Building SUSY models II: using superfields

Seminar on Supersymmetry and its breaking

Matteo Zortea

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Coordinated by prof. Jörg Jäckel

Main points of the talk

- Apply the superspace formalism and show why it is useful to build SUSY theories
- Principles to construct SUSY lagrangians
- SUSY gauge theories: QED and QCD
- SUSY predictions: particles, interaction, masses, ...
- N=1 nonrenormalization theorem

How to build SUSY invariant lagrangians

We want our theory to be SUSY invariant, that is

$$\left(\epsilon\hat{Q}+\epsilon^{\dagger}\hat{Q}\right)S=\left(\epsilon\hat{Q}+\epsilon^{\dagger}\hat{Q}\right)\int dx^{\mu}\mathcal{L}=0$$

We know that this condition is met if, under a given transformation

$$\mathcal{L}\rightarrow\mathcal{L}+\partial_{\mu}f$$

Hence our goal is to build a lagrangin which transforms in this way, using **Chiral** ($\bar{D}_{\dot{\alpha}}\Phi=0$ or $D_{\alpha}\Phi^{*}=0$) and **Vector** ($V=V^{*}$) superfields

$$\Phi(x,\theta,\bar{\theta})=\phi(x)+i\bar{\theta}\bar{\sigma}^{\mu}\theta\partial_{\mu}\phi(x)+\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_{\mu}\partial^{\mu}\phi(x)+\sqrt{2}\theta\psi(x)+$$

$$-\frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^{\mu}\partial_{\mu}\psi(x)+\theta\theta\underbrace{F(x)}_{\text{F term}}$$

$$V\left(x,\theta,\bar{\theta}\right)=a+\theta\xi+\bar{\theta}\xi^{*}+\theta\theta b+\bar{\theta}\bar{\theta}b^{*}+\bar{\theta}\bar{\sigma}^{\mu}\theta A_{\mu}+$$

$$+\bar{\theta}\bar{\theta}\theta\left(\lambda-\frac{i}{2}\sigma^{\mu}\partial_{\mu}\xi^{*}\right)+\theta\theta\bar{\theta}\left(\lambda^{*}-\frac{i}{2}\sigma^{\mu}\partial_{\mu}\xi\right)+\theta\theta\bar{\theta}\bar{\theta}\underbrace{\left(\frac{1}{2}D+\frac{1}{4}\partial_{\mu}\partial^{\mu}a\right)}_{\text{D term}}$$

How to build SUSY invariant lagrangians

General superfield expansion

$S\left(x,\theta,\bar{\theta}\right) = a + \theta\xi + \bar{\theta}\chi^{\dagger} + \theta\theta b + \bar{\theta}\bar{\theta}c + \bar{\theta}\bar{\sigma}^{\mu}\theta v_{\mu} + \bar{\theta}\bar{\theta}\theta\eta + \theta\theta\bar{\theta}\zeta^{\dagger} + \theta\theta\bar{\theta}\bar{\theta}d$

For the chiral field $\Phi(x,\theta\bar{\theta})$ components one has the following transformation properties

$$\delta\phi = \sqrt{2}\zeta\psi$$

$$\delta\psi_{\alpha} = -\sqrt{2}F\zeta_{\alpha} - i\sqrt{2}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\zeta}^{\dot{\alpha}}\partial_{\mu}\phi$$

$$\boxed{\delta_{\epsilon}F = -i\epsilon^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\psi}$$

For the vector field $V(x,\theta,\bar{\theta})$ components one has

$$\sqrt{2}\delta_{\epsilon}a = \epsilon\xi + \epsilon^{\dagger}\xi^{\dagger} \qquad \sqrt{2}\delta_{\epsilon}\xi_{\alpha} = 2\epsilon_{\alpha}b - \left(\sigma^{\mu}\epsilon^{\dagger}\right)_{\alpha}(A_{\mu} + i\partial_{\mu}a)$$

$$\sqrt{2}\delta_{\epsilon}b = \epsilon^{\dagger}\lambda^{\dagger} - i\epsilon^{\dagger}\sigma^{\mu}\partial_{\mu}\xi \qquad \sqrt{2}\delta_{\epsilon}A^{\mu} = i\epsilon\partial^{\mu}\xi - i\epsilon^{\dagger}\partial^{\mu}\xi^{\dagger} + \epsilon\sigma^{\mu}\lambda^{\dagger} - \epsilon^{\dagger}\bar{\sigma}^{\mu}\lambda$$

$$\sqrt{2}\delta_{\epsilon}\lambda_{\alpha} = \epsilon_{a}D + \frac{i}{2}\left(\sigma^{\mu}\sigma^{\nu}\epsilon\right)_{\alpha}\left(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}\right)$$

$$\boxed{\sqrt{2}\delta_{\epsilon}D = -i\epsilon\sigma^{\mu}\partial_{\mu}\lambda^{\dagger} - i\epsilon^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\lambda}$$

