

# Building SUSY models II: using superfields

Seminar on Supersymmetry and its breaking

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# Main points of the talk

- Apply the superspace formalism and show why it is useful to build SUSY theories
- Principles to construct SUSY lagrangians
- SUSY gauge theories: QED and QCD
- SUSY predictions: particles, interaction, masses, ...
- $N=1$  nonrenormalization theorem

# How to build SUSY invariant lagrangians

We want our theory to be SUSY invariant, that is

$$\left( \epsilon \hat{Q} + \epsilon^\dagger \hat{Q}^\dagger \right) S = \left( \epsilon \hat{Q} + \epsilon^\dagger \hat{Q}^\dagger \right) \int dx^\mu \mathcal{L} = 0$$

We know that this condition is met if, under a given transformation

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu f$$

Hence our goal is to build a lagrangian which transforms in this way, using

**Chiral** ( $\bar{D}_{\dot{\alpha}} \Phi = 0$  or  $D_\alpha \Phi^* = 0$ ) and **Vector** ( $V = V^*$ ) superfields

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\phi(x) + \sqrt{2}\theta\psi(x) +$$

$$-\frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \underbrace{\theta\theta F(x)}_{\text{F term}}$$

$$V(x, \theta, \bar{\theta}) = a + \theta\xi + \bar{\theta}\xi^* + \theta\theta b + \bar{\theta}\bar{\theta}b^* + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu +$$

$$+ \bar{\theta}\bar{\theta}\theta \left( \lambda - \frac{i}{2}\sigma^\mu\partial_\mu\xi^* \right) + \theta\theta\bar{\theta} \left( \lambda^* - \frac{i}{2}\sigma^\mu\partial_\mu\xi \right) + \underbrace{\theta\theta\bar{\theta}\bar{\theta} \left( \frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a \right)}_{\text{D term}}$$

# How to build SUSY invariant lagrangians

General superfield expansion

$$S(x, \theta, \bar{\theta}) = a + \theta\xi + \bar{\theta}\chi^\dagger + \theta\theta b + \bar{\theta}\bar{\theta}c + \bar{\theta}\bar{\sigma}^\mu\theta v_\mu + \bar{\theta}\bar{\theta}\theta\eta + \theta\theta\bar{\theta}\zeta^\dagger + \theta\theta\bar{\theta}\bar{\theta}d$$

For the chiral field  $\Phi(x, \theta, \bar{\theta})$  components one has the following transformation properties

$$\delta\phi = \sqrt{2}\zeta\psi$$

$$\delta\psi_\alpha = -\sqrt{2}F\zeta_\alpha - i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}\partial_\mu\phi$$

$$\delta_\epsilon F = -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi$$

For the vector field  $V(x, \theta, \bar{\theta})$  components one has

$$\sqrt{2}\delta_\epsilon a = \epsilon\xi + \epsilon^\dagger\xi^\dagger \quad \sqrt{2}\delta_\epsilon\xi_\alpha = 2\epsilon_\alpha b - (\sigma^\mu\epsilon^\dagger)_\alpha (A_\mu + i\partial_\mu a)$$

$$\sqrt{2}\delta_\epsilon b = \epsilon^\dagger\lambda^\dagger - i\epsilon^\dagger\sigma^\mu\partial_\mu\xi \quad \sqrt{2}\delta_\epsilon A^\mu = i\epsilon\partial^\mu\xi - i\epsilon^\dagger\partial^\mu\xi^\dagger + \epsilon\sigma^\mu\lambda^\dagger - \epsilon^\dagger\bar{\sigma}^\mu\lambda$$

$$\sqrt{2}\delta_\epsilon\lambda_\alpha = \epsilon_a D + \frac{i}{2}(\sigma^\mu\sigma^\nu\epsilon)_\alpha (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\sqrt{2}\delta_\epsilon D = -i\epsilon\sigma^\mu\partial_\mu\lambda^\dagger - i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\lambda$$

# How to build SUSY invariant lagrangians

Idea  $\rightarrow$  "select" the components of the fields that transforms as total derivatives

$$\mathcal{L}[\Phi, V] = \alpha [V]_D + \beta [\Phi]_F + \gamma [\Phi^*]_F$$

How can we "pick" only the terms we need?  $\rightarrow$  Grassman integration

$$\begin{aligned}\int d^2\theta \theta^\alpha f(x, \bar{\theta}) &= f(x, \bar{\theta}) \delta_{\alpha,2} \\ \int d^2\theta d^2\bar{\theta} \bar{\theta}^\alpha \theta^\beta f(x) &= f(x) \delta_{\alpha,2} \delta_{\beta,2}\end{aligned}$$

Hence we are interested in

$$\begin{aligned}[\Phi]_F &= \int d^2\theta \Phi + \text{total derivative} \\ [V]_D &= \int d^2\theta d^2\bar{\theta} V(x, \theta, \bar{\theta}) + \text{total derivative}\end{aligned}$$

# How to build SUSY invariant lagrangians

Let us focus for a moment on the D term.

