FINANCIAL ENGINEERING
Project Report 6

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Contents

Case Study: Structured Bond

In this assignment we had to focus on the the Bank XX issuing a structured bond on the 16th of February of 2024 at 10:45 C.E.T., whose hedging termsheet is described in the annex, and could be described in the following way:

- PARTY A:
  Party A is the Bank XX, which quarterly pays the Euribor 3m + spol, that in this case is 2%.

- PARTY B:
  Party B,that is an I.B., pays at the start date $X\%$ of the Principal Amount and quarterly the coupon, except for the first quarter that pays just 3%. The coupon is given by:
  - Euribor 3m + 1.10% capped at 4.30% up to (and including) the 5th year
  - Euribor 3m + 1.10% capped at 4.60% After 5y and up to (and including) the 10th year
  - Euribor 3m + 1.10% capped at 5.10% after the 10th year.

The computation time for all the code is at least 40 sec.

1 Bootstrap

The first thing we had to do was to Bootstrap the market discounts for the $16^{th}\ February$ 2024. We wanted to have a complete set of swap rates, expiring each year, from 2 years up to 50 years. We took the exact date for each year, taking into account European holidays, we computed the mean between the bid and the ask values, and we interpolated them, with a spline interpolation, in order to find the value for each date of the swaps. After this procedure we were able to use our function `bootstrap.m` in order to find all the dates, up to 50 years, and the corresponding discounts.
We obtained the following values for discounts and zero rates:
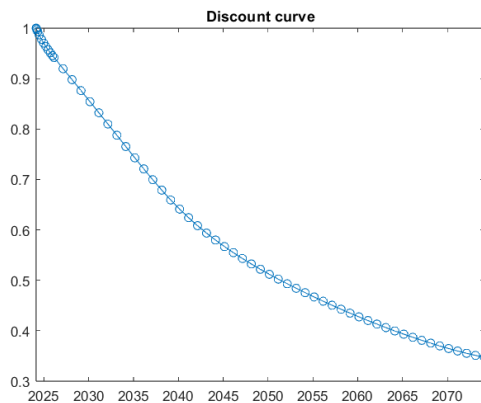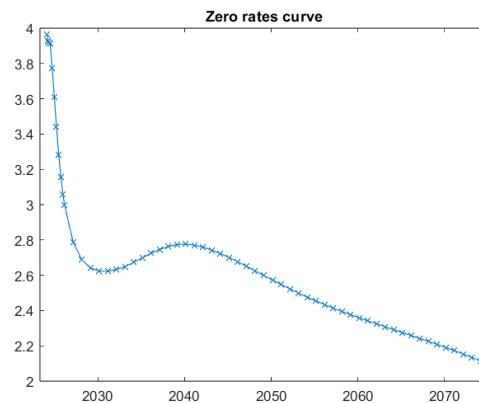

Figure 1: Discounts

Figure 2: Zero rates

From the Zero Rates curve we can notice a behaviour that is aligned with what is actually happening, since in the last year there has been a significant growth in the rates, so the market is expecting for them to drop in the nearby future as the shape suggest.

2 Pricing

2.1 Computing Spot Volatilities

The aim of this section was to firstly calibrate the spot volatilities based on the given flat volatilities. We were provided with a series of Maturities (up to 30 years), Strikes (given in percentage) and the corresponding flat volatilities.
The calibration is based on the price of different caps, therefore we started by computing the price of all the caps associated to the flat volatilities that were given. We created a vector containing all the dates of the caplets, (every quarter, from 1 year up to 30 years), and we priced each cap (each one corresponding to a flat volatility in the table) using the Bachelier equation:

$$Caplet_i(T_0) = B(T_0, T_{i+1})\delta_i([L_i(T_0) - K]N(d^n) + \sigma_i\sqrt{T_i - T_0}\phi(d^n))$$

where

$$\delta_i = \delta(T_i, T_{i+1})$$
$$L_i(T_0) = L(T_0, T_i, T_{i+1})$$
$$d^n = \frac{L_i(T_0) - K}{\sigma_i\sqrt{T_i - T_0}}$$

σ_i is the value of the flat volatilies from the table (the one already integrated)

(We used act360 convection for the yearfrac in the caplet and act365 for the time to maturity)
The formula is used to price a single Caplet, so in order to price the cap at a certain year we summed all the prices of the caplets up to that year. We had to take into account the fact that in some cases there was a jump of more than an year in the dates contained in the table, in that case we considered more than 4 caplets using the maturity of the cap. We plotted the curve that represents Cap prices, as a function of both the strikes and maturities:

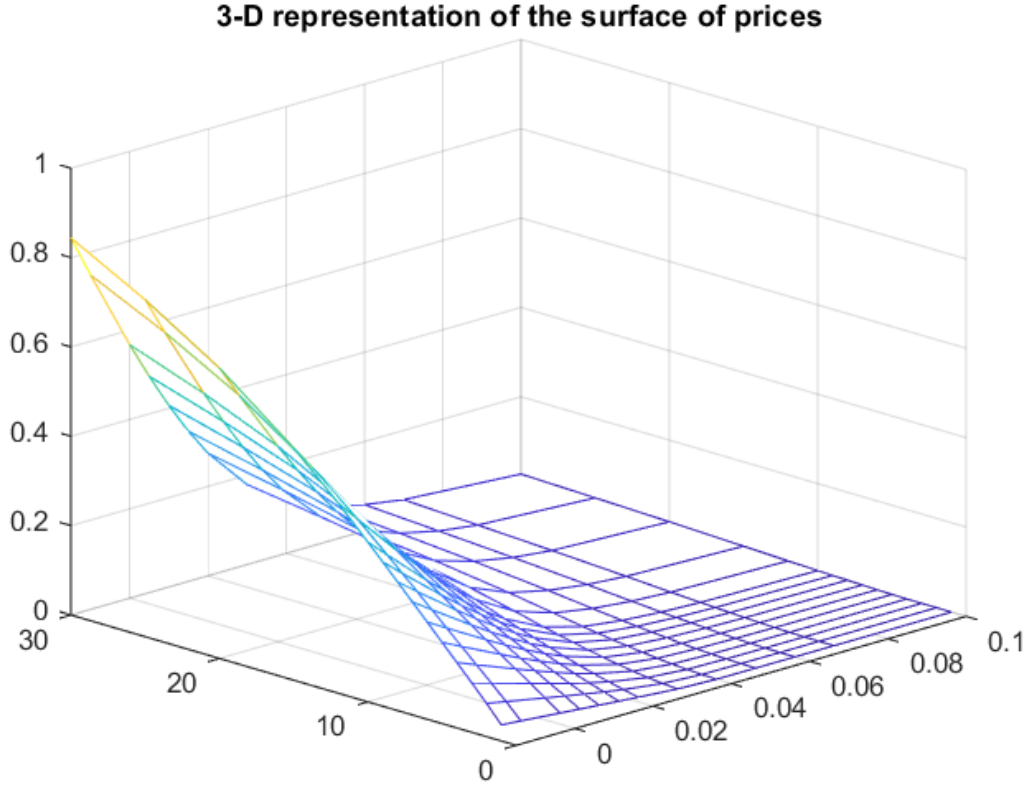**3-D representation of the surface of prices**

Figure 3

We notice that the surface of the prices of the cap is quite smooth, the price increase with the increase of the expiry as expected since the payoff could be greater. On the other hand, we can see that for a cap out of money (with strike greater), the price get smaller, since the probability that we exercise the option will be smaller.
Having the prices of the caps we were able to proceed with the calibration. The first assumption we made is that the volatilities for the first year are constant, therefore we imposed the same vol for the first three dates. For each remaining date of a caplet we wanted to find the corresponding spot volatilities, to do so we exploited the following relation:

$$\Delta C = Cap(T_\alpha) - Cap(T_\beta) = \sum_{T_\alpha \leq T_i < T_\beta} caplet(i, \sigma_i)$$

The difference between the prices of two Cap is equal to the sum of the caplets priced considering the dates between the expiries of the two Caps.
By solving this equation and by imposing a linear correlation between the volatilities, with respect to the dates, we were able to construct the matrix of spot volatilities. Also in this case, we took into account that there can be more than 4 caplets between one cap quoted and the next one. We built

a function computing_sigma.m in order to create the linear system that should be solve to find the spot volatilities.

From the calibration_vol.m we get one matrix 15x13 with the spot volatilities corresponding to the maturity of each cap and one 119Xx13 with the spot volatilities for every quarter (ie each caplet) expect the first one. We made a 3D-plot of both the flat and the spot volatilities to compare their behaviour:
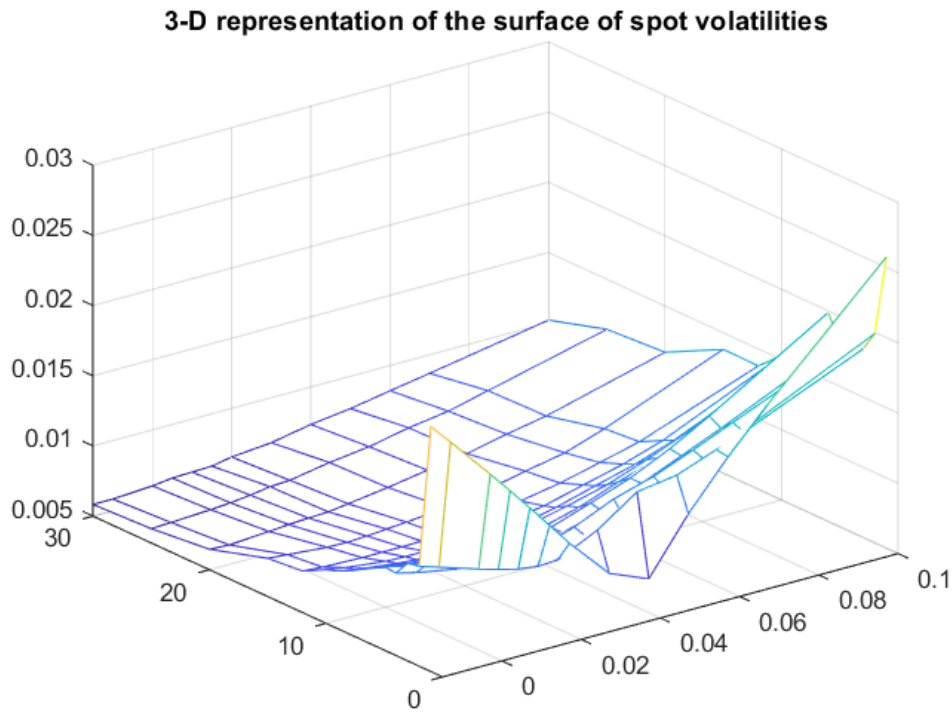
3-D representation of the surface of spot volatilities