How to build SUSY invariant lagrangians

Idea → "select" the components of the fields that transforms as total derivatives

$$\mathcal{L}[\Phi, V] = \alpha\ [V]_D + \beta\ [\Phi]_F + \gamma\ [\Phi^*]_F$$

How can we "pick" only the terms we need? → Grassman integration

$$\int d^2\theta\ \theta^\alpha\, f(x, \bar{\theta}) = f(x, \bar{\theta})\, \delta_{\alpha,2}$$

$$\int d^2\theta d^2\bar{\theta}\ \bar{\theta}^\alpha \theta^\beta f(x) = f(x)\, \delta_{\alpha,2}\, \delta_{\beta,2}$$

Hence we are interested in

$$[\Phi]_F = \int d^2\theta \Phi + \text{total derivative}$$

$$[V]_D = \int d^2\theta d^2\bar{\theta} V(x, \theta, \bar{\theta}) + \text{total derivative}$$

How to build SUSY invariant lagrangians

Let us focus for a moment on the D term.

A vector field can be obtained from 2 chiral superfields by taking the product $\Phi^*\Phi$

$$\Phi(x,\theta,\bar\theta) = \phi(x) + i\bar\theta\bar\sigma^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial_\mu\partial^\mu\phi(x) + \sqrt{2}\theta\psi(x)$$
$$-\frac{i}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + \theta\theta F(x)$$
$$\Phi^{*i}\Phi_j = \phi^{*i}\phi_j + \sqrt{2}\theta\psi_j\phi^{*i} + \sqrt{2}\theta^\dagger\psi^{\dagger i}\phi_j + \theta\theta\phi^{*i}F_j + \theta^\dagger\theta^\dagger\phi_j F^{*i}$$
$$+\theta^\dagger\bar\sigma^\mu\theta\left[i\phi^{*i}\partial_\mu\phi_j - i\phi_j\partial_\mu\phi^{*i} - \psi^{\dagger i}\sigma_\mu\psi_j\right]$$
$$+\frac{i}{\sqrt{2}}\theta\theta\theta^\dagger\bar\sigma^\mu\left(\psi_j\partial_\mu\phi^{*i} - \partial_\mu\psi_j\phi^{*i}\right) + \sqrt{2}\theta\theta\theta^\dagger\psi^{\dagger i}F_j$$
$$+\frac{i}{\sqrt{2}}\theta^\dagger\theta^\dagger\theta\sigma^\mu\left(\psi^{\dagger i}\partial_\mu\phi_j - \partial_\mu\psi^{\dagger i}\phi_j\right) + \sqrt{2}\theta^\dagger\theta^\dagger\theta\psi_j F^{*i}$$
$$+\theta\theta\theta^\dagger\theta^\dagger\Bigg[F^{*i}F_j - \frac{1}{2}\partial^\mu\phi^{*i}\partial_\mu\phi_j + \frac{1}{4}\phi^{*i}\partial^\mu\partial_\mu\phi_j + \frac{1}{4}\phi_j\partial^\mu\partial_\mu\phi^{*i}$$
$$+\frac{i}{2}\psi^{\dagger i}\bar\sigma^\mu\partial_\mu\psi_j + \frac{i}{2}\psi_j\sigma^\mu\partial_\mu\psi^{\dagger i}\Bigg]$$

How to build SUSY invariant lagrangians

Now "select" the SUSY invariant component (D component)

$$[\Phi^*\Phi]_D = \int d^2\theta d^2\bar{\theta}\ \Phi^*(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta})$$

$$\Big[F^*F - \frac{1}{2}\partial^\mu\phi^{*i}\partial_\mu\phi_j + \frac{1}{4}\phi^{*i}\partial^\mu\partial_\mu\phi_j + \frac{1}{4}\phi_j\partial^\mu\partial_\mu\phi^{*i}$$

$$+\frac{i}{2}\psi^{\dagger i}\bar{\sigma}^\mu\partial_\mu\psi_j + \frac{i}{2}\psi_j\sigma^\mu\partial_\mu\psi^{\dagger i}\Big] =$$

$$= -\partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + F^*F + \text{total derivative}$$

Hence

$$\boxed{S = \int dx^\mu \mathcal{L} = \int dx^\mu - \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + F^*F}$$

$\rightarrow$ **Free Wess-Zumino model!**

How to build SUSY invariant lagrangians

In order to add SUSY invariant interactions, let us recall the definitions of chiral superfields

