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Measuring portfolio credit risk correctly: Why parameter uncertainty matters

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ABSTRACT

Why should risk management systems account for parameter uncertainty? In addressing this question, the paper lets an investor in a credit portfolio face non-diversifiable uncertainty about two risk parameters – probability of default and asset-return correlation – and calibrates this uncertainty to a lower bound on estimation noise. In this context, a Bayesian inference procedure is essential for deriving and analyzing the main result, i.e. that parameter uncertainty raises substantially the tail risk perceived by the investor. Since a measure of tail risk that incorporates parameter uncertainty is computationally demanding, the paper also derives a closed-form approximation to such a measure.

1. Introduction

Measures of credit risk are often based on an analytic model and on the assumption that the parameters of this model are known with certainty. In turn, it is common practice for risk management systems to rely on such measures because of their tractability, even though it is attained by ignoring *estimation noise*. This practice may impair severely the quality of risk management systems because, besides credit-risk factors, estimation noise is another important determinant of uncertainty about potential losses.

In order to substantiate this claim, this paper generalizes the popular asymptotic single risk factor (ASRF) model of portfolio credit risk by allowing for noisy estimates of two key parameters: probability of default (PD) and asset-return correlation.[1] Applied to a stylized empirical framework, the generalized model delivers two alternative measures of tail risk that help underscore the importance of estimation noise. The first measure is a *naive* value-at-risk (VaR) of the portfolio, which accounts for the credit-risk factor but treats point estimates of the PD and asset-return correlation as equal to the true values of the respective parameters. The second is the *correct* VaR measure, which accounts not only for the credit-risk factor that influences the naive VaR but also for parameter uncertainty.

In principle, the correct VaR not only differs from the naive one but also does not match the *actual* risk that an investor is exposed to. The reason is that measuring the actual VaR of a portfolio requires knowledge of the true values of the risk parameters. By contrast, the correct VaR reflects the level of tail risk *perceived* by an investor, given his imperfect information about risk parameters.

Quantifying the correct VaR on the basis of a Bayesian inference procedure and comparing it to the naive one reveals that ignoring estimation noise should be expected to lead to a substantial understatement of the level of tail risk perceived by an investor. In the benchmark specification – where an investor in a homogeneous portfolio estimates the PD and asset-return correlation at 1% and 20%, respectively, on the basis of data covering 200 obligors over 10 years – the correct VaR is 27% higher than the corresponding naive VaR. This result is striking, not least because the underlying stylized empirical framework incorporates a lower bound on the amount of estimation noise.

In addition, accounting for estimation noise dampens (correctly) the sensitivity of VaR measures to changes in parameter estimates. The flip side is that, by abstracting from estimation noise, naive VaRs overstate the information content of parameter estimates. This manifests itself in that the difference between the correct and naive VaRs – i.e. the *add-on* induced by parameter uncertainty – decreases (increases) by less than the naive VaR when changes in parameter estimates suggest lower (higher) tail risk of the portfolio.

In comparison to Löffler (2003) and Gössl (2005) – which also incorporate estimation noise in measures of portfolio tail risk – this

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[1] The paper abstracts from issues arising from *model* uncertainty. Kerkhof et al. (2010), Hsien-Hsing et al. (2009) and Tarashev and Zhu (2008a) study the impact of model mis-specification on measures of portfolio risk.

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paper conducts the analysis in a more transparent framework that allows for comparing in a straightforward fashion the importance of different sources of noise. Namely, a lengthening of the time series of the available data from 5 to 10 or from 10 to 20 years is seen to reduce the correct VaR add-on by a factor of two. In comparison, similar changes to the size of the cross section have a markedly smaller impact on the add-on. This finding is rooted in the standard assumption that credit risk is driven by asset returns that are serially uncorrelated but are correlated across obligors, which implies that increasing the time series of the data brings in more information than expanding the cross section.

The transparent framework of this paper also helps to analyze the trade-off between accuracy and reduction of the computational burden. This trade-off underscores the advantages of an approximate VaR measure, which exists in closed form and accounts for uncertainty about the PD but needs to be adjusted to reflect noise in observed asset returns. Given a judicious adjustment for such noise, this measure approximates the correct VaR quite well and alleviates substantially the computational burden.

That said, owing to the underlying Bayesian inference procedure, the computational burden is substantial for both the correct and the approximate VaR measures. This procedure is instrumental for capturing an important empirical regularity. Namely, over a realistic range of parameter values, higher levels of the PD and asset-return correlation are associated with greater noise in the associated estimates. This dependence between estimation noise and true parameter values, which is missed if one circumvents Bayesian inference, raises the probability that the true PD and correlation are bigger than their point estimates and, consequently, raises the correct VaR.

From a general point of view, this paper provides support to a call made by Borio and Tsatsaronis (2004) for including "measurement error information" in financial reporting. In deriving an ideal information set for attaining efficiency of the financial system, that paper emphasizes measurement error – a specific example of which is estimation error – as a piece of information that is of natural interest to risk managers and supervisors alike. From this perspective, the results derived below highlight specific scenarios in which knowledge of measurement error is indeed highly valuable as such error accounts for much of the uncertainty about potential credit losses.

The present paper differs in an important way from a number of recent articles – e.g. Tarashev and Zhu (2008a) and Heitfield (2008) – that have analyzed estimation noise in the context of portfolio credit risk.[2] Conditioning on hypothesized true parameter values, these articles study the extent to which ignoring estimation noise introduces errors in VaR measures.[3] However, the articles do not demonstrate how to *incorporate* the inevitable uncertainty about the true parameter values in measures of portfolio credit risk. In terms of the terminology introduced here, these articles quantify the discrepancy between the naive and actual VaRs but do not derive correct VaRs.

The rest of the paper is organized as follows: Section 2 describes the model and then derives alternative measures of portfolio VaR. These measures are considered in the context of an empirical framework that is outlined in Section 3. In turn, Section 4 presents and analyzes the quantitative results, paying particular attention to the trade-off between accuracy and computational complexity. Finally, Section 5 provides two extensions of the baseline analysis.

2. Stylized credit portfolio

The impact of parameter uncertainty on measures of tail risk is analyzed on the basis of a stylized credit portfolio. There are n exposures in this portfolio and all of them are of equal size, which is set to $1/n$. The analysis considers the limit $n \rightarrow \infty$, in which the portfolio is referred to as *asymptotic* or "perfectly fine-grained".[4]

The portfolio is also *homogeneous* in the sense that all of the exposures have the same credit characteristics. These characteristics are captured by two parameters: the degree to which a common risk factor affects an obligor's assets, denoted by ρ^*; and the obligor's probability of default, PD^*.[5] When parameter estimates are based on portfolios observed at different points in time (see Section 3.1), the true values of ρ^* and PD^* are assumed to remain constant across such portfolios.

The role of the two credit-risk parameters is rooted in the stochastic process followed by the value of the assets of each obligor, i:

$$\ln(V_{i,\tau}) = \ln(V_{i,\tau-\Delta}) + \mu^*\Delta + \sigma^*\sqrt{\Delta}\Big(\sqrt{\rho^*}M_\tau + \sqrt{1-\rho^*}Z_{i,\tau}\Big), \qquad (1)$$

$$\text{where } M_\tau \sim N(0,1), \quad Z_{i,\tau} \sim N(0,1),$$
$$Cov(M_\tau, Z_{i,\tau}) = 0, \quad Cov(Z_{i,\tau}, Z_{j,\tau}) = 0 \text{ for all } i \text{ and } j \neq i$$

and Δ denotes the period, in years, between two observations. There are two serially uncorrelated factors: one *common* to *all* obligors in the portfolio, M, and one specific to this obligor, Z_i. The drift of the asset value, the volatility of asset returns and the correlation between the asset returns of any two obligors – determined by μ^*, $\sigma^* > 0$ and $\rho^* \in [0,1]$, respectively – are the same for all i.

Obligor i defaults if and only if $\ln(V_{i,\tau})$ is below some threshold, D_i^*. In line with Merton (1974), default events are assumed to occur only at the end, $t \in \{1, 2, \ldots, T\}$, of non-overlapping and adjacent *1-year* periods, which may be longer than the periods between two consecutive observations of the obligors' assets, i.e. $1 \geqslant \Delta$. Then, assuming that the loss-given-default on each exposure is unity, Eq. (1) implies that the loss on this portfolio over the next year t, is given by $L_{n,t}$:

$$L_{n,t} = \sum_i^n U_{i,t}, \text{ where} \qquad (2)$$
$$U_{i,t} = \begin{cases} 1/n & \text{if } \sqrt{\rho^*}M_t + \sqrt{1-\rho^*}Z_{i,t} < \Phi^{-1}(PD^*), \\ 0 & \text{otherwise,} \end{cases}$$
$$\Phi^{-1}(PD^*) = \left(D_i^* - \ln(V_{i,t-1}) - \mu^*\right)/\sigma^*$$

and PD^* is the unconditional 1-year probability of default, which is assumed to be the same across exposures (requiring that so is $D_i^* - \ln(V_{i,t-1})$). The expression $\Phi^{-1}(PD^*)$ is henceforth referred to as the (standardized) default point.[6]

An investor is interested in the maximum portfolio loss that is exceeded within a year with probability α, i.e. in the 1-year VaR at the $(1-\alpha)$ confidence level. It will be assumed that the investor

[2] Of the related articles, only Löffler (2003) analyzes uncertainty about PD on the basis of actual default data and does so via non-parametric bootstrap. See Lando and Skodeberg (2002) and Hanson and Schuermann (2006) for an extensive analysis of bootstrap approaches to the derivation of PD confidence intervals and Cantor et al. (2008) for an application of such an approach to a large dataset. The analysis below, just like Heitfield (2008) and Tarashev and Zhu (2008a), circumvents the use of bootstrap methods by assuming that the functional form, albeit not the parameter values, of the data-generating process is known.