Figure 4: Flat volatilities curve

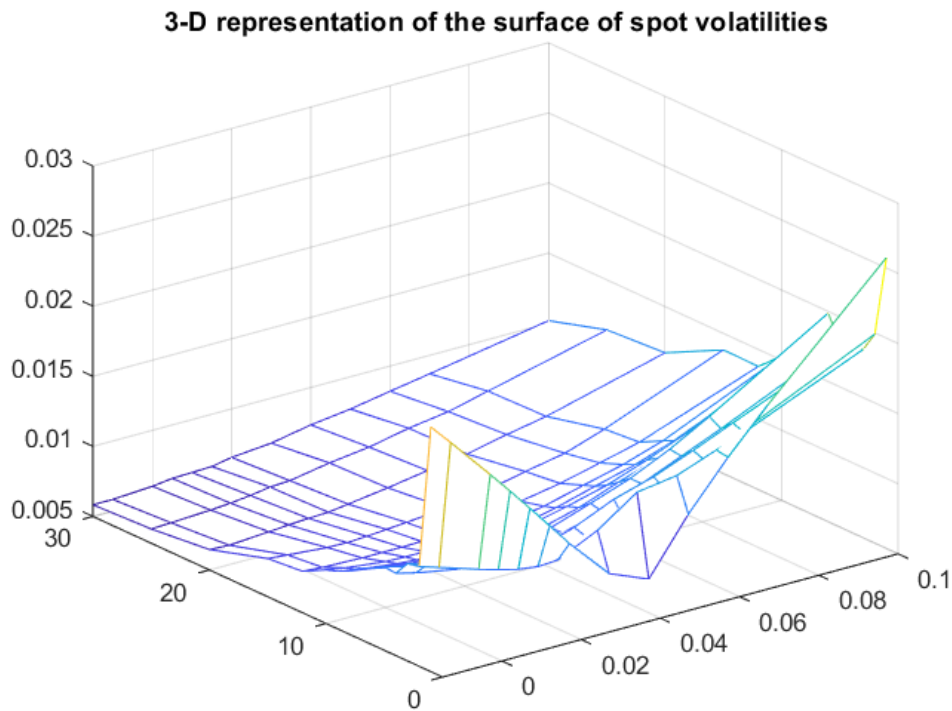**3-D representation of the surface of spot volatilities**

Figure 5: Spot volatilities curve

The main thing we can notice is that the behaviour of the spot volatilities is very similar to the flat volatilities, they both decrease with respect to the maturity. We observe that the volatilities for the cap in and out of the money are greater than the one close to the at the money cap.

2.2 Determine the Upfront

After having computed the spot volatilities we calculated the Upfront $X\%$ paid by the I.B., by imposing the NPV equal to zero, that is the result of putting equal the following relations:

- Party A:

$$\sum_{i=0}^{N-1} B(t_0, t_{i+1}) \cdot \text{spol} + 1 - B(t_0, t_N)$$

- Party B:

$$X\% + \sum_{i=0}^{N-1} \mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) \cdot \bar{c}\right]$$

Where we have:

- spol = spread over LIBOR, in our case 2%

- $\bar{c}$ is the coupon paid each quarter by B, which could also be expressed as:

  - For the first quarter: $\bar{c} = 3\%$
  - For the first 5 years (excluding the first quarter): $\bar{c} = \min(L(t_i, t_{i+1}) + 1.1\%, 4.3\%)$
  - For the years after the fifth and up to the tenth: $\bar{c} = \min(L(t_i, t_{i+1}) + 1.1\%, 4.6\%)$
  - For the last 5 years: $\bar{c} = \min(L(t_i, t_{i+1}) + 1.1\%, 5.0\%)$

Further more the expected value of the discounted $\bar{c}$ (after the first quarter) could be written as:

$$s \cdot B(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) + B(t_0, t_i) - B(t_0, t_{i+1}) - \text{Caplet}_i(k)$$

where s equals 1.1% and k, that is the strike of each caplet, is 3.2% for the caplets up to the 5th year, 3.5% for the caplets after the 5th year up to the 10th and 4% for the remaining caplets up to the 15th year.