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \qquad D_{\alpha}\Phi^* = 0$$

Noote that any analytic function of chiral superfields is in turn a chiral superfield (power series expansion and product rule).
$\Rightarrow$ Write our chiral term of N fields as

$$W(\{\Phi_k\}) = \sum_i^N M_i\Phi_i + \sum_{i,j}^N \frac{1}{2!}M_{ij}\Phi_i\Phi_j + \sum_{i,j,k}^N \frac{1}{3!}y_{ijk}\Phi_i\Phi_j\Phi_k$$

Higher order terms are non-renormalisable.
Reality of the action requires us to take $W + W^*$

How to buld SUSY invariant lagrangians

$$\mathcal{L}_{WZ}\left(\{\Phi_i\},\{\Phi_i^*\}\right)=\mathcal{L}_{WZ,D}+\mathcal{L}_{WZ,F}=$$
$$=\left[\Phi^{*i}\Phi^i\right]_D+\left[W(\{\Phi_i\})\right]_F+\left[W^*(\{\Phi_i^*\})\right]_F=$$
$$=\boxed{\int d^2\theta\ \left(-\frac{1}{4}\overline{DD}\Phi^{*i}\Phi_i+W(\{\Phi_i\})\right)+\int d^2\bar{\theta}\ W^*(\{\Phi_i^*\})}$$

Equations of motion varying w.r.t. Φ_i and Φ_i^*

$$0=-\frac{1}{4}\overline{DD}\Phi^{*i}+\frac{\delta W}{\delta\Phi_i}$$
$$0=-\frac{1}{4}DD\Phi_i+\frac{\delta W^*}{\delta\Phi^{*i}}$$

Wess-Zumino model

We are now interested in studying better the components of Φ, hence let us focus in the case $i = j$

$$\mathcal{L}(\Phi, \Phi^*) = \dots$$

Abelian gauge theories

Let us introduce (abelian) gauge interactions.

Let us start with U(1) global symmetry

$$\Phi_i \to e^{iq_i\Lambda_i}\Phi_i$$

The kinetic part of the lagrangian is always invariant

$$\mathcal{L}_K = \mathcal{L}_{WZ,D} = \int d^2\theta d^2\bar{\theta}\Phi^*\Phi = \int d^2\theta - \frac{1}{4}\overline{DD}\Phi^*\Phi$$

The interaction part

$$\mathcal{L}_{int} = \mathcal{L}_{WZ,F} = \int d^2\theta \frac{1}{2}\sum_{ij}\phi_i\phi_j + \frac{1}{3!}\sum_{ijk}\phi_i\phi_j\phi_k + \text{complex. conj.}$$

requires

$$m_{ij} = 0 \qquad \text{or} \qquad y_{ijk} = 0$$

whenever

$$q_i + q_j \neq 0 \qquad \text{or} \qquad q_i + q_j + q_k \neq 0$$

Abelian gauge theories

Promote to a local gauge symmetry

$$\Lambda \to \Lambda(x, \theta, \bar{\theta})$$

- Gauge parameter is now a supergauge field $\Lambda = \Lambda(x, \theta, \bar{\theta})$
- Promote derivatives to covariant derivatives $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$
- We need $\Lambda(x, \theta, \bar{\theta})$ to be a left-chiral superfield if we want Φ' to be a left-chiral superfield (chain rule).
- Thus this causes a problem in the kinetic term because Λ^* is a right-chiral superfield hence obviously $\Phi'^*\Phi' \neq \Phi^*\Phi$
- The problem is analogous to the kinetc term in "normal" when $\partial_u \phi^* \partial^\mu \phi$ was not gauge invariant
- Solution: add a term that compensate the gauge for the non invariant terms

Abelian gauge theories

$$\Phi^+\Phi \to \Phi^* e^{-i\Lambda^*(x)} e^{i\Lambda(x)} \Phi$$

$\Rightarrow$ need an object A such that $A' = e^{i\Lambda *} A e^{-i\Lambda}$ and the quantity $\Phi^* A \Phi *$ is then gauge invariant.
Remember that a vector superfield V transforms according to

$$V \to V' = V + i\Lambda * -\Lambda$$

$$\Rightarrow \qquad A = e^V \text{ is what we need.}$$

Now we are just left with finding the gauge-invariant strength field term (euivalent of $F_{\mu\nu}F^{\mu\nu}$)

