Report file - Problem Set #9

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November 26, 2024

Abstract

This is the report for the problem set #9. Since the problem set is composed of two exercises, we divide the report into two sections, one for each problem. The scripts (labelled as ps_9.'problem number') and the raw file of the images are in this directory.

1 Problem 1

1.1 Formulation of the problem

We are asked to code a solver for the Harmonic oscillator problem:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x, \tag{1}$$

the anharmonic oscillator with equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x^3, \tag{2}$$

and the *van der Pol* oscillator:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \mu(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2 x = 0. \tag{3}$$

1.2 Computational methods

First, we rewrite the various problems in terms of two coupled first-order equations by defining $\frac{\mathrm{d}x}{\mathrm{d}t} = v$:

- Harmonic oscillator:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\omega^2 x \end{cases} \tag{4}$$

- Anharmonic oscillator:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\omega^2 x^3 \end{cases} \tag{5}$$

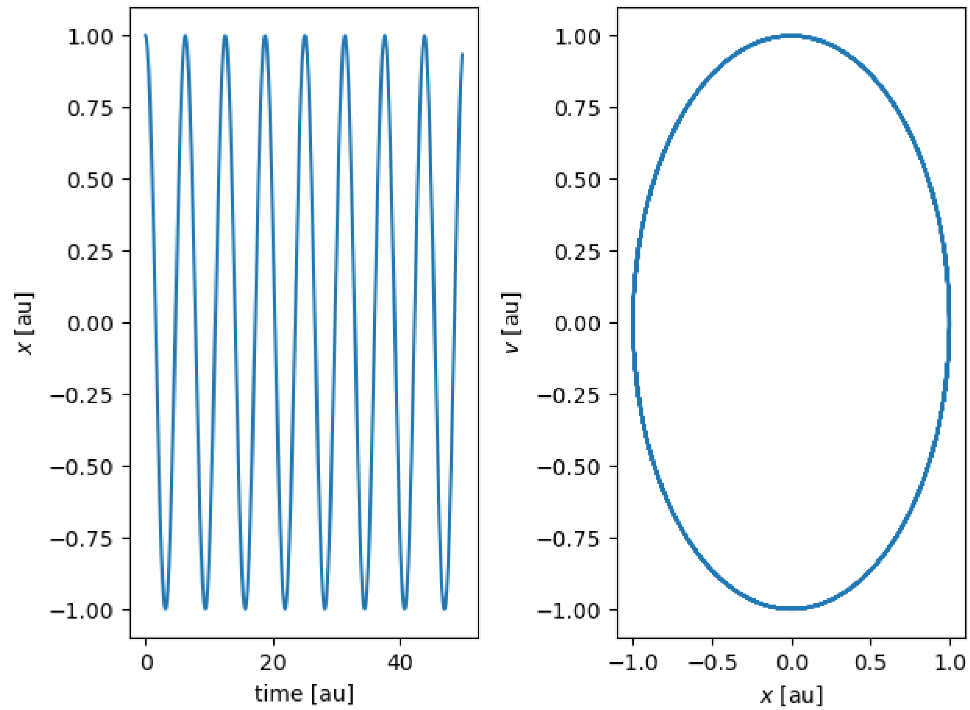

Figure 1: Solution (both position and phase space plots) of the harmonic problem with $\omega = 1$ and initial conditions $x = 1$ and $v = 0$.

- Van der Pol oscillator:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = v \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \mu(1 - x^2)v - \omega^2 x = 0 \end{cases} \tag{6}$$

Then we solve the linearized systems using the fourth-order Runge-Kutta method (applying slight modifications to the code present in the Jupiter notebook of the class).

1.3 Results

We report the results (both position and phase space plots) of the harmonic oscillator with $\omega = 1$ and initial conditions $x = 1, 2$ and $v = 0$ respectively in figs. 1 and 2, from which we see that the period doesn't change when we double the amplitude of the oscillations. Quantitatively, we can look for the frequencies most represented in the FFT of the two signals to be $f = 0.16$ in both cases).

