Laboratory of Physics III

Experience 5:
Oscillators And The Van der Pol Equation

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Introduction

Electronic oscillators are circuits designed to obtain an oscillating voltage as ouput, that is a voltage periodic in time. Common examples of such waveforms are sine waves or square waves.

Oscillation, however, is a far more general phenomena that we can observe in nature on different time scales without even noticing it. For example, crystals oscillators are components than can produce an almost constant oscillation via the piezioelectric effect, and they are commonly found in technological devices as watches, radio and smartphones, but they can also be found in more sophisticated fields like electronics (e.g. for clock generation). Another typical example of an oscillator is the heart, whose typical oscillating frequency is automatically regulated by our body between 50bpm and 200bpm depending on the blood demand. Although very different, a lot oscillating systems often share one feature, which is that they are described by a famous equation known as the *Van der Pol* equation.

In this experience we have studied three different types of oscillators made with operational amplifiers. The first two of them are different implementations of a *Wien's bridge oscillator*, which, as we will see, is an example of the Van der Pol equation at work. The last one, known as *relaxation oscillator*, is instead dependent on an unstable feedback mechanism, and outputs a square wave.

1 Wien's bridge oscillators

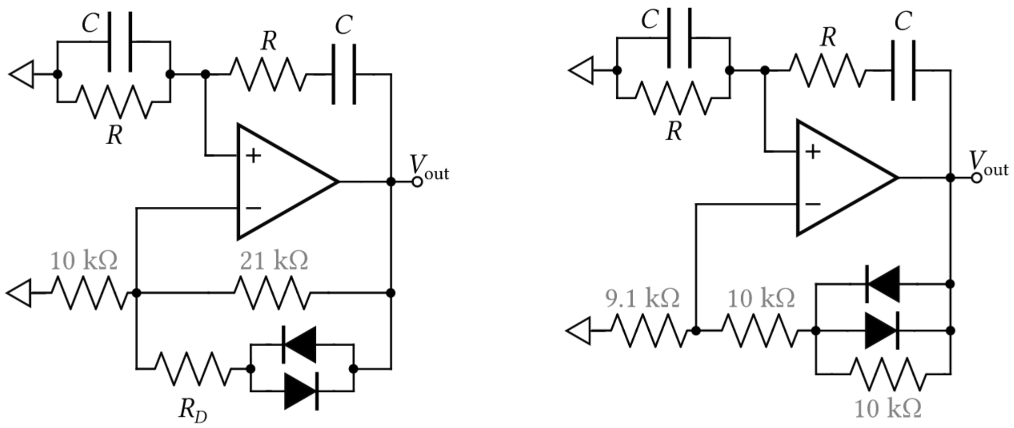

Figure 1: Two different implementations of a Wien's bridge oscillator. Both circuits are well described by the Van der Pol equation.

1.1 Circuit analysis and explanation

The base idea behind these circuits is that the two diodes behave like a "dynamic resistance" that varies with the output voltage. Let's call this resistance $\rho(w)$ where w is the voltage difference on the two diodes. In both circuits the negative feedback loop can be thought as a resistance R_x which is itself dynamic, since it depends directly on ρ. In particular, let it be for the two circuits

$$R_{x,1} = R_1 \parallel (R_D + \rho) \qquad R_{x,2} = R1 + (\rho \parallel R_D) \tag{1}$$

where in the first circuit R_1 denotes the $21\,\mathrm{k\Omega}$ resistance, while in the second R_1 denotes the $10\,\mathrm{k\Omega}$ resistance connected to the negative input of the op-amp and R_D is the $10\,\mathrm{k\Omega}$ resistance in parallel with the diodes. Both circuits can then be schematized as in figure 2.

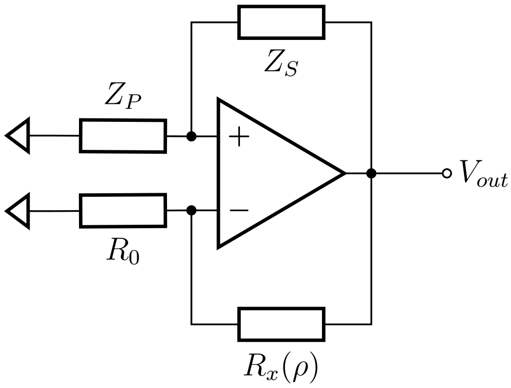

Figure 2: Equivalent representation of both Wien's bridge oscillator circuits in terms of the dynamic resistance R_x. The dynamic resistance is different in the two circuits (see figure 1). $Z_P = R \parallel \frac{1}{sC}$ and $Z_S = R + \frac{1}{sC}$.

