Correlating Lévy processes with Self-Decomposability: Applications to Energy Markets *

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Abstract

Based on the concept of self-decomposability we extend some recent multidimensional Lévy models built using multivariate subordination. Our aim is to construct multivariate Lévy processes that can model the propagation of the systematic risk in dependent markets with some stochastic delay instead of affecting all the markets at the same time. To this end, we extend some known approaches keeping their mathematical tractability, study the properties of the new processes, derive closed form expressions for their characteristic functions and detail how Monte Carlo schemes can be implemented. We illustrate the applicability of our approach in the context of gas, power and emission markets focusing on the calibration and on the pricing of spread options written on different underlying commodities.

Keywords: Multivariate Lévy Processes, Self-Decomposability, Monte Carlo, FFT, Energy Markets, Spread Options.

1 Introduction

During the last decades a lot of efforts have been done in financial modeling to go beyond the Black and Scholes [3] framework. The Black-Scholes (BS) formula is widely used by practitioners, but its limitations are well-known and over the years several researchers - Merton [20], Madan and Seneta [19] and Heston [15] among others - have proposed more sophisticated models to overcome its shortcomings. On the other hand, their main focus was the single asset modeling framework.

In a multi-asset market one has to take care of the modeling of the dependence structure, which can easily become a challenging task. Mainly, one comes up against three issues:

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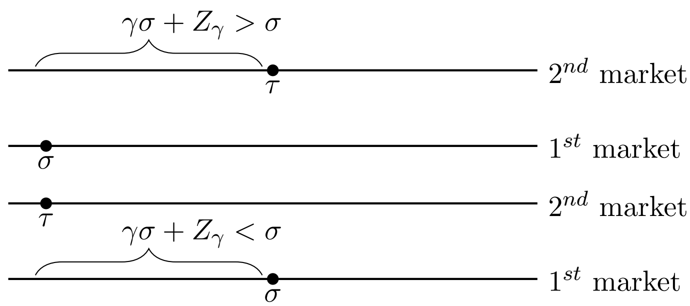

Figure 1: Representation of possible realizations of stochastic delay between two events at σ and τ.

- How to extend a univariate model to a multivariate setting preserving mathematical tractability?
- How to calibrate this model?
- Which techniques can be used for derivative pricing?

Beyond the Gaussian world, some choices have been proposed to model dependence in the context of Lévy processes. Among others, Cont and Tankov [11], Cherubini et al. [10], Panov and Samarin [23], Panov and Sirotkin [24] have discussed the use of Lévy copulas or of Lévy series representations.

In this study, we address the three issues above in the context of multi-dimensional processes using multivariate subordination. To this end, several approaches are available in the literature: for instance, Barndorff-Nielsen et al. [2] have introduced multivariate subordination and have provided general results and applications. In the same spirit, in a series of papers Semeraro [31], Luciano and Semeraro [18], Ballotta and Bonfiglioli [1], Buchmann et al. [4, 5] have proposed models based on subordination to introduce dependence among Lévy processes. The common idea of these papers is to define multivariate processes that are the sum of an independent process and a common one. For example Ballotta and Bonfiglioli [1] define a multivariate process $\mathbf{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ in the following way:

$$\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_n(t))^T = (X_1(t) + a_1 Z(t), \ldots, X_n(t) + a_n Z(t))^T, \tag{1}$$

where $Z = \{Z(t); t \geq 0\}$, $X_j = \{X_j(t); t \geq 0\}$, $j = 1, \ldots, n$ are independent Lévy processes. From a financial standpoint, the common process Z can be viewed as a systemic risk, whereas the independent processes X_j can be considered as the idiosyncratic components.

On the other hand, particularly in illiquid markets, it is not so rare to observe that some news or shocks in a certain market do not have a simultaneous impact on the other ones. One rather observes a sort of "delay in the propagation of the information" across markets namely, a "delay in reacting to the given shock".

In this study we capture this last feature by adding one parameter to the approaches mentioned above and at the same time retain mathematical tractability. Our applications are relative to the energy markets nevertheless this technique can be applied to other contexts for instance, to credit risk.

Following the ideas of Cufaro Petroni and Sabino [13, 14], one can assume that a certain market shock occurs at a random time σ and that has effect on a related market at time τ. Cufaro Petroni and Sabino [14] describe this type of interaction as *synaptic risk* that can also be seen in terms of random delays. With the additional condition that σ and τ follow the same (marginal) distribution with different parameters, a natural way to introduce dependence between these two stochastic times is to set:

$$\tau = \gamma\sigma + Z_\gamma,$$

where $\gamma > 0$, $Z_\gamma \geq 0$ almost surely, and σ and Z_γ are independent. It turns out that the parameter γ is related to the linear correlation between σ and τ and Z_γ plays the role of a delay. Clearly if $\gamma > 1$, we have $\tau > \sigma$ almost surely, and this situation is shown in the top picture of Figure 1. On the other hand, if $0 < \gamma < 1$ it might happen that $\tau < \sigma$ and in this case the interpretation is the following: the *sudden event* first occurs in the latter market at time τ and then we observe its effect on the former market at time $\sigma > \tau$. This latter case is illustrated at the bottom picture of Figure 1. For example, σ can represent the default event of a bank and τ is the default of another related bank or company after a random time Z_γ.

The modeling above is then strictly related to the mathematical concept of self-decomposability (*sd* hereafter), based on which, following the approach proposed by Sabino and Petroni [28], we extend the multivariate Lévy models presented by Semeraro [31], Luciano and Semeraro [18] and Ballotta and Bonfiglioli [1] in order to include a "delay in the propagation of the impact" of the systemic risk component. It is worthwhile mentioning that an alternative approach to obtain multivariate Lévy processes can be found in Buchmann et al. [4], Michaelsen and Szimayer [22], Michaelsen [21], Buchmann et al. [5] and Buchmann et al. [6] relying on the notion of *weak-subordination*.

As far as the second, calibration issue is concerned, we show how general techniques, such as Non-Linear-Least-Square (NLLS) or Generalized Method of Moments (GMM), can be adopted in our framework, implementing a two-step method as the one presented by Ballotta and Bonfiglioli [1].

Moreover, we derive the characteristic functions (*chf*'s) of the log-prices in closed form, therefore we can tackle the third and last issue of the derivative pricing based on the Fourier transform methods introduced in Hurd and Zhou [16], Pellegrino [25] and Caldana and Fusai [8]. Finally, standard path generation schemes can be adapted to our models, allowing a numerical pricing via Monte Carlo simulations.

The article is organized as follow: in Section 2 we give the basic notions that we need in the sequel and we describe an economic interpretation of proposed modeling framework. In Sections 3 we extend the models of Semeraro [31], Luciano and Semeraro [18] and Ballotta and Bonfiglioli [1] using *sd* subordinators, whereas in Sections 4 and 5 we calibrate these new models on power, gas and emissions forward markets and we price spread options. Section 6 concludes the paper with an overview of possible future inquiries. Finally, all proofs are given in Appendix A.

2 Preliminaries

In this section we introduce the fundamental concepts we need in the sequel: *sd* laws and Brownian subordination. We look at *sd* as a simple way to model correlated *rv*'s and we use this notion to build dependent stochastic processes in continuous time. We define *sd*-subordinators and accordingly use the subordination technique to obtain dependent subordinated Brownian Motions (*BM*).

We recall that a law with probability density (*pdf*) $f(x)$ and *chf* $\varphi(u)$ is said to be *sd* (see Sato [30] or Cufaro Petroni [12]) when for every $0 < a < 1$ we can find another law with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that:

$$\varphi(u) = \varphi(au)\chi_a(u). \tag{2}$$

We will accordingly say that a random variable (*rv*) X with *pdf* $f(x)$ and *chf* $\varphi(u)$ is *sd* when its law is *sd*: looking at the definition, this means that, for every $0 < a < 1$, we can always find two *independent* *rv*'s, Y (with the same law of X) and Z_a (here called *a-remainder*), with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that:

$$X \stackrel{d}{=} aY + Z_a. \tag{3}$$

It is easy to see that a plays the role of correlation coefficient between X and Y, therefore, following the idea introduced in Section 1, we build stochastic Lévy processes starting from *sd* *rv*'s. To this end, it is well-known that a *sd* law is also infinitely divisible (*id*) and that for a given $a \in (0,1)$ the law of Z_a is *id* (see Sato [30, Proposition 15.5]), therefore we can construct their associated Lévy process.
A common way to build new Lévy processes is to use non-decreasing Lévy processes, called subordinators, to *time-change* known ones (see for instance Cont and Tankov [11]). If the time-change is done on a *BM* this operation is called Brownian subordination.

Definition 2.1. *Consider a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*,* $\mu \in \mathbb{R}$ *and* $\sigma \in \mathbb{R}^+$*. Let* $W = \{W(t); t \geq 0\}$ *be a* BM *and let* $G = \{G(t); t \geq 0\}$ *be a subordinator. A subordinated* BM $X = \{X(t); t \geq 0\}$ *with drift is defined as:*

$$X(t) = \mu G(t) + \sigma W(G(t)). \tag{4}$$

We define *sd* subordinators as follows.

