The Variance Gamma ++ Process

Matteo Gardini[*] Piergiacomo Sabino[†] Emanuela Sasso[‡]

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Abstract

All the reported results are instrumental to the study of the process we call Variance Gamma++ process.

Tutti i risultati sono ricavati con la prametetrizzazione delle densità nel senso più generale possibile, ovvero senza richiedere sui subordinatori G siano tali che $\mathbb{E}\left[G(t)\right] = t$. Potremmo imporla in un secondo momento, magari quando parliamo delle applicazioni finanziarie.

1 Introduction

In this section we give a complete characterization of a new Lévyprocess that we call the Variance Gamma ++ ($VG++$) process. The Variance Gamma process presented by Madan and Seneta [15] is the Lévy process obtained by subordinating a Brownian motion with a Gamma process. The process we obtain has nice mathematical properties: for instance the Lévy measure, the density and the characteristic functions are available in a closed form. All these facts allows us to use some very efficient numerical techniques to price derivatives within the Variance Gamma model.

A very interesting result is that the Gamma law is self-decomposable as was shown by Grigelionis [8]: from an intuitive point of view, this means that the random variable can be decomposed as the "weighted sum" of two independent random variables where one of them has the same law of the original random variable whereas the other one has a law that has to be found but it can be proved that this law is infinitely divisible (Sato [18, Proposition 15.5])). This means that we can build the Lévy process associated to the law of this infinitely divisible random variable. As will become clear later, the process we obtain has almost surely increasing paths and hence it is what is called a subordinator. It follows that such a process can be used for Brownian subordination and this leads us to define a new Lévy process that we will call $VG++$ process.

The goal of this report is to give a full characterization of the this new process and to show a possible application to finance.

[*]Department of Mathematics, University of Genoa, Via Dodecaneso 16146, Genoa, Italy, email gardini@dima.unige.it

[†]Quantitative Risk Management, E.ON SE, Brüsseler Platz 1, 45131 Essen, Germany, email piergiacomo.sabino@eon.com

[‡]Department of Mathematics, University of Genoa, Via Dodecaneso 16146, Genoa, Italy, email sasso@dima.unige.it

2 The $VG++$ process

In this section we give a series of results that fully characterize the $VG++$ process.

2.1 Self-decomposable laws

Definition 2.1. *A random variable X is said to have a self decomposable law if for all $a \in (0,1)$ there exist a random variable Y with the same law of X and a random variable Z_a independent of Y such that*

$$X \stackrel{d}{=} aY + Z_a.$$

We call Z_a the a-remainder.

If we denote by $\phi_X(u)$ the characteristic function of X and by $\phi_{Z_a}(u)$ the one of Z_a we have that:

$$\phi_X(u) = \phi_X(au)\,\phi_{Z_a}(u)\,. \tag{1}$$

It can be proved that the law of the a-remainder of a self-decomposable random variable is infinitely divisible (see Sato [18, Proposition 15.5]). For this reason we can construct the process Lévy process $Z_a = \{Z_a(t); t \geq 0\}$ associated to the law of the a-remainder.

2.2 The process Z_a

It can be shown that the law of Gamma random variables with parameters $\alpha > 0$ and $\beta > 0$ is self-decomposable (see Grigelionis [8]) and hence the law of its a-remainder Z_a is infinitely divisible.

Definition 2.2. *Let X be a random variable with a Gamma distribution of parameter $\alpha > 0$ and $\beta > 0$ and we write $X \sim \Gamma(\alpha, \beta)$. We say that Z_a has a Gamma++ law and we write $Z_a \sim \Gamma^{++}(\alpha, \beta)$.*

Proposition 2.1. *DA RIMUOVERE: discende dalla definizione. If $X \sim \Gamma(\alpha, \beta)$, then the characteristic function of Z_a is given by:*

$$\phi_{Z_a}(u) = \left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha}. \tag{2}$$

Proof. Using Equation (16) we have that if $X \sim \Gamma(\alpha, \beta)$ then its characteristic function is given by:

$$\phi_X(u) = \left(\frac{\beta}{\beta - iu}\right)^{\alpha}.$$

Using Equation (1) we obtain:

$$\phi_{Z_a}(u) = \frac{\left(\frac{\beta}{\beta - iu}\right)^{\alpha}}{\left(\frac{\beta}{\beta - iua}\right)^{\alpha}} = \left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha}.$$

$\blacksquare$

Starting from the characteristic function we can retrieve the moment generating function and hence we can compute all the moments of Z_a. In particular its mean the variance are given by:

$$\mathbb{E}\left[Z_a\right] = (1-a)\,\frac{\alpha}{\beta} \qquad Var\left[Z_a\right] = \left(1-a^2\right)\frac{\alpha}{\beta^2}.$$

These expression are very useful to check the correctness of a possible simulation algorithm or for parameter estimation purposes using, for example, the generalized method of moments.

Before proceeding to obtain the probability density function of Z_a we define what we mean by Polya distribution is and we state a useful Lemma (see Sabino and Cufaro-Petroni [17]).

Definition 2.3. *A discrete random variable S is said to be be distributed according to a Polya (or Negative Binomial) with parameters $\alpha > 0$ and $p \in (0,1)$, if its probability mass function has the following form:*

$$\mathbb{P}\left(\{S=k\}\right) = \binom{\alpha+k-1}{k}(1-p)^{\alpha}\,p^{k}, \qquad k=0,1,\ldots$$

where:

$$\binom{\alpha}{k} = \frac{\alpha\,(\alpha-1)\ldots(\alpha-k+1)}{k!}, \qquad \binom{\alpha}{0} = 1.$$

We write $S \sim \overline{\mathfrak{B}}\left(\alpha,p\right)$.

Lemma 2.2. *Consider a discrete random variable $S \sim \overline{\mathfrak{B}}\left(\alpha,p\right)$ and a sequence $\{X_i\}_{i\in\mathbb{N}}$ of iid random variables, independent of S, with characteristic function $\phi_X\left(u\right)$, for $u \in \mathbb{R}$. Define a new random variable:*

$$Z = \sum_{i=1}^{S} X_i,$$

and $Z=0$ whenever $S=0$. The characteristic function of Z is given by:

$$\phi_Z\left(u\right) = \left(\frac{1-p}{1-p\phi_X\left(u\right)}\right)^{\alpha}. \tag{3}$$

Proof. . Togliere:già presente in Sabino and Cufaro-Petroni [17].

$$\begin{aligned}
\mathbb{E}\left[e^{iuZ}\right] &= \mathbb{E}\left[e^{iu\sum_{i=1}^{S}X_i}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iu\sum_{i=1}^{s}X_i}\,\middle|\, S=s\right]\right] \\
&= \mathbb{E}\left[\phi_X\left(u\right)^{S}\right] = \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}(1-p)^{\alpha}\,p^{k}\phi_X\left(u\right)^{k} \\
&= (1-p)^{\alpha}\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\left(p\phi_X\left(u\right)\right)^{k} \\
&= (1-p)^{\alpha}\,\frac{1}{\left(1-p\phi_X\left(u\right)\right)^{\alpha}} = \left(\frac{1-p}{1-p\phi_X\left(u\right)}\right)^{\alpha}.
\end{aligned}$$

Observing that:

$$|z| = |p\phi_X(u)| \le p\,|\phi_X(u)| \le p < 1.$$

we applied the relation:

$$\sum_{n=0}^{\infty} \binom{n+\alpha}{n} z^n = \frac{1}{(1-z)^{\alpha+1}}, \qquad z \in \mathbb{C},\ |z| < 1,$$

to $z = p\phi_X(u)$. ■

Corollary 2.3. *If $X \sim \Gamma(\alpha, \beta)$, then the a-remainder*

$$Z_a \stackrel{d}{=} \begin{cases} \sum_{i=1}^{S} X_i, & \text{when } S > 0 \\ 0, & \text{when } S = 0 \end{cases}$$

when $X_i \sim \mathcal{E}(\beta/a)$ iid and $S \sim \overline{\mathfrak{B}}(\alpha, 1-a)$. In particular $Z_{a|S=s} \sim \Gamma(s, \beta/a)$, when $s > 0$.

Proof. . Togliere:già presente in Sabino and Cufaro-Petroni [17]. It is sufficient compare the characteristic function of Z_a in Proposition 2.1 with (3).

$$\begin{aligned}
\phi_Z(u) &= \left(\frac{1-p}{1-p\left(\frac{\beta/a}{\beta/a - iu}\right)} \right)^{\alpha} = \left(\frac{(1-p)(\beta/a - iu)}{\beta/a - iu - p\beta/a} \right)^{\alpha} \\
&= \left(\frac{\beta/a - iu - p\beta/a + iup}{\beta/a(1-p) - iu} \right)^{\alpha} = \left(\frac{\beta/a(1-p) - iu(1-p)}{\beta/a(1-p) - iu} \right)^{\alpha} \\
&\stackrel{p=1-a}{=} \left(\frac{\beta - iua}{\beta - iu} \right)^{\alpha} = \phi_{Z_a}(u).
\end{aligned}$$

If $S = s$ then Z_a is the sum of s independent exponential random variable with rate β/a. Using the characteristic functions of exponential random variable, it follows that $Z_a|_{S=s} \sim \Gamma(s, \beta/a)$.

■

Proposition 2.4. *The probability density function $dg_a(x)$ of $Z_a \sim \Gamma^{++}(\alpha, \beta)$ is given by:*

$$dg_a(x) = a^{\alpha}\delta_0(x) + \sum_{n\ge 1} \binom{\alpha+n-1}{n} a^{\alpha}(1-a)^n f_{n,\beta/a}(x)\, \mathbb{1}_{(0,\infty)}(x)\, dx \tag{4}$$

where $\delta(x)$ is the Dirac function, $f_{n,\beta/a}(x)$ is the probability density function of an Erlang law with parameters n and β/a which is given by:

$$f_{n,\beta/a}(x) = \left(\frac{\beta}{a}\right)^n \frac{x^{n-1} e^{-bx/a}}{(n-1)!} \mathbb{1}_{[0,\infty)}(x).$$

Proof. . Togliere:già presente in Sabino and Cufaro-Petroni [17]. The characteristic function of Z_a is given by Proposition 2.1.

$$\begin{aligned}
\left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha} &= \left(\frac{a\left(\beta - iua\right)}{a\beta - iua + \beta - \beta}\right)^{\alpha} = \left(\frac{a\left(\beta - iua\right)}{\beta - iua - \left(1-a\right)\beta}\right)^{\alpha} \\
&= \left(\frac{a}{1 - \left(1-a\right)\frac{\beta}{\beta - iua}}\right)^{\alpha} = \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k} a^{\alpha}\left(1-a\right)^{k}\left(\frac{\beta}{\beta - iua}\right)^{k} \\
&= a^{\alpha} + \sum_{k\geq 1}\binom{\alpha+k-1}{k} a^{\alpha}\left(1-a\right)^{k}\underbrace{\left(\frac{\beta}{\beta - iua}\right)^{k}}_{\phi_Y(u)}
\end{aligned}$$

where we used the identity we derived in the proof of Lemma 2.2 together with the fact that $\binom{\alpha+n}{0} = 1$. We observe that the therm $\phi_Y\left(u\right)$ is the characteristic function of an Erlang random variable $Y \sim \mathcal{E}\left(k, \beta/a\right)$, whereas a^{α} is the characteristic function of a random variable with all the mass concentrated at the origin[1]. If we anti-transform the characteristic function we get Equation (4). ■

Remark. Observe that the probability density function of the random variable Z_a is a mixture of Erlang, where the mixing density is a Polya distribution, plus a degenerate law at $x = 0$.

As we stated before if a random variable has a self-decomposable law then the law of its a-remainder is infinitely divisible. For this reason we can construct the Lévy process $Z_a = \{Z_a(t); t \geq 0\}$ associated to the a-remainder of a self-decomposable law. By the result of Corollary 2.3 we have that:

$$Z_a(t) \stackrel{d}{=} \begin{cases} \sum_{i=1}^{S(t)} X_i, & \text{when } S(t) > 0, \\ 0, & \text{when } S(t) = 0 \end{cases}, \tag{5}$$

where $X_i \sim \mathcal{E}(\beta/a)$ *iid* and $S = \{S(t); t \geq 0\}$ is a Polya process such that for each $t \geq 0$, $S(t) \sim \overline{\mathfrak{B}}\left(\alpha t, 1-a\right)$. Since the Polya distribution is infinitely divisible the Polya process is a Lévy process.

As next step we derive the characteristic Lévy triplet of the process Z. It is well know that if a law has a self-decomposable law, then its Lévy measure has the following form:

$$\nu(x) = \frac{k(x)}{|x|}$$

where $k(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ (see Cont and Tankov [3, Proposition 15.3]). The following proposition (see Cufaro-Petroni and Sabino [4]) allows us to compute the Lévy triplet of the process Z_a.

[1]Let Y be a random variable with probability density function given by $\delta_0\left(x\right)$, which is the Dirac function at $x = 0$. Then:

$$\mathbb{E}\left[e^{iuY}\right] = \int_{-\infty}^{\infty}\delta_0\left(x\right)e^{iux}dx = 1.$$

Proposition 2.5. *. Togliere:già presente in Sabino and Cufaro-Petroni [17] e aggiungere nella dimostrazione della successiva proposizione la referenza. Consider a self-decomposable law with Lévy triplet (γ, σ, ν), there $\sigma > 0$ is the diffusion and ν is the Lévy measure. Then for every $a \in (0, 1)$ the law of its a-remainder has Lévy triplet $(\gamma_a, \sigma_a, \nu_a)$:*

$$\begin{aligned}
\gamma_a &= \gamma (1-a) - a \int_{\mathbb{R}} sign(x) \left(\mathbb{1}_{|x|\le 1/a} - \mathbb{1}_{|x|\le 1}\right) k(x)\, dx, \\
\sigma_a &= \sigma \sqrt{1-a^2}, \\
\nu_a(x) &= \nu(x) - \frac{\nu(x/a)}{a}.
\end{aligned}$$

Proposition 2.6. *Consider the process Z_a. Then the following hold:*

(i) The characteristic triplet $(\gamma_a, \sigma_a, \nu_a)$ of Z_a is given by:

$$\begin{aligned}
\gamma_a &= \left(1 - e^{-\beta}\right) - a\left(1 - e^{-\beta/a}\right), \\
\sigma_a &= 0, \\
\nu_a(x) &= \frac{\alpha}{x}\left(e^{-\beta x} - e^{-\beta x/a}\right) \mathbb{1}_{(0,\infty)}(x).
\end{aligned}$$

(ii) Z_a has finite variation and, in particular, it is a subordinator.

