Conditional Expectation

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Abstract

All that you want to know about conditional expectation. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the conditional expectation with respect a σ-algebra generated by a partition of Ω and then we extend the construction to a general sub-σ-algebra of $\mathcal{G} \subseteq \mathcal{F}$.

Keywords: Conditional expectation, σ-algebras, Radon-Nycodim theorem.

1 The setting

Assume that we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$. Under this hypotesis I can define the *conditional probability* as:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}. \tag{1}$$

This is a new probability measure on $(\Omega, \mathcal{F})$ and hence I can define an expected value with respect this new measure, using the (equivalent) symbols:

$$\mathbb{E}[f|A], \quad \mathbb{E}_A[f], \quad \int_\Omega f dP_A,$$

where $f : \Omega \to \mathbb{R}$ is a $\mathcal{F}$-measurable function. Recall from measure theory that the following holds:

$$\mathbb{E}[\mathbb{1}_B] = \mathbb{P}(B), \quad \text{for } B \in \mathcal{F},$$

and hence I want a definition which mocks the previous relation, namely:

$$\mathbb{E}[\mathbb{1}_B|A] = P(B|A)$$

It turns out that a good definition is the following:

$$\mathbb{E}[f|A] = \frac{\int_{\omega\in\Omega} f(\omega) \cdot \mathbb{1}_A(\omega) d\mathbb{P}(\omega)}{\mathbb{P}(A)}.$$

Observe that this quantity is a number and that if $f = \mathbb{1}_B$ for $B \in \mathcal{F}$ we have:

$$\mathbb{E}[f|A] = \frac{\int_A \mathbb{1}_B d\mathbb{P}}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \mathbb{P}(B|A).$$

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2 Conditional expectation with respect a σ-algebra generated by a partition of Ω

Assume that we have a partition of Ω, namely a collection of elements $\{A_n\}_{n\geq 1}$ such that:

- $A_n \subseteq \Omega$.
- $\bigcup_{n\geq 1} A_n = \Omega$.
- $A_j \cap A_i = \emptyset$ for $i \neq j$.

We can define $\mathcal{G} = \sigma(A_n, n \geq 1)$ the σ-algebra generated by the partition.

2.1 Preliminary results

First we prove the following lemma.

Lemma 2.1. *Define $\mathcal{H} = \left\{\bigcup_{j\in M} A_j : M \subseteq \mathbb{N}\right\}$. Then $\mathcal{H}$ is a σ-algebra.*

Proof. Let be $H \in \mathcal{H}$. Since $\{A_n\}_{n\geq 1}$ is a partition we have that:

$$H^C = \left(\bigcup_{j\in M} A_j\right)^C = \bigcup_{j\in\mathbb{N}\setminus M} A_j \in \mathcal{H}.$$

If $\{H_m\}_{m\in M}$ with $H_m \in \mathcal{H}$, with $M \subseteq \mathbb{N}$ then:

$$H = \bigcup_{m\in M} H_m = \bigcup_{m\in M} \left\{\bigcup_{j\in M_m} A_j\right\} \in \mathcal{H},$$

since this is a countable union of countable unions and hence it is a countable union of sets of the σ-algebra $\mathcal{H}$ and hence it belongs to $\mathcal{H}$. ■

The following proposition characterizes the sets $B \in \mathcal{G}$: they can be written as countable unions of elements of the partition.

Proposition 2.2. *$B \in \mathcal{G}$ if and only if $\exists M \subseteq \mathbb{N}$ such that $B = \bigcup_{n\in M} A_n$.*

Proof. We have to prove both implications.

- $\Rightarrow$: Assume that $B \in \mathcal{G}$.

  Define $\mathcal{H} = \left\{\bigcup_{j\in M} A_j : M \subseteq \mathbb{N}\right\}$. Clearly we have that $\mathcal{H} \subseteq \mathcal{G}$, since its elements are countable unions of A_j and countable unions belongs to $\mathcal{G}$ bu construction.

  It turns out (Lemma 2.1) that $\mathcal{H}$ is a σ-algebra containing the partition $\{A_n\}_{n\geq 1}$. But $\mathcal{G}$ is the smallest σ-algebra containing the partition (by definition of generate σ-algebra) and hence $\mathcal{G} \subseteq \mathcal{H}$. Hence $\mathcal{G} = \mathcal{H}$.

- $\Leftarrow$: Assume $\exists M \subseteq \mathbb{N}$ such that $B = \bigcup_{n\in M} A_n$. $A_n \in \mathcal{G}$ by construction of $\mathcal{G}$, and $\mathcal{G}$ is a σ-algebra. Then $B \in \mathcal{G}$ since it is a countable union of elements of the σ-algebra.

$\blacksquare$

So far, we have proved that any element of the σ-algebra $\mathcal{G}$ can be written as countable union of elements of the partition. We have the following.

