Exchange option pricing under variance gamma-like models

Matteo Gardini[*] Piergiacomo Sabino[†‡]

December 1, 2023

Abstract

In this article we focus on the pricing of exchange options when the risk-neutral dynamic of log-prices follows either the well-known variance gamma or the recent variance gamma++ process introduced in Gardini et al. [19]. In particular, for the former model we can derive a Margrabe's type formula whereas, for the latter one we can write an "integral free" formula. Furthermore, we show how to construct a general multidimensional versions of the variance gamma++ processes preserving both the mathematical and numerical tractability.
Finally we apply the derived models to German and French energy power markets: we calibrate their parameters using real market data and we accordingly evaluate exchange options with the derived closed formulas, Fourier based methods and Monte Carlo techniques.

Keywords: Lévy Processes; Exchange Options; Energy Markets; Derivative Pricing

1 Introduction

Spread options are widely used in many fields of finance, in particular in energy markets, and their payoff depends on the difference of the value of two risky underlying assets instead of one. The spread might be between spot and futures prices, between currencies, interest rates or commodities, among others. As mentioned, this type of options is particularly frequent in energy markets and the name changes depending on the commodity one is dealing with. The most notable examples are the crack, crush, and spark spreads, which measure profits in the oil, soybean, and gas markets, respectively. The interested reader can refer to Carmona and Durrleman [10] for a detailed discussion on this topic. By denoting $S_i = \{S_i(t); t \geq 0\}, i = 1, 2$, the evolution of the value of two risky assets respectively, and labelling the strike price with K, the payoff of a spread option at maturity T is given by

$$\Phi(T) = (S_2(T) - S_1(T) - K)^+ . \tag{1}$$

[*]Department of Mathematics, University of Genoa, Via Dodecaneso 16146, Genoa, Italy, email gardini@dima.unige.it

[†]Quantitative Risk Management, E.ON SE, Brüsseler Platz 1, 45131 Essen, Germany, email piergiacomo.sabino@eon.com

[‡]Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 FI-00014 Finland, email piergiacomo.sabino@helsinki.fi

The spread option valuation has been widely investigated: the first attempt to properly compute the fair price of such a contract can be found in Margrabe [35] where the author finds a closed formula assuming a Black and Scholes [6] market and $K = 0$. Unfortunately, so far, no closed form for the more general case $K \neq 0$ is known. Nevertheless, many approximate formulas have been proposed over the years. One which is commonly used has been proposed by Kirk [26] and only provides a good approximation for small values of K. Many other alternatives are available in the literature under the assumption of a the Black-Scholes (BS) market, see for instance Bjerksund and Stensland [5] and Carmona and Durrleman [10]. They can be also extended taking a multi-asset spread (see for instance S. J. Deng and Zhou [44], Pellegrino and Sabino [42, 43]) or a non-Gaussian market (see for instance Pellegrino [41]). On the other hand, it is well know that price dynamics present jumps, fat tails, volatility clusters or even stochastic volatility and many other stylized facts which cannot be properly modeled by Gaussian processes. One approach to address some of the issues mentioned above is to rely on Lévy processes which have gained popularity because of their mathematical tractability also from the numerical point of view (see Sato [46] and Applebaum [2] for an extensive discussion and Cont and Tankov [14] for applications to financial markets). On the order hand, if one leaves the Gaussian framework things might be hard to handle. Nevertheless, in some cases, such as the variance gamma (VG) process (Madan and Seneta [34]), the jump diffusion models proposed by Merton [37] and Kou [27] and the variance gamma++ (VG++) process recently investigated by Gardini et al. [19], closed formula expressions for European options valuation can be derived. For more general models or complicated derivative structures one has to resort to numerical methods, for instance based on Monte Carlo simulations (Glasserman [20]), PDE's (Seydel [49] and Cont and Tankov [14]), Fourier methods (see Carr and Madan [11], Lewis [28], Attari [3], Lord et al. [29]), or on the knowledge of the expression of the characteristic function (see Fang [16]). Therefore, the toolkit is rich to properly model single asset dynamics. Unfortunately, this is not always the case for the multidimensional framework. Modeling, calibration and pricing with multivariate Lévy processes are challenging tasks, several attempts are available in the literature: one example is the copula approach proposed by Cherubini et al. [12] and Cont and Tankov [14] which sometimes is difficult to handle. Moreover, in a multidimensional framework, numerical methods based on PDE or on the Fourier transform gain computational complexity. Finally, the calibration step, which is essential, might be very unstable. To address these issues Schoutens [47], Semeraro [48], Luciano and Semeraro [32] and Ballotta and Bonfiglioli [4] proposed a framework, based on multivariate Brownian subordination, which leads to a tractable class of multidimensional Lévy models both from a numerical and a theoretical point of view. However, in such frameworks, closed formula expressions for spread options are rare (see for instance Cufaro Petroni and Sabino [15]) and numerical methods, such as those proposed by Choi [13], Hurd and Zhou [23], Van Belle et al. [51] and Caldana and Fusai [9], should be used. Nevertheless, in some very particular cases, closed form expressions for very simple exchange options can be obtained giving a computational advantage and a useful benchmark to test numerical algorithms. In addition, they can be used in numerical inversion to guess the so called "implied parameters" as observed by Carmona and Durrleman [10].

The first contribution of this article is the derivation of an analytic expression for the exchange options under a bi-dimensional version of the VG model which improves the results of Hürlimann [24] and Van Belle et al. [51]. The second contribution is the finding of an integral free Margrabe's style formula for the VG++ model introduced by Gardini et al. [19] which can be considered an extension of the VG model. To this end, we derive a multidimensional versions of the VG++ process, we briefly study its properties, we provide closed form expressions of their characteristic function and that of the linear correlation coefficient. Accordingly, we compare the performance of the new formulas.

The article is organized as follows: in Sections 2 and 3 we recall the Margrabe formula and extend it to the VG setting. In Section 4 we recall the notion of self-decomposability and we briefly introduce the VG++ process. These concepts are preparatory for the derivation of the integral free pricing formula for exchange options in the VG++ framework presented in Section 5. Then, in Section 6 we derive multidimensional versions of the VG++ process, we briefly study its properties, provide closed form expressions of their characteristic function and that of the linear correlation coefficient. Section 7 contains the applications to concrete problems: we consider electricity markets and calibrate the model parameters to real data and price exchange options written on futures power German and France calendar contracts using the new closed formulas, Monte Carlo methods and Fourier techniques. Section 8 concludes the paper and discusses possible extensions along with further inquires.

2 The Margrabe's formula

In this section we briefly recall some well-known results which are preparatory for the sequel. Consider a market with two risky assets, S_1 and S_2 and the money market account M, with the following dynamics under the risk neutral measure $\mathbb{Q}$

$$dM(t) = rM(t)dt, \quad dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dW_i^{\mathbb{Q}}(t), \quad i = 1, 2, \tag{2}$$

where $W_1^{\mathbb{Q}}$ and $W_2^{\mathbb{Q}}$ are Brownian motions such that $\mathbb{E}\left[W_1^{\mathbb{Q}}(t)W_2^{\mathbb{Q}}(t)\right] = \rho t$. Observe that in Equation (2) the drift is denoted by μ_i. Consider the pricing problem of an exchange option, whose payoff at maturity T is given by

$$\Phi(T) = \left(S_2(T) - S_1(T)\right)^+, \tag{3}$$

namely a spread option with a zero strike price. By using the change of numéraire technique, presented in Shreve [50], taking S_2 as numéraire, it is easy to show that the value of this contract $V(0)$ at time $t = 0$ is given by

$$V(0) = e^{(\mu_2 - r)T} S_2(0)\mathcal{N}(d_1) - e^{(\mu_1 - r)T} S_1(0)\mathcal{N}(d_2), \tag{4}$$

where

$$d_1 = \frac{\log\left(\frac{S_2(0)}{S_1(0)}\right) + \frac{1}{2}\bar{\sigma}^2 T}{\bar{\sigma}\sqrt{T}}, \quad d_2 = d_1 - \bar{\sigma}\sqrt{T}, \quad \bar{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2. \tag{5}$$

Of course when $\mu_1 = \mu_2 = r$ one gets the well known formula which was derived by Margrabe [35]. Clearly, since we mentioned that we are working under $\mathbb{Q}$, we could have set written the equations above taking $\mu_1 = \mu_2 = r$. Nevertheless, we prefer keeping this notation since it will be easier to adapt the following discussion. If the value of the strike price K is different from zero the payoff at time T is given by Equation (1) and no closed pricing formula is known therefore, numerical techniques or approximations must be adopted to properly evaluate the derivative.

It is customary to model the asset prices by the so called Lévy exponentiation framework, presented in Cont and Tankov [14]. The idea is to model the asset price process $S = \{S(t); t \geq 0\}$ as $S(t) = e^{rt+X(t)}$, where $X = \{X(t); t \geq 0\}$ is a Lévy Process. If X is a Brownian motion with drift, we talk about Brownian exponentiation and we reduce to the original BS framework. In particular, for $T > 0$, we assume that

$$M(T) = M(0)\exp(rT), \quad S_i(T) = S_i(0)\exp\left(\omega_i T + rT + \theta_i T + \sigma_i W_i^{\mathbb{Q}}(T)\right), \quad i = 1, 2,$$

where $\mathbb{E}\left[W_1^{\mathbb{Q}}(t)W_2^{\mathbb{Q}}(t)\right] = \rho t$ and ω_1 and ω_2 are the drift corrector parameters[1]: they are chosen so that the discounted price processes are martingales. Applying Itō's Lemma we obtain the dynamic of S_1 and S_2 which is given by

$$\frac{dS_i(t)}{S_i(t)} = \left(r + \omega_i + \theta_i + \frac{\sigma_i^2}{2}\right)dt + \sigma_i dW_i^{\mathbb{Q}}(t), \quad i = 1, 2.$$

In order to price exchange options we can use Equation (4) where we take:

$$\mu_i = r + \omega_i + \theta_i + \frac{\sigma_i^2}{2}, \quad i = 1, 2.$$

This formula allows us to easily cope with the more general problem of pricing an exchange option under a time changed stochastic process like, for example, the VG model introduced by Madan and Seneta [34], which overcomes some limits of the standard BS approach.

3 Margrabe's formula under the VG model

In this section, we derive a Margrabe style option pricing formula assuming that the asset log-price dynamic follows a VG process. Let $G = \{G(t); t \geq 0\}$ be a gamma subordinator with law $\Gamma(\alpha t, \beta)$, we model the evolution of the price of the risky assets under the risk neutral measure as follows

$$S_i(T) = e^{\omega_i T + rT + \theta_i G(T) + \sigma_i W_i(G(T))}, \quad i = 1, 2,$$

where $\mathbb{E}[W_1(t)W_2(t)] = \rho t$ and:

$$\omega_i = \alpha t \cdot \log\left(1 - \frac{\theta_i}{\beta} - \frac{\sigma_i^2}{2\beta}\right).$$

Note that the two dynamics share the same subordinator G. This choice ensures that the couple $(\log(S_1(t)), \log(S_2(t)))$ is a Lévy process. If we use two different subordinators, G_1 and G_2, there is no guarantee that the couple is a Lévy process, as observed in Buchmann et al. [7], Buchmann et al. [8] and Michaelsen and Szimayer [38].