(iii) Z_a has finite activity and hence it is a compound Poisson process with intensity $\lambda = \alpha \log(1/a)$ and jumps' distribution $f(x)$ given by:

$$f(x) = \int_1^{1/a} \frac{1}{y \log(1/a)} \cdot \beta y e^{-\beta x y} dy.$$

Proof. (i) This is a direct consequence of Proposition 2.5.

$$\begin{aligned}
\sigma_a &= 0, \\
\nu_a(x) &= \frac{\alpha}{x} e^{-\beta x} \mathbb{1}_{(x,\infty)}(x) - \frac{1}{a}\left(a \cdot \frac{\alpha}{x} e^{-\beta x/a}\right) \mathbb{1}_{(x,\infty)}(x) = \frac{\alpha}{x}\left(e^{-\beta x} - e^{-\beta x/a}\right) \mathbb{1}_{(x,\infty)}(x)
\end{aligned}$$

The computation of γ_a requires a small effort:

$$\begin{aligned}
\gamma_a &= \gamma(1-a) - a \int_0^{1/a} \alpha e^{-\beta x} dt + a \int_0^1 \alpha e^{-\beta x} dx \\
&= \gamma(1-a) - a\alpha \left(-\frac{e^{-\beta x}}{\beta}\bigg]_0^{1/a} + \frac{e^{-\beta x}}{\beta}\bigg]_0^1 \right) \\
&= \gamma(1-a) + \frac{a\alpha}{\beta}\left(e^{\beta/a} - e^{-\beta}\right) \\
&= \frac{\alpha}{\beta}\left(1 - e^{-\beta}\right)(1-a) + \frac{a\alpha}{\beta}\left(e^{\beta/a} - e^{-\beta}\right) \\
&= \frac{\alpha}{\beta}\left(1 - e^{-\beta} - a\left(1 - e^{-\beta/a}\right)\right) \\
&= \frac{\alpha}{\beta}\left(1 - a + a e^{-\beta/a} - e^{-\beta}\right) \\
&= \frac{\alpha}{\beta}\left(\left(1 - e^{-\beta}\right) - a\left(1 - e^{-\beta/a}\right)\right) \ge 0.
\end{aligned}$$

(ii) By Cont and Tankov [3, Proposition 3.9] a Lévy process with characteristic triplet (A, ν, γ) is of finite variation if and only if:

$$A = 0 \quad \text{and} \quad \int_{|x|\leq 1} |x| \nu (dx) < \infty.$$

$A = \sigma_a = 0$ and the computation of the integral is straightforward:

$$\begin{aligned}\int_{|x|\leq 1} |x| \nu_a (dx) &= \int_0^1 \alpha t \left(e^{-\beta x} - e^{-\beta x/a}\right) dx \\ &= \frac{\alpha}{\beta}\left(1 - e^{-\beta} - a\left(1 - e^{-\beta/a}\right)\right) < \infty\end{aligned}$$

By Cont and Tankov [3, Proposition 3.10] since $\sigma_a = 0$, $\nu_a ((-\infty, 0]) = 0$ and $b = \gamma - \int_0^1 x\nu_a (x) \geq 0$ it follows that Z_a is a subordinator.

(iii) To show that Z_a has finite activity we have to show that $\nu_a (\mathbb{R}) < \infty$. This is a direct consequence of Gradshteyn and Ryzhik [7, 3.434] which states:

$$\int_0^\infty \frac{e^{-zx} - e^{-wx}}{x} dx = \log\left(\frac{w}{z}\right), \quad [\Re (z) > 0,\ \Re (w) > 0]. \tag{6}$$

Then:

$$\nu_a (\mathbb{R}) = \alpha \int_{-\infty}^{\infty} \frac{e^{-\beta x} - e^{-\beta x/a}}{x} \mathbb{1}_{(0,\infty)} (x)\, dx = \log\left(\frac{1}{a}\right) < \infty.$$

Therefore, Z_a has finite activity.
By Cont and Tankov [3, Proposition 3.6] it follows that any finite activity Lévy process can be viewed as a compound Poisson process with intensity λ and jumps' distribution $f(x)$. The characteristic function of compound Poisson process at t is given by (see Cont and Tankov [3, Propostion 3.4])

$$\mathbb{E}\left[e^{iuX(t)}\right] = \exp\left\{t\lambda \int_{\mathbb{R}} \left(e^{iux} - 1\right) f (dx)\right\}.$$

By Cont and Tankov [3, Proposition 3.5] the Lévy measure of a compound Poisson process is given by $\nu (dx) = \lambda f (dx)$. Define $\Lambda = \log (1/a)$, it follows that:

$$\begin{aligned}\nu_a (x) &= \Lambda\alpha \cdot \frac{1}{\Lambda x}\left(e^{-\beta x} - e^{-\beta/ax}\right) = \Lambda\alpha \cdot \frac{1}{\Lambda x}\int_1^{1/a} -\beta x e^{-\beta xy} dy \\ &= \Lambda\alpha \int_1^{1/a} -\frac{\beta}{\Lambda} e^{-\beta xy} dy = \Lambda\alpha \int_1^{1/a} \frac{\beta}{\log a} e^{-\beta xy} dy \\ &= \Lambda\alpha \int_1^{1/a} \frac{\beta y}{y \cdot \log a} e^{-\beta xy} dy \\ &= \underbrace{\Lambda\alpha}_{\lambda} \cdot \underbrace{\int_1^{1/a} \frac{1}{y \log a} f_{\mathcal{E}} (x|\mu = \beta y)\, dy}_{f(x)},\end{aligned}$$

where $f_{\mathcal{E}} (x|\mu)$ denotes the probability density function of an exponential distribution with parameter $\mu > 0$.

■

Claim *(iii)* of Proposition 2.6 states that the jumps' measure can be seen as a mixture of a exponential density with stochastic rate given by βY where Y is a random variable which probability density function is given by $g_Y(y) = \frac{1}{y \log a}\mathbb{1}_{[1,1/a]}(y)$. The distribution function of Y is given by:

$$F_Y(x) = \frac{1}{\log a}\int_1^x \frac{1}{y}dy = \frac{\log x}{\log a}.$$

Simulation of a random variable with density

$$f(x) = \int_1^{1/a} \frac{1}{y\log(1/a)\cdot \beta y e^{-\beta x y} dy},$$

can be done using Algorithm 1.

Algorithm 1 Jumps's simulation of $Z_a(t)$.

1: Simulate $u \sim U[0,1]$.
2: Set $y = a^u$.
3: Simulate $J \sim \mathcal{E}(\beta y)$.

Algorithm 2 Simulation of $Z_a(t)$.

1: Simulate $n \sim Poisson(\alpha t \log(1/a))$
2: Simulate n *iid* random variables $\{J_i\}_{i=1}^n$ by using Algorithm 1.
3: Set $Z_a(t) = \sum_{i=0}^n J_i$.

The simulation of $Z_a(t)$ is shown by Algorithm 2. It is enough to simulate a compound Poisson process which intensity is given by $\alpha t\log(1/a)$ and jumps which density is $f(x)$.

Nevertheless the previous algorithm is not the only one we can use to simulate the random variable $Z_a(t)$. By Equation (5) we observe that it can be simulated as a stochastic sum of independent random variables with exponential law with parameter β/a where the number of terms $S(t) \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$. Algorithm 3 shows this procedure.

Algorithm 3 Simulation of $Z_a(t)$.

1: Simulate $s \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$.
2: Simulate s *iid* random variables $\{x_i\}_{i=1}^s$ where $X_i \sim \mathcal{E}(\beta/a)$.
3: Set $Z_a(t) = \sum_{i=0}^s x_i$.

2.3 Variance Gamma++ process

Since $Z_a = \{Z_a(t); t\geq 0\}$, $Z_a(t) \sim \Gamma^{++}(\alpha t, \beta)$ is a subordinator, we can use it for Brownian subordination. Consider a Brownian motion $W = \{W(t); t \geq 0\}$, with drift $\theta \in \mathbb{R}$, diffusion $\sigma \in \mathbb{R}^+$ and independent of Z_a. The process $X = \{X(t); t\geq 0\}$ defined as:

$$X(t) = \theta Z_a(t) + \sigma W(Z_a(t)), \qquad t \geq 0. \tag{7}$$

is called $VG++$ process.

Proposition 2.7. *For $u \in \mathbb{R}$, the characteristic function of X at time t is given by:*

$$\phi_{X(t)}(u) = \phi_{Z_a(t)}\left(\theta u + iu^2 \frac{\sigma^2}{2}\right) = \left(\frac{\beta - i\left(\theta u + iu^2\sigma^2/2\right)a}{\beta - i\left(\theta u + iu^2\sigma^2/2\right)}\right)^{\alpha t} \tag{8}$$

Proof. Remember that the characteristic function of a normal random variable with mean θ and variance σ^2 is given by:

$$\phi(u) = e^{i\theta u - \frac{\sigma^2 u^2}{2}}.$$

Using Proposition 2.1 we get that

$$\phi_{Z(t)}(u) = \exp\left\{t \log\left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha}\right\}, \tag{9}$$

and therefore:

$$\begin{aligned}
\mathbb{E}\left[e^{iuX(t)}\right] &= \mathbb{E}\left[e^{iu(\theta Z_a(t) + \sigma W(Z_a(t)))}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{iu(\theta z(t) + \sigma W(z(t)))}\middle| Z_a(t)\right]\right] \\
&= \mathbb{E}\left[e^{iu\theta Z_a(t) - \frac{\sigma^2}{2}u^2 Z_a(t)}\right] = \mathbb{E}\left[e^{i\left(u\theta + i\frac{\sigma^2}{2}u^2\right)Z_a(t)}\right] = \phi_{Z_a(t)}\left(u\theta + i\frac{\sigma^2}{2}u^2\right) \\
&= \exp\left\{\log\left(\frac{\beta - i\left(u\theta + i\frac{\sigma^2}{2}u^2\right)a}{\beta - i\left(u\theta + i\frac{\sigma^2}{2}u^2\right)}\right)^{\alpha t}\right\} = \left(\frac{\beta - i\left(\theta u + iu^2\sigma^2/2\right)a}{\beta - i\left(\theta u + iu^2\sigma^2/2\right)}\right)^{\alpha t}.
\end{aligned}$$

$\blacksquare$

Proposition 2.8. *The $VG++$ process defined in (7) can be written as difference of two independent processes $Z_{a_p} = \left\{Z_{a_p}(t); t \geq 0\right\}$ and $Z_{a_n} = \left\{Z_{a_n}(t); t \geq 0\right\}$ where $Z_{a_n}(t) \sim \Gamma^{++}(\alpha t, \beta_n)$ and $Z_{a_p}(t) \sim \Gamma^{++}(\alpha t, \beta_p)$.*

Proof. We show that for all $t \geq 0$ the $X(t) = \theta Z_a(t) + \sigma W(Z_a(t))$ can be written as difference of two independent subordinators.

$$\phi_{X(t)}(u) = \phi_{Z(t)}\left(u\theta + \frac{iu^2\sigma^2}{2}\right) = \frac{\left(\frac{1}{1 - \frac{i}{\beta}\left(u\theta + \frac{iu^2\sigma^2}{2}\right)}\right)^{\alpha t}}{\left(\frac{1}{1 - \frac{ia}{\beta}\left(u\theta + \frac{iu^2\sigma^2}{2}\right)}\right)^{\alpha t}} = \frac{A}{B}$$

First, we consider the term A:

$$A = \left(\frac{1}{1 - \frac{i}{\beta}\left(u\theta + \frac{iu^2\sigma^2}{2}\right)}\right)^{\alpha t} = \left(\frac{1}{1 - \frac{iu}{\beta_p}}\right)^{\alpha t}\left(\frac{1}{1 + \frac{iu}{\beta_n}}\right)^{\alpha t}.$$

If one observe that:

$$1 - iu\frac{\theta}{\beta} - i^2u^2\frac{\sigma^2}{2\beta} = 1 - iu\left(\frac{1}{\beta_p} - \frac{1}{\beta_n}\right) - iu^2\frac{1}{\beta_p\beta_n},$$

then:

$$\frac{\theta}{\beta} = \frac{1}{\beta_p} - \frac{1}{\beta_n}, \qquad \frac{1}{\beta_p \beta_n} = \frac{\sigma^2}{2\beta}.$$

Solving the previous system with respect to β_p and β_n and considering only the positive solution we have that:

$$\beta_n = \frac{\sqrt{\theta^2 + 2\sigma^2\beta} + \theta}{\sigma^2}, \qquad \beta_p = \frac{\sqrt{\theta^2 + 2\sigma^2\beta} - \theta}{\sigma^2}.$$

The term B can be decomposed in an analogous way leading to:

$$\tilde{\beta}_n = \frac{\sqrt{\theta^2 + 2\sigma^2\beta/a} + \theta}{\sigma^2}, \qquad \tilde{\beta}_p = \frac{\sqrt{\theta^2 + 2\sigma^2\beta/a} - \theta}{\sigma^2}.$$

It follows that:

$$\phi_{X(t)} = \frac{\left(\frac{1}{1-iu/\beta_p}\right)^{\alpha t}\left(\frac{1}{1+iu/\beta_n}\right)^{\alpha t}}{\left(\frac{1}{1-iu/\tilde{\beta}_p}\right)^{\alpha t}\left(\frac{1}{1+iu/\tilde{\beta}_n}\right)^{\alpha t}} = \left(\frac{1 - iu\left(\frac{\beta_p}{\tilde{\beta}_p}\right)/\beta_p}{1 - iu/\beta_p}\right)^{\alpha t}\left(\frac{1 + iu\left(\frac{\beta_n}{\tilde{\beta}_n}\right)/\beta_n}{1 + iu/\beta_n}\right)^{\alpha t} \tag{10}$$

Observe that $0 < \beta_p/\tilde{\beta}_p < 1$ and hence we can define $a_p = \beta_p/\tilde{\beta}_p$ and $a_n = \beta_n/\tilde{\beta}_n$ and we obtain that:

$$\phi_{X(t)}(u) = \left(\frac{1 - iua_p/\beta_p}{1 - iu/\beta_p}\right)^{\alpha t}\left(\frac{1 + iua_n/\beta_n}{1 + iu/\beta_n}\right)^{\alpha t}.$$

This is the characteristic function of the difference of two independent random variables $Z_{a_p}(t) \sim \Gamma^{++}_{a_p}(\alpha t, \beta_p)$ and $Z_{a_n}(t) \sim \Gamma^{++}_{a_n}(\alpha t, \beta_n)$. Therefore the process X can be expressed as difference of two independent subordinators $Z_{a_p} = \left\{Z_{a_p}(t); t \geq 0\right\}$ and $Z_{a_n} = \{Z_{a_n}(t); t \geq 0\}$. ■

Proposition 2.9. *The Lévy measure of the $VG++$ process X is given by:*

$$\nu(x) = \left(\alpha x^{-1} e^{-x\beta_p} - \alpha x^{-1} e^{-x\beta_p/a_p}\right) \mathbb{1}_{(0,\infty)}(x) + \left(-\alpha x^{-1} e^{x\beta_n} + \alpha x^{-1} e^{x\beta_n/a_n}\right) \mathbb{1}_{(-\infty,0]}(x).$$

Moreover, the process X is of finite variation and, in particular, of finite activity.