Proposition 2.3. *$h : \Omega \to \mathbb{R}$ is $\mathcal{G}$-measurable if and only if h is constant on each A_i of the partition.*

Proof. We have to prove both implications.

- $\Leftarrow$: Since h is constant over the elements of the partition we can assume that it has the following form:
$$h(\omega) = \sum_{j \in \mathbb{N}} h_j \mathbb{1}_j(\omega)$$
Let $B \in \mathcal{B}(\mathbb{R})$ and take $x \in B$ and observe that $B = \bigcup_{x \in B} \{x\}$. We have only two possibilities: $h^{-1}(\{x\}) \in A_j$ for some j or $h^{-1}(\{x\}) \in \emptyset$. Hence:
$$h^{-1}(B) = \bigcup_{j \in M} A_j$$
for some countable set $M \subset \mathbb{N}$, and hence $h^{-1}(B) \in \mathcal{G}$ since it is the union of elements of the partition.

- $\Rightarrow$: Assume h is $\mathcal{G}$-measurable, hence:
$$h^{-1}(B) \in \mathcal{B}, \quad \forall B \in \mathcal{B}(\mathbb{R}).$$
Define the image of h as:
$$B_0 = \{x \in \mathbb{R} : \exists \omega \in \Omega : h(\omega) = x\}$$
Fix x in B_0 hence $\{x\} \in \mathcal{B}(\mathbb{R})$ and hence $h^{-1}(\{x\}) \in \mathcal{G}$ and then it is a union of elements of the partition.
$$h^{-1}(\{x\}) = \bigcup_{j \in M_x} A_j.$$
If we take two different x and y, clearly we have that $M_x \cap M_y = \emptyset$. M_x is a partition of $\mathbb{N}$. We can construct a injection between B_0 and the union of M_x by taking:
$$x \mapsto \min\{M_x\}$$
Hence we have that B_0 is countable.

  Finally we have:
$$h(\omega) = \sum_{x \in B_0} x \mathbb{1}_{h^{-1}(\{x\})}(\omega) = \sum_{x \in B_0} x \mathbb{1}_{\bigcup_{j \in M_x} A_j}(\omega)$$

Observe that if you take two elements $\omega_1, \omega_2 \in A_{j_0}$ they are mapped on the same value x, and hence this means that h is constant over elements of the partition.

There is also another way to prove this fact, by contradiction. Assume that $\omega_1, \omega_2 \in A_{j_0}$ such that $h(\omega_1) = c_1 \neq c_2 = h(\omega_2)$. We have that:

$$h^{-1}(\{c_1\}) = \{\omega \in \Omega : h(\omega) = c_1\} \in \mathcal{G}$$

But then we have:

$$\begin{aligned} h^{-1}(\{c_1\}) &\subseteq A_{j_0}, \\ h^{-1}(\{c_1\}) &\neq A_{j_0}, \end{aligned}$$

which is a contradiction since $h^{-1}(\{c_1\})$ cannot be written as union of the elements of the partition. (Draw a picture to convince yourself).

■

2.2 The construction with respect to the sub σ-algebra $\mathcal{G}$

We consider $(\Omega, \mathcal{F}, \mathbb{P})$, $f : \Omega \to \mathbb{R}$, $\mathcal{F}$-measurable and integrable and $\mathcal{G}$ the σ-algebra generated by the partition $\{A_j\}_{j\in\mathbb{N}}$. We define the following function:

$$\mathbb{E}[f|\mathcal{G}](\omega) = \sum_{j\in\mathbb{N}} \mathbb{E}[f|A_j] \mathbb{1}_{A_j}(\omega)$$

We have the following theorem:

Theorem 2.4. *Under the hypothesis of the construction of* $\mathbb{E}[f|\mathcal{G}](\omega)$*:*

- $\mathbb{E}[f|\mathcal{G}](\omega)$ *is* $\mathcal{G}$*-measurable.*
- *Given* g $\mathcal{G}$*-measurable, bounded and* fg *integrable then:*

$$\int fg d\mathbb{P} = \int \mathbb{E}[f|\mathcal{G}] g d\mathbb{P}$$

Observe that the identity reminds the usual definition of conditional expectation if $g = \mathbb{1}_B$ for $B \in \mathcal{G}$

Proof. We prove both statements.

- $\mathbb{E}[f|\mathcal{G}](\omega)$ is $\mathcal{G}$-measurable. Consider $\omega \in A_j$. $\mathbb{E}[f|A_j]$ is a number and hence the function $\mathbb{E}[f|\mathcal{G}](\omega)$ is constant over the elements of the partition. By Proposition 2.3 we have that it is $\mathcal{G}$-measurable.

