff change by a redefinition of the couplings, eventually including also beyond tree level transformations.

Finally, we also want to look at more complex observables such as the fermionic physical mass and the bosonic renormalised mass. They are reported in figure 5.9.

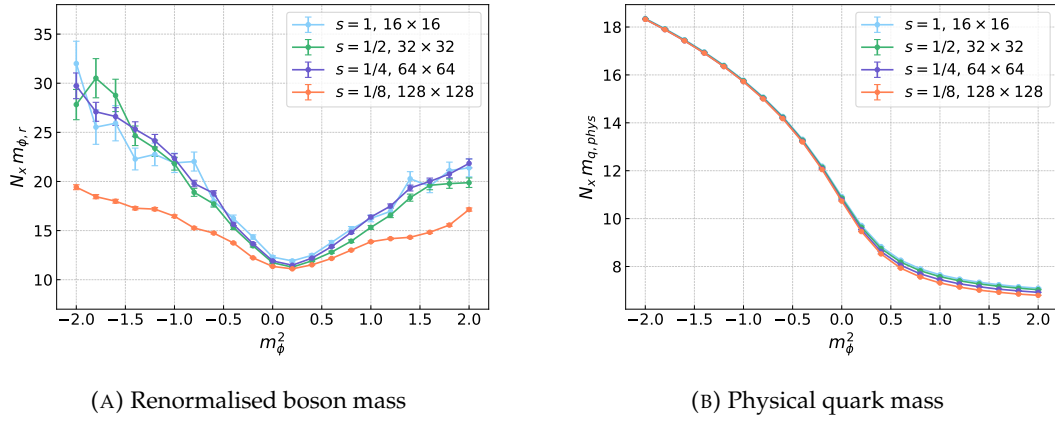


FIGURE 5.9: Renormalised bosonic mass $m_{\phi,r}$ and pole fermionic mass $m_{q,phys}$ for various values of the noise fraction s .

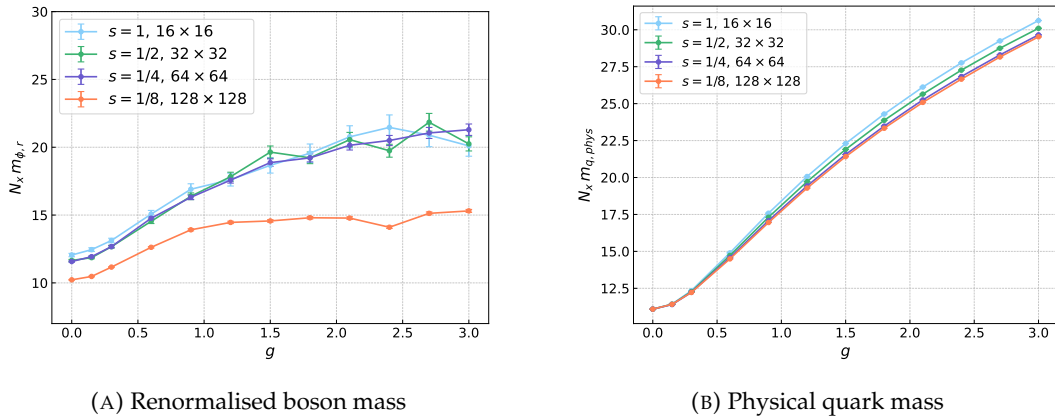


FIGURE 5.10: Renormalised bosonic mass $m_{\phi,r}$ and pole fermionic mass $m_{q,phys}$ for various values of the noise fraction s .

As one can see, there is a clear deviation for $m_{\phi,r}$ at the third block-spin iteration. This is indeed a delicate issue that resides in the use of sharp cutoffs to regulate the noise term. In fact, as shown in section 2.3.2, the noise-to-noise correlation function expressed in momentum space exhibits an oscillating behaviour described by a Bessel function, which was calculated and shown in figure 2.3.

