Computational Finance 2024/2025

Energy Finance

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Abstract

The aim of this project is to calibrate an HJM model on German electricity swaps and price structured pay-offs options by means of Monte Carlo simulation.
We firstly briefly introduce the concept of swap contract in the energy market. There are three fundamental dates in this type of agreements: the first one is the date in which the contract is stipulated, which we usually tend to consider as $t = 0$, we then have τ_1 which is the moment in which the owner of the contract receives a constant flow of the commodity, having payed a fixed payment for unit. Lastly, we consider τ_2 which is the maturity of the contract.
Our objective is to price this type of contracts using the Heath-Jarrow-Turnbull approach.
RunProject5.m is the main code of the project.
Total computation time: **8.5 seconds**

1 Data

In order to complete all of the computations we were given a dataset containing different information about the German swaps.
First of all we had the **prices** of multiple **swaps**, with the corresponding expiry date. From a first analysis we did not find a specific behavior of the prices related to their expiry.
Since our computations specifically focused on 4Q25 German power swap, we had access to the **implied volatility curve** for these specific contracts. We were indeed given the value of the volatility for a certain amount of strikes and tenors (which is the fraction of the year that is between τ_1 and τ_2. We can see the behavior of the implied volatilities in the following figure:

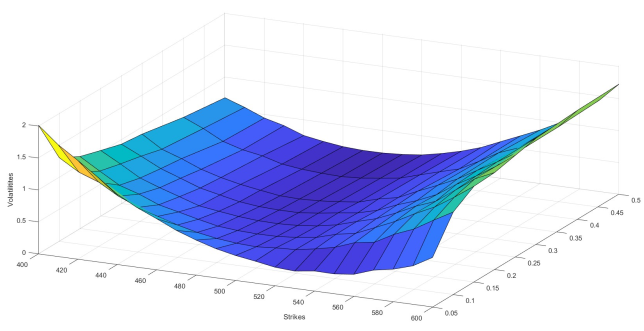

Figure 1: implied volatily curve

Lastly, we had the **discount factors** for some of the dates that are included in our time interval $[\tau_1, \tau_2]$. Since we did not have the discounts corresponding to all the dates we needed, we interpolated the values, to do so we obviously exploited the zero rates using the formula:

$$B(0, \tau_1, t) = e^{-r(0,t)*\tau_2}$$

We obtained the following discount curve:

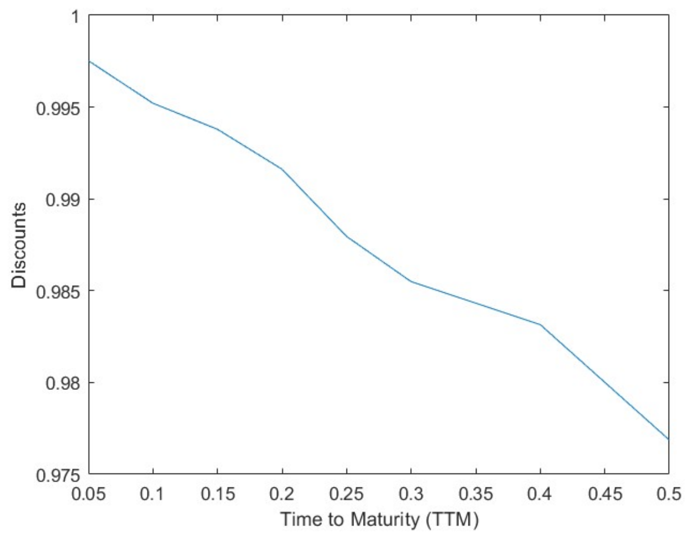

Figure 2: Discounts curve

1.a Data cutting

Initially, we attempted to calibrate the model using the entire dataset, including all strike prices. However, this approach resulted in poor calibration, as the prices of out-of-the-money (OTM) options were significantly out of scale. To address this issue, we decided to exclude strikes greater than **550** due to their illiquidity and the fact that their errors were excessively high.
This adjustment consistently reduced the mean squared error (MSE) and enabled better calibration. Since one of our goals is to price at-the-money (ATM) options accurately, we proceeded in this way. Although in-the-money (ITM) options can also lead to mispricing, they were not removed because we wanted to retain as much information as possible and their prices were not significantly different.

