An application of multivariated Lévy subordinated processes to electricity and natural gas markets

Matteo Gardini*

November 11, 2019

Abstract

We use Multivariate subordination techniques to model correlated products in Energy Markets: we got from existing litterature some multivariate Lévy models and we test them to price Spread Options written on Power and Natural Gas Forward. We show what models can be used what should be avoided in differente cases.

1 Introduction

Energy markets and maily powe markets are relatively new financial markets. They have features make them completely different from other markets. For example spot prices exhibit strong seasonality, long-time mean reversion and spikes. Moreover, mainly in recent years, spot electricity powe can become negative, a feature rarely observed in other financial markets. Forward dynamic, instead, exibits a strong correlation, sometimes jumps and presents non Gaussian log-returns. During the previous years a lot of models were presented to model energy markets. The literature for spot model is quite extensive: one of the most known works is by Schwartz and Smith [26] who proposed a model that was adapted to electricity contest by Kluge [21]. Other very famous works are for example by Cartea and Figueroa [12], Lucia and Schwartz [23] and Burger and Muller [8].
Famous works in which the autors model forward dynamics are by, for example, Cartea and Villaplana [13], [14] and Benth [5]. In Benth's book the famous Heat-Jarrow-Mortorn [18] framework is adapted to energy markets. Another standard approach, adopted in industries, is based on Black model [6] while another possible approach was proposed by Carmona and Durrleman in [9]. All these works are somehow developed in Gaussian framework. This approach allows relatively easy computations, closed form option formula and other nice featurer. However energy markets, as other markets, are far to be Gaussian [15]. In univariate setting a lot of different model can be developed and used

*ERG Spa, Genoa and Mathematical Departement University of Genoa. For comments and suggestions write to gardini@dima.unige.it

with pricing and hedging purposes: stochastic volatility models as proposed by Heston [19] and Bates [4] or time change models as proposed by Carr, Madan and Chang [11] and Barndorff-Nielsen [2] are awaiable and permit to go beyond Gaussian framework. The problem is that extending this models to multivariate framework is not as easy as in Gaussian case. Some techniques are provided in [29].
An important works on Multivariate Lévy models is by Luciano and Schoutens [24]. Based on their ideas, Semeraro in [27] developed a framework for a Multivariate Lévy subordination that was extended in [25]. In [1] proposed another multivariated Lévy models based on the same concepts of the previous articles. All these models admits Lévy margins (such as Variance-Gamma or Normal Inverse Gaussian) and a richer dependence structure than the Gaussian case.

The aim of this article is to apply this models, that have been tested in other financial contests [1] to Power Energy Markets, and see if they adeguate for these markets.

The article is organized as follows. In Section 2 we briefly introduce the models while in Section 3 we apply them for Power Forward market modelling. In Section 4 we sketch how to calibrate them and in Section 5 we applied them to European Power and Gas markets.

2 Models

In this section we define the models we use in the paper. The first was was proposed by Semeraro [27] and we denote it by S. An extension of this model was presented by Semeraro and Luciano in [25] and we denote it by LS. The last model we analized was proposed by Ballotta and Bonfiglioli in [1] and we refer to it as the BB model. All these model can be defined for general subordinator classes. We select a gamma multivariated subordinator, as defined in [27] leading to Variance Gamma margins.
We present all models in bivariate version but they can be easly extended to any finite dimension.

2.1 S-model

We follow what is presented in [27].
Define a process $\mathbf{G}(t)$:

$$\mathbf{G}(t) = \begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix} = \begin{pmatrix} X_1(t) + \alpha_1 Z(t) \\ X_2(t) + \alpha_2 Z(t) . \end{pmatrix}$$

where:

$$\begin{aligned} X_j(t) &\sim \Gamma\left(\left(\frac{b}{\alpha_j} - a\right)t, \frac{b}{\alpha_j}\right), \quad j = 1, 2. \\ Z(t) &\sim \Gamma(at, b) \end{aligned}$$

where: $a, b > 0$ and $0 < \alpha_j < \frac{b}{a}$ for $j = 1, 2$. Using properties of Gamma distribution it is easy to show that:

$$G_j(t) \sim \Gamma\left(\frac{tb}{\alpha_j}, \frac{b}{\alpha_j}\right), \quad j = 1, 2.$$

It is possible to show that the process $\mathbf{G}(t)$ is a multivariated subordinator. See [3] for a demostration of this fact and for more properties.

Since $\mathbf{G}(t)$ is a subordinator we can use it to subordinate a multivariate Brownian Motion $\mathbf{B}(t)$ with indipendent components $B_1(t)$ and $B_2(t)$. As stated in [7] and proved in [3] the subordinated Brownian Motion $\mathbf{B}(\mathbf{G}(t))$ is still a Lévy process if and only if the components $B_1(t)$ and $B_2(t)$ are independent. Moreover $\mathbf{G}(t)$ and $\mathbf{B}(t)$ are assumed to be indipendent.
We can define the $\mathbf{Y}(t)$:

$$\mathbf{Y}(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} = \begin{pmatrix} \mu_1 G_1(t) + \sigma_1 B_1(G_1(t)) \\ \mu_2 G_2(t) + \sigma_1 B_2(G_2(t)) \end{pmatrix} \tag{1}$$

In this process one can show that the margins are Variance Gamma processes and it's possible to show that:

$$\rho_{\mathbf{Y}(t)} = \frac{\mu_1\mu_2\alpha_1\alpha_2 a}{b\sqrt{(b\sigma_1^2 + \mu_1^2\alpha_1)(b\sigma_2^2 + \mu_2^2\alpha_2)}} \tag{2}$$

Note that this model has a limit: if $\mu_1 = \mu_2 = 0$ then the correlation $\rho_{\mathbf{Y}(t)}$ is zero even a dependence between components exists.