A vector field can be obtained from 2 chiral superfields by taking the product  $\Phi^* \Phi$

$$\begin{aligned}
 \Phi(x, \theta, \bar{\theta}) &= \phi(x) + i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\phi(x) + \sqrt{2}\theta\psi(x) \\
 &\quad - \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \theta\theta F(x) \\
 \Phi^{*i}\Phi_j &= \phi^{*i}\phi_j + \sqrt{2}\theta\psi_j\phi^{*i} + \sqrt{2}\theta^\dagger\psi^\dagger{}^i\phi_j + \theta\theta\phi^{*i}F_j + \theta^\dagger\theta^\dagger\phi_jF^{*i} \\
 &\quad + \theta^\dagger\bar{\sigma}^\mu\theta\left[i\phi^{*i}\partial_\mu\phi_j - i\phi_j\partial_\mu\phi^{*i} - \psi^\dagger{}^i\sigma_\mu\psi_j\right] \\
 &\quad + \frac{i}{\sqrt{2}}\theta\theta\theta^\dagger\bar{\sigma}^\mu\left(\psi_j\partial_\mu\phi^{*i} - \partial_\mu\psi_j\phi^{*i}\right) + \sqrt{2}\theta\theta\theta^\dagger\psi^\dagger{}^iF_j \\
 &\quad + \frac{i}{\sqrt{2}}\theta^\dagger\theta^\dagger\theta\sigma^\mu\left(\psi^\dagger{}^i\partial_\mu\phi_j - \partial_\mu\psi^\dagger{}^i\phi_j\right) + \sqrt{2}\theta^\dagger\theta^\dagger\theta\psi_jF^{*i} \\
 &\quad + \theta\theta\theta^\dagger\theta^\dagger\left[F^{*i}F_j - \frac{1}{2}\partial^\mu\phi^{*i}\partial_\mu\phi_j + \frac{1}{4}\phi^{*i}\partial^\mu\partial_\mu\phi_j + \frac{1}{4}\phi_j\partial^\mu\partial_\mu\phi^{*i}\right. \\
 &\quad \left.+ \frac{i}{2}\psi^\dagger{}^i\bar{\sigma}^\mu\partial_\mu\psi_j + \frac{i}{2}\psi_j\sigma^\mu\partial_\mu\psi^\dagger{}^i\right]
 \end{aligned}$$

# How to build SUSY invariant lagrangians

Now "select" the SUSY invariant component (D component)

$$\begin{aligned} [\Phi^* \Phi]_D &= \int d^2\theta d^2\bar{\theta} \Phi^*(x, \theta, \bar{\theta}) \Phi(x, \theta, \bar{\theta}) \\ &= \left[ F^* F - \frac{1}{2} \partial^\mu \phi^{*i} \partial_\mu \phi_j + \frac{1}{4} \phi^{*i} \partial^\mu \partial_\mu \phi_j + \frac{1}{4} \phi_j \partial^\mu \partial_\mu \phi^{*i} \right. \\ &\quad \left. + \frac{i}{2} \psi^{\dagger i} \bar{\sigma}^\mu \partial_\mu \psi_j + \frac{i}{2} \psi_j \sigma^\mu \partial_\mu \psi^{\dagger i} \right] = \\ &= -\partial^\mu \phi^* \partial_\mu \phi + i \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^* F + \text{total derivative} \end{aligned}$$

Hence

$$S = \int d^4x \mathcal{L} = \int d^4x \left[ -\partial^\mu \phi^* \partial_\mu \phi + i \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^* F \right]$$

→ **Free Wess-Zumino model!**

# How to build SUSY invariant lagrangians

In order to add SUSY invariant interactions, let us recall the definitions of chiral superfields

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \quad D_{\alpha}\Phi^* = 0$$

Note that any analytic function of chiral superfields is in turn a chiral superfield (power series expansion and product rule).