Proof. Since $X(t)$ can be written as sum of two independent processes Z_{a_p} and Z_{a_n}, which Lévy measures are respectively:

$$\nu_{a_p}(x) = \frac{\alpha}{x}\left(e^{-\beta_p x} - e^{-\beta_p x/a_p}\right) \mathbb{1}_{(0,\infty)}(x),$$
$$\nu_{a_n}(x) = \frac{\alpha}{-x}\left(e^{\beta_n x} - e^{\beta_n x/a_n}\right) \mathbb{1}_{(-\infty,0]}(x)$$

by Cont and Tankov [3, Theorem 4.2] the first part of the thesis follows.
Observe that:

$$\int_0^1 \alpha\left(e^{-\beta_p x} - e^{-\beta_p/a_p x}\right) dx < \infty,$$
$$\int_{-1}^0 -\alpha\left(e^{\beta_n x} - e^{\beta_n/a_n x}\right) dx < \infty$$

and therefore $\int_{|x|\le 1} |x|\nu(dx) < \infty$. Applying Cont and Tankov [3, Theorem 4.1] we have that $b = \gamma - \int_{|x|\le 1} x\nu(dx) = 0$. By Cont and Tankov [3, Theorem 4.2] we get that the diffusion term of X, A_X, is equal to zero. This last result together with the fact that $\int_{|x|\le 1} |x|\nu(dx) < \infty$ shows that the $VG++$ process is of finite variation.

By using Equation (6) it is easy to check that $\nu(\mathbb{R}) < \infty$ and hence the $VG{+}{+}$ process is of finite activity. ■

Proposition 2.10. *The first four cumulants of the process X at time $t \ge 0$ are given by:*

$$c_1(X(t)) = \mathbb{E}[X(t)] = \alpha t\left(\frac{1}{\beta_p} - \frac{1}{\tilde{\beta}_p} - \frac{1}{\beta_n} + \frac{1}{\tilde{\beta}_n}\right),$$
$$c_2(X(t)) = Var[X(t)] = \alpha t\left(\frac{1}{\beta_p^2} - \frac{1}{\tilde{\beta}_p^2} + \frac{1}{\beta_n^2} - \frac{1}{\tilde{\beta}_n^2}\right),$$
$$c_3(X(t)) = 2\alpha t\left(\frac{1}{\beta_p^3} - \frac{1}{\tilde{\beta}_p^3} - \frac{1}{\beta_n^3} + \frac{1}{\tilde{\beta}_n^3}\right),$$
$$c_4(X(t)) = 6\alpha t\left(\frac{1}{\beta_p^4} - \frac{1}{\tilde{\beta}_p^4} + \frac{1}{\beta_n^4} - \frac{1}{\tilde{\beta}_n^4}\right),$$

where β_p, $\tilde{\beta}_p$, β_n, $\tilde{\beta}_n$ are the same as the ones defined in Proposition 2.8.

Proof. The proof is an immediate consequence of Cont and Tankov [3, Proposition 3.3] applied to the $VG++$ process and of the following relation (see Gradshteyn and Ryzhik [7, 2.321]):

$$\int x^n e^{ax} = e^{ax}\left(\frac{x^n}{a} + \sum_{k=1}^{n} (-1)^k \frac{n(n-1)\dots(n-k+1)}{a^{k+1}} x^{n-k}\right),$$

for $a \in \mathbb{R}$ and $n \in \mathbb{N}$. ■

Proposition 2.11. *The probability density function of the $VG{+}{+}$ process $X = \{X(t); t \ge 0\}$ at $t \ge 0$ is given by:*

$$f_{X(t)}(x) = a^{\alpha t}\delta_0(x) + \sum_{k\ge 1}\binom{\alpha t + k - 1}{k} a^{\alpha t}(1-a)^k f^{VG}_{k,\beta/a}(x). \tag{11}$$

where $\delta_0(x)$ is the Dirac function and $f^{VG}(k, \beta/a)$ is the probability density function of a Variance Gamma law with parameters $k \in \mathbb{N}$ and β/a.

Proof. This result can be proved in two ways.

(i) The first proof relies upon in observing that the random variable $Z(t)$ consists in a normal variance mixture where the mixing density is given by $g_a(x)$ derived in Proposition 2.4. We have that:

$$\begin{aligned}
f_{X(t)}(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left\{-\frac{(x-\theta s)^2}{2\sigma^2 s}\right\} g_a(s)\,ds \\
&= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left\{-\frac{(x-\theta s)^2}{2\sigma^2 s}\right\} a^{\alpha t}\delta_0(s)\,ds \\
&+ \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left\{-\frac{(x-\theta s)^2}{2\sigma^2 s}\right\} \sum_{n\geq 1}\binom{\alpha t+n-1}{n} a^{\alpha t}(1-a)^n f_{n,\beta/a}(s)\,ds \\
&= I_1 + I_2.
\end{aligned}$$

We first compute I_2[2]:

$$\begin{aligned}
I_2 &= \int_0^\infty \sum_{n\geq 1} \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left\{-\frac{(x-\theta s)^2}{2\sigma^2 s}\right\} \binom{\alpha t+n-1}{n} a^{\alpha t}(1-a)^n f_{n,\beta/a}(x)\,ds \\
&= \sum_{n\geq 1}\binom{\alpha t+n-1}{n} a^{\alpha t}(1-a)^n \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left\{-\frac{(x-\theta s)^2}{2\sigma^2 s}\right\} f_{n,\beta/a}(s)\,ds
\end{aligned}$$

The inner integral gives the probability density function of a Variance Gamma whit parameters $n \in \mathbb{N}$ and β/a and its probability density function is given by Proposition D.4. It follows that:

$$\begin{aligned}
I_2 =& \sum_{n\geq 1}\binom{\alpha t+n-1}{n} a^{\alpha t}(1-a)^n \\
& K_{n-\frac{1}{2}}\left(|x|\frac{\sqrt{2\sigma^2\beta/a+\theta^2}}{\sigma^2}\right) \frac{\exp\left(\theta x/\sigma^2\right)}{\sqrt{2\pi\sigma^2}} \frac{(\beta/a)^n}{\Gamma(n)} \left(2\sigma^2\beta+\theta^2\right)^{\frac{1}{4}-\frac{n}{2}} 2|x|^{n-\frac{1}{2}} \\
=& \sum_{n\geq 1}\binom{\alpha t+n-1}{n} a^{\alpha t}(1-a)^n f^{VG}_{n,\beta/a}(x).
\end{aligned}$$

To compute the integral I_1 observe that:

$$I_1 = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left\{-\frac{(x-\theta s)^2}{2\sigma^2 s}\right\} a^{\alpha t}\delta_0(s)\,ds,$$

is the probability density function of a normal variance mixture where the mixing is a Dirac measure computed at $x = 0$. This can be computed considering the random variable $Y = \theta S + \sigma\sqrt{S}X$ where $X \sim \mathcal{N}(0,1)$ and S is the degenerated distribution at $x = 0$. Compute the characteristic function $\phi_Y(u)$ of Y:

$$\phi_Y(u) = \mathbb{E}\left[e^{iuY}\right] = \mathbb{E}\left[e^{iu\theta S+\sigma\sqrt{S}X}\right] = \mathbb{E}\left[e^{iuS\theta-\frac{\sigma^2u^2}{2}S}\right] = e^{iu0} = 1.$$

[2]By monotone convergence theorem you can exchange the serie with the integral.

Remember that $\phi_Y(u) = \int_{-\infty}^{\infty} e^{iux} f(x)\, dx$. Then we can invert the Fourier Transform and obtain:

$$f(x) = \delta(x)\, \mathbb{1}_{x=0}.$$

Finally, we conclude that:

$$I_1 = a^{\alpha t} \delta(x)\, \mathbb{1}_{x=0}.$$

(ii) The second proof uses the characteristic functions of the processes X at time t which is given by Equation (8). Following exactly the same steps of the proof of Proposition 2.4 we can write:

$$\begin{aligned}
\phi_{X(t)}(u) &= \left(\frac{\beta - i\left(\theta u + iu^2\sigma^2/2\right) a}{\beta - i\left(\theta u + iu^2\sigma^2/2\right)} \right)^{\alpha t} = \left(\frac{a}{1 - (1-a)\frac{\beta}{\beta - ia(\theta u + iu^2\sigma^2/2)}} \right)^{\alpha t} \\
&= \sum_{k=0}^{\infty} \binom{\alpha t + k - 1}{k} a^{\alpha t} (1-a)^k \left(\frac{\beta}{\beta - ia\left(\theta u + iu^2\sigma^2/2\right)} \right)^k \\
&= a^{\alpha t} + \sum_{k \geq 1} \binom{\alpha t + k - 1}{k} a^{\alpha t} (1-a)^k \left(\frac{\beta}{\beta - ia\left(\theta u + iu^2\sigma^2/2\right)} \right)^k.
\end{aligned}$$

One can recognize that $X(t)$ is a mixture of Variance Gamma random variables where the weights are given by a Polya distribution plus a degenerate distribution at $x = 0$. By taking the inverse Fourier transform you get the probability density function which is given by (11).

■

Proposition 2.12. *Consider the $VG{+}{+}$ process X defined in* (7). *Let $S = \{S(t); t \geq 0\}$ be a Polya process where $S(t) \sim \overline{\mathfrak{B}}(\alpha t, 1-a)$, $(I_k)_{k\geq 1}$ and $(J_k)_{k\geq 1}$ a sequence of independent identically distributed random variables, with I_k independent of J_k for all $k \geq 1$, where $I_k \sim exp\left(\tilde{\beta}_p\right)$, $J_k \sim exp\left(\tilde{\beta}_n\right)$ where $\tilde{\beta}_n$ and $\tilde{\beta}_p$ are defined as in Equation* (10). *Define $\delta_k = I_k - J_k$ and the process $C = \{C(t); t \geq 0\}$ be:*

$$C(t) = \sum_{k=0}^{S(t)} \delta_k, \quad C(t) = 0 \text{ when } S(t) = 0.$$

Then:

$$X(t) \stackrel{d}{=} C(t).$$

Proof. First we prove that the $VG++$ process at time t can written as Polya sum of independent random variables. For $u \in \mathbb{R}$, consider the characteristic function $\phi_{X(t)}(u)$ at time t of the $VG{+}{+}$ process given in (8) and define $g(u) = i\left(\theta u + iu^2\sigma^2/2\right)$. We have:

$$\begin{aligned}
\phi_{X(t)}(u) &= \left(\frac{1}{\frac{\beta - g(u)}{\beta - ag(u)}} \right)^{\alpha t} = \left(\frac{a}{\frac{a\beta + \beta - \beta a g(u)}{\beta - ag(u)}} \right)^{\alpha t} = \left(\frac{a}{1 - (1-a)\frac{\beta}{\beta - ag(u)}} \right)^{\alpha t} \\
&\stackrel{a=1-p}{=} \left(\frac{1-p}{1 - p\frac{1}{1 - \frac{a}{\beta} g(u)}} \right)^{\alpha t} = \left(\frac{1-p}{1 - p\varphi(u)} \right)^{\alpha t},
\end{aligned}$$

where:
$$\varphi(u) = \frac{1}{1-\frac{a}{\beta}g(u)} = \frac{\beta/a}{\beta/a - iu\theta + u^2\sigma^2/2}.$$

Therefore, the process $X(t)$ at time t can be represented as a Polya sum of independent random variables which characteristic function given by $\varphi(u)$. We can write:

$$\varphi(u) = \frac{1}{1-\frac{iua\theta}{\beta}-\frac{i^2u^2\sigma^2}{2}}$$

and the denominator can be decomposed as:

$$1-\frac{iua\theta}{\beta}-\frac{i^2u^2\sigma^2}{2} = \left(1-\frac{iu}{\tilde{\beta}_p}\right)\left(1+\frac{iu}{\tilde{\beta}_n}\right) = 1-iu\left(\frac{1}{\tilde{\beta}_p}-\frac{1}{\tilde{\beta}_n}\right)-i^2u^2\frac{1}{\tilde{\beta}_p\tilde{\beta}_n}.$$

By imposing:
$$\frac{1}{\tilde{\beta}_p}-\frac{1}{\tilde{\beta}_n} = \frac{a\theta}{\beta}, \qquad \frac{1}{\tilde{\beta}_p\tilde{\beta}_n} = \frac{2\beta}{a\sigma^2}$$

and solving with respect to $\tilde{\beta}_n$ and $\tilde{\beta}_p$ and considering only positive solutions we have:

$$\tilde{\beta}_p = \frac{\sqrt{\theta^2+2\sigma^2-\beta/a}-\theta^2}{\sigma^2}, \quad \tilde{\beta}_n = \frac{\sqrt{\theta^2+2\sigma^2-\beta/a}+\theta^2}{\sigma^2}.$$

Then $\varphi(u)$ can be written as:

$$\varphi(u) = \frac{1}{1-\frac{iu}{\tilde{\beta}_p}} \cdot \frac{1}{1+\frac{iu}{\tilde{\beta}_n}},$$

which is the characteristic function of the difference of two independent exponential random variables with parameters given by $\tilde{\beta}_p$ and $\tilde{\beta}_n$ respectively.
By computing the characteristic function of $C\left(t\right)$ it is easy to check that:

$$\phi_{C(t)}(u) = \phi_{X(t)}(u),$$

and therefore $X(t) \stackrel{d}{=} C(t)$. ■

2.4 An option pricing formula under $VG++$ model

Suppose to model the risky asset process $S = \{S(t); t \geq 0\}$ by Lévy exponentiation, as proposed by Cont and Tankov [3]:

$$S\left(t\right) = S\left(0\right)e^{rt+\omega t+\theta Z_a(t)+\sigma W(Z_a(t))},$$

with:
$$\omega = \log\left(\frac{\beta-\left(\theta+\sigma^2/2\right)}{\beta-a\left(\theta+\sigma^2/2\right)}\right)^{\alpha},$$

Proposition 2.13. *The price of an European call option with stike price K, maturity T at time 0 is given by:*

$$C\left(0,K\right)=C\left(0\right)a^{\alpha T}+\sum_{n\geq1}\binom{\alpha T+n-1}{n}\left(1-a^{n}\right)a^{\alpha T}C_{n,\beta/a}^{VG}(0,K),\tag{12}$$

where

$$C\left(0\right)=\max\left(S(0)e^{\omega T}-e^{-rT}K,0\right)$$

and $C_{n,\beta/a}^{VG}(0,K)$ is the price of a Call option with strike K and maturity T under the Variance Gamma model with parameters n and β/a.

