NB.05.F1

July 23, 2024

1 Theorem

The variety $\mathcal{L}$ is an irreducible hypersurface.

Here we provide some computations that aid the proof of the result.

[1]:
```
load("basic_functions.sage")
```

[2]:
```
P1 = vector(S, (A1, B1, C1))
P2 = vector(S, (A2, B2, C2))
P3 = u1*P1+u2*P2
M = condition_matrix([P1, P2, P3], S, standard="all")
```

The following columns of M are linearly dependent: 0, 1, 2, 4, 5, 7 or 1, 2, 3, 5, 6, 8 or 4, 5, 6, 7, 8, 9. We can verify this directly, as follows:

[3]:
```
for cols in [
    [0, 1, 2, 4, 5, 7],
    [1, 2, 3, 5, 6, 8],
    [4, 5, 6, 7, 8, 9]
]:
    Ma = M.matrix_from_columns(cols)
    assert(
        (
            S.ideal(Ma.minors(6))
        ).is_zero()
    )
```

or we can see the dependencies of the columns in a more explicit way. We select the 10 columns of M:

[4]:
```
c = {}
for i in range(10):
    c[i] = M.matrix_from_columns([i])
```

We call α, β, γ the entries of $P_1 \times P_2$

[5]:
```
alpha, beta, gamma = tuple(wedge_product(P1, P2))
```

We call N_1 and N_2 the following matrices:

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \alpha & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

[6]:
```
N1 = matrix([[alpha, 0, 0], [0, beta, 0], [0, 0, gamma]])
N2 = matrix([[0, alpha, 0], [gamma, 0, 0], [0, 0, beta]])
```

Then we see that * c0, c1, c2, c4, c5, c7 * c1, c2, c3, c5, c6, c8 * c4, c5, c6, c7, c8, c9

are linearly dependend:

[7]:
```
L1 = c[0].augment(c[2]).augment(c[7])
L2 = c[1].augment(c[4]).augment(c[5])
assert(
    (
        (L1*N1 + 2*L2*N2)*wedge_product(P1, P2)
    ).is_zero()
)
```

[8]:
```
L1 = c[1].augment(c[3]).augment(c[8])
L2 = c[2].augment(c[5]).augment(c[6])
assert(
    (
        (L1*N1 + 2*L2*N2)*wedge_product(P1, P2)
    ).is_zero()
)
```

[9]:
```
L1 = c[4].augment(c[6]).augment(c[9])
L2 = c[5].augment(c[7]).augment(c[8])
assert(
    (
        (L1*N1 + 2*L2*N2)*wedge_product(P1, P2)
    ).is_zero()
)
```