Let's look at the maximum correlation captured by the model. Suppose that margins parameters $(\mu_i, \sigma_i, \alpha_i)$ $i = 1, 2$ are fixed look at Equation (2). We have that the maximum value for $\rho_{\mathbf{Y}(t)}$ is obtained if:

$$a_{max} = \min\left(\frac{b}{\alpha_1}, \frac{b}{\alpha_2}\right). \tag{3}$$

The correlation becomes:

$$\rho_{\mathbf{Y}(t)} = \frac{\mu_1\mu_2\alpha_{min}}{\sqrt{(b\sigma_1^2 + \mu_1^2\alpha_1)(b\sigma_2^2 + \mu_2^2\alpha_2)}}$$

and the maximum value for correlation is now obtained letting $b \to 0$:

$$\rho_{\mathbf{Y}(t)}^{max} = \lim_{b\to 0} \rho_{\mathbf{Y}(t)} = \sqrt{\frac{\alpha_{min}}{\alpha_{max}}} \tag{4}$$

2.2 LS-model

Semeraro and Luciano proposed in [25] an extension of S-model.
The generalization is based on a decomposition of the process $\mathbf{Y}(t)$ defined in (1), which is proven in [28]. The following decomposition holds:

$$\mathbf{Y} \stackrel{d}{=} \mathbf{Y}^X + \mathbf{Y}^{\alpha Z} \tag{5}$$

where $\mathbf{Y}^X = (B_1(X_1), B_2(X_2))$ and $\mathbf{Y}^{\alpha Z} = \left(\tilde{B}_1(\alpha_1 Z), \tilde{B}_2(\alpha_2 Z)\right)$. $\mathbf{B}(t)$ and $\tilde{\mathbf{B}}(t)$ are indipendent Brownian Motions with indipendent components. Moreover $\mathbf{Y}^X$ and $\mathbf{Y}^{\alpha Z}$ are indipendent. This decomposotion give us a way for correlatin Brownian Motions remaining in Lévy setting. We can consider correlation in the Y^Z component. Let $\mathbf{Y}^X$ as above and let $\mathbf{B}^\rho$ a multidimensional Brownian motion with drifts $\mu_j\alpha_j$ $j=1,2$, correlation ρ_{12} and diffusions $\sigma_j\sqrt{\alpha_j}$ $j=1,2$. Let:

$$\mathbf{Y}_\rho^Z = \mathbf{B}^\rho\left(Z\left(t\right)\right).$$

Now we can define:

$$\mathbf{Y}_\rho = \mathbf{Y}^X + \mathbf{Y}_\rho^Z. \tag{6}$$

In [25] it is shown that the process $\mathbf{Y}$ and the process $\mathbf{Y}_\rho$ has the same margins in law. But in this case we have correlation in Brownian Motion in the $\mathbf{Y}_\rho^Z$. component. The correlation $\rho_{\mathbf{Y}_\rho(t)}$ is given by:

$$\rho_{\mathbf{Y}_\rho(t)} = \frac{\rho\sigma_1\sigma_2\sqrt{\alpha_1}\sqrt{\alpha_2}ab + \mu_1\mu_2\alpha_1\alpha_2 a}{b\sqrt{\left(b\sigma_1^2+\mu_1^2\alpha_1\right)\left(b\sigma_2^2+\mu_2^2\alpha_2\right)}} \tag{7}$$

From Equation (7) we have that if $\mu_1=\mu_2=0$ the correlation $\rho_{\mathbf{Y}_\rho(t)}$ is not zero.

Repeting the same analysis done in the previous Section for S-model we have we have maximum correlation in two cases:

$$\rho^{max}_{\mathbf{Y}_\rho(t)} = \lim_{\substack{\rho\to1\\ b\to+\infty}} \rho_{\mathbf{Y}_\rho(t)} = \sqrt{\frac{\alpha_{min}}{\alpha_{max}}}.$$

and

$$\rho^{max}_{\mathbf{Y}_\rho(t)} = \lim_{\substack{\rho\to1\\ b\to0}} \rho_{\mathbf{Y}_\rho(t)} = \sqrt{\frac{\alpha_{min}}{\alpha_{max}}}.$$

2.3 BB-model

In this Section we present another model developed by Ballotta and Bonfiglioli in [1].

Let $Z\left(t\right), X_1\left(t\right), X_2\left(t\right)$ be indipendent Lévy processes and consider:

$$\boldsymbol{Y}\left(t\right) = \left(Y_1\left(t\right), Y_2\left(t\right)\right) = \left(X_1\left(t\right)+a_1Z\left(t\right), X_2\left(t\right)+a_2Z\left(t\right)\right)$$

with $a_j\in\mathbb{R}$ $j=1,2$. It can be proven [1] that the resulting process $\boldsymbol{Y}\left(t\right)$ is a Lévy process. The key idea of the model is to impose some convolution restrictions such that the process $\boldsymbol{X}\left(t\right)+Z\left(t\right)$ has the same distribution of $\boldsymbol{Y}\left(t\right)$. Let $\phi_W\left(u\right)$ denote the characteristic exponent of the Lévy process W_t, i.e. :

$$\varphi_{W_t}\left(u\right) = \mathbb{E}\left[e^{iuW_t}\right] = e^{t\phi(u)}. \tag{8}$$