[3] Bongaerts and Charlier (2009) perform a similar exercise in the context of regulatory capital assessments.

[4] See Gordy and Lütkebohmert (2007) for an analysis of real-life departures from perfect granularity.

[5] In this paper, "obligor" and "exposure" are used as close synonyms.

[6] In order to streamline the analysis, this setup abstracts from some important aspects of portfolio credit risk. In addition to adopting the afore mentioned assumption of perfect granularity, the setup rules out stochastic shocks to loss-given-default and exposure-at-default, which are modelled as correlated with the common default-risk factor by Kupiec (2008). The setup also abstracts from cross-sectional dispersion of credit-risk parameters, which is analyzed by Tarashev and Zhu (2008a).

knows how the portfolio loss is determined, i.e. knows the model in (2) and the distribution of the credit-risk factors, M and Z_i. However, the investor has to estimate the homogeneous asset-return correlation, ρ^*, and probability of default, PD^*.

The following three subsections outline three alternative measures of portfolio VaR that make different uses of the information available to the investor. The first measure is the naive VaR, which treats the point estimates of the asset-return correlation and probability of default – denoted by $\hat{\rho}$ and $\widehat{PD}$ – as equal to the true parameter values. The second measure equals the correct VaR perceived by the investor. This measure also relies on the point estimates $\hat{\rho}$ and $\widehat{PD}$ but, in addition, incorporates the investor's uncertainty about the true parameter values. The third measure, which the paper considers only in passing, has the same functional form as the naive VaR but, instead of the estimates $\hat{\rho}$ and $\widehat{PD}$, incorporates conservative values of the asset-return correlation and the probability of default. The relationship between these values and the size of the investor's uncertainty about the true parameters is determined in an ad hoc way.

2.1. Portfolio VaR under the ASRF model

In deriving the naive VaR, one abstracts from parameter uncertainty, which reduces the above setup to a special case of the popular asymptotic single risk factor (ASRF) model. In this case, obligor-specific risk factors influence only the level of expected portfolio losses. Building on this result, Gordy (2003) demonstrates that the VaR of an ASRF portfolio at the $(1-\alpha)$ confidence level equals the sum of exposure-specific expected losses, conditional on the common risk factor, M, being at the αth quantile of its distribution.[7] Given that the forecast horizon is 1 year, expression (2) implies that this boils down to:

$$\begin{aligned} VaR^{naive} &= \lim_{n\to\infty} E(L_n|M = \Phi^{-1}(\alpha), PD^* = \widehat{PD}, \rho^* = \hat{\rho}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\Phi\left(\frac{\Phi^{-1}(\widehat{PD}) - \sqrt{\hat{\rho}}\Phi^{-1}(\alpha)}{\sqrt{1-\hat{\rho}}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(\widehat{PD}) - \sqrt{\hat{\rho}}\Phi^{-1}(\alpha)}{\sqrt{1-\hat{\rho}}}\right) \end{aligned} \quad (3)$$

where $\Phi(\bullet)$ denotes the standard normal CDF and time subscripts have been suppressed in order to alleviate the notation. Owing to the assumed parameter homogeneity and constant portfolio size (normalized to unity), the conditional expectation on the first line of (3) equals the average of equal terms and, thus, is independent of the number of exposures, n. Since this makes the limit redundant, it is henceforth dropped.

2.2. Portfolio VaR with parameter uncertainty

Since it abstracts from estimation noise, the naive VaR in (3) fails to reflect the fact that the *correct* VaR, i.e. the one perceived by the investor, treats *candidate* values of the PD and asset-return correlation, PD^c and ρ^c, as random variables. The joint probability density of these random variables is assumed to be well-defined, continuous and bounded away from zero everywhere on its support. The next section presents a concrete empirical framework, in which this assumption is borne out.

Parameter uncertainty implies that the investor faces *multiple common* risk factors. These are the common credit-risk factor, M, and the common "estimation-risk" factors, PD^c and ρ^c. Since the uncertainty about M refers to the future while the uncertainty embedded in ρ^c and PD^c is driven by past data, the assumed serial independence of M implies that it is independent of ρ^c and PD^c.

Importantly, the presence of multiple risk factors violates a key assumption of the ASRF model, implying that the formula in (3) is no longer valid. Instead, it is necessary to consider the following conditional expected loss on the portfolio:

$$E(L_n|M, PD^c, \rho^c) = \Phi\left(\frac{\Phi^{-1}(PD^c) - \sqrt{\rho^c}M}{\sqrt{1-\rho^c}}\right) \equiv E(L|M, PD^c, \rho^c), \quad (4)$$

which is a random variable. Then, let $Q_{1-\alpha}(E(L|M, PD^c, \rho^c))$ denote the $(1-\alpha)$ quantile of this conditional expectation and let $Q_{1-\alpha}(L_n)$ stand for the correct VaR of the portfolio at the $(1-\alpha)$ confidence level. This notation helps formulate Proposition 1 (proved in Appendix A), which implies that the distribution of the random variable in (4) is sufficient for deriving the correct VaR of an *asymptotic* portfolio:

Proposition 1. *As* $n \to \infty, Q_{1-\alpha}(L_n) \to Q_{1-\alpha}(E(L|M, PD^c, \rho^c)) \equiv VaR^{correct}$.

Sections 4 and 5 below will refer to $VaR^{correct} - VaR^{naive}$ as the VaR add-on.

It is important to note that there would generally not be an analytic expression for the correct VaR measure, even in the limit $n \to \infty$. This might make it tempting to consider alternative measures that focus directly on conservative values of the credit- and estimation-risk factors. Specifically, in the light of the VaR confidence level, such an alternative could condition on the α quantile of the common credit factor and the $(1-a)$ quantiles of the probability of default and correlation candidates (henceforth, $PD^c_{1-\alpha}$ and $\rho^c_{1-\alpha}$):

$$\begin{aligned} VaR^{alt} &= E\left(L|M = \Phi^{-1}(\alpha), PD^c = PD^c_{1-\alpha}, \rho^c = \rho^c_{1-\alpha}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(PD^c_{1-\alpha}) - \sqrt{\rho^c_{1-\alpha}}\Phi^{-1}(\alpha)}{\sqrt{1-\rho^c_{1-\alpha}}}\right). \end{aligned} \quad (5)$$

Albeit computationally more efficient, VaR^{alt} turns out to be materially larger than the correct VaR, $Q_{1-\alpha}(E(L|M, PD^c, \rho^c))$. To see why, note that the correct VaR would coincide with VaR^{alt} if the estimation-risk factors, PD^c and ρ^c, and (the negative of) the credit-risk factor, M, were comonotonic random variables.[8] Since the risk factors are *not* comonotonic, however, there are diversification benefits that depress the correct VaR measure but do not affect VaR^{alt}. The difference between the two alternative measures is quantified in Sections 4.1 and 4.2 below.

3. Empirical framework

This section outlines the derivation of the correct VaR measure, $Q_{1-\alpha}(E(L|M, PD^c, \rho^c))$. In addition to the distribution of the common credit-risk factor, this measure incorporates the joint probability distribution of the two estimation-risk factors, which reflects features of the data used for estimation and the inference procedure. These features are assumed to be such as to ensure that the correct VaR measure entails reasonable computational burden and, at the same time, is informative about the impact of estimation noise in real-life applications.

3.1. Stylized dataset

The investor observes asset returns and default rates, which are delivered by the data-generating process specified in (1) and (2). It

[7] In general terms, the qth quantile of the distribution of a generic random variable Y is defined as $Q_q(Y) \equiv \inf\{y : \Pr(Y \leqslant y) \geqslant q\}$.

[8] When this is the case, conditional on the common credit-risk factor being equal to the qth quantile of its distribution, the PD and correlation candidates are at the $(1-q)$th quantiles of their respective distributions with probability 1. See McNeil et al. (2005) for a general discussion of comonotonicity.

will be assumed that these data cover T independent cohorts, each one of which comprises N obligors. Each cohort is followed for 1 year $t \in \{1, \ldots, T\}$ and, for each month in this year, the investor observes the assets of the N obligors (implying that the length of the period between two consecutive asset observations is $\Delta = 1/12$). At the end of each year t, the investor also observes the default rate in the cohort. In line with the investor's portfolio, all obligors in each cohort are characterized by an asset-return correlation that equals ρ^* and a probability of default that equals PD^* at the beginning of the relevant year.