Proof. Consider $X(T)=\theta Z_a(T)+\sigma W(Z_a(T))$ which probability density function $f_{X(T)}(x)$ is given by Equation (11). The value of the call Option at $t=0$ can be computer by taking the discounted expected value under the risk-neutral measure.

$$\begin{aligned}
C\left(0,T\right)&=e^{-rT}\mathbb{E}\left[(S(T)-K)^{+}\right]=e^{-rT}\int_{-\infty}^{\infty}\left(S(0)e^{rT+\omega T+x}-K\right)^{+}f_{X(T)}(x)dx\\
&=e^{-rt}\int_{-\infty}^{\infty}\left(S(0)e^{rT+\omega T+x}-K\right)^{+}a^{\alpha T}\delta_{0}\left(x\right)\\
&+e^{-rT}\int_{-\infty}^{\infty}\left(S(0)e^{rT+\omega Tx}-K\right)^{+}\cdot\left(\sum_{n\geq1}\binom{\alpha T+n-1}{n}a^{\alpha T}\left(1-a\right)^{n}f_{n,\beta/a}^{VG}\left(x\right)\right)dx\\
&=\underbrace{a^{\alpha T}\left(S(0)e^{\omega T}-e^{-rT}K\right)^{+}}_{C(0)}\\
&+\sum_{n\geq1}\binom{\alpha T+n-1}{n}a^{\alpha T}\left(1-a\right)^{k}\underbrace{\int_{-\infty}^{\infty}\left(S(0)e^{rT+\omega T+x}-K\right)^{+}f_{k,\beta/a}(x)dx}_{C_{n,\beta/a}^{VG}(0,T)}
\end{aligned}$$

where in the last step we used the monotone convergence theorem to interchange the order of the integral and summation. ■

Remark. The option price given by Equation (12) can be computed in very efficient way using the results provided by Sexton and Hanzon [19] which are summarized in Appendix F.4. Since the shape parameter n is an integer the computation of $C_{n,\beta/a}^{VG}(0,T)$ is easier than the case in which the shape parameter of the Gamma distribution is a real number. In particular, we do not need to compute any integral when we evaluate $C_{n,\beta/a}^{VG}(0,T)$ because this term can be simply obtained as matrix products which is usually faster than numerical integration.

2.5 $VG++$ backward simulation

Definition 2.4. *A continuous random variable X is said to have a Beta distribution with parameters $\alpha>0$, $\beta>0$ if its probability density function $f\left(x;\alpha,\beta\right)$ with support on $[0,1]$ is given by:*

$$f\left(x;\alpha,\beta\right)=\frac{x^{\alpha-1}\left(1-x\right)^{\beta}}{B\left(\alpha,\beta\right)}$$

where $\mathrm{B}(\alpha,\beta)$ *is the Beta function and it is given by:*

$$\mathrm{B}(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}\,du.$$

We write $X\sim Beta(\alpha,\beta)$*.*

Definition 2.5. *A discrete random variable* X *is said to have a Binomial distribution with parameter* $n\in\mathbb{N}$ *and* $p\in[0,1]$ *if its probability mass function* $f(k;n,p)$ *is given by:*

$$f(k;n,p)=\binom{p}{k}p^k(1-p)^k.$$

We write $X\sim Bin(n,p)$*.*

Definition 2.6. *A discrete random variable* X *is said to have a Beta Binomial distribution with parameters* $\alpha>0$*,* $\beta>0$ *and* $n\in\mathbb{N}_0$ *ans support* $k=\{0,1,\ldots,n\}$ *if its probability mass function is given by:*

$$f(k|n,\alpha,\beta)=\binom{n}{k}\frac{\mathrm{B}(\alpha+k,\beta+n-k)}{\mathrm{B}(\alpha,\beta)},$$

where $\mathrm{B}(x,y)$ *is the Beta function. We state this fact by* $X\sim\mathcal{B}(\alpha,\beta,n)$*.*

Proposition 2.14 (Polya Bridge)**.** *Consider a process* $S=\{S(t);t\geq 0\}$ *such that* $S(0)=0$ *a.s. and* $S(t)\sim\overline{\mathfrak{B}}(\alpha t,1-a)$ *with probability mass function:*

$$\mathbb{P}(S(t)=K)=\binom{\alpha t+k-1}{k}a^{\alpha t}(1-a)^k,$$

where:

$$\binom{\alpha}{0}=1,\qquad\binom{\alpha}{k}=\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}.$$

For $0<t\leq T$*, define:*

$$\mathbb{P}\left(S_{tT}^{(k)}=j\right):=\mathbb{P}\left(S(t)=j|\,S(T)=k\right)$$

Then:

$$\mathbb{P}\left(S_{tT}^{(k)}=j\right)=\binom{k}{j}\frac{\mathrm{B}(\alpha t+j,\alpha(T-t)+k-j)}{\mathrm{B}(\alpha t,\alpha(T-t))}$$

and therefore $S_{tT}^{(k)}$ *has a Beta Binomial distribution* $\mathcal{B}(\alpha t,\alpha(T-t),k)$*.*

Proof. The proof is the following simple computation.

$$\begin{aligned}
\mathbb{P}\left(S_{tT}^{k}=j\right)&=\frac{P(S(t)=j,S(T)=k)}{\mathbb{P}(S(T)=k)}=\frac{\mathbb{P}(S(t)=j)\,\mathbb{P}(S(T-t)=k-j)}{\mathbb{P}(S(T)=k)}\\
&=\frac{\binom{\alpha t+j-1}{j}\binom{\alpha(T-t)+k-j-1}{k-j}}{\binom{\alpha T+k-1}{k}}=\frac{\frac{(\alpha t+j-1)(\alpha t+j-2)\ldots(\alpha t)}{j!}\cdot\frac{(\alpha(T-t)+k-j-1)(\alpha(T-t)+k-j-2)\ldots\alpha(T-t)}{(k-j)!}}{\frac{(\alpha T+k-1)(\alpha T+k-2)\ldots\alpha T}{k!}}\\
&=\binom{k}{j}\frac{(\alpha t+j-1)(\alpha t+j-2)\ldots\alpha t\cdot(\alpha(T-t)+k-j-1)\,(\alpha(T-t)+k-j-2)\ldots\alpha(T-t)}{(\alpha T+k-1)(\alpha T+k-2)\ldots\alpha T}\\
&=\binom{k}{j}\frac{\Gamma(\alpha t+j)}{\Gamma(\alpha t)}\frac{\Gamma(\alpha(T-t)+k-j)}{\Gamma(\alpha(T-t))}\frac{\Gamma(\alpha T)}{\Gamma(\alpha T+k)}\\
&=\binom{k}{j}\frac{\mathrm{B}(\alpha t+j,\alpha(T-t)+k-j)}{\mathrm{B}(\alpha t,\alpha(T-t))}
\end{aligned}$$

where we used the relations:

$$(\alpha t+j-1)(\alpha t+j-2)\dots\alpha t=\frac{\Gamma(\alpha t+j)}{\Gamma(\alpha t)},\qquad \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\mathrm{B}(x,y).$$

■

Proposition 2.15. *Consider a Gamma process $G=\{G(t);t\geq 0\}$ where $G(t)\sim\Gamma(t,\beta/a)$, $\beta>0$, $a\in(0,1)$. Consider a Polya process $S=\{S(t);t\geq 0\}$ such that $S(t)\sim\overline{\mathfrak{B}}(\alpha t,1-a)$ and define the process $Y=\{Y(t);t\geq 0\}$ as:*

$$Y(t)=G(S(t)),\qquad t\geq 0.$$

Then:

$$Z_a(t)\overset{d}{=}Y(t),\qquad t\geq 0,$$

where Z_a is the Lévy process associated to the a-remainder of a Gamma law with parameters α and β, as defined by (5)*.*

Proof. We compute the characteristic function of $Y(t)$ for $u\in\mathbb{R}$.

$$\begin{aligned}\mathbb{E}\left[e^{iuY(t)}\right]&=\mathbb{E}\left[\mathbb{E}\left[e^{iuG(S(t))}\middle|S(t)\right]\right]=\mathbb{E}\left[\left(\frac{\beta}{\beta-iua}\right)^{S(t)}\right]\\&=\mathbb{E}\left[\left(\frac{\beta/a}{\beta/a-iu}\right)^{S(t)}\right].\end{aligned}\tag{13}$$

From Corollary 2.3 we have that

$$Z_a(t)=\sum_{n=0}^{S(t)}E_n$$

where E_n are *iid* random variables with exponential law with parameter β/a. The characteristic function of $Z_a(t)$ is given by:

$$\begin{aligned}\mathbb{E}\left[e^{iuZ_a(t)}\right]&=\mathbb{E}\left[e^{iu\sum_{n=0}^{S(t)}E_n}\right]=\mathbb{E}\left[\mathbb{E}\left[e^{iu\sum_{n=0}^{S(t)}E_n}\middle|S(t)\right]\right]\\&=\mathbb{E}\left[\prod_{n=0}^{S(t)}\mathbb{E}\left[e^{iuE_1}\right]\right]=\mathbb{E}\left[\prod_{n=0}^{S(t)}\frac{\beta/a}{\beta/a-iu}\right]=\mathbb{E}\left[\left(\frac{\beta/a}{\beta/a-iu}\right)^{S(t)}\right]\end{aligned}$$

which is the same as Equation (13). Since the characteristic function characterizes the distribution we can conclude that $Z_a(t)\overset{d}{=}Y(t)$. ■

Suppose that the value of the process Z_a at time T is given and is equal to s_T. Therefore, we have $Z_a(T)=s_T$ and that $Z_a(0)=0$. It follows that the value of the process Z_a at time $t\in(0,T)$ can be simulated by generating first the value of the Polya process S at time t using a Polya bridge between T and 0 and hence a the value of the Gamma process

Algorithm 4 Backward simulation of Z_a.

1: Simulate $s_T \sim \overline{\mathfrak{B}}\left(\alpha T, 1-a\right)$.
2: Simulate $z \sim \Gamma\left(s_T, b/a\right)$ and set $Z_a\left(T\right) = z$.
3: Consider $t \in (0, T)$ and $p \sim Beta\left(\alpha t, \alpha\left(T-t\right)\right)$.
4: Simulate $s_t \sim Bin\left(s_T, p\right)$.
5: Simulate $\beta \sim Beta\left(s_t, s_T - s_t\right)$.
6: Set $Z_a\left(t\right) = Z_a\left(T\right) \cdot \beta$.

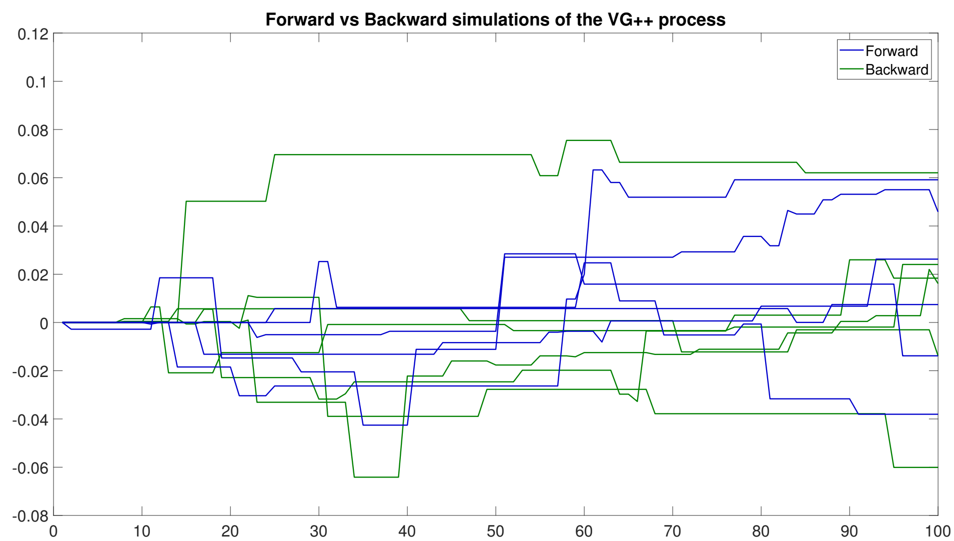

Figure 1: Path of the $VG{+}{+}$ process X obtained using forward and backward simulations.

at $G\left(S(t)\right)$ by using a Gamma bridge at $S(t) \in (0, S(T))$. This procedure is displayed in the Algorithm 4.

Once that we are able to simulate backward the process Z_a the backward simulation of the $VG++$ process X defined by (7), which value at T is $X(T) = x_T$ and $X(0) = 0$, can be obtained by simulating the Brownian bridge at time $Z_a\left(t\right)$ between 0 and $Z_a(T)$. The procedure to simulate the $VG++$ process X backward in time at time $Z_a(t)$ is given by Algorithm 5.

Algorithm 5 Backward simulation of X.

1: Set $X(0) = 0$ and $Z_a(0) = 0$.
2: Simulate $Z_a\left(T\right)$ and $Z_a\left(t\right)$ using Algorithm 4.
3: Simulate $x_T \sim \mathcal{N}\left(\theta Z_a(T), \sigma^2 Z_a(T)\right)$.
4: Simulate $x_t \sim \mathcal{N}\left(x_T \frac{Z_a(t)}{Z_a(T)}, \frac{Z_a(t)(Z_a(T)-Z_a(t))}{Z_a(T)}\sigma^2\right)$.
5: Set $X(t) = x_t$.