So we chose the distributions of $\boldsymbol{Y}\left(t\right), \boldsymbol{X}\left(t\right), Z\left(t\right)$ from the same distribution and then we impose the following convolution conditions:

$$\phi_{Y_j}\left(u\right) = \phi_{X_j}\left(u\right) + \phi_Z\left(a_ju\right)\ \ j=1,2. \tag{9}$$

Taking a tempered stable subordinator[1] $G(t)$ with $\alpha \in [0,1)$ and characteristic function:

$$\phi_G = \frac{\alpha - 1}{\alpha k}\left[\left(1 - \frac{iuk}{1-\alpha}\right)^{\alpha} - 1\right] \tag{10}$$

and considering a subordinated Brownian Motion $X_1(t), X_2(t), Z(t)$ indipendent and of the same family with:

- X_1 with drift β_1 and volatility γ_1 subordinator params ν_1
- X_2 with drift β_2 and volatility γ_2 subordinator params ν_2
- Z with drift β_Z and volatility γ_Z subordinator params ν_Z

the $\boldsymbol{X}(t)$ is a subordinate brownian motion of the same family of X_1, X_2, Z with drift $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and volatility $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ if convolutions conditions (9) are satisfied. If we use (10) and (9) then we have the following:

$$\begin{cases} k_i\theta_i = \nu_Z a_i \beta_Z & j = 1,2 \\ k_i\sigma_i^2 = \nu_Z a_i^2 \gamma_Z^2 & j = 1,2 \end{cases} \tag{11}$$

and consequently:

$$\theta_j = \beta_j + a_j\beta_Z \quad \sigma_j^2 = \gamma_j^2 + a_j^2\gamma_Z^2 \quad k_j = \nu_j\nu_Z/\left(\nu_j + \nu_Z\right) \quad j = 1,2 \tag{12}$$

The correlation between components of $\mathbf{X}(t)$ has is the following general expression:

$$\rho_{\mathbf{Y}(t)} = \frac{a_1 a_2 Var\left[Z(t)\right]}{\sqrt{Var\left[Y_1(t)\right]}\sqrt{Var\left[Y_2(t)\right]}} \tag{13}$$

and for Variance Gamma margins becomes:

$$\rho_{\mathbf{Y}(t)} = \frac{a_1 a_2\left(\gamma_Z^2 + \beta_Z^2\nu_Z^2\right)}{\sqrt{\sigma_1^2 + \theta_1^2 k_1}\sqrt{\sigma_2^2 + \theta_2^2 k_2}} \tag{14}$$

Manipulating (13) using (11) it is possible to show that the maximum value for $\rho_{\mathbf{Y}(t)}$ is reached for $\nu_1, \nu_2 \to 0$. This force $\nu_Z \to 0$ and $\rho_{\mathbf{Y}(t)}$ to the maximum value $\rho_{\mathbf{Y}(t)}^{max}$. Since ν_1 and ν_2 are not subject to any constraint the can be adjusted to fit a given correlation value. Numerical experiment will make this analysis clearer.

3 Financial Model

In this section we use process $\boldsymbol{Y}(t)$ to model Power Forward markets. We define an exponential Lévy model as proposed, for example, in [29].
Suppose to be in univariate setting with the Lévy process $Y(t)$: we model the Forward price F_t as:

[1] Remember that Variance Gamma is a special cale of this process where $\alpha = 0$.

$$F_t = F_0 e^{rt+\omega t+Y_t} \tag{15}$$

where r is the risk-free rate and ω is the drift corrector. By introducing a new source of randomness the market is incomplete and we don't have a unique equivalent martingale measure. For this reason we need a drift corrector so that the expected value of the Forward price F_t be a martingale. We require that:

$$\mathbb{E}[F_t] = F_0 e^{rt}.$$

Using 15 and remembering the definition of characteristic exponent $\phi(u)$ of the Lévy process $Y(t)$ we can compute ω by the following:

$$\omega = -\phi(-i) \tag{16}$$

For the Variance Gamma process with parameters (μ, σ, ν) is shown [29] that the characteristic exponent is:

$$\phi(u) = -\frac{1}{\nu}\log\left(1 + \frac{u^2\sigma^2\nu}{2} - i\mu\nu u\right)$$

and thus using (16) we have that:

$$\omega = \frac{1}{\nu}\log\left(1 - \frac{\sigma^2\nu}{2} - \mu\nu\right).$$

The multivariate with dimension n model is immediate defining:

$$F_t^i = F_0^i e^{rt+\omega_i t+Y_t^i} \quad i = 1, \cdots, n.$$

4 Calibration

We use a standard calibration procedures as in [1] and [25]. Since derivatives written on more products are very illiquid and sometimes they doesn't even exists, we use a two step procedure. First we calibrate margins parameters on simple vanilla products: this products are usually liquid in many markets. Then we calibrate other parameters fitting the correlation matrix.

