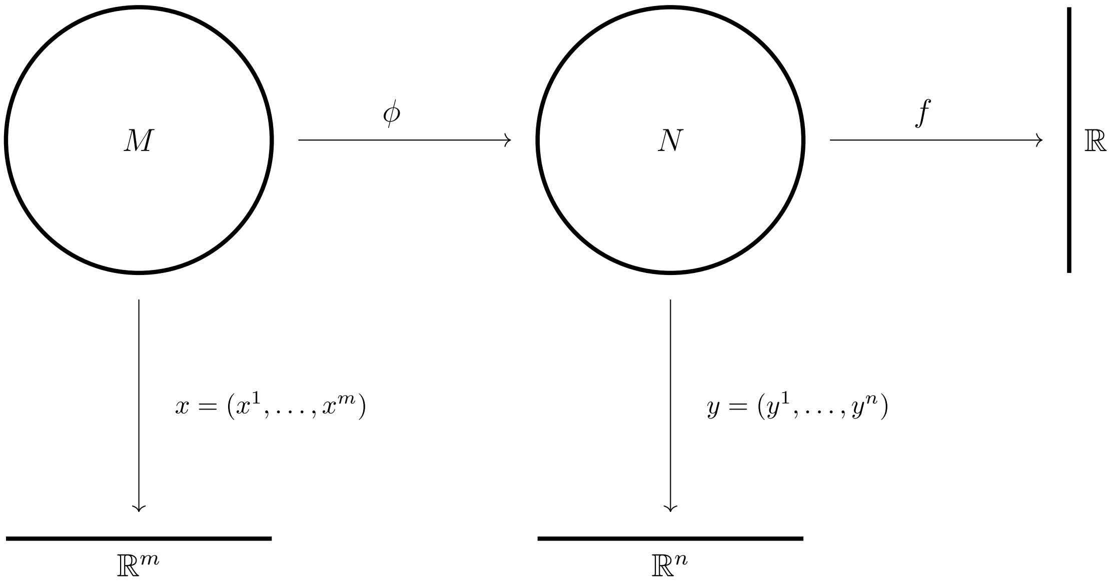
Push-forward and pull-back

Pull-back of a function



Let us consider two manifolds M and N, respectively m-dimensional and n-dimensional.
Let us then consider a mapping between the two manifolds $\phi : M \to N$ and a function $f : N \to \mathbb{R}$.
Let us now consider a vector $V \in T_pM$ where $p \in M$. The vector V is "a derivation" operator acting on smooth functions on M. Given a set of coordinate functions $x = (x^1, \dots, x^m)$ on M, a coordinate representation of V is $V = V^\mu \frac{\partial}{\partial x^\mu}$.
By definition the differential operator V can act only on functions on M. Thus it is possible to make it act on f by *pulling f back* to M trough ϕ.
Indeed the application $f \circ \phi$ maps a point of M in $\mathbb{R}$, hence such a function can be differentiated by X.
The operation of *pulling f back* to M is called *pull-back* of f. More precisely, the pull-back of any smooth function on N is a function

$$\phi^* : \mathcal{S}(N) \to \mathcal{S}(M)$$

where $\mathcal{S}(A)$ indicates the set of smooth functions acting on A, and its action is defined as

$$V(\phi^* f)_p = V(f \circ \phi)$$

where the subscript p means that it acts on M (from now on this will be omitted considering it implicit).
In coordinates $f(\phi(p)) = f(\phi(x^{-1}(\mathbf{x}))) \Rightarrow f = f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^m$. Hence

$$V(\phi_* f) = V^\mu \frac{\partial f(\mathbf{x})}{\partial x^\mu}$$

Push-forward of a vector

The same result above can also be seen as the result of the action of another operator, namely the *push-forward*, which we denote as ϕ_*, whose role is to *push V forward* to N, so that $\phi_* V \in T_{\phi(p)}N$ and its action on a function $f : N \to \mathbb{R}$ is defined by requesting that

$$(\phi_* V)_{\phi(p)}\, f = V(\phi^* f)_p = V(f \circ \phi) \tag{1}$$

where, again, the subscript $\phi_(p)$ indicates that $\phi_* V$ is a vector in $T_{\phi(p)}N$ (and again from later on the subscript will be dropped and considered implicit).
What we did previousle was extending f to M and working in the manifold M. Here, instead, we are extending V to N and working on N.
Now let us introduce the charts $x = (x^1, \dots, x^m)$ in M and $y = (y^1, \dots, y^m)$ in N. A coordinate representation of V is $V = V^\mu \frac{\partial}{\partial x^\mu}$.
Since $\phi_* V$ is a vector in $T_{\phi(p)}N$ a coordinate representation is $\phi_* V = (\phi_* V)^\mu \frac{\partial}{\partial y^\mu}$.
To get each component $(\phi_* V)^\alpha$ we must make $\phi_* V$ act on each coordinate y^α so that