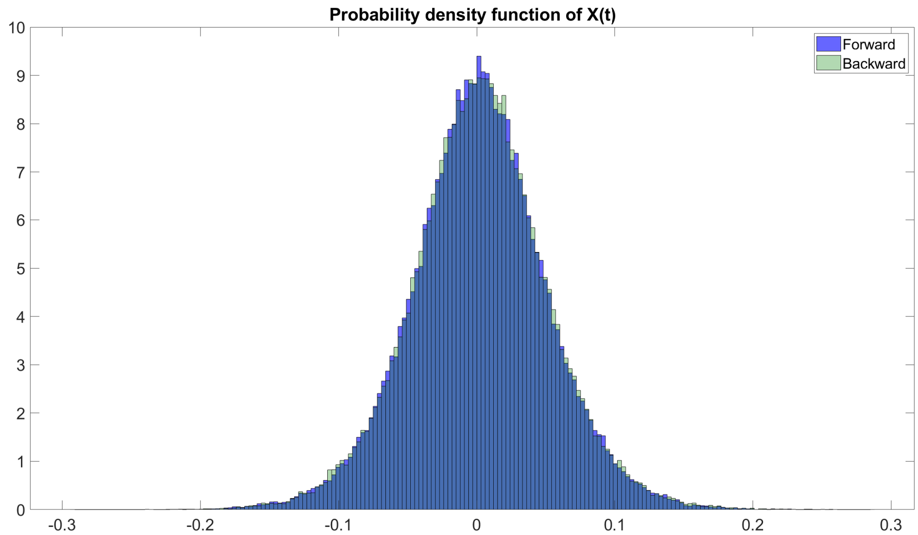

Figure 2: Distribution of the process X at time $t > 0$ using backward and forward simulations.

3 Numerical applications

In this section we show some numerical application of the results we derived in the previous section. First we compare three different methods for options pricing, then we discuss about backward simulation for American derivatives pricing and, finally, we discuss about calibration.

3.1 Three different pricing methods

In this section we compare three different pricing methods for Vanilla options. In Section 2.4 we derived an closed pricing formula to price European Call options. As observed such a formula is a weighted sum European Call options under the Variance Gamma model where the shape parameter of the gamma subordinator is an integer. This last fact makes sure that any numerical integration is needed to compute the option prices. Therefore we have a very efficient formula for European derivatives.

In many applications the pricing of exotic derivatives is needed. In order to evaluate such contracts a Monte Carlo scheme is needed for path simulations. In Section 2.2 we showed how the subordinator $Z_a = \{Z_a(t); t \geq 0)\}$ can be simulated. Once that the process Z_a has been simulated the numerical scheme to simulate a subordinated Brownian Motion can be obtained by standard algorithm such the one is presented in Cont and Tankov [3, Algorithm 6.10]. Finally the option price can be obtained by taking the expected value under the risk-neutral measure of the discounted payoff.

Finally, Proposition 2.7 provides us the characteristic function of the $VG{+}{+}$ at time t. For this reason, we can use a wide series of methods based on Fourier inversion to compute the option pricing for European derivatives. Among the others we remember Carr and Madan [2], Lewis [11], Fang and Oosterlee [5] and Lord et al. [13]. In this section we focus

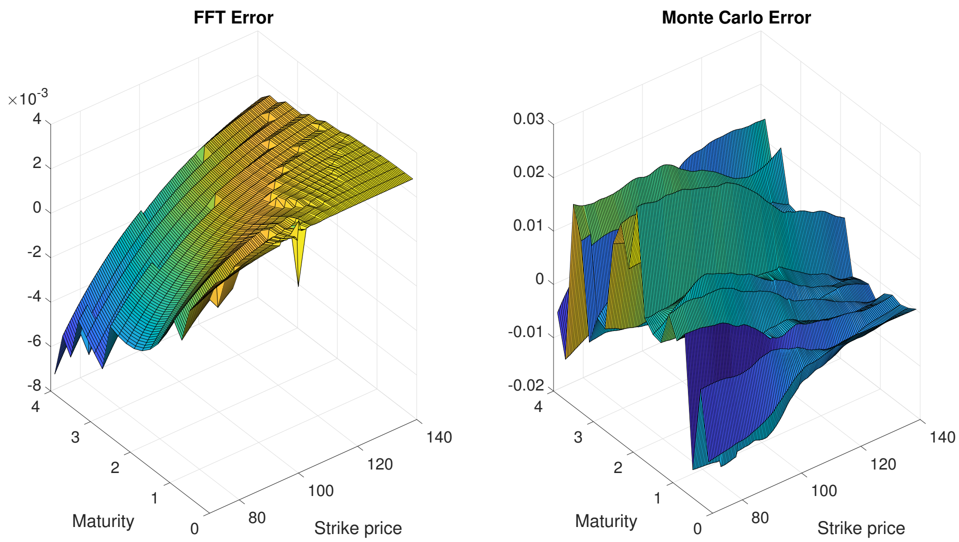

Figure 3: Fourier and Monte Carlo methods error for different values of maturities and strike prices.

only on the method proposed by Carr and Madan [2] which is the most famous one and the first to appear.
In our analysis we used the set of parameters we reported in Table 1. For the Monte Carlo algorithm we used 10^6 simulation and we imposed that $\beta = (1-a)\,\alpha$ that guaranties that $\mathbb{E}\left[Z_a(t)\right] = t$. In the exact formula the infinite sum is truncated as soon as the $n+1$ term of the sum gives a contribution that is smaller that the 0.01% of the sum up to step n.

$S(0)$	r	σ	θ	a	α
100	0.01	0.2	-0.1436	0.5	10

Table 1: Set of parameters for the numerical experiment.

In Figure 3 we compare the error with respect to the closed formula of the Fourier and Monte Carlo methods. We observe that both methods produce small errors and hence we conclude that they are suitable for option pricing.

3.2 Calibration and Financial Applications

In this section we show how the derived model can be applied to a real financial market. We show how to get the best set of parameters $\Theta = (\theta, \sigma, \alpha, a)$ that "properly fits our model" according to a given set of data. For historical calibration we use the *Maximum Likelihood Estimator* (*MLE*) and the *Generalized Method of Moments* (*GMM*) whilst we combine the Fourier technique with a classical Non-Linear Least Square approach for the market calibration. Finally we conclude the section by pricing an American Put option using the algorithm proposed by Langville and Stewart [10] with backward simulations we discussed in Section 2.5. For this purpose we refer to the Italian Energy power market and, in particular, we focus on the forward market.

The data-set is composed by Forward quotations from 23 August 2017 to 27 December 2019 of Calendar 2020. Moreover, for market calibration we refer to the call options written on the Italian calendar 2020 with quoted on the 19th Novembre 2019 and expiration on December 2020.

First we discuss historical calibration. Essentially, two techniques are used in finance for this purpose: the *MLE* and the *GMM* methods. The first one requires to be able to write the likelihood $\mathcal{L}(\Theta)$ and this required the knowledge of the transition densities of the process. When we deal with Lévy processes the problem is considerably simplified because we work with independent identically distributed random variables and this fact allows us to write the likelihood $\mathcal{L}(\Theta)$ as product of transition densities $f(\Theta|\, x_i)$ of the process from time t to $t+\Delta t$:

$$\mathcal{L}(\Theta) = \prod_{i=1}^{n} f(\Theta|\, x_i), \tag{14}$$

where $x = (x_1, x_2, \ldots, x_n)$ is the data sample we observed. The probability density function of the $VG++$ process is given by Proposition 2.11 and therefore one can easily write the likelihood function and numerically maximize its logarithm with respect the set of parameter Θ.

The *GMM* techniques needs known expression for the theoretical moment of a given random variable, to be applied. Such moments depend on the parameter set Θ and the goal is to find the best set of parameters that minimize "a sort of distance" between the theoretical moments from the numerical ones which can be easily computer form the given observations x. Proposition 2.10 gives a closed expression for the first four cumulants. Recalling that the first cumulant is the mean, the second one is the variance and that skewness $s(X)$ and kurtosis $k(X)$ can be retrieved by cumulants by using the relations:

$$s(X) = \frac{c_3(X)}{c_2(X)^{3/2}}, \qquad k(X) = \frac{c_4(X)}{c_2(X)^2},$$

one can solve the non linear system of four equations in four variables and find the suitable set of parameters Θ.

Historical calibration and simulations are generally suitable for risk-management purposes. If the goal is derivative pricing a market calibration must be considered. Assuming we observe n quoted products in the market[3] $\{C_i\}_{i=1}^n$ the idea is to find the set of parameters Θ such that if we use it within the model, quoted market prices are replicated "as much as possibile". This leads to the following constrained non-linear least squares (*NLLS*) problem:

$$\arg\min_{\Theta} \sum_{i=1}^{n} \left(C_i - C_i(\Theta)\right)^2$$

where $C_i(\Theta)$ is the price obtained by using the model. Usually to efficiently solve this problem, a fast method, such as FFT or a closed formula, to compute $C_i(\Theta)$ is required. In

[3]Usually vanilla products, such as European call or put options are quoted and liquid for many markets whereas more complex derivatives are traded over the counter.

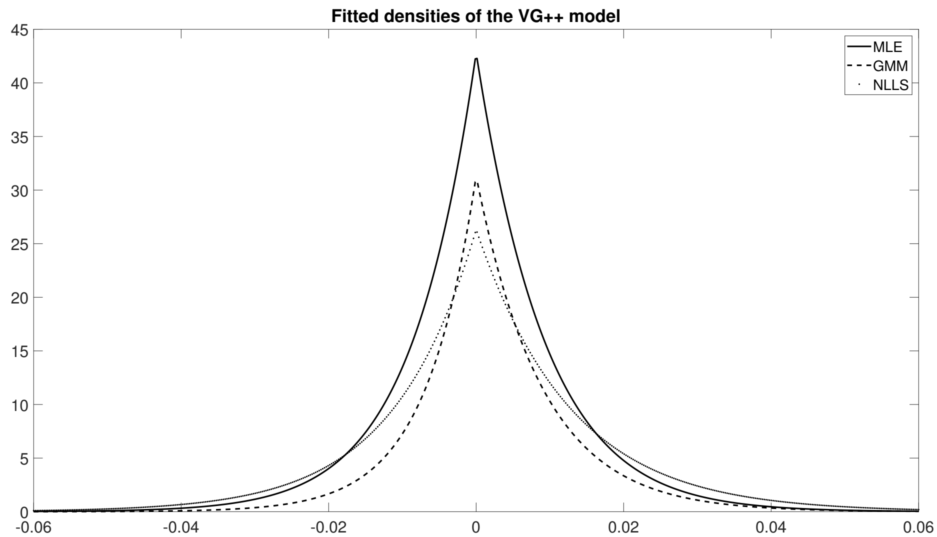

Figure 4: Fitted probability densities of the $VG++$ process obtained by using *MLE*, *GMM* and *NLLS* method.

our numerical experiment we use the FFT method proposed by Carr and Madan [2] since it is allows to obtain options value for different strike prices at the same time. In Table 2 we reported the parameters we obtain using the three different calibration techniques whereas in Figure 4 we draw the corresponding densities[4].

Method	σ	θ	a	α
MLE	0.1660	0.2619	0.5390	581.055
GMM	0.1224	0.1181	0.6915	466.350
NLLS	0.2041	0.2355	0.7126	724.484

Table 2: Set of parameters Θ obtained with the different methods.

Once we get the set of parameter Θ that properly fits the market by *NLLS* technique, we can price derivatives. For illustration purposes we use backward simulation techniques that we introduced in Section 2.5 combined with the Least-Square Monte Carlo technique presented by Longstaff and Schwartz [12] to price an American put options written on the power forward Italian calendar for different values of the strike prices K. The results are reported in Figure 5. As expected, we observe that the value of the American Put option is higher that the respective European version and, moreover, it is never lower that its payoff as observed, for example, in Seydel [20]. Backward simulations are not necessary faster than forward simulations (see Sabino [16]) but we do not need to store all the simulation in memory since at each time step t_j of the Least-Square Monte Carlo technique we need only the simulations at time t_j and t_{j+1}.

[4]Remember that the density has a non-zero mass at point $x = 0$.

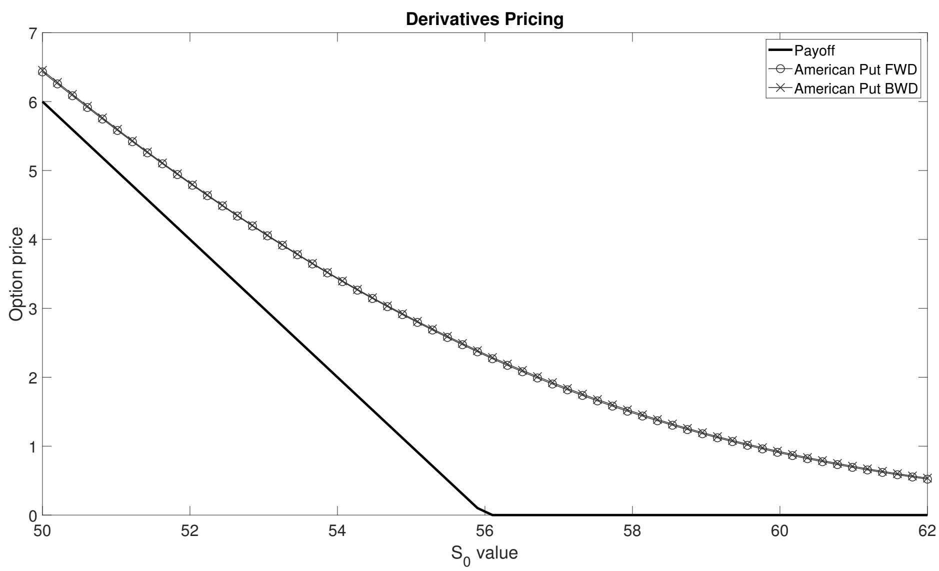

Figure 5: Price of the American Put option with different values of starting point $F(0)$ using Least-Square Monte Carlo with forward and backward simulations.

Appendices

In this Appendices we report some results that are useful to understand the report.