4.1 Margins parameter calibration

A good overview on calibration problem can be found in [29]. We note that in our models margins parameters does not depends on parameters used to introduce dependence. So we can calibrate margins and correlation parameters separately.
Suppose we can observe on the market n vanilla contract prices denoted with C_i con $i = 1, \cdots, n$. Denoted with $\boldsymbol{\theta}$ the vector of unknown parameters and with $C_i^{\boldsymbol{\theta}}(K,T)$ the price in output from the optimal $\boldsymbol{\theta}^*$ can be found solving:

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \left(C_i^{\boldsymbol{\theta}}(K,T) - C_i\right)^2. \tag{17}$$

Since calibration requires to compute a large number of prices for differente contracts and for different values of $\boldsymbol{\theta}$ we need a very fast option pricing method.

For this purpose methods based on FFT, as proposed in [10] and [22], techniques can be used.

Once we fitted margins parameters we have to fit correlation structure. Note that in Semeraro and in Semeraro-Luciano model we can choose $b = 1$ as observed in [25]: ideed we fit only the variance of $Z(t)$ and so $b = 1$ is not restrictive. The idea for correlation fitting is in observing the historical realizations of two (or more) underlying. Two different approaches can be used. The first one is simply to observe the correlation of log-returns in the market ρ_M and try to minimize

$$\boldsymbol{\eta}^* = \arg\min_{\boldsymbol{\eta}} \left(\rho_{\mathbf{Y}(t,\eta)} - \rho_M\right)^2. \tag{18}$$

where η is the set of parameters defining $\rho_{\mathbf{Y}(t,\eta)}$ i.e. the correlation of the model as computed by (2),(7) or (14). Some constraints on the parameters can be added.

Another possible approach is to use the GMM as presented in [17]. Since characteristic function $\varphi_{\boldsymbol{Y}(t)}(u_1, u_2)$ for the process $\boldsymbol{Y}(t)$ are avaible in closed form (see [27], [25] and [1]) it is possible to compute the moment generating function $M(u_1, u_2)$ and then computing the mixed moments:

$$\mathbb{E}\left[Y_1^n Y_2^m\right] = \left.\frac{\partial M^{m+m}(u_1, u_2)}{\partial u_1^n \partial u_2^m}\right|_{\substack{u_1=0\\u_2=0}}. \tag{19}$$

Then the Generalized Mehod of Moments is applied and vector $\boldsymbol{\eta}^*$ estimated. Both methods has advantages and drawbacks. The problem in (18) is undeterminated because we have only one equation and several parameters. In (19) numerical higher moments suffers of precision if the dataset is not large enoght. See Appendix A for a short comparison of the two methods.

5 Numerical Results

This section is made up of two parts: In the firstone we apply the three previous models to the German and French Forward Power Market and we compare results. Then we price a Cross-Border option written on this underlyings. In the second part of this section we repeat the same analysis using as underlyings German Power Forward and Coal Forward. The we price a Dark-Spread option.

5.1 Power Forward Markets and Cross-Border option pricing

The dataset[2] is composed as follow:

- Forward quotation from 25 April 2017 to 21 December 2019 of Calendar 2019 Power Forward. A Forward Calendar 2019 contract is a contract between two parties to buy or sell a specific volume of energy at fixed price for all the hours of 2019. Calendar Power Forward in German and France are stated with DEBY and F7BY respectively.

[2]Data Source: www.eex.com.

- Call Option on Power Forward 2019 quotations for both countries with settlement date 12 Novembre 2018. We used strikes is a range of $\pm 10[EUR/MWh]$ around the settlement price of the Forward contract, i.e. we exclude deep ITM and OTM options.

- We assume risk-free rate $r = 0.015$.

The first step is margin's parameters calibration. We approach the problem as descripted in Section 4 and we use a FFT approach for pricing as proposed in [10] with dumping parameter $\alpha = 0.75$. We have also used Lewis' approach which does require the introduction of a dumping parameter and we obtained the same results.

Optimal margins parameters are shown in Table 1. Here we note that margin's parameters for the three models are the same: indeed we imposed conditions leading the process $\boldsymbol{Y}(t)$ having the same margins distribution.

Using the optimal set of parameters $\boldsymbol{\theta}^*$ we obtained a good fit of market option prices as shown in Figure 1. Relative error for Call Option C_i, $E_{r,i}$, is computed as in (20):

$$E_{r,i} = 100 \cdot \frac{\left(C_i - C_i^{\boldsymbol{\theta}^*}(K,T)\right)}{C_i} \tag{20}$$

Parameter	Value
μ_1	1.03
μ_2	0.43
σ_1	0.30
σ_2	0.32
ν_1	0.01
ν_2	0.02

Table 1: Fitted margins parameters values for different models: Semeraro (S), Luciano-Semeraro (LS) and Ballotta-Bonfiglioli (BB).

Fitted margins parameters we have to fit dependence structure: using Forward quotations we computed a very hight correlation $\rho_M = 0.94$ which is typical in Forward Power Markets. We solved Problem 18 defined in Section 4. Fitted dependence parameters are shown in Table 2.

In Table 3 log-returns correlation caought is shown. Observe that only the BB-model is able to catch market correlation. S-model and LS-model has natural bounds on maximum value of correlation, as discussed in Section 2: as discussed in [25] we assume $b = 1$.

Suppose that a company want to price a Cross-Border Option. A Cross-Border option is an option written on both underlyings with payoff $\phi_T\left(F_T^1, F_T^2\right)$ given by:

$$\phi_T\left(F_T^1, F_T^2\right) = \left(F_T^1 - F_T^2 - K\right)^+$$

where F_t^1 is F7BY and F_t^2 is DEBY. We price this contract using the three models and comparing the prices. We choose as maturity one six month $T = 0.5$

Parameter	Value
a	45.054
b	1

S-model

Parameter	Value
a	45.054
b	1
ρ	1

LS-model

Parameter	Value
α_1	0.6297
α_2	0.6809
β_Z	0.4501
γ_Z	0.4656
ν_Z	0.0445

BB-model

Table 2: Fitted dependence parameters for the three models.