As a consequence, correlation functions are affected by this behaviour. This holds

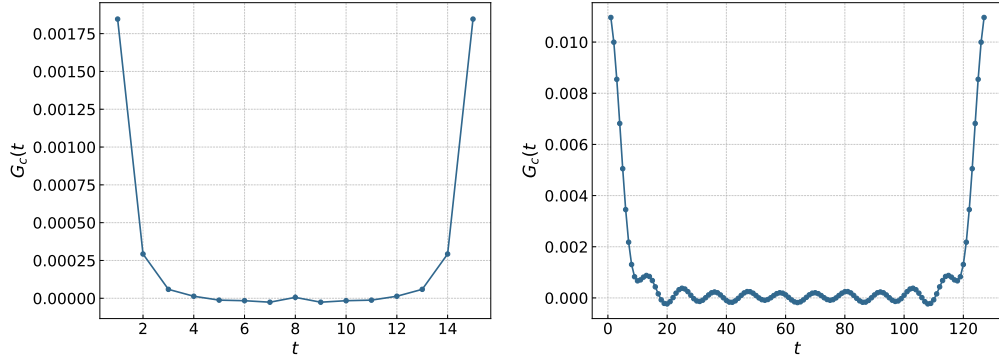


FIGURE 5.11: (Left) Connected two-points function of the original theory on the 16×16 lattice. (Right) Connected two-points function after three block spin-iterations at cutoff fraction $s = 1/8$, on a 128×128 lattice. The use of the sharp cutoff as a regulating function results in a disturbance of the signal of the bosonic two-points function. The Bessel function profile which was shown in figure 2.3 induces errors in the calculation of the renormalised mass.

particularly for the connected two-points function as shown in figure 5.11. This explains the big discrepancy encountered in the renormalised mass in figures 5.9 and 5.10. This error cannot be solved by simply going to higher order correction to the rescaling procedure, but can only be fixed by changing the regulating term, since it is an intrinsic characteristic of the latter. Possible alternative regulators are discussed in [49].

For what concerns the physical quark mass, which also shows some discrepancy for $m_\phi^2 > 0$, we need to investigate further to understand the cause of the problem.

Another source of error is the following: since the regulating term applies strictly speaking, only to the noise term, after performing a block-spin transformation the quantum theory is set back at its original scale, while the classical one lies at a finer spacing.

Even with the presence of these problems, it is nevertheless remarkable that observables such as the order parameters are not significantly altered in the coarse-graining procedure.

Chapter 6

Conclusions and outlook

Appendix A

Conventions

Spinor representation

We choose the following representation of the two-dimensional γ matrices

$$\gamma_0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

They satisfy the Euclidean Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$$

and they have the following properties

$$\gamma_\mu^\dagger = \gamma_\mu, \quad \gamma_\mu^2 = \mathbb{1}.$$

We then introduce γ_5 as

$$\gamma_5 = i\gamma_0\gamma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

with the following properties

$$\gamma_5^\dagger = \gamma_5, \quad \gamma_5^2 = \mathbb{1} \quad (\gamma_5)^2 = \mathbb{1}, \quad \{\gamma_5, \gamma_\mu\} = 0$$

Fourier transform

The Fourier transformation of a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is given by

$$\bar{f}(p) = \mathcal{F}[f](p) = \int_x e^{-ixp} f(x)$$

and its inverse is

$$f(x) = \mathcal{F}^{-1}[\bar{f}](x) = \frac{1}{(2\pi)^d} \int_p e^{ixp} \bar{f}(p)$$

Appendix B

Wilson fermions

The naïve discretisation of the fermionic action for free fermions yields the Dirac operator components

$$\hat{D}_{nm} = \sum_{\mu} \gamma_{\mu} \frac{\delta_{n,n+\mu} - \delta_{n,n-\mu}}{2} + m_q \delta_{n,m},$$

which have the momentum representation

$$\hat{D} = \hat{m}_q + i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a).$$

The last expression can be easily inverted

$$\hat{D}^{-1}(p) = \frac{\hat{m}_q - i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a)}{\hat{m}_q^2 + \sum_{\mu} \sin^2(p_{\mu} a)}.$$

For $m_q = 0$, one can easily see that, in addition to $\hat{p}_{\mu} = 0$, all the edges of the Brillouin zone $\hat{p}_{\mu} \in \{\pm\pi/a, \pm\pi/a\}$ are poles. They indeed represent additional fermions, which come from the discretisation of the action. These additional degrees of freedom are called doublers and represent unphysical particles, at least if one's goal is to recover the original continuum theory. If m_q is finite,