The stylized dataset warrants four remarks. First, the assumed frequencies of asset and default rate observations are intended to capture common practice. Reportedly, in order to filter out high-frequency noise, practitioners base their estimates of asset-return correlations on weekly or monthly time series of assets that are derived from daily equity prices and/or CDS spreads.[9] In turn, yearly observations of 1-year default rates are standard ex post measures of short-term credit risk.

Second, despite the maintained focus on an asymptotic portfolio, the numerical results below reflect datasets with finite cross sections. This corresponds to the likely real-life case in which the investor has access to data on only a subset of the exposures. That said, the benchmark numerical exercise in this paper uses a portfolio of $N = 200$ exposures and it turns out that expanding such a cross section leads to small declines of estimation noise, which have a limited impact on the VaR perceived by the investor (see Sections 4.1 and 4.2).

Third, the benchmark exercise incorporates 1-year default rates realized over $T = 10$ years and asset returns realized over $T = 120$ months. To put the length of these time series into perspective, note that the Basel II accord requires regulated institutions to base their PD estimates for corporate exposures on at least 5 years of data.

Fourth, changes in credit-risk outlooks are a frequent real-life phenomenon, which might invalidate the assumption that a fixed number of obligors, N, feature the same parameters, ρ^* and PD^*, over several years. A time invariant N is, however, in line with the illustrative nature of the analysis.

3.2. Bayesian inference procedure

The investor addresses estimation noise via a Bayesian inference procedure. Namely, the investor does not know the true values of the asset-return correlation and the probability of default, ρ^* and PD^*, but holds *prior* beliefs about the probability distribution of the candidate values, ρ^c and PD^c. Then, on the basis of the data described in Section 3.1 and knowledge of the true structure of the data-generating process, the investor derives point estimates of ρ^* and PD^* and uses these estimates to update the prior beliefs into *posterior* probability distributions of ρ^c and PD^c.

Assumed features of the data lead to an important difference between the inference about ρ^c and that about PD^c. Namely, given that asset returns are observed directly, default data do not provide any additional information as regards the asset-return correlation. At the same time, information about this correlation improves the estimate of PD^* for any observed default rate. The inference procedure incorporates this by treating the point estimate of PD^* as a function of the correlation candidate: $\widehat{PD}(\rho^c)$.

In terms of notation, the posterior density of correlation candidates equals:

$$h(\rho^c|\hat{\rho}) = \frac{g(\rho^c)f(\hat{\rho}|\rho^c)}{\int g(\rho^c)f(\hat{\rho}|\rho^c)d\rho^c}, \tag{6}$$

where $g(\bullet)$ is the prior PDF of ρ^c and $f(\bullet|\rho^c)$ is the conditional PDF of the correlation estimator. In turn, the posterior density of PD candidates is given by:

$$\tilde{h}(PD^c|\widehat{PD}(\rho^c), \rho^c) = \frac{\tilde{g}(PD^c)\tilde{f}(\widehat{PD}(\rho^c)|PD^c, \rho^c)}{\int \tilde{g}(PD^c)\tilde{f}(\widehat{PD}(\rho^c)|PD^c, \rho^c)dPD^c}, \tag{7}$$

where $\tilde{g}(\bullet)$ is the prior PDF of PD^c and, conditional on PD and correlation candidates, $\tilde{f}(\bullet|PD^c, \rho^c)$ is the PDF of the probability-of-default estimate. For given correlation estimate $\hat{\rho}$ and default rate, the conditional densities in (6) and (7) define the *joint* posterior density of ρ^c and PD^c.

Practical applications would typically adopt a specific parameterization of probability density functions (and assume that it is known to the investor). The parameterization adopted for the numerical results in this paper (outlined in Appendix B) possesses two noteworthy features. First, it is assumed that parameters are estimated on the basis of *minimum-variance unbiased* estimators that attain the respective *Cramer-Rao lower bounds* (see Appendix C). Second, $\tilde{h}(\bullet|\widehat{PD}(\rho^c), \rho^c)$ is calibrated in such a way that the implied posterior distribution of the default point $\Phi^{-1}(PD^c)$ is normal.

3.3. Deriving the correct VaR measure

The discussion in Sections 3.1 and 3.2 leads directly to the derivation of the correct VaR measure. To derive this measure, the investor makes use of: (i) panel datasets of asset returns and default rates; (ii) knowledge of the joint posterior distribution of the three risk factors. This knowledge comprises the posterior densities of correlation and PD candidates – $h(\bullet|\bullet)$ and $\tilde{h}(\bullet|\bullet,\bullet)$, respectively – and the independent, standard normal distribution of the credit-risk factor, M.

The investor first quantifies the probability distribution of the conditional expected loss, $E(L|M, PD^c, \rho^c)$, via the following simulation procedure:

1. The data on asset returns produce a point estimate of their correlation, $\hat{\rho}$.
2. A candidate ρ^c is drawn from the posterior density $h(\bullet|\hat{\rho})$.[10]
3. Conditioning on ρ^c, a point estimate of the probability of default, $\widehat{PD}(\rho^c)$, is obtained from data on default rates via maximum likelihood estimation.[11]
4. Conditioning on ρ^c and $\widehat{PD}(\rho^c)$, a value of $\Phi^{-1}(PD^c) - \sqrt{\rho^c}M$ is drawn from the normal PDF implied by the joint normality of $\Phi^{-1}(PD^c)$ and M.
5. Repeating step 4 a large number of times delivers the probability distribution of the conditional expectation $E(L|M, PD^c, \rho^c)$ for the given ρ^c and default rates.
6. Repeating steps 2–5 delivers the distribution of $E(L|M, PD^c, \rho^c)$ for any point estimate $\hat{\rho}$ and observed default rates.

Then, by Proposition 1, the $(1-\alpha)$ quantile of the so-derived distribution equals the correct VaR of the portfolio at the $(1-\alpha)$ confidence level.

3.4. Assuring comparability among the different VaR measures

Each set of measures – used in the numerical examples below for comparing naive, correct and alternative VaRs – is underpinned

[9] In line with such practice, Moody's KMV publishes monthly estimates of the market value of the assets of the firms in its database. See Heitfield (2008) and Tarashev and Zhu (2008b) for further detail.

[10] A discretization of $h(\cdot|\hat{\rho})$ considerably speeds up the numerical simulation. A description of a particular discretization procedure is available from the author upon request.

[11] Most of the analysis below refers to the Cramer-Rao lower bounds on the variance of parameter estimators, while keeping the exact specification of these estimators in the background. That said, simulation exercises reveal that the maximum-likelihood estimator of the probability of default attains the Cramer-Rao lower bound and is largely consistent with the assumed shape of $\tilde{f}(\bullet|PD^c, \rho^c)$.

by a *single* dataset. Given the data description above, this implies that, when a naive and correct VaRs are compared to each other, they are based on the same correlation estimate, $\hat{\rho}$. Then, given $\hat{\rho}$ and an observed default rate, calculating the naive VaR on the basis of expression (3) requires a point estimate of the default probability. This estimate, $\widehat{PD}$, is derived via Step 3 in Section 3.3 when the true value of the asset-return correlation is set equal to $\hat{\rho}$: i.e. $\widehat{PD} = \widehat{PD}(\hat{\rho})$. In turn, given the same $\hat{\rho}$ and observed default rate, the correct VaR is derived via the procedure outlined in Section 3.3. Finally, the alternative VaR is based on the marginal posterior distributions of the risk factors, which are an intermediate output of the same procedure.

Note that a given correlation estimate, $\hat{\rho}$, and given sizes of the dataset, N and T, imply a one-to-one correspondence between the observed default rate and the PD estimate. Thus, each numerical example, which gives rise to different but comparable VaR measures, can be defined by the four-tuple $\{\hat{\rho}, \widehat{PD}, N, T\}$. Such definitions are used in Sections 4 and 5 below.

3.5. Discussion of the empirical framework

This section revisits important aspects of the empirical framework. First, some of the framework's underlying assumptions – e.g. homogeneity of exposures, time-invariant risk parameters of the obligors in the data, a convenient parameterization of probability densities – are key for deriving correct VaR measures that are not prohibitively burdensome to calculate. The reason is that these assumptions limit the number of the parameters of interest to two – i.e. the common PD and asset-return correlation – and help circumvent inference about "nuisance" parameters, which are not central to the problem at hand (such as the drift and volatility of asset returns).[12] To see the importance of this implication, note first that the calculation of a single VaR measure under the adopted framework requires roughly 6 days of computer time (on a Pentium(R) 4 CPU 3.20 GHz machine with 2 GB of RAM). Moreover, as indicated by the procedure described in Section 3.3, the number of simulations grows exponentially in the number of parameters relevant for deriving a VaR.

A second important aspect of the framework is that some of the simplifying assumptions are likely to depress the correct VaR measure. For example, the assumptions that the available estimators attain the Cramer-Rao lower bound, that exposures are homogeneous and that the size of the cross section of the data is fixed over time limit significantly the amount of estimation noise allowed to affect the investor's perception of risk. As a result, the correct VaR derived under such simplifying assumptions should be treated as a lower bound on the VaRs faced by investors in real-life portfolios, where many of these assumptions are violated.