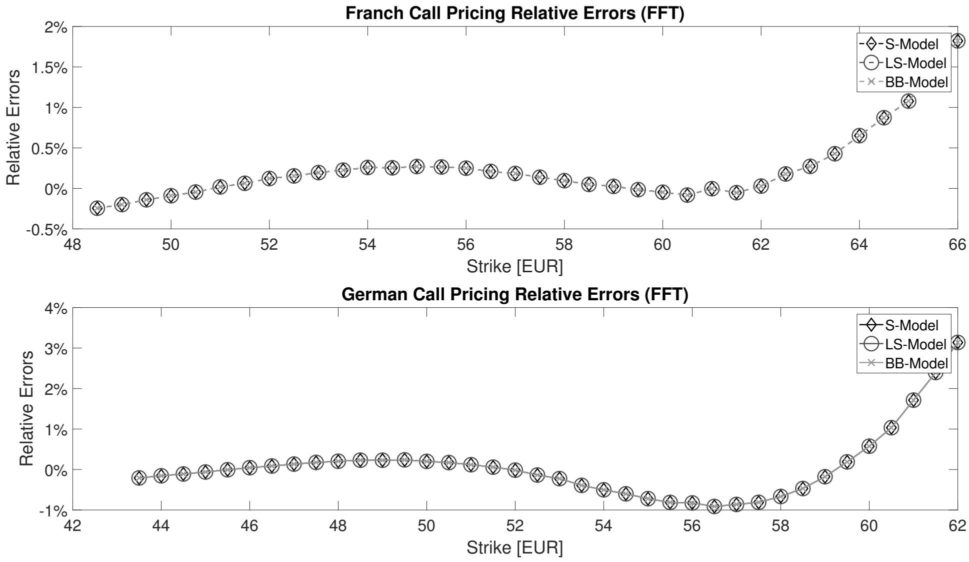

Figure 1: Pricing relative errors of the three models S, LS and BB. Pricing errors are very small and identical in all cases: this is due to the fact that fitted parameters shown in Table 1 are the same for the three models.

Parameter	Value
ρ_M	0.94
ρ_S	0.049
ρ_{LS}	0.71
ρ_{BB}	0.94

Table 3: Log-returns models correlation caught from the models compared with log-returns market correation.

and a strip of strikes $\boldsymbol{K} = [K_{min}, \cdots, K_{max}]$ with $K_{min} = 0$ and $K_{max} = 12$. In order to price this contract we use a Monte Carlo algorithm with $N_{sim} = 10^6$ simulations. Algorithm to generate a Gamma Variable are avaiable in the all most important software as MATLAB, PYTHON and R. And Algorithm to simulate the multivariate Variance Gamma process $\boldsymbol{Y}(t)$ can be easly derived starting from the once for univariate Variance Gamma presented in [29].
In Figure 2 results are dispayed. It is clear that option prices in S and LS-model are higher than in BB-model due to the fact that correlation ρ captured by the model is lower. This lead to a wider spread and, consequently, to a larger payoff at maturity. We can conclude that only BB-model is adeguate to price Cross-Border options in Power markets because it is able to replicate the hight correlation between commodities. Using S or LS model the contract may result overpriced.

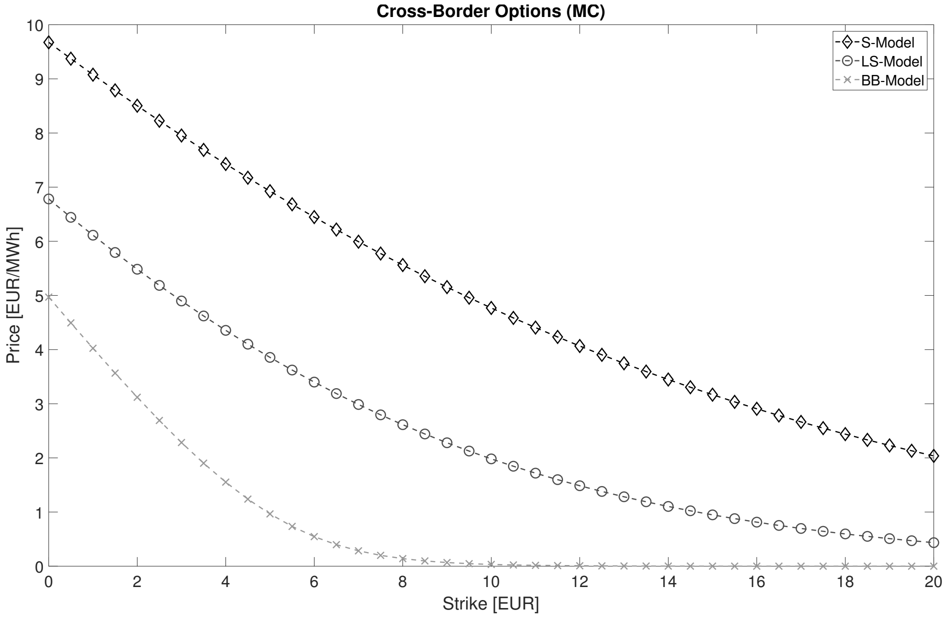

Figure 2: Premium of Cross-Border Options with different strikes in S-model, LS-model and BB-model.