Proof (I consider the first period up to 5y)
I consider only one coupon.

$$\mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) \cdot \bar{c}\right] =$$

$$= \mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) \cdot \min(L(t_i, t_{i+1}) + 1.1\%, 4.3\%)\right] =$$

$$= 1.1\% \cdot \mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1})\right] + \mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) \cdot \min(L(t_i, t_{i+1}), 3.2\%)\right] =$$

If I consider a generic underlying S_t and $\mathbb{E}\left[D(t_0, t_{i+1}) \cdot \min(S_t, K)\right]$ I can prove that the payoff can be replicated by *Underlying + Call(S_t,K)* where K is the strike. So:

$$= 1.1\% \cdot \mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1})\right] + \mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) \cdot L(t_i, t_{i+1}))\right]$$

$$+ \mathbb{E}\left[D(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) \cdot (L(t_i, t_{i+1}), 3.2\%)^+\right] =$$

Since the third term represents a call with respect to $L(t_i, t_{i+1})$, then I can use the Caplet formula.

$$= 1.1\% \cdot B(t_0, t_{i+1}) \cdot \delta(t_i, t_{i+1}) + B(t_0, t_i) - B(t_0, t_{i+1}) - \text{Caplet}_i(k)$$

So putting all together, simplifying the discounts we get the following formula to price the Upfront (we consider the first Libor rate because not all the discounts are simplified since the first IB quarter coupon is completely fixed):

$$X\% = \sum_{i=0}^{N-1} \text{spol} \cdot \delta(t_i, t_{i+1}) \cdot B(t_0, t_{i+1})$$

$$- \sum_{i=1}^{N-1} s \cdot \delta(t_i, t_{i+1}) \cdot B(t_0, t_{i+1})$$

$$- 3\% \cdot \delta(t_0, t_1) \cdot B(t_0, t_1) + \delta(t_i, t_1) \cdot L(t_0, t_1) \cdot B(t_0, t_1) + \sum_{i=1}^{3} \text{fwd\_Cap}_i$$

where each fwd_Cap$_i$ is the sum of the Cap of every year, that are the sum of the respectively caplets with the strike mentioned before: fwd_Cap$_1$ is the sum of the Caplets of the first 5 years, fwd_Cap$_2$ of the Caplets after the 5th to the 10th and fwd_Cap$_3$ of the Caplets of the last 5 years.

The 3 types of caplets are calculated with the function `Pricing_Caplet` that implements the Bachelier method, that is:

$$\text{Caplet}_i(T_0) = B(T_0, T_{i+1}) \cdot \delta(T_i, T_{i+1}) \cdot \left([\text{L}_i(T_0) - \text{K}] \cdot N(d^n) + \sigma_i \cdot \sqrt{T_i - T_0} \cdot \phi(d^n) \right)$$

$$\text{with } d^n = \frac{\text{L}_i(T_0) - \text{K}}{\sigma_i \cdot \sqrt{T_i - T_0}}$$

Notice that in order to find the right σ_i we had to interpolate the Volatility Matrix (Spot or Flat) in the desired strike using a Spline interpolation.

Considering a notional of € 50,000,000, the result that we got by doing this and using the spot volatilities is:

$X\% = $ **€ 9,433,119** or in percentage **18.866**%.

3 Delta Bucket Sensitivities

The aim of this part is to compute and analyze the Delta-bucket sensitivities of the structured product.

3.1 Shift before spline interpolation

We started from the original data given, in which the swap dates were up to 50 years, but did not contain all the years in between. We added, one at a time, 1 bps to the values contained in the rates set. For each shifted curve we did the same computations as in point a), in this way we obtained a new curve of discounts and zero rates, with values for swaps from 1 year up to 50 years (using spline interpolation). We then proceeded to evaluate the NPV for each shifted curve, using spot volatilities. We then computed the difference between the NPV found from the shifted curve and the original NPV. This difference is exactly the delta-bucket sensitivity found using a numerical approach.

With this computation we obtained a vector of 28 sensitivities, the first 4 ones are related to the depos, then the following 7 to futures and lastly the rest are connected to the swaps, with dates coinciding with the one from the original bootstrap.

This procedure was implemented in the function: *delta_bucket2.m*. We plotted the results:
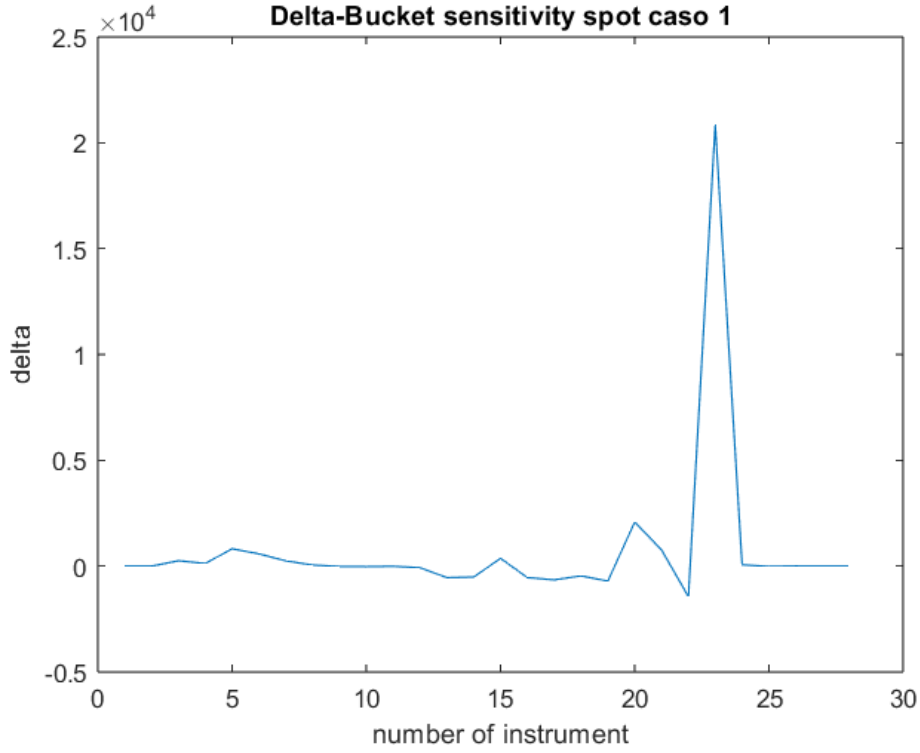
Figure 6: Spot volatilities Delta-Bucket

