Lévy processes and approximation with compound Poisson process: theory, implementation and code.

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Abstract

In this article, referring to the book of Cont and Tankov [2], we detailed show how to approximate an infinite activity Lévy process with a compound Poisson process. We also provide an application to the gamma process and we implement the algorothm in Python.

1 Introduction

In this article we provide a simple example of the approximation of an infinite activity Lévy process by a compound Poisson process. We mainly refer to Cont and Tankov [2, Chapter 6.3]. We try also to make this work self-contained, providing all the necessary elements we need to understand this topic. We will show an application fo this approximation to the gamma process and we also provide the Python code which implements from scratch the simulation algorithm.

2 Preliminaries

In this section we introduce all the theory we need to better understand the example we provide in Section 3. The interested reader can refer to Cont and Tankov [2] and Schoutens [6] to better understand applications of Lévy processes to finance or to Sato [5] and Applebaum [1] for a more theoretical discussion of the topic.

2.1 Poisson processes and its counting measure

Definition 2.1. *Cont and Tankov [2, Definition 2.2] Let $E \subset \mathbb{R}^d$. A Radon measure on $(E, \mathcal{B}(E))$ is a measure μ such that for every compact[1] measurable set $B \in \mathcal{B}(\mathbb{R}^d)$, $\mu(B) < \infty$.*

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[1] A compact subset of $\mathbb{R}^d$ is simply a bounded closed subset.

Consider now a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Starting from a given positive Radon measure μ on a general measurable space $(E, \mathcal{E})$ we can construct a Poisson random measure which is an integer valued measure on $\Omega \times \mathcal{E} \to \mathbb{N}$ "counting the number of events occurring in a given element of the σ-algebra $\mathcal{E}$". The measure is Random since it depends on $\omega \in \Omega$.

In order to better understand this concept, we first consider a simple example, by referring to the Poisson process (see Cont and Tankov [2, Chapter 2.5]).

Definition 2.2. *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter $\lambda > 0$ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $(N_t)_{t\geq 0}$ defined by:*

$$N_t = \sum_{n\geq 1} \mathbb{1}_{T_n \leq t} \tag{1}$$

is called Poisson process with intensity λ.

The Poisson process is a counting process: it counts the number of random times T_n which occur in the time interval $[0, t]$. Hence:

$$N_t = \#\left\{i \geq 1, T_i \in [0, t]\right\}.$$

The jumps times $T_1, T_2, \ldots$ form a random configuration of points in $[0, \infty)$ and the process N_t counts the number of such points in the interval $[0, t]$. This counting procedure can be used to define a measure M on $\mathbb{R}^+$: for any measurable set $A \subset \mathbb{R}^+$ let:

$$M(\omega, A) = \#\left\{i \geq 1, T_i \in A\right\}. \tag{2}$$

With fixed $\omega \in \Omega$, namely fixing the random variables $T_i(\omega)$, $M(\omega, \cdot)$ is a positive, integer valued measure and $M(\cdot, A)$ is finite with probability 1 for every bounded set A (this follow from the first point of Cont and Tankov [2, Proposition 2.12]). Note that $M(\omega, \cdot)$ depends on ω: it is thus a *random measure*.

Observe also that, from Cont and Tankov [2, Proposition 2.12] we have that N_t follows a Poisson distribution with parameter λt. Hence

$$\mathbb{E}\left[M(\cdot, [s, t])\right] = \lambda(t - s),$$

or for a more general measurable set $A \subset \mathbb{R}^+$

$$\mathbb{E}\left[M(\cdot, A)\right] = \lambda|A|,$$

where $|A|$ denotes the Lebesgue measure of A. M is called the random jump measure[2] associated toe the Poisson process $\{N_t\}_{t\geq 0}$.

The Poisson process can be expressed in terms of the random measure M in the following way:

$$N_t(\omega) = M(\omega, [0, t]) = \int_{[0,t]} M(\omega, ds).$$

[2] Actually, it is a Poisson random measure and it is just a special case of the construction we derive in Section 2.3

Remark 1. The notation $\int_{[0,t]} M(\omega, ds)$ means that you are counting the events occurring on the interval $[0,t]$ according to the random measure M. Observe that this quantity is a random variable (it depends on ω). Intuitively, starting from a general random measure M for each $t > 0$ it defines a random variable and hence it can be used to define a stochastic process.

Remark 2. The properties of the Poisson process translate into the following properties for M. For disjoints intervals $[t_1, t_1'], \ldots, [t_n, t_n']$:

- $M([t_k, t_k'])$ is the number of jumps of the Poisson process in $[t_k, t_k']$: it is a Poisson random variable with parameter $\lambda(t_k' - t_k)$.
- If $j \neq i$: $M([t_k, t_k'])$ and $M([t_i, t_i'])$ are independent random variables.
- More generally for any measurable set A, $M(\cdot, A)$ follows a Poisson distribution with parameter $\lambda|A|$ where $|A|$ is the Lebesgue measure of A.

Hence we have a random configuration of points in $\mathbb{R}^+$ and for any $A \subset \mathbb{R}^+$ measurable the measure M counts the number of these random points which belongs to A.

We have defined the Poisson process, which induces a counting (random) measure M on $\mathbb{R}^+$ which, for every measurable $A \subset \mathbb{R}^+$, counts the number of random points belonging to A.

2.2 Integration with respect a counting measure

In this section we give an intuition of what does it means integrate with respect an integer measure. Consider $(E, \mathcal{E}, \nu)$, $E \subset \mathbb{R}$, with $\nu : E \to \mathbb{N}$ be a counting measure, namely for each A measurable subset of E $\nu(A)$ represents the cardinality of A. Consider $f : E \to \mathbb{R}$. We want to define $\int_E f d\nu$. Assume f is a simple function of the form:

$$f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_n}.$$

Then we define the integral in the usual way:

$$\int_E f d\nu = \sum_{i=1}^{n} c_i \nu(A_i).$$

We have that this is equal to:

$$\int_E f d\nu = \sum_{k \geq 1} f(x_k).$$

where $E = \bigcup_{k \geq 1} \{x_k\}$. Hence by proceeding in the usual way we can define the integral for f positive and hence for general functions $f : E \to \mathbb{R}$.