We report the results (both position and phase space plots) of the anharmonic oscillator with $\omega = 1$ and initial conditions $x = 1, 2$ and $v = 0$ respectively in figs. 3 and 4, from which we see that the oscillations get faster at higher amplitudes.

We report the results (both position and phase space plots) of the vander Pol oscillator with $\omega = 1$ and $\mu \in \{1, 2, 4\}$ initial conditions $x = 1$ and $v = 0$ respectively in figs. 5 to 7. For easier

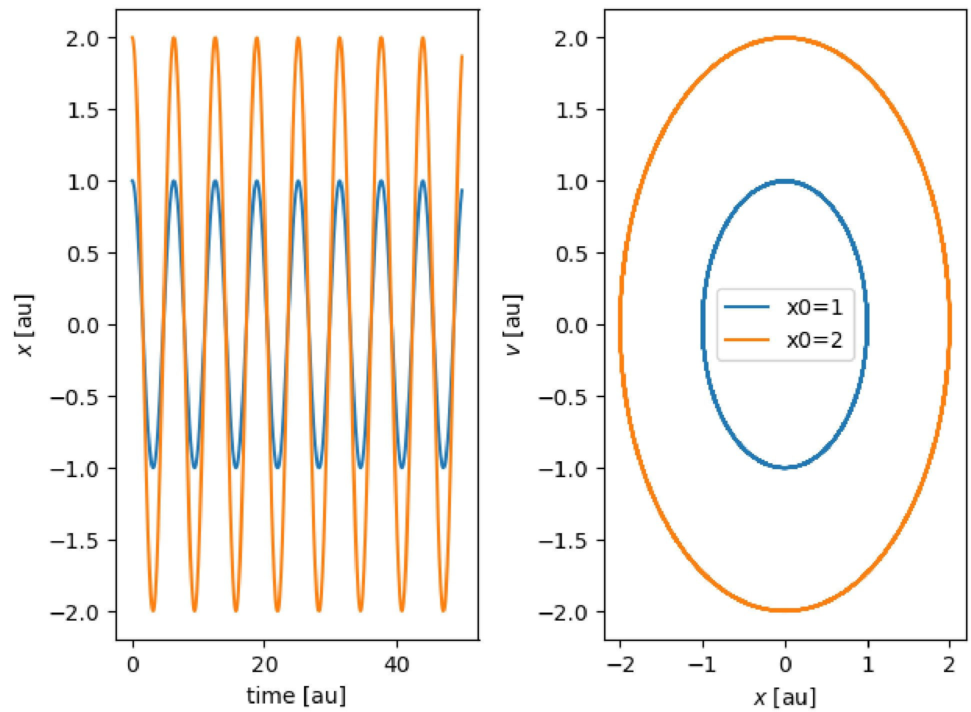

Figure 2: Solution (both position and phase space plots) of the harmonic problem with $\omega = 1$ and initial conditions $x = 1$ (blue line) or $x = 2$ (orange) and $v = 0$. We visually deduce that the period doesn't change when we double the amplitude of the oscillations.