1.b HJM issues

The HJM model typically assume that option volatility depends on the tenor and that log-returns follow a normal distribution. However, real market data often exhibit traits like negative skewness and fat tails and the model struggle to account for. Moreover, using the Black model to calculate Swaptions prices from implied volatilities can introduce inefficiencies as it relies on overly simplified assumptions.
On the other hand, although this model is relatively simple, its main advantage is that it is easy to calibrate.

2 Range for the model parameters

The aim of this section is to find the admissible range for the model parameters of an HJM model for the 4Q25 German power swap. In order to accomplish our goal we started from equation 6.9 in Benth 2008:

$$F(t,\tau_1,\tau_2) = F(0,\tau_1,\tau_2)\exp\Big(\int_0^t A(u,\tau_1,\tau_2)\,du \\ + \sum_{k=1}^p \int_0^t \Sigma_k(u,\tau_1,\tau_2)\,dW_k(u) + \sum_{j=1}^n \int_0^t \Upsilon_j(u,\tau_1,\tau_2)\,dJ_j(t)\big)$$

This equation represents the risk-neutral price dynamics of the swap. $A(u,\tau_1,\tau_2)$ is the drift, $\Upsilon_j(u,\tau_1,\tau_2)$ represents the jump sizes of the additive processes, and $\Sigma_k(u,\tau_1,\tau_2)$ are the volatilities of the Brownian motions, which in this analysis we consider constant.

Since we are considering the 4Q25 German power swap, we have $n = 0$ and $p = 2$, the price equation becomes:

$$\begin{aligned} F(t,\tau_1,\tau_2) &= F(0,\tau_1,\tau_2)\cdot\exp\left(\int_0^t A(u,\tau_1,\tau_2)\,du + \sum_{k=1}^{2}\int_0^t \Sigma_k(u,\tau_1,\tau_2)\,dW_k(u)\right) \\ &= F(0,\tau_1,\tau_2)\cdot\exp\left(\int_0^t A(u,\tau_1,\tau_2)\,du + \Sigma_1 W_1(t) + \Sigma_2 W_2(t)\right) \end{aligned}$$

Where the last summation in the formula is equal to 0 since there are no jumps.
We need the explicit dynamics of the related forward to be set under the risk-neutral probability Q, so we impose the martingality condition. To prove this condition we exploit Proposition 6.3 in Benth 2008, which is satisfied since the process does not have any jumps.
Another condition that needs to be satisfied is the drift condition, since we need to ensure the absence of arbitrage for each swap contract. This condition is satisfied if:

$$\Sigma_1 \geq 0 \text{ and } \Sigma_2 \geq 0$$

3 Condition for futures prices to be martingales

To demonstrate the feasibility of deriving an explicit condition on the drift, let us recall the Drift Condition general form as reported by Proposition 6.3 in Benth, 2008:

$$\int_0^t A(u,\tau_1,\tau_2) + \frac{1}{2}\sum_{k=1}^{p}\Sigma_k^2(u,\tau_1,\tau_2)\,du + \sum_{j=1}^{n}\int_0^t \Upsilon_j(u,\tau_1,\tau_2)\,d\gamma_j(u)$$

$$+ \int_0^t \int_{\mathbb{R}} \left(\exp(\Upsilon_j(u,\tau_1,\tau_2)z) - 1 - \Upsilon_j(u,\tau_1,\tau_2)z\mathbf{1}_{\{|z|<1\}}\right)\nu_j(dz,du) = 0.$$

Starting from the previous formula, considering constant volatility and absence of jumps, we can easily rewrite the drift condition as:

$$\int_0^t A(u,\tau_1,\tau_2)\,du = -\frac{1}{2}\sum_{k=1}^{p}\int_0^t \Sigma_k^2(u,\tau_1,\tau_2)\,du.$$