2.3 Poisson random measure

Is it possible, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a measurable space $(E, \mathcal{E}, \mu)$, with $E \subset \mathbb{R}^d$ and μ a (positive) Radon measure, to construct an integer valued random measure:

$$\begin{aligned} M: \ & \Omega \times \mathcal{E} \to \mathbb{N}, \\ & (\omega, A) \mapsto M(\omega, A). \end{aligned}$$

such that its properties generalize the properties we obtained when we consider the random measure associated to a Poisson process? The answer is yes, and it turns out that the random measure we obtained in the previous chapter is just a special case of this construction when we take $E = \mathbb{R}^+$ and μ as the Lebesgue measure on $\mathbb{R}^+$.

First we define what a Poisson random measure is.

Definition 2.3. *Cont and Tankov [2, Definition 2.18] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $E \subset \mathbb{R}^d$ and μ a given (positive) Radon measure on $(E, \mathcal{E})$. A Poisson random measure on E with intensity measure μ is an integer valued random measure:*

$$\begin{aligned} M: \ & \Omega \times \mathcal{E} \to \mathbb{N}, \\ & (\omega, A) \mapsto M(\omega, A). \end{aligned}$$

such that:

1. *For (almost all) $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on E: for any bounded measurable $A \subset E$, $M(\cdot, A) < \infty$ is an integer-valued random variable.*
2. *For each measurable set $A \subset E$, $M(\cdot, A) = M(A)$ is a Poisson random variable with parameter $\mu(A)$:*

$$\forall k \in \mathbb{N}: \quad \mathbb{P}(M(A) = k) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}.$$

3. *For disjoint measurable sets $A_1, \ldots, A_n \in \mathcal{E}$, the variables $M(A_1), \ldots, M(A_n)$ are independent.*

The following proposition shows how to construct a Poisson random measure starting from a general Radon measure μ.

Proposition 2.1. *Cont and Tankov [2, Proposition 2.18] For any Radon measure μ on $E \subset \mathbb{R}^d$, there exists a Poisson random measure M on E with intensity μ.*

Proof. We consider only the case when $\mu(E) < \infty$. For the other case see Cont and Tankov [2, Proposition 2.18].

1. Consider a sequence of *iid* random variables $\{X_i\}_{i \geq 1}$ so that $\mathbb{P}(X_i \in A) = \frac{\mu(A)}{\mu(E)}$.
2. Take $M(\cdot, E)$ be a Poisson random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with mean $\mu(E)$ and independent of the X_i.

3. Define:

$$M(\cdot, A) = \sum_{i=1}^{M(\cdot,E)} \mathbb{1}_A(X_i).$$

M is a Poisson random measure with intensity measure μ. For simplicity denote $M(A) = M(\cdot, A)$ and $M(E) = M(\cdot, E)$.

1. Fix $\omega \in \Omega$. $M(\omega, A) = \sum_{i=1}^{M(\omega,E)} \mathbb{1}_A(X_i(\omega)) < \infty$ since the sum is finite. $M(\omega, A) \in \mathbb{N}$ by definition.

2. Fix $k \in \mathbb{N}$.

$$\begin{aligned}
\mathbb{P}(M(A) = k) &= \mathbb{P}\left(\sum_{i=1}^{M(E)} \mathbb{1}_A(X_i)\right) \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{1}_A(X_i = k | M(E) = n)\right) \mathbb{P}(M(\cdot, E) = n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} \left(\frac{\mu(A)}{\mu(E)}\right)^k \left(1 - \frac{\mu(A)}{\mu(E)}\right)^{n-k} \frac{\mu(E)^n}{n!} e^{-\mu(E)} \\
&= \frac{\mu(A)^k}{k!} e^{-\mu(A)}.
\end{aligned}$$

3. Independence follows from the fact that the joint probability factorizes, namely if $A_1 \cap A_2 = \emptyset$ then:

$$\begin{aligned}
&\mathbb{P}(M(A_1) = k, M(A_2) = m) \\
&= \sum_{n \geq k+m} \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{1}_{A_1}(X_i) = k, \sum_{i=1}^{n} \mathbb{1}_{A_2}(X_i) = m\right) \mathbb{P}(M(E) = n) \\
&= \sum_{n \geq m+k} \left(\frac{\mu(A_1)}{\mu(E)}\right)^k \left(\frac{\mu(A_2)}{\mu(E)}\right)^m \binom{n}{k} \binom{n-k}{m} \left(1 - \frac{\mu(A_1)}{\mu(E)} - \frac{\mu(A_2)}{\mu(E)}\right)^{n-k-m} \frac{e^{-\mu(E)}\mu(E)^n}{n!} \\
&= \frac{\mu(A_1)^k}{k!} \frac{\mu(A_2)^m}{m!} e^{-\mu(A_1)} e^{-\mu(A_2)} \\
&= \mathbb{P}(M(A_i) = k) \cdot \mathbb{P}(M(A_j) = m).
\end{aligned}$$

■

The above proof shows that any Poisson random measure on E can be represented as counting measure associate to a random sequence of points in E: there exists $\{X_n(\omega)\}_{n \geq 1}$ such that:

$$\forall A \in \mathcal{E}, \qquad M(\omega, A) = \sum_{n \geq 1} \mathbb{1}_A(X_n(\omega)) = \sum_{n \geq 1} \delta_{X_n(\omega)}(A).$$

M in thus the sum of Dirac masses located at random points $(X_n)_{n \geq 1}$.

Remark 3. For the Poisson process defined in Equation (1), $E = [0, T] \times \{1\}$ and the sequence is given by $(T_n, 1)_{n \geq 1}$. It represent a sequence of points of size 1 located at time T_n. For an intuitive introduction to Poisson point process see Lanchier [4].

