Covariance of Geometric Brownian Motion

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March 28, 2025

Abstract

Some useful computations.

Keywords: Geometric Brownian Motion, Stochastic Calculus, Stochastic Processes.

1 Computations regarding the hedging problem

Stock process:

$$S(t) = S_0 e^{-\frac{\sigma^2}{2}t + \sigma W(t)}.$$

$$\mathbb{E}\left[S(t)\right] = S_0, \qquad Var\left[S(t)\right] = S_0^2\left(e^{\sigma^2 t} - 1\right),$$
$$cov\left(W(t), W(s)\right) = t \wedge s.$$

We have the following relations. $Y_i \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right)$, $cov\left(Y_i, Y_j\right) = \rho_{ij}\sigma_i\sigma_j$, define $X_i = e^{Y_i}$.

In this case:

$$Var\left[X_i + X_j\right] = Var\left[e^{Y_i} + e^{Y_j}\right] = Var\left[e^{Y_i}\right] + Var\left[e^{Y_j}\right] + 2cov\left[e^{Y_i}, e^{Y_j}\right],$$

and

$$\begin{aligned} cov\left[e^{Y_i}, e^{Y_j}\right] &= \mathbb{E}\left[e^{Y_i + Y_j}\right] - \mathbb{E}\left[e^{Y_i}\right]\mathbb{E}\left[e^{Y_j}\right] = e^{\mathbb{E}[Y_i + Y_j] + \frac{1}{2}Var[Y_j, Y_i]} - \mathbb{E}\left[e^{Y_i}\right]\mathbb{E}\left[e^{Y_j}\right] \\ &= \exp\left\{\mu_i + \mu_j + \frac{1}{2}\left(\sigma_i^2 + \sigma_j^2 + 2\rho\sigma_i\sigma_j\right)\right\} - \exp\left\{\mu_j + \frac{\sigma_j}{2}\right\}\exp\left\{\mu_i + \frac{\sigma_i}{2}\right\} \end{aligned}$$

Now $S(t_1) = S_0 e^{-\frac{\sigma^2}{2}t_1 + \sigma W(t_1)}$ and $S(t_2) = S_0 e^{-\frac{\sigma^2}{2}t_2 + \sigma W(t_2)}$.

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$$Var\left[a_1S(t_1),a_2S(t_2)\right]=a_1^2Var\left[S(t_1)\right]+a_2^2Var\left[S(t_2)\right]+2a_1a_2cov\left[S(t_1),S(t_2)\right].$$

$$\begin{aligned}cov\left[S(t_1),S(t_2)\right]&=\mathbb{E}\left[S_0^2e^{-\frac{\sigma^2}{2}t_1-\frac{\sigma^2}{2}t_2+\sigma W(t_1)+\sigma W(t_2)}\right]-\mathbb{E}\left[S(t_1)\right]\mathbb{E}\left[S(t_2)\right]\\&=S_0^2e^{-\frac{\sigma^2}{2}(t_1+t_2)}\mathbb{E}\left[e^{\sigma W(t_1)+\sigma W(t_2)}\right]-S_0^2.\end{aligned}$$

And:

$$\begin{aligned}\mathbb{E}\left[e^{\sigma W(t_1)+\sigma W(t_2)}\right]&=\exp\left\{\mathbb{E}\left[\sigma W(t_1)+\sigma W(t_2)\right]+\frac{1}{2}Var\left[\sigma W(t_1)+\sigma W(t_2)\right]\right\}\\&=\exp\left\{\frac{\sigma^2}{2}(t_1+t_2)+\sigma^2cov\left(W(t_1),W(t_2)\right)\right\}\\&=\exp\left\{\frac{\sigma^2}{2}(t_1+t_2)+\sigma^2(t_1\wedge t_2)\right\}\end{aligned}.$$

Assume that $t_1\leq t_2$. We have:

$$\begin{aligned}Var\left[a_1S(t_1)+a_2S(t_2)\right]&=a_1^2Var\left[S(t_1)\right]+a_2^2Var\left[S(t_2)\right]+2a_1a_2cov\left[S(t_1),S(t_2)\right]\\&=a_1^2S_0^2\left(e^{\sigma^2t_1}-1\right)+a_2^2S_0^2\left(e^{\sigma^2t_2}-1\right)+2a_1a_2S_0^2\left(e^{\sigma^2t_1}-1\right).\end{aligned}$$

Assuming $t_1\leq t_2\leq t_3$ and using all the previous relations:

$$\begin{aligned}Var\left[S(t_1)+S(t_2)+S(t_3)\right]&=Var\left[S(t_1)\right]+Var\left[S(t_2)\right]Var\left[S(t_3)\right]\\&\quad+2cov\left(S(t_1),S(t_2)\right)-2cov\left(S(t_1),S(t_3)\right)-2cov\left(S(t_2),S(t_3)\right)\\&=S_0^2\left[e^{\sigma^2t_1}-e^{\sigma^2t_2}+e^{\sigma^2t_3}-1\right]\end{aligned}$$

The latter quantity can be smaller that $Var\left[S(t_1)\right]=S_0^2\left(e^{\sigma^2t_1}-1\right)$?

Yes, if and only if $e^{\sigma^2t_3}-e^{\sigma^2t_2}\leq 0$ and this happens only if $t_3\leq t_2$ and this is absurd, since we have assumed $t_2\leq t_3$.

In conclusion, hedging as soon as possible is the strategy such that (under these hypothesis) the risk is reduced.