Classical Mechanics

Given a vector space V, a semi-positive quadratic form (which we call *metric* in physics) naturally gives raise to a dual vector (and the dual space). Indeed let us denote by $g(\circ, \circ)$ the bilinear form associated to the quadratic form. Given two vectors $v, w \in V$ a functional g_w can be defined via $g(w, v) \equiv g_w(v)$.
Remembering that the dual space of a vector space is given by the set of linear functionals acting on the space, one has that $V^* = \{g_w |\ w \in V\}$.
A bilinear form (and hence the quadratic form) is defined by its action on the basis set. For example in the euclidean space with basis $\{e_i\}_i$ one imposes $g(e_i, e_j) = \delta_{ij}$.
One normally says that the euclidean metric components on the cartesian basis is δ_{ij}.
Let us know search for which transformations leave the length of vectors unchanged with the euclidean norm. In other words let us search for transformation matrices M^μ_ν such that given a vector x^ν the length of the vector $x'^\mu = M^\mu_\nu x^\nu$ is the same as the x^ν's one.

$$\delta_{\mu\nu} x^\mu x^\nu = \delta_{\alpha\beta} x'^\alpha x'^\beta = \delta_{\alpha\beta} M^\alpha_\mu M^\beta_\nu x^\mu x^\nu$$

This is satisfied iff

$$M^\mu_\alpha M^\nu_\beta \delta_{\alpha\beta} = M^\mu_\alpha M^\nu_\alpha = \delta_{\mu\nu}$$

which means

$$MM^T = \mathbb{1} \tag{1}$$

If one expresses M as a matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and write outs the last condition in terms of the matrix elements, one gets 6 linearly independet equations that are contraints on the value of M. This means that any matrix that "preserves the euclidean metric" can be described by three parameters and obeys condition 1 (i.e. it is an orthogonal matrix).
The set of matrices with such a property forms a group under the matrix multiplication product and we call such a group $O(3)$ (*orthogonal group*), where the 3 stands for the number of free parameters we have.
Immediately from equation 1 follows that

$$det(MM^T) = det(M)det(M^T) = det^2(M) = det(\mathbb{1}) = 1$$

In other words one has that $det(M) = \pm 1$. This conditions split the elements of $O(3)$ into two disco-connected components (i.e. it does not exist any continuous transformation that bring a matrix with determinant 1 into one with determinant -1). The set of matrices with determinant 1 forms itself a group under the matrix multiplication, which is a subgroup of $O(3)$ which we call $SO(3)$ (*special orthogonal group*). The other component does not constitute a subgroup since it does not contain the identity.