2.4 Building jump processes from Poisson random measure: Marked point process

Consider $(\Omega, \mathcal{F}, \mathbb{P})$, $E = [0,T] \times \mathbb{R}^d \setminus \{0\}$ and a given random measure M on E. As in the proof of Proposition 2.1, the random measure can be describen as the counting measure associate to a configuration of points $(T_n, Y_n) \in E$:

$$M(\omega, A) = \sum_{n \geq 1} \delta_{(T_n(\omega), Y_n(\omega)))}(A), \quad A \in \mathcal{E}.$$

The random sequence (T_n, Y_n) is called marked point process.

Definition 2.4. *A marked point process on* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ *is a sequence* $(T_n, Y_n)_{n \geq 1}$ *where:*

- $(T_n)_{n \geq 1}$ *is an increasing sequence of non anticipating random times with* $T_n \to \infty$ *almost surely when* $n \to \infty$.
- $(Y_n)_{n \geq 1}$ *is a sequence of random variables taking values in* $\mathbb{R}^d \setminus \{0\}$.
- *The value of* Y_n *is revealed at* T_n*:* Y_n *is* $\mathcal{F}_{T_n}$ *measurable.*

The first condition guarantees that the number of events in an interval of the form $[0,T]$ is finite.

Fixed $\omega \in \Omega$, $M(\omega, \cdot)$ defines a measure on E. Hence we can construct integrals with respect this measures. Consider $f : E \to \mathbb{R}$.

- $f : E \to \mathbb{R}^+$ where $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$, with $c_i \geq 0$ and $A_i \subset E$ disjoint measurable subset of E. Define the integral as:

  $$M(f) = \sum_{i=1}^n c_i M(A_i).$$

  Observe that $M(f)$ **is a random variable** with expectation given by[3]:

  $$\mathbb{E}[M(f)] = \sum_{i=1}^n c_i \mu(A_i).$$

  For each $\omega \in \Omega$ you have a different configuration of points $(T_n(\omega), Y_n(\omega))_{n \geq 1}$ in E. For each measurable $A_i \subset E$ you simply count the number of points belonging to A_i, by using M. If you change ω the configuration of points changes and also $M(\omega, A_i)$ changes and hence the value of the integral $M(f)$ consequently.

  If $f_n = \sum_{j=1}^{m(n)} c_j \mathbb{1}_{A_j}$ is simple we can define the integral with respect to μ as:

  $$\mu(f_n) = \sum_{j=1}^{m(n)} c_j \mu(A_j),$$

[3]Remember that a Poisson random measure has intensity measure μ defined on $(E, \mathcal{E})$.

and hence we can define $\mu(f) = \sup\{\mu(\phi), \phi \text{ simple function}, \phi \leq f\}$ for a general positive function $f : E \to [0, \infty)$.

It follows that:

$$\mathbb{E}[M(f_n)] = \sum_{i=1}^{m(n)} c_i \mu(A_i) = \mu(f_n).$$

- For $f : E \to [0, \infty)$ you can take $(f_n)_{n\geq 1}$ simple increasing to f and define:

$$M(f) = \lim_{n\to\infty} M(f_n).$$

$M(f)$ is a random variable since it is the limit of random variables. It follows that:

$$\mathbb{E}[M(f)] = \mathbb{E}\left[\lim_{n\to\infty} M(f_n)\right] = \lim_{n\to\infty} \mathbb{E}[M(f_n)] = \lim_{n\to\infty} \mu(f_n) = \mu(f),$$

where the limit is to be considered as the one we use to define the integral of f. Expectation and limit can be exchanged because of the monotone convergence theorem.

- If $f : E \to R$ it can be decompose as: $f = f_+ - f_-$. We also request that:

$$\mu(|f|) = \int_{[0,T]\times\mathbb{R}^d\setminus\{0\}} |f(s, y)|\mu(ds \times dy) < \infty.$$

Thanks to this condition the random variables $M(f_+)$ and $M(f_-)$ have finite expectation and hence are almost surely finite. We can define $M(f) = M(f_+) - M(f_-)$. $M(f)$ is a random variable and its expectation is given by:

$$\mathbb{E}[M(f)] = \mu(f) = \int_{[0,T]\times\mathbb{R}^d\setminus\{0\}} f(s, y)\mu(ds \times dy).$$

Integrating f with respect to M up to time t yields to a nonanticipating stochastic process:

$$X_t = \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} f(s, y)M(ds \times dy) = \sum_{\{n,T_n\in[0,t]\}} f(T_n, Y_n).$$

Hence, starting from a Poisson random measure we have constructed a jump process. $(X_t(f))_{t\in[0,T]}$ is a jump process whose jumps happen at random time T_n and have amplitude given by $f(T_n, Y_n)$.

2.5 Jump measure of a cadlag process

So far we have seen that, given a point process we can construct a Poisson random measure (see proof of proposition 2.1).

Given a cadlag process $(X_t)_{t\in[0,T]}$ with values in $\mathbb{R}^d$ one can associate a random measure J_X on $[0, T]\times\mathbb{R}^d$ called the jump measure in the following way. Since X is a cadlag process it has a countable number of jumps: $\{t \in [0, T],\ \Delta X_t = X_t - X_{t-} \neq 0\}$ is countable and its elements can be arranged in a sequence $\{T_n\}_{n\in\mathbb{N}}$ which are the (random) jump rimes of X. At time T_n, the process X has a discontinuity $Y_n = X_{T_n} - X_{T_n-} \in \mathbb{R}^d \setminus \{0\}$.

Hence $(T_n, Y_n)_{n\geq 1}$ defines a marked point process on $[0,T]\times\mathbb{R}\setminus\{0\}$ which contains all the information about the jumps of the process X: jumps time T_n and jump sizes Y_n. The associated random measure, which is denoted by J_X, is called **jump measure** of the process X:

$$J_X(\omega,\cdot)=\sum_{n\geq 1}\delta_{(T_n(\omega),Y_n(\omega))}=\sum_{t\in[0,T],\Delta X_t\neq 0}\delta_{t,\Delta X_t}.$$

Intuitively, for any measurable set $A\subset\mathbb{R}^d$:

$$J_X([0,T]\times A)=\text{number of jumps of } X \text{ occurring between 0 and } t \text{ whose amplitude belongs to } A.$$

The random measure J_X does not contains information about the continuous part of X, but it describes only its jumps, in particular when they occur and how big they are.