5.2 Natural Gas and Power markets and Spark Spread Option pricing

The aim of this section is to price a spread option written on two underlyings: one is the Natural Gas TTF and the other one is DEBY. We have Forward products with delivery January 2020 and options writte on this Forward. The dataset[3] is compose as follow:

- Forward quotation from 1 July 2019 to 9 September 2019 of Montly January 2020 Power Germany and Natural Gas TTF. A Forward Month 2020 contract is a contract between two parties to buy or sell a specific volume of commodity at fixed price for all hours of January 2020.
- Call Options on Power and TTF Forward with settement date 9 September 2019. Again we used strikes is a range of $\pm 10[EUR/MWh]$ around the settlement price of the Forward contract.
- We assume, again, $r = 0.015$.

Margins parameters are listed in Table 4. In Table 5 correlations are shown. In this case we can see that all three models can reply the market correlation of log-returns. All the models, except S-model, are adeguate to price Spread Options because the dependence of log-returns is well fitted. Also in this case for Semeraro and Semeraro-Luciano model we fixed $b = 1$ as suggested in [25]. In Figure 3 we show option prices of several Spark-Spread Options computed with the three models. Pricing was performed using Monte Carlo methods with $N_{sim} = 5 \cdots 10^6$ and maturity $T = 0.5$.

[3]Datasource: The Ice: www.theice.com.

Parameter	Value
μ_1	0.48
μ_2	0.23
σ_1	0.43
σ_2	0.33
ν_1	0.07
ν_2	0.06

Table 4: Fitted margins parameters values for different models: Semeraro (S), Luciano-Semeraro (LS) and Ballotta-Bonfiglioli (BB).

Parameter	Value
ρ_M	0.54
ρ_S	0.04
ρ_{LS}	0.54
ρ_{BB}	0.54

Table 5: Log-returns models correlation caught from the models compared with log-returns market correation.

6 Conlcusion and Further developments

In this article we analyzed three different models proposed in litterature based on multivariate Lévy subordination. All models can be calibrated fitting the margins and then the dependence structure.
In case of Power Forward Markets the only model that can be used to price Cross Border Options in the BB-model, derived in [1] because this is the only one that car replicate the hight correlation inherent in this market.
Pricing Spark Spread option can be done using all the three models: this is due to the fact that correltion between log-returns of Power Forward Germany and Natural Gas TTF is lower that correlation between electricity products and so it is easier to replicate it.

In this paper we used as subordinator a Gamma Process that leads to a Variance Gamma process for log-return. Nevertheless theory in [27] can be extended also for other processes such as the well know Normal Inverse Gaussian process as can be found in [1] and [25]. All the steps in calbration are the same: chacacteristic functions and simulation algorithm for NIG process can be found in [29].
For sake of semplicity we used Monte Carlo algorthm for pricing but is possible to apply a FFT techniques also for Spread Options Pricing, as it was shown in [20].

Generally speaking we can conclude that all these models can be used in Energy Markets but must be handled carefully, due to hight correlation in these markets.

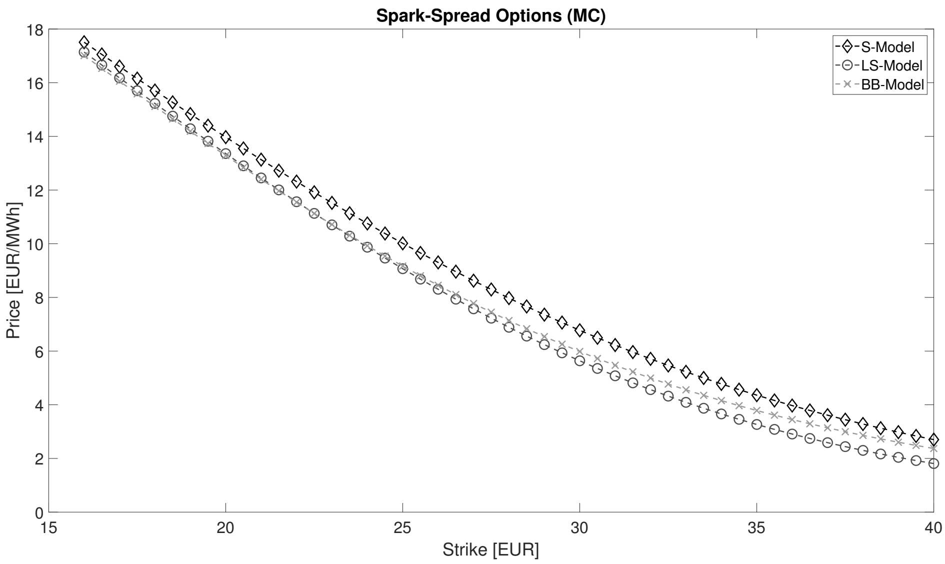

Figure 3: Premium of Spark-Spread Options with different strikes in S-model, LS-model and BB-model.