Definition 2.2 (Self-decomposable subordinators)**.** *Let* $\tilde{H}_1$ *and* $\tilde{H}_2$ *be* $\mathbb{P}$-a.s. *non-negative rv's with sd laws and consider the Lévy processes* $H_i = \{H_i(t); t \geq 0\}$ *and* $Z_a = \{Z_a(t); t \geq 0\}$ *such that* $(H_i(1)) \stackrel{d}{=} \tilde{H}_i, i = 1, 2$ *and* $Z_a(1) \stackrel{d}{=} \tilde{Z}_a$*. A sd subordinator* $\boldsymbol{H} = \{\boldsymbol{H}(t); t \geq 0\}$*,* $\boldsymbol{H}(t) = (H_1(t), H_2(t))$ *is defined as:*

$$H_2(t) = aH_1(t) + Z_a(t). \tag{5}$$

Note that the process H_2 defined in (5) is a Lévy process because it is a linear combination of two independent Lévy processes (Cont and Tankov [11, Theorem 4.1]).

The construction in Equation (5) has a clear financial interpretation. The stochastic time processes H_1, H_2 "run together" and their difference $H_2(t) - H_1(t)$ can be viewed as a random delay or the interaction between the flow of information across the two markets. This mechanism is described by the parameter a and by the term $Z_a(t)$ and of course the difference is not always positive, therefore it can provide a useful tool to capture those market situations where events are correlated, interdependent and do not occur at the same time. Figure 2 illustrates different paths of the process $\boldsymbol{H}$ varying the parameter a; one can observe that if $a \to 1$, the processes H_1 and H_2 are essentially indistinguishable. The former construction can also be extended to the case $n > 2$ as it will be shown in the sequel.

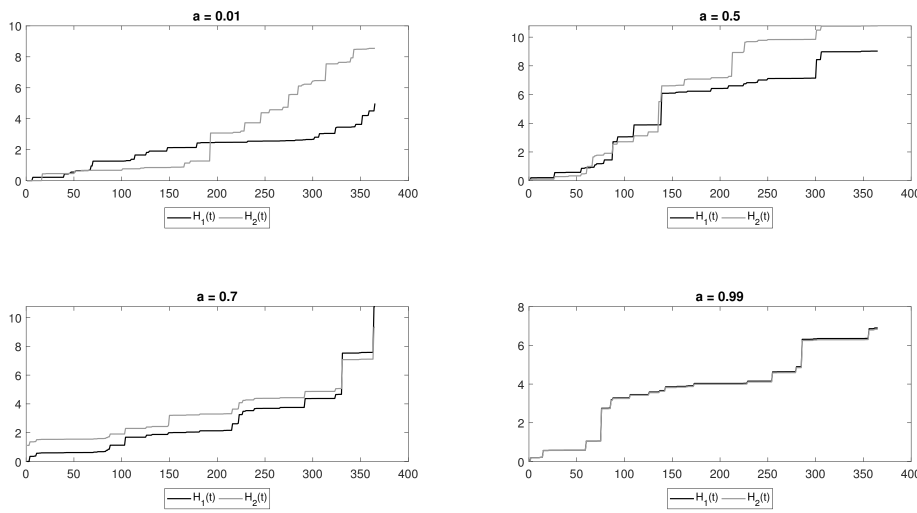

Figure 2: Dependent subordinators H_1 and H_2 with different values of a.

3 Model extensions with Self-Decomposability

In this section we extend the models presented by Semeraro [31], Luciano and Semeraro [18] and Ballotta and Bonfiglioli [1] using the *sd* subordinators introduced in Section 2.

3.1 Extension of the Semeraro's Model

In this section we illustrate the extension of the model proposed by Semeraro [31] using *sd* subordinators.

Definition 3.1 (*sd*-Semeraro Model)**.** *Let* $I_j = \{I_j(t); t \geq 0\}$ $j = 1, 2$ *be independent subordinators, and* H_1, H_2 *be the sd subordinators defined in Equation* (5)*, independent of* I_j*. Define now the subordinator* $G_j = \{G_j(t); t \geq 0\}$ *as:*

$$G_j(t) = I_j(t) + \alpha_j H_j(t), \quad j = 1, 2, \tag{6}$$

with $\alpha_j \in \mathbb{R}^+$*. Consider* $\mu_j \in \mathbb{R}$*,* $\sigma_j \in \mathbb{R}^+$ *and* $W_j = \{W_j(t); t \geq 0\}$*, namely independent BM's. Finally, define the subordinated* BM *with drift* $Y = \{Y_j(t); t \geq 0\}$ *as follows:*

$$Y_j(t) = \mu_j G_j(t) + \sigma_j W_j(G_j(t)), \quad j = 1, 2. \tag{7}$$

Note that the "delay" is defined at the level of subordinators G_j and it is given by the process $\boldsymbol{H}$. Moreover, it is easy to check that, since I_j and H_j are *sd* and independent the subordinator G_j is *sd*.

It is worthwhile noting that the process $\boldsymbol{Y} = \{(Y_1(t), Y_2(t)); t \geq 0\}$ is Lévy because W_1 and W_2 are independent as observed in Barndorff-Nielsen et al. [2] and in Buchmann et al. [7].

The joint *chf* of the process defined in (7) has the following closed form.

Proposition 3.1 (Characteristic Function)**.** *For $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$, the joint chf $\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u})$ of the process $\boldsymbol{Y}$ at time $t \geq 0$ defined in* (7) *is given by:*

$$\begin{aligned} \phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = & \phi_{I_1(t)}\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right)\phi_{I_2(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\phi_{Z_a(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right) \\ & \phi_{H_1(t)}\left(\alpha_1\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right) + a\alpha_2\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\right). \end{aligned} \tag{8}$$

Remark. Taking the limit for $a \to 1$ in (8) we have that:

$$\begin{aligned} \lim_{a\to 1}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = & \phi_{I_1(t)}\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right)\phi_{I_2(t)}\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right) \\ & \phi_{H_1(t)}\left(\alpha_1\left(u_1\mu_1 + i\frac{\sigma_1^2 u_1^2}{2}\right) + \alpha_2\left(u_2\mu_2 + i\frac{\sigma_2^2 u_2^2}{2}\right)\right), \end{aligned}$$

which coincides with the *chf* in Semeraro [31].

Starting from the explicit expression of the *chf* one can easily compute the linear correlation coefficient.

Proposition 3.2 (Correlation)**.** *The correlation at time $t \geq 0$, denoted by $\rho_{Y_1(t),Y_2(t)}$, is given by:*

$$\rho_{Y_1(t),Y_2(t)} = \frac{\mu_1\mu_2\alpha_1\alpha_2 a Var\left[H_1(t)\right]}{\sqrt{Var\left[Y_1(t)\right]Var\left[Y_2(t)\right]}}. \tag{9}$$

We observe that the linear correlation above is lower than the one obtained by Semeraro [31], which is retrieved in the limit $a \to 1$ when H_1 and H_2 are indistinguishable. Indeed, in the original setting the systemic risk component is modeled using a common subordinator while instead we use two processes, H_1, H_2.

3.1.1 2D - Variance-Gamma

So far we analyzed the general model without assuming a particular form for the law of any of the processes involved. Since the gamma law is *sd* it is a suitable candidate for our construction. We recall that a gamma law has *pdf* $f(\alpha, \beta; x)$ and *chf* given by:

$$\begin{aligned} f(\alpha, \beta; x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x>0}(x), \\ \phi_X(u) &= \left(1 - \frac{iu}{\beta}\right)^{-\alpha}, \end{aligned} \tag{10}$$

with $\alpha, \beta \in \mathbb{R}^+$. It is well-known that if $X \sim \Gamma(\alpha, \beta)$ and $c > 0$, then $cX \sim \Gamma\left(\alpha, \frac{\beta}{c}\right)$ and if $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$ are independent, then $X + Y \sim \Gamma(\alpha_1 + \alpha_2, \beta)$. Now taking in (6)

$$I_j \sim \Gamma\left(A_j, \frac{B}{\alpha_j}\right), \quad H_j \sim \Gamma(A, B), \quad j = 1, 2$$

and noting that $\alpha_j H_j \sim \Gamma\left(A, \frac{B}{\alpha_j}\right)$ we have

$$G_j \sim \Gamma\left(A_j + A, \frac{B}{\alpha_j}\right), \quad j = 1, 2.$$

It is custom to consider subordinators such that $\mathbb{E}[G_j] = 1$ and $\mathbb{E}[G_j(t)] = t$, therefore for $A_j, A, B, \alpha_j \in \mathbb{R}^+$ it results

$$\frac{1}{A_j + A} = \frac{\alpha_j}{B}, \quad j = 1, 2, \tag{11}$$

$$0 < \alpha_j \leq \frac{B}{A}, \quad j = 1, 2. \tag{12}$$

Remark. Under the condition (11), we have that:

$$1 = \alpha_1 \frac{(A_1 + A)}{B} = \alpha_2 \frac{(A_2 + A)}{B},$$

hence the parameter B is somehow redundant so that we can assume $B = 1$.

The following corollaries are a direct application of Propositions 3.1 and 3.2.

