Lab 3 - Experience 4

Feedback and oscillators

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Introduction

In this experience we implemented two oscillating circuits by using operational amplifiers and we studied the stability of each configuration. In particular we first analysed a damped harmonic oscillator whose damping ratio was controlled by two resistors, and then implemented an auto-sustained harmonic oscillator described by the Van der Pol equation.

A damped harmonic oscillator is usually described by the differential equation

$$y''(t) + 2K\omega_0 y'(t) + \omega_0^2 y(t) = 0 \tag{1}$$

The value of K is crucial to determine the behaviour of the system: this parameter is often called *damping ratio.* Mathematically speaking the value of K determines the nature of the zeros of the related characteristic equation (distinct and real, coincident or complex conjugate) and hence determining the form of the solution. Physically speaking the three different natures of the zeros result in three different behaviours of the system. Once that initial conditions are set:

- $K > 1 \to$ distinct and real zeros $\to$ overdamped system. The general solution is a linear combination of exponential decays. After a perturbation the system reaches equilibrium without oscillating.
- $K = 1 \to$ coincident real zeros $\to$ critically damped system. The general solution is again a combination of exponential decays, both with the same time parameter. After a perturbation the system reaches equilibrium in the fastest way without oscillating.
- $0 < K < 1 \to$ complex conjugate zeros $\to$ underdamped system. The general solution is a linear combination of products of an exponential decay and an oscillating term. In this case the system oscillates before reaching equilibrium when moved from a resting point.
- $K < 0 \to$ unstable system. The general solution is an exponential with positive exponent. After a perturbation the system explodes, never reaching a resting point.

1 Damped harmonic oscillator

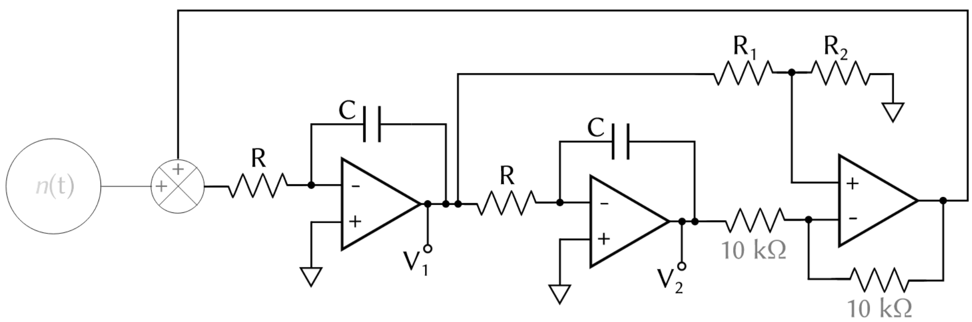

Figure 1: Damped harmonic oscillator circuit

The first configuration we implemented was the damped harmonic oscillator via the circuit in figure 1. The first two operational amplifiers work as integrators (transfer function $\frac{1}{sRC}$ in the Laplace domain), while the last op-amp is an amplifier.
Firstly, we analysed the transfer function of the circuit "disconnecting" the output from the input, i.e. the open loop function. By considering op-amps as ideal components the output signal of the circuit is simply the product of the transfer function of each stage, so that

$$G(s) = -\left[\frac{1}{s^2\tau^2} + \frac{2}{s\tau}\frac{R1}{R1+R2}\right] \tag{2}$$

where $\tau = RC$. Let's define this expression as the **open loop gain** of the circuit.

The closed loop function can then be derived by analogy with a general system as the one in figure 2, interpreting expression 2 as the product $A(s)B(s)$. The **closed loop** gain (that is, the gain with the output connected to the input) is then $\frac{1}{1-G_{OL}(s)}$.

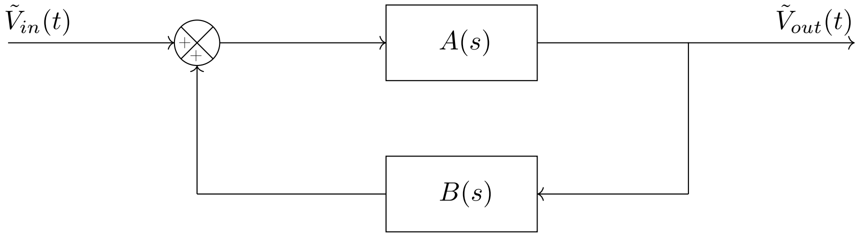

Figure 2: Block diagram of a general feedback system. The product $A(s)B(s)$ is called open-loop gain, while $[1 - A(s)B(s)]^{-1}$ is the closed loop gain