Applying Millman's theorem on both the inverting and non-inverting input of the op-amp, one can obtain the following relation (in the Laplace domain)

$$\Delta V = \Delta V_0 + (V_+ - V_-) = \Delta V_0 + V_{out}\left(\frac{Z_p}{Z_p + Z_s} - \frac{R_0}{R_0 + R_x(\rho)}\right) \tag{2}$$

where R_0 is the $10\,\mathrm{k\Omega}$ resistance in the first circuit and the $9.1\,\mathrm{k\Omega}$ resistance in the second circuit; ΔV_0 is a constant term deriving from the initial conditions, due to the Laplace transform.
Imposing that $V_{out} = A\Delta V$, in the limit $|A| \to +\infty$ the previous relation implies that

$$V_{out} = \frac{\Delta V_0}{\frac{Z_p}{Z_p+Z_s} - \frac{R_0}{R_0+R_x(\rho)}} \tag{3}$$

By substituting in this equation the expression $\frac{Z_p}{Z_p+Z_s} = \frac{s\tau}{s^2\tau^2+3s\tau+1}$, where $\tau = RC$, and rearranging terms, we obtain

$$s^2 V_{out} - \frac{1}{R_0\tau}\left(R_x(\rho) - 2R_0\right) V_{out} + \frac{1}{\tau^2} V_{out} \approx 0$$

where we used the fact that $\Delta V_0 \approx 0$. We now define a new parameter $\omega_s \equiv 1/\tau$ and antitransform the above expression

$$\ddot{V}_{out} - \frac{\omega_s}{R_0}\left(R_x(\rho) - 2R_0\right)\dot{V}_{out} + w_s^2 V_{out} = 0 \tag{4}$$

The above expression is similar to the one that describes a Van der Pol oscillator. To get the Van der Pol equation explicitly one should express $R_x(\rho)$ in terms of V_{out} and show that there's a quadratic relation such that $R_x - R_0 \approx C_1 - C_2\, V_{out}^2$. The calculations to obtain such a relation are very tedious and complicated: we prefer to evaluate the last equation in a more intuitive manner, describing what happens as $R_x(\rho)$ changes due to

V_{out}.

As an example we choose the first circuit only, and we explicitly express the dynamic feedback resistance from equation 1

$$R_{x,1}(\rho) = \frac{R_1 \left(R_D + \rho\right)}{R_1 + R_D + \rho} \tag{5}$$

To obtain a relation for ρ as a function of the diode voltage w we note that the current that passes trough the diodes is, according to the Shockley model

$$I_D = I_{D,1} - I_{D,2} = I_s \left(e^{\beta w/n} - e^{-\beta w/n}\right) = 2I_s \sinh\left(\beta w/n\right) \tag{6}$$

so that $\rho = w/I_s(w) = w/(2I_s \sinh\left(\beta w/n\right)$.

We can insert this expression in equation 5 and make a plot of its value as a function of w (figure 3).

