Building supersymmetric models using superfields

Seminar on Supersimmetry

University of Heidelberg, Summer Semester 2022

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Part on $\mathcal{L} \to \mathcal{L} + \partial_\mu f$
The two main ingredients of a supersymmetric theory in the superspace formalism, are the chiral and vector superfields. Let us briefly recall some of their properties, useful for the subsequent reasonings.

A left-chiral superfield Φ (right-chiral superfield χ) is obtained by imposing the constrain $\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0$ ($D_\alpha \chi(x,\theta,\bar{\theta}) = 0$), and a general expansion in powers of $\theta, \bar{\theta}$ reads

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = \varphi(x) + i\bar{\theta}\bar{\sigma}^\mu\theta\partial_\mu\varphi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\varphi(x) + \sqrt{2}\theta\psi(x) + \\ -\frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi(x) + \theta\theta F(x)\end{aligned} \tag{1}$$

The hermitian conjugate of a left-chiral superfield $\Phi^\dagger$ is a right-chiral superfield and vice-versa. The field ϕ entering equation **??** is the bosonic field of the theory, ψ is its fermionic supersymmetric partner and F will simply turn out to be unphysical. This will become clearer when looking at the interactions between the fields and at the equations of motion.
Under a SUSY transformation the components transform as

$$\delta_\epsilon\phi = \epsilon\psi \qquad \delta_\epsilon\psi_\alpha = -i\left(\sigma^\mu\epsilon^\dagger\right)_\alpha\partial_\mu\phi + \epsilon_\alpha F, \qquad \boxed{\delta_\epsilon F = -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi}$$

The transformation law of F is of particular interest, since it is precisely a total derivative, that is what one looks for to build invariant lagrangians.

A vector superfield V is obtained by imposing the reality condition $V = V^*$, and an expansion in powers of $\theta, \bar{\theta}$ reads

$$\begin{aligned}V\left(x,\theta,\bar{\theta}\right) = a + \theta\xi + \bar{\theta}\xi^\dagger + \theta\theta b + \bar{\theta}\bar{\theta}b^\dagger + \bar{\theta}\bar{\sigma}^\mu\theta A_\mu + \\ +\bar{\theta}\bar{\theta}\theta\left(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\xi^\dagger\right) + \theta\theta\bar{\theta}\left(\lambda^\dagger - \frac{i}{2}\sigma^\mu\partial_\mu\xi\right) + \theta\theta\bar{\theta}\bar{\theta}\left(\frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a\right)\end{aligned}$$

Here A_μ will be the spin-1 gauge field, with λ beeing its fermionic supersymmetric partner. The field D will dropout when looking at the equations of mottion, while all the others degrees of freedom can be supergagued away by going in the Wess-Zumino gauge.
Under a SUSY transformation the components transform as

$$\begin{gathered}\sqrt{2}\delta_\epsilon a = \epsilon\xi + \epsilon^\dagger\xi^\dagger \qquad \sqrt{2}\delta_\epsilon\lambda_\alpha = \epsilon_a D + \frac{i}{2}\left(\sigma^\mu\sigma^\nu\epsilon\right)_\alpha\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\ \sqrt{2}\delta_\epsilon b = \epsilon^\dagger\lambda^\dagger - i\epsilon^\dagger\sigma^\mu\partial_\mu\xi \qquad \sqrt{2}\delta_\epsilon\xi_\alpha = 2\epsilon_\alpha b - \left(\sigma^\mu\epsilon^\dagger\right)_\alpha\left(A_\mu + i\partial_\mu a\right) \\ \sqrt{2}\delta_\epsilon A^\mu = i\epsilon\partial^\mu\xi - i\epsilon^\dagger\partial^\mu\xi^\dagger + \epsilon\sigma^\mu\lambda^\dagger - \epsilon^\dagger\bar{\sigma}^\mu\lambda \\ \boxed{\sqrt{2}\delta_\epsilon D = -i\epsilon\sigma^\mu\partial_\mu\lambda^\dagger - i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\lambda}\end{gathered}$$

Here the term of our interest is the field D because it transforms precisely as a total derivative, as desired.
Hence a lagrangian of the type

$$\mathcal{L} = \int d^4x\, F + F^* + D$$

would certainly be SUSY invariant due to the transformation properties of the selected fields.