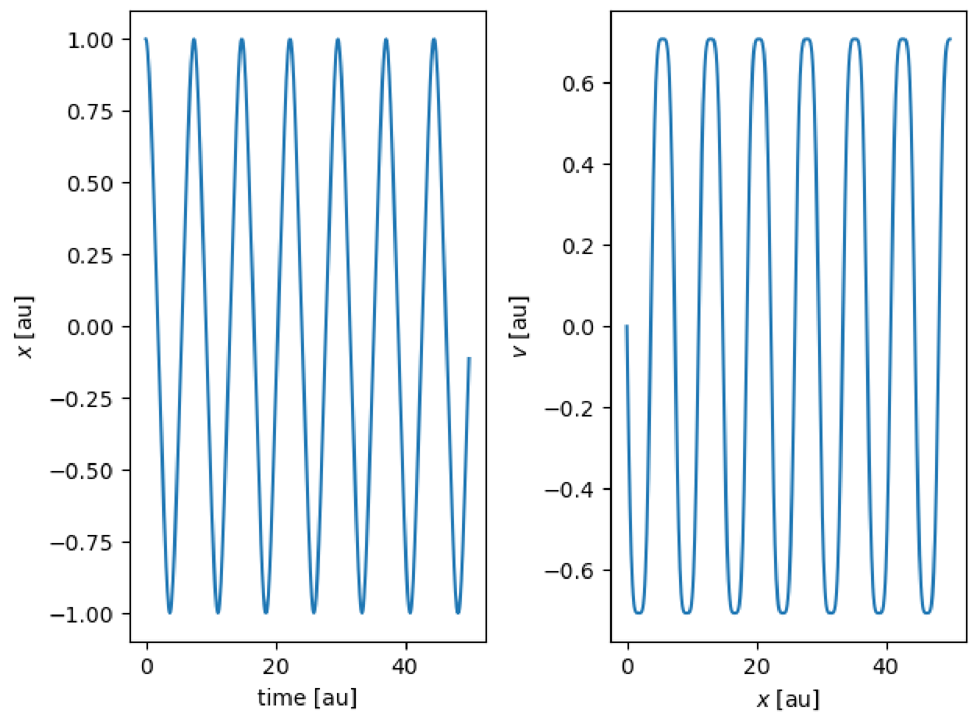

Figure 3: Solution (both position and phase space plots) of the harmonic problem with $\omega = 1$ and initial conditions $x = 1$ and $v = 0$.

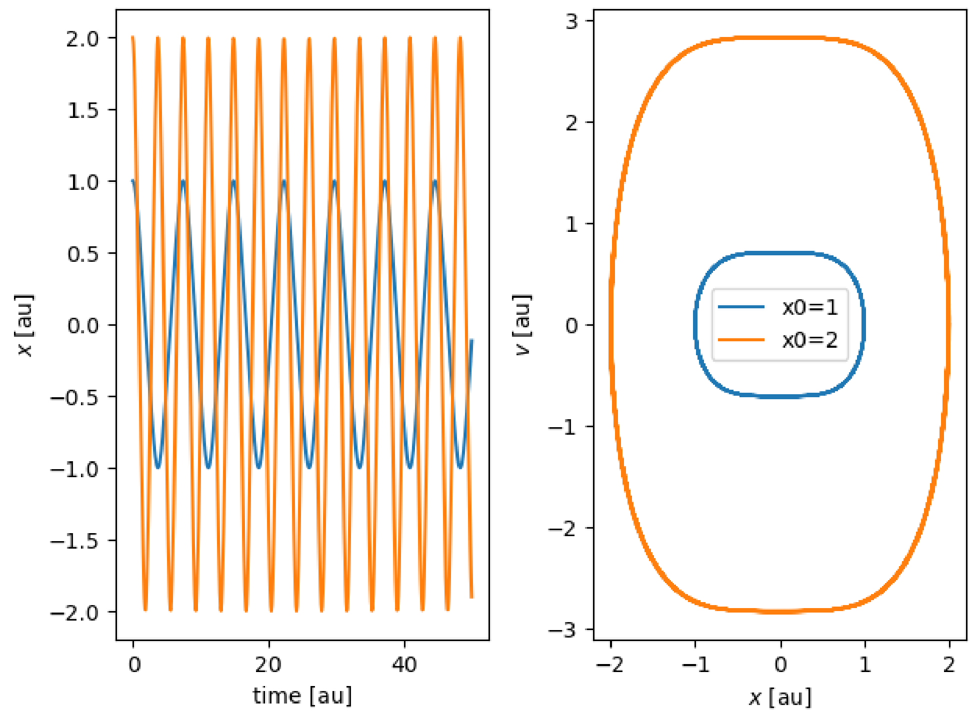

Figure 4: Solution (both position and phase space plots) of the harmonic problem with $\omega = 1$ and initial conditions $x = 1$ (blue line) or $x = 2$ (orange) and $v = 0$. We visually deduce that the oscillations get faster at higher amplitudes.