$$(\phi_* V) y^\alpha = \left((\phi_* V)^\mu \frac{\partial}{\partial y^\mu} \right) y^\alpha = (\phi_* V)^\mu = (\phi_* V)^\mu \frac{\partial y^\alpha}{\partial y^\mu} = (\phi_* V)^\mu \delta^\alpha_\mu = (\phi_* V)^\alpha$$

In other words we extracted the components α of the vector $\phi_* V \in T_{\phi(p)}N$.
By using the definition given by 1 and the coordinate representation of V

$$\begin{aligned}(\phi_* V)^\alpha = (\phi_* V)\, y^\alpha = V(y^\alpha \circ \phi) = V^\mu \frac{\partial}{\partial x^\mu}(y^\alpha \circ \phi) = \\ = V^\mu \frac{\partial}{\partial x^\mu}(y^\alpha(\phi(p))) = V^\mu \frac{\partial}{\partial x^\mu}(y^\alpha(\phi(x^{-1}(\mathbf{x}))))\end{aligned} \tag{2}$$

where $\mathbf{x} = x(p) = (x^1, \ldots, x^m) \in \mathbf{R}^m$.
The last member of equation 1 tells us that we can consider y^α as a function of $\mathbf{x}$.
Hence $y^\alpha = y^\alpha(\mathbf{x})$. Finally, equation 2 reduces to

$$(\phi_* V)^\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}$$

Sounds familiar? The function ϕ_* acts on V as a jacobian matrix [1].
Since $\phi_* V \in T_{\phi(p)}N$ we can make it act on a differential form $\omega \in T^*_{\phi(p)}N$.

$$\phi_* V = (\phi_* V)^\mu \frac{\partial}{\partial y^\mu}$$

$$\omega = \omega^\mu dy^\mu$$

$$(\phi_* V)(\omega) = \omega(\phi_* V) = \langle \phi_* V, \omega \rangle = V^\mu \omega^\nu \left\langle \frac{\partial}{\partial y^\mu}, dy^\nu \right\rangle = V^\mu \omega^\mu$$

Pull-back of a differential form

The result we have just obtained can be seen in another way. Indeed, instead of *pushing V forward* to N (working on N), we can *pull ω back* to V (working on N). More precisely we can introduce a new differential form $\phi^* \omega \in T^*_p M$ by requesting that

$$\langle \phi_* V, \omega \rangle = \langle V, \phi^* \omega \rangle$$

The map $\phi^* : T^*_{\phi(p)}N \to T^*_p M$ is called *pull-back* of ω.
This nicely shows that the pull-back ϕ^* and the push-forward ϕ_* are dual functions one to the other.

Integral curves and flows

Let us consider a vector field χ on M, that is a a function that assigns a vector to each point of the manifold

$$\chi : M \to TM$$

$$p \to v \in T_p M$$

Given a set of coordinates in a neighbourhood of a point $p \in M$, the vector defined by χ at the point p is $\chi(p) = X^\mu(x(p)) \frac{\partial}{\partial x^\mu}|_p$.
The vector field χ defines a curve σ (namely the *integral curve* of χ) trough the relation

$$\frac{d}{dt}\sigma(t) = \chi(\sigma(t)) \tag{3}$$

which in coordinates reads

$$\frac{d}{dt} x^\mu(\sigma(t)) = X^\mu(x(\sigma(t))) \qquad \mu = 1, \ldots, m$$

We will mean this last equation when we write

$$\frac{d}{dt}\sigma^\mu(t) = X^\mu(\sigma(t))$$

Equation 3 means that the integral curve of a vector field χ is the one whose tanget vector to σ at each point $p \in M$ is the vector given by $\chi(p)$. The general solution to the ODE is a family of functions. One can identify a particular solution by specifying a the value of the curve at a given point, e.g. $\sigma(t = 0) = p \in M$. We will indicate such a particular solution as $\sigma(t, p)$. The function $\sigma(t, p)$ can be also seen as a function of the point in the manifold p

$$\sigma_t(p) : M \to M$$

We call such a function *flow* generated by χ.

[1] Note that since we are working on a manifold, this holds only locally! And this is why the push-forward is the generalization of the jacobian matrix. The two things coincide when we work in $\mathbb{R}^n$.

Lie derivative