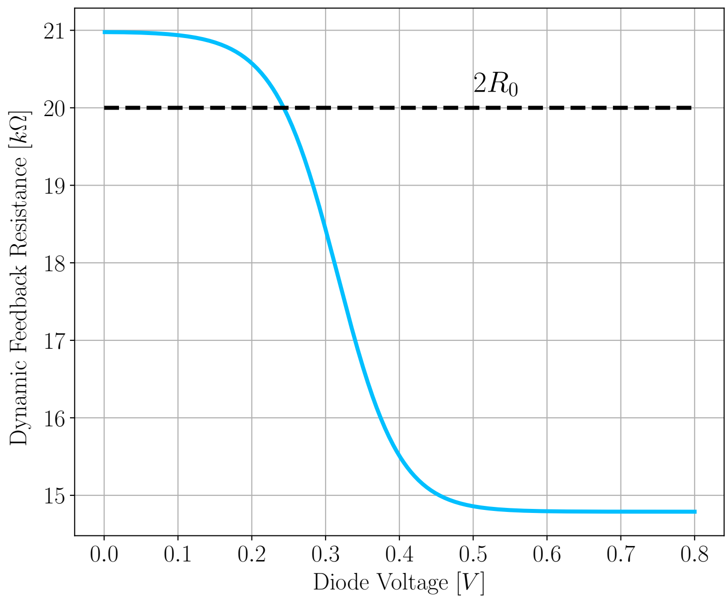

Figure 3: The dynamic feedback resistance R_x decreases as the voltage across the diode increases. When the value of the dynamic resistance becomes lower than $2R_0$ the coefficient of $\dot{V}_{out}$ in the equation 4 becomes negative (dissipative) leading to oscillation. The precise values are indicative since the dynamic resistance is strictly dependent on the diode's saturation current, a parameter we did not estimate precisely.

The dynamic resistance $R_x(\rho)$ tends to decrease as w increases (so when V_{out} increases). This can be explained in the following way: the diode's dynamic resistance $\rho(w)$ decreases as w increases, so that for small V_{out} the resistance R_x (that is, the parallel between R_1 and $R_D + \rho$) is dominated by R_1. As V_{out} increases, the conducting diode's intrinsic resistance decreases and the value of $R_x(\rho)$ decreases as a consequence. The main role of the diodes here is to lower the feedback resistance as V_{out} grows (in module).

To be more quantitative in the description, let's return to equation 4. The maximum value for $R_x(\rho)$ is for $w = 0\,\mathrm{V}$, for which we have $R_x^{min}(\rho) \approx R_1$. The coefficient of the first derivative of V_{out}, $R_x(\rho) - 2R_0$, is then initially positive and "pushes" the output, until a certain point is reached when $R_x(\rho) = 2R_0$. After that point, the coefficient becomes negative and it pushes in the opposite direction of growth (it can be thought as a resistant force applied to a moving body), hence slowing down V_{out} growth and inverting its direction. Since we assumed $R_x(\rho)$ proportional to ${V_{out}}^2$ [1] there is a complete symmetry between positive and negative values of V_{out}, so the system has the same behaviour also for $V_{out} < 0$.

R_D determines the time at which the system starts being damped because the greater R_D, then the greater $R_D + \rho$ and hence the smaller the feedback resistance $R_x(\rho)$.

1.2 Data analysis

We assumed that both circuits are in good approximation described be the Van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{7}$$

where μ is a typical parameter that fix the intensity of the (non-linear) attenuation.

One can extract a class of solutions of equation 7 with a perturbative approach as follows. By observing the behaviour of the equation with $\mu = 0$ we can assume a solution of the form

$$x(t) = \sum_{n=0}^{+\infty} \mu^n a_n \cos(\omega_n t + \phi_n)$$

The first order perturbation is thus

$$x(t) = a_0 \cos(\omega_0 t) + a_1 \mu \cos(\omega_1 t + \phi_1) + \mathcal{O}(\mu^2)$$

By inserting this expression in the equation 7 and imposing the equality on the coefficients one can show that a first-order approximated solution is

$$x(t) = 2\cos(\omega_s t) - \frac{\mu}{4\omega_s} \sin 3\omega_s t + \mathcal{O}(\mu^2) \tag{8}$$

Our general approach to analyse the circuits was to extract the output waveform and fit with a model of the type

$$y(t) = A\cos(\omega_s t + \phi_0) + B\cos(3\omega_s t + \phi_1) + C$$

with A, ϕ_0, B, ϕ_1, C fit parameters and ω_s fixed by known values of the components. We provided one example plot of the fit for each configuration in figure 4. We then evaluated the ratio B/A to show that the $3\omega_s$ component is small compared to the ω_s. For the two chosen configurations we obtained respectively

$$B_1 = 0.68\%\, A_1 \qquad B_2 = 2.36\%\, A_2$$

[1] This is not strictly necessary. In fact we know that R_x dependence on V_{out} is due to ρ, but ρ is an odd function of w (and hence of V_{out}) so that the dynamic damping is independent of V_{out} sign.