⇒ Write our chiral term of  $N$  fields as

$$W(\{\Phi_k\}) = \sum_i^N a_i \Phi_i + \sum_{i,j}^N \frac{1}{2!} m_{ij} \Phi_i \Phi_j + \sum_{i,j,k}^N \frac{1}{3!} y_{ijk} \Phi_i \Phi_j \Phi_k$$

Higher order terms are non-renormalisable.

Reality of the action requires us to take  $W + W^*$



# How to build SUSY invariant lagrangians

$$\begin{aligned}\mathcal{L}_{WZ}(\{\Phi_i\}, \{\Phi_i^*\}) &= \mathcal{L}_{WZ,D} + \mathcal{L}_{WZ,F} = \\ &= [\Phi^{*i}\Phi^i]_D + [W(\{\Phi_i\})]_F + [W^*(\{\Phi_i^*\})]_F = \\ &= \boxed{\int d^2\theta \left( -\frac{1}{4}\overline{D}D\Phi^{*i}\Phi_i + W(\{\Phi_i\}) \right) + \int d^2\bar{\theta} W^*(\{\Phi_i^*\})}\end{aligned}$$

Equations of motion varying w.r.t.  $\Phi_i$  and  $\Phi_i^*$

$$\begin{aligned}0 &= -\frac{1}{4}\overline{D}D\Phi^{*i} + \frac{\delta W}{\delta\Phi_i} \\ 0 &= -\frac{1}{4}D\overline{D}\Phi_i + \frac{\delta W^*}{\delta\Phi^{*i}}\end{aligned}$$

# Wess-Zumino model

We are now interested in studying better the components of  $\Phi$ , hence let us focus on the case  $i = j$  with  $a_i = a$ ,  $m_{ij} = m$ ,  $y_{ijk} = y$  and the superpotential

$$W(\Phi) = \frac{1}{2}m\Phi\Phi + \frac{1}{3!}\Phi\Phi\Phi$$

where we dropped the linear term

$$\begin{aligned}\mathcal{L}(\Phi, \Phi^*) &= \mathcal{L}_{WZ,D} + \mathcal{L}_{WZ,F} = \\ &= F^\dagger F + (\partial_\mu \phi)(\partial^\mu \phi)^\dagger + \frac{i}{2}\psi\sigma^\mu(\partial_\mu \bar{\psi}) - \frac{i}{2}(\partial_\mu \psi)\sigma^\mu \bar{\psi} + \\ &\quad - m\phi F - \frac{m}{2}(\psi\psi) - \frac{y}{2}\phi\phi F - \frac{y}{2}\phi(\psi\psi) + \text{h.c.}\end{aligned}$$

E.o.m. for  $F$  (analogous for  $F^*$ ) is

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu F)} - \frac{\partial \mathcal{L}}{\partial F} = -\frac{\partial \mathcal{L}}{\partial F} = -F^\dagger + m\phi + \frac{y}{2}\phi\phi$$

$\Rightarrow$  algebraic equation  $\Rightarrow F, F^*$  are unphysical.

# Wess-Zumino model

$$F^* = m\phi + \frac{y}{2}\phi\phi \quad F = m\phi^* + \frac{y}{2}\phi\phi$$

One can rewrite the terms containing  $F, F^*$  as

$$F^\dagger F - \left( m\phi F + \frac{y}{2}\phi\phi F + hc \right) = - \left| m\phi + \frac{y}{2}\phi\phi \right|^2 = - \left| \frac{\partial W(\phi)}{\partial \phi} \right|^2$$

that is, the superpotential evaluated at the scalar field  $\phi$ ! The lagrangian then becomes

$$\begin{aligned} \mathcal{L}_{WZ} = & (\partial_\mu \varphi) (\partial^\mu \varphi)^\dagger + \frac{i}{2} \psi \sigma^\mu (\partial_\mu \bar{\psi}) - \frac{i}{2} (\partial_\mu \psi) \sigma^\mu \bar{\psi} \\ & - |M|^2 \varphi \varphi^\dagger - \frac{|y|^2}{4} \varphi \varphi \varphi^\dagger \varphi^\dagger - \left( \frac{M}{2} \psi \psi + \frac{M \cdot y}{2} \varphi \varphi \varphi^\dagger + \frac{y}{2} \varphi \psi \psi + \text{h.c.} \right) \end{aligned}$$