What we can notice from this plot is the fact that since the first year does not really matter in our computations the value of the first elements is equal to 0. The same holds for the last elements, since we are analysing a 15 year contract, what happens after that date does not have an influence. We notice how the last value is exactly equal to 0 and the other last ones are really close to 0, they are not exactly 0 since we are using a spline interpolation.

3.2 Shift after spline interpolation

In this case we considered the shift after the spline interpolation, therefore we added 1bp to each rate, one at a time, and we did not change the other ones as in the previous case.
The difference from the previous computations is the number of elements we obtained in the vector of the delta-bucket. In this case we end up with a vector of 60 elements, the first 4 are related to the depos, the following 7 to the futures and the remaining to the swaps. This procedure was implemented in the function: *delta_bucket.m*. We plotted the results:
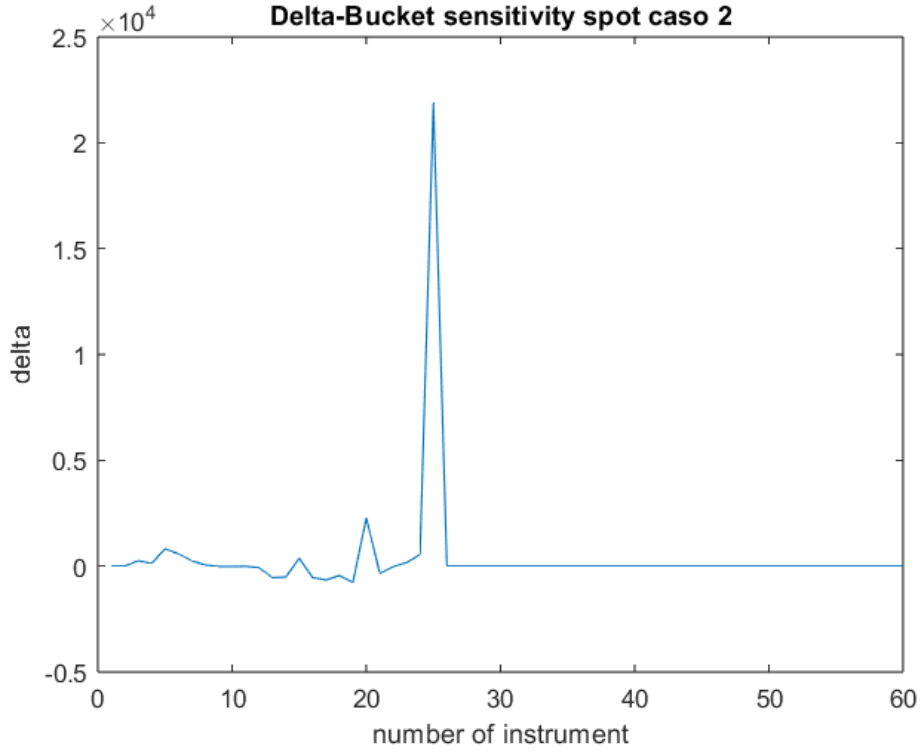
Figure 7: Spot volatilities Delta-Bucket

There are some differences and similarities with the results obtained in the previous computations. We can notice how the first 11 elements of both vectors are the same, since the difference in the two procedures is contained just in the swaps. The main difference is the fact that with the second method we obtain a vector whose elements become 0 after the 15 years.

We decided to use the second approach since we noticed that in point f, if we used the first method, we obtained a strange behaviour regarding the coarse-grained bucket values, in this way we had more clear results.

4 Total Vega

In this part we wanted to evaluate the total Vega of the certificate, we wanted to see how much the shift in the volatilities could effect the value of the certificate. We worked by shifting all the flat volatilities up by 1bp, we then calibrated the spot volatilities based on the shifted flat vol, lastly we computed the total Vega of the certificate by evaluating the NPV in the case of the shifted volatilities and subtracting to it the orginal NPV. We obtained the following results:

- Total Vega Spot Vol : **55831.933**

The result takes into account the Notional

5 Vega Bucket Sensitivities

We then analysed Vega-bucket sensitivities. We proceeded similarly to the previous point, the only difference is in the way we shifted the volatilities curve. We started from the original table which gives us flat volatilities, at some given years for given strikes. We shifted, a group at a time, all the vol which had the same dates. For every shifted volatilities curve we calibrated the spot volatilities, and we lastly found the Vega as done before (as the difference between the NPV obtained by shifting the volatilities and the original NPV). The vector in this case is the following:

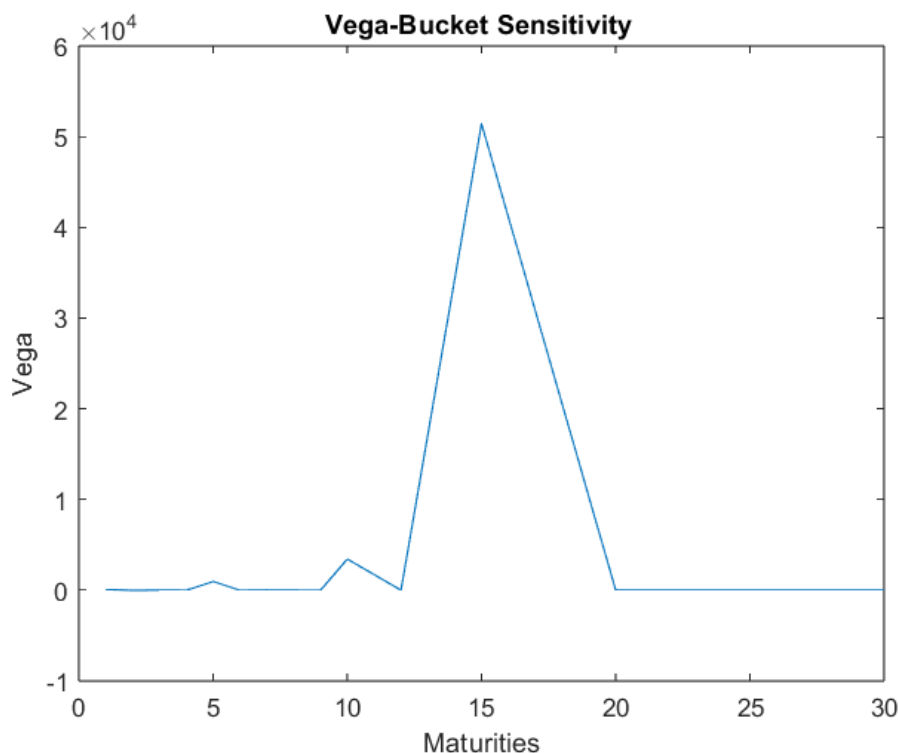|	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	12y	15y	20y	25y	30y
Vega	88.19	-18.83	13.78	-8.75	955.34	-12.75	17.73	-1.90	-8.69	3430.70	0.0	51415.98	0	0	0

Figure 8: Flat volatilities curve

We can notice that the highest values are the one corresponding to the shift in the curve at 5 years, 10 years and 15 years, these dates are the most significant one in the certificate we are analysing. We can also notice that after 15 years the values become 0, this is always connected to the fact that our certificate is defined up to 15 years.

Moreover, the sum of the vega bucket sensitivities, which is equal to **55870.777**, is close to the Total Vega and this is coherent with the theory.

6   Hedging Delta bucket (coarse-grained buckets)

In this point the aim was to hedge the delta risk using the Swaps considering 4 coarse grained buckets (0-2years; 2-5years; 5-10years, 10-15years).

In Risk Management it's common to divide the time horizon using the coarse grained buckets because the traders think to the movement of rates in blocks. Indeed there is evidence to say that usually several rates have the same behaviour and so they move together: for this reason it's better to work in terms of aggregate sensitivities. We used the Swaps payer in order to hedge the risk coming from the underlying since they are liquid instruments in which we can control the maturity and for which if the rates increase the NPV increases.

The first thing that we did was to build the weights of each macro bucket, so in the function `weights_delta` we implemented the following weights:
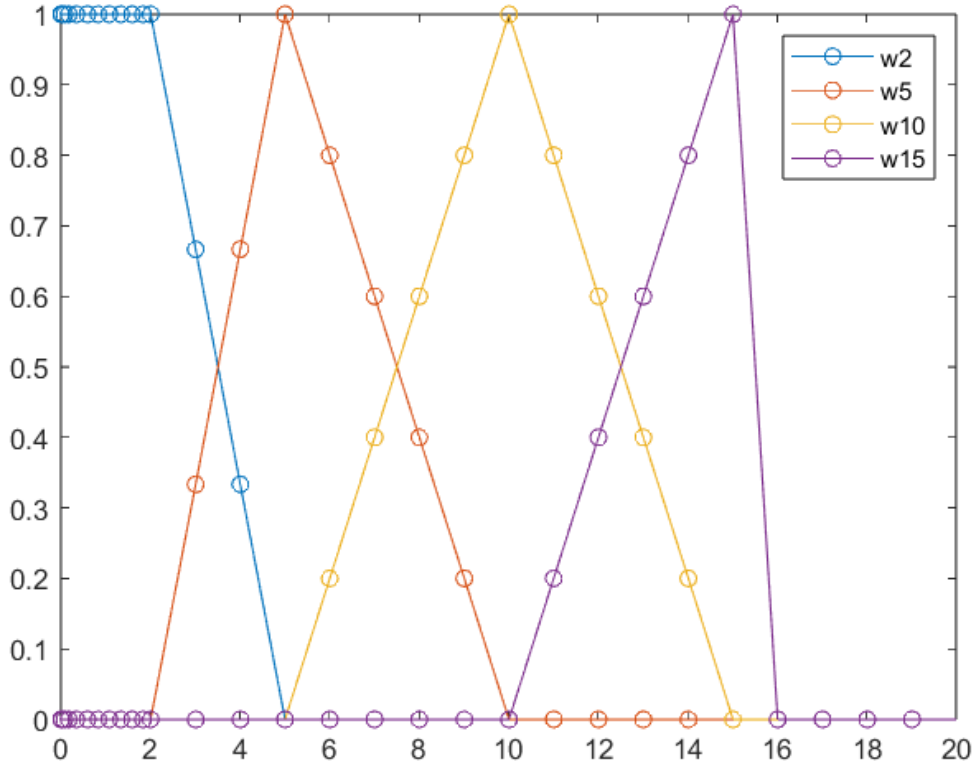
Figure 9: Spot volatilities Delta-Bucket