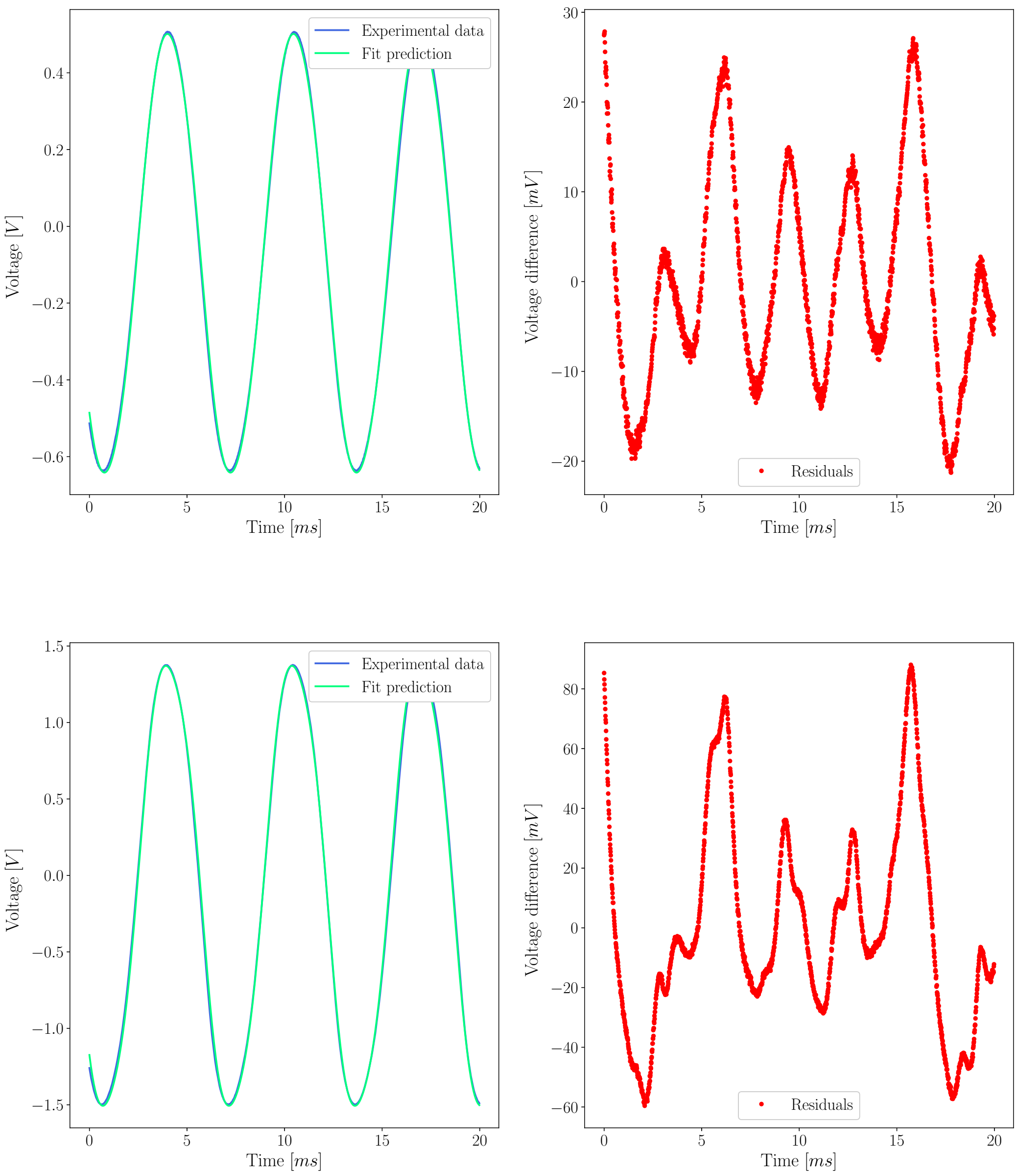

Figure 4: Comparison between measured data and fitted curve for the two Wien's bridge oscillator circuits. The chosen model was the one that can be extracted with a perturbative approach to the Van der Pol equation as explained in section 1.2. The first row refers to the first configuration of the circuit, the second to the other one (see circuits in figure 1).