[1] In BS model we have that $\omega_i = -\left(\theta_i + \frac{\sigma_i^2}{2}\right)$, for $i = 1, 2$.

Remark 1. In financial modeling it is customary to impose that $\mathbb{E}\left[G(t)\right] = t$, which means that, on average, the stochastic time runs as fast as the deterministic one. This condition entails some parameter constrains: for example, if $G(t) \sim \Gamma\left(\alpha t, \beta\right)$, we have $\mathbb{E}\left[G(t)\right] = \alpha t/\beta$ and hence $\alpha = \beta$. Nevertheless, we keep our setting as general as possible but adopt such an assumption in the numerical section.

Remark 2. We recall that if $Y_i = \left\{W_i\left(G(t)\right); t \geq 0\right\}$ is a subordinated Brownian motion with drift θ_i, diffusion σ_i and the subordinator is $G = \{G(t); t \geq 0\}$, the correlation coefficient at time $t \geq 0$ is given by:

$$\rho_{Y_i(t),Y_j(t)} = \frac{\theta_i\theta_j Var\left[G(t)\right] + \rho\sigma_i\sigma_j\mathbb{E}\left[G(t)\right]}{\sqrt{\sigma_i^2\mathbb{E}\left[G(t)\right] + \theta_i^2 Var\left[G(t)\right]}\sqrt{\sigma_j^2\mathbb{E}\left[G(t)\right] + \theta_j^2 Var\left[G(t)\right]}}, \tag{6}$$

where $\rho = \mathbb{E}\left[W_i(1)W_j(1)\right]$. In the multidimensional Variance Gamma, where Y_i is a $VG\left(\theta_i, \sigma_i, \alpha_i, \beta_i\right)$ process, Equation (6) reduces to:

$$\rho_{Y_i(t),Y_j(t)} = \frac{\theta_i\theta_j\frac{\alpha}{\beta^2} + \rho\sigma_i\sigma_j\frac{\alpha}{\beta}}{\sqrt{\sigma_i^2\frac{\alpha}{\beta} + \theta_i^2\frac{\alpha}{\beta^2}}\sqrt{\sigma_j^2\frac{\alpha}{\beta} + \theta_j^2\frac{\alpha}{\beta^2}}},$$

Under the usual risk-neutral argument, the price of an exchange option at time $t = 0$ is

$$\begin{aligned}
V(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(S_2(T) - S_1(T)\right)^+\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\left(S_2(0)e^{\omega_2 T + \theta_2 g + \sigma_2 W_2(g)} - S_1(0)e^{\omega_1 T + \theta_1 g + \sigma_1 W_1(g)}\right]\right)^+ \middle| G(T) = g\right]\Bigg] \\
&= \mathbb{E}^{\mathbb{Q}}\left[S_2(0)e^{\omega_2 T + \left(\theta_2 + \frac{\sigma_2^2}{2}\right)G(T)}\mathcal{N}\left(d_1(G(T))\right) - S_1(0)e^{\omega_1 T + \left(\theta_1 + \frac{\sigma_1^2}{2}\right)G(T)}\mathcal{N}\left(d_2(G(T))\right)\right],
\end{aligned}$$

where in the last step, we used the Margrabe's formula derived in Equation (4) and where

$$d_1(g) = \frac{\log\left(\frac{S_2(0)}{S_1(0)} + \left(\omega_2 - \omega_1\right)T + \left(\theta_2 - \theta_1\right)g + \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)g + \frac{1}{2}\bar{\sigma}g\right)}{\bar{\sigma}\sqrt{g}}, \tag{7}$$

$$d_2(g) = d_1(g) - \bar{\sigma}\sqrt{g}, \tag{8}$$

where $\bar{\sigma}$ is defined as in Equation (5). Since the probability density function of $\Gamma\left(\alpha T, \beta\right)$ is given by

$$f\left(x; \alpha T, \beta\right) = \frac{\beta^{\alpha T}}{\Gamma\left(\alpha T\right)}x^{\alpha T - 1}e^{-\beta x}\mathbb{1}_{x>0},$$

we get:

$$\begin{aligned}
V(0) = &\underbrace{\int_0^\infty S_2(0)e^{\omega_2 T + \left(\theta_2 + \frac{\sigma_2^2}{2}\right)g}\mathcal{N}\left(d_1(g)\right)\frac{\beta^{\alpha T}}{\Gamma\left(\alpha T\right)}g^{\alpha T - 1}e^{-\beta g}dg}_{I_1} \\
&- \underbrace{\int_0^\infty S_1(0)e^{\omega_1 T + \left(\theta_1 + \frac{\sigma_1^2}{2}\right)g}\mathcal{N}\left(d_2(g)\right)\frac{\beta^{\alpha T}}{\Gamma\left(\alpha T\right)}g^{\alpha T - 1}e^{-\beta g}dg}_{I_2}.
\end{aligned} \tag{9}$$

Defining

$$\Psi\left(a,b;\gamma\right)=\int_{0}^{\infty}\mathcal{N}\left(\frac{a}{\sqrt{u}}+b\sqrt{u}\right)\frac{u^{\gamma-1}}{\Gamma\left(\gamma\right)}e^{-u}du,$$

and focusing on the computation of I_1 we have that

$$I_1=S_2(0)\beta^{\alpha T}e^{\omega_2 T}\int_0^{\infty}\frac{1}{\Gamma\left(\alpha T\right)}g^{\alpha T-1}e^{-\left(\beta-\left(\theta_2+\frac{\sigma_2^2}{2}\right)\right)g}$$
$$\mathcal{N}\left(\frac{\log\left(\frac{S_2(0)}{S_1(0)}\right)+\left(\omega_2-\omega_1\right)T}{\bar{\sigma}\sqrt{g}}+\frac{\theta_2-\theta_1+\frac{1}{2}\left(\sigma_2^2-\sigma_1^2+\bar{\sigma}^2\right)}{\bar{\sigma}}\sqrt{g}\right)dg.$$

Now defining

$$A=\beta-\left(\theta_2+\frac{\sigma_2^2}{2}\right),\qquad u=Ag,$$

we get

$$I_1=S_2(0)\beta^{\alpha T}e^{\omega_2 T}\int_0^{\infty}\frac{1}{\Gamma\left(\alpha T\right)}\frac{1}{A^{\alpha}}u^{\alpha-1}e^{-u}\mathcal{N}\left(\frac{\tilde{a}}{u}+\tilde{b}\sqrt{u}\right)du,$$

where

$$\tilde{a}=\frac{\log\left(\frac{S_2(0)}{S_1(0)}+\left(\omega_2-\omega_1\right)T\right)\sqrt{A}}{\bar{\sigma}},\qquad \tilde{b}=\frac{\theta_2-\theta_1+\frac{1}{2}\left(\sigma_2^2-\sigma_1^2+\bar{\sigma}^2\right)}{\bar{\sigma}\sqrt{A}}. \tag{10}$$

Finally, we obtain:

$$I_1=S_2(0)\frac{\beta^{\alpha T}e^{\omega_2 T}}{A^{\alpha T}}\Psi\left(\tilde{a},\tilde{b},\alpha T\right).$$

The second integral I_2 can be computed in a similar way. defining

$$C=\beta-\left(\theta_1+\frac{\sigma_1^2}{2}\right),$$

and

$$\tilde{c}=\frac{\log\left(\frac{S_2(0)}{S_1(0)}+\left(\omega_2-\omega_1\right)T\right)\sqrt{C}}{\bar{\sigma}},\qquad \tilde{d}=\frac{\theta_2-\theta_1+\frac{1}{2}\left(\sigma_2^2-\sigma_1^2+\bar{\sigma}^2\right)}{\bar{\sigma}\sqrt{C}},$$

we obtain that

$$I_2=S_1(0)\frac{\beta^{\alpha T}e^{\omega_1 T}}{C^{\alpha T}}\Psi\left(\tilde{c},\tilde{d},\alpha T\right).$$

Finally, the price of the exchange option in the VG model is given by

$$V(0)=S_2(0)\frac{\beta^{\alpha T}e^{\omega_1 T}}{A^{\alpha}}\Psi\left(\tilde{a},\tilde{b};\alpha T\right)-S_1(0)\frac{\beta^{\alpha T}e^{\omega_1 T}}{C^{\alpha}}\Psi\left(\tilde{c},\tilde{d};\alpha T\right). \tag{11}$$

Incidentally, we observe that the derivation method for the Equation (11) is similar to the one used in Madan and Seneta [34] to derive the European call pricing formula.

The function $\Psi\left(a,b;\gamma\right)$ can be written in terms of the modified Bessel function of the second type (see Abramowitz and Stegun [1, pag. 374]) and of the confluent hypergeometric function of two variables introduced by Humbert [22] hence, it can be efficiently computed,

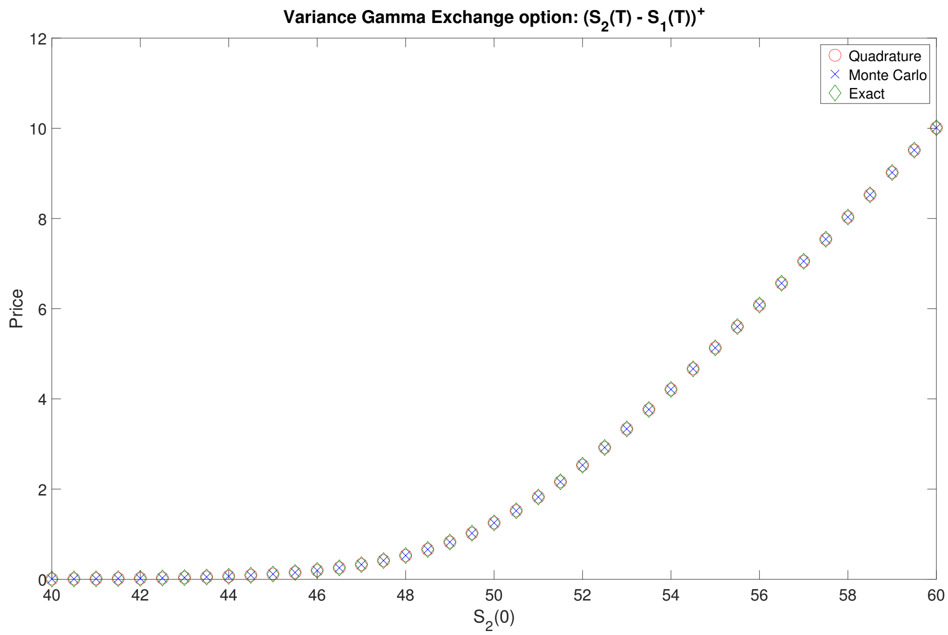

Figure 1: Price of an exchange option priced by using quadrature, Monte Carlo simulation and the closed formula. $S_1(0) = 50$, $S_2(0) \in [40, 60]$, $\mu_1 = -0.55$, $\mu_2 = 0.10$, $\sigma_1 = 0.25$, $\sigma_2 = 0.15$, $r = 0.05$, $T = 2$, $\alpha = 1$, $\beta = 8$.

as shown in Madan et al. [33]. Under the additional hypothesis $S_1(0) = S_2(0)$ and $\mu_1 = \mu_2$, Van Belle et al. [51] have shown that a closed form option pricing formula for exchange options can be obtained in terms of the gamma and of the hypergeometric functions (see Abramowitz and Stegun [1, pag 556]). It is easy to check that such a result is a particular case of the more general formula presented in Equation (11).