That said, a third important aspect of the framework relates to the inference procedure, which has been designed to insulate the precision of parameter estimates from some highly stylized aspects of the dataset. Namely, when estimating the probability of default, the investor is not allowed to use information that is contained in observed asset returns but is missed by the estimate of their correlation (see Section 3.2). Thus, the investor is not allowed to exploit the fact that, since all obligors are assumed to be ex ante homogeneous, the minimum asset value among surviving obligors and the maximum asset value among defaulting obligors bound from below and above the possible values of the default point. Given the sizes of the datasets examined in this paper, the two bounds are typically so close to each other as to effectively *reveal* the default point and, thus, the PD.

The inference procedure adopted here abstracts from this clearly unrealistic implication. Indeed, Heitfield (2008) reports that, in real-life applications – where (i) obligors are heterogeneous and (ii) asset-return and default-rate data cover different sets of obligors – the common practice is to use asset-returns data in order to estimate the *average* correlation and to use the default data and correlation estimate in order to estimate the *average* PD but *not* to use asset-returns data for a direct estimation of the default point. The empirical framework in this paper emulates this common practice.

A fourth important aspect of the empirical framework relates to the assumption that prior beliefs about the credit-risk parameters are diffuse. Although this assumption is quite in line with the level of generality in this paper, it could easily be replaced with other similarly acceptable alternatives. Section 5.1 below derives such alternatives on the basis of long historical data on default rates and then studies their implications for portfolio VaR.

Fifth, the assumption that asset returns are observed directly masks an important challenge associated with correlation estimates. In practice, asset returns are subject to observation noise because they need to be extracted from other variables, such as stock prices and CDS spreads.[13] As shown in Section 5.2 below, abstracting from observation noise in asset returns can bias the VaR measure. The section also discusses conditions under which the bias can be positive or negative, and proposes a correction.

4. Results

The main result is that ignoring parameter uncertainty leads to a significant understatement of the portfolio tail risk perceived by the investor. Importantly, this result is derived within a stylized empirical framework, which, as argued above, delivers a *lower* bound on the impact of parameter uncertainty on VaR measures. In qualitative terms, the result is robust to changes in the point estimates of the credit-risk parameters, $\widehat{PD}$ and $\hat{\rho}$, and in the size of the dataset, N and T.

A numerical example, reported in the top panel of Table 1, helps fix ideas. Consider the benchmark case in which an investor obtains point estimates $\widehat{PD} = 1\%$ and $\hat{\rho} = 20\%$ on the basis of panel data on (monthly) asset returns and (yearly) default rates that are observed for $T = 10$ years and $N = 200$ obligors. If the investor is interested in the portfolio VaR at the 99.9% confidence level and ignores estimation noise, he/she uses Eq. (3) and calculates a *naive* VaR that equals 14.55 cents on the dollar. However, Proposition 1 implies that estimation noise requires an "add-on" – on top of the naive VaR – that equals 3.93 cents on the dollar (see right-hand panel of Table 1). Thus, the *correct*VaR (14.55 + 3.93 = 18.48 cents) is 27% higher than the naive one.

A comparison between the top and bottom right-hand panels of Table 1 reveals that the correct VaR measure is less sensitive to changes in the parameter estimates than the naive VaR. Changing the PD estimate in the benchmark case to $\widehat{PD} = 5\%$ results in the correct VaR measure rising from 18.48 cents on the dollar to 43.37 (i.e. 38.44 + 4.93) cents, or by a factor of 2.35. At the same time the naive VaR rises by a factor of 2.64, from 14.55 to 38.44 cents on the dollar. Further, this difference in sensitivities to changing parameter estimates becomes more pronounced as the sample size (N and/or T) decreases and, thus, parameter uncertainty increases.

From a different perspective, reducing either the cross section of the data or its time series leads to higher uncertainty-induced VaR add-ons but the effect of the latter reduction is considerably more

[12] See Heitfield (2008) for a setup, in which it is necessary to make inference about nuisance parameters.

[13] See Forte and Peña (2009) and Tarashev and Zhu (2008b) for analyses of the information content of stock prices and CDS spreads.

Table 1
Impact of estimation noise on portfolio VaR (in per cent).

	Naive VaR	Noise in $\widehat{PD}$ only; add-ons to naive VaR			Noise in $\hat{\rho}$ only; add-ons to naive VaR			Noise in both $\widehat{PD}$ and $\hat{\rho}$: add-ons to naive VaR		
		$N = 50$	200	1000	$N = 50$	200	1000	$N = 50$	200	1000
$\widehat{PD} = 1\%$; $\hat{\rho} = 20\%$										
T = 5 years (= 60 months)	14.55	10.77	6.41	4.98	0.91	0.82	0.81	13.10	8.20	7.00
T = 10 years (= 120 months)		5.07	3.17	2.49	0.42	0.41	0.40	5.85	3.93	3.22
$\widehat{PD} = 5\%$; $\hat{\rho} = 20\%$										
T = 5 years (= 60 months)	38.44	8.92	6.71	5.17	1.94	1.79	1.77	11.37	9.08	7.36
T = 10 years (= 120 months)		4.71	3.71	2.90	0.97	0.89	0.88	5.97	4.93	3.96

Note: Results refer to the 99.9% VaR of an asymptotic homogeneous portfolio. Panel headings refer to particular point estimates of the probability of default, $\widehat{PD}$ and asset-return correlation, $\hat{\rho}$. Row headings, T, refer to the length of the time series underpinning $\widehat{PD}$ (years) and $\hat{\rho}$ (months). Column headings, N, refer to the number of obligors in the dataset underpinning $\widehat{PD}$ and $\hat{\rho}$.

important (see Table 1). Starting with the benchmark case and then cutting the length of the time series in half (to $T = 5$ years of data) more than doubles the VaR add-on (to 8.20 cents on the dollar). By contrast, decreasing the size of the cross section by a factor of four (to $N = 50$ obligors) raises the VaR add-on by 50% (to 5.85 cents).

The remainder of this section analyzes in some detail the impact of parameter uncertainty on the correct VaR and, in the process, provides explanations for the above results. The first subsection considers uncertainty stemming only from the estimation of the asset-return correlation, under the assumption that PD^* is observed directly. The advantage of focusing on this case stems from the existence of an analytical expression for the dependence of correlation uncertainty on the size of the dataset and the true parameter values. Then, the second subsection focuses on the impact of uncertainty about the PD, assuming that the asset-return correlation ρ^* is observed directly. Even though a measure of PD uncertainty can be obtained only numerically, there is a closed-form expression for the VaR that incorporates such a measure. The third subsection provides numerical results regarding the correct VaR when both ρ^* and PD^* need to be estimated. The last subsection proposes a closed-form approximation to this VaR.

4.1. Correlation uncertainty

Uncertainty about the asset-return correlation leads to a small VaR add-on. To parallel the above benchmark example, suppose that the investor knows with certainty that the true $PD^* = 1\%$ and obtains the point estimate $\hat{\rho} = 20\%$ on the basis of $T = 120$ months of data covering the asset returns of $N = 200$ obligors. As reported in the top middle panel of Table 1, the resulting VaR add-on is 0.41 cents on the dollar. This is roughly 2.8% of the naive VaR, which ignores parameter uncertainty altogether, and slightly more than 10% of the add-on that incorporates noise in both the correlation and PD estimates.

Further numerical results, reported in Table 2, allow for analyzing the importance of alternative drivers of correlation uncertainty.[14] In accordance with lessons learned from examining Table 1, the VaR add-on induced by noise in the correlation estimate depends little on the size of the cross-section, N, but is quite sensitive to the length of the sample period, T.

Heuristically, this result reflects the fact that, even though an increase of the sample size makes estimates converge to the true parameter values, correlation among observations "slows down" and might eventually halt the convergence. This convergence is underpinned by the Law of Large Numbers (LLN). The intuition behind LLN is that, when the sample size increases, there is a greater chance that estimation noise is "averaged out" and, as a result, point estimates become more precise. In the present context, however, LLN "works" fully in the time dimension (where there is serial independence)[15] but only up to a point in the cross section (where the common credit-risk factor leads to a positive correlation among exposures). Increasing either N or T increases the likelihood that obligor-specific noise is averaged out. As far as the common factor is concerned, however, this likelihood increases with T but *not* with N.

This intuition finds its concrete expression in the Cramer-Rao lower bound on the variance of the noise in correlation estimates (see Appendix C):

$$\sigma^2_\rho(\rho^*; N, T) = \frac{2(1-\rho^*)^2(1+(N-1)\rho^*)^2}{TN(N-1)}, \tag{8}$$

which decreases to 0 as $T \to \infty$ but to $2(1-\rho^*)^2(\rho^*)^2/T > 0$, as $N \to \infty$. This is illustrated in Figs. 1 and 2, which plot, respectively, the impact of increasing the time series and the cross-section of the data on $\sigma_\rho(\rho^*; N, T)$.