2 Relaxation Oscillator

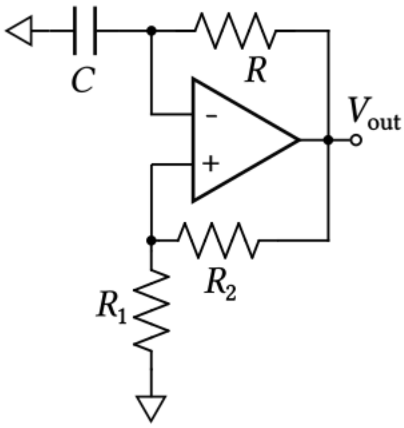

Figure 5: Relaxation oscillator circuit

Let us know concentrate on another oscillating circuit, which is commonly called *relaxation oscillator*, represented in figure 5. Differently from Wien's oscillator, this one doesn't produce a sinusoidal wave, but rather an almost-perfect square wave[2], as we will see in the analysis section below.

2.1 Circuit analysis

In this section, we'll try to understand how the circuit works and what are the characteristics of the output wave. Since we will use it a lot, let us call

$$r = \frac{R_1}{R_1 + R_2} \tag{9}$$

First of all, let's start assuming that some little electrical noise is present in the input pins of the op-amp, and that the potential of the positive one, V_+ is slightly higher than that of the negative one, V_-. We can assume it without loss of generality, since, if the opposite was true, the following process would just be translated in time but wouldn't change in its concept. What will then happen to V_{out}? Let's recall that

$$V_{out} = A\,(V_+ - V_-) = A\Delta V \qquad \text{with } |A| \simeq 10^8 \text{ to } 10^{15} \tag{10}$$

so V_{out} will be equal to a little positive ΔV (presumably $\simeq$ µV), amplified by an incredibly great factor A. This will make V_{out} rise quickly and greatly, until it will reach the maximum possible value of V_{sat}. This is expected to be equal to $+V_{cc}$, i.e. the power supply positive potential. As an immediate consequence, V_+ will find itself at rV_{out} because of the presence of a voltage divider, while V_-, at the very beginning, will still be at about ground level, because the capacitor hasn't had time yet to charge itself. So to recapitulate, at $t = 0$ (stage 1 in figure 6), we will have

$$V_{out} = V_{sat}, \qquad V_+ = rV_{out} = rV_{sat}, \qquad V_- = 0 \tag{11}$$

After this initial condition, the capacitor will start to charge itself, and V_- will slowly rise, trying to bring its potential near to V_{out}'s one. In doing so, it will obviously follow the well-know exponential curve

$$V_-(t) = V_{sat}\left(1 - e^{-t/\tau}\right) \qquad \text{with } \tau = RC \tag{12}$$

[2]At least, under reasonable conditions.

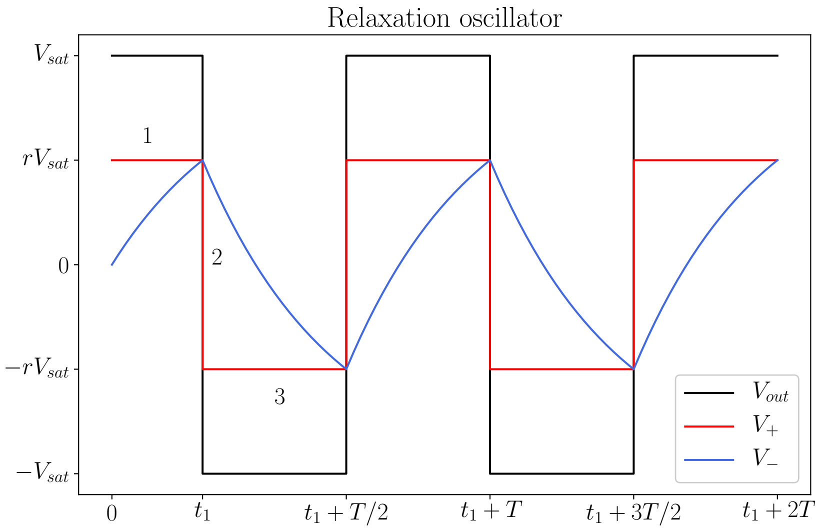

Figure 6: Evolution in time of the relevant potentials in the relaxations oscillator.