The idea proposed by Wilson to remove the doublers is to redefine the action as

$$S_W = S - \frac{r}{2} \sum_{m,n} \hat{\psi}(m) \hat{\square} \hat{\psi}(n),$$

where

$$(\hat{\square})_{mn} = \sum_{\mu} \frac{\delta_{m,n+\hat{\mu}} + \delta_{m,n-\hat{\mu}} - 2\delta_{mn}}{2},$$

and $r \in [0, 1]$ is a free parameter. In this project we always consider $r = 1$. The Wilson-Dirac operator assumes the form

$$\hat{D} = \hat{m}_q + \sum_{\mu} \gamma_{\mu} \frac{\delta_{m,m+\hat{\mu}} - \delta_{m,m-\hat{\mu}}}{2} + \frac{r}{2} \sum_{\mu} \frac{\delta_{m,n+\hat{\mu}} + \delta_{m,n-\hat{\mu}} - 2\delta_{mn}}{2}.$$

and its momentum space representation is

$$\begin{aligned}
\hat{D}(p) &= \hat{m}_q + i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a) + \frac{r}{2} \sum_{\mu} [\cos(p_{\mu} a) - 1] \\
&= \hat{m}_q + i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a) + r \sum_{\mu} \frac{\cos(p_{\mu} a) - 1}{2} \\
&= \hat{m}_q + i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a) + r \sum_{\mu} \sin^2\left(\frac{p_{\mu} a}{2}\right).
\end{aligned}$$

The last expression can be inverted straightforwardly

$$\begin{aligned}
\hat{D}^{-1}(p) &= \frac{\hat{m}_q + \frac{2r}{a} \sum_{\mu} \sin^2\left(\frac{p_{\mu} a}{2}\right) - i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a)}{\left[\hat{m}_q + \frac{2r}{a} \sum_{\mu} \sin^2\left(\frac{p_{\mu} a}{2}\right)\right]^2 + \sum_{\mu} \sin^2(p_{\mu} a)} \\
&= \frac{\hat{M}(p) - i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu} a)}{\hat{M}^2(p) + \sum_{\mu} \sin^2(p_{\mu} a)},
\end{aligned}$$

where we defined $\hat{M}(p) \equiv \hat{m}_q + \frac{2r}{a} \sum_{\mu} \sin^2\left(\frac{p_{\mu} a}{2}\right)$.

For every $\hat{p}_{\mu} \neq (\pm\pi/a, \pm\pi/a)$, one has that $a \rightarrow 0$ implies $\hat{M}(p) \rightarrow \hat{m}_q$.

Instead, for every momentum vector that lies at the edge of the Brillouin zone $M(p) \rightarrow \infty$ as $a \rightarrow 0$. This removes the fermion doublers in the continuum limit.

A notable problem is that one then loses chiral symmetry since the Wilson term is clearly not invariant under neither the continuous transformation

$$\psi \rightarrow e^{i\theta\gamma_5} \psi \quad \bar{\psi} \rightarrow -\bar{\psi} e^{i\theta\gamma_5},$$

nor the discrete

$$\psi \rightarrow \gamma_5 \psi \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5.$$

It is in fact proven that one either completely removes the doublers or breaks chiral symmetry [51].

Appendix C

Computational strategies

The application of coloured noise, and in particular the cooling technique, demand for an efficient volume scaling of the simulation, as pointed out in section 4.1. The code for this project was developed accordingly.

While a full documentation of the code esulates from the scope of this thesis, we want to provide some details on the strategies that were adopted.

The code for this project has been developed to run on a Graphical Processing Unit (GPU). This ensures a massive parallelisation for operations such as scalar products and reduction operations such as summing components of a vector, which are extensively used in the course of the project.