A comparison among the top, middle and bottom panels of Table 2 reveals that the correct VaR add-on (driven only by noise in the correlation estimate) increases as the point estimate of correlation increases within the considered range. If $N = 200$ and $T = 120$ months, for example, this add-on increases from 0.26 to 0.41 and then to 0.50 cents on the dollar as the point estimate, $\hat{\rho}$, increases from 10% to 20% and 30%. A key reason for this is that the variance of the noise in correlation estimates increases as the true correlation increases from 0% to (roughly) 50% (see Figs. 1 and 2). Concretely, by Eq. (8),

$$\frac{d\sigma^2_\rho(\rho^*; N, T)}{d\rho^*} > 0 \quad \text{for } \rho^* \in \left[0, 0.5\frac{N-2}{N-1}\right]. \tag{9}$$

This points to a pitfall in measuring VaR without taking into account the dependence of the size of estimation uncertainty on the true value of the correlation. If this dependence were ignored, the posterior density of correlation candidates would coincide with $f(\bullet|\rho^* = \hat{\rho})$, which is the PDF of the correlation estimator when the true parameter, ρ^*, happens to be at the point estimate, $\hat{\rho}$. In this case, the Bayesian updating procedure would be redundant and, thus, the computational burden would be substantially re-

[14] A "correct" add-on in Table 2 is one that incorporates fully correlation uncertainty but also reflects the assumption that the PD is known. Similar conventions are adopted in Tables 3 and 5.

[15] In real-life applications, risk factors should be expected to exhibit serial correlation, the importance of which has been recently brought to the fore by amendments to the Basel Capital Accord requiring institutions to derive charges for incremental risk (see Dunn, 2007). In the context of this paper, numerical simulations reveal that serial correlation of the common risk factor slows down the reduction in estimation noise brought about by a lengthening of the data time series. Ceteris paribus, the greater amount of estimation noise would reinforce the paper's main conclusion that parameter uncertainty has a material impact on the correct VaR.

Table 2
Impact of noise in the correlation estimate on portfolio VaR (in per cent).

	Naive	Sloppy add-on			Correct add-on			Alternative sloppy add-on			Alternative add-on		
		$N=50$	$N=200$	$N=1000$	$N=50$	$N=200$	$N=1000$	$N=50$	$N=200$	$N=1000$	$N=50$	$N=200$	$N=1000$
$\hat{\rho}=10\%; PD^*=1\%$													
T = 60 months	7.75	0.27	0.21	0.19	0.66	0.56	0.52	4.65	3.99	3.82	5.53	4.95	4.79
T = 120 months		0.14	0.11	0.10	0.32	0.26	0.25	3.11	2.69	2.58	3.53	3.16	3.06
$\hat{\rho}=20\%; PD^*=1\%$													
T = 60 months	14.55	0.40	0.34	0.34	0.91	0.82	0.81	8.59	7.97	7.81	9.79	9.31	9.18
T = 120 months		0.22	0.19	0.18	0.42	0.41	0.40	5.78	5.38	5.27	6.36	6.03	5.94
$\hat{\rho}=30\%; PD^*=1\%$													
T = 60 months	22.44	0.55	0.44	0.43	0.95	0.95	0.95	12.38	11.81	11.66	13.19	12.80	12.69
T = 120 months		0.32	0.25	0.24	0.50	0.50	0.50	8.38	8.01	7.91	8.83	8.54	8.47

Note: Results refer to the 99.9% VaR of an asymptotic homogeneous portfolio. Panel headings refer to a particular point estimate of the asset-return correlation, $\hat{\rho}$, and a particular true value of the probability of default, PD^*. Row headings, T, and column headings, N, refer, respectively, to the size of the time series and cross section underpinning $\hat{\rho}$. "Naive" = a VaR measure that abstracts from estimation noise. Add-ons are defined as follows: "sloppy" = the difference between (i) a VaR measure that ignores the dependence of estimation noise on the value of the true parameter and (ii) the naive VaR measure; "correct" = the difference between (i) the correct VaR measure (under the assumption that PD^* is observed directly) and (ii) the naive VaR measure; "alternative sloppy" = the difference between (i) an alternative VaR measure based on the 99.9th quantile of a probability distribution of correlation candidates that ignores the dependence of estimation noise on the true value of the asset-return correlation and (ii) the naive VaR measure; "alternative" = the difference between (i) an alternative VaR measure based on the 99.9th quantile of the true posterior distribution of correlation candidates and (ii) the naive VaR measure.

duced. However, as indicated by expression (9), a higher ρ^* is associated with greater dispersion in $f(\bullet|\rho^*)$. Thus, a given point estimate, $\hat{\rho}$, is more likely to be associated with $\rho^* > \hat{\rho}$ than with $\rho^* < \hat{\rho}$, suggesting that the posterior density of correlation candidates, $h(\bullet|\hat{\rho})$, would have a more pronounced right skew and a higher mean than $f(\bullet|\rho^*=\hat{\rho})$. When priors are uniform, this is indeed the case, irrespective of the fact that the estimator of the asset-return correlation is unbiased.

Thus, ignoring that estimation noise depends on the true parameter value leads to a mis-specified posterior distribution that attributes too much (little) probability mass to low (high) correlation candidates. In turn, since higher values of the asset-return correlation imply higher tail risk, this mis-specification leads to an understatement of the VaR perceived by the investor. The magnitude of such an understatement is illustrated in Table 2 by the difference between correct VaR add-ons and "sloppy VaR" add-ons, which are based on the (wrong) assumption that estimation uncertainty is independent of the true ρ^*. For all considered sample sizes (N, T) and values of the point estimate, $\hat{\rho}$, the former add-ons are roughly *twice* the size of the latter.

That said, when the point estimate of correlation increases, the VaR add-on induced by noise in this estimate *decreases* as a *share* of the naive VaR. Continuing with the above example, in which $N=200$ and $T=120$ months, the add-on decreases from 3.4% of the naive VaR when $\hat{\rho}=10\%$ to 2.2% of the naive VaR when $\hat{\rho}=30\%$. The flipside of this result is that the naive VaR, which abstracts from the noise in parameter estimates and, thus, overstates their information content, is more sensitive to changes in these estimates than the correct VaR (i.e. the naive VaR plus the add-on).

The underlying framework allows for a straightforward derivation of VaR^{alt} add-ons that focus directly on specific quantiles of the distribution of ρ^c and the common credit-risk factor, M (recall Eq. (5)). The two right-most panels of Table 2 quantify two versions of these add-ons. One of them arises when the dependence of estimation noise on the true parameter value is taken into account ("alternative add-on") and the other one when it is not ("alternative sloppy add-on"). As anticipated in Section 2.2, the alternative add-ons, ranging between 35% and 71% of the corresponding naive VaR, are substantially larger than the correct add-ons.

4.2. PD uncertainty

When one considers solely uncertainty about the probability of default, the analytic challenges are the reverse of these encountered in the previous subsection. Namely, the Cramer-Rao lower bound on the variance of the noise in PD estimates does not exist in closed form (see Appendix C) but, for a given value of this lower bound, the VaR measure itself does. In order to see the latter implication, let the true correlation be observed without noise ($\hat{\rho}=\rho^*$) and recall that the common credit-risk factor, M, and the default point, $\Phi^{-1}(PD^c)$, are jointly normal (and mutually independent). By expression (2), this means that the investor effectively faces a *single common risk factor*, which equals $\Phi^{-1}(PD^c) - \sqrt{\hat{\rho}}M$ and has a normal distribution. As a result, the ASRF model is applicable, implying that the portfolio VaR at the $(1-\alpha)$ confidence level equals:

$$VaR(\widehat{PD}, \hat{\rho}) = \Phi\left(\frac{\mu_D(\widehat{PD}, \hat{\rho}) - \sqrt{\hat{\rho} + \sigma_D^2(\widehat{PD}, \hat{\rho})}\,\Phi^{-1}(\alpha)}{\sqrt{1-\hat{\rho}}}\right), \tag{10}$$

where $\mu_D(\widehat{PD}, \hat{\rho})$ and $\sigma_D^2(\widehat{PD}, \hat{\rho})$ denote the mean and variance of the posterior distribution of candidate default points. Expression (10) leads to the VaR add-ons reported in the left-hand panels of Table 1 as well as in Table 3.

A comparison between the left-hand and middle panels of Table 1 points to three important differences between the impact of PD uncertainty on the VaR perceived by the investor and the corresponding impact of correlation uncertainty. First, the VaR add-on induced by uncertainty in the PD estimate is much higher. If, for example, $\rho^*=20\%$ and the point estimate $\widehat{PD}=1\%$ is obtained from data covering $N=200$ obligors over $T=10$ years, the correct add-on is 3.17 cents on the dollar, or 22% of the naive VaR. This add-on is almost eight times larger than the corresponding add-on induced by correlation uncertainty.

