OU-processes: mathematical notes and correlated spot-forward model

Matteo Gardini*

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Abstract

Some notes on OU-processes and martingale conditions. Also a small insight on spot price model correlated to a forward price model. Concise, no verbose things and somehow informal.

Keywords: Ornstein-Uhlenbeck processes, martingales, Itō's calculus

1 Solution of Ornstein-Uhlenbeck process

$$dX(t) = \theta\left(\mu - X(t)\right)dt + \sigma dW(t).$$

Consider $f(X(t), t) = X(t)e^{\theta t}$ and apply Itō's lemma:

$$\begin{aligned} df(X(t), t) &= \theta e^{\theta t} X(t) dt + e^{\theta t} dX(t) \\ &= e^{\theta t}\left[\theta X(t)dt + \theta\left(\mu - X(t)\right)dt + \sigma dW(t)\right] \\ &= e^{\theta t}\left[\theta \mu dt + \sigma dW(t)\right]. \end{aligned}$$

Integrate between s and t and you get:

$$X(t)e^{\theta t} - X(s)e^{\theta s} = \int_s^t \theta \mu e^{\theta z} dz + \int_s^t e^{\theta z} \sigma dW(z),$$

solve it:

$$X(t) = \mu + \left[X(s) - \mu\right] e^{-\theta(t-s)} + \sigma \int_s^t e^{-\theta(t-z)} dW(z).$$

From this it is immediate that **OU process is not a martingale**.

$$\mathbb{E}\left[X(t)|\mathcal{F}(s)\right] = \mu + \left[X(s) - \mu\right] e^{-\theta(t-s)} \neq X(s).$$

*For comments and suggestions: matteo.gardini88@gmail.com

Nothing prevents you from defining a centered version of the OU process. Find $h(t)$ such that $\mathbb{E}\left[X(t)+h(t)\right]=0$.

$$\mathbb{E}\left[X(t)+h(t)\right]=\mu+\left[X(0)-\mu\right]e^{-\theta t}+h(t)=0,$$

hence $h(t)=\left[\mu-X(0)\right]e^{-\theta t}-\mu$. With this choice the expected value of X is constant over time but the process is not a martingale.

It is easy to construct a process with zero mean but which is not a martingale. Take Z a *rv* with Bernoulli law and $\mathbb{P}\left[Z=1\right]=\mathbb{P}\left[Z=-1\right]=\frac{1}{2}$ and define $X(t)=tZ$. Clearly $\mathbb{E}\left[X(t)\right]=0$ but if you choose $\mathcal{F}(s)=\sigma(Z)$ for all $s\geq 0$ you have:

$$\mathbb{E}\left[X(t)|\mathcal{F}(s)\right]=tZ=X(t)\neq X(s)=sZ,$$

and hence X is not a martingale even if it has constant expected value.

2 Simulation of the Ornstein-Uhlenbeck process

Example 2.1. *Consider the process:*

$$X(t)=\mu+\left[X(0)-\mu\right]e^{-\theta t}+\sigma\int_0^t e^{-\theta(t-z)}dW(z).$$

with $X(0)=0$ and $\mu=0$:

$$X(t)=\sigma\int_0^t e^{-\theta(t-z)}dW(z).$$

Set:

$$S(t)=\exp\left\{X(t)\right\}$$

and simulate the path of S on a equispaced time grid with $\Delta t=t_{i+1}-t_i$ as $S(t_{i+1}+\Delta t)$ from the value of $S(t_i)=s$.

A first way in which you can simulate $S(t)$ is by simulating $X(t)$ and hence taking the exponential: this trivially works by definition.

But you might also want to simulate $S(t+\Delta t)$ from $S(t)$ as is typically done with the geometric Brownian motion with drift μ and volatility σ:

$$S(t+\Delta t)=S(t)\exp\left\{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z\right\},$$

with $Z\sim\mathcal{N}(0,1)$. What does it mean? It simply means that you have to find the conditional distribution of $S(t+\Delta t)$ given $S(t)=s$: this is the meaning of the empirical sentence "simulate the next step of $S(t)$ from its previous value".