In Figure 1 we compare three different methods to compute the exchange option price under the VG model: a standard Monte Carlo method (see Cont and Tankov [14]), a quadrature method to directly integrate Equation (9) and finally the closed formula in Equation (11). As discussed by Loregian et al. [30] and Mercuri et al. [36], the most appropriate integration quadratures for integrals with support $[0, +\infty)$ are the Gauss-Laguerre and the Generalized Gauss-Laguerre. In particular, the numerical procedure discussed by Mercuri et al. [36] can be easily applied to numerically compute the integrals in Equations (9) and (23). Another way to deal with the numerical computation of such integrals is to use Gauss-Kronrod quadrature which has been deeply investigated by Kahaner and Monegato [25] and Monegato [39] among the others. We observe that all methods produce the same results, as expected. Of course, the direct integration approach should be implemented carefully since the integrand function is quite complex. In Table 1 we compare the computational time of both the Monte Carlo approach with 10^6 simulations and the closed formula. We observe that the closed formula is approximately one hundred times faster than the Monte Carlo method and therefore such an approach is the preferable one[2].

[2] The computational performance has been measured using MATLAB on a PC with an Intel Core i5-10210U 2.11 GHz processor.

Algorithm	Computational time (s)
Monte Carlo	$2.60 \cdot 10^{-1}$
Closed formula	$3.91 \cdot 10^{-3}$

Table 1: Computational time to price an exchange option with the closed formula and $N_{sim} = 10^6$ Monte Carlo simulations: the resulting time of the Monte Carlo is the average of 100 runs.

4 Self-decomposability and the VG++ process

Gardini et al. [19] have studied a new Lévy process called the VG++ process. Its construction relies upon the notion of self-decomposability which on the other hand can be also used to build multidimensional Lévy processes including stochastic delay in the propagation of market news, as discussed in Gardini et al. [17, 18].

In defining the VG++ process the main idea is to replace the gamma subordinator by another subordinator strictly related to it which in turn, has finite activity and the step-wise increasing trajectories. Therefore, seeing the subordinator as the evolution of the number of transactions, the probability of having no transaction in a finite period of time will not be in general zero.

We recall that a random variable X is said to have a self-decomposable law if for all $a \in (0, 1)$ we can always find two independent random variables Y, with the same law as X, and Z_a such that in distribution we have

$$X \stackrel{d}{=} aY + Z_a. \tag{12}$$

We refer to Z_a as the a-remainder of the self-decomposable law. Accordingly, denoting $\phi_X(u)$ the characteristic function of X and by $\phi_{Z_a}(u)$ that of Z_a we have that

$$\phi_X(u) = \phi_X(au)\,\phi_{Z_a}(u)\,. \tag{13}$$

The law of Z_a turns out to be infinitely divisible (see Sato [46, Proposition 15.5]) and one can construct the associated Lévy process $Z_a = \{Z_a(t); t \geq 0\}$ which is actually a subordinator that can be used for Brownian subordination.

The gamma distribution is a well-known self-decomposable law (see Grigelionis [21]) and the distribution $\Gamma^{++}(a, \alpha, \beta)$ of its a-remainder Z_a is infinitely divisible. The VG++ process is then defined as follows

Definition 4.1. *Consider a Brownian motion $W = \{W(t); t \geq 0\}$, with drift $\theta \in \mathbb{R}$, diffusion $\sigma \in \mathbb{R}^+$ independent of Z_a^{++}. We call the process $X = \{X(t); t \geq 0\}$ defined as*

$$X(t) = \theta Z_a(t) + \sigma W(Z_a(t))\,, \quad t \geq 0 \tag{14}$$

VG++ process.

Such a process has been studied in details in Gardini et al. [19] and it has several nice properties. It can be written as difference of two Z_a^{++} processes with suitable parameters, it is of finite activity and therefore of finite variation. In particular, the probability density function of X at time $t > 0$ and its characteristic function can be written in closed form. In addition, the path-simulation is very efficient. Accordingly, all the common numerical techniques for calibration and pricing can be adopted. Finally, a numerically convenient integral-free closed form option pricing formula for European call (and put) options is available.

From an economical point, the VG++ process can be used to model illiquid markets. Indeed, the value of the parameters a and α are related to the liquidity activity of the market. Taking the change $\Delta X = X(t) - X(t-1)$ of the log-price over the time interval Δt, it can be shown that the probability that the increment equals zero over the time interval Δt is strictly larger than zero and, more precisely, it is given by

$$P\left(\Delta X = 0\right) = a^{\alpha \Delta t},$$

since the density of the VG++ process has an atom in zero. This is the main financial difference from the standard VG process which does imply that non-zero trading activity takes place in every time interval. On the other hand, the VG++ model inherits the mathematical tractability of the standard VG process which is, in any case, recovered when a tends to zero.

It turns out that an integral free closed formula for exchange options can obtained for the VG++ model as well, which will be detailed in the next section.

5 An integral free formula for exchange options under the VG ++ process

In this section we derive the explicit formula for exchange options under the VG++ process but first recall some useful relations.

Following Humbert [22], we define the confluent hypergeometric function of two variables x and y as

$$\Phi\left(\alpha, \beta, \gamma; x, y\right) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}\left(1-u\right)^{\gamma-\alpha-1}\left(1-ux\right)^{-\beta} e^{uy} du, \tag{15}$$

and observe that if $\alpha = n, \beta = 1-n$ and $\gamma = 1+n$ for $n \in \mathbb{N}$, Equation (15) becomes:

$$\Phi\left(n, 1-n, 1+n; x, y\right) = n \int_0^1 u^{n-1}\left(1-ux\right)^{n-1} e^{uy} du. \tag{16}$$

It turns out that the integral in Equation (16) can be computed exactly, as it will be shown later. $\Phi\left(1+n, 1-n, 2+n; x, y\right)$ can be computed in a similar way.

Another important formula relative to the Bessel function of the second kind K_ν when $\nu = n + \frac{1}{2}$ (see Abramowitz and Stegun [1, pag. 444]) is

$$\sqrt{\frac{\pi}{2x}} K_{n+\frac{1}{2}}(x) = \left(\frac{\pi}{2}\right) e^{-x} \sum_{k=0}^{n} \left(n+\frac{1}{2}, k\right)\left(2x\right)^{-k}, \tag{17}$$

where

$$\left(n+\frac{1}{2}, k\right) = \frac{(n+k)!}{k!\Gamma\left(n-k+1\right)}.$$

Finally, the following trivial identity will be used

$$K_{n-\frac{1}{2}}(x) = K_{(n-1)+\frac{1}{2}}(x), \quad n \in \mathbb{N}.$$

5.1 Computation of $\Phi\,(n, 1-n, 1+n; x, y)$

There are several possible ways to compute Equation (16) analytically. By using the Fubini's theorem we observe that

$$\begin{aligned} I_2 &= \int_0^1 (u(1-ux))^{n-1}\, e^{uy} du = \int_0^1 (u(1-ux))^{n-1} \sum_{m=0}^{\infty} \frac{(uy)^m}{m!} du \\ &= \sum_{m=0}^{\infty} \frac{y^m}{m!} \int_0^1 u^{m+n-1}\,(1-ux)^{n-1}\, du, \end{aligned} \tag{18}$$

and hence the problem reduces to compute an integral of the form

$$I_3 = \int u^p\,(1-ux)^q\, du = I_{3,1} + I_{3,2} + I_{3,3} + C,$$

where $p, q \in \mathbb{N}$ and C is real constant. By iteratively using the integrating by parts rule simple computations show that

$$\begin{aligned} I_{3,1} &= \frac{u^{p+1}}{p+1}\,(1-ux)^q\,, \\ I_{3,2} &= \sum_{j=2}^{q} x^{j-1}\,(1-ux)^{q+1-j}\, \frac{\prod_{k=2}^{j}(q+2-k)}{\prod_{k=1}^{j}(p+j)}, \\ I_{3,3} &= x^q q! \frac{u^{p+q+1}}{(p+1)(p+2)\ldots(p+q)(p+q+1)}. \end{aligned}$$

and hence we obtain an analytical expression for Equation $\Phi\,(n, 1-n, 1+n; x, y)$.

On the other hand, relying upon the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

by the linearity of the integral, it follows that

$$\begin{aligned} I_2 &= \int_0^1 \left(u - u^2 x\right)^{n-1} e^{uy} du = \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^1 u^{n-1-k} \left(-u^2 x\right)^k e^{uy} du \\ &= \sum_{k=0}^{n-1} (-1)^k\, x^k \binom{n-1}{k} \int_0^1 u^{n-1+k} e^{uy} du. \end{aligned} \tag{19}$$

Since:

$$\int_0^1 u^m e^{uy} du = e^{uy} \cdot \left\{ \sum_{k=0}^{m} (-1)^k \binom{m}{k} k! \frac{u^{m-k}}{y^{k+1}} \right\},$$

we can conclude that

$$\Phi\left(n, 1-n, 1+n; x, y\right) = n \sum_{k=0}^{n-1} \left\{ (-1)^k x^k \binom{n-1}{k} \cdot e^{uy} \cdot \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} j \frac{u^{m-j}}{y^{j+1}} \right) \right\}.$$

5.2 Exchange option Pricing

In this section we derive a Margrabe's style formula for exchange option for the VG++ model. For $T > 0$, we model the two risky asset S_1,S_2 under measure $\mathbb{Q}$ as follows

$$S_i(T) = S_i(0) e^{\omega_i T + rT + \theta_i Z_a(T) + \sigma_i W_i(Z_a(T))}, \quad i = 1, 2, \tag{20}$$

where $W_1 = \{W_1(t); t \geq 0\}$ and $W_2 = \{W_2(t); t \geq 0\}$ are two standard Brownian motions, $\mathbb{E}\left[W_1(t)W_2(t)\right] = \rho t$ and $Z_a = \{Z_a(t); t \geq 0\}$ is a $\Gamma^{++}\left(a, \alpha t, \beta\right)$ subordinator, as discussed in Section 4. Observe that we use the subordinator Z_a to time change both Brownian motions W_1 and W_2 as proposed by Luciano and Schoutens [31]: this guarantees to remain in a Lévy framework also when W_1 and W_2 are correlated. The same approach is also discussed in in Madan and Seneta [34] and Sato [46]. Furthermore, from Equation (6) if $Y_i = \{W_i\left(Z_a(t)\right); t \geq 0\}$ is a VG++$(\theta_i, \sigma_i, \alpha, \beta, a)$ process we have that:

$$\rho_{Y_i(t), Y_j(t)} = \frac{\theta_i \theta_j \frac{(1-a^2)\alpha}{\beta^2} + \rho \sigma_i \sigma_j \frac{(1-a)\alpha}{\beta}}{\sqrt{\sigma_i^2 \frac{(1-a)\alpha}{\beta} + \theta_i^2 \frac{(1-a^2)\alpha}{\beta^2}} \sqrt{\sigma_j^2 \frac{(1-a)\alpha}{\beta} + \theta_j^2 \frac{(1-a^2)\alpha}{\beta^2}}}.$$