All quantities involving the jumps of X can be computed by integrating various functions against J_X. For example if $f(t,y)=y^2$ we have:

$$\int_{[0,T]\times\mathbb{R}} y^2 J_X(dt\times dy)=\sum_{t\in[0,T]}(\Delta X_t)^2.$$

These expression may involve infinite sums and hence its convergence must be defined in some sense.

Example 2.1. *Consider a Poisson process. It jumps measure is given by:*

$$J_N=\sum_{n\geq 1}\delta_{(T_n,1)},$$

namely:

$$J_N([0,t]\times A)=\begin{cases}\#\{i\geq 1,\ T_i\in[0,t]\} & \text{if } 1\in A\\ 0 & \text{if } 1\notin A.\end{cases}\tag{3}$$

2.6 Lévy processes: compound Poisson process

In this section we briefly outline some important properties of Lévy processes. The interested reader car refer to Cont and Tankov [2, Chapter 3].

There is an important connection between Lévy processes and infinitely divisible laws.

Proposition 2.2. *Let $(X_t)_{t\geq 0}$ be a Lévy process. Then for every t, X_t has an infinitely divisible distribution. Conversely if F is an infinitely divisible distribution, then there exists a Lévy process $(X_t)_{t\geq 0}$ such that the distribution of X_1 is given by F.*

Moreover, the characteristic function of a Lévy process has specific form.

Proposition 2.3. *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. There exists a continuous function $\psi:\mathbb{R}^d\to\mathbb{R}$ called the characteristic exponent of X, such that:*

$$\mathbb{E}\left[e^{iu\cdot X_r}\right]=\exp\left\{t\psi(z)\right\},\quad z\in\mathbb{R}^d.$$

A particular Lévy process is called the Compound Poisson process.

Definition 2.5. *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $(X_t)_{t\geq 0}$ defined as:*

$$X_t = \sum_{i=1}^{N_t} Y_i, \tag{4}$$

where jumps sizes Y_i are iid with distribution f and $\{N(t), t \geq 0\}$ is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$.

The characteristic function of a compound Poisson process is given by (see Cont and Tankov [2, Proposition 3.4]):

$$\mathbb{E}\left[e^{iu\cdot X_t}\right] = \exp\left\{t\lambda \int_{\mathbb{R}^d} \left(e^{iu\cdot x} - 1\right) f(dx)\right\}, \quad u \in \mathbb{R}^d. \tag{5}$$

Introducing a new measure $\nu(A) = \lambda f(A)$ Formula (6) can be rewritten as:

$$\mathbb{E}\left[e^{iu\cdot X_t}\right] = \exp\left\{\lambda \int_{\mathbb{R}^d} \left(e^{iu\cdot x} - 1\right) \nu(dx)\right\}, \quad u \in \mathbb{R}^d. \tag{6}$$

ν is called Lévy measure of the process $(X_t)_{t\geq 0}$.

Since a compound Poisson process $(X_t)_{t\geq 0}$ is a cadlag continuous time jump process we can associate to it its jump measure, as observed in the previous section. This measure J_X fully describes the structure of jumps. It is a random measure on $[0,\infty) \times \mathbb{R}^d$ and for any measurable set $B \subset \mathbb{R}^d \times [0,\infty)$ we have:

$$J_X(B) = \#\left\{(t, X_t - X_{t-}) \in B\right\}.$$

In particular, for any measurable set $A \subset \mathbb{R}^d$, $J_X\left([t_1,t_2] \times A\right)$ counts the number of jumps of X in the interval $[t_1,t_2]$ which size belongs to A.

Proposition 2.4. *Let $(X_t)_{t\geq 0}$ be a compound Poisson process with intensity λ and jump size distribution f. Its jump measure J_X is a Poisson random measure on $\mathbb{R}^d \times [0,\infty)$ with intensity measure $\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt$.*

This proposition suggest to interpret the Lévy measure as the average number of jumps per unit of time.

Definition 2.6. *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$. The measure ν on $\mathbb{R}^d$ defined by:*

$$\nu(A) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\right\}\right], \quad A \in \mathcal{B}\left(\mathbb{R}^d\right).$$

is called Lévy measure of X. $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A.

Proposition 2.4 implies that every compound Poisson process can be represented in the following form:

$$X_t = \sum_{s\in[0,t]} \Delta X_s = \int_{[0,t]\times\mathbb{R}^d} x J_x\left(ds \times dx\right), \tag{7}$$

where J_X is a Poisson random measure with intensity measure $\nu(dx)dt$. This is a special case of the **Lévy-Itō decomposition**: we have only rewritten the process X as the sum of its jumps. Since a compound Poisson process has almost surely a finite number of jumps in interval $[0,t]$ the stochastic integral is just a finite sum and hence there are no convergence problems.

2.7 Lévy-Itō decomposition

Consider $\left\{X_t^0\right\}_{t\geq 0}$ represented as is Equation (7). Consider its Lévy measure defined by:

$$\nu(A) = \mathbb{E}\left[\#\left\{t\in[0,1] : \Delta X_t^0 \neq 0, \Delta X_t^0 \in A\right\}\right], \quad A\in\mathcal{B}\left(\mathbb{R}^d\right).$$

Take a Brownian motion with drift $\gamma t + W_t$, independent from X_0. $X_t = X_t^0 + \gamma t + W_t$ defines another Lévy process which can be represented as:

$$X_t = \gamma t + W_t + \sum_{s\in[0,t]} \Delta X_s = \gamma t + W_t + \int_{[0,t]\times\mathbb{R}^d} x J_x\left(ds\times dx\right), \tag{8}$$

where J_X is a Poisson random measure on $[0,\infty)\times\mathbb{R}^d$ with intensity measure $\nu(dx)dt$.