Corollary 3.3. *The chf of the 2D Variance-Gamma case* $\phi_{\mathbf{Y}(t)}(\boldsymbol{u})$ *in* (8) *can be computed by taking*

$$\phi_{H_j(t)}(u) = \left(1 - i\frac{u}{B}\right)^{-tA}, \quad j = 1, 2,$$

$$\phi_{I_j(t)}(u) = \left(1 - \alpha_j i\frac{u}{B}\right)^{-tA_j}, \quad j = 1, 2,$$

$$\phi_{Z_a(t)}(u) = \frac{\phi_{H_1(t)}(u)}{\phi_{H_1(t)}(au)} = \left(\frac{B - iu}{B - iau}\right)^{-tA}. \tag{13}$$

Corollary 3.4. *The linear correlation coefficient at time* $t \geq 0$ *of the 2D Variance-Gamma case is given by:*

$$\rho_{Y_1(t), Y_2(t)} = \frac{\mu_1 \mu_2 \alpha_1 \alpha_2 a A}{\sqrt{\sigma_1^2 + \mu_1^2 \alpha_1} \sqrt{\sigma_2^2 + \mu_2^2 \alpha_2}}.$$

3.2 Extension of the Semeraro-Luciano's Model

A typical application of the dynamics illustrated so far, is to model log-returns. To this end, the model presented in Luciano and Semeraro [18] is a modification of the approach in Semeraro [31] motivated by that fact that it has been observed that the latter approach does not reproduce certain levels of correlation (see Wallmeier and Diethelm [32]).

Following the same route of Section 3.1, in the sequel we illustrate the extension of the model of Luciano and Semeraro [18] with *sd*-subordinators.

Definition 3.2 (*sd*-Luciano and Semeraro's model)**.** *Let $I_j = \{I_j(t); t \geq 0\}$, $j = 1, 2$, be subordinators and let $H_1 = \{H_1(t); t \geq 0\}$ and $H_2 = \{H_2(t); t \geq 0\}$ be two sd subordinators independent of I_j. Define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ as follows:*

$$\boldsymbol{Y}(t) = \left(\begin{array}{l} \mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) + \alpha_1 \mu_1 H_1(t) + \sqrt{\alpha_1}\sigma_1 W_1^{\rho}(H_1(t)) \\ \mu_2 I_2(t) + \sigma_2 W_2(I_2(t)) + \alpha_2 \mu_2 H_2(t) + \sqrt{\alpha_2}\sigma_2 \left(W_2^{\rho}(aH_1(t)) + \tilde{W}(Z_a(t)) \right) \end{array} \right), \tag{14}$$

where $\mu_j \in \mathbb{R}$, $\sigma_i \in \mathbb{R}^+$ and $\alpha_j \in \mathbb{R}^+$. $\boldsymbol{W} = \{(W_1(t), W_2(t)); t \geq 0\}$ is a standard BM *with independent components, $\boldsymbol{W}^{\rho} = \{(W_1^{\rho}(t), W_2^{\rho}(t)); t \geq 0\}$ is a standard* BM *such that:*

$$\mathbb{E}\left[dW_1^{\rho}(t)\, dW_2^{\rho}(t)\right] = \rho dt$$

and $\tilde{W} = \left\{\tilde{W}(t); t \geq 0\right\}$ is independent of$\boldsymbol{W}$ and $\boldsymbol{W}^{\rho}$.

Unlike the process derived in the previous section, it can be shown that, $\boldsymbol{Y}$ is not a Lévy process because the covariance matrix of increments $cov\left(\boldsymbol{Y}(t-s), \boldsymbol{Y}(s)\right)$ is not zero $\forall s < t$, even though its margins clearly are Lévy. Moreover, it can be proven that the process $\boldsymbol{Y}$ is not even a Markov process.

Following the same route of the last section we find the closed forms of the *chf* and the correlation coefficient.

Proposition 3.5 (Characteristic Function)**.** *The joint chf $\phi_{\boldsymbol{Y}^{\rho}(t)}(\boldsymbol{u})$ of the process $\boldsymbol{Y}$ at time $t \geq 0$ defined in* (14) *is given by:*

$$\begin{aligned} \phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = & \phi_{I_1(t)}\left(u_1\mu_1 + \frac{i}{2}\sigma_1^2 u_1^2\right) \phi_{I_2(t)}\left(u_2\mu_2 + \frac{i}{2}\sigma_2^2 u_2^2\right) \\ & \phi_{H_1(t)}\left(\frac{i}{2}u_1^2\alpha_1\sigma_1^2(1-a) + \boldsymbol{u}^T\boldsymbol{\mu} + \frac{i}{2}\boldsymbol{u}^T a\Sigma\boldsymbol{u}\right) \phi_{Z_a(t)}\left(u_2\mu_2\alpha_2 + \frac{i}{2}u_2^2\alpha_2\sigma_2^2\right), \end{aligned}$$

where $\boldsymbol{\mu} = [\alpha_1\mu_1, a\alpha_2\mu_2]$ and

$$\Sigma = \begin{bmatrix} \alpha_1\sigma_1^2 & \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho \\ \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho & \alpha_2\sigma_2^2 \end{bmatrix}.$$

Proposition 3.6 (Correlation)**.** *The correlation at time $t \geq 0$, $\rho_{Y_1(t),Y_2(t)}$ is given by:*

$$\rho_{Y_1(t),Y_2(t)} = \frac{a\left(\mu_1\mu_2\alpha_1\alpha_2 Var\left[H_1(t)\right] + \rho\sigma_1\sigma_2\sqrt{\alpha_1\alpha_2}\mathbb{E}\left[H_1(t)\right]\right)}{\sqrt{Var\left[Y_1(t)\right] Var\left[Y_2(t)\right]}}. \tag{15}$$

3.2.1 2D - Variance-Gamma

It is possible to build a 2D-Variance Gamma process by taking:

$$I_j \sim \Gamma\left(A_j, \frac{B}{\alpha_j}\right),\ H_j \sim \Gamma(A, B),\ j = 1, 2,$$

therefore

$$I_j + \alpha_j H_j \sim \Gamma\left(A_j + A, \frac{B}{\alpha_j}\right), \ j = 1, 2.$$

Formulas for the linear correlation coefficient and the *chf* for the 2D Variance Gamma case can be derived following the same arguments of Section 3.1.1.

Corollary 3.7. *The linear correlation coefficient at time $t \geq 0$ in 2D Variance-Gamma case is given by:*

$$\rho_{Y_1(t),Y_2(t)} = \frac{a\left(\mu_1\mu_2\alpha_1\alpha_2 A + \rho A\sigma_1\sigma_2\sqrt{\alpha_1\alpha_2}\right)}{\sqrt{\sigma_1^2 + \mu_1^2\alpha_1}\sqrt{\sigma_2^2 + \mu_2^2\alpha_2}}.$$

The *chf* can be obtained combining Corollary 3.3 with Proposition 3.5.

3.3 Extension of the Ballotta-Bonfiglioli's Model

The construction of dependent Lévy processes described by Ballotta and Bonfiglioli [1] is slightly different from what we have seen so far because the dependence is not introduced at the level of the subordinators. The dependent processes are build via linear transformation of subordinated *BM*'s of the same type plus a common factor (see Section 1). Some convolution conditions on parameters guarantee that the resulting processes are of the same type of those of the linear combination. In the section we outline how to extend this model with *sd* subordinators.

Definition 3.3 (*sd*-Ballotta and Bonfiglioli's model)**.** *Consider a sd subordinator $\boldsymbol{H} = \{(H_1(t), H_2(t)); t \geq 0\}$ and define the process $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ as:*

$$\boldsymbol{Y}(t) = (Y_1(t), Y_2(t)) = (X_1(t) + a_1 R_1(t), X_2(t) + a_2 R_2(t)), \tag{16}$$

where $a_1, a_2 \in \mathbb{R}$, $X_j = \{X_j(t); t \geq 0\}$ is a subordinated BM *with parameters $(\beta_j, \gamma_j, \nu_j)$, $j = 1, 2$, $\beta_j \in \mathbb{R}$ is the drift, $\gamma_j \in \mathbb{R}^+$ is the diffusion and $\nu_j \in \mathbb{R}^+$ is the variance of the subordinator at time* 1*. In addition, $G_j = \{G_j(t); t \geq 0\}$ represents the subordinator of X_j, with G_1 and G_2 independent, such that*

$$X_j(t) = \beta_j G_j(t) + W_j(G_j(t)), \quad j = 1, 2.$$

Finally, $R_1 = \{R_1(t); t \geq 0\}$ and $R_2 = \{R_2(t); t \geq 0\}$ are given by

$$\begin{aligned} R_1(t) &= \beta_{R_1} H_1(t) + \gamma_{R_1} W(H_1(t)), \\ R_2(t) &= \beta_{R_2} H_2(t) + \gamma_{R_2}\left(W(aH_1(t)) + \tilde{W}(Z_a(t))\right), \end{aligned} \tag{17}$$

where $W = \{W(t); t \geq 0\}$ and $\tilde{W} = \left\{\tilde{W}(t); t \geq 0\right\}$ are independent BM*'s and $\beta_{R_j} \in \mathbb{R}$ and $\gamma_{R_j} \in \mathbb{R}^+$.*

In order to derive the *chf* of the process, we first prove the following lemma.