We recall that the density of $Z_a \sim \Gamma^{++}\left(a, \alpha, \beta\right)$ is given by

$$h_a(x) = a^\alpha \delta_0(x) + \sum_{n \geq 1} \binom{\alpha + n - 1}{n} a^\alpha \left(1 - a\right)^n f_{n, \beta/a}(x) \mathbb{1}_{x \geq 0},$$

where, for $\alpha > 0$, we consider the generalized binomial coefficient defined as

$$\binom{\alpha}{k} = \frac{\alpha\left(\alpha - 1\right) \dots \left(\alpha - k + 1\right)}{k!}, \quad \binom{\alpha}{0} = 1,$$

and

$$f_{n, \beta/a} = \left(\frac{\beta}{a}\right)^n x^{n-1} \frac{e^{-\beta x/a}}{\Gamma\left(n\right)},$$

which is the density of an Erlang distribution with parameters $n \in \mathbb{N}$ and β/a. Proceeding by conditioning on $Z_a(T)$, as done in the VG case, we have that the value of the exchange option $V(0)$ at time $t = 0$ is given by

$$\begin{aligned} V(0) &= e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[\left(S_2(T) - S_1(T)\right)^+ \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[S_2(0) e^{\omega_2 T + \left(\theta_2 + \frac{\sigma_2^2}{2}\right) Z_a(T)} \mathcal{N}\left(d_1\left(Z_a(T)\right)\right) - S_1(0) e^{\omega_1 T + \left(\theta_1 + \frac{\sigma_1^2}{2}\right) Z_a(T)} \mathcal{N}\left(d_2\left(Z_a(T)\right)\right) \right], \end{aligned}$$

where $d_1(x)$ and $d_2(x)$ are given by Equation (8).

We use the linearity of expected value and observe that we can focus only on the computation of the first expectation, since the one of the second term is similar. Omitting the the subscripts we have that

$$\mathbb{E}^{\mathbb{Q}}\left[S(0)e^{\omega T+\theta Z_a(T)+\frac{\sigma^2}{2}Z(T)}\mathcal{N}\left(d_1\left(Z_a(T)\right)\right)\right] = S(0)e^{\omega T}\int_0^{\infty} e^{\theta x+\frac{\sigma^2}{2}x}\mathcal{N}\left(d_1(x)\right)h_a(x)dx$$
$$= \psi \cdot a^{\alpha T}S(0)e^{\omega T} + S(0)e^{\omega T}\sum_{n\geq 1}\binom{\alpha T+n-1}{n}a^{\alpha T}\left(1-a\right)^n \underbrace{\int_0^{\infty} e^{\theta x+\frac{\sigma^2}{2}x}f_{n,\beta/a}(x)\mathcal{N}\left(d_1(x)\right)dx}_{I_1}, \tag{21}$$

where

$$\psi = \begin{cases} 1 & \text{if } \log\left(S_2(0)/S_1(0)\right)+\left(\omega_2-\omega_1\right)T>0, \\ 0 & \text{otherwise.}\end{cases} \tag{22}$$

Focusing on the computation of the term I_1

$$I_1 = \int_0^{\infty}\left(\frac{\beta}{a}\right)^n\frac{1}{\Gamma(n)}e^{-\left(\beta/a-\left(\theta+\frac{\sigma^2}{2}\right)x\right)}x^{n-1}\mathcal{N}\left(d_1(x)\right)dx, \tag{23}$$

and defining $A_j = \frac{\beta}{a}-\left(\theta+\frac{\sigma^2}{2}\right)$ and $u = A_j x$ we have that

$$I_1 = \left(\frac{\beta}{aA_j}\right)^n\int_0^{\infty}\mathcal{N}\left(\frac{\tilde{a}}{u}+\tilde{b}\sqrt{u}\right)\frac{u^{n-1}}{\Gamma(n)}e^{-u}du = \left(\frac{\beta}{aA_j}\right)^n\Psi\left(\tilde{a},\tilde{b};n\right).$$

for properly defined values of $\tilde{a}$ and $\tilde{b}$, given by Equations (10).

In particular, as shown in Madan et al. [33], $\Psi\left(\tilde{a},\tilde{b};n\right)$ can be expressed in terms of the modified Bessel function of the second kind $K_\nu(z)$ and $\Phi\left(\alpha,\beta,\gamma;x,y\right)$ where α, β and γ assume integer values. $\Psi\left(a,b;\gamma\right)$ has the following expression

$$\begin{aligned}\Psi\left(a,b;\gamma\right) = & \frac{c^{\gamma+1/2}\exp\left[sign\left(a\right)c\right]\left(1+u\right)^{\gamma}}{\sqrt{2\pi}\Gamma\left(\gamma\right)\gamma}\cdot K_{\gamma+1/2}\left(c\right) \\ & \Phi\left(\gamma,1-\gamma,1+\gamma,\frac{1+u}{2},-sign\left(a\right)c\left(1+u\right)\right) \\ & - sign\left(a\right)\frac{c^{\gamma+1/2}\exp\left[sign\left(a\right)c\right]\left(1+u\right)^{\gamma+1}}{\sqrt{2\pi}\Gamma\left(\gamma\right)\left(1+\gamma\right)}\cdot K_{\gamma-1/2}\left(c\right) \\ & \Phi\left(1+\gamma,1-\gamma,2+\gamma,\frac{1+u}{2},-sign\left(a\right)c\left(1+u\right)\right) \\ & + sign\left(a\right)\frac{c^{\gamma+1/2}\exp\left[sign\left(a\right)c\right]\left(1+u\right)^{\gamma}}{\sqrt{2\pi}\Gamma\left(\gamma\right)\gamma}\cdot K_{\gamma-1/2}\left(c\right) \\ & \Phi\left(\gamma,1-\gamma,1+\gamma,\frac{1+u}{2},-sign\left(a\right)c\left(1+u\right)\right),\end{aligned}$$

where $c = |a|\sqrt{2+b^2}$ and $u = \frac{b}{\sqrt{2+b^2}}$. $K_\nu(z)$ can be computed using Equation (17) when $\nu = n \pm 1/2$, whereas, the confluent hypergeometric function in Equation (15) has an explicit expression as observed in Section 5.1. Summarizing all the previous results we have the following proposition.

Proposition 5.1. *The price $V(0)$ of an exchange option at time $t=0$ with payoff at maturity T given by $\Phi(T) = (S_2(T) - S_1(T))^+$ under the dynamics in Equation (20) is given by*

$$\begin{aligned} V(0) = S_2(0)e^{\omega_2 T} &\left(\psi \cdot a^{\alpha T} + \sum_{n\geq 1} \binom{\alpha T + n - 1}{n} a^{\alpha T} (1-a)^n \, \Psi\left(\tilde{a}, \tilde{b}; n\right) \frac{1}{A^n}\right) \\ - S_1(0)e^{\omega_1 T} &\left(\psi \cdot a^{\alpha T} + \sum_{n\geq 1} \binom{\alpha T + n - 1}{n} a^{\alpha T} (1-a)^n \, \Psi\left(\tilde{c}, \tilde{d}; n\right) \frac{1}{C^n}\right), \end{aligned} \tag{24}$$

where

$$A = \frac{a}{\beta}\frac{1}{A_2}, \qquad C = \frac{a}{\beta}\frac{1}{A_1}$$

ψ is defined as in Equation (22), and

$$\tilde{a} = \frac{\log\left(\frac{S_2(0)}{S_1(0)} + (\omega_2 - \omega_1)T\right)\sqrt{A_2}}{\bar{\sigma}}, \qquad \tilde{b} = \frac{\theta_2 - \theta_1 + \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2 + \bar{\sigma}^2\right)}{\bar{\sigma}\sqrt{A_2}},$$

$$\tilde{c} = \frac{\log\left(\frac{S_2(0)}{S_1(0)} + (\omega_2 - \omega_1)T\right)\sqrt{A_1}}{\bar{\sigma}}, \qquad \tilde{d} = \frac{\theta_2 - \theta_1 + \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2 + \bar{\sigma}^2\right)}{\bar{\sigma}\sqrt{A_1}},$$

and

$$A_1 = \frac{\beta}{a} - \left(\theta_1 + \frac{1}{2}\sigma_1^2\right), \quad A_2 = \frac{\beta}{a} - \left(\theta_2 + \frac{1}{2}\sigma_2^2\right), \quad \bar{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

Remark 3. We stress that the pricing formula for the exchange option $V(0)$ in Proposition 5.1 does not require numerical integration since $\Psi(x, y; n)$ can be computed analytically for $n \in \mathbb{N}$ as shown in Section 5.1.

5.3 Numerical test of the option pricing formula

In this section we compare the price of an exchange option calculated with the exact formula of the previous section to those ones obtained with Monte Carlo and Fourier-based techniques.

We focus on the bi-variate VG++ process for S_1 and S_2 defined in Equation (20) and we define

$$Y_i(t) = \theta_i Z_a(T) + \sigma_i W_i(Z_a(T)), \quad i = 1, 2.$$

The generation of the skeleton of each component is straightforward given the simulation of Z_a which can be easily obtained relying on the method illustrated in Sabino and Petroni [45]. Namely,

$$Z_a \stackrel{d}{=} \begin{cases} \sum_{i=1}^{S} X_i, & \text{when } S > 0 \\ 0, & \text{when } S = 0 \end{cases}$$

where $X_i \sim \mathcal{E}(\beta/a)$ iid and $S \sim \overline{\mathfrak{B}}(\alpha, 1-a)$. In particular $Z_{a|S=s} \sim \Gamma(s, \beta/a)$, when $s > 0$.

Several FFT-based methods are available in the literature, among other for instance the one proposed in Hurd and Zhou [23] or Caldana and Fusai [9] or that based on the cosine expansions proposed by Pellegrino [41]. All these approaches require the characteristic function of the log-distribution of prices under the risk-neutral measure in a closed form. To this end, it is possible to show that the characteristic function of $\boldsymbol{Y}(t) = (Y_1(t), Y_2(t)))$ is given by

$$\phi_{\boldsymbol{Y}(t)} = \phi_{Z(t)} \left(\boldsymbol{\theta}^T \boldsymbol{u} + \frac{i}{2} \boldsymbol{u}^T \Sigma \boldsymbol{u} \right),$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2]$, $\boldsymbol{u} = [u_1, u_2]$ and

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$

The prices of the exchange options for different values of a are shown in Figure 2. Here we compare the analytic formula with the Monte Carlo scheme and the approximate formula for spread options proposed by Caldana and Fusai [9], which is simpler than the one proposed by Hurd and Zhou [23] since it requires a single Fourier transform inversion.