Then we had to associate these weights to the sensitivities of the 4 coarse grained bucket because our aim was to use 4 Swaps, all with different maturities and Notional, in order to be hedged in every macro bucket. We used 4 swap with maturities 15y, 10y, 5y and 2y and we started hedging with the longest swap. So we had to solve this system in order to find the only unknowns variables, the 4 notionals:

$$\begin{cases} N_1 \cdot \text{DV01\_w15\_swap15} + \text{DV01\_w15} = 0; \\ N_1 \cdot \text{DV01\_w10\_swap15} + N_2 \cdot \text{DV01\_w10\_swap10} + \text{DV01\_w10} = 0; \\ N_1 \cdot \text{DV01\_w5\_swap15} + N_2 \cdot \text{DV01\_w5\_swap10} + N_3 \cdot \text{DV01\_w5\_swap5} + \text{DV01\_w5} = 0; \\ N_1 \cdot \text{DV01\_w2\_swap15} + N_2 \cdot \text{DV01\_w2\_swap10} + N_3 \cdot \text{DV01\_w2\_swap5} + N_4 \cdot \text{DV01\_w2\_swap2} + \text{DV01\_w2} = 0; \end{cases}$$

where:

- N_1, N_2, N_3, N_4 are the notionals corresponding to the 15-year, 10-year, 5-year, and 2-year Swaps, respectively.

- $\text{DV01\_w}_i\text{\_swap}_i$ is the total DV01 of the Swaps already weighted with the corresponding weights.

- DV01_w_i is the total DV01 of our portfolio, without the Swaps, weighted with the corresponding weights.

We can see that in this way we are hedging every macro bucket just using 4 Swaps and considering the effects of their sensitivities with the respective weight of the macro buckets: we can approach the problem in this way because the DV01 swaps for the 2y, 5y and 10y, for example, are equal to zero with respect to the 10y-15y weights since their maturity is smaller.
So, we can create a simple linear system which computes the Notionals.
The DVO1 used for just our portfolio, so not considering the Swaps, was the one computed before in the function `delta_bucket` in the subsection 3.2 multiplied by the corresponding weights.
Concerning the DV01 of the Swaps we did a similar thing computing the 4 Sensitivities, one for each macro bucket, using the function `Deltabucket_Swap_1`. Here, as before, we shifted one by one the rates of 1bp and for each bucket we calculated the new NPV shifted and the DV01 was the difference

of this shifted NPV and the unmodified one.

The NPVs of the Swap were calculated with the function `SWAPpricing` where was implemented the difference between the NPV of the floating leg and of the Fixed leg (we are considering a Swap payer position), i.e.:

$$NPV = NPV_{\text{float}} - NPV_{\text{fixed}}$$
$$\text{where}$$
$$NPV_{\text{float}} = 1 - B(t_0, t_N)$$
$$NPV_{\text{fixed}} = s \sum_{i=0}^{N-1} B(t_0, t_{i+1}) \delta(t_i, t_{i+1})$$

where s is the Swap Rate, and in our case we considered the mid rate between the bid and ask swap rate, corresponding at each expiry that we used, so the 2-year, 5-year, 10-year, and the 15-year.

In every equation, we multiply the $DV01_{\text{Swap}}$ with the corresponding weights.

The result that we got is the following:

N_1	**-18560323.18€**
N_2	**-1032837.38€**
N_3	**2915170.13€**
N_4	**-7450143.48€**

We can see that the N_1, N_2 and N_4 notionals are negative, in particular the first one is very high in absolute value: this is coherent because the DV01-bucket 15y of the swap is positive and it us way greater than the other ones; this impacts in the computation.

6.1 Point c - Spline interpolation effect

At the line 310 of the main *runAssignment6_Group4.m* there is a commented part: we put there the coarse-grained bucket computed in the alternative case, so shifting the rates before the spline interpolation of the swap.

We obtained that the coarse-grained delta buckets of the swap are different from the previous case: for example, the DV01_w5_swap_2y coarse-grained bucket is different from zero and this is due to the spline interpolation effect since in this case the interpolation consider all the curve and the shift affects all the curve in this case.

In this case we would have to compute a 4x4 linear system to consider all the risks.