Then, considering our case: $p = 2$ and constant volatility, we obtain the following explicit condition on the drift:

$$\int_0^t A(u,\tau_1,\tau_2)\,du = -\frac{1}{2}(\Sigma_1^2 + \Sigma_2^2)\cdot t.$$

4 Calibration with constant volatilities

After outlining the form and parameters of our model for pricing the swap, it is necessary to calibrate the model to align optimally with observed market prices. Our approach consists in calibrating the model on the entire surface of 4Q25 German call option prices by minimizing the distance between model and market prices.

Regarding the market prices of the German option for 4Q25, we implemented a function named `calibration_const` which exploits the `blackformula` function based on Black's model applied to price futures options. This function calibrates constant volatility parameters for the Black model in order to compute option prices based on the calibrated volatilities. We provided the function with constant volatility parameters, forward price at time $t = 0$, strike prices, time to maturities, and discount factors.

Fortunately, in the case of no jumps, as in the model in Point I, a formula similar to the one developed by Black and Scholes can be applied. This formula is known as the Black-76. Let us define the following quantities:

$$\hat{\sigma} = \sqrt{\sum_{k=1}^{p}\int_0^T \Sigma_k^2(u,\tau_1,\tau_2)\,du}$$

$$d_2 = \frac{\ln\left(\frac{F(0,\tau_1,\tau_2)}{K}\right) - \frac{1}{2}\hat{\sigma}^2}{\hat{\sigma}}, \quad d_1 = d_2 + \hat{\sigma}$$

$$C(0,T,K,\tau_1,\tau_2) = e^{-rT}\left[F(0,\tau_1,\tau_2)\Phi(d_1) - K\Phi(d_2)\right].$$

In the case of constant volatilities and $p = 2$, the quantity $\hat{\sigma}$ can be written explicitly as:

$$\hat{\sigma} = \sqrt{(\Sigma_1^2 + \Sigma_2^2)T}.$$

We first calibrated our model on the dataset of strikes using the `fmincon` function provided by

MATLAB. We passed through the function that we implemented to compute the model prices and used the MSE between market and model prices as our objective function to minimize. Furthermore, inside the minimization itself, we imposed a constraint to ensure that the parameters remain positive.

The surfaces are calibrated using all volatilities for strikes below 550.Larger strikes are not required for the project's objectives, allowing us to focus on a more localized calibration of the relevant surface rather than a global one.

As our starting point, we sampled two points from a random uniform distribution ranging from zero to the maximum observed volatility. This procedure leads to the following values:

$$\Sigma_1 = 0.2577, \quad \Sigma_2 = 0.2640, \quad \text{MSE} = 318.8313.$$

Which produces the surfaces:

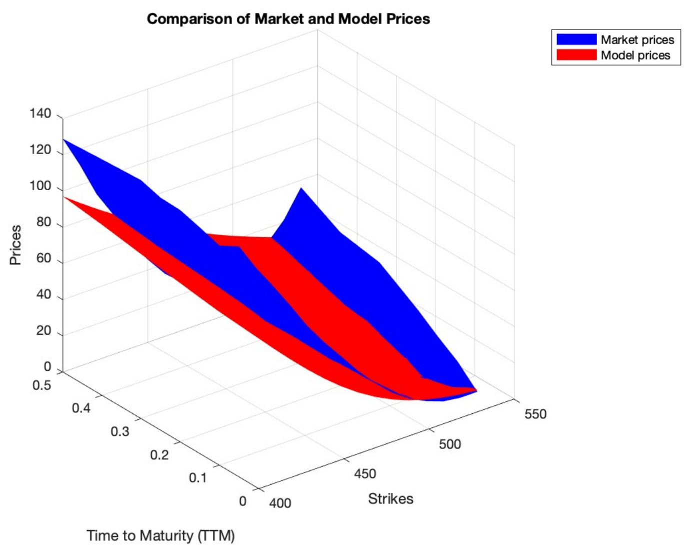

Figure 3: Comparison between Market Prices (blue) and Model Prices (red)

