NB.03.F1

July 23, 2024

1 Proposition

Let P_1, P_2, P_4 be three distinct points of the plane. Then: * $5 \leq \operatorname{rk}\Phi(P_1, P_2, P_4) \leq 6$; * if $\operatorname{rk}\Phi(P_1, P_2, P_4) = 5$, then P_1, P_2, P_4 are aligned and the line joining them is tangent to the isotropic conic in one of the three points.

[5]:
```
load("basic_functions.sage")
```

1.1 Proof of $\operatorname{rk}\Phi(P_1, P_2, P_4) \geq 5$

We distinguish two cases: $P_1 = (1 : 0 : 0)$ and $P_1 = (1 : i : 0)$.

1.1.1 Case $P_1 = (1 : 0 : 0)$

We define the three points.

[31]:
```
P1 = vector((1, 0, 0))
P2 = vector((A2, B2, C2))
P4 = vector((A4, B4, C4))
```

We define the matrix of conditions of P_1, P_2, and P_4.

[32]:
```
M = condition_matrix([P1, P2, P4], S, standard="all")
```

We compute the ideal of minors of order 5 of M.

[33]:
```
J5 = S.ideal(M.minors(5))
```

[34]:
```
J5 = J5.saturation(
    S.ideal(matrix([P1, P2]).minors(2))
)[0].saturation(
    S.ideal(matrix([P1, P4]).minors(2))
)[0].saturation(
    S.ideal(matrix([P2, P4]).minors(2))
)[0].radical()
```

J_5 is the ideal (1), so the matrix M cannot have rank < 5.

[35]:
```
assert(J5 == S.ideal(S.one()))
```

1.1.2 Case $P_1 = (1 : i : 0)$

We define the three points.

[6]:
```
P1 = vector((1, ii, 0))
P2 = vector((A2, B2, C2))
P4 = vector((A4, B4, C4))
```

We define the matrix of conditions of P_1, P_2, and P_4.

[51]:
```
M = condition_matrix([P1, P2, P4], S, standard="all")
```

We compute the ideal of minors of order 5 of M.

[52]:
```
J5 = S.ideal(M.minors(5))
```

[53]:
```
J5 = J5.saturation(
    S.ideal(matrix([P1, P2]).minors(2))
)[0].saturation(
    S.ideal(matrix([P1, P4]).minors(2))
)[0].saturation(
    S.ideal(matrix([P2, P4]).minors(2))
)[0].radical()
```

J_5 is the ideal (1), so the matrix M cannot have rank < 5.

[54]:
```
assert(J5 == S.ideal(S.one()))
```

1.2 Proof of $\operatorname{rk}\Phi(P_1, P_2, P_4) = 5$ if and only if P_1, P_2, and P_4 are aligned and the line joining them is tangent to the isotropic conic in one of the three points.

We distinguish two cases: $P_1 = (1 : 0 : 0)$ and $P_1 = (1 : i : 0)$.

1.2.1 Case $P_1 = (1 : 0 : 0)$

We define the three points.

[7]:
```
P1 = vector((1, 0, 0))
P2 = vector((A2, B2, C2))
P4 = vector((A4, B4, C4))
```

We define the matrix of conditions of P_1, P_2, and P_4.

[8]:
```
M = condition_matrix([P1, P2, P4], S, standard="all")
```

We compute the ideal of minors of order 6 of M.

[9]:
```
J6 = S.ideal(M.minors(6))
```

[12]:
```
J6 = J6.saturation(
    S.ideal(matrix([P1, P2]).minors(2))
)[0].saturation(
```

```
    S.ideal(matrix([P1, P4]).minors(2))
)[0].saturation(
    S.ideal(matrix([P2, P4]).minors(2))
)[0]
```

We compute the primary decomposition of J_6

[13]:
```
pd = J6.radical().primary_decomposition()
```

We claim we have only two possibilities: * either P_1, P_2, P_4 are aligned and the line is tangent to the isotropic conic in P_2 (hence P_2 orthogonal to P_4, P_2 orthogonal to P_2 and P_1, P_2, P_4 aligned):

[44]:
```
H6 = S.ideal(
    scalar_product(P1, P2),
    scalar_product(P2, P2),
    det(matrix([P1, P2, P4]))
).saturation(
    S.ideal(list(P2))
)[0].radical()
PD1 = H6.primary_decomposition()
```

- or P_1, P_2, P_4 are aligned and the line is tangent to the isotropic conic in P_4 (hence P_1 orthogonal to P_4, P_4 orthogonal to P_4 and P_1, P_2, P_4 aligned):

[48]:
```
K6 = S.ideal(
    scalar_product(P1, P4),
    scalar_product(P4, P4),
    det(matrix([P1, P2, P4]))
).saturation(
    S.ideal(list(P4))
)[0].radical()
PD2 = K6.primary_decomposition()
```

We check that indeed these are the only two possibilities.

[49]:
```
assert(Set(pd) == Set(PD1 + PD2))
```

1.2.2 Case $P_1 = (1 : i : 0)$

We define the three points.

[6]:
```
P1 = vector((1, ii, 0))
P2 = vector((A2, B2, C2))
P4 = vector((A4, B4, C4))
```

We define the matrix of conditions of P_1, P_2, and P_4.

[51]:
```
M = condition_matrix([P1, P2, P4], S, standard="all")
```

We compute the ideal of minors of order 5 of M.

[ ]:
```
J6 = S.ideal(M.minors(6))
```

[ ]:
```
J6 = J6.saturation(
    S.ideal(matrix([P1, P2]).minors(2))
)[0].saturation(
    S.ideal(matrix([P1, P4]).minors(2))
)[0].saturation(
    S.ideal(matrix([P2, P4]).minors(2))
)[0].radical()
```

When J_6 is satisfied, we have that P_1, P_2, P_4 are aligned and the line $P_1 \vee P_2 \vee P_4$ is tangent to the isotropic conic in P_4.

[67]:
```
K6 = S.ideal(
    scalar_product(P1, P2),
    scalar_product(P1, P4),
    matrix([P1, P2, P4]).det()
).saturation(
    S.ideal(list(P4))
)[0].radical()
```

[68]:
```
assert(J6 == K6)
```