Abelian gauge theories

Let us start by defining the two chiral fields

$$\mathcal{W}_\alpha = -\frac{1}{4}\overline{DD}D_\alpha V, \quad \overline{\mathcal{W}}_{\dot\alpha} = -\frac{1}{4}DD\bar{D}_{\dot\alpha}V$$

where V is a vector field.
The gauge-invariant dynamical term (equivalent to $F_{\mu\nu}F^{\mu\nu}$) is

$$[W]_F + [\bar{W}]_F = \int d^2\theta W_\alpha W^\alpha + \int d^2\bar\theta \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}$$

The explicit derivation is quite long (it will be put in the report appendix) but we can make two checks to get more convinced

- Check that it is indeed gauge invariant
- Check that it contains the "normal" gauge strength field $F_{\mu\nu}F^{\mu\nu}$ after integrating out θ and $\bar\theta$

Abelian gauge theories

To see that it is gauge invariant:

$$
\begin{aligned}
\mathcal{W}_\alpha \to -\frac{1}{4}\overline{DD}D_\alpha\left[V + i\left(\Omega^* - \Omega\right)\right] &= \mathcal{W}_\alpha + \frac{i}{4}\overline{DD}D_\alpha\Omega \\
&= \mathcal{W}_\alpha - \frac{i}{4}\bar{D}^{\dot\beta}\left\{\bar{D}_{\dot\beta}, D_\alpha\right\}\Omega \\
&= \mathcal{W}_\alpha + \frac{1}{2}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar{D}^{\dot\beta}\Omega \\
&= \mathcal{W}_\alpha
\end{aligned}
$$

Abelian gauge theories

Remember that in the Wess-Zumino gauge the field expansion takes the form

$$V(y,\theta,\bar{\theta}) = \bar{\theta}\bar{\sigma}^{\mu}\theta A_{\mu}(y) + \bar{\theta}\bar{\theta}\theta\lambda(y) + \theta\theta\bar{\theta}\lambda^{\dagger}(y) + \frac{1}{2}\theta\theta\theta\bar{\theta}\bar{\theta}\left[D(y) + i\theta_{\mu}A^{\mu}(y)\right]$$

Hence

$$W = \dots$$

Finally

$$\frac{1}{4}\left[\mathcal{W}\mathcal{W}\right]_F + \frac{1}{4}\left[\overline{\mathcal{W}\mathcal{W}}\right]_F = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\lambda^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\lambda + \frac{1}{2}D^2$$

We recovered the desired term $F_{\mu\nu}F^{\mu\nu}$, but what are the other two terms?

Abelian gauge theories

The term

$$i\lambda^* \bar{\sigma}^\mu \partial_\mu \lambda$$

is just the superpartner of the photon, the *photino*!

For the other term, we can show that it plays no physical role (as the F term in the chiral fields). To do this we need to spot all the D dependence in our lagrangian.

Remembering that up to now our lagrangian is

$$\mathcal{L} = \frac{1}{4}\left[\mathcal{W}\mathcal{W}\right]_F + \frac{1}{4}\left[\overline{\mathcal{W}\mathcal{W}}\right]_F + \left[\Phi^* e^V \Phi\right]_D + \left[W(\Phi)\right]_F + \left[\bar{W}(\Phi^*)\right]_F$$

once can note that the only other dependence on D is in $\left[\Phi^* e^V \Phi\right]_D = \Phi^* \Phi D$. Hence the equation of motions for D are

$$0 = \frac{\partial \mathcal{L}}{\partial D} = D + \Phi * \Phi$$

Abelian gauge theories

Putting all together we get the SUSY QED lagrangian

$$\mathcal{L} = \dots$$

To this we add another SUSY and supergauge invariant term $2[kV]_D = 2kD$ (e.o.m. is still algebraic). This term is called *Fayet-Iliopoulos* and it will play an important role in the spontaneous SUSY breaking (next talks)

$$\boxed{\mathcal{L}_{SQED} = \dots}$$

Non abelian gauge theories

Now extend to generic gauge theories, in particular SU(n). In spacetime:

- $n^2 - 1$ generators $T_1, \dots T_{n^2-1}$ (gauge fields)
- for $n = 3$ we have 8 fields (gluons)
- $[T_a, T_b] = i\, f_{abc}\, T_c$ where f_{abc} are called structure constants
- $U \in SU(n) \Rightarrow U = \exp(igA^a T^a)$
- Covariant derivatives in spacetime are
  $D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igA^a_u T^a$

In spacetime, guided by the fact that for $U(1)$ we had

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{i}{e}\,[D_\mu, D_\nu]$$

we *define* our field strength tensor for a general gauge symmetry via

$$F_{\mu\nu} \equiv \frac{i}{g}\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]$$

$\Rightarrow$ how to extend to superspace?

Non abelian gauge theories

Our starting point is again a term like

$$\Phi^* e^A \Phi$$

$\rightarrow$ we need to figure out how to set A