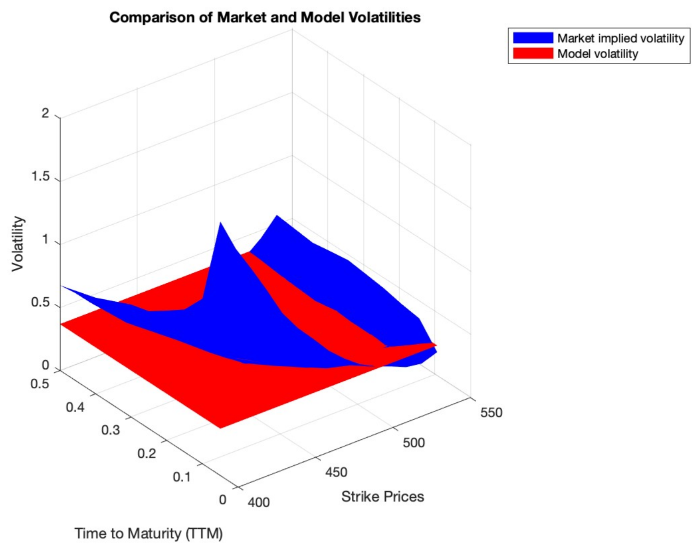

Figure 4: Comparison between Market implied Volatilities (blue) and Model implied Volatilities(red)

Even with its own limitations, we are satisfied with the obtained model. The resulting MSE is indeed rather reasonable and does not considerably exceed the market prices' order of magnitude. Moreover, we recall the strong simplifications imposed on the HJM model from Point I, which contribute to reducing correctness and accuracy in price computations.

The prices calibration is quite good in this case, but the volatilities one can certainly be improved: indeed it is too flat with respect to the market one.

Since $\hat{\sigma}$ considers also the time term, in order to compare it with the market implied volatility, we divide it with the respectively $\sqrt{TTM}$ and in this particular case we obtain a completely flat model volatility surface, since the parameters are constant.

5 Calibration with one time dependent parameter

In this case, we considered one of the two integrals as time-dependent, while the other was defined in the same way as in the previous section.

To maintain the most general approach, we did not calibrate the model to a specific function of time, such as a step-wise function or a particular deterministic function.

Instead, we considered the complete integral defined as:

$$\int_0^T \Sigma_1^2(u, \tau_1, \tau_2) du$$

It can be noticed that the integrand function is positive, so the integral, with respect to T, is an increasing function. As a consequence, we calibrated the model using:

$$\int_{T_n}^{T_{n+1}} \Sigma_1^2(u, \tau_1, \tau_2) du$$

where $n = 0, \ldots, N$, $t_0 = 0$, and $t_N = T$.
More specifically, for each maturity, we considered the integral from 0 to the Time to Maturity (TTM) using the command *cumsum*, which defines the necessary vector of integrals, and we defined $\hat{\sigma}$ as in the previous section as the square of the sum of the integrals.

$$\Sigma_2 = 0.0465 \qquad \text{MSE} = 283.5906$$

Table 1: One constant volatility case

TTM	$\int_{T_n}^{T_{n+1}} \Sigma_1^2(u, \tau_1, \tau_2) du$	$\int_0^{TTM} \Sigma_1^2(u, \tau_1, \tau_2) du$	$\Sigma_2^2 * TTM$	$\hat{\sigma}$
0.050	0.027	0.027	0.0001	0.163
0.100	0.002	0.028	0.0002	0.169
0.150	0.003	0.031	0.0003	0.178
0.200	0.004	0.035	0.0004	0.188
0.250	0.004	0.038	0.0005	0.197
0.300	0.004	0.042	0.0006	0.206
0.400	0.001	0.043	0.0009	0.209
0.500	0.003	0.046	0.0010	0.216

As expected, volatility increases with the TTM and the MSE decreases with respect to the previous case. Moreover, it can be observed that the integral is much more relevant than the constant parameter: this is due to the fact that the integral is more precise and it contains more information in this way.
It is interesting that the first volatility increment explains most of the variance, while the other increments are way smaller than the first one.