However, it will not eventually reach its intended value V_{sat}, because at some instant t_1, V_- will become bigger than V_+, and so ΔV will be < 0. This will cause a sudden drop in V_{out}, which, according to eq. 10, will try to reach great negative values. Again, it will stop at the lowest possible potential $-V_{sat} \simeq -V_{cc}$; accordingly, V_+ will simultaneously drop to $-rV_{sat}$ (stage 2 in figure 6). The time t_1 at which that will happen can be obtained by imposing

$$V_-(t_1) = V_{sat}\left(1 - e^{-t_1/\tau}\right) = rV_{sat} \quad \Rightarrow \quad t_1 = \tau \log\left(\frac{1}{1-r}\right) \tag{13}$$

Afterwards, V_- will be dragged down, trying to reach V_{out}'s potential, and again it will do so with an exponential decay (stage 3 in figure 6):

$$V_-(t) = V_{sat}(1+r)e^{-(t-t_1)/\tau} - V_{sat} \tag{14}$$

This exponential decay will stop at a certain instant t_2 as V_- will become lower than $V_+ = -rV_{sat}$, because it will cause ΔV to become positive again, and V_{out} will rise again up to V_{sat}. From this moment on, the cycle will continue, with V_- oscillating with exponential decays between $\pm rV_{sat}$, V_+ describing a square wave between $\pm rV_{sat}$ and V_{out} describing a square wave between $\pm V_{sat}$, as represented in figure 6. The period of the square waves is determined by the characteristic time of the exponential decay, τ. Half of a period, $T/2$, can be determined by imposing for instance

$$V_{sat}(1+r)e^{-T/2\tau} - V_{sat} = -rV_{sat} \tag{15}$$

finding

$$T = 2\tau \log\left(\frac{1+r}{1-r}\right) \tag{16}$$

2.2 Experimental evaluation

In the laboratory, we chose 3 values for the ration r, namely 0.1, 1 and 10, and measured period and pp-amplitude of the square wave in the output V_{out}. Figure 7 reports an example of the output waveforms that we saw.

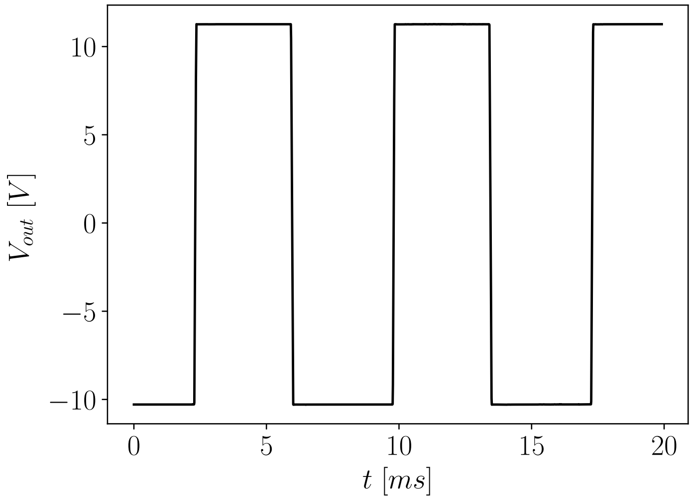

Figure 7: Relaxation oscillator output waveform with $r = 1$, $R = 100\,\mathrm{k}\Omega$ and $C = 100\,\mathrm{nF}$.

With regards to the amplitude, we just compared the measured value with the expected value, which is $V_{sat_{pp}} = 2V_{cc} = 24\,\mathrm{V}$. We found a slightly different value, i.e. $V_{out_{pp}} = 20.7\,\mathrm{V}$, but this can be explained by non-idealities of the op-amp.

With regards to the period, because the theoretical period follow eq. 16, we plotted the measured and theoretical periods against the quantity $\log\left(\frac{1+r}{1-r}\right)$. However, they resulted to be completely incompatible. Because we saw that the problem was mainly the angular coefficient of the theoretical line, we tried to modify eq. 16 and found out that by replacing 2τ with $\tau/2$, data an model were in good agreement. The comparison between the two formulas is represented in figure 8. As one can see, in the rightmost