Grassman integration provides a natural way to select such components. Indeed

$$\int d^2\theta\, \Phi(x,\theta,\bar\theta) = \frac{1}{4}\,\bar\theta\bar\theta\partial_\mu\partial^\mu\phi(x) - \frac{i}{\sqrt{2}}\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + F(x) = F(x) + \text{total derivative} \equiv [\Phi]_F$$

$$\int d^2\theta d^2\bar\theta\, V(x,\theta,\bar\theta) = \frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a = \frac{1}{2}D + \text{total derivative} \equiv [V]_D$$

The first meaningful supersymmetric lagrangian can be obtained by noting that the product $\Phi^\dagger\Phi$ is real, hence a vector field. In particular, this implies that one can select the D component $\left[\Phi^\dagger\Phi\right]_D$ via Grassman integration to obtain a SUSY invariant lagrangian.
Let us write the product $\Phi^\dagger\Phi$ in components and let us in particulat look at the D term (i.e. the coefficient of $\theta\theta\bar\theta\bar\theta$, compare with expansion of V)

$$\Phi^\dagger\Phi = greatmess$$

$$\boxed{\mathcal{L} = \int d^4x\,\left[\Phi^\dagger\Phi\right]_D = \int d^4x\left(-\partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar\sigma^\mu\partial_\mu\phi + F^*F\right)}$$

This is the lagrangian of the *Free Wess-Zumino model.*
We now want to proceed by adding interactions to this model. In order to do this one can observe that products of fields of the same chirality like $\Phi\Phi$, $\Phi^\dagger\Phi^\dagger$ are in turn chiral fields, since $D_{\dot\alpha}\Phi\Phi = (D_{\dot\alpha}\Phi)\,\Phi + \Phi(D_{\dot\alpha}\Phi) = 0$ ($D_\alpha\Phi^\dagger\Phi^\dagger = 0$ follows analogously). More in general any holomorphic function of fields of the same chirality is in turn a chiral field with the same chirality, as it can easililly be seen by expanding it in power series.
Thus a function of the type

$$W[\Phi] = a\,\Phi + \frac{1}{2!}\,m\,\Phi\Phi + \frac{1}{3!}\,y\Phi\Phi\Phi \tag{2}$$

is certainly a left-chiral field, hence its F component is SUSY invariant. We call $W[\Phi]$ the *superpotential.* The lagrangian

$$\begin{aligned}\mathcal{L} &= \left[\Phi^\dagger\Phi\right]_D + [W[\Phi]]_F + \left[W^\dagger[\Phi^\dagger]\right]_F = \\ &= \int d^2\theta d^2\bar\theta\Phi^\dagger\Phi + \int d^2\theta\, W[\Phi] + \int d^2\bar\theta\, W^\dagger[\Phi^\dagger]\end{aligned}$$

is thus supersymmetric invariant. By using the property $\overline{DD}(\theta\theta) = -\frac{1}{4} = DD(\bar\theta\bar\theta)$ one can rewrite the D term as an F term and the lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \int d^2\theta\left(-\frac{1}{4}\,\overline{DD}(\Phi^\dagger\Phi) + W[\Phi]\right) + \int d^2\bar\theta W^\dagger[\Phi^\dagger] \\ &= \int d^2\theta\, W[\Phi] + \int d^2\bar\theta\left(-\frac{1}{4}\,DD(\Phi^\dagger\Phi) + W^\dagger[\Phi^\dagger]\right)\end{aligned}$$

This allows to derive the equations of motions for the superfields straightforwardly

$$\overline{DD}\Phi^\dagger = 4\frac{\delta W}{\delta\Phi} \qquad\qquad DD\Phi = 4\frac{\delta W^\dagger}{\delta\Phi^\dagger}$$