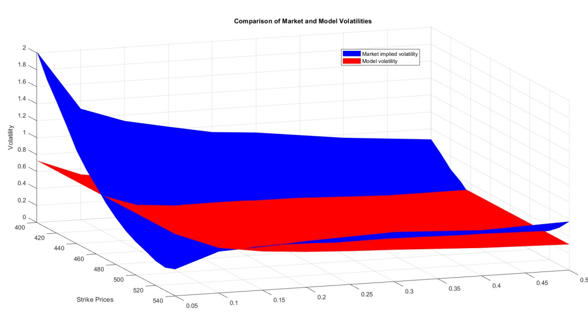

Figure 5: Comparison between Market implied Volatilities (blue) and Model implied Volatilities (red)

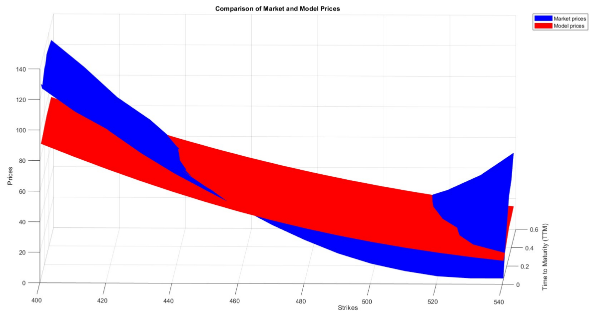

Figure 6: Comparison between Market Prices (blue) and Model Prices (red)

First, it can be noticed that the prices are similar to those in the constant case, indicating no significant improvement.
On the other hand, the implied volatility surface is better calibrated compared to the first case. In general, a slight enhancement can be observed, as will be discussed in the conclusion of the Section 6.
The main limitation lies in the small number of points used for calibration, which results in a relatively flat price surface. Similarly, the volatility surface appears flat, but this is due to the fact that $\hat{\sigma}$ remains constant for each strike.

6 Calibration with two time dependent parameters

In this section, we considered both the integrals as time-dependent in order to improve our model. As before, we did not calibrate the model to a specific function of time, but we considered the complete integrals.
First, it can be observed that the two integrals are identical, which is reasonable since they are defined in the same way. Moreover, $\hat{\sigma}$ remains consistent in both cases, leading us to expect values very close to those of the previous prices.
As described in Point 4, the first integral captures the majority of the volatility.

$$\text{MSE} = 283.5906$$

Table 2: All time dependent case

TTM	$\int_{T_n}^{T_{n+1}} \Sigma_1^2(u,\tau_1,\tau_2)du$	$\int_{T_n}^{T_{n+1}} \Sigma_2^2(u,\tau_1,\tau_2)du$	$\hat{\sigma}$
0.050	0.0134	0.0134	0.163
0.100	0.0009	0.0009	0.169
0.150	0.0015	0.0015	0.178
0.200	0.0019	0.0019	0.188
0.250	0.0018	0.0018	0.197
0.300	0.0018	0.0018	0.206
0.400	0.0006	0.0006	0.209
0.500	0.0014	0.0014	0.216

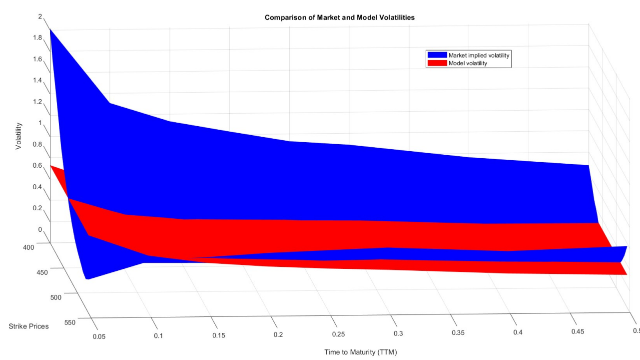

Figure 7: Comparison between Market implied Volatilities (blue) and Model implied Volatilities (red)