A Brownian Bridge

Consider a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with its natural filtration $\{\mathcal{F}_t; t \geq 0\}$ and consider $0 < u < s < t$. Assume also that: $W(u) = x$, $W(t) = y$ and that we want to recover the law of $W(s)$. As shown in Glasserman [6] we have that:

$$\begin{pmatrix} W(s) \\ W(u) \\ W(t) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & u & s \\ u & u & u \\ s & u & t \end{pmatrix}\right)$$

Proposition A.1. *Assume that $W = \{W(t); t \geq 0\}$ be a Brownian motion with drift $\mu \in \mathbb{R}$ and diffusion $\sigma \in \mathbb{R}^+$. For $0 < u < s < t$, the distribution of $W(s)$, given $W(u) = x$ and $W(t) = y$ is normal with mean and variance given by:*

$$\begin{aligned} \mathbb{E}\left[W(s)|\, W(u) = x, W(t) = y\right] &= \frac{(t-s)\,x + (s-u)\,y}{t-u}, \\ Var\left[W(s)|\, W(u) = x, W(t) = y\right] &= \sigma^2 \frac{(s-u)\,(t-s)}{(t-u)}. \end{aligned}$$

Proof. The proof relies upon the following result. Assume that the random vector $\boldsymbol{X} \in \mathbb{R}^n$ is given by:

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$$

where $X_1 \in \mathbb{R}^p$, $X_2 \in \mathbb{R}^q$, $\Sigma_{11} \in \mathbb{R}^{p \times p}$, $\Sigma_{21} = \Sigma_{21} \in \mathbb{R}^{p \times q}$ and $\Sigma_{22} \in \mathbb{R}^{q \times q}$. Then:

$$X_1|\, X_2 = x \sim \mathcal{N}\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}\,(x - \mu_2)\right), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

By taking $\mu_1 = 0$, $\mu_2 = (0,0)^T$ and

$$\begin{aligned} \Sigma_{11} &= s, \\ \Sigma_{12} &= (u, s)\,, \\ \Sigma_{21} &= \Sigma_{12}^T \\ \Sigma_{22} &= \begin{pmatrix} u & u \\ u & t \end{pmatrix} \end{aligned}$$

the result follows by straightforward computation. ■

B The Gamma Distribution

We assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed.

Definition B.1 (Gamma distribution)**.** *Let $X : \Omega \to \mathbb{R}$ be a random variable. We say that X has a Gamma distribution if its probability density function has the following form:*

$$f(x) = \frac{\beta}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x>0} \tag{15}$$

where $\alpha, \beta \in \mathbb{R}^+$ and Γ is the gamma function at $z > 0$:

$$\Gamma(z) = \int_0^\infty y^{z-1} e^{-z} dy.$$

We use the notation: $X \sim \Gamma(\alpha, \beta)$.

Proposition B.1. *Given $X \sim \Gamma(\alpha, \beta)$, its characteristic function $\phi(u) = \mathbb{E}[e^{iuX}]$ at $u \in \mathbb{R}$ is given by:*

$$\phi(u) = \left(1 - \frac{iu}{\beta}\right)^{-\alpha} = \left(\frac{\beta}{\beta - iu}\right)^{\alpha}. \tag{16}$$

Proposition B.2. *The gamma law is infinitely divisible.*

C The Gamma process

Definition C.1 (Gamma process)**.** *Let* $X = \{X(t); t \geq 0\}$ *be a stochastic process. We say that* X *is a Gamma process if* $X(t) \sim \Gamma(\alpha t, \beta)$*.*

Proposition C.1. *For* $u \in \mathbb{R}$*, The characteristic function of the gamma process at time* $t \geq 0$ *is given by:*

$$\varphi_{X(t)}(u) = \mathbb{E}[e^{iuX(t)}] = e^{t\psi(u)}$$

where $\psi(u)$ *is called the characteristic exponent of* X *and it is given by:*

$$\psi(u) = -\frac{1}{2}u^2\sigma^2 + i\gamma u + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1}\right) \nu(dx), \tag{17}$$

where:

$$\begin{aligned}
\gamma &= \frac{\alpha t}{\beta}(1 - e^{-\beta}), \\
\sigma &= 0, \\
\nu(x) &= \alpha x^{-1} e^{-\beta x} \mathbb{1}_{x>0}.
\end{aligned}$$

Proof. Using (16) it is easy to see that the characteristic exponent of X is given by:

$$\psi(u) = \log\left(1 - \frac{iu}{\beta}\right)^{-\alpha}.$$

For $z \in \mathbb{C}$ such that $\Re e(z) \leq 0$ and $\alpha, \beta \in \mathbb{R}^+$ the following Frullani equation holds:

$$\frac{1}{\left(1 - \frac{z}{\beta}\right)^{\alpha}} = e^{-\int_0^\infty (1-e^{zx})\alpha x^{-1} e^{-\beta x} dx} \tag{18}$$

For $z = iu$ and taking the logarithm both side in (18) and using (17) we get:

$$\psi(u) = \log\left(1 - \frac{iu}{\beta}\right)^{-\alpha} = -\int_0^\infty (1 - e^{iux})\alpha x^{-1} e^{-\beta x} dx. \tag{19}$$

Equating (19) and (17) we get:

$$\begin{aligned}
\psi(u) &= -\frac{1}{2}\sigma^2 u^2 + iu\left(\gamma - \int_{-1}^{1} x\nu(dx)\right) + \int_{\mathbb{R}} \left(e^{iux} - 1\right) \nu(dx) \\
&= -\int_0^\infty (1 - e^{iux})\alpha x^{-1} e^{-\beta x} dx = \int_0^\infty (e^{iux} - 1)\alpha x^{-1} e^{-\beta x} dx.
\end{aligned}$$

It follows that $\sigma^2 = 0$, $\nu(x) = \alpha x^{-1} e^{-\beta x} \mathbb{1}_{x>0}$ and:

$$\gamma = \int_0^1 x\alpha t x^{-1} e^{-\beta x} dx = \frac{\alpha t}{\beta}\left(1 - e^{-\beta}\right).$$

This is the characteristic triple of the Gamma process. ∎

Since $\sigma^2 = 0$, $\gamma \geq 0$ and $\nu((-\infty, 0]) = 0$ it follows that the gamma process is a subordinator (Cont and Tankov [3, Proposition 3.10]). Moreover, since it is a subordinator it is of finite variation (see Cont and Tankov [3, Proposition 3.9]). By Cont and Tankov [3, Corollary 3.1] its characteristic function at t can be expressed as:

$$\phi_{X(t)}(u) = \exp\left\{t\left(ibu + \int_{\mathbb{R}} \left(e^{iux} - 1\right)\nu\left(dx\right)\right)\right\}$$

where $b = \gamma - \int_{|x|\leq 1} |x| \nu\left(dx\right)$. For the gamma process we have that:

$$b = \gamma - \int_0^1 x\alpha t x^{-1} e^{-\beta x} dx = 0.$$

C.1 The Gamma Bridge

In this section we sketch how to construct a Gamma Bridge.

Proposition C.2. *Consider a Gamma process $G = \{G(t); t \geq 0\}$ such that $G(t) \sim \Gamma\left(\alpha t, \beta\right)$ and let be $0 \leq t < s < T$. Suppose that the value of the process G at times t and T is given and that $G(t) = g_t$ and $G(T) = g_t$. Then:*

$$\frac{G(s) - G(t)}{G(T) - G(t)} \sim B\left(\alpha\left(s-t\right), \alpha\left(T-s\right)\right)$$

where where B denotes the Beta distribution. Therefor we have that:

$$G(s) = g_t + \left(g_T - g_t\right)\beta,$$

where $\beta \sim B\left(\alpha\left(s-t\right), \alpha\left(T-s\right)\right)$.

D Variance Gamma process

Definition D.1. *Consider the gamma process $G = \{G(t); t \geq 0\}$ such that $G(t) \sim \Gamma(\alpha t, \beta)$ and consider a Brownian motion W with drift $\theta \in \mathbb{R}$ and diffusion $\sigma \in \mathbb{R}^+$. The process $X = \{X(t); t \geq 0\}$ defined as:*

$$X(t) = \theta G(t) + \sigma W(G(t)) \quad t \geq 0, \tag{20}$$

is called Variance Gamma process.

Proposition D.1. *For $u \in \mathbb{R}$ the characteristic function of X at time t is given by:*

$$\phi_{X(t)}(u) = \left(1 - \frac{i}{\beta}\left(u\theta + iu^2\frac{\sigma^2}{2}\right)\right)^{-\alpha t}. \tag{21}$$

Proposition D.2. *A Variance Gamma process X can be written as the difference of two independent gamma subordinators.*

Proof.

$$\begin{aligned}\phi_{X(t)}(u) &= \left(\frac{1}{1 - \frac{i}{\beta}\left(\theta u + i\frac{\sigma^2}{2}u^2\right)}\right)^{\alpha t} = \left(\frac{1}{1 - \frac{i}{\beta_1}u}\right)^{\alpha t}\left(\frac{1}{1 + \frac{i}{\beta_2}u}\right)^{\alpha t} \\ &= \left(\frac{1}{1 - iu\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) + u^2\frac{1}{\beta_1\beta_2}}\right)^{\alpha t}.\end{aligned}$$

We get:

$$\frac{\theta}{\beta} = \frac{1}{\beta_1} - \frac{1}{\beta_2}, \qquad \frac{1}{\beta_1\beta_2} = \frac{\sigma^2}{2\beta}$$

and solving for β_1 and β_2 and considering only the positive solutions we get:

$$\begin{aligned}\beta_1 &= \frac{\sqrt{\theta^2 + 2\sigma^2\beta} - \theta}{\sigma^2}, \\ \beta_2 &= \frac{\sqrt{\theta^2 + 2\sigma^2\beta} + \theta}{\sigma^2}.\end{aligned}$$

Then we have that the characteristic function of the Variance Gamma process X can be written as:

$$\begin{aligned}\phi_{X(t)}(u) &= \left(\frac{1}{1 - \frac{i}{\beta}\left(\theta u + i\frac{\sigma^2}{2}u^2\right)}\right)^{\alpha t} = \left(\frac{1}{1 - \frac{i}{\beta_1}u}\right)^{\alpha t}\left(\frac{1}{1 + \frac{i}{\beta_2}u}\right)^{\alpha t} \\ &= \phi_{Y(t)}(u)\phi_{Z(t)}(-u)\end{aligned}$$

where $Y(t) \sim \Gamma(\alpha t, \beta_1)$ and $Z(t) \sim \Gamma(\alpha t, \beta_2)$ and $Y(t)$ and $Z(t)$ are independent. Then $X(t)$ is the difference of two independent random variable with gamma law. We can conclude that the Variance Gamma process can be written as difference of two independent gamma process.

This gives us a good way to compute the Lévy measure of the Variance Gamma process. If two independent real values Lévyprocesses Y and Z has Lévy triplet $(\gamma_1, \Sigma_1, \nu_1)$ and $(\gamma_2, \Sigma_2, \nu_2)$ respectively, then the process $X = Y + Z$ is a Lévy process and its Lévy measure ν is given by:

$$\nu(B) = \nu_1(B) + \nu_2(B) \quad \forall B \in \mathcal{B}(\mathbb{R})$$

It immediately follows that the Lévy measure of the Variance Gamma process X is given by:

$$\nu(x) = \underbrace{\frac{\alpha}{x} e^{-\left(-\frac{\theta}{\sigma^2} + \frac{\sqrt{\theta^2 + 2\sigma^2\beta}}{\sigma^2}\right)x} \mathbb{1}_{\{x>0\}}}_{\nu_1(x)} + \underbrace{\frac{\alpha}{-x} e^{-\left(\frac{\theta}{\sigma^2} + \frac{\sqrt{\theta^2 + 2\sigma^2\beta}}{\sigma^2}\right)-x} \mathbb{1}_{\{x<0\}}}_{\nu_2(x)}$$

Let be:

$$A = \frac{\theta}{\sigma^2}$$
$$B = \frac{\sqrt{\theta^2 + 2\sigma^2\beta}}{\sigma^2}$$

then we have:

$$\nu(x) = \frac{\alpha}{x} e^{-(B-A)x} \mathbb{1}_{\{x>0\}} + \frac{\alpha}{-x} e^{-(B+A)x} \mathbb{1}_{\{x<0\}} = \frac{\alpha}{|x|} e^{Ax - B|x|}.$$

Since $\Sigma_1 = \Sigma_2 = 0$ then $\Sigma = 0$, by Cont and Tankov [3, Example 4.1] and since $\int_{|x|\le 1} |x| \nu(x) < \infty$, then the process X is of finite variation. Then its characteristic function has the following form:

$$\phi_{X(t)}(u) = \exp\left\{ t\left(ibu + \int_{\mathbb{R}} \left(e^{iux} - 1\right) \nu(dx) \right) \right\}$$

and since X is the sum of two independent stochastic processes with characteristic triplet $\left(\int_0^1 x\nu_1(dx), 0, \nu_1(x)\right)$ and $\left(\int_{-1}^0 x\nu_2(dx), 0, \nu_2(x)\right)$ its characteristic function can be written as:

$$\begin{aligned}\phi_{X(t)}(u) &= \exp\left\{ t\left(\int_{\mathbb{R}} \left(e^{iux} - 1\right) \left(\nu_1(dx) + \nu_2(dx)\right) \right) \right\} \\ &= \exp\left\{ t\left(\int_{\mathbb{R}} \left(e^{iux} - 1\right) \nu(dx) \right) \right\}.\end{aligned}$$

it follows that $b = 0$ and hence $\gamma = \int_{|x|\le 1} |x| \nu(x)$. ∎

Remark. To find the value of γ for the variance gamma process we can use a simplified version of Cont and Tankov [3, Theorem 4.2]. If two independent processes X, Y are of finite variation with Lévy triplet given by $(0, b_1, \nu_1)$ and $(0, b_2, \nu_2)$ then $X + Y$ has the following characteristic triplet: $(0, b_1 + b_2, \nu_1 + \nu_2)$. Since $b_1 = b_2 = 0$ then $b = 0$ and hence $\gamma = \int_{|x|\le 1} |x| \nu(x)$.