The open loop gain is quite simple and the stability can be easily evaluated by inspection of the denominator of the closed loop gain. Defining $K \equiv R1/(R1 + R2)$ we have

$$F(s) = 1 - G_{OL}(s) = \frac{s^2\tau^2 + sK\tau s + 1}{s^2\tau^2}$$

The above expression has two complex conjugate zeros $s_{1,2} = K\tau\left(-1 \pm \sqrt{1 - \frac{1}{K^2}}\right)$ both in the left half plane because $K < 1$, hence assuring stability.

We shall now change the view over the circuit in order to analyse it from a different perspective. Let's think about $V_2(t)$ as a function $y(t)$ (referring to figure 1). Remembering the Laplace transform property $\mathcal{L}\left(\int_{t_0}^{t} f(t')dt'\right) = \frac{\tilde{f}(s)}{s}$, we can use the time domain version of the transfer function in equation 2 to write a differential equation for $y(t)$:

$$\ddot{y}(t) + 2\frac{1}{\tau}\frac{R_2}{R_1 + R_2}\dot{y}(t) + \frac{y(t)}{\tau^2} \qquad \longleftrightarrow \qquad \ddot{y}(t) + 2K\omega_0\dot{y}(t) + \omega_0^2 y(t) = 0 \tag{3}$$

We can now see the full parallelism with what described in the introduction. In particular it is possible now to understand the physical meaning of the parameters $K \equiv \frac{R_2}{R_1+R_2}$ and $\omega_0 \equiv \frac{1}{\tau}$. The value of the voltage divider K determines the intensity of the damping while the value of R and C determines the decay characteristic time. Of course $K < 1$, consequently assuring an oscillating solution, modulated by an exponential decay.

1.1 Stability analysis via Nyquist criterion

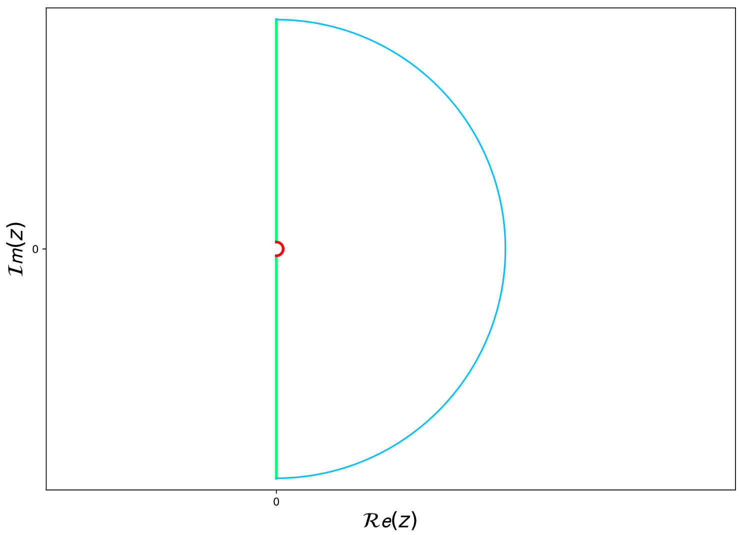

Figure 3: Modified Nyquist path. The deformation of the curve around the origin avoids the (double) pole.

We tried to evaluate stability via the Nyquist criterion too, a method that follows as a consequence of Cauchy's argument principle. Cauchy's argument principles states that the following relation holds

$$Z - P = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz$$

where Z and P are respectively the number of zeros and poles of the function $f(s)$ and Γ is a closed curve. We can use this argument to extract information about poles and zeros of our transfer function. Let's consider the function $F(s) = 1 - G_{OL}(s)$ as our function so that the zeros of $F(s)$ correspond to the poles of the closed loop function. In our case Cauchy's argument principle assures that

$$Z - P = \frac{1}{2\pi i} \int_{\Gamma} \frac{F'(z)}{F(z)} \, dz = -\frac{1}{2\pi i} \int_{\Gamma} \frac{G'_{OL}(z)}{1 - G_{OL}(z)} \, dz$$

where Γ is the circuit in figure 3. We took this curve for 2 reasons:

- We want to "scan" the entire right-half-plane (we are interested in positive ω)
- We want to avoid the (double) pole in the origin

We then perform a change of variable $G_{OL}(z) = -v$ obtaining

$$Z - P = \frac{1}{2\pi i} \int_{\Gamma'} \frac{dv}{v + 1}$$

where Γ' denotes the curve Γ mapped through the change of variable $v = -G_{OL}(s)$. This integral can now be evaluted with the residue theorem: the number of windings of the curve Γ' around the point -1 is equal to $Z - P$, that is, the difference of the number of zeros and poles in the function F.[1]
The mapped curve Γ' is reported in figure 4 and zoomed in figure 5: it's a pacman!