The superpotential in equation ?? can be straightforwardly generalised to an arbitrary number of chiral superfields

$$W[\Phi_i] = \sum_i a_i\,\Phi_i + \frac{1}{2!}\,m_{ij}\,\Phi_i\Phi_j + \frac{1}{3!}\,y_{ijk}\Phi_i\Phi_j\Phi_k$$

Note that in principle we are allowed to go to higher order in the power sum, but higher order terms would lead to a non renormalizable theory. To show this, let us do some dimensional analysis.
From the superfield expansion ?? we note that the superfield and the bosonic field carry the same dimension $[\Phi] = [\varphi] = E^1$. If we want ψ to be the fermionic superpartner of ϕ then it must be a spinor with the usual dimension $[\psi] = E^{1/2} = [\psi^\dagger]$. This implies that it must be $[\theta] = E^{1/2} = [\bar{\theta}]$. As known, the lagrangian must have dimension E^4, hence so it must be the dimension of $[W]_F = \int d^2\theta W[\Phi]$. Remembering that integrals in Grassman variables "act as derivatives", we conclude that the dimension of W must be E^3. Finally, by looking at ??, one has that $[a] = E^2, [m] = E^1, [y] = E^0$, confirming that higher order terms would be non renormalizable. A similar reasoning forbids forbids also terms like $(\Phi^\dagger\Phi)(\Phi^\dagger\Phi)$ which would require a coupling constant with negative mass dimension.

Let us now write down the lagrangian in terms of components of the fields $\Phi, \Phi^\dagger$

$$\begin{aligned}\mathcal{L} = \mathcal{L}_{WZ,D} + \mathcal{L}_{WZ,F} =& \\ = F^*F + (\partial_\mu\varphi)\,(\partial^\mu\varphi)^* + \frac{i}{2}\psi\sigma^\mu\left(\partial_\mu\psi^\dagger\right) - \frac{i}{2}\,(\partial_\mu\psi)\,\sigma^\mu\psi^\dagger +& \\ -m\varphi F - \frac{m}{2}(\psi\psi) - \frac{y}{2}\varphi\varphi F - \frac{y}{2}\varphi(\psi\psi) + \text{ h.c.}&\end{aligned}$$

This allows us to derive the equations of motions for F and F^* straightforwardly

$$\begin{aligned}F^* &= m\varphi + \frac{y}{2}\varphi\varphi \\ F &= m\varphi^* + \frac{y}{2}\varphi^*\varphi^*\end{aligned}$$

These are algebraic equation which allow one to express directly F as a function of φ, φ^*. By inserting the expressions in the lagrangian one gets

$$\begin{aligned}\mathcal{L}_{\text{WZ}} = (\partial_\mu\varphi)\,(\partial^\mu\varphi)^* + \frac{i}{2}\psi\sigma^\mu\left(\partial_\mu\psi^\dagger\right) - \frac{i}{2}\,(\partial_\mu\psi)\,\sigma^\mu\psi^\dagger \\ - |M|^2\varphi\varphi^* - \frac{|y|^2}{4}\varphi\varphi\varphi^*\varphi^* - \left(\frac{M}{2}\psi\psi + \frac{M\cdot y}{2}\varphi\varphi\varphi^* + \frac{y}{2}\varphi\psi\psi + \text{ h.c.}\right)\end{aligned} \tag{3}$$

This expression allows one to study the interactions and draw Feynman diagrams. Before doing so, let us give a couple of comments

- In the last step we inserted the equations of motions in the lagrangian: this step should be taken with some care. In principle, the action can be evaluated for any configuration of the fields $\Phi(x, \theta, \bar{\theta}), \Phi^\dagger(x, \theta, bar\theta)$, with values of F that can be completely unrelated to the values of φ^*. When we look at the equations of motions $\delta S = 0$ we are contraining the action to be a local extremum. Thus the lagrangian ?? is not a general one but rather one that extremizes the action, which would correspond to physical processes in the classical framework. One typically says that it is the "on shell" expression.