Lemma 3.8. *The chf of the process defined in* (17) *at time* $t \geq 0$ *is given by:*

$$\begin{aligned}\phi_{\boldsymbol{R}(t)}\left(\boldsymbol{u}\right) =& \phi_{H_1(t)}\left(u_1\beta_{R_1} + u_2\beta_{R_2}a + \frac{i}{2}\left(u_1^2\gamma_{R_1}^2 + 2u_1u_2\gamma_{R_1}\gamma_{R_2}a + u_2^2 a\gamma_{R_2}^2\right)\right)\\ & \phi_{Z_a(t)}\left(u_2\beta_{R_2} + \frac{i}{2}u_2^2\gamma_{R_2}^2\right),\end{aligned}$$

where $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$*.*

Proposition 3.9 (Characteristic Function)**.** *The chf of* $\boldsymbol{Y}(t)$ *defined in* (16) *is given by:*

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}\left(\boldsymbol{u}\right) =& \phi_{G_1(t)}\left(\beta_1 u_1 + \frac{i}{2}u_1^2\gamma_1^2\right)\\ & \phi_{G_2(t)}\left(\beta_2 u_2 + \frac{i}{2}u_2^2\gamma_2^2\right)\\ & \phi_{\boldsymbol{R}(t)}\left(\boldsymbol{a}\circ\boldsymbol{u}\right),\end{aligned} \tag{18}$$

where $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$*,* $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ *and* $\circ$ *denotes the Hadamard product.*

Remark. It is easy to verify that:

$$\begin{aligned}\lim_{\substack{a\to 1\\ \beta_{R_1},\beta_{R_2}\to\beta_Z\\ \gamma_{R_1},\gamma_{R_2}\to\gamma_Z}} \phi_{\boldsymbol{Y}(t)}\left(u_1, u_2\right) =& \phi_{G_1(t)}\left(\beta_1 u_1 + \frac{i}{2}u_1^2\gamma_1^2\right)\\ & \phi_{G_2(t)}\left(\beta_2 u_2 + \frac{i}{2}u_2^2\gamma_2^2\right)\phi_{Z(t)}\left(\beta_Z\left(a_1u_1 + a_2u_2\right) + \frac{i}{2}\left(a_1u_1 + a_2u_2\right)^2\gamma_Z^2\right),\end{aligned}$$

which is the *chf* of the original model by Ballotta and Bonfiglioli [1]. It is worthwhile noting that the process $\boldsymbol{R}$ is not Lévy because its increments are not independent, therefore neither $\boldsymbol{Y}$ is a Lévy process. Moreover, it can be proven that the process $\boldsymbol{Y}$ it is not even a Markov process.

Finally, the correlation coefficient is given by the following proposition.

Proposition 3.10. *The correlation coefficient at time* $t \geq 0$ *of the process* $\boldsymbol{Y}$ *defined in* (16) *is given by:*

$$\rho_{Y_1(t),Y_2(t)} = \frac{a_1a_2a\left(\beta_{R_1}\beta_{R_2}Var\left[H_1(t)\right] + \gamma_{R_1}\gamma_{R_2}\mathbb{E}\left[H_1(t)\right]\right)}{\sqrt{Var\left[Y_1(t)\right]}\sqrt{Var\left[Y_2(t)\right]}}. \tag{19}$$

3.3.1 Convolution Conditions

It is possible to verify that, if $X_j = \{X_j(t); t \geq 0\}$ and $R_j = \{R_j(t); t \geq 0\}$, $j = 1, 2$, are subordinated *BM*'s with subordinators from the same family, then $Y_j = \{Y_j(t); t \geq 0\}$ is a subordinated process of the same type of X_j and R_j if the following convolution conditions hold (see Ballotta and Bonfiglioli [1]):

$$\nu_R := \nu_{R_1} = \nu_{R_2}, \tag{20}$$

$$\begin{cases} \alpha_j \mu_j = \nu_R a_j \beta_{R_j} & j = 1, 2, \\ \alpha_j \sigma_j^2 = \nu_R a_j^2 \gamma_{R_j}^2 & j = 1, 2 \end{cases} \tag{21}$$

then

$$\mu_j = \beta_j + a_j \beta_{R_j}, \quad \sigma_j^2 = \gamma_j^2 + a_j^2 \gamma_{R_j}^2, \quad \alpha_j = \nu_j \nu_R / (\nu_j + \nu_R).$$

3.3.2 2D - Variance-Gamma

Based on the results of the previous section we can construct a 2D - Variance-Gamma using gamma subordinators as follows.

- Take a gamma subordinator $H_1 = \{H_1(t); t \geq 0\}$ such that $H_1(t) \sim \Gamma\left(\frac{t}{\nu_R}, \frac{1}{\nu_R}\right)$ and set $H_2(t) = aH_1(t) + Z_a(t)$.
- Consider $R_j = \{R_j(t); t \geq 0\}$, $j = 1, 2$ as the subordinated BM (with drift β_{R_j} and diffusion γ_{R_j}) obtained using the gamma subordinator H_j just defined.
- Let $X_j = \{X_j(t); t \geq 0\}$ be a subordinated BM (with drift β_j and diffusion γ_j) obtained using a gamma subordinator $G_j = \{G_j(t); t \geq 0\}$ such that $G(t) \sim \Gamma\left(\frac{t}{\nu_j}, \frac{1}{\nu_j}\right)$ for $j = 1, 2$.
- Set $Y_j(t) = X_j(t) + a_j R_j(t)$.

It is straightforward to verify that the process $Y(t)$ is a 2D process with Variance Gamma marginals with parameters $(\mu_j, \sigma_j, \alpha_j)$, $j = 1, 2$, that fulfill the requirements (21).

The joint *chf* $\phi_{\boldsymbol{Y}(t)}(u_1, u_2)$ can be obtained plugging the *chf* of the gamma law in (18) in addition, applying Proposition 3.10, the correlation coefficient results

$$\rho_{Y_1(t), Y_2(t)} = \frac{a_1 a_2 a \left(\beta_{R_1} \beta_{R_2} \nu_R + \gamma_{R_1} \gamma_{R_2}\right)}{\sqrt{\sigma_1^2 + \mu_1^2 \alpha_1} \sqrt{\sigma_2^2 + \mu_2^2 \alpha_2}}.$$

4 Financial Applications

In this section we show concrete applications of the models presented in Section 3 to energy markets. Similarly to Cont and Tankov [11] we model the dynamics of energy forward contracts with exponential processes, whose components are Lévy processes, based on dynamics of the type of $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ derived in Section 3[1]. The evolution of the forward price $F_j(t)$, $j = 1, 2$ at time t is defined as

$$F_j(t) = F_j(0) e^{\omega_j t + Y_j(t)}, \tag{22}$$

non-arbitrage conditions can be obtained setting

$$\omega_j = -\varphi_j(-i), \tag{23}$$

[1]Note that even if $\boldsymbol{Y}$ may not be a Lévy process, its components are Lévy process. For this reason, even if $\boldsymbol{F} = \{(F_1(t), F_2(t)); t \geq 0\}$ is not an exponential Lévy process, the dynamics of the forward price $F_j(t)$, $j = 1, 2$ is an exponential Lévy process.

where $\varphi_j(u)$ denotes the characteristic exponent of the process Y_j.

We adopt the same two-steps calibration procedure of Luciano and Semeraro [18]; to this end, it is worthwhile noticing that the marginal distributions do not depend on the parameters required to model the dependence structure. The vector of the marginal parameters $\boldsymbol{\theta}^*$ is obtained solving the following optimization problem

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \left(C_i^{\boldsymbol{\theta}}(K,T) - C_i \right)^2, \tag{24}$$

where $C_i, i = 1, \ldots, n$ are the values of n quoted vanilla products and $C_i^{\boldsymbol{\theta}}(K,T), i = 1, \ldots, n$ are the relative model prices. Once we have fitted $\boldsymbol{\theta}^*$ we have to calibrate the remaining parameters for the dependence structure. Generally derivatives written on multiple underlying assets are not very liquid and market quotes are rarely available, therefore the vector $\boldsymbol{\eta}^*$ is estimated fitting the correlation matrix on historical data. The expression of the theoretical correlation matrix has been derived in Section 3.

For the first step we have combined the NLLS approach with the FFT method proposed by Carr and Madan [9] (the version proposed by Lewis [17] returns similar results), whereas for the second one we have used the plain NLLS method for the minimization of the distance between the theoretical and the observed correlation coefficient.

As far as the path generation of the skeleton of the 2D-Variance Gamma processes is concerned, the only non-standard step is the simulation of the process $Z_a = \{Z_a(t); t \geq 0\}$. On the other hand, Sabino and Petroni [28, 29] have shown that the a-remainder Z_a of a gamma distribution $\Gamma(\alpha, \lambda)$ can be exactly simulated knowing that

$$Z_a = \sum_{j=1}^{S} X_j,$$

where

$$S \sim \mathfrak{B}(\alpha, 1-a) \quad X_j \sim \mathfrak{E}(\lambda/a) \quad X_0 = 0 \quad \mathbb{P}-a.s..$$

Here $\mathfrak{B}(\alpha, p)$ denotes a Polya distribution with parameters α and p and $\mathfrak{E}(\lambda)$ denotes an exponential distribution with rate parameter λ.