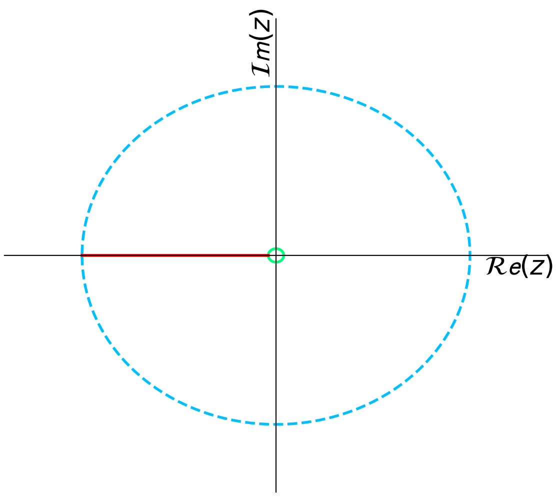

Figure 4: Mapped curve Γ'. The modified Nyquist path Γ in figure 3 has been mapped trough the function $G_{OL}(s)$ that represents the open loop gain of the circuit.

We know that P=0 because of the choice of Γ.[2] The curve Γ' does not encircle the point -1 so $Z - P = 0$ and $Z = 0$ as expected.

1.2 Measurements and data analysis

We studied the circuit in figure 1 for 4 different values of the ratio

$$\epsilon = \frac{R_2}{R_1} \tag{4}$$

[1]In complex analysis this formula is known as the winding number of the curve Γ' around the point -1
[2]So the course of Metodi was useful!

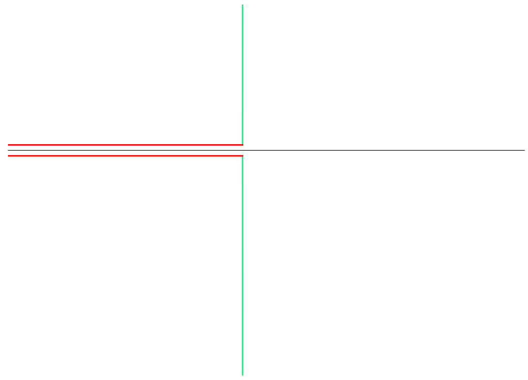

Figure 5: Zoom around the junction between two pieces of the curve Γ'

We can see that changing the value of ϵ affects the dumping factor of the system. Indeed, the damping factor, which can be directly deduced by looking at eq. 3, is

$$K = \frac{\epsilon}{1+\epsilon} \tag{5}$$

One can easily demonstrate by solving the equation of the damped oscillator that it's characteristic time of decay, τ_D is such that

$$\frac{1}{\tau_D} = K\omega_0 = \omega_0 \frac{\epsilon}{1+\epsilon} \tag{6}$$

We chose to change the value of ϵ by keeping $R_1 = 1\,\mathrm{M\Omega}$ and changing R_2, in order to the factor $\epsilon/(1+\epsilon)$ to be 10^{-1}, 10^{-2}, 10^{-3} and 10^{-4}. Unfortunately, we mistakenly used $R_1 = 10\,\mathrm{M\Omega}$ and realized it only during the data analysis.[3] As a consequence, the factor $\epsilon/(1+\epsilon)$ will be 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}; it's very unfortunate that we don't have a measure for 10^{-1} but the data will tell us a lot anyway.

On the oscilloscope we measured 5 couples of time and voltage on the peaks of the damped sinusoids, and then extrapolated the value of the decay time τ_D with an exponential fit. The value of $1/\tau_D$ has then been represented against $\epsilon/(1+\epsilon)$, and the result is visible in figure 6 both in linear and log scale.