From a practical point of view, since the evaluation of the call option via Equation (24) requires the computation of an infinite sum, some truncation method is required. Since a theoretical analysis of the rate of convergence of the series in Equation (24) is hard to tackle, we give a flavor of the rate of convergence from a numerical point of view. First we compute the value of the option $V(0)^N$ via Equation (24) by truncating the series at a very large N. In order to be sure that convergence has been reached we compare this value with the one we obtained by the Monte Carlo integration and by the Fourier method. Then, we compare the value of the of the call option $V(0)^n$ which is obtained by truncating the series at $n \in \{1, \ldots, N\}$. Finally we compute the absolute percentage error

$$\epsilon_n = \frac{|V(0)^N - V(0)^n|}{V(0)^n}$$

and we look for the value N^* such that $\epsilon_{N^*} < 0.1\%$. If we fix all parameters and we let one of them varies we have the results in Table 2. We note that the number of terms we need to compute to have a good convergence of the series is acceptable for different values of the parameters. Furthermore, numerical tests have shown that numerical problems arise if n increases since the computation of the modified Bessel function of the second type becomes problematic. Fortunately, n remains "small enough" for a wide range of parameters which cover many market conditions.

6 Multidimensional modeling

In this section we go beyond the modeling approach used in the previous section which relies upon a single subordinator (see Luciano and Schoutens [31]). Such an approach can be extended using a multidimensional subordinator while still remaining in a Lévy

a	N^*	θ_1	N^*	T	N^*	σ_1	N^*	α	N^*	ρ	N^*
0.10	94	-0.20	14	0.50	11	0.10	14	0.10	4	0.10	13
0.20	44	-0.16	13	1.00	14	0.18	14	0.64	9	0.19	13
0.30	27	-0.11	12	1.50	16	0.26	14	1.19	11	0.28	13
0.40	19	-0.07	11	2.00	18	0.33	13	1.73	13	0.37	13
0.50	14	-0.02	10	2.50	20	0.41	13	2.28	14	0.46	13
0.60	10	0.02	9	3.00	21	0.49	12	2.82	16	0.54	14
0.70	8	0.07	8	3.50	23	0.57	12	3.37	17	0.63	14
0.80	5	0.11	7	4.00	25	0.64	12	3.91	18	0.72	14
0.85	4	0.16	7	4.50	26	0.72	12	4.46	19	0.81	14
0.90	3	0.20	6	5.00	28	0.80	11	5.00	20	0.90	14

Table 2: Needed number of terms N^* of the series in order to obtain a percentage error $\epsilon_n < 0.1\%$ with different parameters. When fixed, the parameters are $r = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\theta_1 = -0.2012$, $\theta_2 = -0.1712$, $S_1(0) = 100$, $S_2(0) = 105$, $T = 1$, $\alpha = 2$, $\rho = 0.8$.

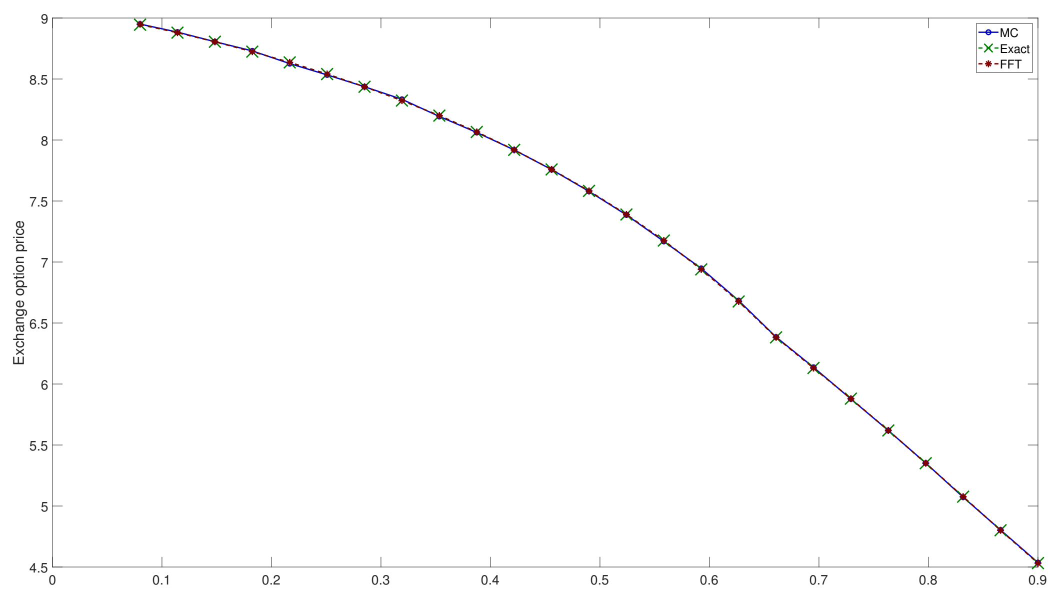

Figure 2: Exchange option pricing in the VG++ model computed by exact formula, Monte Carlo simulations and FFT method. $r = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\theta_1 = -0.2012$, $\theta_2 = -0.1712$, $S_1(0) = 100$, $S_2(0) = 105$, $T = 1$, $\alpha = 2$, $\rho = 0.8$. Number of Monte Carlo simulations $N = 10^7$.

framework. Successful approaches have been proposed by Semeraro [48], Luciano and Semeraro [32], Ballotta and Bonfiglioli [4] and taken up by Gardini et al. [17, 18] to introduce delays in market news propagation. In order to generalize the results of Section 5 we recall some properties of the $\Gamma^{++}(a, \alpha, \beta)$ distribution which are directly inherited from the gamma law (see Sabino and Petroni [45]).

$$
\begin{aligned}
&Z_a \sim \Gamma^{++}(a, \alpha, \beta), && cZ_a \sim \Gamma^{++}\left(a, \alpha, \frac{\beta}{c}\right), \ c > 0, \\
&Z_{a,i} \sim \Gamma^{++}(a, \alpha_i, \beta), && \sum_{i=1}^{n} Z_{a,i} \sim \Gamma^{++}\left(a, \sum_{i=1}^{n} \alpha_i, \beta\right).
\end{aligned}
$$

Furthermore, the characteristic function of $Z_a \sim \Gamma^{++}(a, \alpha, \beta)$ is given by

$$
\phi_{Z_a}(u) = \left(\frac{\beta - iua}{\beta - iu}\right)^{\alpha}, \quad u \in \mathbb{R},. \tag{25}
$$

Moreover, for $c \in \mathbb{R}^+$, we have that $\phi_{cZ_a}(u) = \phi_{Z_a}(cu)$.

Unlike the model we introduced in Section 5.2, due to the presence of more than one subordinator, closed pricing formula as the one we presented in Proposition 5.1 are not available for the forthcoming models and numerical techniques, such as Monte Carlo simulations or Fourier methods, should be used in order to properly evaluate vanilla financial derivatives.

6.1 Semeraro's approach

We adapt the approach illustrated in Semeraro [48], in the context of the VG++process. We consider independent subordinators $I_j = \{I_j(t); t \geq 0\}$, $j = 1, \ldots, n$ also independent from $Z_a = \{Z_a(t); t \geq 0\}$ and, for $\alpha_j \geq 0$, $j = 1, \ldots, n$, we consider $G_j = \{G_j(t); t \geq 0\}$ defined as

$$
G_j(t) = I_j(t) + \alpha_j Z_a(t), \quad j = 1, \ldots, n. \tag{26}
$$

By choosing:

$$
\begin{aligned}
I_j(t) &\sim \Gamma^{++}\left(a, A_j t, \frac{B}{\alpha_j}\right), \quad A_j, B > 0, \\
Z_a(t) &\sim \Gamma^{++}(a, At, B), \quad A, B > 0,
\end{aligned} \tag{27}
$$

we have that

$$
G_j(t) \sim \Gamma^{++}\left(a, (A_j + A)\, t, \tfrac{B}{\alpha_j}\right).
$$

Then, given a standard Brownian motion with drift $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and volatility $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ we can define a subordinated Brownian motion $\boldsymbol{Y} = \{(Y_1(t), \ldots, Y_n(t)), t \geq 0\}$ as

$$
Y_j(t) = \mu_j G_j(t) + \sigma_j W_j(G_j(t)), \quad j = 1, \ldots, n.
$$

The characteristic function of $\boldsymbol{Y}$ at time t can be easily computed and is given by

$$\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y}\rangle}\right]=\prod_{j=1}^{n}\phi_{I_j(t)}\left(u_j\mu_j+i\frac{\sigma_j^2}{2}u_j^2\right)\cdot\phi_{Z_a(t)}\left(\alpha_j\left(u_j\mu_j+i\frac{\sigma_j^2}{2}u_j^2\right)\right),$$

whereas, the linear correlation coefficient at time t has the following form

$$\rho_{Y_i(t),Y_j(t)}=\frac{\mu_i\mu_j\alpha_i,\alpha_j Var\left[Z_a(t)\right]}{\sqrt{Var\left[Y_i(t)\right]Var\left[Y_j(t)\right]}}. \tag{28}$$

Linear correlation coefficient is important for the model calibration while instead, the knowledge of the characteristic function allows us to use pricing methods based on Fourier techniques. It is worth noting that the model we have just proposed is not always able to properly catch the correlation observed in the market, especially if the drift μ_j is small. For this reason, Luciano and Semeraro [32] introduced an extended version of this model which aims at better fitting the market correlation even if the drift μ_j vanishes.

6.2 Luciano-Semeraro's approach

In this section we adapt the approach of Luciano and Semeraro [32] which in contrast, introduces correlated Brownian motions (hereafter labeled LS). Consider mutually independent subordinators $I_i=\{I_i(t);t\geq 0\}$, $i=1,\ldots,n$ and another independent subordinator $Z_a=\{Z_a(t);t\geq 0\}$. Define the multidimensional process $\boldsymbol{Y}=\{Y_1,\ldots,Y_n\}$ as follows

$$\boldsymbol{Y}(t)=\begin{pmatrix}\mu_1 I_1(t)+\sigma_1 W_1(I_1(t))+\alpha_1\mu_1 Z_a(t)+\sqrt{\alpha_1}\sigma_1 W_1^{\rho}(Z_a(t))\\ \ldots\\ \mu_n I_n(t)+\sigma_n W_n(I_n(t))+\alpha_n\mu_n Z_a(t)+\sqrt{\alpha_n}\sigma_n W_n^{\rho}(Z_a(t))\end{pmatrix}, \tag{29}$$

where $I_j(t)$ $j=1,\ldots,n$ and $Z_a(t)$ are distributed as in Equation (27). As shown by Luciano and Semeraro [32], the process $\boldsymbol{Y}$ is a Lévy process. The expression of its characteristic function and that of its linear correlation coefficient are known in closed form and they are respectively given by

$$\phi_{\boldsymbol{Y}(t)(\boldsymbol{u})}=\prod_{i=1}^{n}\phi_{I_i(t)}\left(u_i\mu_i+\frac{i}{2}\sigma_i^2u_i^2\right)\phi_{Z_a(t)}\left(\boldsymbol{u}^T\boldsymbol{\mu}+\frac{i}{2}\boldsymbol{u}^T\Sigma\boldsymbol{u}\right),$$

where $\boldsymbol{\mu}=[\alpha_1\mu_1,\ldots,\alpha_n\mu_n]$ and

$$\Sigma=\begin{bmatrix}\alpha_1\sigma_1^2 & \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho_{1,2} & \ldots & \sqrt{\alpha_1\alpha_n}\sigma_1\sigma_n\rho_{1,n}\\ \vdots & \ddots & \ddots & \vdots\\ \sqrt{\alpha_n\alpha_1}\sigma_n\sigma_1\rho_{n,1} & \sqrt{\alpha_n\alpha_2}\sigma_n\sigma_2\rho_{n,2} & \ldots & \alpha_n\sigma_n^2\end{bmatrix},$$

and

$$\rho_{Y_i(t),Y_j(t)}=\frac{\mu_i\mu_j\alpha_i\alpha_j Var\left[Z_a(t)\right]+\rho\sigma_i\sigma_j\sqrt{\alpha_i\alpha_j}\mathbb{E}\left[Z_a(t)\right]}{\sqrt{Var\left[Y_i(t)\right]Var\left[Y_j(t)\right]}}.$$

In this case we observe that, even if the drift μ_j vanishes the linear correlation coefficient can still be different from zero. Furthermore, if the correlation $\rho=0$ we obtain the same expression of the original model proposed by Semeraro [48].