Remark. Using Cont and Tankov [3, Theorem 4.2] that if a Brownian motion with drift $\theta \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ is subordinated with a subordinator with Lévy triplet $(0, 0, \rho)$ (i.e. with zero drift) the Lévy measure of the subordinated process is given by:

$$\nu(x) = \int_0^\infty \frac{e^{-\frac{(x-\theta t)^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}} \rho(dt)$$

and this means that, if the subordinator is a gamma process:

$$\int_0^\infty \frac{e^{-\frac{(x-\theta t)^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}} \cdot \alpha t^{-1} e^{-\beta t} dt = \frac{\alpha}{|x|} e^{Ax - B|x|} \tag{22}$$

Proposition D.3. *For $z \in \mathbb{C}$, $z = x + iy$ such that $\arg(z)$ denotes the argument of a complex number i.e. that value $\varphi \in \mathbb{R}$ such that, for $r = \sqrt{x+iy}$ we have:*

$$z = r(\cos\varphi + i\sin\varphi) = e^{i\pi\phi}.$$

For $|\arg(z)| \leq \frac{\pi}{2}$, $\mathfrak{Re} z^2 > 0$ and $\nu \in \mathbb{C}$ we have that the Bessel Modified Function with of the Second type is given by:

$$K_\nu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^\nu \int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right) \frac{1}{\nu^{\nu+1}} dt. \tag{23}$$

Moreover:

$$K_{-\nu}(z) = K_\nu(z). \tag{24}$$

Relation (23) can be found in Gradshteyn and Ryzhik [7, WA 203(15)], whereas (24) can be fount in Gradshteyn and Ryzhik [7, WA 93(8)].

Proposition D.4. *The probability density function of a random variable X with Variance Gamma law is given by:*

$$f_X(x) = K_{\alpha-\frac{1}{2}}\left(|x|\frac{\sqrt{2\sigma^2\beta + \theta^2}}{\sigma^2}\right) \frac{\exp\left(\theta x/\sigma^2\right)}{\sqrt{2\pi\sigma^2}} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(2\sigma^2\beta + \theta^2\right)^{\frac{1}{4}-\frac{\alpha}{2}} 2|x|^{\alpha-\frac{1}{2}} \tag{25}$$

Proof. Remember that a Variance Gamma random variable X is defined as the normal variance-mean mixture where the mixing density is the gamma distribution. X is defined by:

$$X = \theta V + \sigma\sqrt{V}Y, \tag{26}$$

where $V \sim \Gamma(\alpha, \beta)$ and $Y \sim \mathcal{N}(0, 1)$ are independent random variables. The probability density function of a normal variance-mean mixture with mixing probability density g is:

$$f_X(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\theta t)^2}{2\sigma^2 t}} g(t) dt$$

which, in the Variance Gamma case, becomes:

$$\begin{aligned}
f_X(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\theta t)^2}{2\sigma^2 t}} \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \exp\left(-\frac{-x^2}{2\sigma^2 t} + \frac{\theta x}{\sigma^2} - \frac{\theta^2 t}{2\sigma^2} - \beta x\right) \frac{1}{t^{1+\frac{1}{2}-\alpha}} dt \\
&= \frac{\exp\left(\theta x/\sigma^2\right)}{\sqrt{2\pi\sigma^2}} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \exp\left(-\left(\frac{\beta + \theta^2}{2\sigma^2}\right) t - \frac{2x^2}{4t\sigma^2}\right) dt.
\end{aligned}$$

Define the following quantities:

$$\begin{aligned} a &= \frac{2\sigma^2\beta+\theta^2}{2\sigma^2}, & b &= 2x^2/\sigma^2, \\ \nu &= \frac{1}{2}-\alpha, & ab &= \frac{2\sigma^2\beta+\theta^2}{\sigma^4}x^2, \end{aligned} \tag{27}$$

we get:

$$f_X(x) = \frac{\exp\left(\theta x/\sigma^2\right)}{\sqrt{2\pi\sigma^2}}\frac{\beta^\alpha}{\Gamma(\alpha)}\underbrace{\int_0^\infty \exp\left(-at-\frac{b}{4t}\right)\frac{1}{\nu^{t^{\nu+1}}}dt}_{I}.$$

Now we compute I: by a variable change $s = at$ we get:

$$\begin{aligned} I &= \int_0^\infty \exp\left(-s-\frac{ba}{4s}\right)\frac{a^{\nu+1}}{s^{\nu+1}}\frac{ds}{a} = a^\nu\int_0^\infty \exp\left(-s-\frac{ba}{4s}\right)\frac{1}{s^{\nu+1}}ds \\ &= a^\nu K_\nu\left(\sqrt{ab}\right)\cdot 2\left(\frac{1}{2}\sqrt{ab}\right)^{-\nu} = K_\nu\left(\sqrt{ab}\right)\cdot\underbrace{2a^\nu\left(\frac{1}{2}\sqrt{ab}\right)^{-\nu}}_{C}. \end{aligned}$$

Using relations (27) we get:

$$\begin{aligned} C &= \left(\frac{2\sigma^2\beta+\theta^2}{2\sigma^2}\right)^{\frac{1}{2}-\alpha}\cdot 2\cdot 2^{\frac{1}{2}-\alpha}\left(\frac{\left(2\sigma^2\beta+\theta^2\right)}{\sigma^2}x\right)^{\frac{1}{2}-\alpha} \\ &= \left(2\sigma^2\beta+\theta^2\right)^{\frac{1}{2}-\alpha}\left(2\sigma^2\right)^{\alpha-\frac{1}{2}}\cdot 2^{\frac{1}{2}-\alpha}\cdot 2\cdot|x|^{\alpha-\frac{1}{2}}/\sigma^{2\alpha-1}\left(2\sigma^2\beta+\theta^2\right)^{\frac{\alpha}{2}-\frac{1}{4}} \\ &= \left(2\sigma^2\beta+\theta^2\right)^{1/2-\alpha+\alpha/2-1/4}|x|^{\alpha-1/2}\cdot 2^{\alpha-1/2}\cdot\sigma^{2\alpha-1}\cdot 2^{1/2-\alpha}\cdot 2\cdot\sigma^{1-2\alpha} \\ &= 2\cdot|x|^{\alpha-1/2}\left(2\sigma^2\beta+\theta^2\right)^{1/4-\alpha/2} \end{aligned}$$

Using I with C and recalling that $\nu = 1/2-\alpha$ and that $K_{-\nu}(z) = K_\nu(z)$ we finally get (25). ■

For Variance Gamma process $X = \{X(t); t \geq 0\}$ the probability density function at time $t \geq 0$ is given by:

$$f_{X(t)}(x) = K_{\alpha t-\frac{1}{2}}\left(|x|\frac{\sqrt{2\sigma^2\beta+\theta^2}}{\sigma^2}\right)\frac{\exp\left(\theta x/\sigma^2\right)}{\sqrt{2\pi\sigma^2}}\frac{\beta^{\alpha t}}{\Gamma(\alpha t)}\left(2\sigma^2\beta+\theta^2\right)^{\frac{1}{4}-\frac{\alpha t}{2}}2|x|^{\alpha t-\frac{1}{2}}$$

By imposing the condition that $\mathbb{E}[G(t)] = t$ we have that $\alpha = \beta$ and we set $\alpha = \beta = \frac{1}{\nu}$ where $\nu = Var[G(1)]$. With this two condition we have that the probability density function of the Variance Gamma process at time t is given by:

$$f_{X(t)}(x) = \frac{2\exp\left(\theta x/\sigma^2\right)}{\nu^{\frac{1}{\nu}}\sqrt{2\pi\sigma^2}\Gamma(\frac{t}{\nu})}\left(\frac{x^2}{2\sigma^2/\nu+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}}K_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{\sqrt{x^2\left(2\sigma^2/\nu+\theta^2\right)}}{\sigma^2}\right),$$

which coincides with Madan et al. [14, Theorem (23)].

E Variance Gamma process and option pricing

We model the risk asset S as a stochastic process $S = \{S(t); t \geq 0\}$ by Lévy exponentiation:

$$S(t) = S(0)e^{rt+\omega t+\theta G(t)+\sigma W(G(t)}, \quad S(0) = S_0, \quad G(t) \sim \Gamma(\alpha t, \beta) \tag{28}$$

where r is the risk free rate and ω is chosen such that $e^{-rt}S(t)$ is a martingale. This last condition is obtained by:

$$\mathbb{E}[S(t)] = S(0)e^{rt} \implies \mathbb{E}\left[e^{\underbrace{\theta G(t) + \sigma W(G(t)}_{X(t)}}\right] = e^{-\omega t},$$

which means that:

$$\mathbb{E}\left[e^{X(t)}\right] = \phi_{X(t)}(-i) = e^{-\omega t} \tag{29}$$

Using (29) and (21) we obtain:

$$\omega = \log\left(1 - \frac{\theta}{\beta} - \frac{\sigma^2}{2\beta}\right)^{\alpha}.$$

E.1 Call option: an explicit formula

Let $S = \{S(t); t \geq 0\}$ be the risky asset process modeled as in (28). For the sake of simplicity we assume that $G(t) \sim \Gamma(\alpha, \beta)$. (We will impose at the end the condition $\alpha = t/\nu$ and $\beta = 1/\nu$, in order to have $\mathbb{E}[G(t)] = t$). Consider now a Call Option with maturity T which payoff is given by $(S(T) - K)^+$. We compute the value of the Call option with strike K at time $t = 0$ $C(0, K)$ using the risk-neutral pricing formula:

$$C(t,K) = \mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] = \mathbb{E}\left[e^{-rT}\mathbb{E}\left[\left(S(0)e^{\overbrace{rT + \omega T + \theta g + \sigma W(g)}^{X(T)}} - K\right)^+ \middle| G(T) = g\right]\right]$$

Conditially to $G(T) = g$ we have that $X(T) \sim \mathcal{N}\left(rT + \omega T + \theta g, \sigma^2 g\right)$. Now we compute the inner expected value. For $Z \sim \mathcal{N}(0, 1)$:

$$\begin{aligned}
c(g) =& e^{-rT}\mathbb{E}\left[\left(S(0)e^{rT+\omega T+\theta g+\sigma\sqrt{g}Z} - K\right)^+\right] = e^{-rT}\int_0^{\infty}\left(S(0)e^{rT+\theta g+\sigma\sqrt{g}z+\omega T} - K\right)^+\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= e^{-rT}\int_{-\infty}^{\infty}\left(S(0)e^{rT+\theta g+\sigma\sqrt{g}z+\omega T} - e^{\log K}\right)^+\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= e^{-rT}\left(S(0)e^{rT+\theta g+\omega T}\int_{(-\log(S(0)/K)-rT-\theta g-\omega T)}^{\infty} e^{\sigma\sqrt{g}z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz\right. \\
&\left.-K\int_{(-\log(S(0)/K)-rT-\theta g-\omega T)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz\right).
\end{aligned}$$

Define:

$$d_2(g) = \frac{\log\left(S(0)/K + rT + \theta g + \omega T\right)}{\sigma\sqrt{g}}, \qquad d_1(g) = d_2(g) + \sigma\sqrt{g},$$

we have that:

$$c\left(g\right) = S(0)e^{\theta\frac{\sigma^2}{2}g+\omega T}\mathcal{N}\left(d1(g)\right) - Ke^{-rT}\mathcal{N}\left(d_2\left(g\right)\right) \tag{30}$$

where:

$$\mathcal{N}(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz.$$

Recall the density of a Gamma function (15) we have:

$$C\left(0, K\right) = \int_0^\infty c\left(g\right)\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}g^{\alpha-1}e^{-\beta g}dg \tag{31}$$

This integral can be computed numerically. Nevertheless a sem-closed formula can be obtained using the confluent hypergeometric function of two variables, introduced by Humbert [9], which is defined by:

$$\Phi\left(\alpha, \beta, \gamma, x, y\right) = \frac{\Gamma\left(\gamma\right)}{\Gamma\left(\alpha\right)\Gamma\left(\gamma-\alpha\right)}\int_0^1 u^{\alpha-1}\left(1-u\right)^{\gamma-\alpha-1}\left(1-ux\right)^{-\beta}e^{uy}du \tag{32}$$

Now we compute the integral (31). Define:

$$\Psi\left(a, b; \gamma\right) := \int_0^\infty \mathcal{N}\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right)\frac{u^{\gamma-1}}{\Gamma\left(\gamma\right)}e^{-u}du. \tag{33}$$

Substituting (30) in (31) we get:

$$C\left(0, K\right) = \underbrace{S(0)\int_0^\infty e^{\omega T + \left(\theta+\sigma^2/2\right)g}\mathcal{N}\left(d_1(g)\right)\frac{\beta^\alpha}{\Gamma\left(\alpha\right)}g^{\alpha-1}e^{-g\beta}dg}_{I_1}$$
$$-\underbrace{Ke^{-rT}\int_0^\infty \mathcal{N}\left(d_2(g)\right)\frac{\beta^\alpha}{\Gamma\left(\alpha\right)g^{\alpha-1}e^{-\beta g}}dg}_{I_2}.$$

We compute I_1 and I_2 separately.

$$I_1 = \beta^\alpha S(0)e^{\omega T}\int_0^\infty e^{-\overbrace{\left(\beta-\left(\theta+\sigma^2/2\right)\right)}^{A}}\frac{g^\alpha}{\Gamma\left(\alpha\right)}\mathcal{N}\left(\frac{\log\left(S(0)/K\right) + \left(r+\omega\right)T}{\sigma\sqrt{g}} + \frac{\left(\theta+\sigma^2\right)\sqrt{g}}{\sigma}\right)dg$$
$$\overset{Ag=u}{=} \beta^\alpha S(0)e^{\omega T}\int_0^\infty e^{-u}\frac{1}{A^\alpha}\frac{u^{\alpha-1}}{\Gamma\left(\alpha\right)}\mathcal{N}\left(\frac{\log\left(S(0)/K\right) + \left(r+\omega\right)T}{\sigma}\frac{\sqrt{A}}{\sqrt{u}} + \frac{\left(\theta+\sigma^2\right)\sqrt{u}}{\sigma\sqrt{A}}\right)du.$$

Define the following quantities:

$$\tilde{a} = \frac{\log\left(S(0)/K\right) + \left(r+\omega\right)T}{\sigma}\sqrt{A}, \qquad \tilde{b} = \frac{\theta+\sigma^2}{\sigma\sqrt{A}},$$

we get:

$$I_1 = \frac{\beta^\alpha S(0) e^{\omega T}}{A^\alpha} \int_0^\infty e^{-u} \frac{u^{\alpha-1}}{\Gamma(\alpha)} \mathcal{N}\left(\frac{\tilde{a}}{\sqrt{u}} + \tilde{b}\sqrt{u} du\right) \stackrel{(33)}{=} \frac{\beta^\alpha S(0) e^{\omega T}}{A^\alpha} \Psi(\tilde{a}, \tilde{b}; \alpha).$$