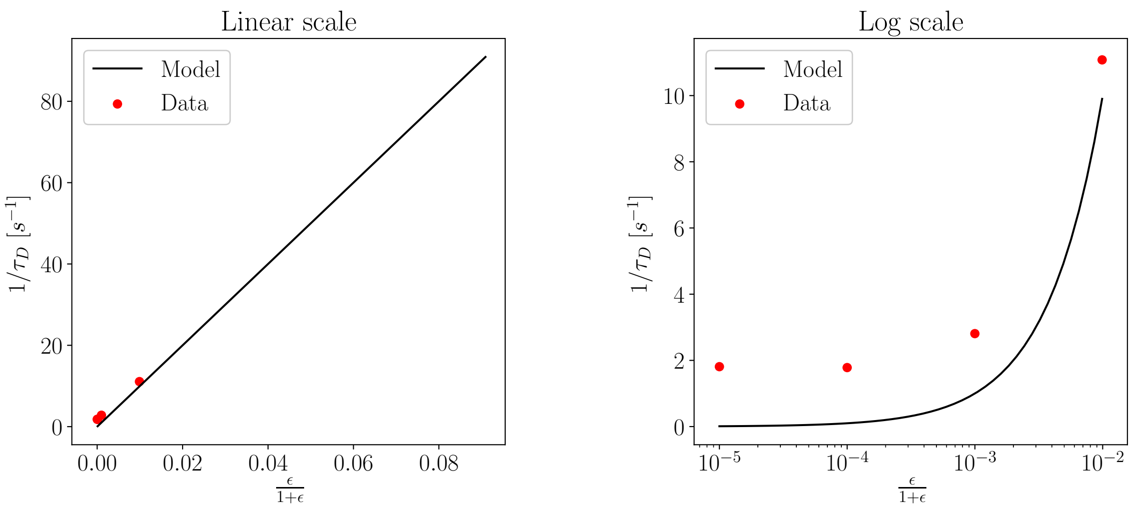

Figure 6: Data and model for the exponential decay time, in linear and log scale.

The expected function, ideally, is a line with slope ω_0 and zero intercept. One can see that the slope between the points is in good agreement with the model, but the intercept is clearly not zero, as is

[3] Green and blue look incredibly similar...

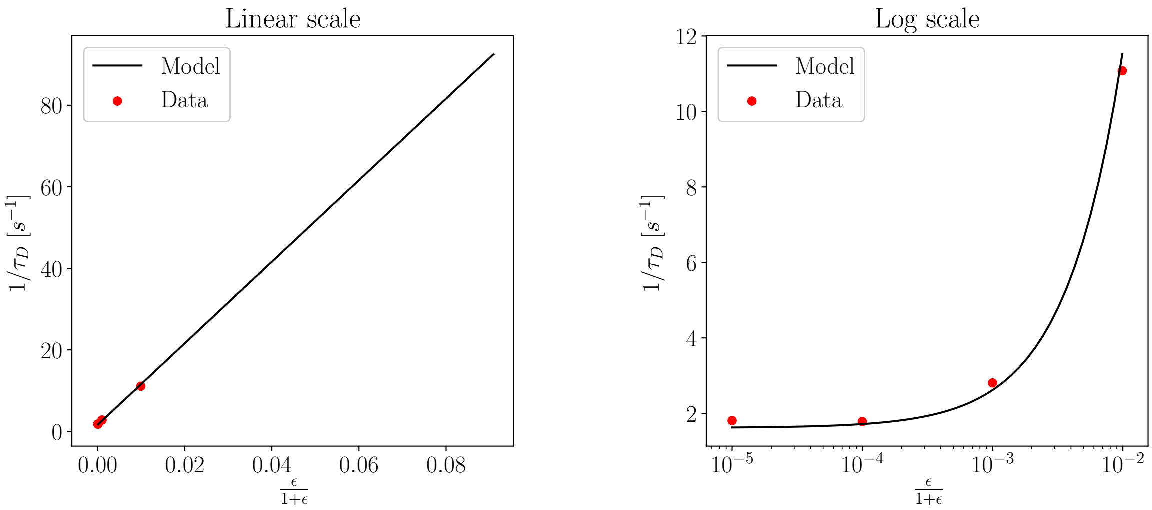

Figure 7: Data and model for the exponential decay time, in linear and log scale, with the r_0 correction.