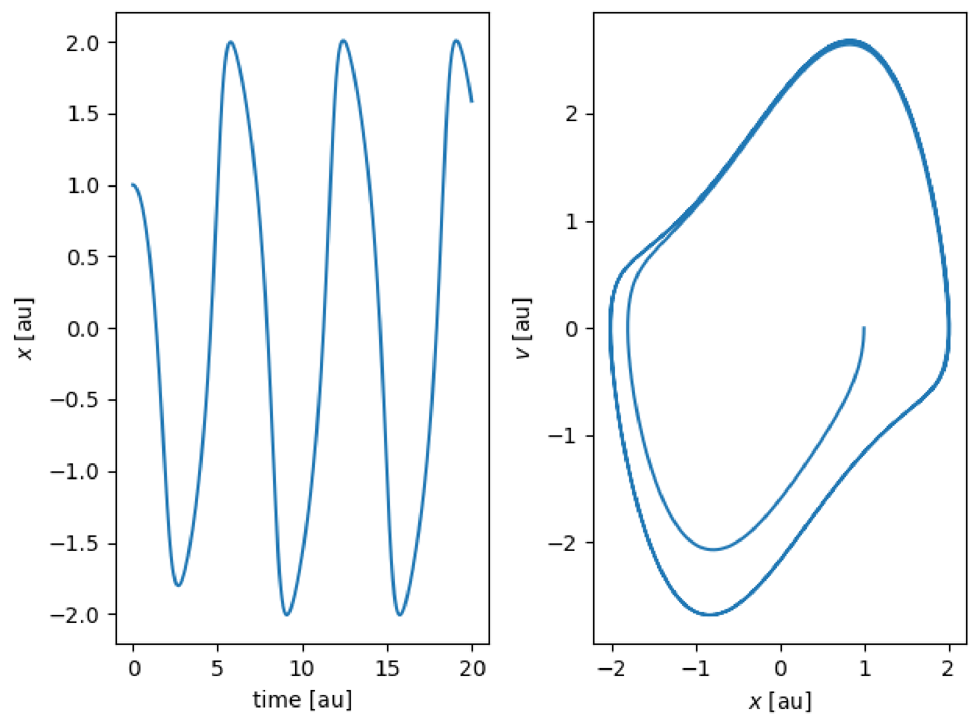

Figure 5: Solution (both position and phase space plots) of the van der Pol problem with $\omega = 1$, $\mu = 1$ and initial conditions $x = 1$ and $v = 0$.

comparison, we plotted the three different phase space trajectories on the same graph in fig. 8.

2 Problem 2

2.1 Formulation of the problem

We are asked to code a solver for the problem of computing the trajectory of a spherical projectile without neglecting air resistance:

$$F = -\frac{1}{2}\pi R^2 \rho C v^2 \hat{v} \tag{7}$$

where R is the sphere's radius, ρ is the density of air, v is the velocity, C is the so-called *coefficient of drag* and $\hat{v}$ denotes the velocity **versor** (same direction and verse but unit module).

The equation of motion can be deduced by Newton's law $F = ma$ (remembering to consider gravity $F_g = -mg\hat{y}$ by dividing both sides by m and projecting in the two directions $\hat{v} = \dot{x}/\sqrt{\dot{x}^2+\dot{y}^2}\hat{x} + \dot{y}/\sqrt{\dot{x}^2+\dot{y}^2}\hat{y}$:

$$\begin{cases} \ddot{x} = -\dfrac{\pi R^2 \rho C}{2m}\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)^2 \dot{x}/\sqrt{\dot{x}^2+\dot{y}^2} = -\dfrac{\pi R^2 \rho C}{2m}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2} \\ \ddot{y} = -g - \dfrac{\pi R^2 \rho C}{2m}\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)^2 \dot{y}/\sqrt{\dot{x}^2+\dot{y}^2} = -g - \dfrac{\pi R^2 \rho C}{2m}\dot{y}\sqrt{\dot{x}^2+\dot{y}^2} \end{cases} \tag{8}$$