and one can prove that in the case of  $N$  fields it can be written as

$$\begin{aligned} \mathcal{L}_{WZ} = & (\partial_\mu \varphi_i) (\partial^\mu \varphi_i)^\dagger + \frac{i}{2} \psi_i \sigma^\mu (\partial_\mu \bar{\psi}_i) - \frac{i}{2} (\partial_\mu \psi_i) \sigma^\mu \bar{\psi}_i \\ & - \sum_i \left| \frac{\partial W(\varphi_i)}{\partial \varphi_i} \right|^2 - \frac{1}{2} \left( \frac{\partial^2 W(\varphi_i)}{\partial \varphi_i \partial \varphi_j} \right) \psi_i \psi_j - \frac{1}{2} \left( \frac{\partial^2 W^\dagger(\varphi_i)}{\partial \varphi_i^* \partial \varphi_j^*} \right) \bar{\psi}_i \bar{\psi}_j \end{aligned}$$

# Wess-Zumino model

$$\mathcal{L}_{\text{WZ}} = (\partial_\mu \varphi) (\partial^\mu \varphi)^\dagger + \frac{i}{2} \psi \sigma^\mu (\partial_\mu \bar{\psi}) - \frac{i}{2} (\partial_\mu \psi) \sigma^\mu \bar{\psi} \\ - |M|^2 \varphi \varphi^\dagger - \frac{|y|^2}{4} \varphi \varphi \varphi^\dagger \varphi^\dagger - \left( \frac{M}{2} \psi \psi + \frac{M \cdot y}{2} \varphi \varphi \varphi^\dagger + \frac{y}{2} \varphi \psi \psi + \text{h.c.} \right)$$

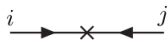
Let us give a look at the interactions



(a)



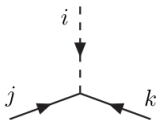
(b)



(c)



(d)



(a)



(b)



(c)



# Abelian gauge theories

Let us introduce (abelian) gauge interactions.

Let us start with U(1) global symmetry

$$\Phi_i \rightarrow e^{iq_i \Lambda_i} \Phi_i$$

The kinetic part of the lagrangian is always invariant

$$\mathcal{L}_K = \mathcal{L}_{WZ,D} = \int d^2\theta d^2\bar{\theta} \Phi^* \Phi = \int d^2\theta -\frac{1}{4} \overline{D D} \Phi^* \Phi$$

The interaction part

$$\mathcal{L}_{int} = \mathcal{L}_{WZ,F} = \int d^2\theta \frac{1}{2} \sum_{ij} \phi_i \phi_j + \frac{1}{3!} \sum_{ijk} \phi_i \phi_j \phi_k + \text{complex. conj.}$$

requires

$$m_{ij} = 0 \quad \text{or} \quad y_{ijk} = 0$$

whenever

$$q_i + q_j \neq 0 \quad \text{or} \quad q_i + q_j + q_k \neq 0$$

# Abelian gauge theories

Promote to a local gauge symmetry

$$\Lambda \rightarrow \Lambda(x, \theta, \bar{\theta})$$

- Gauge parameter is now a supergauge field  $\Lambda = \Lambda(x, \theta, \bar{\theta})$
- Promote derivatives to covariant derivatives  $\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$
- We need  $\Lambda(x, \theta, \bar{\theta})$  to be a left-chiral superfield if we want  $\Phi'$  to be a left-chiral superfield (chain rule).
- Thus this causes a problem in the kinetic term because  $\Lambda^*$  is a right-chiral superfield hence obviously  $\Phi'^* \Phi' \neq \Phi^* \Phi$
- The problem is analogous to the kinetic term in "normal" when  $\partial_\mu \phi^* \partial^\mu \phi$  was not gauge invariant
- Solution: add a term that compensate the gauge for the non invariant terms

# Abelian gauge theories

$$\Phi^\dagger \Phi \rightarrow \Phi^* e^{-i\Lambda^*(x)} e^{i\Lambda(x)} \Phi$$

$\Rightarrow$  need an object  $A$  such that  $A' = e^{i\Lambda^*} A e^{-i\Lambda}$  and the quantity  $\Phi^* A \Phi$  is then gauge invariant.

Remember that a vector superfield  $V$  transforms according to

$$V \rightarrow V' = V + i\Lambda * -\Lambda$$

$$\Rightarrow \quad A = e^V \text{ is what we need.}$$

Now we are just left with finding the gauge-invariant strength field term (equivalent of  $F_{\mu\nu} F^{\mu\nu}$ )



# Abelian gauge theories

Let us start by defining the two chiral fields

$$\mathcal{W}_\alpha = -\frac{1}{4}\overline{D}\overline{D}D_\alpha V, \quad \overline{\mathcal{W}}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}} V$$

where  $V$  is a vector field.