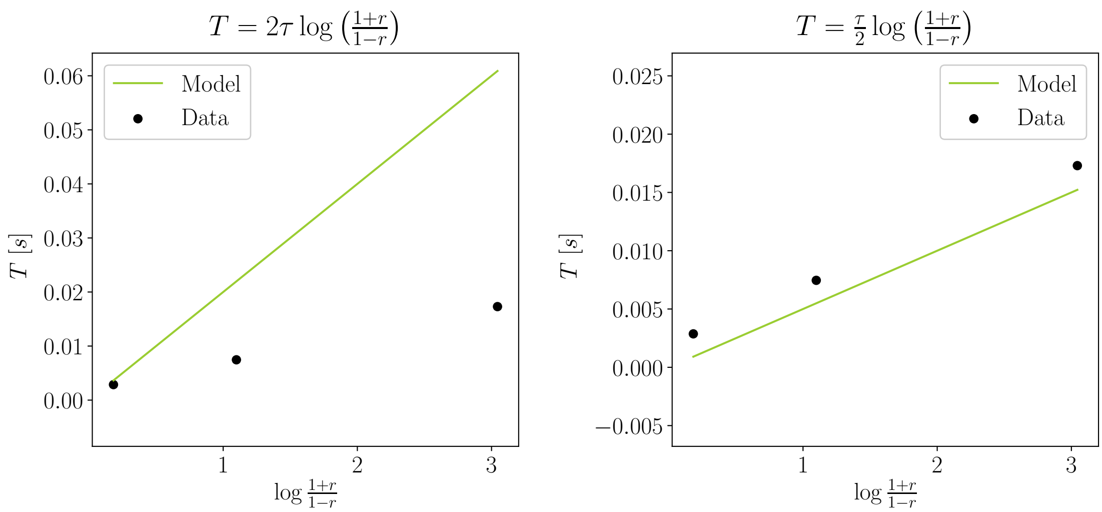

Figure 8: Period of the relaxed oscillator: comparison between two models. The first figures follows eq. 16, while the second one substituted 2τ with $\tau/2$.

figure the slope of the model agrees with the slope between the points; there is a small

offset between points and theoretical line, but this can be easily taken into account by considering non-idealities of the op-amp.
We weren't able to find a satisfying explanation for this evident deviation from the theoretical model, and it's hard to use errors during the measurements as an excuse.

Lastly, we changed the value of R and C to reduce a lot the period of oscillation in order to see the effect. The output waveform is reported in figure 9. The period of the oscillation, this time, perfectly follows eq. 16 (with 2τ!), but as one can see the output wave is all but square. We suppose that the reason is that the time that the op-amp needs to transition from V_{sat} to $-V_{sat}$ cannot be zero; in this conditions, where the period is in the order of $\simeq 100\,\mu s$, that time is expected to be relevant.

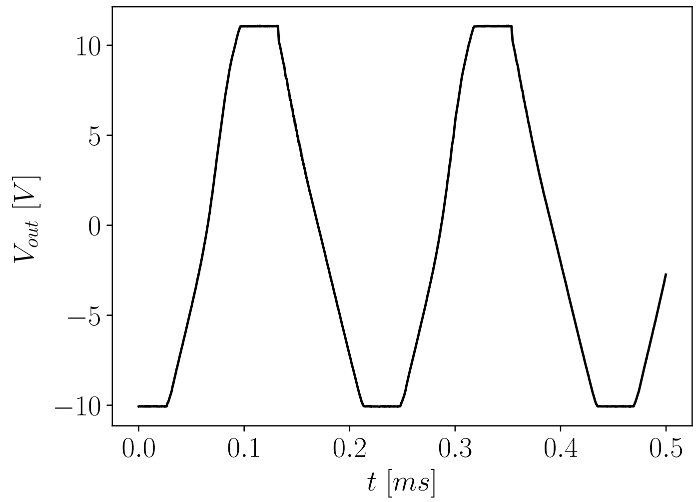

Figure 9: Relaxation oscillator output waveform with $r = 1$, $R = 10\,k\Omega$ and $C = 10\,nF$.

As a last comment, note that this circuit works as expected because it is *unstable*!

Conclusion

This experience has been very useful to study the implementation and behaviour of oscillating systems. Even though we haven't derived explicitly the Van der Pol equation for the Wien's bridge oscillators, we provided a model that enabled us to describe the behaviour of the system in a semi-quantitative manner and understand the physics behind it in a clear way.
With regards to the relaxation oscillator, we learnt how a square wave can be implemented by exploiting instability and saturation in operational amplifiers. Square waves, although may not seem as *noble* as sinusoidal waves, have nevertheless very important applications, like the implementation of a CPU clock.