The difference between these add-ons is a natural consequence of the fact that the uncertainty about the probability of default is greater than the uncertainty about the asset-return correlation. In turn, the relative size of the two types of uncertainty is mostly driven by the maintained assumption that there are fewer data points for default rates than for asset returns (see Section 3.1).[16] For an implication of this assumption, suppose that the true parameter values are $\rho^*=20\%$ and $PD^*=1\%$, and the sample is of size $N=200$, $T=10$. In this case, the Cramer-Rao lower bound on the standard deviation of the PD estimate is 0.47%, i.e. almost half of the true parameter value (see the second panel of Table

[16] Another important assumption in this context is that asset returns are observed directly. This assumption is relaxed in Section 5.2.

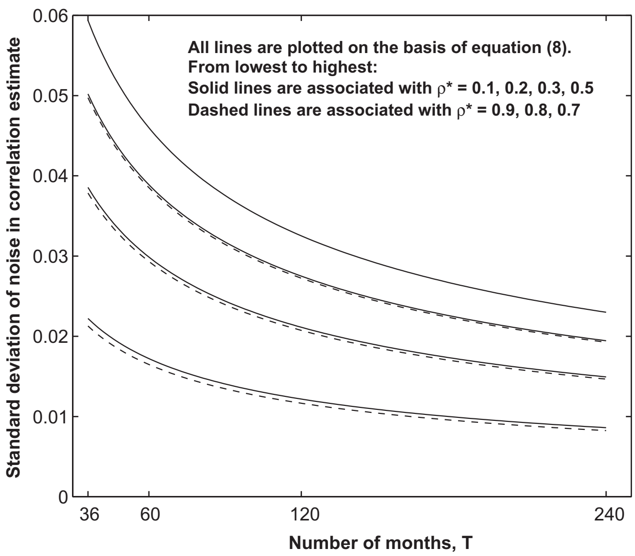

Fig. 1. Benefit of lengthening the time series (for $N = 200$ firms).

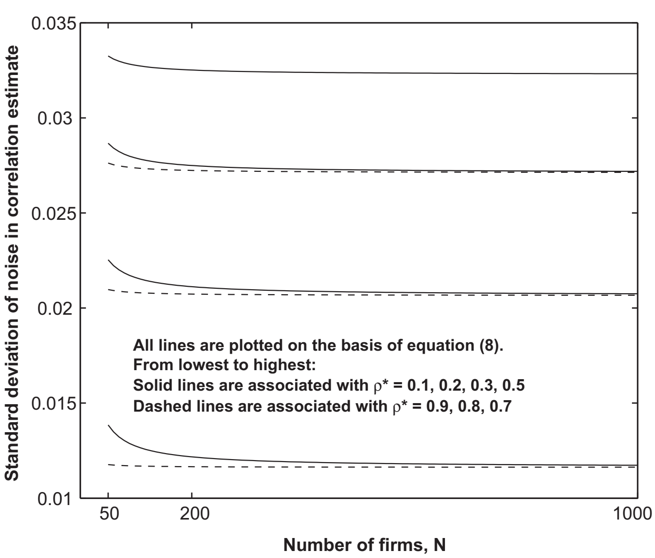

Fig. 2. Benefit of increasing the cross section (for $T = 120$ months).

4). By contrast, the Cramer-Rao lower bound on the standard deviation of the correlation estimate is 2.1%, which is slightly more than one-tenth of the true parameter value (see Fig. 2).

The second noteworthy difference from the case of correlation uncertainty is that the VaR add-on induced by PD uncertainty is more sensitive to changes in the size of the cross section. For example, Table 3 reports that, when $\rho^* = 20\%, \widehat{PD} = 1\%$ and $T = 10$, reducing the cross section of the dataset from $N = 200$ to 50 obligors increases the VaR add-on by 60% (from 3.17 to 5.07 cents on the dollar). By contrast, the corresponding change is virtually nil in the context of correlation uncertainty (recall Table 2). Furthermore, raising N from 200 to 1000 obligors depresses the VaR add-on associated with PD uncertainty by a non-negligible 21%. That said, at 2.49 cents on the dollar or 17% of the naive VaR, the VaR add-on remains substantial even when $N = 1000$, i.e. when there are data on all the exposures in a virtually asymptotic portfolio.[17]

Third, only the analysis of PD uncertainty provides examples that a rise in the point estimate, which increases the naive VaR, can depress the correct add-on. If, for example, $N = 50$ and the time series covers $T = 5$ years, a rise in the point estimate from $\widehat{PD} = 1\%$ to $\widehat{PD} = 5\%$ leads to a decline in the add-on from 10.77 to 8.92 cents on the dollar (see the left-hand panels of Table 1). This is another manifestation of the fact that, since it accounts for noise in parameter estimates, the correct VaR is less sensitive to changes in these estimates than the naive VaR, which abstracts from estimation noise. This phenomenon is more pronounced in the context of PD uncertainty because this uncertainty is greater than that about the correlation.

Besides these three differences, there are a number of qualitative similarities between the effects of PD and correlation uncertainty on the correct VaR. In particular, a comparison between Tables 2 and 3 reveals similar consequences of: (i) changing the time span of the data, (ii) ignoring the dependence of estimation noise on the true parameter value, and (iii) focusing directly on specific quantiles of the credit- and estimation-risk factors.

4.3. Combining the two sources of uncertainty

The interaction between noise in the estimator of the asset-return correlation and noise in the PD estimator inflates the VaR perceived by the investor. In order to see what drives this result, it is useful to step back and recall the sequential estimation procedure. As outlined in Section 3.3, the investor first derives the posterior distribution of candidate values for the correlation coefficient. Then, conditioning on such a candidate value and the observed default rates, the investor obtains a point estimate of the PD.

Under the adopted empirical framework and for the considered numerical examples, a higher correlation candidate induces a higher point estimate of the PD. In turn, this positive relationship imputes a positive correlation in the joint posterior distribution of the candidate values ρ^c and PD^c, which acts to increase the correct VaR (refer to Table 1). In the benchmark example, where the point estimates are $\widehat{PD} = 1\%$, $\hat{\rho} = 20\%$ and there are $T = 10$ years of data on $N = 200$ obligors, the correct VaR add-on (3.93 cents on the dollar) is 10% larger than the sum of the two add-ons associated, respectively, with noise only in the PD or the correlation estimate (3.17 + 0.41 = 3.58 cents on the dollar). Across the other examples in the table, this discrepancy tends to range between 7% and 14%.

4.4. A closed-form approximation of the VaR measure

An important by-product of the analysis is the finding that the closed-form approximate measure in expression (10), which incorporates only uncertainty about the PD, goes a long away in accounting for the correct VaR. This is clearly seen by referring to Table 1 and constructing VaR measures by summing the corresponding naive VaRs and the add-ons reported in the left- and right-hand panels. For the point estimates and sample sizes considered for this table, VaR measures that incorporate only PD uncertainty tend to understate the corresponding correct VaRs by 5% or less.

Importantly, the closed-form expression in (10) has two general computational advantages. First, it reduces the computational burden by limiting the Bayesian inference procedure to a single parameter, PD^*. Second, for a given posterior distribution of PD candidates – and, thus, of candidate default points – this expression delivers a VaR measure directly, without necessitating numerical simulations.

5. Extensions

This section considers two extensions of the estimation procedure, which address aspects of the investor's information set that

[17] Tarashev and Zhu (2008a) demonstrate that a portfolio of 1000 homogeneous exposures can be safely treated as perfectly fine-grained (or asymptotic).

Table 3
Impact of noise in the PD estimate on portfolio VaR (in per cent).

	Naive	Sloppy add-on			Correct add-on			Alternative sloppy add-on			Alternative add-on		
		$N = 50$	$N = 200$	$N = 1000$	$N = 50$	$N = 200$	$N = 1000$	$N = 50$	$N = 200$	$N = 1000$	$N = 50$	$N = 200$	$N = 1000$
$\rho^* = 20\%; \widehat{PD} = 1\%$													
T = 5 years	14.55	5.56	3.34	2.48	10.77	6.41	4.98	31.57	23.78	20.09	39.07	28.07	23.43
T = 10 years		2.92	1.73	1.29	5.07	3.17	2.49	22.04	16.40	13.91	25.11	18.40	15.40
$\rho^* = 20\%; \widehat{PD} = 5\%$													
T = 5 years	38.44	6.38	4.82	4.25	8.92	6.71	5.17	30.77	27.02	25.48	32.52	27.96	24.91
T = 10 years		3.31	2.48	2.18	4.71	3.71	2.90	22.62	19.68	18.49	23.76	20.63	18.50

Note: Results refer to the 99.9% VaR of an asymptotic homogeneous portfolio. Panel headings refer to a particular point estimate of the probability of default, $\widehat{PD}$, and a particular true value of the asset-return correlation, ρ^*. Row headings, T, and column headings, N, refer, respectively to the size of the time series and cross section underpinning $\widehat{PD}$. "Naive" = a VaR measure that abstracts from estimation noise. Add-ons are defined as follows: "sloppy" = the difference between (i) a VaR measure that ignores the dependence of estimation noise on the value of the true parameter and (ii) the naive VaR measure; "correct" = the difference between (i) the correct VaR measure (under the assumption that ρ^* is observed directly) and (ii) the naive VaR measure; "alternative sloppy" = the difference between (i) an alternative VaR measure based on the 99.9th quantile of a probability distribution of PD candidates that ignores the dependence of estimation noise on the true value of the probability of default and (ii) the naive VaR measure; "alternative" = the difference between an alternative VaR measure based on the 99.9th quantile of the true posterior distribution of PD candidates and (ii) the naive VaR measure.

the analysis has so far abstracted from. The first subsection illustrates how richer prior beliefs about credit-risk parameters can affect the VaR measure. Then, the second subsection shows that noise in observed asset returns complicates materially the inference procedure and that such noise can be addressed in a computationally efficient way.