Since we have derived the *chf*'s of the log-process in closed form, in alternative to the MC schemes we can adopt Fourier methods. Different Fourier based techniques are available for the option pricing in a multivariate setting (see for example Hurd and Zhou [16], Pellegrino [25] and Caldana and Fusai [8]). In this section we use the method proposed by Caldana and Fusai [8] which gives a good approximation for spread-options prices and has the advantage to require only one Fourier inversion.

The remaining part of the section is split into two parts: in the first one we apply our models to the German and French power forward markets and in the second part we focus on the German power forward market and to the TTF natural gas forward market. In the first case we have selected markets that are strongly positively correlated due to the configuration of European electricity network, whereas in the second case, the correlation between markets is still positive, because natural gas can be used to produce electricity,

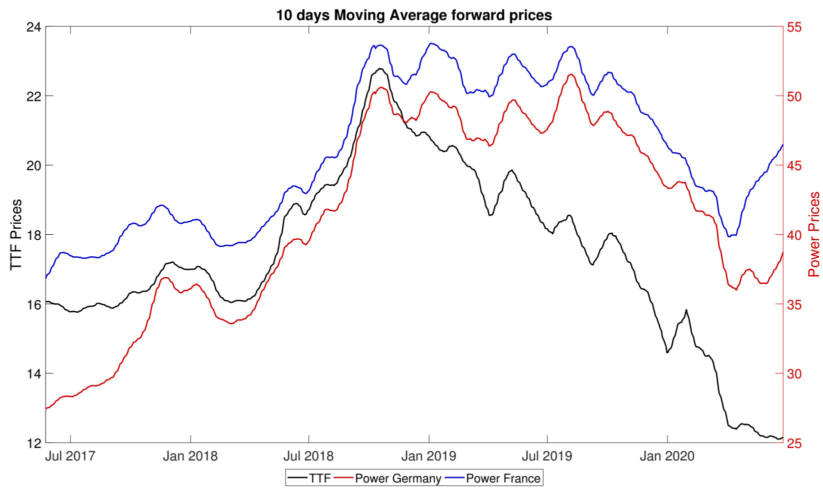

Figure 3: German, French and natural gas TTF forward market.

but it is not as high as in the first one. This gives us the opportunity to test our models for different levels of correlations.
Moreover, as shown in Figure 3, due to the particular structure of European electricity grid, power markets are very interconnected and usually react “in the same way at the same time”, whereas the reaction of the power market to a shock in the natural gas market (and vice versa) is more likely to occur with a stochastic delay. Based on these observations, we expect that the value of the parameter a will be very close to one in the first example and will have a lower value in the second one.

For the sake of concision we use the following notation:

- (SSD): sd-Semeraro's model presented in Section 3.1.
- ($LSSD$): sd-Luciano and Semeraro's model presented in Section 3.2.
- ($BBSD$): sd-Ballotta and Bonfiglioli's model presented in Section 3.3.

In our experiments we price spread options on future prices, denoted $F_i(t), i = 1, 2$, whose payoff is given by:

$$\Phi_T = (F_1(T) - F_2(T) - K)^+ .$$

It is customary to reserve the name *cross-border* or *spark-spread* option if the futures are relative to power or gas markets, respectively. In all our experiments we use the MC technique with $N_{sim} = 10^6$ simulations and the Fourier-based method proposed by Caldana and Fusai [8].

4.1 Application to the German and French Power Markets

In order to calibrate our models we need the quoted prices of the derivatives contracts written on each of the two forward products and their joint historical time series. The data-set[2] we have relied upon is composed as follows:

- Forward quotations from 25 April 2017 to 12 November 2018 of calendar 2019 power forward. A forward calendar 2019 contract is a contract between two counterparts to buy or sell a specific volume of energy in MWh at fixed price for all the hours of 2019. calendar power forward in German and France are stated respectively with DEBY and F7BY.

- Call options on power forward 2019 quotations for both countries with settlement date 12 November 2018. We used strikes in a range of $\pm 10\,[EUR/MWh]$ around the settlement price of the forward contract, i.e. we exclude deep ITM and OTM options.

- We assume a risk-free rate $r = 0.015$.

- The estimated historical correlation between markets is $\rho_{mkt} = 0.94$.

From Table 1 we see that all models provide the same set of marginal parameters. In the lower box of Figure 4 we report the percentage error ϵ_i defined as:

$$\epsilon_i = \frac{C_i^{\theta}\left(K, T\right) - C_i}{C_i}.$$

We can observe that this error is very small and our models are able to replicate market prices for different values of the strike price K.

If we look at the fitted correlation the situation is slightly different. The *SSD* model presented in Section 3.1 fits a correlation that is roughly zero, therefore it is not recommendable for *cross-border* option pricing because it overestimates the derivative price as shown in the upper picture in Figure 4. In contrast, the correlation estimated selecting the *LSSD* is very close to the one observed in the market, as we can see from Table 3; for this reason the *LSSD* model is appropriate to price *cross-border* options. Moreover, the *BBSD* model derived in Section 3.3 provides an even better fit of market correlation and therefore we conclude that the *BBSD* model is the best one for the valuation of *cross-border* options. An additional comparison among the models is illustrated in the upper part of Figure 4: the option prices returned by the *BBSD* model are the lowest ones due to the highest value of fitted correlation.

Finally, we remark that, as we expected, the fitted value for the *sd* parameters a is very close to one for all the three settings. This fact does not come as a surprise because the German and French forward markets are so strictly interconnected that whenever an event occurs in a market it has an immediate impact on the other one. As already mentioned, if $a \to 1$ we obtain the original models of Semeraro [31], Luciano and Semeraro [18] and Ballotta and Bonfiglioli [1]. For this reason, for *cross-order* options, there is not an essential difference between original models and those presented in Section 3.

4.2 Application to the German Power and the TTF Gas Markets

In this section we consider the German power forward market and the TTF gas forward market. These two markets are not as positively correlated as two purely power markets.

[2] Data Source: www.eex.com.

Model	μ_1	μ_2	σ_1	σ_2	α_1	α_2
SSD	0.40	0.61	0.31	0.32	0.02	0.02
LSSD	0.40	0.61	0.31	0.32	0.02	0.02
BBSD	0.40	0.61	0.31	0.32	0.02	0.02

Table 1: Fitted marginal parameters for the German and French power markets.

Parameter	Value
A	41.89
B	1.00
a	0.99
ρ_{mod}	0.05

Table 2: *SSD*

Parameter	Value
A	42.31
B	1.00
ρ	1.00
a	0.99
ρ_{mod}	0.92

Table 3: *LSSD*

Parameter	Value	Parameter	Value
β_1	-0.00	β_{R_2}	0.85
β_2	0.09	γ_{R_1}	0.50
γ_1	0.00	γ_{R_2}	0.47
γ_2	0.10	ν_R	0.02
ν_1	1.01	a	0.99
ν_2	0.14	ρ_{mod}	0.94
β_{R_1}	0.62		

Table 4: *BBSD*

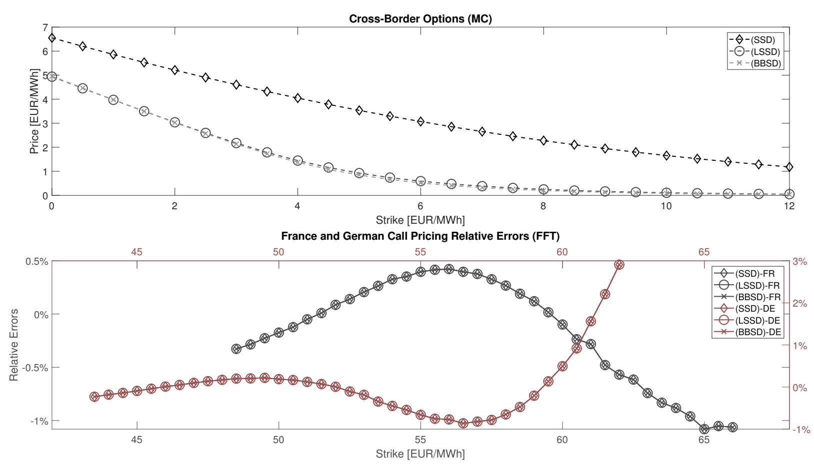

Figure 4: Percentage errors and cross-border option prices.

The data-set[3] we relied upon consists of:

- Forward quotations from 1 July 2019 to 09 September 2019 relative to January 2020 for the power forward in Germany and the gas TTF forward.
- Call options relative to January 2020 for both power and gas with settlement date 9 September 2019. As done before, we use strikes prices K in a range of $\pm 10\,[EUR/MWh]$ around the settlement price of the forward contract, i.e. we exclude deep ITM and OTM options.
- We assume a risk-free rate $r = 0.015$.
- The estimated historical correlation between log-returns is $\rho_{mkt} = 0.54$.