To find $S(t+\Delta t)|S(t)=s$ we investigate $X(t+\Delta t)|X(t)=\log s$ and from the previous section we have that:

$$X(t+\Delta t)=X(t)e^{-\theta\Delta t}+\sigma\int_t^{t+\Delta t}e^{-\theta(t+\Delta t-z)}dW(z). \tag{1}$$

and hence:

$$X(t+\Delta t)|X(t) = \log s \sim \mathcal{N}\left(e^{-\theta\Delta t}X(t), \frac{\sigma^2}{2\theta}\left(1-e^{-2\theta\Delta t}\right)\right).$$

Now:

$$\begin{aligned}S(t+\Delta t) - S(t) &= e^{X(t+\Delta t)} - e^{X(t)} = e^{X(t)+X(t+\Delta t)-X(t)} - e^{X(t)} \\ &= e^{X(t)}\left[e^{X(t+\Delta t)-X(t)} - 1\right] = S(t)\left[e^{X(t+\Delta t)-X(t)} - 1\right]\end{aligned}$$

which leads to:

$$S(t+\Delta t) = S(t)e^{X(t+\Delta t)-X(t)}$$

Manipulating Equation (1) we have:

$$X(t+\Delta t) - X(t) = X(t)\left[e^{-\theta\Delta t} - 1\right] + \sigma\int_t^{t+\Delta t} e^{-\theta(t+\Delta t - z)}dW(z),$$

and hence:

$$X(t+\Delta t) - X(t) \sim \mathcal{N}\left(X(t)\left[e^{-\theta\Delta t}-1\right], \frac{\sigma^2}{2\theta}\left(1-e^{-2\theta\Delta t}\right)\right) \tag{2}$$

The numerical scheme is:

$$S(t+\Delta t) = S(t)\exp\left\{\log S(t)\left[e^{-\theta\Delta t}-1\right] + \sqrt{\frac{\sigma^2}{2\theta}\left(1-e^{-2\theta\Delta t}\right)}\,Z\right\}$$

where $Z \sim \mathcal{N}(0,1)$. The variance of $S(t)$ can be computed, to check that the simulation scheme has been properly implemented: see Figure 1.

$$Var\left[S(t)\right] = S(0)^2\left[\exp\{\eta(t)\} - 1\right]\exp\{\eta(t)\}$$

where:

$$\eta(t) = \frac{\sigma^2}{2\theta}\left(1-e^{-2\theta t}\right).$$

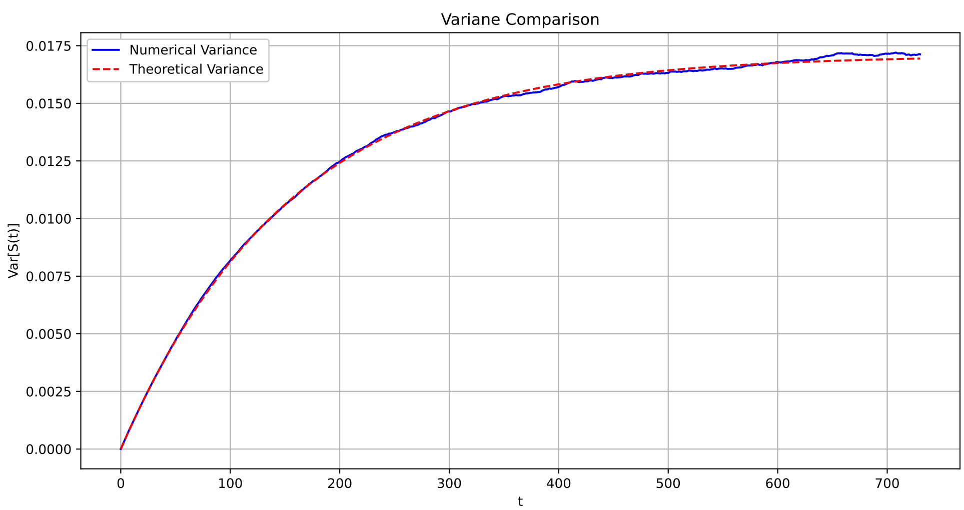

Figure 1: Numerical comparison of theoretical and numerical variance. $\theta = 1.2$ and $\sigma = 0.2$.

3 A possible model for spot price

Here we depict a possible model for correlated spot/forward prices in energy markets. This aims not to be a detailed discussion but just a simple sketch.

3.1 Stochastic Leibniz integral rule

The deterministic formula

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

hols also (under certain hypothesis) in a stochastic setting. In particular by setting:

$$g(t) = \int_0^t f(s,t)dW(s),$$

we get:

$$dg(t) = \left(\int_0^t \frac{\partial f}{\partial t}(s,t)dW(s)\right)dt + f(t,t)dW(t).$$

3.2 Some mathematics about the spot price model

Define the stochastic process for prices

$$S(t) = \exp\left\{h(t) + \sigma \int_0^t e^{-\theta(t-u)}dW(u)\right\}.$$

By choosing $h(t) = \frac{\sigma^2}{4\theta}\left(1 - e^{-2\theta t}\right)$ we obtain that $\mathbb{E}\left[S(t)\right] = 1$ for all $t \geq 0$.