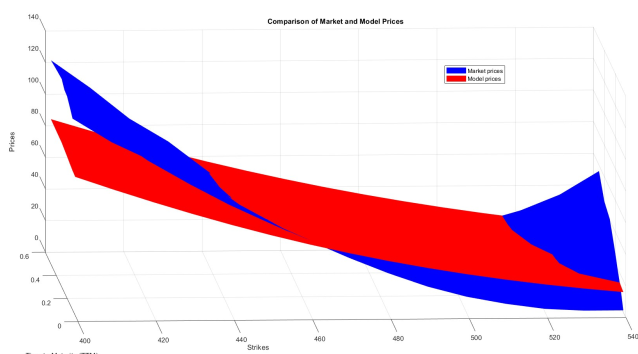

Figure 8: Comparison between Market Prices (blue) and Model Prices (red)

The graphs and the MSE are identical to those in the previous section, indicating that no significant improvements were achieved compared to the single constant case.

6.a Comparison between cases

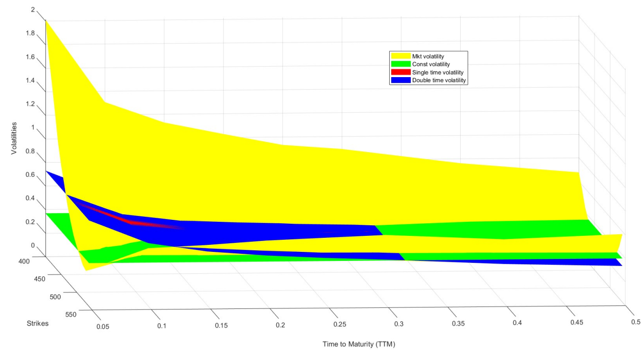

Figure 9: Comparison between Market implied Volatilities (yellow), First Model implied Volatilities (green), Second Model implied Volatilities (red) and Third Model implied Volatilities (blue)

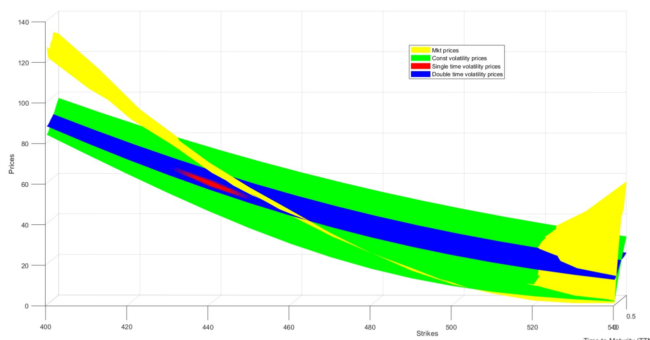

Figure 10: Comparison between Market Prices (yellow), First Model Prices (green), Second Model Prices (red) and Third Model Prices (blue)

In conclusion, the time-dependent models give similar results since the graphs coincide and the prices in both cases differ only at the fourth decimal digit.
Moreover, the time-dependent volatility surfaces align more closely with the market surface: indeed, the volatilities obtained in the constant parameters case are noticeably flatter.
On the other hand, the prices are described in a similar way by all the models, but they are slightly better captured by the time-dependent models.
Overall, the differences between models and market data remain significant due to the inherent limitations of the HJM model and the limited number of samples used.

7 Down & In Call Pricing

The aim of this section is to price a down-and-in call option with a maturity of 6 months, strike price $K = 500$, and barrier $L = 450$, using the calibrated parameters from the previous sections.