[3]Data Source: www.eex.com and www.theice.com

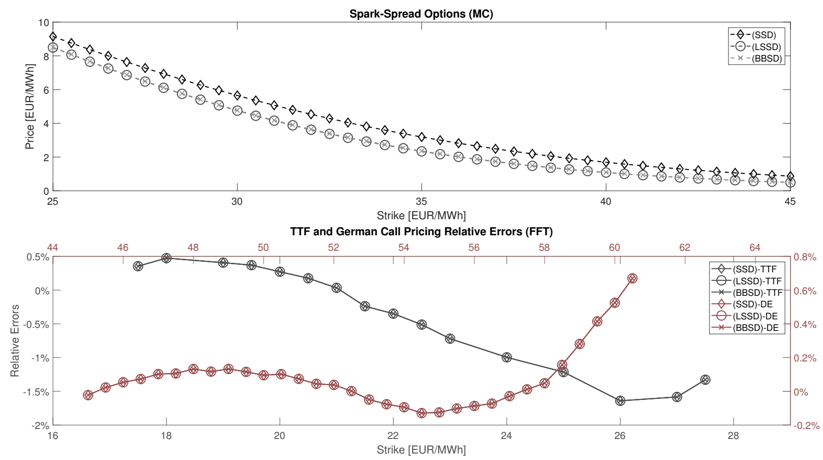

Figure 5: Percentage errors and spark-spread option prices.

In the picture at the bottom of Figure 5 we can see that all the models provide a good fit of the quoted market options because the error ϵ is very small. In Figure 5 the picture at the top shows that the *SSD* model overprices the *spark-spread* option due to the fact that the estimated correlation is close to zero. In contrast, both models *LSSD* and *BBSD* are able to capture the market correlation and return a lower spread option price. The fitted parameters are shown in Table 5 and we observe that the *sd* parameter a is not as close to one as it was in the previous example.

This result can be explained by the fact that approximately 25% of electricity in Germany is produced using natural gas, hence a certain downward or upward change of the natural gas price will not immediately affect the power prices. Moreover, in contrast to electricity, natural gas can be stored and players of gas markets can subscribe swing contracts to protect against natural gas price movements. Of course, if the natural gas price shock persists for some time it will impact electricity prices as well.

Model	μ_1	μ_2	σ_1	σ_2	α_1	α_2
SSD	0.46	0.24	0.43	0.33	0.08	0.05
LSSD	0.46	0.24	0.43	0.33	0.08	0.05
BBSD	0.46	0.24	0.43	0.33	0.08	0.05

Table 5: Fitted marginal parameters for the power and gas forward markets.

Parameter	Value
A	12.36
B	1.00
a	0.99
ρ_{mod}	0.04

Table 6: *SSD*

Parameter	Value
A	9.89
B	1.00
ρ	0.89
a	0.90
ρ_{mod}	0.57

Table 7: *LSSD*

Parameter	Value	Parameter	Value
β_1	0.13	β_{R_2}	0.29
β_2	0.12	γ_{R_1}	0.47
γ_1	0.23	γ_{R_2}	0.29
γ_2	0.23	ν_R	0.11
ν_1	0.28	a	0.90
ν_2	0.12	ρ_{mod}	0.54
β_{R_1}	0.47		

Table 8: *BBSD*

5 Modeling, calibration and pricing for $n \geq 3$

So far, we have considered only the bivariate case. In this section we outline how to extend our approach when the number of commodities is larger than two and we provide a numerical experiment for $n = 3$. From a mathematical point of view, the procedure consists in defining the process $\boldsymbol{H} = \{\boldsymbol{H}(t); t \geq 0\}$, where now $\boldsymbol{H}(t) = (H_1(t), H_2(t), \ldots, , H_n(t))$, as follows:

$$\begin{aligned}
&H_1(t), \\
&H_2(t) = a_1 H_1(t) + Z_{a_1}(t), \\
&\ldots \\
&H_n(t) = a_{n-1} H_1(t) + Z_{a_{n-1}}(t),
\end{aligned} \tag{25}$$

$a_1, \ldots, a_{n-1} \in (0, 1)$ and $Z_{a_1}(t) \ldots, Z_{a_n}(t)$ and $H_1(t)$ are independent, therefore we need $n-1$ parameters $a_j,\ j = 1, \ldots, n-1$. Based on the definition above, the extension of the models presented in Section 3 is straightforward and it is briefly discussed in the following subsections.

5.1 *sd*-Semeraro (SSD)

Consider a n-dimensional subordinator $\boldsymbol{I} = \{(I_1, \ldots, I_n); t \geq 0\}$ with independent components, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ and $\boldsymbol{H}$ defined in Equation (25). The multivariate subordinator $\boldsymbol{G} = \{\boldsymbol{G}(t); t \geq 0\}$ is now defined as

$$G_j(t) = I_j(t) + \alpha_j H_j(t), \quad j = 1, \ldots, n.$$

Taking a n-dimensional *BM* $\boldsymbol{W} = \{\boldsymbol{W}(t); t \geq 0\}$ with independent components we then define the n-dimensional process $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ as

$$Y_j(t) = \mu_j G_j(t) + \sigma_j W(G_j(t)), \quad j = 1, \ldots, n,$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}_+^n$. $\boldsymbol{I}$, $\boldsymbol{H}$ and $\boldsymbol{W}$ are mutually independent.

5.2 *sd*-Luciano-Semeraro (LSSD)

As far as the extension to a n-dimensional *sd*-Semeraro-Luciano model is concerned, we consider a multivariate subordinator $\boldsymbol{I} = \{(I_1, \ldots, I_n); t \geq 0\}$ with independent components and the process $\boldsymbol{H}$ defined in Equation (25). Then, we define $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ as:

$$\boldsymbol{Y}(t) = \begin{pmatrix}
\mu_1 I_1(t) + \sigma_1 W_1(I_1(t)) + \alpha_1 \mu_1 H_1(t) + \sqrt{\alpha_1}\sigma_1 W_1^{\rho}(H_1(t)) \\
\ldots \\
\mu_j I_j(t) + \sigma_j W_j(I_j(t)) + \alpha_j \mu_j H_j(t) + \sqrt{\alpha_j}\sigma_j \left(W_j^{\rho}(a_{j-1} H_1(t)) + \tilde{W}_1\left(Z_{a_{j-1}}(t)\right)\right) \\
\ldots \\
\mu_n I_n(t) + \sigma_n W_n(I_n(t)) + \alpha_n \mu_n H_n(t) + \sqrt{\alpha_n}\sigma_n \left(W_n^{\rho}(a_{n-1} H_1(t)) + \tilde{W}_{n-1}\left(Z_{a_{n-1}}(t)\right)\right)
\end{pmatrix}, \tag{26}$$

where $a_j \in (0,1)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n_+$; $\boldsymbol{W} = \{\boldsymbol{W}(t); t \geq 0\}$ and $\tilde{\boldsymbol{W}} = \left\{\tilde{\boldsymbol{W}}(t); t \geq 0\right\}$ are *BM*'s with independent components, $\boldsymbol{W}^\rho = \{\boldsymbol{W}^\rho(t); t \geq 0\}$ is a *BM* with correlated components, moreover $\boldsymbol{I}$, $\boldsymbol{H}$, $\boldsymbol{W}, \boldsymbol{W}^\rho$ and $\tilde{\boldsymbol{W}}$ are mutually independent.

5.3 *sd*-Ballotta-Bonfiglioli (BBSD)

Following the same route, we can extend the *BBSD* to the case in which the market present $n \geq 3$ risky assets. Consider the process $\boldsymbol{H} = \{\boldsymbol{H}(t); t \geq 0\}$ defined in Equation (25) and take $\boldsymbol{R} = \{\boldsymbol{R}(t); t \geq 0\}$ defined as:

$$\boldsymbol{R}(t) = \left(\begin{array}{l} R_1(t) = \beta_{R_1} H_1(t) + \gamma_{R_1} W(H_1(t)) \\ \cdots \\ R_j(t) = \beta_{R_j} H_2(t) + \gamma_{R_j} W(a_j H_1(t)) + \tilde{W}_j\left(Z_{a_j}(t)\right) \\ \cdots \\ R_n(t) = \beta_{R_n} H_n(t) + \gamma_{R_n} W(a_n H_1(t)) + \tilde{W}_n\left(Z_{a_{n-1}}(t)\right) \end{array} \right),$$

where $a_j \in (0,1)$, $j = 1, \ldots, n-1$, $\boldsymbol{\beta_R} = (\beta_{R_1}, \ldots, \beta_{R_n}) \in \mathbb{R}^n$, $\boldsymbol{\gamma_R} = (\gamma_{R_1}, \ldots, \gamma_{R_n}) \in \mathbb{R}^n_+$. In addition, take two independent processes, a one-dimensional BM $W = \{W(t); t \geq 0\}$ and an independent multivariate *BM* $\tilde{\boldsymbol{W}} = \left\{\tilde{\boldsymbol{W}}(t); t \geq 0\right\}$ with independent components. Consider now a n-dimensional subordinator $\boldsymbol{G} = \{\boldsymbol{G}(t); t \geq 0\}$ with independent components, a standard *BM* $\boldsymbol{W} = \{\boldsymbol{W}(t); t \geq 0\}$ with independent components, and independent of $\boldsymbol{G}$, and define the process $\boldsymbol{X} = \{\boldsymbol{X}(t); t \geq 0\}$, the subordinated n-dimensional *BM* with drift $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n_+$, by:

$$X_j(t) = \beta_j G_j(t) + \sigma_j W_j(G_j(t)) \quad j = 1, \ldots, n.$$

Finally, we define the general n-variate process $\boldsymbol{Y} = \{\boldsymbol{Y}(t); t \geq 0\}$ as follows:

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{\alpha} \cdot \boldsymbol{R}, \quad Y_j(t) = X_j(t) + \alpha_j R_j(t), \quad j = 1, \ldots, n,$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$.