evident especially from the log scale plot. This is due to the fact that eq. 6 is ideal and, in reality, it should include another term that limits the value of τ_D when $\epsilon = 0$. A more correct relationship should then be

$$\frac{1}{\tau_D} = K\omega_0 = \omega_0 \left(\frac{\epsilon}{1+\epsilon} + r_0 \right) \tag{7}$$

where the factor r_0 is due to the non-ideality of the op-amp. Because it is an unknown factor, we extrapolated it with a fit and found

$$r_0 \simeq 1.6 \tag{8}$$

which means that, when $\epsilon = 0$, τ_D is not infinite but rather

$$\tau_D\,(\epsilon = 0) \simeq 0.6\,\text{seconds} \tag{9}$$

The new graphs with the model following eq. 7 are represented in figure 7.

2 Harmonic Oscillator

2.1 Circuit analysis

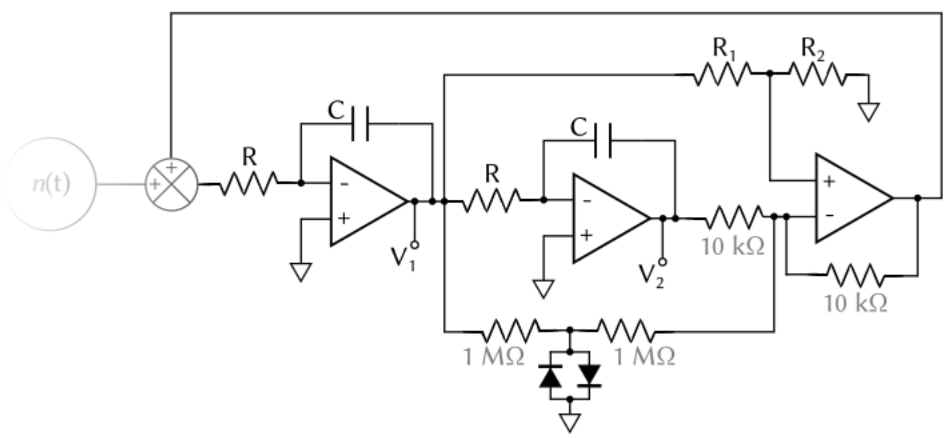

Figure 8: Harmonic oscillator

Although stable, the previous circuit isn't much useful, because most of the time we are interested in a *perfect* harmonic oscillator, i.e. something that remains in a state of perpetual oscillation. The damped oscillator circuit can be transformed in such an harmonic oscillator thanks to a little modification, visible in figure 8: before the second integrator, let us introduce another branch where the key part is the pair of diodes oppositely polarized. The pair of diodes will introduce non-linearity in the circuit, and this will do the magic we need to make an harmonic oscillator. Let us briefly explain the main ideas behind the circuit.
The current in each of the two diodes follows the Shockley equation, namely

$$I = I_s \left(e^{\beta \Delta V / n} - 1 \right) \tag{10}$$

Where ΔV is by definition the potential difference between the end of the diode in direct bias; so pay attention that ΔV will have opposite sign in the two diodes. I_s, β and n are characteristic constants of the diodes.
One then inserts the Shockely equation into the equations of the circuit and resolves them. It is important to mention that, in order to do the circuit analysis, we considered

$$V_+ \simeq V_- \simeq 0 \tag{11}$$

We can do this approximation for two reasons: the first is that we assume for the moment that the circuit is stable, so that we can say $V_+ \simeq V_-$; secondly, we assume that

$$R_2 << R_1 \tag{12}$$

as will actually be the case in our circuit. If so, then the potential of V_+ is approximately at ground level. As we will see, the assumption stated in eq. 12 will be fundamental for obtaining perpetual oscillation.
Now, made the those assumptions, let us call $V_1(t) = u(t)$, R_0 the two 1 MΩ resistors, R_f the two 10 kΩ resistors, $\omega_s = 1/(RC)$ and $r = R_2/(R_1 + R_2)$. One can obtain that $u(t)$ follows the following equation:

$$\ddot{u} - \omega_s \left(\frac{R_f}{R_0} \chi - 2r - 3 \frac{R_f}{R_0} \xi u^2 \right) \dot{u} + \omega_s^2 u = 0 \tag{13}$$

where χ and ξ are constants that depend on the diode parameters β, I_s and n and on R_0.
It looks like a very big and complicated equation, but look at the first and last terms: they're the terms of the equation of an harmonic oscillator. However, the term in between is the most interesting one. When u^2 is small, the coefficient of $\dot{u}$ is dominated by