Now the question is: is $S = \{S(t); t \geq 0\}$ a martingale?
Let $X(t) = \sigma \int_0^t e^{-\theta(t-u)}dW(u)$ its differential is given by:

$$dX(t) = \sigma dW(t) - \left(\sigma\theta \int_0^t e^{-\theta(t-u)}dW(u)\right)dt,$$

and hence:

$$\begin{aligned} dS(t) &= S(t)\left(h'(t)dt + dX(t) + dX(t)^2\right) \\ &= S(t)\left(h'(t)dt + \sigma dW(t) - \theta X(t)dt + \frac{1}{2}\sigma^2 dt\right) \\ &= S(t)\left(\left(h'(t) - \theta X(t) + \frac{1}{2}\sigma^2\right)dt + \sigma dW(t)\right), \end{aligned}$$

which is not a martingale because of the non zero drift term.

The model we propose is similar to this one so it is possible that the spot price in this case is not a martingale.

3.3 The spot model

Assume to be able to model the forward price $F(t,T)$ of the rolling product $M+1$ as follows:

$$F(t,T) = F(t_0,T)Z(t,T)$$

where $Z(t,T)$ is a positive martingale such that $\mathbb{E}\left[Z(t,T)|\mathcal{F}(0)\right] = 1$. A common choice for $Z(t,T)$ would be $Z(t,T) = \exp\left\{-\frac{1}{2}\sigma^2 t + W(t)\right\}$, but we do not assume this. We assume that (somehow the forward simulations are given from, for example, the model proposed by Gardini and Santilli [2]).

Here the idea is simple: create a process $S(t)$ which logarithm reverts toward $\log(Z(t,T))$ and such that the average is equal to one. Multiply that process for your value on the delivery period $[\tau_s, \tau_e]$ and get spot simulations which are correlated to $F(t,T)$, by construction.

We need a spot process to be positive (this is the reason why we model the logarithm) and hence we define for $t_0 \le u \le t \le T$:

$$\log S(t) = \left(\log S(u) - \log Z(u,T)\right) e^{-a(t-u)} + \log Z(t,T) + \sigma_s \int_u^t e^{-a(t-u)} dW(u),$$

where the integral term distributes as $\mathcal{N}\left(0, \bar{\sigma}^2(t)\right)$ where:

$$\bar{\sigma}^2(t) = \frac{\sigma_s^2}{2a}\left(1 - e^{-2a(t-u)}\right).$$

We impose: $\log S(t_0) = \log F(t_0,T) = 0$ and we define:

$$Y(t_1) = \log Z(t_1,T) + \bar{\sigma}(t)Z_1$$

where $Z_1 \sim \mathcal{N}(0,1)$. Finally:

$$S(t_1) = \exp\left\{Y(t_1)\right\}$$

Once again, we can set a drift corrector $h(t)$ such that $E\left[S(t)\right] = 1$ by taking:

$$\mathbb{E}\left[e^{\log Z(t_1,T) + \bar{\sigma}(t)Z_1 + h(t)}\right] = 1.$$

Hence we get: $h(t) = -\frac{\bar{\sigma}^2(t)}{2}$.

Given the simulation of the forward price $F(t,T)$ and hence the simulation of $Z(t,T)$ I have created a mean reverting process reverting to the martingale $Z(t,T)$. It is a sort of mean reverting process reverting to a stochastic quantity given by $Z(t,T)$. Thanks to the drift correction it is guaranteed that $\mathbb{E}\left[S(t)\right] = 1$ and hence I can center the simulation on the desired level, for example on $F(t,\tau_s,\tau_e)$ namely the level of the forward price delivery between τ_s and τ_e.

We suggest you to generate daily shocks (i.e. to simulate the average daily price) for hourly power spot simulations (as usual) and hence introduce a hourly shape to re-scale the simulation to hourly granularity.