The gauge-invariant dynamical term (equivalent to  $F_{\mu\nu}F^{\mu\nu}$ ) is

$$[W]_F + [\bar{W}]_F = \int d^2\theta W_\alpha W^\alpha + \int d^2\bar{\theta} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}$$

The explicit derivation is quite long (it will be put in the report appendix) but we can make two checks to get more convinced

- Check that it is indeed gauge invariant
- Check that it contains the "normal" gauge strength field  $F_{\mu\nu}F^{\mu\nu}$  after integrating out  $\theta$  and  $\bar{\theta}$

To see that it is gauge invariant:

$$\begin{aligned}\mathcal{W}_\alpha &\rightarrow -\frac{1}{4}\overline{D}D D_\alpha [V + i(\Omega^* - \Omega)] = \mathcal{W}_\alpha + \frac{i}{4}\overline{D}D D_\alpha \Omega \\ &= \mathcal{W}_\alpha - \frac{i}{4}\bar{D}^{\dot{\beta}} \left\{ \bar{D}_{\dot{\beta}}, D_\alpha \right\} \Omega \\ &= \mathcal{W}_\alpha + \frac{1}{2}\sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \bar{D}^{\dot{\beta}} \Omega \\ &= \mathcal{W}_\alpha\end{aligned}$$

# Abelian gauge theories

Remember that in the Wess-Zumino gauge the field expansion takes the form

$$V(y, \theta, \bar{\theta}) = \bar{\theta} \bar{\sigma}^\mu \theta A_\mu(y) + \bar{\theta} \bar{\theta} \theta \lambda(y) + \theta \theta \bar{\theta} \lambda^\dagger(y) + \frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} [D(y) + i \theta_\mu A^\mu(y)]$$

Hence

$$W = \dots$$

Finally

$$\frac{1}{4} [\mathcal{W}\mathcal{W}]_F + \frac{1}{4} [\overline{\mathcal{W}\mathcal{W}}]_F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda + \frac{1}{2} D^2$$

We recovered the desired term  $F_{\mu\nu} F^{\mu\nu}$ , but what are the other two terms?

# Abelian gauge theories

The term

$$i\lambda^* \bar{\sigma}^\mu \partial_\mu \lambda$$

is just the superpartner of the photon, the *photino*!

For the other term, we can show that it plays no physical role (as the  $F$  term in the chiral fields). To do this we need to spot all the  $D$  dependence in our lagrangian.

Remembering that up to now our lagrangian is

$$\mathcal{L} = \frac{1}{4} [\mathcal{W}\mathcal{W}]_F + \frac{1}{4} [\overline{\mathcal{W}\mathcal{W}}]_F + [\Phi^* e^V \Phi]_D + [W(\Phi)]_F + [\bar{W}(\Phi^*)]_F$$

once can note that the only other dependence on  $D$  is in

$[\Phi^* e^V \Phi]_D = \Phi^* \Phi D$ . Hence the equation of motions for  $D$  are

$$0 = \frac{\partial \mathcal{L}}{\partial D} = D + \Phi * \Phi$$

Putting all together we get the SUSY QED lagrangian

$$\mathcal{L} = \dots$$

To this we add another SUSY and supergauge invariant term  $2[kV]_D = 2kD$  (e.o.m. is still algebraic). This term is called *Fayet-Iliopoulos* and it will play an important role in the spontaneous SUSY breaking (next talks)

$$\mathcal{L}_{SQED} = \dots$$

# Non abelian gauge theories

Now extend to generic gauge theories, in particular  $SU(n)$ . In spacetime:

- $n^2 - 1$  generators  $T_1, \dots, T_{n^2-1}$  (gauge fields)
- for  $n = 3$  we have 8 fields (gluons)
- $[T_a, T_b] = i f_{abc} T_c$  where  $f_{abc}$  are called structure constants
- $U \in SU(n) \Rightarrow U = \exp(igA^a T^a)$
- Covariant derivatives in spacetime are
$$D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igA_\mu^a T^a$$

In spacetime, guided by the fact that for  $U(1)$  we had

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e} [D_\mu, D_\nu]$$

we *define* our field strength tensor for a general gauge symmetry via

$$F_{\mu\nu} \equiv \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$$

$\Rightarrow$  how to extend to superspace?

# Non abelian gauge theories

Our starting point is again a term like

$$\Phi^* e^A \Phi$$

→ we need to figure out how to set  $A$