Given a general Lévy process X_t, we can still define its Lévy measure ν as above. $\nu(A)$ is still finite for any compact set A not containing 0: if this were not true the process would have an infinite number of jumps of finite size on $[0,T]$ which contradicts the cadlag property. So ν defines a Radon measure on $\mathbb{R}\setminus\{0\}$. But ν is not necessarily a finite measure: the above restriction still allows it to blow up at zero and X may have an infinite number of small jumps on $[0,T]$. In this case the sum of the jumps becomes an infinite series and its convergence imposes some conditions on the measure ν, under which we obtain a decomposition of X similar to the one above.

Proposition 2.5. *Lévy-Itō decomposition Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ and ν its Lévy measure.*

- *ν is a Radon measure on $\mathbb{R}^d\setminus\{0\}$ and verifies:*

$$\int_{|x|\leq 1} |x|^2\nu(dx) < \infty, \qquad \int_{|x|>1} \nu(dx) < \infty.$$

- *The jump measure of X, denoted by J_X, is a Poisson random measure on $[0,\infty)\times\mathbb{R}^d$ with intensity measure $\nu(dx)dt$.*

- *There exists a vector γ and a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ with covariance matrix A such that:*

$$X_t = \gamma t + B_t + X_t^l + \lim_{\epsilon\to 0} \tilde{X}_t^\epsilon.$$

where:

$$X_t^l = \int_{|x|\geq 1, s\in[0,t]} x J_X\left(ds\times dx\right),$$
$$\tilde{X}_t^\epsilon = \int_{\epsilon\leq|x|\leq 1, s\in[0,t]} x\left\{J_X\left(ds\times dx\right) - \nu(dx)ds\right\}.$$

All these terms are independent and the convergence of the last term in almost sure and uniform in t in $[0,T]$.

(ν, A, γ) is unique and it is called the Lévy triplet of the process X.

The so called Lévy-Khinchin representation follows free from this result.

Proposition 2.6. *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic triplet (A, ν, γ). Then:*

$$\mathbb{E}\left[e^{iz\cdot X_t}\right] = e^{t\psi(z)},\ z \in \mathbb{R}^d, \tag{9}$$

$$\psi(z) = -\frac{1}{2} z \cdot Az + i\gamma \cdot z + \int_{\mathbb{R}^d} \left(e^{iz\cdot x} - 1 - iz \cdot x \mathbb{1}_{|x|\leq 1}\right) \nu(dx) \tag{10}$$

γ is not the drift and it depends on the choice of the truncation function of the large jumps, which is this case is given by $g(x) = \mathbb{1}_{|x|\leq 1}$.
When $\nu(\mathbb{R}) = \infty$ (infinite activity case), the set of jump times or every trajectory of the Lévy process is countably finite and dense in $[0, \infty)$. Intuitively, and infinite activity Lévy process has a countably infinite number of small jumps in each time interval $[t_1, t_2]$.

2.8 Path properties of Lévy processes

Properties of sample paths of a Lévy process X can be deduced by analytical properties of its characteristic triplet (A, γ, ν).

Proposition 2.7. *Lévy process with piece-wise constant trajectories*
A Lévy process has piecewise constant trajectories if and only if is characteristic triplet satisfies the following conditions: $A = 0$, $\int_{\mathbb{R}^d} \nu(dx) < \infty$ and $\gamma = \int_{|x|\leq 1} x\nu(dx)$ or, equivalently, if its chracteristic exponent is of the form:

$$\psi(z) = \int_{-\infty}^{\infty} \left(e^{iux} - 1\right) \nu(dx), \quad \textit{with } \nu(\mathbb{R}) < \infty.$$

Remark 4. Recall that almost all trajectories of a Lévy process are piecewise contant if and only if it is of compound Poisson type (see Cont and Tankov [2, Proposition 3.3]).

A Lévy process is said to be of finite variation if its trajectories are functions of finite variation with probability 1.

Proposition 2.8. *Lévy process of finite variation A Lévy process is of finite variation if and only if its characteristic triplet (A, γ, ν) satisfies:*

$$A = 0, \quad \int_{|x|\leq 1} |x|\nu(dx) < \infty.$$

In this case we have also that if $\{X_t\}_{t\geq 0}$ is a Lévy process with characteristic function $(\nu, 0, \gamma)$:

$$X_t = bt + \int_{[0,t]\times\mathbb{R}^d} xJ_X(ds \times dx) = bt + \sum_{s\in[0,t],\ \Delta X_s \neq 0} \Delta X_s.$$

and its characteristic function can be expressed as:

$$\mathbb{E}\left[e^{iz\cdot X_t}\right] = \exp\left\{ib\cdot z + \int_{\mathbb{R}^d}\left(e^{iz\cdot x}-1\right)\nu(dx)\right\},$$

where $b = \gamma - \int_{|x|\leq 1} x\nu(dx)$. Note that the characteristic triplet is not given by $(b,0,\nu)$ but by $(\gamma,0,\nu)$. In fact, γ is not an intrinsic quantity and depends on the truncation function used in the Lévy-Khinchin representation while bt as an intrinsic interpretation as the continuous part of X.