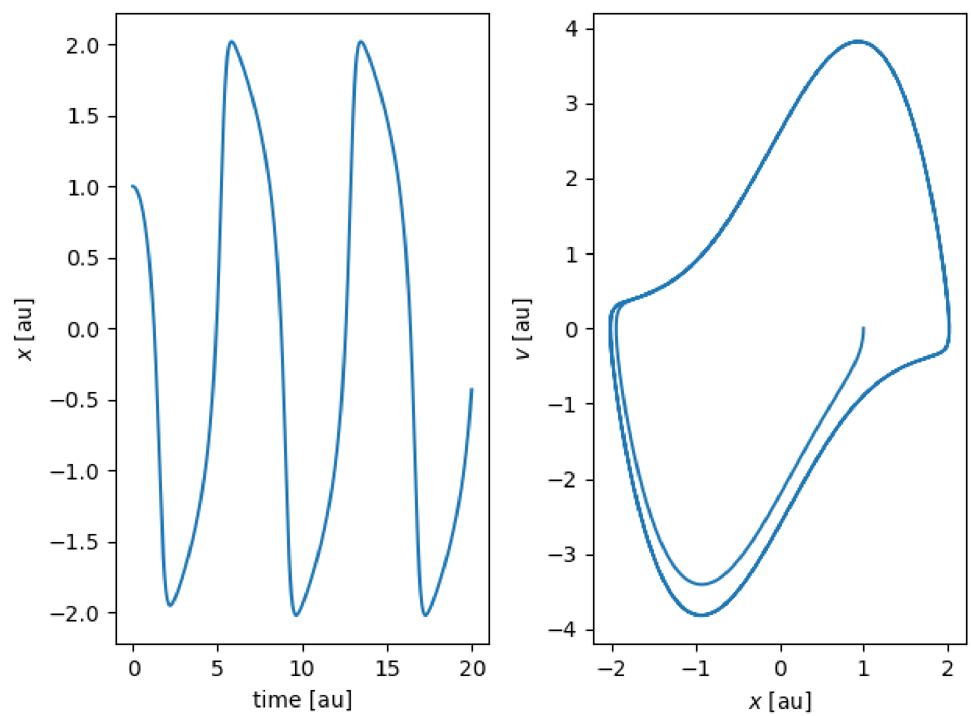

Figure 6: Solution (both position and phase space plots) of the van der Pol problem with $\omega = 1$, $\mu = 2$ and initial conditions $x = 1$ and $v = 0$.

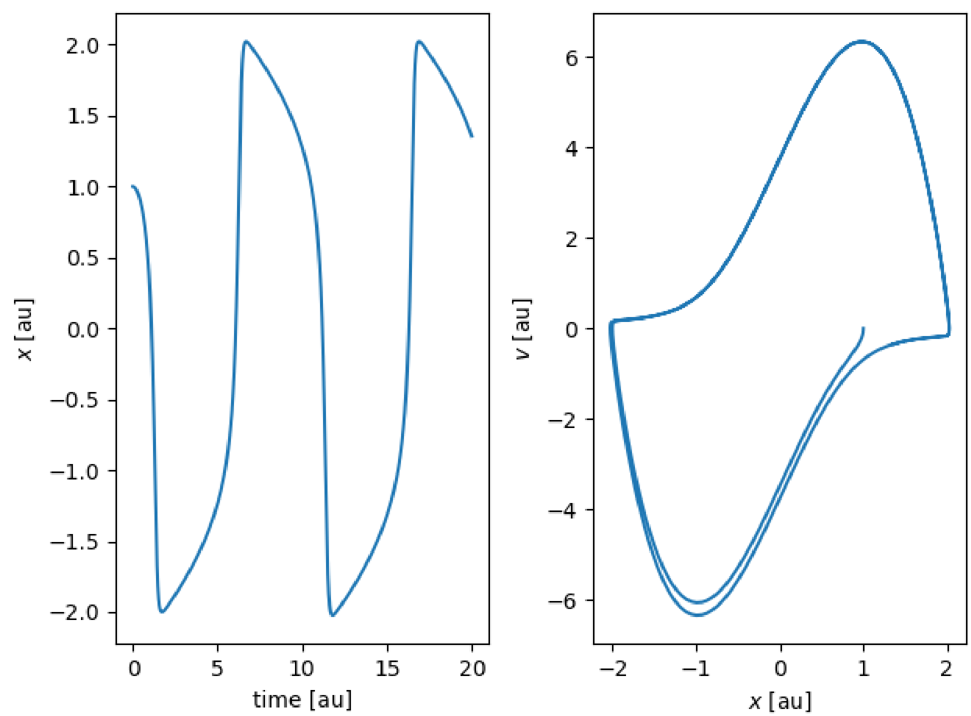

Figure 7: Solution (both position and phase space plots) of the van der Pol problem with $\omega = 1$, $\mu = 4$ and initial conditions $x = 1$ and $v = 0$.