A similar procedure can be adopted to compute I_2 which leads to:

$$\begin{aligned} I_2 &= K e^{-rT} \int_0^\infty \mathcal{N}\left(\frac{\log(S(0)/K) + (\omega + r) T}{\sigma\sqrt{g}} + \frac{\theta}{\sigma}\sqrt{g}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} g^{\alpha-1} e^{-\beta g} dg \\ &\stackrel{\beta g = u}{=} K e^{-rT} \int_0^\infty \mathcal{N}\left(\frac{\log(S(0)/K) + (\omega + r) T}{\sigma} \frac{\sqrt{\beta}}{\sqrt{u}} + \frac{\theta}{\sigma\sqrt{\beta}}\sqrt{u}\right) \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u} du. \end{aligned}$$

If we define:

$$\tilde{c} = \frac{\log(S(0)/K) + (r + \omega) T\sqrt{\beta}}{\sigma} \qquad \tilde{d} = \frac{\theta}{\sigma\sqrt{\beta}},$$

we get:

$$I_2 = K e^{-rT} \Psi(\tilde{c}, \tilde{d}; \alpha).$$

The we can conclude that the price of the Call option at time $t = 0$ is given by:

$$C(0, K) = \frac{\beta^\alpha S(0) e^{\omega T}}{A^\alpha} \Psi(\tilde{a}, \tilde{b}; \alpha) - K e^{-rT} \Psi(\tilde{c}, \tilde{d}; \alpha) \tag{34}$$

It can be shown that the quantity $\Psi(a, b; \gamma)$ can be expressed in terms of confluent hypergeometric function of two variables (32) and in terms of the Bessel Modified function of the second type $K_\nu(z)$ by the following relation:

$$\begin{aligned} \Psi(a, b; \gamma) = & \frac{c^{\gamma+1/2} \exp[sign(a)\, c]\,(1+u)^\gamma}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \cdot K_{\gamma+1/2}(c) \\ & \Phi\left(\gamma, 1-\gamma, 1+\gamma, \frac{1+u}{2}, -sign(a)\, c\,(1+u)\right) \\ & - sign(a) \frac{c^{\gamma+1/2} \exp[sign(a)\, c]\,(1+u)^{\gamma+1}}{\sqrt{2\pi}\Gamma(\gamma)(1+\gamma)} \cdot K_{\gamma-1/2}(c) \\ & \Phi\left(1+\gamma, 1-\gamma, 2+\gamma, \frac{1+u}{2}, -sign(a)\, c\,(1+u)\right) \\ & + sign(a) \frac{c^{\gamma+1/2} \exp[sign(a)\, c]\,(1+u)^\gamma}{\sqrt{2\pi}\Gamma(\gamma)\gamma} \cdot K_{\gamma-1/2}(c) \\ & \Phi\left(\gamma, 1-\gamma, 1+\gamma, \frac{1+u}{2}, -sign(a)\, c\,(1+u)\right), \end{aligned}$$

where $c = |a|\sqrt{2+b^2}$ and $u = \frac{b}{\sqrt{2+b^2}}$. Since we are dealing with a Gamma Process and since we want that $\mathbb{E}[G(t)] = t$ then we impose the following conditions:

$$\alpha = \frac{t}{\nu}, \qquad \beta = \frac{1}{\nu},$$

where $\nu > 0$ and represents the variance of G at time $t = 1$.

F Variance Erlang distribution: derivation and option pricing

For this section we follow Sexton and Hanzon [19].

F.1 2-EPT distributions

The class of Exponential Polynomial Trigonometric (EPT) functions $f : [0, \infty) \to \mathbb{R}$ is given by:

$$f(x) = \mathfrak{Re}\left(\sum_{k=1}^{K} p_k(x) e^{\mu_k x}\right)$$

where $\mathfrak{Re}(z)$ denotes the real part of a complex number $z \in \mathbb{C}$, $p_k(x)$ is poynomial with complex coefficient for each $k = 1, \ldots, K$ and $\mu_k \in \mathbb{C}$ for $k = 1, 2, \ldots, K$. And EPT function defined on the positive real line can be represented in the following form:

$$f(x) = \boldsymbol{c} e^{\boldsymbol{A} x} \boldsymbol{b}, \qquad x \geq 0$$

where $\boldsymbol{A}$ is a nxn matrix, $\boldsymbol{c}$ is $1xn$ vector and $\boldsymbol{b}$ is a $nx1$ vector. We consider probability density functions which can be written as two separate EPT functions:

$$f(x) = \begin{cases} \boldsymbol{c}_N e^{\boldsymbol{A}_N x} \boldsymbol{b}_N, & x \geq 0 \\ \boldsymbol{c}_P e^{\boldsymbol{A}_P x} \boldsymbol{b}_P, & x > 0. \end{cases}$$

For all the details and properties of 2-EPT functions see Sexton and Hanzon [19].

F.2 Variance Gamma as 2-EPT function

Variance Gamma density can be views as an 2-EPT function. The density of a random variable X with Variance Gamma law is given by:

$$f_X(x; C, G, M) = \frac{(GM)^C}{\sqrt{\pi}\Gamma(C)} \exp\left(\frac{(G-M)x}{2}\right)\left(\frac{|x|}{G+M}\right)^{C-\frac{1}{2}} K_{C-\frac{1}{2}}\left(\frac{(G+M)|x|}{2}\right)$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind and $C, G, M > 0$. Its characteristic function has an even simpler formula:

$$\phi_X(u) = \left(\frac{GM}{GM + (M-G)iu + u^2}\right)^C$$

If $C \in N$, using Abramowitz and Stegun [1, pag. 443] which is:

$$\sqrt{\frac{\pi}{2x}} K_{n+\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right) e^{-x} \sum_{k=0}^{n} \left(n + \frac{1}{2}, k\right) (2z)^{-k}$$

where

$$\left(n + \frac{1}{2}, k\right) = \frac{(n+k)!}{k!\Gamma(n-k+1)}.$$

Define $z := \frac{(G+M)|x|}{2}$

$$
\begin{aligned}
f_X(x) &= \frac{(GM)^C}{\sqrt{\pi}\Gamma(c)} \exp\left(\frac{(G-M)x}{2}\right)\left(\frac{|x|}{G+M}\right)^{C-\frac{1}{2}} K_{C-\frac{1}{2}}\left(\frac{(G+M)|x|}{2}\right) \\
&= \frac{(GM)^C}{\sqrt{\pi}\Gamma(c)} \exp\left(\frac{(G-M)x}{2}\right)\left(\frac{|x|}{G+M}\right)^{C-\frac{1}{2}} \sqrt{\frac{2z}{\pi}}\sqrt{\frac{\pi}{z}} K_{C-\frac{1}{2}}(z) \\
&= \frac{(GM)^C}{\sqrt{\pi}\Gamma(c)} \exp\left(\frac{(G-M)x}{2}\right)\left(\frac{|x|}{G+M}\right)^{C-\frac{1}{2}} \sqrt{\frac{2z}{\pi}}\frac{\pi}{z} e^{-z} \sum_{k=0}^{C-1}\left(C-1-\frac{1}{2},k\right)(2z)^{-k} \\
&= \frac{(GM)^C}{(C-1)!} \exp\left(\frac{(G-M)x}{2} - \frac{(G+M)|x|}{2}\right) \sum_{k=0}^{C-1}\left(C-1-\frac{1}{2},k\right)|x|^{C-1}(G+M)^{-k-C} \\
&= \exp\left(\frac{(G-M)x}{2} - \frac{(G+M)|x|}{2}\right) \underbrace{\frac{(GM)^C}{(C-1)!}\sum_{k=0}^{C-1}\frac{(C-1+k)!\,(G+M)^{-C-k}|x|^{C-1-k}}{(C-1-k)!k!}}_{p(x)}
\end{aligned}
$$

We can split the density around the origin and we get:

$$
f_X(x) = \begin{cases} \exp(Gx)\,\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!(C-1-s)!} & x \le 0 \\ \exp(-Mx)\,\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!(C-1-s)!} & x > 0. \end{cases} \tag{35}
$$

Observe that the polynomial part of (35) are identical for all x and this implies that $\boldsymbol{c}_N = \boldsymbol{c}_P$ and $\boldsymbol{b}_N = \boldsymbol{b}_P$. By setting:

$$
\begin{aligned}
\boldsymbol{c} &= (c_0, \dots, c_{S-1}), & \boldsymbol{c} \in \mathbb{R}^{1\times C} \\
c_s &= \frac{(MG)^C}{(C-1)!}\frac{(2(C-1)-s)!\,(G+M)^{-2C+1+s}}{(C-1-s)!}, & s \in (0,\dots,C-1)
\end{aligned}
$$

Similarly $\boldsymbol{b} = (1,0,\dots,0)^T$ a $C\times 1$ column vector whereas $\boldsymbol{a}$ is given by:

$$
\boldsymbol{a} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0\cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}
$$

and finally we get that $p(x) = \boldsymbol{c}e^{-\boldsymbol{a}x}\boldsymbol{b}$. This leads us to write:

$$
f_X(x; C, G, M) = \begin{cases} \boldsymbol{c}e^{Gx}e^{-\boldsymbol{a}x}\boldsymbol{b} & x \le 0 \\ \boldsymbol{c}e^{-Mx}e^{\boldsymbol{a}x}\boldsymbol{b} & x > 0. \end{cases}
$$

$$
f_X(x; C, G, M) = \begin{cases} \boldsymbol{c}e^{(G\boldsymbol{I}-\boldsymbol{a})x}\boldsymbol{b} & x \le 0 \\ \boldsymbol{c}e^{(-M\boldsymbol{I}+\boldsymbol{a})x}\boldsymbol{b} & x > 0. \end{cases}
$$

Define $\boldsymbol{A}_N = G\boldsymbol{I} - \boldsymbol{a}$ and $\boldsymbol{A}_P = -M\boldsymbol{I} + \boldsymbol{a}$, therefore:

$$f_X(x; C, G, M) = \begin{cases} \boldsymbol{c}e^{\boldsymbol{A}_N x}\boldsymbol{b} & x \leq 0 \\ \boldsymbol{c}e^{\boldsymbol{A}_P x}\boldsymbol{b} & x > 0. \end{cases}$$

F.3 The price process

We model out underlying asset S as:

$$S(t) = S(0)e^{rT+\omega T+X(T)}, \quad S(0) = S_0$$

where $T \geq 0$. If we want that our discounted process is a martingale under the risk-neutral measure $\mathbb{Q}$ we must require that:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\omega T X(T)}\right] = 1$$

and this leads to:

$$\omega = C \log\left(\left(1 - \frac{1}{M}\right)\left(1 + \frac{1}{G}\right)\right).$$

We add the constrain that $CT \in \mathbb{N}$ and we observe that ω is defined only if $M > 1$. If $CT \in \mathbb{N}$ the a closed formula for a Call option with maturity T can be derived (In the original article you have $\tau = T - t$, which is the time to maturity, instead of T: here we considered $t = 0$ and hence τ and T coincides).

F.4 A closed formula for Call option pricing

Consider a Call option with strike price K, maturity T. The value of the underlying asset at $t = 0$ is $S(0)$ and we consider a constant risk free rate $r \geq 0$. Define:

$$d = \log\left(\frac{S(0)}{K}\right) + (r + \omega)T.$$

The price of the Call $C(0, K)$ option, where $X(T)$ has a infinitely divisible distribution with 2-EPT density distribution with realizations $(\boldsymbol{A}_N, \boldsymbol{b}_N, \boldsymbol{c}_N, \boldsymbol{A}_P, \boldsymbol{b}_P, \boldsymbol{c}_P)$, is given by:

- If $d > 0$:

$$\begin{aligned} C(0, K) = S(0)e^{\omega T}\left(\boldsymbol{c}_N(\boldsymbol{A}_N + \boldsymbol{I})^{-1}\right)\boldsymbol{b}_P - \boldsymbol{c}_N(\boldsymbol{A}_N \ + \boldsymbol{I})^{-1} e^{-(\boldsymbol{A}_N+\boldsymbol{I})d}\boldsymbol{b}_N \\ - \boldsymbol{c}_P(\boldsymbol{A}_p + \boldsymbol{I})^{-1}\boldsymbol{b}_P - Ke^{-rT}\left(1 - \boldsymbol{c}_N\boldsymbol{A}_N^{-1}e^{-\boldsymbol{A}_N d}\boldsymbol{b}_N\right) \end{aligned}$$

- If $d \leq 0$:

$$C(0, K) = -S(0)e^{\omega T}\boldsymbol{c}_P(\boldsymbol{A}_P + \boldsymbol{I})^{-1} e^{-(\boldsymbol{A}_p+\boldsymbol{I})d}\boldsymbol{b}_p + Ke^{-rT}\boldsymbol{c}_P\boldsymbol{A}_P^{-1}e^{-\boldsymbol{A}_p d}\boldsymbol{b}_P.$$

F.5 From C, G, M to $\alpha, \beta, \sigma, \theta$

We defined the Variance Gamma process as in Equation (20) which characteristic function at time t is given by (21). Observe that the characteristic function, which depends on $\theta, \sigma, \beta, \alpha$, of the Variance Gamma process can be rewritten as:

$$\phi_{X(t)} = \left(1 - \frac{1}{\beta}\left(u\theta + i\frac{\sigma^2}{2}u^2\right)\right)^{-\alpha T} = \left(\frac{2\frac{\beta}{\sigma^2}}{2\frac{\beta}{\sigma^2} - iu\frac{2\theta}{\sigma^2} + u^2}\right)^{\alpha T},$$

that has to be compared to:

$$\phi_{X(t)}(u) = \left(\frac{GM}{GM + (M - G)\, iu + u^2}\right)^{C}.$$

This leads to

$$GM = 2\frac{\beta}{\sigma^2}$$
$$M - G = -2\frac{\theta}{\sigma^2}$$

and finally:

$$G = \frac{1}{\sigma^2}\left(\theta + \sqrt{\theta^2 + \beta\sigma^2}\right)$$
$$M = \frac{\sqrt{\theta^2 + \beta\sigma^2}}{\sigma^2} - \frac{\theta}{\sigma^2}$$

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