Choice of the spinor basis

The choice of the chiral basis for the spinors illustrated in appendix A was done for computational performance purposes. To see this, one has to look at the explicit expression of the Wilson projectors in the chiral basis:

$$\begin{aligned}\Gamma_{+\hat{0}} &= (1 + \gamma_0) \psi = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, & \Gamma_{-\hat{0}} &= (1 - \gamma_0) \psi = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \\ \Gamma_{+\hat{1}} &= (1 + \gamma_1) \psi = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, & \Gamma_{-\hat{1}} &= (1 - \gamma_1) \psi = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.\end{aligned}$$

Note that all the matrices are degenerate. This implies that when the Wilson projectors are applied to a vector, only half of the components have to be computed, and the other can be deduced since they are either zero or linearly dependent. Explicitly:

$$\begin{aligned}(1 - \gamma_0) \psi(n + \hat{0}) &= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\psi_2 \end{bmatrix}, \\ (1 + \gamma_0) \psi(n - \hat{0}) &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 2\psi_1 \\ 0 \end{bmatrix}, \\ (1 - \gamma_1) \psi(n + \hat{1}) &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \psi_1 - \psi_2 \\ -\psi_1 + \psi_2 \end{bmatrix} \equiv \begin{bmatrix} \eta \\ -\eta \end{bmatrix}, \\ (1 + \gamma_1) \psi(n - \hat{1}) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \psi_1 + \psi_2 \\ \psi_1 + \psi_2 \end{bmatrix} \equiv \begin{bmatrix} \chi \\ \chi \end{bmatrix}.\end{aligned}$$

Conjugate Gradient algorithm

One can note that for the purpose of computing the fermionic contribution to the drift force and the extraction of the physical quark mass from the correlator (details in section 3.2 and section 3.4), we do not need the full inverse of the Dirac operator, but only the inverse matrix applied to a vector. Hence it is sufficient to compute

$$\psi = D^{-1} \eta. \quad (\text{C.1})$$

Computing ψ via equation (C.1) is equivalent to solve the linear system $D\psi = \eta$, which can be done efficiently by employing a method for sparse matrices such as Conjugate Gradient (CG) as explained in the following way.

We want to solve the equation

$$D \psi = \eta.$$

CG requires the matrix to be hermitian while D is only γ^5 -hermitian, namely

$$\gamma_5 D \gamma_5 = D^\dagger.$$

One can thus solve the linear system

$$(DD^\dagger) \xi = \eta,$$

and then obtain ψ by multiplying the solution ξ by D^\dagger since

$$D^\dagger \xi = D^\dagger (DD^\dagger)^{-1} \eta = D^{-1} \eta = \psi. \quad (\text{C.2})$$

Analogously, one can calculate

$$\chi = (D^\dagger)^{-1} \eta$$

by solving

$$(D^\dagger D) \xi = \eta,$$

and then applying D to the result.

Bilinear noise scheme

When computing the drift for the Langevin evolution, one needs to evaluate the trace of the Dirac operator, as shown in section 3.2.

We compute the trace by employing the bilinear noise scheme. For this, one needs a set of N vectors of normally distributed random complex numbers, namely vectors η_i where each component η_i^α satisfies

$$\langle \eta_i^\alpha \rangle = 0, \quad \langle \eta_i^\alpha \eta_j^\beta \rangle = \delta_{i,j} \delta^{\alpha\beta}.$$

The trace of the matrix is then computed as

$$\text{Tr } A = \frac{1}{N} \lim_{N \rightarrow \infty} \sum_i^N \eta_i^\dagger A_{ij} \eta_j \quad (\text{C.3})$$

In our case, we can first compute

$$|\psi\rangle = D^{-1} |\eta\rangle = D^\dagger \underbrace{(DD^\dagger)^{-1} |\eta\rangle}_{\text{CG}}$$

by means of the Conjugate Gradient algorithm, and then obtain the trace as

$$\text{Tr } A = \langle \eta | \psi \rangle.$$

Not that the series (C.3) requires in principle an infinite number of vectors to evaluate the trace exactly. In practice we truncate it to $N = 1$. The average over Monte Carlo samples will eventually converge nevertheless to the right result in the limit of infinite samples.

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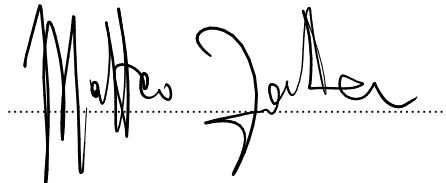
List of Abbreviations

RG	R enormalisation G roup
fRG	F unctional R enormalisation G roup
UV	U ltraviolet
IR	I nfrared
QFT	Q uantum F ield T heory

Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27.11.2023

A handwritten signature in black ink, reading "Matteo Zeller", written over a horizontal dotted line.