NB.07.F4

July 23, 2024

1 Configuration (C_6)

[2]:
```
load("basic_functions.sage")
```

1.1 Six points in configuration (C_6)

We assume the collinearities given by:

$(1, 2, 3), (1, 4, 5), (2, 4, 6), (3, 5, 6)$

We construct $P_2, \dots, P_5$, four generic points and we define P_1 and P_6 in such a way that $P_1, \dots, P_6$ are in configuration (C_6).

We verify that P_1 and P_6 are always be defined.

If config (C_6) is realizable by eigenpoints, then $\delta_1(P_2, P_3, P_4), \delta_1(P_3, P_2, P_5)$, $\delta_1(P_5, P_3, P_4), \delta_1(P_4, P_5, P_2)$, $\delta_1(P_1, P_3, P_5), \delta_1(P_6, P_4, P_5)$ must be zero.

We compute these polynomials and we simplify them, since two of them have some factors which are surely not zero.

[41]:
```
## we construct 6 points in configuration (6)
## P2, P3, P4, P5 are generic, so there are no collinearities among them.

P2 = vector(S, (A2, B2, C2))
P3 = vector(S, (A3, B3, C3))
P4 = vector(S, (A4, B4, C4))
P5 = vector(S, (A5, B5, C5))

## P1 and P6 are as follows:
P1 = vector(S, list(intersection_lines(P2, P3, P4, P5)))
P6 = vector(S, list(intersection_lines(P2, P4, P3, P5)))

# P1 and P6 are always defined:
J1 = S.ideal(list(P1))
J1 = J1.saturation(S.ideal(matrix([P2, P3]).minors(2)))[0]
J1 = J1.saturation(S.ideal(matrix([P4, P5]).minors(2)))[0]
J1 = J1.saturation(matrix([P2, P4, P5]).det())[0]
assert(J1 == S.ideal(1))
```

```
J1 = S.ideal(list(P6))
J1 = J1.saturation(S.ideal(matrix([P2, P3]).minors(2)))[0]
J1 = J1.saturation(S.ideal(matrix([P3, P5]).minors(2)))[0]
J1 = J1.saturation(matrix([P2, P4, P5]).det())[0]
assert(J1 == S.ideal(1))

d1 = delta1(P2, P3, P4)
d2 = delta1(P3, P2, P5)
d3 = delta1(P5, P3, P4)
d4 = delta1(P4, P5, P2)
d5 = delta1(P1, P3, P5)
d6 = delta1(P6, P4, P5)

## we simplify d1, ..., d6:
## we note indeed that delta1(P6, P4, P5) is divisible by
## det(matrix([P3, P4, P5]))*det(matrix([P2, P4, P5]))
## and  delta1(P1, P3, P5) is divisible by:
## det(matrix([P3, P4, P5]))*det(matrix([P2, P3, P5]))

assert(d6.quo_rem(det(matrix([P3, P4, P5]))*det(matrix([P2, P4, P5])))[1] == 0)
d6 = d6.quo_rem(det(matrix([P3, P4, P5]))*det(matrix([P2, P4, P5])))[0]

assert(d5.quo_rem(det(matrix([P3, P4, P5]))*det(matrix([P2, P3, P5])))[1] == 0)
d5 = d5.quo_rem(det(matrix([P3, P4, P5]))*det(matrix([P2, P3, P5])))[0]
```

Now we define the ideal J generated by $d_1, \dots, d_6$ and we saturate it with polynomials which cannot be zero, and we get the ideal (1), so (C_6) cannot be realized.

[42]:
```
J = S.ideal(d1, d2, d3, d4, d5, d6)
J = J.saturation(det(matrix([P2, P3, P4])))[0]
J = J.saturation(det(matrix([P2, P3, P5])))[0]
J = J.saturation(det(matrix([P3, P4, P5])))[0]

## we get that now J = (1):
assert(J == S.ideal(1))
```