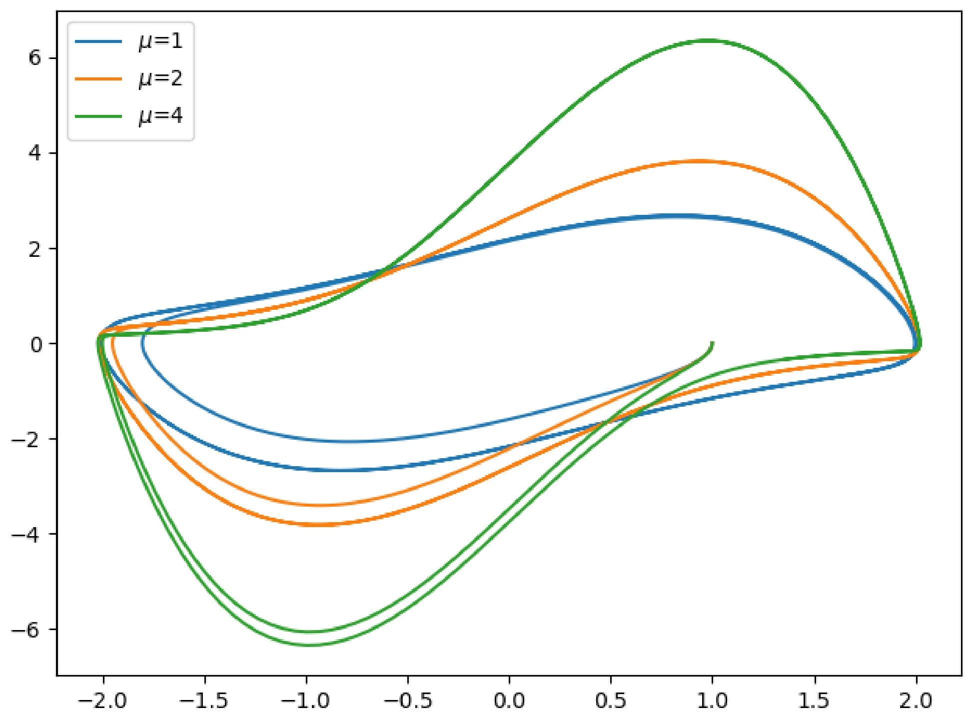

Figure 8: Phase space plot of the van der Pol problem with $\omega = 1$, $\mu \in \{1, 2, 4\}$ and initial conditions $x = 1$ and $v = 0$.

2.2 Computational methods

First, by rescaling the variables $t \to t'/T,\ x \to x'g/T^2$ we show that the problem is governed by the combination $A := R^2\rho C g T^2/m$:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} \to \frac{\mathrm{d}\big(x'g/T^2\big)}{\mathrm{d}(t'/T)} = \frac{\dot{x}'g}{T}, \quad \ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} \to \frac{\mathrm{d}(\dot{x}'g/T)}{\mathrm{d}(t'/T)} = \ddot{x}'g \tag{9}$$

which, can be plugged into eq. (8) to obtain:

$$\begin{cases} \ddot{x}'g = -\dfrac{\pi R^2\rho C}{2m}\Big(\dot{x}'\dfrac{g}{T^2}\Big)\sqrt{\Big(\dot{x}'\dfrac{g}{T}\Big)^2 + \Big(\dot{y}'\dfrac{g}{T}\Big)^2} \Rightarrow \ddot{x}' = -\dfrac{\pi}{2}A\dot{x}'\sqrt{(\dot{x}')^2 + (\dot{y}')^2} \\ \ddot{y}'g = -g - \dfrac{\pi R^2\rho C}{2m}\Big(\dot{y}'\dfrac{g}{T^2}\Big)\sqrt{\Big(\dot{x}'\dfrac{g}{T}\Big)^2 + \Big(\dot{y}'\dfrac{g}{T}\Big)^2} \Rightarrow \ddot{y}' = -1 - \dfrac{\pi}{2}A\dot{y}'\sqrt{(\dot{x}')^2 + (\dot{y}')^2} \end{cases} \tag{10}$$

where in the "⇒" step we divided both sides by g and recognized the definition of A. From here onward we drop the prime symbols.

Then, we rewrite the various problems in terms of four coupled first-order equations by defining $\dot{x} = v_x,\ \dot{y} = v_y$:

$$\begin{cases} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{v}_x = -\dfrac{\pi}{2}Av_x\sqrt{v_x^2 + v_y^2} \\ \dot{v}_x = -1 - \dfrac{\pi}{2}Av_x\sqrt{v_x^2 + v_y^2} \end{cases} \tag{11}$$

Finally, we solve the linearized system using the fourth-order Runge-Kutta method (applying slight modifications to the code present in the Jupiter notebook of the class).

2.3 Results

In fig. 9, we report the trajectory of a projectile with parameters:

$$m = 1\text{kg}, \quad \rho = 1,22\text{kg m}^{-3}, \quad R = 8\text{cm}, \quad C = 0.47, \tag{12}$$

and initial conditions:

$$x(0) = y(0) = 0, \quad v(0) = 100(\cos(30°)\hat{x} + \sin(30°)\hat{y})\text{m s}^{-1} \tag{13}$$

. We conclude that the projectile lands at:

$$x = 242.65\text{m} \tag{14}$$

We then study how the landing spot depends on the mass (by keeping all the other parameters and initial conditions as in eqs. (12) and (13)). We report our results in figs. 10 and 11: we see that the distance travelled increases with the mass.

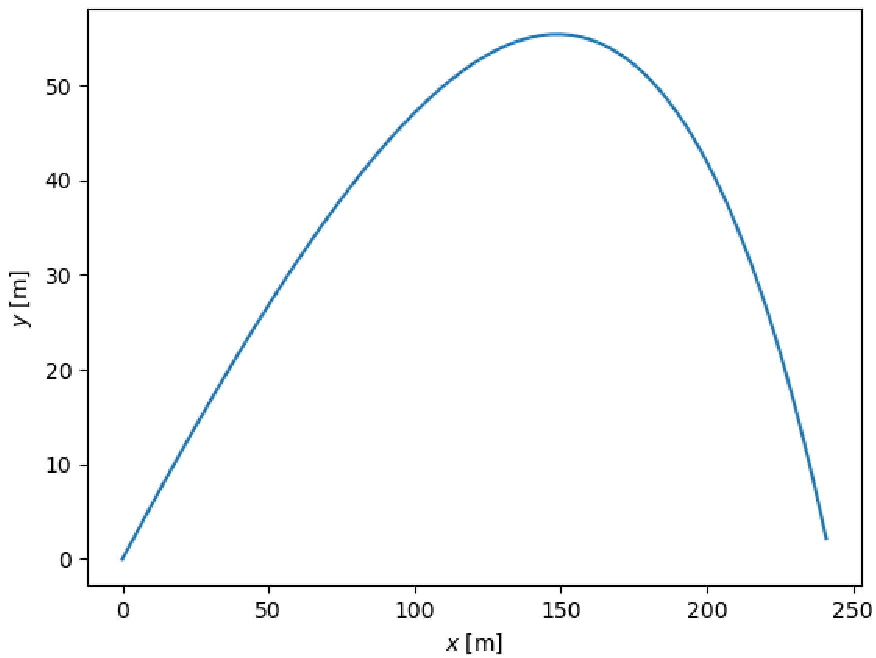

Figure 9: Trajectory of a projectile with parameters and initial conditions described by eqs. (12) and (13).