5.1. Prior beliefs about probability of default

The results reported in Section 4 are based on diffuse priors regarding the asset-return correlation and the PD. The benefit of assuming such priors is that they render transparent the transition from the conditional PDF of the estimator to the posterior PDF of parameter candidates (see Eqs. (6) and (7)). That said, an investor's information about a parameter may go beyond the information contained in the dataset that he/she uses in order to obtain a point estimate of this parameter. In other words, the prior belief may not be diffuse, which could affect the parameter's posterior distribution and, consequently, the perceived VaR in important ways.

Consider an investor who has a short forecast horizon and is interested in the VaR of a portfolio of homogeneous obligors that have a particular credit rating. Suppose further that the investor observes a rather long time series of historical 1-year default rates in the same rating class. Given the investor's short horizon and assuming rather frequent, albeit persistent, changes in the (average) PD within the rating class, the investor might derive a posterior distribution of PD candidates that relies more heavily on recent default rates. In the light of Eq. (7), one way of doing this is to let all of the observed default rates determine prior beliefs but to derive a point estimate of the PD only on the basis of the last several default rates.

The extent to which prior beliefs affect the VaR measure depends on the extent to which they are in accordance with the point estimate. To take a concrete example, suppose that the investor bases his/her prior on the 1-year default-rates of all corporate obligors rated BB by Moody's from 1990 to 2007. The average of these default rates is 0.98% and their standard deviation is 1.18%.[18] Suppose further that the investor obtains a point estimate $\widehat{PD} = 1\%$ on the basis of default data covering $N = 200$ BB-rated obligors over $T = 5$ years and observes directly the true correlation $\rho^* = 20\%$. Since, in this example, the prior mean and the point estimate are quite close, the main effect of the prior is to tighten the posterior distribution of PD candidates relative to that implied by a diffuse prior.

[18] The data source is the Credit Risk Calculator database of Moody's Investors Service.

Table 4
Errors in PD estimators (Cramer-Rao lower bounds on the standard deviation of unbiased estimators, in per cent).

	$N = 50$ obligors	$N = 200$ obligors	$N = 1000$ obligors
$\rho^* = 10\%; PD^* = 1\%$			
T = 5 years	0.76	0.51	0.41
T = 10 years	0.54	0.36	0.29
T = 20 years	0.38	0.26	0.21
$\rho^* = 20\%; PD^* = 1\%$			
T = 5 years	0.88	0.67	0.57
T = 10 years	0.62	0.47	0.40
T = 20 years	0.44	0.33	0.28
$\rho^* = 30\%; PD^* = 1\%$			
T = 5 years	1.02	0.81	0.69
T = 10 years	0.72	0.57	0.49
T = 20 years	0.51	0.41	0.35
$\rho^* = 10\%; PD^* = 5\%$			
T = 5 years	2.04	1.63	1.50
T = 10 years	1.44	1.15	1.06
T = 20 years	1.02	0.82	0.75
$\rho^* = 20\%; PD^* = 5\%$			
T = 5 years	2.56	2.20	2.06
T = 10 years	1.81	1.56	1.46
T = 20 years	1.28	1.10	1.03
$\rho^* = 30\%; PD^* = 5\%$			
T = 5 years	2.94	2.54	2.11
T = 10 years	2.08	1.79	1.49
T = 20 years	1.47	1.27	1.05

Note: Panel headings indicate the true values of credit-risk parameters.

Being tantamount to less parameter uncertainty, this results in a VaR add-on that equals 3.25 cents on the dollar, down from 6.41 cents under a diffuse prior (refer to Table 3).

Importantly, the result could be quite different if, keeping all else constant, the investor's dataset covered B-rated corporate obligors. A point estimate $\widehat{PD} = 1\%$ for B-rated obligors is admittedly extreme but not unreasonable, given that the default rate of such obligors averaged 1.01% between 2003 and 2007. However, such a point estimate would be quite at odds with a prior based on the default history of B-rated corporate obligors since 1990: default rates in this history average 5% and have a standard deviation of 4%. Against such a prior, the point estimate appears overly optimistic. Consequently, the implied posterior distribution attributes more probability mass to high PD candidates than a posterior based on a diffuse prior. Not surprisingly then, the end result is a VaR add-on of 9.15 cents on the dollar, up from 6.41 cents under a diffuse prior.

The above two illustrative examples indicate that departures from the assumption of diffuse priors can lead to substantial changes in the correct VaR. These changes may stem from an improved precision of the information set, as most clearly seen in the example with BB-rated obligors, or from an alignment of posterior beliefs with long-term default experience, as seen in the example with B-rated obligors. That said, a rigorous analysis of how prior beliefs are determined would have to be based explicitly on an intertemporal learning process, which is beyond the scope of this paper.

5.2. Noise in observed asset returns

In real-life applications, asset returns – or, more generally, the stochastic drivers of default events – would be observed with noise. Depending on whether the noise is idiosyncratic (driven by obligor-specific imperfections in the measurement of assets' market value) or systematic (a result, for example, of mapping equity prices of different obligors into corresponding asset values via the same mis-specified model), it could lead to a downward or upward bias in correlation estimates.

To see why, generalize (1) to account for observation noise, which is denoted by $U_{i\tau}$:

$$\begin{aligned}
\ln(V_{i,\tau}) - \ln(V_{i,\tau-1}) &= \mu^*\Delta + \sigma^*\sqrt{\Delta}\sqrt{1-\psi^*}\left(\sqrt{\rho^*}M_\tau + \sqrt{1-\rho^*}Z_{i,\tau}\right) \\
&\quad + \sigma^*\sqrt{\Delta}\sqrt{\psi^*}U_{it},
\end{aligned}$$
$$U_{i\tau} = \sqrt{\lambda^*}M_\tau^U + \sqrt{1-\lambda^*}Z_{i\tau}^U,$$
$$M_\tau^U \sim N(0,1) \text{ is independent of } Z_{i\tau}^U \sim N(0,1),$$
$$\psi^* \in [0,1], \quad \lambda^* \in [0,1],$$

where ψ^* and λ^* control, respectively, the amount of observation noise and its systematic component. Then note that the true correlation of *observed* asset returns equals

$$\rho^{*,OBS} = (1-\psi^*)\rho^* + \lambda^*\psi^*, \tag{11}$$

where ρ^* continues to indicate the true correlation of *actual* asset returns. All else constant, $\rho^{*,OBS}$ increases in the systematic component of the observation noise, λ^*. The level of $\rho^{*,OBS}$ also increases in the overall amount of noise, ψ^*, if the systematic component is large enough, i.e. if $\lambda^* > \rho^*$.

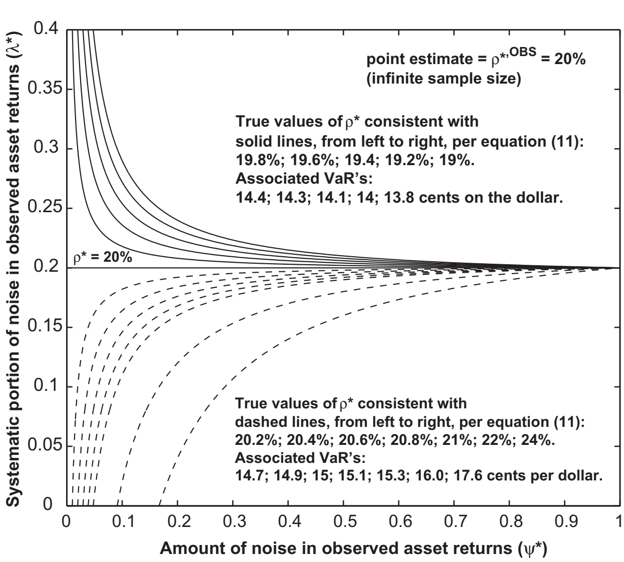

Fig. 3. Iso-correlation curves.

An important implication of (11) is that, if the data comprise only noisy observations of asset returns, the true correlation of actual asset returns is *un*identifiable. Namely, for any given $\rho^{*,OBS}$, one can pick any $\rho^* \in [0,1]$ and find a *continuum* of different pairs (λ^*, ψ^*) that render $\rho^{*,OBS}$ and ρ^* mutually consistent. This is illustrated in Fig. 3.