$$- \omega_s \left(\frac{R_f}{R_0} \chi - 2r \right) \tag{14}$$

Now let's recall our central assumption stated in eq. 12, $r << 1$. If this is enough to assure that $2r << \chi R_f / R_0$ (as will be the case), then the whole term is negative, and this, in turn, makes eq. 13 not a *damped* oscillator, but rather an *exploding* oscillator! But don't worry, because when u rises, then the term in u^2 becomes relevant again, and it flips the sign of the whole coefficient of $\dot{u}$, limiting the rise of u. This equation is so important and ubiquitous in real harmonic oscillator systems, that one usually renames some constants and changes variable in order to rewrite it in the form

$$\ddot{x} - \omega_s \mu \dot{x} (1 - x^2) + \omega_s^2 x = 0 \tag{15}$$

where $\mu = R_f \chi / R_0 - 2r$.

Now, to calculate the exact form of $x(t)$ one must solve eq. 15. It is a hard task, but one can use the perturbation theory using μ as perturbation parameter and search for approximated solutions. One then finds that a first-order approximated solution is given by

$$x(t) = 2 \cos(\omega_s t) - \frac{1}{4\omega_s} \mu \sin(3 \omega_s t) \tag{16}$$

2.2 Experimental evaluation

In the lab we realized the circuit with

$$R = 100\,\mathrm{k\Omega} \qquad C = 10\,\mathrm{nF} \qquad R_1 = 1\,\mathrm{M\Omega} \qquad R_2 = 100\,\Omega \tag{17}$$

so that the condition $2r << \chi R_f/R_0$ is satisfied.

We measured the signals V_1 and V_2 as in figure 8. Without giving any energy to circuit except to the op-amps, we saw that V_1 and V_2 at the beginning were sinusoidal waves with amplitude rising in time, but then the amplitude stabilized at a certain level, and the the signals remained two perpetual sinusoidal waves! We verified that there was a phase shift of $\pi/2$ between them, with V_2 in advance w.r.t. V_1. We also measured their frequency, finding

$$\omega_s = (0.997 \pm 0.002) \times 10^3 \text{ rad s}^{-1} \tag{18}$$

which is consistent with the expected value $\omega_s = (RC)^{-1} = 10^3$ rad s^{-1}.

We then tried to alter some parameters of the circuits to see and interpret the change in the behaviour. We first tried to change the inverting gain of the op-amp, substituting the second of the two $10\,\mathrm{k}\Omega$ resistors with $R_{f2} = 20\,\mathrm{k}\Omega$, so that $R_{f2}/R_{f1} = 2$. Looking at the calculations that lead to Van der Pol's equation (that we didn't report here to avoid taking too much space), one notices that this has the primary effect of changing the fundamental frequency of the oscillation, which becomes

$$\omega_s^{'} = \omega_s * \sqrt{2} \simeq 1.414 \times 10^3 \text{ rad s}^{-1} \tag{19}$$

The resulting effect was indeed a reduction in the observed frequency of the signals, which we measured to be

$$\omega_s^{'} = (1.475 \pm 0.004) \times 10^3 \text{ rad s}^{-1} \tag{20}$$

which is not completely compatible with the expected value, but is quite close. The phase shift between the two measured signals continued to be $\pi/2$.

We then tried to change the value of C in both the integrator, using $C = 1\,\mathrm{nF}$ instead of $C = 10\,\mathrm{nF}$. This change should result in an increase of ω_s, which should now be $\omega_s^{''} = 10^4\,\mathrm{rad/s}$. The measured frequency is

$$\omega_s^{''} = (1.02 \pm 0.01) \times 10^4\,\mathrm{rad/s} \tag{21}$$

which is in good agreement.

Lastly, we changed the value of R decreasing is significantly, using $R = 100\,\Omega$. The observed result was the clipping of V_1 and V_2, which is explained by the fact that reducing R increased the current flowing into the op-amp, so that V_{out} reaches saturation with the maximum possible voltage V_{sat}.

Conclusion

This experience has let us familiarize with the theory of stability and it implications. We learnt how to realize a damped oscillator using op-amps as integrators and we verified that the circuit's behaviour was consistent with the differential equation that we found. In addition, we also discovered how to realize a self-sustaining perpetual oscillator, studying the theory behind Van der Pol's equation and experimenting by changing some parameters of the circuit.