Proposition 2.9. *Subordinators Let be $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. The following conditions are equivalent:*

- *$X_t \geq 0$ a.s. for some $t \geq 0$.*
- *$X_t \geq 0$ a.s. for every $t \geq 0$.*
- *Sample paths of X are almost surely non decreasing: $s \leq t \implies X_s \leq X_t$.*
- *The characteristic triplet of $(X_t)_{t\geq 0}$ satisfies the following conditions: $A = 0$, $\nu\left((-\infty,0]\right) = 0$, $\int_0^\infty (x\wedge 1)\,\nu(dx) < \infty$ and $b \geq 0$. Hence X has no diffusion component, only positive jumps of finite variation and positive drift.*

3 Approximation of an infinite activity Lévy process by a compound Poisson

Let $(X_t)_{t\geq 0}$ be an infinite activity Lévy process with characteristic triplet $(0,\nu,\gamma)$. From the Lévy-Itō decomposition we know that X can be represented as a sum of compound Poisson process and an almost sure limit of compensated compound processes:

$$X_t = \gamma t + \sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|\geq 1} + \lim_{\epsilon\to 0} N_t^\epsilon,$$

where

$$N_t^\epsilon = \sum_{s\leq t}\Delta X_s \mathbb{1}_{\epsilon\leq|\Delta X_s|\leq 1} - t\int_{\epsilon\leq|x|\leq 1} x\nu(dx).$$

Therefor a natural idea is to approximate X with:

$$X_t^\epsilon = \gamma t + \sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|\geq 1} + N_t^\epsilon.$$

For a more detailed exposition see Cont and Tankov [2, Chapter 6.3].

3.1 The finite variation case

Assume that X is a Lévy process of finite variation. Hence we have:

$$A = 0 \int_{|x|\leq 1}|x|\nu(dx) < \infty \qquad\qquad \gamma,$$

$$\mathbb{E}\left[e^{izX_t}\right] = \exp\left\{ibz + \int_{\mathbb{R}^d}(e^{izx}-1)\nu(dx)\right\},$$

with $b = \gamma - \int_{|x|\leq 1} x\nu(dx)$.

Hence we have that:

$$\begin{aligned}
X_t^\epsilon &= \gamma t + \sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|\geq 1} + N_t^\epsilon \\
&= \gamma t + \sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|\geq 1} + \sum_{s\leq t}\Delta X_s \mathbb{1}_{\epsilon\leq|\Delta X_s|<1} - t\int_{\epsilon\leq|x|<1} \\
&+ t\int_{0<|x|<\epsilon} x\nu(dx) - t\int_{0<|x|<\epsilon} x\nu(dx) \\
&= bt + \sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|\geq\epsilon} + \int_0^t\int_{|x|<\epsilon} x\nu(dx)ds.
\end{aligned}$$

Recall that we can write:

$$\sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|<\epsilon} = \int_{[0,t]\times\mathbb{R}^d} x\mathbb{1}_{|x|<\epsilon} J_X(ds\times dx),$$

and that $\mathbb{E}\left[J(\cdot,A)\right] = \mu(A)$. Hence we have that:

$$\mathbb{E}\left[\sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|<\epsilon}\right] = \int_{[0,t]\times\mathbb{R}^d} x\mathbb{1}_{|x|<\epsilon}\nu(dx)ds$$

and finally:

$$X_t^\epsilon = bt + \sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|\geq\epsilon} + \mathbb{E}\left[\sum_{s\leq t}\Delta X_s \mathbb{1}_{|\Delta X_s|<\epsilon}\right].$$

In the finite variation case the approximation is constructed by replacing small jumps with their expectation. In the next section we show the complete procedure to approximate a gamma process with a compound Poisson process. Other examples can be found in Cont and Tankov [2, Chapter 6.3].

3.2 An example: the gamma process

Consider the Gamma process, which characteristic triplet is given by:

$$\gamma = \frac{\alpha t}{\beta}\left(1-e^{-\beta}\right), \quad \sigma = 0, \quad \nu(x) = \alpha x^{-1}e^{-\beta x}\mathbb{1}_{x\geq 0}.$$

Its characteristic exponent is given by:

$$\psi(u) = \int_0^\infty \left(e^{iux}-1\right)\alpha x^{-1}e^{-\beta x}dx = \log\left(1-\frac{iu}{\beta}\right)^{-\alpha}.$$

with $b = \gamma - \int_{|x|\leq 1} x\nu(dx) = 0$. The process is of finite variation, hence we use the approximation introduced in the previous section:

$$X_t^\epsilon = bt + \sum_{s\leq t} \Delta X_s \mathbb{1}_{|\Delta X_s|\geq\epsilon} + \mathbb{E}\left[\sum_{s\leq t} \Delta X_s \mathbb{1}_{|\Delta X_s|<\epsilon}\right]. \tag{11}$$

We have to investigate the two terms. The first one is a compound Poisson process (since we have truncated the original Lévy measure around the origin). The second term must be computed, possibly explicitly.

First we truncate the Lévy measure:

$$\nu^\epsilon(x) = \alpha \frac{e^{-\beta x}}{x} \mathbb{1}_{x\geq\epsilon}(x).$$

Since for a compound Poisson process the intensity measure of the jump measure J_X is given by:

$$\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt,$$

where $f(dx)$ is the distribution (*cdf*)[4] of jumps, we can obtain the intensity $\lambda(\epsilon)$ by integrating the Lévy measure over $\mathbb{R}$.

$$\lambda(\epsilon) = \int_{\mathbb{R}} \nu^\epsilon(dx) = \alpha E_1(\beta\epsilon),$$

where $E_1(z)$ is the Exponential integral defined by:

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt, \quad |\arg z| \leq \pi.$$

On the other hand, the *pdf* $p^\epsilon(x)$ of jumps can be obtained by:

$$p^\epsilon(dx) = \frac{\nu^\epsilon(dx)}{\lambda(\epsilon)}.$$

In order to simulate from jumps following a distribution with *pdf* given by p^ϵ, one can observe that:

$$p^\epsilon(x) = \frac{\alpha e^{-\beta x}}{x} \mathbb{1}_{x\geq\epsilon}(x) \frac{1}{U(\epsilon)} \leq \beta e^{-\beta x} \frac{\alpha}{\epsilon\beta U(\epsilon)} = g(x)M.$$

Hence the *pdf* p^ϵ is bounded by the *pdf* of and exponential random variable with parameter β, $\mathcal{E}(\beta)$, multiplied by a constant term $M = \frac{\alpha}{\epsilon\beta U(\epsilon)}$. For this reason, in order to sample jumps, a simple acceptance rejection method can be implemented (see Devroye [3]).