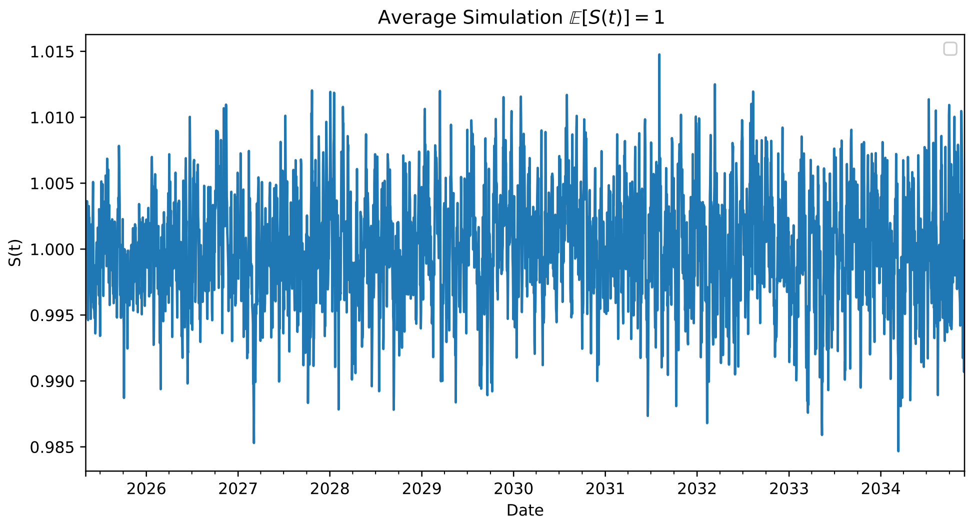

Figure 2: Simulation of $S(t)$ (before centering on a given level) and check that $\mathbb{E}\left[S(t)\right] = 1$ by using the drift-correct $h(t)$.

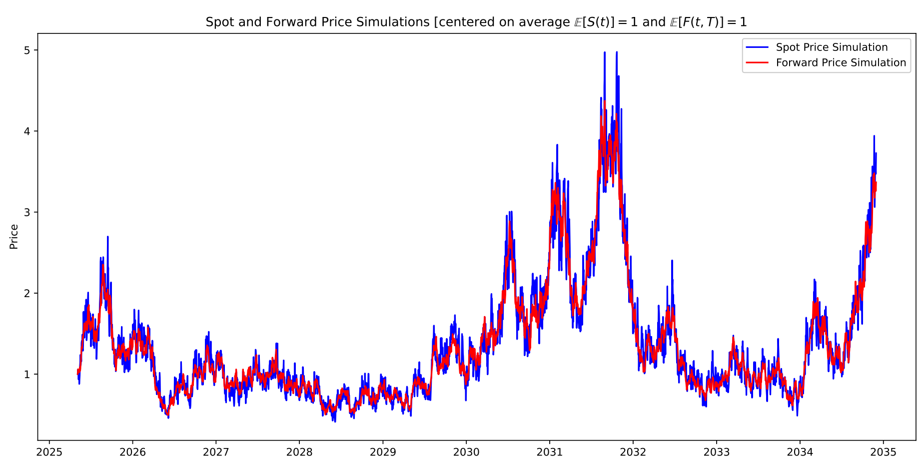

Figure 3: Daily spot and forward prices before centering o a given level (namely: $\mathbb{E}\left[S(t)\right] = 1$ and $\mathbb{E}\left[F(t,T)\right] = 1$): a realization of spot prices correlated to the one of the forward price $F(t,T)$, obtained via the Heath-Jarrow-Morton model proposed by Benth et al. [1] and Gardini and Santilli [2].

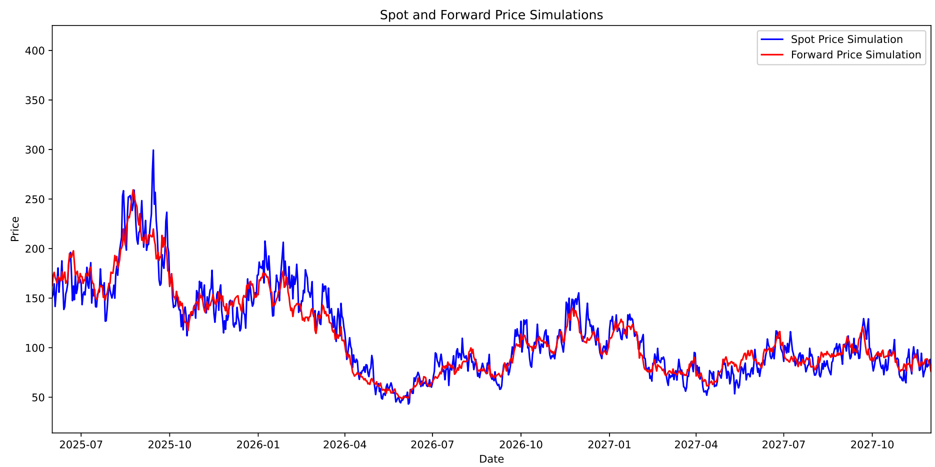

Figure 4: Daily spot and forward prices: a realization of spot prices correlated to the one of the forward price $F(t, T)$: obtained via the Heath-Jarrow-Morton model by Benth et al. [1] and Gardini and Santilli [2].