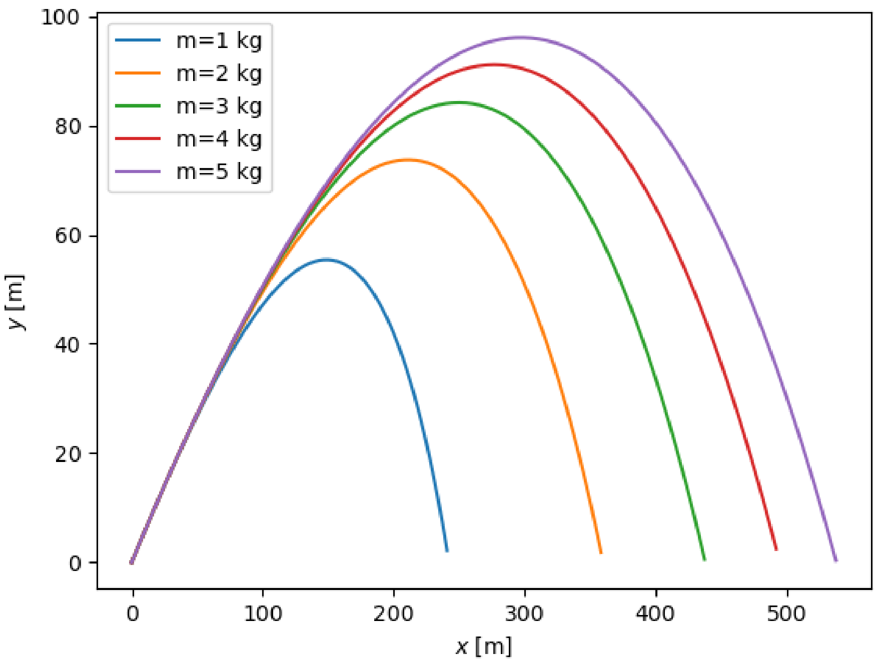

Figure 10: Plot of multiple projectiles with fixed parameters and initial conditions as in eqs. (12) and (13) except for a varying mass.

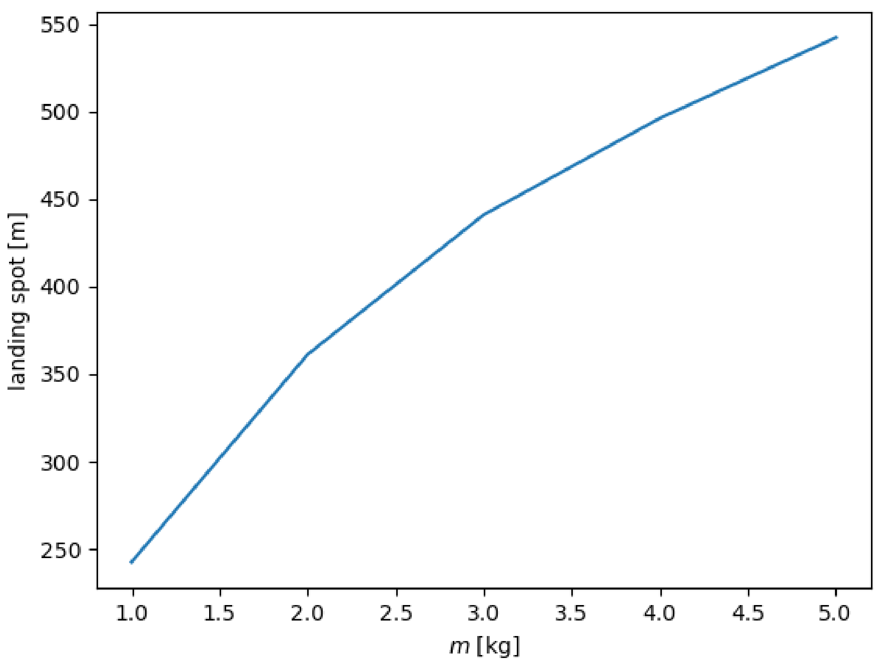

Figure 11: Plot of the landing spot dependence on the mass of the projectile (all the other parameters and initial conditions being those of eqs. (12) and (13))