- First we have to check that $\mathbb{E}[f|\mathcal{G}]\, g$ is integrable. Remember that $\mathbb{E}[f|A_j]$ is a number and use its definition. We have that:

$$\begin{aligned}\int |\mathbb{E}[f|\mathcal{G}]| d\mathbb{P} &= \int |\sum_{j\in\mathbb{N}} \mathbb{E}[f|A_j]\, \mathbb{1}_{A_j}| d\mathbb{P} \leq \sum_{j\in\mathbb{N}} \int |\mathbb{E}[f|A_j]| \mathbb{1}_{A_j} d\mathbb{P} \\ &= \sum_{j\in\mathbb{N}} |\mathbb{E}[f|A_j]| \mathbb{P}(A_j) = \sum_{j\in\mathbb{N}} \int |f| \mathbb{1}_{A_j} d\mathbb{P} = \int |f| d\mathbb{P} < \infty.\end{aligned}$$

  Once that we have this we can conclude that:

$$\int |\mathbb{E}[f|\mathcal{G}]\, g| d\mathbb{P} \leq M \int |\mathbb{E}[f|\mathcal{G}]| d\mathbb{P} < \infty.$$

- Now we have to show that the equality holds.

  First consider $g = \mathbb{1}_B$ with $B \in \mathcal{G}$ and hence $B = \bigcup_{j\in M} A_j$ by Proposition 2.2. Now, it is a matter of computation (we use the dominated convergence theorem):

$$\begin{aligned}\int g\mathbb{E}[f|\mathcal{G}]\, d\mathbb{P} &= \int \sum_{j\in M} \mathbb{1}_{A_j} \mathbb{E}[f|\mathcal{G}]\, d\mathbb{P} = \sum_{j\in M} \int \mathbb{1}_{A_j} \mathbb{E}[f|\mathcal{G}]\, d\mathbb{P} \\ &= \sum_{j\in M} \int \mathbb{E}[f|A_j]\, d\mathbb{P} = \sum_{j\in M} \mathbb{E}[f|A_j]\, \mathbb{P}(A_j) \\ &= \sum_{j\in M} \int f \mathbb{1}_{A_j} d\mathbb{P} = \int \sum_{j\in M} f \mathbb{1}_{A_j} d\mathbb{P} = \int f \mathbb{1}_B d\mathbb{P} = \int f g d\mathbb{P}.\end{aligned}$$

  Now we can consider the general case in which g is $\mathcal{G}$-measurable. Since it is $\mathcal{G}$-measurable we have that it has the form:

$$g(\omega) = \sum_{j\in\mathbb{N}} g_j \mathbb{1}_{A_j}(\omega).$$

  Again, by using the dominated convergence theorem and the fact that $\mathbb{E}[f|\mathcal{G}]$ is integrable, we have:

$$
\begin{aligned}
\int g\mathbb{E}\left[f|\mathcal{G}\right] d\mathbb{P} &= \int \left(\sum_{j\in\mathbb{N}} g_j \mathbb{1}_{A_j}\right) \mathbb{E}\left[f|\mathcal{G}\right] d\mathbb{P} \\
&= \int \left(\sum_{j\in\mathbb{N}} g_j \mathbb{1}_{A_j} \mathbb{E}\left[f|\mathcal{G}\right]\right) d\mathbb{P} \overset{DCT}{=} \sum_{j\in\mathbb{N}} \left(\int g_j \mathbb{1}_{A_j} \mathbb{E}\left[f|\mathcal{G}\right] d\mathbb{P}\right) \\
&= \sum_{j\in\mathbb{N}} g_j \left(\int \mathbb{1}_{A_j} \mathbb{E}\left[f|\mathcal{G}\right] d\mathbb{P}\right) \overset{A_j\in\mathcal{G}}{=} \sum_{j\in\mathbb{N}} g_j \left(\int f \mathbb{1}_{A_j} d\mathbb{P}\right) \\
&= \int \sum_{j\in\mathbb{N}} g_j f \mathbb{1}_{A_j} d\mathbb{P} = \int f \sum_{j\in\mathbb{N}} g_j \mathbb{1}_{A_j} d\mathbb{P} \\
&= \int fgd\mathbb{P}.
\end{aligned}
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Hence, we have constructed a random variable $\mathbb{E}\left[f|\mathcal{G}\right]$ such that it is $\mathcal{G}$-measurable and such that for g bounded and $\mathcal{G}$-measurable:

$$\int g\mathbb{E}\left[f|\mathcal{G}\right] d\mathbb{P} = \int fgd\mathbb{P}.$$

Observe that if $g = \mathbb{1}_B$ with $B \in \mathcal{G}$ we have the usual property of the conditional expectation:

$$\int_B \mathbb{E}\left[f|\mathcal{G}\right] d\mathbb{P} = \int_B fd\mathbb{P}.$$

3 The construction with respect a general sub σ-algebra

References