Now, what we have to do is to compute the second term in Equation (11). This is very easy and we get:

$$\mathbb{E}\left[\sum_{s\leq t} \Delta X_s \mathbb{1}_{|\Delta X_s|<\epsilon}\right] = \int_0^t \int_0^\epsilon \frac{x\alpha e^{-\beta x}}{x} dx ds = tb_\epsilon,$$

[4] June 25, 2025: Probably here you have to use the distribution of jumps and not the *pdf* but it appears it is the same. Check it.

Algorithm 1 Acceptance-rejection method

Sample $y \sim \mathcal{E}(\beta)$.
Sample $u \sim U[0,1]$
if $u \geq \frac{p^{\epsilon}(y)}{Mg(y)}$ **then**
    Return u as a valid sample from $p^{\epsilon}(x)$.
else
    Repeat.
end if

where $b_{\epsilon} = \frac{\alpha}{\beta}\left[1 - e^{-\beta\epsilon}\right]$.

Finally, the Compound Poisson process with intensity $\lambda(\epsilon)$, jumps distribution $p^{\epsilon}(x)$ and drift b_{ϵ} can be easily simulated by using Cont and Tankov [2, Algorithm 6.2] which is reported in Algorithm 2.

Algorithm 2 Algorithm for compound Poisson process (with drift b)

1: Simulate a random variable N from Poisson distribution with parameter $\lambda(\epsilon)T$. N gives the total number of jumps in the interval $[0,T]$.
2: Simulate N independent *rv* U_i, uniformly distributed on the interval $[0,T]$. These variables correspond to the jump times.
3: Simulate jump sizes: N independent *rv* Y_i with law $\frac{\nu(dx)}{\lambda}$. The trajectory is given by:

$$X_t = bt + \sum_{i=1}^{N} \mathbb{1}_{U_i<t} Y_i.$$

In the next section, we report the python code which implements this approximation and we compare it with the simulation of the gamma process obtained by simulating the increments.

4 Python Code Example

The Python code[5] is the following

```
import numpy as np
import matplotlib.pyplot as plt
import scipy
from scipy.special import exp1
from scipy.special import gamma


def exponential_density(x, mu):
    """
    Density of an exponential random variable
    :param x: point at which the pdf is evaluated
    :param mu: rate > 0
    :return:
    """
    if x > 0:
        return mu*np.exp(-x*mu)
    else:
        return 0


def gamma_density(x, alpha, beta):
    """

    :param x: point at which the pdf is evaluated
    :param alpha: shape parameter
    :param beta: rate parameter
    :return:
    """
    if x > 0:
        return (beta**alpha)/gamma(alpha)*x**(alpha-1)*np.exp(-beta
   *x)
    else:
        return 0


class GammaJumpsDistribution:
    """
    This class implements the distribution of jumps
    """

    def __init__(self, alpha, beta, epsilon):
        self.alpha = alpha
        self.beta = beta
        self.epsilon = epsilon

```

[5]Please note that this code aims not to be a perfect exercise on object oriented programming nor aims at showing how to implement the algorithm following Python best practices. It is just a didactic example, which try to help to reader to move from theory to the numerical implementation.

```
    def _density(self, x):
        """
        Density of jumps.
        :param x: point at which the pdf is evaluated
        :return:
        """

        # This is what in the article is called lambda(epsilon)
        u_epsilon = self.alpha * scipy.special.exp1(self.beta*self.
epsilon)

        if x <= self.epsilon:
            return 0
        else:
            return self.alpha/x*np.exp(-self.beta*x)/u_epsilon

    def density(self, x):
        """
        Vectorized the density function
        :param x: points at which the pdf is evaluated
        :return:
        """
        g = np.vectorize(self._density, otypes=[np.ndarray])
        return g(x)

    def sample(self, n_sim=1):
        """
        Sample from the distribution using an acceptance rejection
method
        :param n_sim: number of sample
        :return:
        """

        u_epsilon = self.alpha * scipy.special.exp1(self.beta *
self.epsilon)
        M = self.alpha/(self.epsilon*self.beta*u_epsilon)
        x = np.zeros(n_sim)

        for i in range(0, n_sim):
            sampled = False
            while not sampled:
                y = np.random.exponential(scale=1/self.beta)
                u = np.random.uniform(low=0, high=1, size=1)
                if exponential_density(y,self.beta) > 0:
                    if u <= self._density(y)/(M*exponential_density
(y,self.beta)):
                        x[i] = y
                        sampled = True

        return x

    def plot_density(self, x, x_sample=None):
```

```
93          """
94          Given empirical data x_sample the function plots and
    histogram of empirical data and compares it with the
95          theoretical probability density function.
96          :param x: point at which the density has to be plotted
97          :param x_sample: empirical data
98          :return:
99          """
100
101         f = self.density(x)
102         plt.plot(x, f)
103
104         if not (x_sample is None):
105             plt.hist(x_sample, bins=500, density=True, edgecolor='
    black', alpha=0.7)
106
107         plt.xlabel('x')
108         plt.ylabel('f(x)')
109         plt.title('pdf')
110         plt.grid(True)
111         plt.show()
112
113
114 class GammaProcess:
115     """
116     This class implements the gamma process
117     """
118
119     def __init__(self, alpha, beta, T):
120         self.alpha = alpha
121         self.beta = beta
122         self.T = T
123         self.sim = None
124         self.time_grid = None
125         self.number_jumps_smaller_than_epsilon = None
126         self.epsilon = 0
127
128     def set_epsilon(self, epsilon):
129         """
130         Set the truncation parameter
131         :param epsilon:
132         :return:
133         """
134         self.epsilon = epsilon
135
136
137     def simulate(self, n_sim, n_steps):
138         """
139         Simulate from a gamma process
140         :param n_sim: number of paths
141         :param n_steps: number of time steps
142         :return:
```

```
        """

        self.time_grid = np.linspace(0, self.T, n_steps)
        X = np.zeros((n_sim, n_steps))

        self.number_jumps_smaller_than_epsilon = np.zeros(n_sim)

        dt = self.T/n_steps

        for i in range(n_sim):
            for j in range(1,n_steps):
                gamma_increments = np.random.gamma(self.alpha*dt,
    1.0 / self.beta)
                if gamma_increments < self.epsilon:
                    self.number_jumps_smaller_than_epsilon[i] +=1

                X[i, j] = X[i, j-1] + gamma_increments

        self.sim = X.transpose()


class CompoundApproximator():
    """
    This class implements the compound Poisson process which
    approximate the gamma process.
    """

    def __init__(self, alpha, beta, epsilon, T):
        self.alpha = alpha
        self.beta = beta
        self.epsilon = epsilon
        self.T = T
        self.sim = None
        self.time_grid = None

    def simulate(self,n_sim,n_steps):
        """
        Simulate from the compound Poisson process
        :param n_sim: number of trajectories
        :param n_steps: number of time steps
        :return:
        """
        intensity = self.alpha * scipy.special.exp1(self.beta *
    self.epsilon)
        Y = GammaJumpsDistribution(self.alpha, self.beta, self.
    epsilon)

        b_epsilon = self.alpha/self.beta*(1-np.exp(-self.beta*self.
    epsilon))

        self.time_grid = np.linspace(0,self.T, n_steps)