5.4 Calibration

The two-steps calibration procedure adopted in the financial applications of Section 4 can be easily adapted for $n \geq 3$; of course, the fit of the marginal distributions is independent of the number of underlying assets or commodities.

On the other hand, the estimation of correlation matrix can be accomplished knowing the theoretical expression of correlation between $Y_j(t)$ and $Y_i(t)$ for all $i, j = 1, \ldots, n$. By direct computation or by deriving the *chf* function of the process $\boldsymbol{Y}$ at time t, it can be shown that the correlation coefficient between $Y_j(t)$ and $Y_i(t)$, $i, j \in 1, \ldots n$ for the *BBSD* model is given by:

$$\rho_{Y_i(t), Y_j(t)} = \frac{\alpha_i \alpha_j \left(\beta_{R_i} \beta_{R_j} a_{i-1} a_{j-1} Var\left[H_1(t)\right] + \gamma_{R_i} \gamma_{R_j} \min(a_{i-1}, a_{j-1}) \mathbb{E}\left[H_1(t)\right]\right)}{\sqrt{Var\left[Y_i(t)\right]} \sqrt{Var\left[Y_j(t)\right]}},$$

with the convention $a_0 = 1$.

Given the theoretical expression of the correlation matrix, one can use the NLLS method or the GMM to find the set of parameters that minimize the distance between the theoretical and the historical correlation matrices. This approach could be applied to an arbitrary number of assets n, nevertheless, it is important to investigate how stable is this methodology and how good is the fit when the dimension of the problem increases. In the next section we illustrate this calibration procedure for $n = 3$, limiting our analysis to the $BBSD$ model for sake of brevity.

5.4.1 A numerical application

In this section we illustrate a financial application with three commodities using the $BBSD$ model only, we omit the application of the other two models for short. We consider future prices of the power Germany $F_1(t)$, the natural gas TTF $F_2(t)$ and the emissions (CO_2) $F_3(t)$. We use the same data set of the previous section with the addition of vanilla call options on CO_2 and the relative forward quotations.
Following the usual two-steps calibration, we first fit the margins on vanilla quoted options, we re-compute the European call options prices and quantify the error against the corresponding market data. In Table 9 we report the mean absolute percentage errors (MAPE). As mentioned, adding new commodities does not impact the robustness of the calibration of the parameters of the marginal distribution. In the second step, we fit the dependence parameters: we observe that the fitted correlation matrix $\tilde{\boldsymbol{\rho}}$ is very similar to the historical one, $\boldsymbol{\rho}$, as reported in Table 10; of course the correlation between power and gas coincides with that estimated in the $n = 2$ case. We can conclude that the fit is also reliable when we deal with $n = 3$ commodities and the parameters can be used to price spread options with a third leg. Derivatives contracts written on these commodities are frequent in the energy sector and it is customary to reserve them the name of *clean-spark-spread* options whose payoff is

$$\Phi_T = \left(F_1\left(T\right) - F_2\left(T\right) - F_3\left(T\right) - K\right)^+ ,$$

we omit this calculation for short.

Commodity	Power Germany	Natural Gas	CO_2
MAPE	0.08%	0.41%	0.27%

Table 9: MAPE of re-computed option prices against the corresponding market data.

$$\boldsymbol{\rho} = \begin{bmatrix} 1.00 & 0.54 & 0.59 \\ 0.54 & 1.00 & 0.68 \\ 0.59 & 0.68 & 1.00 \end{bmatrix}$$

Historical correlation matrix.

$$\tilde{\boldsymbol{\rho}} = \begin{bmatrix} 1.00 & 0.54 & 0.59 \\ 0.54 & 1.00 & 0.67 \\ 0.59 & 0.67 & 1.00 \end{bmatrix}$$

Fitted correlation matrix.

Table 10: Comparison of the historical correlation matrix against the fitted one.

However, because the number of parameters rapidly increases as n increases, both the quality and the robustness of the fit of the correlation will be affected if n is large. This fact has been already observed and detailed in Ballotta and Bonfiglioli [1] and Luciano and Semeraro [18] relatively to the original models. For this reasons, we are inclined to conclude that all presented models should be applied only when the number of assets is relatively small in order to avoid biased results in the option pricing.

6 Conclusions

Based on the concept of self-decomposability we have presented a new method to build dependent stochastic processes that are, at least, marginally Lévy. Such processes are constructed using Brownian subordination via what we call *sd* subordinators which are derived from self-decomposable laws. In financial markets, it is common to describe the sources of risk in terms of a systematic risk component and an idiosyncratic component, the former one being viewed as a common factor affecting all the markets at the same time. With our approach, making use of the self-decomposability, the newly built processes somehow generalize this description because the common factor is replaced by *dependent* factors whose impact is propagated with a stochastic delay.

We have used this approach to extend the recent works based on multivariate subordinators presented by Semeraro [31], Luciano and Semeraro [18] and Ballotta and Bonfiglioli [1] and we have shown that our methodology retains a high degree of mathematical tractability, as the multivariate characteristic function is always available in closed form along with the formula for the linear correlation coefficient. These results are instrumental to design Monte Carlo schemes, Fourier techniques and to calibrate the models to real data in energy markets, and finally to price cross border and spark spread options.

We have focused on the German and French power markets, on the gas TTF forward market and on the emissions forward market. We have calibrated our models using a two steps calibration technique, consisting in fitting the marginal parameters on quoted vanilla products in the first step and of then estimating the correlation on the historical realizations. Our numerical experiments have shown that our proposed models can capture even high values of correlations between these commodities.

These approaches, and the relative developed numerical techniques, have been applied to energy markets with two and three commodities only. Nevertheless, our modeling framework is general enough and can be applied to an arbitrary number of assets. Moreover, such a framework can be used, for example, in equity derivatives, with an arbitrary number of stocks, or in credit risk to model a chain of defaults caused by a market shock that propagates with a stochastic delay.

Furthermore, although most of our results are general, we have focused on gamma self-decomposable subordinators. It will be worthwhile investigating the case of Inverse Gaussian processes, and therefore Normal Inverse Gaussian processes, in more detail, deriving for instance, an efficient path-generation technique. Finally, a topic deserving further investigation is the time-reversal simulation of such processes in order to efficiently price other contracts like swings and storages via backward simulation as detailed in Pellegrino and Sabino [26] and Sabino [27].

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A Appendix

A.1 Proof of Proposition 3.1 (See page 6)

Proof. Substituting the expression of $Y_j(t)$, conditioning with respect $G_j(t)$ and based on the fact that $W_j(t)$ are independent we get:

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) =&\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y}(t)\rangle}\right] = \mathbb{E}\left[e^{iu_1Y_1(t)+iu_2Y_2(t)}\right]\\ =&\mathbb{E}\left[e^{i\left(u_1\mu_1+i\frac{\sigma_1^2u_1^2}{2}\right)G_1(t)}e^{i\left(u_2\mu_2+i\frac{\sigma_2^2u_2^2}{2}\right)G_2(t)}\right].\end{aligned}$$

Using the definition of $G_j(t)$ we have:

$$\begin{aligned}\phi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = \mathbb{E}\Bigg[&e^{i\left(u_1\mu_1+i\frac{\sigma_1^2u_1^2}{2}\right)I_1(t)}e^{i\left(u_2\mu_2+i\frac{\sigma_2^2u_2^2}{2}\right)I_2(t)}e^{i\left(u_2\mu_2+i\frac{\sigma_2^2u_2^2}{2}\right)\alpha_2Z_a(t)}\\ &e^{i\left(\left(u_1\mu_1+i\frac{\sigma_1^2u_1^2}{2}\right)\alpha_1+\left(u_2\mu_2+i\frac{\sigma_2^2u_2^2}{2}\right)\alpha_2a\right)H_1(t)}\Bigg]\end{aligned}$$

and the proof is concluded observing that $I_j(t)$, $H_1(t)$ and $Z_a(t)$ are mutually independent. □

A.2 Proof of Proposition 3.2 (See page 6)

Proof. We need to compute:

$$cov\left(Y_1(t), Y_2(t)\right) = \mathbb{E}\left[Y_1(t)\,Y_2(t)\right] - \mathbb{E}\left[Y_1(t)\right]\mathbb{E}\left[Y_2(t)\right].$$

Substituting the expressions of $Y_j(t)$ and $G_j(t)$ and observing that

$$\mathbb{E}\left[H_1(t)\,H_2(t)\right] = aVar\left[H_1(t)\right],$$

the proof is concluded after simple algebra. □

A.3 Proof of Proposition 3.5 (See page 8)

Proof. Rewrite $\boldsymbol{Y}(t)$ as:

$$\boldsymbol{Y}(t) = \boldsymbol{Y_I}(t) + \boldsymbol{Y_H}(t),$$

where:

$$\boldsymbol{Y_I}(t) = \begin{pmatrix} \mu_1I_1(t)+\sigma_1W_1(I_1(t)) \\ \mu_2I_2(t)+\sigma_2W_2(I_2(t)) \end{pmatrix}$$

and:

$$\boldsymbol{Y_H}(t) = \begin{pmatrix} \alpha_1\mu_1H_1(t)+\sqrt{\alpha_1}\sigma_1W_1^{\rho}(H_1(t)) \\ \alpha_2\mu_2H_2(t)+\sqrt{\alpha_2}\sigma_2\left(W_2^{\rho}(aH_1(t))+\tilde{W}(Z_a(t))\right) \end{pmatrix}.$$