As mentioned in Remark 1, in financial modeling it is customary to impose that the stochastic time, in average, run as fast as the deterministic time. Setting $G_j(t) = I_j(t) + \alpha_j Z_a$ and imposing that $\mathbb{E}[G_j(t)] = t$, together with the fact that if $Z_a \sim \Gamma^{++}(a, \alpha, \beta)$ then $\mathbb{E}[Z_a] = (1-a)\frac{\alpha}{\beta}$, we get to the following relation

$$A_j = \frac{B}{\alpha_j(1-a)} - A, \quad A_j > 0, \quad j = 1, \ldots, n.$$

6.3 Ballotta-Bonfiglioli's approach

Alternatively to the previous approaches, the dependence can be also introduced at the level of subordinated Brownian motions as proposed by Ballotta and Bonfiglioli [4] (hereafter labeled BB) . This approach is intuitive, but some attention is required in order to guarantee that the sum of two subordinated Brownian motions is still a subordinated Brownian motion of the same family. In order to ensure that, one has to impose some convolution conditions. Unfortunately, in some case, they can be difficult to handle, especially from a numerical point of view.

6.3.1 Convolution conditions

In this section first we investigate the convolution conditions which guarantee that a linear combination of two random variables with a VG++ keeps the same law. Accordingly, we use such conditions in order to build a multidimensional version of the VG++ process.

Consider three independent subordinators G_Y, G_X and G_Z with Γ^{++} law such that at time $t \geq 0$:

$$\begin{aligned}
G_Y(t) &\sim \Gamma^{++}(a, A_y t, B_y), \quad A_y, B_y > 0, \\
G_X(t) &\sim \Gamma^{++}(a, A_x t, B_x), \quad A_x, B_x > 0, \\
G_Z(t) &\sim \Gamma^{++}(a, A_z t, B_z), \quad A_z, B_z > 0, .
\end{aligned}$$

and three independent Brownian motions W_x, W_y and W_z. We define then, the processes X, Y and Z as follows

$$\begin{aligned}
Y(t) &= \mu G_Y(t) + \sigma W_y(G_Y(t)), \quad \mu \in \mathbb{R},\ \sigma > 0, \\
X(t) &= \beta G_X(t) + \gamma W_x(G_X(t)), \quad \beta \in \mathbb{R},\ \gamma > 0, \\
Z(t) &= \beta_z G_Z(t) + \gamma_z W_z(G_Z(t)), \quad \beta_z \in \mathbb{R},\ \gamma_z > 0.
\end{aligned}$$

Finally, we define the process $\hat{Y} = X + a_1 Z$, with $a_1 \in \mathbb{R}$. The goal is to find some relations between the parameters such that $Y(t) \stackrel{d}{=} \hat{Y}(t),\ \forall t \geq 0$. Following the approach of Ballotta and Bonfiglioli [4] and using (25) we equate the characteristic exponents, obtaining

$$A_y \log\left(\frac{1 - iya/B_y}{1 - iy/B_y}\right) = A_x \log\left(\frac{1 - ixa/B_x}{1 - ix/B_x}\right) + A_z \log\left(\frac{1 - iza/B_z}{1 - iz/A_z}\right),$$

where

$$
\begin{aligned}
y &= u\mu + iu^2\frac{\sigma^2}{2},\\
x &= u\beta + iu^2\frac{\gamma^2}{2},\\
z &= u\beta_z a_1 + iu^2 a_1^2\frac{\gamma_z^2}{2}.
\end{aligned}
$$

The first condition we need to impose is $A_y = A_x + A_z$ which leads to

$$
\begin{aligned}
A_y \log\left(\frac{1-iya/B_y}{1-ix/B_y}\right) = A_y \log\left(\frac{1-ixa/B_x}{1-ix/B_x}\right) - A_z \log\left(\frac{1-ixa/B_x}{1-ix/B_x}\right)\\
+ A_z \log\left(\frac{1-iza/B_z}{1-iz/A_z}\right).
\end{aligned}
$$

The second one is

$$
\begin{aligned}
\frac{y}{B_y} &= \frac{x}{B_x},\\
\frac{x}{B_x} &= a_1\frac{z}{B_z}.
\end{aligned}
$$

From these equations we obtain the following conditions

$$
\begin{aligned}
\beta &= \frac{B_x}{B_z}a_1\beta_z, \qquad & \gamma^2 &= \frac{B_x}{B_z}\gamma_z a_1^2,\\
\mu &= \frac{B_y}{B_x}\gamma^2\beta, \qquad & \sigma^2 &= \frac{B_y}{B_x}\gamma^2,
\end{aligned}
$$

and therefore

$$
\mathbb{E}\left[Y(t)\right] = \frac{A_y}{B_y}\mu = \beta\frac{A_x}{B_x} + a_1\beta_z\frac{A_z}{B_z}.
$$

Finally, it is easy to check that

$$
Var\left[Y(t)\right] = \mu^2\left(1-a^2\right)\frac{A_y}{B_y^2} + \sigma^2\left(1-a^2\right)\frac{A_y}{B_y}.
$$

Consequently, we have $Y(t) \stackrel{d}{=} \tilde{Y}(t),\ \forall t \geq 0$ and a tool to construct multidimensional Lévy processes as proposed by Ballotta and Bonfiglioli [4]. In addition to previous relation we impose that $\mathbb{E}\left[G_Y(t)\right] = t$, which leads to the additional condition $B_y = (1-a)\,A_y$.

6.3.2 A multidimensional VG++ process

We briefly recall how the VG++ process is constructed. Consider the subordinator $Z = \{Z_a(t); t \geq 0\}$ where $Z_a(t) \sim \Gamma^{++}(a, At, B)$ and a Brownian motion $W = \{W(t); t \geq 0\}$ with drift θ and diffusion γ then the VG++ process X with parameters β, γ, A, B, a is defined by:

$$
X(t) = \beta Z_a(t) + \gamma W\left(Z_a(t)\right),\ t \geq 0.
$$

According to the convolution conditions of the previous section we can introduce a multi-dimensional version of the VG++ as follows.

Model	Parameters	Parameters constraints
VG	$\mu_1,\, \mu_2,\, \sigma_1,\, \sigma_2,\, \alpha,\, \beta,\, a$	$\alpha = \beta$
VG++	$\mu_1,\, \mu_2,\, \sigma_1,\, \sigma_2,\, \alpha,\, \beta,\, a$	$\alpha = (1-a)\beta$
VG++LS	$\mu_1,\, \mu_2,\, \sigma_1,\, \sigma_2,\, A,\, B,\, a,$ $A_1,\, A_2,\, \alpha_1,\, \alpha_2,\, \rho$	$A_j = \frac{B}{\alpha_j(1-a)} - A,\ j=1,2$
VG++ BB	$\mu_1,\, \mu_2,\, \sigma_1,\, \sigma_2,\, A_z,\, B_z,$ $a,\, A_{x_1},\, A_{x_2},\, B_{x_1},\, B_{x_2},$ $a_1,\, a_2,\, \gamma_1,\, \gamma_2,\, A_{y_1},\, A_{y_2},$ $\beta_1,\, \beta_2,\, \gamma_z,\, \beta_z$	$B_{y_j} = (1-a)\, A_{y_j},\ \beta_j = \frac{B_{x_j}}{B_z} a_j \beta_z,$ $\gamma_j^2 = \frac{B_{x_j}}{B_z}\gamma_z^2 a_j^2,\ \mu_j = \frac{B_{y_j}}{B_{x_j}}\beta_j,$ $\sigma_j^2 = \frac{B_{y_j}}{B_z}\gamma^2,\ j=1,2.$

Table 3: *Models parameters and constraints in case of two processes.*

Definition 6.1 (VG++ Ballotta and Bonfiglioli's model)**.** *Let Z_a be a subordinator and define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t)\,; t \geq 0\}$ as*

$$\boldsymbol{Y}(t) = (Y_1(t), \dots, Y_n(t)) = (X_1(t) + a_1 Z(t), \dots, X_n(t) + a_n Z(t)), \tag{30}$$

where $X_j = \{X_j(t)\,; t \geq 0\}$ is a $VG{+}{+}\left(\beta_j, \gamma_j, A_{x_j}, B_{x_j}, a\right)$ process, $Z = \{Z(t)\,; t \geq 0\}$ is a $VG{+}{+}(\beta_z, \gamma_z, A_z, B_z, a)$ process and they are independent. All the subordinate Brownian motions are assumed to be independent and $a_1, \dots, a_n \in \mathbb{R}$.

The resulting process is such that $Y_j(t) \stackrel{d}{=} \mu_j G_{Y_j}(t) + \sigma_j W_y(G_{Y_j}(t))$ where $G_{Y_j}(t) \sim \Gamma^{++}\left(a, A_{y_j}t, B_{y_j}\right)$. As usual, we require that $\mathbb{E}\left[G_{Y_j}(t)\right] = t$ which leads to the relation $B_{y_j} = (1-a)\, A_{y_j}$. Moreover, the convolution conditions of the previous sections are met and guarantee that Y_j is a $VG{+}{+}\left(\mu_j, \sigma_j, A_{y_j}, B_{y_j}, a\right)$ process.

After straightforward computations one can show that the characteristic function of the process $\boldsymbol{Y}$ at time t is given by

$$\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = \prod_{k=1}^{n} \phi_{G_k(t)}\left(\beta_k u_k + \frac{i}{2}u_k^2\gamma_k^2\right) \cdot \phi_{Z_a(t)}\left(\sum_{k=1}^{n} a_k u_k \beta_z + \frac{i}{2}\left(\sum_{k=1}^{n} a_k u_k \gamma_z\right)^2\right).$$

Finally, the linear correlation coefficient at time t has the following expression

$$\rho_{Y_i(t),Y_j(t)} = \frac{a_i a_j \left(\beta_z^2\left(1-a^2\right) A_z/B_z^2 + (1-a)\,\gamma_z^2 A_z/B_z\right)}{\sqrt{\mu_i^2\left(1-a^2\right)\frac{A_{y_i}}{B_{y_i}^2} + \sigma_i^2\left(1-a^2\right)\frac{A_{y_i}}{B_{y_i}}} \cdot \sqrt{\mu_j^2\left(1-a^2\right)\frac{A_{y_j}}{B_{y_j}^2} + \sigma_j^2\left(1-a^2\right)\frac{A_{y_j}}{B_{y_j}}}}.$$

A list of parameters along with the set of constraints for each model, in a two-dimensional case, is shown in Table 3. Clearly, since convolution conditions impose some restrictions to the choice of the parameters a loss of generality in the distributions is introduced. Nevertheless, as we show in Section 7 the fitness on real market data performs well.