```

```
        X = np.zeros((n_sim, n_steps))

        for i in range(n_sim):
            # Generate event times
            num_events = np.random.poisson(intensity * self.T)
            event_times = np.sort(np.random.uniform(0, self.T,
    num_events))

            # Generate random values
            random_values = Y.sample(n_sim=num_events)

            # Compute compound process values
            X[i, :] = b_epsilon*self.time_grid

            for index, jump_time in enumerate(event_times):
                idx_time = self.time_grid >= jump_time
                X[i,idx_time] = X[i,idx_time] + random_values[index
    ]

        self.sim = X.transpose()


if __name__ == "__main__":
    alpha = 5.0
    beta = 2.0
    epsilon = 0.1  # truncation parameter
    T = 2.0  # time horizon of the simulation

    # number of paths and number of time steps
    n_sim = 5000
    n_steps = 500

    # Simulate using the approximation with a compound Poisson
    X_eps = CompoundApproximator(alpha=alpha, beta=beta, epsilon=
    epsilon, T=T)
    print("Approximator sim")
    X_eps.simulate(n_sim=n_sim, n_steps=n_steps)

    # Exact simulation of the gamma process
    print("Exact process sim")
    X = GammaProcess(alpha=alpha, beta=beta, T=T)
    X.set_epsilon(epsilon)
    X.simulate(n_sim=n_sim, n_steps=n_steps)

    # Plot the distribution at time t = T/2 of the compound Poisson
    process and compare it with the one of the gamma
    # process
    data = X.sim[250, :]
    data_eps = X_eps.sim[250, :]

    t = T/n_steps*250
    vectorized_func = np.vectorize(gamma_density)
```

```
    # Plot histograms
    plt.figure()
    plt.hist(data_eps, bins=50, density=True, alpha=0.9, label='
compound Poisson process')
    plt.hist(data, bins=50, density=True, alpha=0.9, label='gamma
process')
    x = np.linspace(np.min(data), np.max(data), 1000)
    y = vectorized_func(x, alpha*t, beta)
    plt.plot(x, y, label="density", color='black', linewidth=2)

    # Add labels, title, and legend
    plt.xlabel('x')
    plt.ylabel('Density')
    plt.title('Distribution of the processes at time t=1')
    plt.legend()

    # Save the figure in EPS format
    plt.savefig('distribution.eps', format='eps')

    # Show the plot
    plt.show()
```

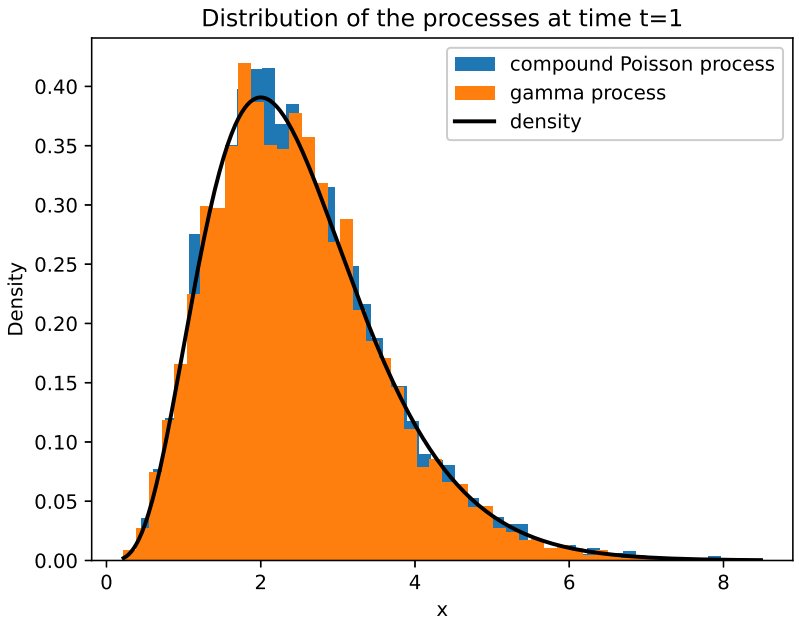

Figure 1: Distribution of the gamma process and of its approximating compound Poisson process at time $t = 1$. The processes have been simulated over the interval $[0, T]$ with $T = 2$. The parameters are: $\alpha = 5$, $\beta = 2$ and truncation level $\epsilon = 0.1$.

References

[1] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2009. doi: 10.1017/CBO9780511809781.

[2] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman and Hall, 2003.

[3] L. Devroye. *Non-Uniform Random Variate Generation*. Springer-Verlag, 1986.

[4] N. Lanchier. *Stochastic Modeling*. Universitext. Springer International Publishing, 2017. ISBN 9783319500379.

[5] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge U.P., Cambridge, 1999.

[6] W. Schoutens. *Lévy Processes in Finance: Pricing Financial Derivatives*. John Wiley and Sons Inc, 2003.