7 Hedge the Vega

We now wanted to hedge the Vega with a 5 years at the money Cap and consequently hedge the delta of our portfolio by using a 5 years swap. We can use the Cap in order to hedge the vega since they are liquid instruments (really cheap transaction cost) and which reacts to changes in volatilities. We began by computing the ATM strike as the mean between the bid and ask for maturity 5 years of the swap taken from the market, we then computed the ATM spot volatilities, for the ATM strikes, by spline interpolation of the previously found spot volatilities.

Since the Cap has maturity of 5 year we uses the 5 year vega bucket sensitivity, found as difference between price of the 5 year Cap ATM, with shifted volatilities, and the price of the 5 year Cap ATM in the basic situation.

Since we also wanted to hedge the total portfolio (certificate + Cap 5y), we used the value of delta-bucket found before for the certificate, the Swap. We found delta-bucket for the Cap, found as difference between price of the 5 year Cap ATM, with shifted bootstrap, and the price of the 5 year Cap ATM in the basic situation. We noticed that the all the delta bucket as explained before can be computed by bumping up directly just the instruments given from the market and then doing the spline interpolation or by interpolating spline, obtaining the full term structure and then bumping up these. We decided to use the first in the second way to keep the code simple, since the difference should be small.

In order to find the notional invested in the Cap and the notional invested in the Swap to have a portfolio fully hedge, we used the command fzero. Here the result:

Cap 5y ATM notional	Swap 5y notional
-3748966.48	**-798608.05**

8   Hedging Vega Coarse Grained Buckets

In this part of the assignment we considered the coarse-grained buckets 0y-5y and 5y-15y and we had to hedge the bond with respect to delta-bucket sensitivity and the vega-bucket sensitivity.
At first, we computed the weights, but in this case with only two buckets and considering the maturity dates. Then, we considered the ATM strikes, which are equal to the 5y and 15y mid Swap rates, and we interpolated the spot volatilities (spline interpolation) to obtain the sigma vectors related to the 2 strikes: in this way we could find the cap prices.

8.1   Vega Hedging

The function *vega_bucket_spline* computed the vega bucket of the two caps: we considered these two particular Caps since the 5y one is not affected by the 5y-15y bucket since its maturity is smaller than the first significant weight.
So we had a simple linear system to solve:

$$\begin{cases} N_1 \cdot \text{Vega\_w15\_bucket\_15y} + \text{w15\_vega} \cdot \text{vega\_bucket\_spot} = 0; \\ N_1 \cdot \text{Vega\_w5\_bucket\_15y} + N_2 \cdot \text{Vega\_w5\_bucket\_5y} + \text{w15\_vega} \cdot \text{vega\_bucket\_spot} = 0; \end{cases}$$

where:

- N_1 and N_2 are the notionals corresponding to the ATM Cap of 5 years and 15 years, respectively.

- Vega_w$_i$_bucket$_i$ represents the weighted vega related to the first 5 years and the years from 5 years to 15 years.

- vega_bucket_spot denotes the vega bucket calculated for the entire period.

- w15_vega denotes the weights related to the period from 5 years to 15 years.

We obtained the following notionals:

Cap 5y ATM notional	Cap 15y ATM notional
-11222262.01	**-44362507.09**

8.2   Delta Hedging

As before, we computed the weight with the bootstrap dates considering the two buckets.
The function *delta_bucket_cap* computed the delta bucket of the two caps.
As before we can neglect some weights...
As before, we obtain a simple system to solve:

$$\begin{cases} N_1 \cdot \text{w15\_delta\_1} \cdot \text{delta\_bucket\_cap\_15y} + N1_{\text{Swap}} \cdot \text{w15\_delta\_1} \cdot \text{delta\_bucket\_swap\_15y} \\ \quad + \text{w15\_delta\_1} \cdot \text{delta\_NPV\_spot1} = 0; \\ N_1 \cdot \text{w15\_delta\_1} \cdot \text{delta\_bucket\_cap\_15y} + N_2 \cdot \text{w15\_delta\_1} \cdot \text{delta\_bucket\_cap\_15y} \\ \quad + N1_{\text{Swap}} \cdot \text{w15\_delta\_1} \cdot \text{delta\_bucket\_swap\_15y} \\ \quad + N2_{\text{Swap}} \cdot \text{w15\_delta\_1} \cdot \text{delta\_bucket\_swap\_5y} \\ \quad + \text{w15\_delta\_1} \cdot \text{delta\_NPV\_spot1} = 0; \end{cases}$$

where:

- $N_{1\,\text{Cap}}$ and $N_{2\,\text{Cap}}$ represents the 2 notionals of the Cap found before.

- $N1_{\text{Swap}}$, $N2_{\text{Swap}}$ are the notionals of the 2 Swap, one of 5y the other of 15y, and it's what we have to find

- w15_delta_1 are the coarse grained weights related to the 5y to 15y

- delta_bucket_cap_15y is the delta of the Cap for the 15y

- delta_NPV_spot1 is the delta bucket calculated above

- delta_bucket_swap_5y represents the delta of the the Swap related to the 5y

- delta_bucket_swap_15y represents the delta of the the Swap related to the 15y

We obtained the following notionals:

Swap 5y notional	Swap 15y notional
-1032837.38	**-18560323.18**