7 Numerical Experiment

In this section we apply the models we have studied so far to real data and investigate their performance. We focus on the multidimensional versions of the VG and VG++ process presented in Section 3 and 5 and on those discussed in Section 6. For each of them we have calibrated the model on market data and priced exchange options using different numerical techniques.

Following Cont and Tankov [14], we model the evolution of the forward price $F_j(t), j = 1, 2$ at time t by the process

$$F_j(t) = F_j(0)e^{\omega_j t + Y_j(t)},$$

where Y_j is a stochastic process. By setting

$$\omega_j = -\phi_j(-i),$$

where $\phi_j(u)$ is denotes the characteristic exponent of the process Y_j, we have that the discounted price processes are martingales with respect the probability measure $\mathbb{Q}$ corresponding to $\omega_j,\ j = 1, 2$. This procedure is known as *mean-correction* of the exponential of the Lévy process and is widely adopted by practitioners (see Schoutens [47]). In particular, the measure $\mathbb{Q}$ is an Equivalent Martingale Measure (EMM) if and only if it is equivalent to the real world probability measure $\mathbb{P}$. In this case, the obtained measure is called *mean correcting martingale measure* (MCMM). Unfortunately, as discussed in Cont and Tankov [14, Section 9.4] and detailed in Yao et al. [52], it can be shown that for a pure jump Lévy process, as the VG++ process, no MCMM can exist. However, it can be proven that in this case there exist an EEM $\mathbb{Q}'$ such that the price of the European call options under $\mathbb{Q}$ and $\mathbb{Q}'$ are identical. In particular Yao et al. [52, Theorem 2.2] implies that although $\mathbb{Q}$ is not an EEM for the VG++ process, the price of the option under $\mathbb{Q}$ is still arbitrage free.

Alternatively to the mean correcting martingale measure one can also apply the Esscher transform (see Cont and Tankov [14, Section 9.5] or Pascucci [40, Section 13.5.3]) which produces an EMM in the exponential Lévy models even for pure jump processes. One can refer to Michaelsen and Szimayer [38] for an application of this technique to option pricing.

In order to calibrate our models we need the quoted prices of the derivatives contracts written on each of the two forward products and their joint historical time series. The data-set[3] we rely upon consists of

- Forward quotations from 4 January 2021 to 19 May 2022 of calendar 2023 power forward. A forward calendar 2023 contract is a contract to buy or sell a specific volume of energy in MWh at fixed price for all the hours of 2023. Calendar power forward in Germany and France are denoted with DEBY and F7BY, respectively.

- Quotations of European call options on power forward 2023 in Germany and France with settlement date 19 May 2022. We use strikes in a range of $\pm 15\,[EUR/MWh]$ around the settlement price of the forward contract, i.e. we excluded deep ITM and OTM options.

[3]Data Source: www.eex.com.

- We assume a risk-free rate $r = 0.015$.
- The estimated historical correlation between markets log-returns is $\rho_{mkt} = 0.96$.

We have adopted the same two-steps calibration procedure of Luciano and Semeraro [32]: to this end, it is worthwhile noticing that the marginal distributions do not depend on the parameters required to model the dependence structure. The vector of the marginal parameters Θ^* is obtained solving the following optimization problem:

$$\Theta^* = \arg\min_{\Theta} \sum_{i=1}^{n} \left(C_i^{\Theta}(K,T) - C_i \right)^2, \tag{31}$$

where $C_i, i = 1, \ldots, n$ are the values of n quoted vanilla products and $C_i^{\Theta}(K,T), i = 1, \ldots, n$ are the relative model prices. Once Θ^* is found we have to calibrate the remaining parameters for the dependence structure. The vector $\boldsymbol{\eta}^*$, that encompasses the dependence parameters, has been estimated fitting the correlation matrix on historical data[4].

Clearly, the model should be fitted on derivative market, but, since the "risk-neutral" empirical correlation matrix is not available, we use the historical one as a proxy for it as proposed, for example, by Luciano and Semeraro [32] and Ballotta and Bonfiglioli [4]. On the other hand, if derivatives products written on more than one underlying asset are quoted, they must be used to properly infer the vector $\boldsymbol{\eta}^*$. The expression of the theoretical correlation matrix for the extension of Luciano and Semeraro [32] and Ballotta and Bonfiglioli [4] in the VG++ framework has been derived in Section 6: the parameters have been chosen so that the correlation is fitted accordingly to the market.

For the first calibration step we have combined the NLLS approach with the FFT method proposed by Carr and Madan [11] (the version proposed by Lewis [28] returns similar results) whereas for the second one, we have used the plain NLLS method for the minimization of the distance between the theoretical and the observed correlation coefficient. The calibrated parameters are reported in Tables 4, 5, 6, 7 and 8 , whereas Table 9 reports the values of the exchange option computed with Monte Carlo, the FFT method and the closed formula expression, when available.

From Table 9 we can observe that all models return very similar prices and that all of them can replicate the observed market log-returns correlation level with the exception of the LS-VG++ model which attains a slightly lower level, $\rho_{Y_1(t),Y_2(t)} = 0.93$. Since the value of the spread option is very sensitive to the correlation, in order to avoid mispricing, one should avoid selecting a model which does not fit it properly.

We remark that we have constructed the VG and VG++ models with a single subordinator whereas, in the extended models LS-VG++ and BB-VG++ we have introduced different dependent subordinators. We observe that the option prices obtained with the different constructions are very similar. This fact is somehow intuitive from an economical point of view. Indeed, the German and French power futures markets are extremely

[4]Since the correlation matrix depends on several parameters, when we fit the correlation coefficients we have more parameters than equations, which leads us to a underdetermined system of equations. One can add more equation by computing and matching higher moments, such as $\mathbb{E}\left[Y_j(t)^n Y_i(t)^m\right],\ m, n \in \mathbb{N}$. Unfortunately imposing these conditions may lead to numerical issues. For this reason, we restricted ourselves to fit only the correlation matrix, accepting that different sets of parameters can replicate the same level of correlation in log-returns.

Parameter	Value
σ_1	0.84
σ_2	0.91
ρ	0.96
$\rho_{Y_1(t),Y_2(t)}$	0.96

Table 4: *BS*

Parameter	Value
μ_1	-0.27
μ_2	-0.24
σ_1	0.98
σ_2	0.92
α	2.04
ρ	0.96
$\rho_{Y_1(t),Y_2(t)}$	0.96

Table 5: *VG*

Parameter	Value
μ_1	-0.28
μ_2	-0.26
σ_1	0.98
σ_2	0.92
α	2.43
β	2.33
a	0.04
ρ	0.96
$\rho_{Y_1(t),Y_2(t)}$	0.96

Table 6: *VG++*

Parameter	Value
μ_1	-0.47
μ_2	-0.35
σ_1	1.06
σ_2	0.96
α_1	1.43
α_2	1.64
a	0.01
A	1.43
B	1224.83
A_1	863.74
A_2	869.83
ρ	0.99
$\rho_{Y_1(t),Y_2(t)}$	0.93

Table 7: *LS-VG++*

Parameter	Value	Parameter	Value
μ_1	-0.38	μ_2	-0.33
σ_1	1.025	σ_2	0.96
A_{y_1}	1.5839	A_{y_2}	1.4439
A_{x_1}	0.141	A_{x_2}	0.0001
B_{x_1}	12.20	B_{x_2}	0.006
B_{y_1}	1.58	B_{y_2}	1.44
β_1	-2.95	β_2	-0.00014
γ_1	2.84	γ_2	0.062
a_1	4.80	a_2	4.74
A_z	1.4438	B_z	62.77
β_z	-3.15	γ_z	1.34
a	0.001	$\rho_{Y_1(t),Y_2(t)}$	0.96

Table 8: *BB-VG++*

intertwined because of the physical electrical grid connections between the two related countries. We could say that the stochastic time "runs in the same way" in both markets and hence, using a single subordinator for price modeling is not a restrictive choice. On the other hand, there may be other market situations in which multi-subordination might be the right approach (see Luciano and Semeraro [32] and Ballotta and Bonfiglioli [4]). In general, when different models lead to comparable results, it is reasonable to select the simpler one and avoid unnecessary complexity.

In Section 4 we have briefly mentioned that the parameter $a \in (0,1)$ can be interpreted as an index of liquidity. Here we observe that a, and hence the probability of having a zero increment in the underlying price over the time interval Δt, is very close to zero in all cases. During the previous years the options power market has lacked of liquidity in many European countries but recently such derivative instruments have become more and more popular leading to an increase in the number of exchanged trades.

Possible limitations in the construction of the multidimensional version of the VG++ processes might arise if we consider markets with different level of liquidities. Indeed, in order to use the summing property of the Γ^{++} law, the value of $a \in (0,1)$ must be the same for all the process we consider. For this reason, such a construction is reasonable if the considered markets have approximately the same level of liquidity. This fact might seem

Model	Monte Carlo	Exact	FFT
BS	80.04	79.92	79.91
VG	81.69	81.68	81.65
VG++	81.62	81.28	81.15
VG++ LS	85.18	-	85.19
VG++ BB	83.15	-	83.29

Table 9: *Exchange option prices*

restrictive but, in the most important European power options markets, almost the same level of liquidity is often observed and hence, the proposed approach results reasonable.

In these applications we focused only on the two dimensional case. Nevertheless, from a theoretical point of view, all the previous models can be easily extended by considering an arbitrary number of assets. Unfortunately, as the number of assets grows the number of parameters rapidly increases, and the calibration procedure becomes unstable, as observed by Luciano and Semeraro [32], leading to hardly usable models for practical purposes, mainly if we consider more than three underlying processes.

In this section we have focused only on exchange options, namely spread options where K is equal to zero. If the strike is not zero, no closed form expression is know, even in the most simple case of the BS model. Nevertheless, one can use the approximate formula proposed by Kirk [26] or the numerical methods proposed by Hurd and Zhou [23], Caldana and Fusai [9] or Pellegrino and Sabino [42].

8 Conclusions

In this article we have firstly discussed the problem of spread option pricing when the dynamic of the log-prices follows a VG or a VG++ process. Focusing on exchange options, we have derived closed form pricing formulas when all the involved Brownian motions are subordinated by a single subordinator. In particular, for the multidimensional version of the VG++ process of Section 5 we have derived an integral free formula. Accordingly, we have investigated the numerical performance of the derived formulas, comparing the results with the ones obtained by Monte Carlo and Fourier techniques. The explicit option pricing formulas are accurate and efficient, leading to computational benefits with respect to the Monte Carlo method, especially in the variance gamma case.