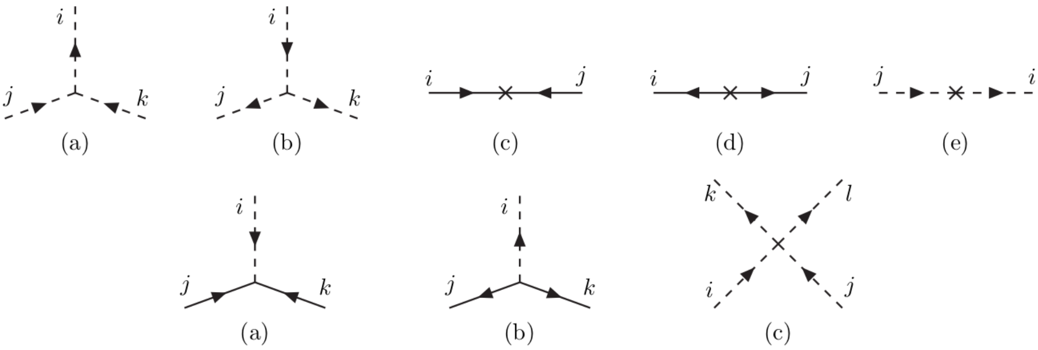

Figure 1: Interactions in the Wess-Zumino model give by the lagrangian **??**

- The lagrangian **??** can be also written in terms of a potential function of the bosonic field φ, φ^* only

$$V(\varphi) \equiv W(\varphi) = a\,\varphi + m\,\varphi^2 + y\,\varphi^3$$

namely the superpotential **??** evaluated at the bosonic field φ. With this definition one can easilly check that

$$F = -\left|\frac{\partial V}{\partial \varphi}\right|^2$$

and

$$\begin{aligned}\mathcal{L}_{\mathrm{WZ}} &= (\partial_\mu \varphi)(\partial^\mu \varphi)^* + \frac{i}{2}\psi\sigma^\mu\left(\partial_\mu \psi^\dagger\right) - \frac{i}{2}(\partial_\mu \psi)\,\sigma^\mu \psi^\dagger \\ &\quad - \left|\frac{\partial V(\varphi)}{\partial \varphi}\right|^2 - \frac{1}{2}\left(\frac{\partial^2 V(\varphi)}{\partial\varphi\partial\varphi}\right)\psi\psi - \frac{1}{2}\left(\frac{\partial^2 V(\varphi^*)}{\partial\varphi^*\partial\varphi^*}\right)\psi^\dagger\psi^\dagger\end{aligned}$$

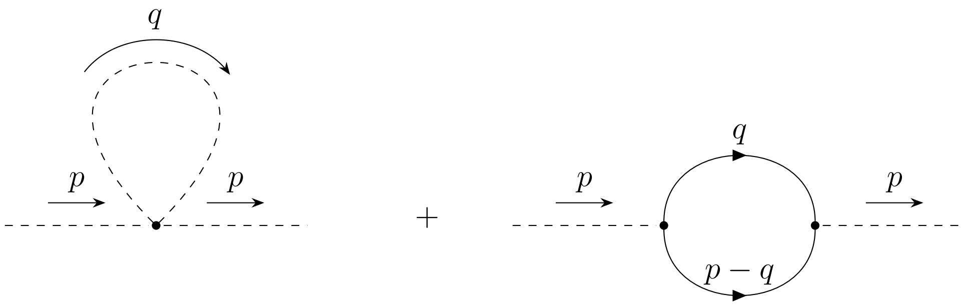

Figure 2: 1-loop corrections to the scalar propagator with the lagrangian give by equation ??????????????????

