NB.04.F3

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1 Proposition

Let $P_1, \ldots, P_5$ be a V-configuration such that it holds

$$\delta_1(P_1, P_2, P_4) = \overline{\delta}_1(P_1, P_2, P_3) = \overline{\delta}_1(P_1, P_4, P_5) = 0$$

Then P_4 is orthogonal to $s_{11}\, P_2 - s_{12}\, P_1$ and one of the four conditions obtained by considering

$$P_3 = (s_{12}^2 + s_{11}s_{22})\, P_1 - 2s_{11}s_{12}\, P_2\,, \quad P_5 = (s_{14}^2 + s_{11}s_{44})\, P_1 - 2s_{11}s_{14}\, P_4\,.$$

and swapping in the latter formulas $2 \leftrightarrow 3$ and $4 \leftrightarrow 5$ holds.

To prove this result, we need to show that none of the following situations may happen: * $\langle P_1, P_1 \rangle = \sigma(P_1, P_2) = 0$ and $P_5 = (s_{14}^2 + s_{11}s_{44})\, P_1 - 2s_{11}s_{14}\, P_4$ * $\langle P_1, P_1 \rangle = \sigma(P_1, P_2) = 0$ and $P_4 = (s_{15}^2 + s_{11}s_{55})\, P_1 - 2s_{11}s_{15}\, P_5$ * $\langle P_1, P_1 \rangle = \sigma(P_1, P_4) = 0$ and $P_3 = (s_{12}^2 + s_{11}s_{22})\, P_1 - 2s_{11}s_{12}\, P_2$ * $\langle P_1, P_1 \rangle = \sigma(P_1, P_4) = 0$ and $P_2 = (s_{13}^2 + s_{11}s_{33})\, P_1 - 2s_{11}s_{13}\, P_3$

[5]:
```
load("basic_functions.sage")
```

We define five points, so that P_1, P_2, and P_3 are aligned and P_1, P_4, and P_5 are aligned.

[6]:
```
P1 = vector(S, (A1, B1, C1))
P2 = vector(S, (A2, B2, C2))
P3 = u1*P1 + u2*P2
P4 = vector(S, (A4, B4, C4))
P5 = v1*P1 + v2*P4
```

[7]:
```
S11 = scalar_product(P1, P1)
S12 = scalar_product(P1, P2)
S13 = scalar_product(P1, P3)
S14 = scalar_product(P1, P4)
S15 = scalar_product(P1, P5)
S22 = scalar_product(P2, P2)
S33 = scalar_product(P3, P3)
S44 = scalar_product(P4, P4)
S55 = scalar_product(P5, P5)
```

1.1 Proof that

1.2

$$\langle P_1, P_1 \rangle = \sigma(P_1, P_2) = 0 \textbf{ and } P_5 = (s_{14}^2 + s_{11}s_{44})\, P_1 - 2s_{11}s_{14}\, P_4$$

1.3 cannot happen.

[13]:
```
I = S.ideal(scalar_product(P1, P1), sigma(P1, P2))
m2 = matrix([(S14^2 + S11*S44)*P1 - 2*S11*S14*P4, P5]).minors(2)
J = S.ideal(m2)
```

[14]:
```
assert(
    (I + J).radical().saturation(
        S.ideal(list(P1))
    )[0].saturation(
        u1*u2*v1*v2
    )[0].saturation(
        matrix([P1, P2, P4]).det()
    )[0] == S.ideal(S.one())
)
```

1.4 Proof that

1.5

$$\langle P_1, P_1 \rangle = \sigma(P_1, P_2) = 0 \text{ and } P_4 = (s_{15}^2 + s_{11}s_{55})\, P_1 - 2s_{11}s_{15}\, P_5$$

1.6 cannot happen.

[15]:
```
I = S.ideal(scalar_product(P1, P1), sigma(P1, P2))
m2 = matrix([(S15^2 + S11*S55)*P1 - 2*S11*S15*P5, P4]).minors(2)
J = S.ideal(m2)
```

[16]:
```
assert(
    (I + J).radical().saturation(
        S.ideal(list(P1))
    )[0].saturation(
        u1*u2*v1*v2
    )[0].saturation(
        matrix([P1, P2, P4]).det()
    )[0] == S.ideal(S.one())
)
```

1.7 Proof that

1.8

$$\langle P_1, P_1 \rangle = \sigma(P_1, P_4) = 0 \text{ and } P_3 = (s_{12}^2 + s_{11}s_{22})\, P_1 - 2s_{11}s_{12}\, P_2$$

1.9 cannot happen.

[17]:
```
I = S.ideal(scalar_product(P1, P1), sigma(P1, P4))
m2 = matrix([(S12^2 + S11*S22)*P1 - 2*S11*S12*P2, P3]).minors(2)
J = S.ideal(m2)
```

[18]:
```
assert(
    (I + J).radical().saturation(
        S.ideal(list(P1))
    )[0].saturation(
        u1*u2*v1*v2
    )[0].saturation(
        matrix([P1, P2, P4]).det()
    )[0] == S.ideal(S.one())
)
```

1.10 Proof that

1.11

$$\langle P_1, P_1 \rangle = \sigma(P_1, P_4) = 0 \text{ and } P_2 = (s_{13}^2 + s_{11}s_{33})\, P_1 - 2s_{11}s_{13}\, P_3$$

1.12 cannot happen.

[19]:
```
I = S.ideal(scalar_product(P1, P1), sigma(P1, P4))
m2 = matrix([(S13^2 + S11*S33)*P1 - 2*S11*S13*P3, P2]).minors(2)
J = S.ideal(m2)
```

[20]:
```
assert(
    (I + J).radical().saturation(
        S.ideal(list(P1))
    )[0].saturation(
        u1*u2*v1*v2
    )[0].saturation(
        matrix([P1, P2, P4]).det()
    )[0] == S.ideal(S.one())
)
```