In a general setting, as discussed earlier, a closed-form expression for the dynamics of the underlying process is available. Given a grid of points $T_n = T_0 + n \cdot \Delta T$, $n = 1, \dots, N$, it is possible to sample the underlying process between each reset date by exploiting the condition on the drift for risk-neutrality:

$$F(T_n, \tau_1, \tau_2) = F(T_{n-1}, \tau_1, \tau_2) \exp\left(-\frac{1}{2} \sum_{k=1}^{p} \int_{T_{n-1}}^{T_n} \Sigma_k^2(u, \tau_1, \tau_2)\, du + \sum_{k=1}^{p} \int_{T_{n-1}}^{T_n} \Sigma_k(u, \tau_1, \tau_2)\, dW_k(u) \right).$$

Thanks to Itô's isometry (since $\Sigma_k(u, \tau_1, \tau_2)$ are deterministic functions of time), the stochastic integral has the following distribution:

$$\int_{T_{n-1}}^{T_n} \Sigma_k(u, \tau_1, \tau_2)\, dW_k(u) \sim N\left(0, \int_{T_{n-1}}^{T_n} \Sigma_k^2(u, \tau_1, \tau_2)\, du\right).$$

Defining

$$\hat{\sigma}(T_{n-1}, T_n) = \sqrt{\sum_{k=1}^{p} \int_{T_{n-1}}^{T_n} \Sigma_k^2(u, \tau_1, \tau_2)\, du},$$

we can simulate the underlying dynamics as:

$$F(t_n, \tau_1, \tau_2) = F(t_{n-1}, \tau_1, \tau_2) \exp\left(-\frac{1}{2} \hat{\sigma}(T_{n-1}, T_n)^2 + \hat{\sigma}(T_{n-1}, T_n) \cdot Z \right),$$

where Z is a standard normal random variable.

The main issue is the choice of reset dates. In the case of calibration with constant parameters, this issue does not arise, as the calibrated parameters are independent of the integration bounds and can be taken outside the integrals. This allows us to use a grid as fine as desired. In the other two cases, however, the calibrated parameters depend on the integration bounds, and we are limited by the illiquidity of the tenors available in the market. Therefore, in these cases, we are constrained to simulate only on these specific reset dates.

We start by simulating for all three cases, with N_simulations = 1e6, using the tenors given by the market, which lead to the following results:

Case	**Price**
Constant Parameters	9.1507
Single Time-Dependent Parameter	1.5597
Double Time-Dependent Parameters	1.5643

Table 3: Down & In Call prices

We observe that the time-dependent prices do not differ significantly, except in the first case. Furthermore, the time-dependent prices show a slight dependence on the number of simulations, as their values vary with an error close to 0.01.

In the case of constant parameters, a closed-form expression exists. By referring to *Arbitrage Theory in Continuous Time* by Tomas Björk (Third Edition, Theorem 18.8) and using the in-out parity, we find that for constant volatility the following holds:

$$\mathrm{Call}_{\mathrm{D\&I}}(F, K, L, T) = \left(\frac{L}{F}\right)^{\frac{2(r-\sigma^2/2)}{\sigma^2}} \mathrm{Call}_{\mathrm{EU}}\left(\frac{L^2}{F}, K, T\right),$$

where the effective volatility is given by:

$$\sigma = \sqrt{\Sigma_1^2 + \Sigma_2^2}.$$

Using this formula, we obtain a price of 17.4732.

This result and the previous ones are consistent with non-arbitrage arguments. The closed-form formula for a European call option yields a price of 42.3865, which is higher than the down-and-in prices, as the European call option is less restrictive regarding the opportunity to receive the payoff.

The three methods underestimate the option value at the given reset dates. However, in the first case, using a finer grid leads to a more accurate price. For instance, with weekly monitoring, the price is 12.3248, and with 500 reset dates, the price is 16.1084. Thus, the model with constant parameters is preferred, as it allows for arbitrarily fine grids.

We emphasize that small calibration errors can strongly impact the option price. For example, using the market's implied volatility for the given strike and maturity yields a price of 2.8345 (using the closed formula). This shows that the price is very sensitive to calibration errors.