The contribution from the first diagram is

$$I_1 = 4\,\frac{i|y|^2}{4}\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2} = -|y|^2\int\frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2-m^2}$$

By performing a Wick rotation $q\to iq$ and putting a cutoff Λ, the integral can be evaluated exactly in polar coordinates

$$I_1 = \frac{i|y|^2}{(2\pi)^3}\int_0^\Lambda dq\int_0^\pi d\varphi_1\int_0^\pi d\varphi_2\,\frac{1}{q^2+m^2}\,q^3 sin^2\varphi_1 sin\varphi_2 =$$
$$= \frac{i|y|^2}{8\pi^2}\int_m^{(\Lambda^2+m^2)^{1/2}} dt\,\frac{t^2-m^2}{t} = \frac{i|y|^2}{16\pi^2}\left(\Lambda^2 - m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right)\right)$$

The contribution from the second diagram is

$$I_2 = -2\left(\frac{iy}{2}\right)\left(\frac{iy^*}{2}\right)\int\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i(\sigma\cdot q+m)}{q^2-m^2}\,\frac{i(\bar\sigma\cdot(p-q)+m)}{(p-q)^2-m^2}\right] =$$
$$-\frac{1}{2}|y|^2\int\frac{d^4q}{(2\pi)^4}\,(-2)\,\frac{q\cdot p - q^2 - 2m^2}{(q^2-m^2)((q-p)^2-m^2)}$$

Where the property $\mathrm{Tr}[\sigma_\mu\bar\sigma_\nu] = -2\eta_{\mu\nu}$ has been used. This integral is not easilly evaluable, but for the current purpose it is sufficient to determine its leading behaviour for $|q|\to+\infty$.
After performing the Wick rotation and writing the integral in polar coordinates, the leading term is

$$I_2 = -\frac{|y|^2}{(2\pi)^3}\int_0^\Lambda dq\int_0^\pi d\varphi_1\int_0^\pi d\varphi_2\,\frac{1}{q^2}\,q^3\sin^2\varphi_1\sin\varphi_2 =$$
$$= -\frac{i|y|^2}{(2\pi)^3}\,\pi\int_0^\Lambda dq\,q = -\frac{i|y|^2}{16\pi^2}\,\Lambda^2$$

This term cancels precisely the quadratic term in ?????? as desired

Gauge theories

We now want to introduce gauge invariance in our theory. At first let us focus on the abelian U(1) group, which will then be generalised to an arbitrary gauge group. Let us consider the Wess-Zumino lagrangian for an arbitrary number of superfields

$$\mathcal{L} = \int d^2\theta d^2\bar\theta\sum_i\Phi_i^\dagger\Phi_i i + \int d^2\theta\sum_i a_i\,\Phi_i + \sum_{ij} m_{ij}\,\Phi_i\Phi_j + \sum_{ijk} y_{ijk}\,\Phi_i\Phi_j\Phi_k + h.c.$$

Let us first consider a global symmetry

$$\Phi \to e^{iq_i\Lambda_i}\Phi \qquad\qquad \Phi^\dagger \to \Phi^\dagger e^{-iq_i\Lambda_i} \tag{4}$$

The kinetic term $\int d^2\theta d^2\bar{\theta} \sum_i \Phi_i^\dagger \Phi_i$ is always invariant but the potential term is not. In fact the linear term is obviously not invariant hence must be dropped. Moreover, whenever $q_i + q_j \neq 0$ we must require $m_{ij} = 0$ and whenever $q_i + q_j + q_k \neq 0$ we must require $y_{ijk} = 0$. Note in particular that diagonal terms in the mass term require $q_i + q_i = 0$ in order to be present.
We now promote the global symmetry to a local symmetry, hence $\Lambda \to \Lambda(x, \theta, \bar{\theta})$. One can now note that the generator is not a normal function but rather a superfield. Note that it cannot be an arbitrary one. In fact gauge symmetries are transformations which do not change the physical state of the system, but rather configurations that differs by a gauge transformation are physically equivalent. This in particular means that a chiral field must be mapped into fields of the same chirality. By looking at equation **??** we immediately conclude that $\Lambda(x, \theta, \bar{\theta})$ must be a left-chiral superfield, which automatically guarantees that $\Lambda^\dagger(x, \theta, \bar{\theta})$ is a right-chiral superfield.