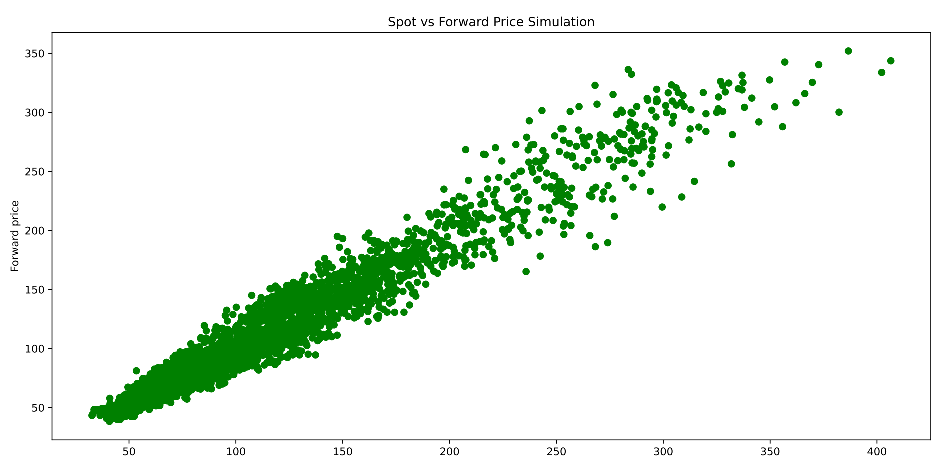

Figure 5: Scatter plot of spot and forward prices. We note that there is a strong dependence between spot and forward prices for the month-ahead product $F(t, T)$.

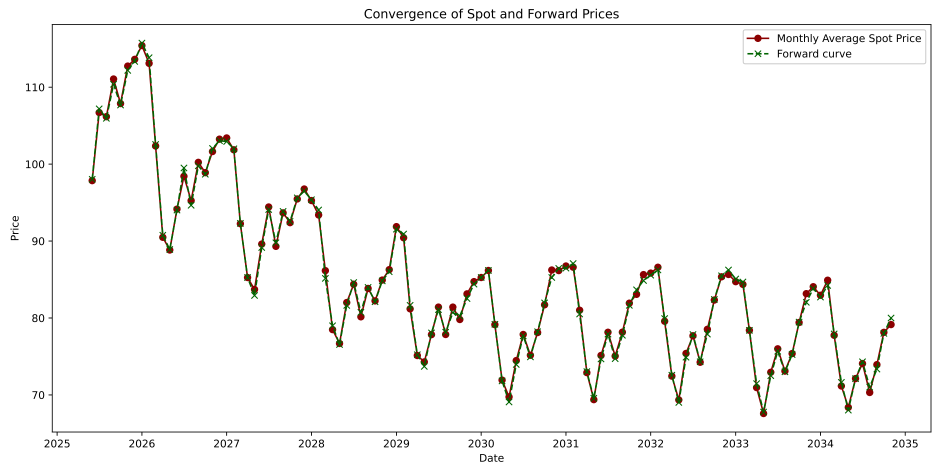

Figure 6: Convergence of monthly spot price $S(t)$ to the monthly forward curve $F(t, \tau_s, \tau_e)$.

If Figure 6 we show that the average of spot price (in this case the monthly spot price) converges to the monthly forward curve (which is the same that was used to simulate the monthly forward price by the model proposed by Benth et al. [1] and Gardini and Santilli [2]). This numerically checks the usual relation given between spot and forward prices (see Benth et al. [1]):

$$\mathbb{E}\left[S(t)\right] = F(t, \tau_s, \tau_e).$$

References

[1] F.E. Benth, J.S. Benth, and S. Koekebakker. *Stochastic Modeling of Electricity and Related Markets*. Number 6811 in World Scientific Books. World Scientific Publishing Co. Pte. Ltd., July 2008. ISBN ARRAY(0x50724998). URL `https://ideas.repec.org/b/wsi/wsbook/6811.html`.

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