As a second contribution, we have proposed more general versions of the multidimensional VG++ model. Instead of using a common subordinator, these versions have been obtained by time changing each Brownian motion with different dependent subordinator relying upon the summing and scaling properties of the Γ^{++} distribution. This new Lévy framework leads to tractable models, both from a theoretical and numerical point of view. For instance, the relative expressions of the characteristic function and of the linear correlation coefficient can be easily derived. Such results are instrumental for the model calibration and for the option pricing based on FFT techniques.

Finally, we have applied all these models to the German and French power futures market. We have calibrated them on vanilla contracts and on historical futures quotations and accordingly have priced exchange options. Under these assumptions we have shown that the models with a single subordinator are sufficiently accurate. This fact has a clear economic interpretation since the German and French electrical grids are strongly connected and therefore, one can assume that the business time modeled by the subordinator "flows equally fast" in both markets. Furthermore, we have observed that the self-decomposability parameter a, that is related to the market liquidity, is very small and reflects the fact that the option market is sufficiently liquid, as has been recently observed also from an empirical point of view. In this work we have mainly focused on the pricing of exchange options nevertheless, all numerical methods could be easily adapted for the pricing of spread options. On the other hand, the investigation of approximate formulas for spread options, for instance, in Kirk's spirit, under the VG and the VG++ models could be of some interest and this will be the subject of a future research.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth dover printing, tenth gpo printing edition, 1964.

[2] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2009. doi: 10.1017/CBO9780511809781.

[3] M. Attari. Option Pricing Using Fourier Transforms: A Numerically Efficient Simplification. *Derivatives*, 2004.

[4] L. Ballotta and E. Bonfiglioli. Multivariate Asset Models Using Lévy Processes and Applications. *The European Journal of Finance*, 13(22):1320–1350, 2013.

[5] P. Bjerksund and G. Stensland. Closed Form Spread Option Valuation. Discussion Papers 2006/20, Norwegian School of Economics, Department of Business and Management Science, December 2006. URL `https://ideas.repec.org/p/hhs/nhhfms/2006_020.html`.

[6] F. Black and M. Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[7] B. Buchmann, B. Kaehler, R. Maller, and A. Szimayer. Multivariate Subordination Using Generalised Gamma Convolutions with Applications to Variance Gamma Processes and Option Pricing. *Stochastic Processes and their Applications*, 127(7):2208–2242, 2017.

[8] B. Buchmann, K. Lu, and D. Madan. Self-Decomposability of Variance Generalised Gamma Convolutions. *Stochastic Processes and their Applications*, 130(2):630–655, 2020. doi: 10.1016/j.spa.2019.02.012.

[9] R. Caldana and G. Fusai. A General Closed-Form Spread Option Pricing Formula. *Journal of Banking & Finance*, 12(37):4863–4906, 2016.

[10] R. Carmona and V. Durrleman. Pricing and Hedging Spread Options. *SIAM Review*, 45:627–685, 2003.

[11] P. Carr and D.B. Madan. Option Valuation Using the Fast Fourier Transform. *Journal of Computational Finance*, 2:61–73, 1999.

[12] U. Cherubini, E. Luciano, and Vecchiato V. *Copula Methods in Finance*. Wiley Finance, 2013.

[13] J. Choi. Sum of all Black–Scholes–Merton Models: An Efficient Pricing Method for Spread, Basket, and Asian Options. *Journal of Futures Markets*, 38(6):627–644, 2018. doi: https://doi.org/10.1002/fut.21909. URL `https://onlinelibrary.wiley.com/doi/abs/10.1002/fut.21909`.

[14] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman and Hall, 2003.

[15] N. Cufaro Petroni and P. Sabino. Pricing Exchange Options with Correlated Jump Diffusion Processes. *Quantitative Finance*, 0(0):1–13, 2018. doi: 10.1080/14697688.2017.1423371.

[16] F. Fang. *The COS Method: An Efficient Fourier Method for Pricing Financial Derivatives*. PhD thesis, Tecnical University of Delft, 2010.

[17] M. Gardini, P. Sabino, and E. Sasso. Correlating Lévy Processes with Self-Decomposability: Applications to Energy Markets. *Decision in Economics and Finance*, 2021.

[18] M. Gardini, P. Sabino, and E. Sasso. A Bivariate Normal Inverse Gaussian Process with Stochastic Delay: Efficient Simulations and Applications to Energy Markets. *Applied Mathematical Finance*, 28(2):178–199, 2021. doi: 10.1080/1350486X.2021.2010106. URL `https://doi.org/10.1080/1350486X.2021.2010106`.

[19] M. Gardini, P. Sabino, and E. Sasso. The Variance Gamma++ Process and Applications to Energy Markets. *Applied Stochastic Models in Business and Industry*, 38(2):391–418, 2022. doi: https://doi.org/10.1002/asmb.2666. URL `https://onlinelibrary.wiley.com/doi/abs/10.1002/asmb.2666`.

[20] P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag New York, 2004.

[21] B. Grigelionis. On the Self-Decomposability of Euler's Gamma Function. *Lithuanian Mathematical Journal*, 43(3):295–305, 2003.

[22] P. Humbert. Ix.—the Confluent Hypergeometric Functions of Two Variables. *Proceedings of the Royal Society of Edinburgh*, 41:73–96, 1922. doi: 10.1017/S0370164600009810.

[23] T.R. Hurd and Z. Zhou. A Fourier Transform Method for Spread Option Pricing. https://arxiv.org/pdf/0902.3643.pdf, 2009.

[24] W. Hürlimann. Margrabe Formulas for a Simple Bivariate Exponential Variance-Gamma Price Process (i) Theory. *International Journal of Scientific and Innovative Mathematical Research*, 1(1):33–44, 2013.

[25] D. K. Kahaner and G. Monegato. Nonexistence of Extended Gauss-Laguerre and Gauss-Hermite Quadrature Rules with Positive Weights. *Z. Angew. Math. Phys.*, 29 (6):983–986, nov 1978. ISSN 0044-2275. doi: 10.1007/BF01590820. URL `https://doi.org/10.1007/BF01590820`.

[26] E. Kirk. Correlations in the Energy Markets. *In Managing Energy Price Risk, Risk Publications and Enron*, pages 71–78, 1995.

[27] S. G. Kou. A Jump-Diffusion Model for Option Pricing. *Manage. Sci.*, 48(8):1086–1101, August 2002. ISSN 0025-1909.

[28] A. Lewis. A Simple Option Formula for General Jump-Diffusion and Other Exponential Lévy Processes. available at http://optioncity.net/pubs/ExpLevy.pdf, 2001.

[29] R. Lord, F. Fang, F. Bervoets, and C.W. Oosterlee. The CONV Method for pricing options. *Proceedings in Applied Mathematics & Mechanics*, 7, 2007.

[30] A. Loregian, L. Mercuri, and E. Rroji. Approximation of the Variance Gamma Model with a Finite Mixture of Normals. *Statistics & Probability Letters*, 82(2):217–224, 2012. ISSN 0167-7152. doi: https://doi.org/10.1016/j.spl.2011.10.004. URL `https://www.sciencedirect.com/science/article/pii/S0167715211003257`.

[31] E. Luciano and W. Schoutens. A Multivariate Jump-driven Financial Asset Model. *Quantitative Finance*, 6(5):385–402, 2006. URL `https://doi.org/10.1080/14697680600806275`.

[32] E. Luciano and P. Semeraro. Multivariate Time Changes for Lévy Asset Models: Characterization and Calibration. *Journal of Computational and Applied Mathematics*, 233(1):1937–1953, 2010.

[33] D. Madan, P. Carr, and E. Chang. The Variance Gamma Process and Option Pricing. *Review of Finance*, 2, 11 1999. doi: 10.1023/A:1009703431535.

[34] D. B. Madan and E. Seneta. The Variance Gamma (V.G.) Model for Share Market Returns. *The Journal of Business*, 63(4):511–524, 1990.

[35] W. Margrabe. The Value of an Option to Exchange One Asset for Another. *Journal of Finance*, 33(1):177–186, 1978.

[36] L. Mercuri, A. Perchiazzo, and E. Rroji. Finite Mixture Approximation of CARMA(p,q) Models. *SIAM Journal on Financial Mathematics*, 12(4):1416–1458, 2021. doi: 10.1137/20M1363248. URL `https://doi.org/10.1137/20M1363248`.

[37] R.C. Merton. Options Pricing when Underlying Shocks are Discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.

[38] M. Michaelsen and A. Szimayer. Marginal Consistent Dependence Modelling using Weak Subordination for Brownian Motions. *Quantitative Finance*, 18(11):1909–1925, 2018.

[39] G. Monegato. An Overview of the Computational Aspects of Kronrod Quadrature Rules. *Numerical Algorithms*, 26:173–196, 2004.

[40] A. Pascucci. *PDE and Martingale Methods in Option Pricing*. Bocconi & Springer Series. Springer Milan, 2011. ISBN 9788847017818. URL `https://books.google.it/books?id=g0LlXe5iN3AC`.

[41] T. Pellegrino. A General Closed Form Approximation Pricing Formula for Basket and Multi-Asset Spread Options. *The Journal of Mathematical Finance*, 6(5):944–974, 2016.

[42] T. Pellegrino and P. Sabino. On the Use of the Moment-matching Technique for Pricing and Hedging Multi-asset Spread Options. *Energy Economics*, 45:172–185, 2014.

[43] T. Pellegrino and P. Sabino. Pricing and Hedging Multiasset Spread Options Using a Three-dimensional Fourier Cosine Series Expansion Method. *The Journal of Energy Markets*, 7(2):71–92, 2014.

[44] M. Li S. J. Deng and J. Zhou. Closed-form Approximations for Spread Option Prices and Greeks. *Journal of Derivatives*, 15(3):58–80, 2008. doi: 10.3905/jod.2008.702506.

[45] P. Sabino and N. Cufaro Petroni. Gamma-related Ornstein–Uhlenbeck Processes and their Simulation. *Journal of Statistical Computation and Simulation*, 0(0):1–26, 2020. doi: 10.1080/00949655.2020.1842408.

[46] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge U.P., Cambridge, 1999.

[47] W. Schoutens. *Lévy Processes in Finance: Pricing Financial Derivatives*. John Wiley and Sons Inc, 2003.

[48] P. Semeraro. A Multivariate Variance Gamma Model For Financial Applications. *International Journal of Theoretical and Applied Finance*, 11(1):1–18, 2008.

[49] R. Seydel. *Tools for Computational Finance*. Universitext (1979). Springer, 2004. ISBN 9783540406044.

[50] S.E. Shreve. *Stochastic Calculus for Finance Volume 2*. Springer, 2004.

[51] J. Van Belle, S. Vanduffel, and J. Yao. Closed-form Approximations for Spread Options in Lévy Markets. *Applied Stochastic Models in Business and Industry*, 35(3):732–746, 2019. doi: https://doi.org/10.1002/asmb.2391. URL `https://onlinelibrary.wiley.com/doi/abs/10.1002/asmb.2391`.

[52] L. Yao, G. Yang, and X. Yang. A Note on the Mean Correcting Cartingale Measure for Geometric Lévy Processes. *Applied Mathematics Letters*, 24(5):593–597, 2011. ISSN 0893-9659. doi: https://doi.org/10.1016/j.aml.2010.11.011. URL `https://www.sciencedirect.com/science/article/pii/S0893965910004301`.