The *chf* is given by:

$$
\begin{aligned}
\phi_{\boldsymbol{Y}(t)}\left(\boldsymbol{u}\right) =&\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y}(t)\rangle}\right] = \mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y}_I(t)+\boldsymbol{Y}_H(t)\rangle}\right]\\
=&\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y_I}(t)\rangle}\right]\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y_H}(t)\rangle}\right].
\end{aligned} \tag{27}
$$

We now compute the two last term separately. Substituting the expression of $\boldsymbol{Y_I}$, conditioning respect $I_j\left(t\right),\ j=1,2$ and remembering that $W_1\left(t\right)$ and $W_2\left(t\right)$ are independent we have:

$$
\begin{aligned}
\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y_I}(t)\rangle}\right] =&\mathbb{E}\left[e^{i\left(u_1\mu_1+\frac{i}{2}u_1^2\sigma_1^2\right)I_1(t)}\right]\mathbb{E}\left[e^{i\left(u_2\mu_2+\frac{i}{2}u_2^2\sigma_2^2\right)I_2(t)}\right]\\
=&\phi_{I_1(t)}\left(u_1\mu_1+\frac{i}{2}\sigma_1^2u_1^2\right)\phi_{I_2(t)}\left(u_2\mu_2+\frac{i}{2}\sigma_2^2u_2^2\right).
\end{aligned} \tag{28}
$$

Following the same approach we can compute the second term, obtaining:

$$
\begin{aligned}
\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y_H}(t)\rangle}\right] =&\mathbb{E}\left[\mathbb{E}\left[e^{iu_1\alpha_1\mu_1H_1(t)+iu_1\sqrt{\alpha_1}\sigma_1W_1^\rho(H_1(t))+iu_2\alpha_2\mu_2aH_1(t)+iu_2\sqrt{\alpha_2}\sigma_2W_2^\rho(aH_1(t))}|H_1\left(t\right)\right]\right.\\
&\left.\mathbb{E}\left[e^{iu_2\alpha_2\mu_2Z_a(t)+iu_2\sqrt{\alpha_2}\sigma_2\tilde{W}(Z_a(t))}|Z_a\left(t\right)\right]\right].
\end{aligned}
$$

Now we compute the inner expected values separately. The second inner expected value is:

$$
\mathbb{E}\left[e^{iu_2\alpha_2\mu_2Z_a(t)+iu_2\sqrt{\alpha_2}\sigma_2\tilde{W}(Z_a(t))}|Z_a\left(t\right)\right] = e^{i\left(u_2\alpha_2+\frac{i}{2}u_2^2\alpha_2\sigma_2\right)Z_a(t)}.
$$

For the second term we have that, since $H_1\left(t\right)$ is known:

$$
\begin{aligned}
&\mathbb{E}\left[e^{iu_1\alpha_1\mu_1H_1(t)+iu_1\sqrt{\alpha_1}\sigma_1W_1^\rho(H_1(t))+iu_2\alpha_2\mu_2aH_1(t)+iu_2\sqrt{\alpha_2}\sigma_2W_2^\rho(aH_1(t))}|H_1\left(t\right)\right]\\
=&\ e^{iu_1\alpha_1\mu_1H_1(t)+iu_2\alpha_2\mu_2aH_1(t)}\mathbb{E}\left[e^{iu_1\sqrt{\alpha_1}\sigma_1W_1^\rho(H_1(t))+iu_2\sqrt{\alpha_2}\sigma_2W_2^\rho(aH_1(t))}|H_1\left(t\right)\right].
\end{aligned}
$$

The only unknown terms is the expected value. We have that:

$$
\mathbb{E}\left[e^{iu_1\sqrt{\alpha_1}\sigma_1W_1^\rho(H_1(t))+iu_2\sqrt{\alpha_2}\sigma_2W_2^\rho(aH_1(t))}|H_1\left(t\right)\right] = e^{-\frac{1}{2}u_1^2\alpha_1\sigma_1^2(1-a)H_1(t)}e^{-\frac{1}{2}a\boldsymbol{u}^T\Sigma\boldsymbol{u}H_1(t)},
$$

where

$$
\Sigma = \begin{bmatrix} \alpha_1\sigma_1^2 & \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho \\ \sqrt{\alpha_1\alpha_2}\sigma_1\sigma_2\rho & \alpha_2\sigma_2^2 \end{bmatrix}
$$

and $\boldsymbol{u}=(u_1,u_2)$. Setting $\boldsymbol{\mu}=(\alpha_1\mu_1,a\alpha_2\mu_2)$ we can conclude that:

$$
\mathbb{E}\left[e^{i\langle \boldsymbol{u},\boldsymbol{Y_H}(t)\rangle}\right] = \phi_{Z_a(t)}\left(u_2\alpha_2+\frac{i}{2}u_2^2\alpha_2\sigma_2\right)\phi_{H_1(t)}\left(\boldsymbol{u}^T\boldsymbol{\mu}+\frac{i}{2}u_1^2\alpha_1\sigma_1^2\left(1-a\right)+\frac{i}{2}a\boldsymbol{u}^T\Sigma\boldsymbol{u}\right). \tag{29}
$$

Combining (28) and (29) in (27) concludes the proof. □

A.4 Proof of Lemma 3.8 (See page 10)

Proof. Replacing the definition of $R_1(t)$ and $R_2(t)$ we get:

$$\begin{aligned}\phi_{\boldsymbol{R}(t)}(\boldsymbol{u}) =& \mathbb{E}\left[e^{iu_1 R_1(t)+iu_2 R_2(t)}\right] \\ =& \mathbb{E}\left[e^{iu_1\beta_{R_1}H_1(t)+iu_2 a\beta_{R_1}H_1(t)+iu_2\beta_{R_2}Z_a(t)}\right. \\ & \left.\mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}\left(W(aH_1(t))+\tilde{W}(Z_a(t))\right)}|H_1(t), Z_a(t)\right]\right].\end{aligned}$$

We compute now the inner expected value:

$$\begin{aligned}&\mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}\left(W(aH_1(t))+\tilde{W}(Z_a(t))\right)}|H_1(t), Z_a(t)\right] \\ &= \mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}W(aH_1(t))}|H_1(t)\right]\mathbb{E}\left[e^{iu_2\gamma_{R_2}\tilde{W}(Z_a(t))}|Z_a(t)\right].\end{aligned}$$

The second computation of the second expected value is immediate:

$$\mathbb{E}\left[e^{iu_2\gamma_{R_2}\tilde{W}(Z_a(t))}|Z_a(t)\right] = e^{-\frac{1}{2}u_2^2\gamma_{R_2}^2 Z_a(t)}.$$

For the first term we have:

$$\mathbb{E}\left[e^{iu_1\gamma_{R_1}W(H_1(t))+iu_2\gamma_{R_2}W(aH_1(t))}|H_1(t)\right] = e^{-\frac{1}{2}\left(u_1^2\gamma_{R_1}^2+2u_1u_2\gamma_{R_1}\gamma_{R_2}a+au_2^2\gamma_{R_2}^2\right)H_1(t)}.$$

The proof is concluded observing that $H_1(t)$ and $Z_a(t)$ are independent. □

A.5 Proof of Proposition 3.9 (See page 10)

Proof. Replacing the expression of Y_j $j = 1, 2$ we have that:

$$\mathbb{E}\left[e^{i\langle\boldsymbol{u},\boldsymbol{Y}(t)\rangle}\right] = \mathbb{E}\left[e^{iu_1X_1(t)}\right]\mathbb{E}\left[e^{iu_2X_2(t)}\right]\phi_{\boldsymbol{R}(t)}(\boldsymbol{a}\circ\boldsymbol{u}).$$

Observe that, conditioning to $G_j(t)$, we have that:

$$\mathbb{E}\left[e^{iu_jX_j(t)}\right] = \mathbb{E}\left[e^{i\left(u_j\beta_j+\frac{i}{2}u_j^2\gamma_j^2\right)G_j(t)}\right] = \phi_{G_j(t)}\left(u_j\beta_j+\frac{i}{2}u_j^2\gamma_j^2\right).$$

This observation jointly with Lemma 3.8 complete the proof. □

A.6 Proof of Proposition 3.10 (See page 10)

Proof. Computing the covariance between $Y_1(t)$ and $Y_2(t)$ we have that:

$$cov(Y_1(t), Y_2(t)) = a_1a_2 cov(R_1(t), R_2(t)). \tag{30}$$

By direct computations one can show that:

$$cov(R_1(t), R_2(t)) = \beta_{R_1}\beta_{R_2}aVar[H_1(t)] + \gamma_{R_1}\gamma_{R_2}a\mathbb{E}[H_1(t)]. \tag{31}$$

where we have used the following property

$$\mathbb{E}[W(H_1(t))W(aH_1(t))] = a\mathbb{E}[H_1(t)].$$

The proof is concluded using (30) and (31). □

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