A Higher Moments Computation

In [27] is derived the close expression of the characteristic function of S-model. Characteristic function $\phi_t(u)$ is related to Moment Generating Function $M_t(u)$ via the following equation:

$$M_t(u) = \phi_t(-iu). \tag{21}$$

Using (21) and the expression of Characteristic Function for S-model we get:

$$\begin{aligned} M_{\boldsymbol{Y}(t)}(u_1, u_2) = & \left(1 - \frac{\alpha_1\left(\mu_1 u_1 + \frac{1}{2}\sigma_1^2 u_1^2\right)}{b}\right)^{-t\left(\frac{b}{\alpha_1} - a\right)} \\ & \left(1 - \frac{\alpha_2\left(\mu_2 u_2 + \frac{1}{2}\sigma_2^2 u_2^2\right)}{b}\right)^{-t\left(\frac{b}{\alpha_2} - a\right)} \\ & \left(1 - \frac{\alpha_1\left(u_1\mu_1 + \frac{1}{2}\sigma_1 u_1^2\right) + \alpha_2\left(u_2\mu_2 + \frac{1}{2}\sigma_2 u_2^2\right)}{b}\right)^{-ta}. \end{aligned} \tag{22}$$

If we define:

$$\begin{aligned} f_i(u_i) &= \frac{\alpha_i}{b}\left(\mu_i u_i + \frac{1}{2}\sigma_i^2 u_i^2\right) \quad i = 1, 2 \\ \gamma_1 &= -t\left(\frac{b}{\alpha_1} - a\right) \\ \gamma_2 &= -t\left(\frac{b}{\alpha_2} - a\right) \\ \gamma_3 &= -ta \end{aligned}$$

we can rewrite (22) as:

$$M_{\boldsymbol{Y}(t)}(u_1, u_2) = (1 - f_1(u_1))^{\gamma_1}(1 - f_2(u_2))^{\gamma_2}(1 - f_1(u_1) - f_2(u_2))^{\gamma_3} \tag{23}$$

In view of future computation it is useful define the following quantity:

$$K_1(u_1, u_2) = \gamma_1(1 - f_1(u_1))^{-1} + \gamma_3(1 - f_1(u_1) - f_2(u_2))^{-1}. \tag{24}$$

Given a Moment Generating Function for a random vector (X, Y) moments can be computed by:

$$m_n = \mathbb{E}[X^n(t)] = M_{(X,Y)}^{(n)}(0) = \left.\frac{d^n M_{(X,Y)}(u, v)}{du^n}\right|_{u=0, v=0}.$$

while mixed moment can be computed by:

$$\mathbb{E}\left[X^l Y^k(t)\right] = \left.\frac{d^n M_{(X,Y)}(u, v)}{du^l dv^k}\right|_{u=0, v=0}.$$

After tedious computatios one can derive closed fortmula for all mixed moments. For example we have:

$$\begin{aligned}\mathbb{E}\left[Y_1^3Y_2\right] =& \left.\frac{\partial^4 M}{\partial u_1^3\partial u_2}\right|_{u_1=0,u_2=0}\\ =& -2f_1''(u_1)\left[\frac{\partial^2 M}{\partial u_1\partial u_2}K(u_1,u_2)+\frac{\partial M}{\partial u_1}\frac{\partial K_1}{\partial u_2}+\frac{\partial M}{\partial u_2}\frac{\partial^2 K_1}{\partial u_1\partial u_2}\right]\\ &- f_1'(u_1)\left[\frac{\partial^3 M}{\partial u_1^2\partial u_2}K_1(u_1,u_2)+\frac{\partial^2 M}{\partial u_1^2}\frac{\partial K_1}{\partial u_2}+2\frac{\partial^2 M}{\partial u_1\partial u_2}\right.\\ &\left.\left.+\frac{\partial M}{\partial u_2}\frac{\partial^2 K_1}{\partial u_1^2}+M(u_1,u_2)\frac{\partial^3 K_1}{\partial u_1^2\partial u_2}\right]\right|_{\substack{u_1=0\\u_2=0}}\end{aligned} \tag{25}$$

How many moments should be selected to perform the Generalized Method of Moment? A first answer to this question is given in [16] where the autor states that under such circumstances, GMM estimation can be consistent but the rate of convergence depends on both the sample size and the number and the quality of the moment conditions. In addition to the usual source of signal from the moment conditions, the variability across moment conditions also plays a role in the asymptotic theory and can influence the point of consistency for the GMM estimator.
In this section we perform a simple numerical experiment: for a given number of scenarios $N_{scen} = 50$, we generate a trajector of $N_{Dates} = 650$ samples of process $\boldsymbol{Y}(t)$ under S-model with a certain set of parameters (randomly generated). Then we fit a and b parameters, supposed known the others, using three mixed moments, six mixed moments and we compute the correlation. Then we compare it with direct log-return correlation fitting. In Figure 4 is shown the fitting error of three methods. Error is computed as:

$$e_a = \rho_M - \rho_S.$$

In this case we see that adding three more moments does not give a so significant improvement to parameters extimation. Best correlation fitting is obtained using direct correlation fitting. Matching mixed moment improves if we increase the size of the sample (Figure 5). Anyway, we reserve to study this topic better on a future research.

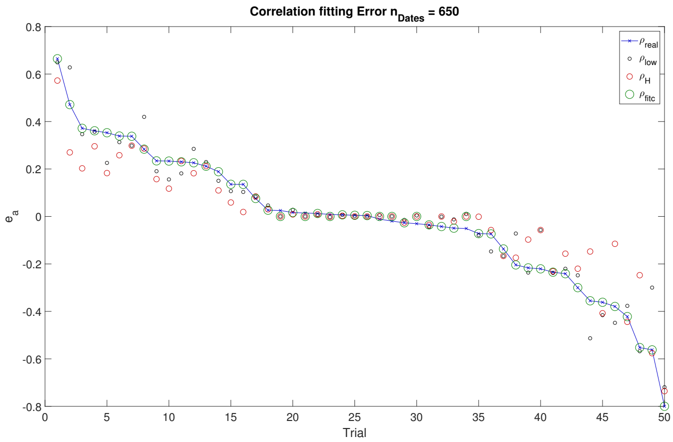

Figure 4: Small sample: 650 observation of correlated process $\boldsymbol{Y}(t)$.

