NB.03.F2

July 23, 2024

1 Proposition

Let P_1, P_2, P_3, P_4 be four distinct points of the plane such that P_1, P_2, P_3 are aligned and let r be the line joining them.

If $\mathrm{rk}\,\Phi(P_1, P_2, P_3, P_4) \leq 7$ then r is tangent to the isotropic conic in one of the three points P_1, P_2, and P_3.

[2]:
```
load("basic_functions.sage")
```

We distinguish two cases: $P_1 = (1 : 0 : 0)$ and $P_1 = (1 : i : 0)$.

1.1 Case $P_1 = (1 : 0 : 0)$

We define four points, so that P_1, P_2, and P_3 are aligned.

[3]:
```
P1 = vector((1, 0, 0))
P2 = vector((A2, B2, C2))
P3 = u1*P1 + u2*P2
P4 = vector((A4, B4, C4))
```

We define the matrix of conditions of P_1, P_2, P_3 and P_4.

[4]:
```
M = condition_matrix([P1, P2, P3, P4], S, standard="all")
```

We compute the ideal of minors of order 8 of M.

[5]:
```
J8 = S.ideal(M.minors(8))
```

We saturate J_8 with respect to the conditions that the points P_1, P_2, P_3, and P_4 are distinct.

[8]:
```
J8 = J8.saturation(
    S.ideal(matrix(S, [P1, P2]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P1, P3]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P2, P3]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P1, P4]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P2, P4]).minors(2))
```

```
)[0].saturation(
    S.ideal(matrix(S, [P3, P4]).minors(2))
)[0]
```

We saturate J_8 with respect to the conditions that the points P_1, P_2, and P_4 are not aligned.

[11]:
```
J8 = J8.saturation(
    S.ideal(matrix(S, [P1, P2, P4]).det())
)[0]
```

The condition imposed by J_8 is equivalent to the one that the line joining P_1, P_2, and P_3 is tangent to the isotropic conic in one of the three points, namely, $\sigma(P_1, P_2) = 0$ and $\langle P_1, P_2 \rangle = \langle P_1, P_3 \rangle = 0$.

[20]:
```
assert(J8 == S.ideal(
    [
        sigma(P1, P2),
        scalar_product(P1, P3)*scalar_product(P1, P2)
    ]
))
```

1.2 Case $P_1 = (1 : i : 0)$

We define four points, so that P_1, P_2, and P_3 are aligned.

[21]:
```
P1 = vector((1, ii, 0))
P2 = vector((A2, B2, C2))
P3 = u1*P1 + u2*P2
P4 = vector((A4, B4, C4))
```

We define the matrix of conditions of P_1, P_2, and P_4.

[22]:
```
M = condition_matrix([P1, P2, P3, P4], S, standard="all")
```

We compute the ideal of minors of order 8 of M.

[23]:
```
J8 = S.ideal(M.minors(8))
```

We saturate J_8 with respect to the conditions that the points P_1, P_2, P_3, and P_4 are distinct.

[24]:
```
J8 = J8.saturation(
    S.ideal(matrix(S, [P1, P2]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P1, P3]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P2, P3]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P1, P4]).minors(2))
)[0].saturation(
    S.ideal(matrix(S, [P2, P4]).minors(2))
)[0].saturation(
```

```
    S.ideal(matrix(S, [P3, P4]).minors(2))
)[0]
```

We saturate J_8 with respect to the conditions that the points P_1, P_2, and P_4 are not aligned.

[25]:
```
J8 = J8.saturation(
    S.ideal(matrix(S, [P1, P2, P4]).det())
)[0]
```

The condition imposed by J_8 is equivalent to the one that the line joining P_1, P_2, and P_3 is tangent to the isotropic conic in one of the three points, namely, $\langle P_1, P_2 \rangle = 0$.

[28]:
```
assert(J8 == S.ideal(
    [
        scalar_product(P1, P2)
    ]
))
```