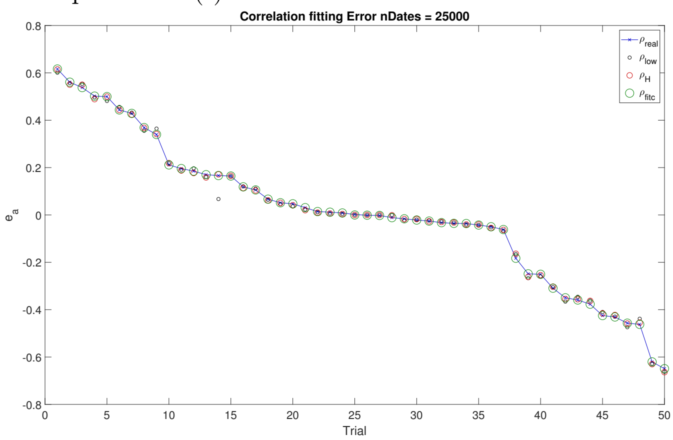

Figure 5: Large sample: 25000 observation of correlated process $\boldsymbol{Y}(t)$.

References

[1] L. Ballotta and E. Bonfiglioli. Multivariate asset models using Lévy processes and applications. 2013.

[2] O. E. Barndorff-Nielsen. Processes of normal inverse gaussian type. *Finance Stochastic*, (2(1)):41–68, 1998.

[3] O.E. Barndorff-Nielsen, J. Pedersen, and K.I. Sato. Multivariate subordination, selfdecomposability and stability. *Advanced Applied Probability*, 33:160–187, 2001.

[4] D. Bates. Jumps and stochastic volatility: the exchange rate processes implicit in deutschemark options. *Rev. Fin. Studies*, (9):69–107, 1996.

[5] F.E. Benth, S.B Benth, and S. Koekebbaker. *Stochastic Modelling of Electricity and Related Markets*. World Scientific, 2009.

[6] F. Black. The pricing of commodity contracts. *Journal Financial Economics*, (3):167–179, 1976.

[7] B. Buchmann, K. Lu, and D.B. Madan. Self-decomposibility of variance generalized gamma convolutions. 2019 in press.

[8] M. Burger and et al. A spot market model for pricing derivatives in electricity markets. *Quantitative Finance*, (4):109–122, 2004.

[9] R. Carmona and V. Durrleman. Pricing and hedging spread options. *SIAM Reviews*, (45):627–685, 2003.

[10] P. Carr and D. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, (2):61–73, 1999.

[11] P. Carr, D. Madan, and Chang E. The variance gamma process and option pricing. *European Finance Review*, (2):79–105, 1998.

[12] A. Cartea and M.G. Figueroa. Pricing in electricity markets: a mean reverting jump diffusion model with seasonality. *Applied Mathematical Finance*, 2005.

[13] A. Cartea and P. Villaplana. Spot price modeling and valutation of eletricity forward contracts; the role of demand and capacity. *Applied Mathematical Finance*, 2008.

[14] L. Clewlow and C. Strickland. A multifactor model for energy deriva- tives (working paper). 2000.

[15] R. Green. Risk management and the normal inverse gaussian distributionevidence from the nordic electricity market (working paper). 2006.

[16] C. Han and P.C.B. Philips. Gmm with many moment conditions. *Econometrica*, 74:147–192, 2006.

[17] L.P. Hansen. Large sample properties of generalized method of moments estimators. *Econometrica*, 50:1029–1054, 1982.

[18] D. Heat, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation. *Econometrica*, (60):77–105, 1992.

[19] S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Fin. Studies*, (6):327–343, 1993.

[20] T.R. Hurd and Z. Zhou. A fourier transform method for spread option pricing.

[21] T. Kluge. *Pricing Swing Options and Other Electricity Derivatives.* PhD thesis, University of Oxford, 2006.

[22] A. Lewis. A simple option formula for general jump-diffusion and other exponential Lévy processes. 2001.

[23] J.J. Lucia and E. Schwartz. Electricity prices and power derivatives: Evidence from the nordic power exchange. *Review of Derivatives Research*, (5):5–50, 2002.

[24] E. Luciano and W. Schoutens. A multivariate jump-driven financial asset model. *Quantitative Finance*, (6), 2005.

[25] E. Luciano and P. Semeraro. Multivariate time changes for Lévy asset models: Characterization and calibration. *Journal of Computational and Applied Mathematics*, 233(1):1937–1953, 2010.

[26] E. Schwartz and J.E. Smith. Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46(7):893–911, 2000.

[27] P. Semeraro. A multivariate variance gamma model for financial applications. *International Journal of Theoretical and Applied Finance*, 11(1):1–18, 2008.

[28] P. Semeraro and E. Luciano. A generalized normal mean variance mixture for return process in finance. *International Journal of Theorical and Applied Finance*, 2008 in press.

[29] P. Tankov and R. Cont. *Financial Modeling with Jump Processes.* Chapman and Hall, 2004.