Given that the asset-return correlation is unidentifiable, an investor could impose a conservative upward adjustment on the point estimate $\hat{\rho}$ and treat the adjusted value as if it were coming from data that are free of observation noise. In the light of Eq. (11), a natural conservative adjustment would be one consistent with no systematic observation noise ($\lambda^* = 0$) and some idiosyncratic observation noise ($\psi^* > 0$). As illustrated by Table 5, such an approach results in significant upward revisions of the VaR measure. Suppose, for example, that the true probability of default $PD^* = 1\%$ is known and the correlation is estimated at $\hat{\rho} = 20\%$ on the basis of data that cover $N = 200$ obligors over $T = 120$ months. Then, setting $\psi^* = 0.05$ (i.e. allowing idiosyncratic observation noise to account for 5% of the variability of observed asset returns) leads to a VaR add-on that is almost three times as high as the add-on obtained for $\psi^* = 0$. However, in order to match the corresponding add-on induced by PD uncertainty – i.e. the add-on obtained when $N = 200, T = 10$ years, $\widehat{PD} = 1\%$ and $\rho^* = 20\%$ is known – it is necessary to set $\psi^* = 0.15$ (recall Table 3).

If one is to incorporate asset-return observation noise in the closed-form VaR approximation (10), it is necessary to abstract from the estimation noise studied in Section 4.1. In other words, it is necessary to treat the point estimate $\hat{\rho}$ as if it were equal to $\rho^{*,OBS}$. Then, Eq. (11) allows to map $\rho^{*,OBS}$ into a correlation of actual asset returns, ρ^*, which can be used directly in (10). Of course, the mapping and, thus, the VaR measure will be affected by the parameterization of the idiosyncratic and systematic observation noise. The effect of different parameterizations is illustrated in the "$T = \infty$" rows in Table 5, which contain VaR add-ons that are based on infinite time series of data and, thus, on perfect knowledge of $\rho^{*,OBS}$.

The discussion in this subsection has so far abstracted from data on default rates. In principle, this is not innocuous because, when asset returns are observed with noise, such data do provide useful information about the asset-return correlation. In the context of the adopted empirical framework, however, using data on default rates in order to make inference about the asset-return correlation would be of little value and would be associated with substantial computational burden. Here is why.

First, the correlation estimate based solely on default data is extremely imprecise, especially if the obligors are of moderate to high credit quality. For example, given a true $PD^* = 1\%$ and a true correlation $\rho^* = 20\%$, default data covering $N = 200$ obligors over $T = 10$ years lead to noise in the most efficient unbiased estimator of ρ^* that has a standard deviation of 12.5 percentage points. To put this into perspective, note that: (i) realistic values of the (average) asset-return correlation are between 5% and 45% and (ii) the standard deviation of a uniform random variable with support from 5% to 45% equals 11.55%.

Second, a VaR measure that incorporates explicitly inference about observation noise in asset returns would face the so-called "curse of dimensionality" . To see why, recall that the VaR measures discussed in Section 4 are based on posterior distributions that condition on two parameter estimates: $\hat{\rho}$ and $\widehat{PD}$. In the presence of observation noise, properly constructed posterior distributions would need to condition on two additional estimates: those of the observation noise parameters ψ^* and λ^*.[19] Since the number

[19] This is because the amount of estimation noise in correlation and PD estimates depends on the *values* of the noise parameters φ^* and ψ^*.

Table 5
Impact of noise in observed asset returns on portfolio VaR (in per cent).

	Naive	Correct add-on									
		N = 50 obligors					N = 200 obligors				
		$\psi^* = 0.0$	$\psi^* = 0.05$	$\psi^* = 0.1$	$\psi^* = 0.15$	$\psi^* = 0.20$	$\psi^* = 0.00$	$\psi^* = 0.05$	$\psi^* = 0.10$	$\psi^* = 0.15$	$\psi^* = 0.20$
$\hat{\rho} = 10\%; PD^* = 1\%$											
T = 60 months	7.75	0.66	1.00	1.44	1.90	2.42	0.56	0.90	1.32	1.79	2.33
T = 120 months		0.32	0.68	1.06	1.53	2.01	0.26	0.64	1.03	1.46	1.95
$T = \infty$		0	0.33	0.71	1.13	1.61	0	0.33	0.71	1.13	1.61
$\hat{\rho} = 20\%; PD^* = 1\%$											
T = 60 months	14.55	0.91	1.71	2.70	3.78	5.06	0.82	1.68	2.59	3.71	4.93
T = 120 months		0.42	1.26	2.19	3.22	4.42	0.41	1.21	2.13	3.22	4.37
$T = \infty$		0	0.78	1.66	2.65	3.80	0	0.78	1.66	2.65	3.80
$\hat{\rho} = 30\%; PD^* = 1\%$											
T = 60 months	22.44	0.95	2.43	4.01	5.88	8.21	0.95	2.39	4.00	5.86	8.09
T = 120 months		0.50	1.84	3.45	5.27	7.39	0.51	1.86	3.40	5.22	7.37
$T = \infty$		0	1.35	2.89	4.66	6.71	0	1.35	2.89	4.66	6.71

Note: The value of ψ^* captures the amount of systematic noise in observed asset returns (refer to Eq. (11) and set $\lambda^* = 0$). See Table 2 for further explanation.

of simulations grows exponentially in the number of parameters that the inference procedure is applied to (recall Section 3.3), doubling the latter number would be prohibitively burdensome.

6. Conclusion

This paper has analyzed the impact of parameter uncertainty on the VaR perceived by an investor in a homogeneous asymptotic credit portfolio. The main conclusion is that this impact is strong for a wide range of portfolio characteristics and for a wide range of dataset sizes. As a useful by-product, the analysis has delivered an approximate VaR measure, which exists in closed form and is, thus, computationally convenient. This measure accounts for PD uncertainty and, given a judicious adjustment for noise in observed asset returns, could approximate well the correct VaR measure.

Relaxing some of the assumptions adopted by this paper would provide for fruitful directions of future research. One is to address rigorously the issue of parameter heterogeneity in the context of credit VaR measures. Given that deriving and simulating the joint posterior distribution of a large number of heterogeneous parameters is likely to impose an insurmountable computational burden, it is important to establish conditions under which it is justifiable to focus solely on noise in the estimator of a representative (e.g. the average) parameter. Another possible direction of future research is to consider cyclical developments in credit conditions, which make credit-risk parameters change over time and, all else equal, impair the estimates of these parameters.

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Appendix A. Proof of Proposition 1

Proposition 1. *Denote the correct VaR of the portfolio at the* $(1-\alpha)$ *confidence level by* $Q_{1-\alpha}(L_n)$. *Define* $X \equiv \{M, PD^c, \rho^c\}$. *The expectation* $E(L_n|X)$ *does not depend on n and can be denoted by* $\widetilde{\Phi}(X)$, *where* $\widetilde{\Phi}(\bullet)$ *is an analytic function. Denote the* $(1-\alpha)$ *quantile of the distribution of* $\widetilde{\Phi}(X)$ *by* $Q_{1-\alpha}(\widetilde{\Phi}(X))$. *As* $n \to \infty, Q_{1-\alpha}(L_n) - Q_{1-\alpha}(\widetilde{\Phi}(X)) \to 0$.

Proof. The proof relies on three aspects of the model:

Aspect 1: The assumed homogeneity of parameters across exposures implies that the following holds trivially:

$$\lim_{n\to\infty} E(L_n|X) = E(L|X) = \Phi\left(\frac{\Phi^{-1}(PD^c) - \sqrt{\rho^c}M}{\sqrt{1-\rho^c}}\right) \equiv \widetilde{\Phi}(X). \tag{12}$$

Aspect 2: The expected loss $E(L|X=x) \equiv \widetilde{\Phi}(x)$ changes continuously in x.
Aspect 3: The PDF of X is well-defined and continuous everywhere on its support.

By the law of iterated expectations, it follows that

$$\lim_{n\to\infty} \Pr(L_n \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X))) = \lim_{n\to\infty} \int_{\mathbb{R}^3} \Pr(L_n \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X))|X=x)dF(x), \tag{13}$$

where $F(x)$ is the CDF of X.

In addition, Proposition 1 in Gordy (2003), which is applicable owing to Aspects 1–3 above, implies that $L_n|x \to \widetilde{\Phi}(x)$ almost surely as $n \to \infty$.

Given this implication, the fact that $F(\bullet)$ is absolutely continuous (by Aspects 2 and 3) and that $\Pr(L_n \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X))|X=\bullet)$ is bounded between 0 and 1, the dominated convergence theorem applies (see Billingsley, 1995). Thus:

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^3} \Pr(L_n \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X))|X=x)dF(x) \\
&= \int_{\mathbb{R}^3} \lim_{n\to\infty} \Pr(L_n \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X))|X=x)dF(x) \\
&= \int_{\mathbb{R}^3} \Pr(\widetilde{\Phi}(x) \leqslant Q_{1-\alpha}(\widetilde{\Phi}(x)))dF(x) \\
&= \Pr(\widetilde{\Phi}(X) \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X))) = 1-\alpha,
\end{aligned} \tag{14}$$

where the second equality follows from Proposition 1 in Gordy (2003). The third equality in expression (14) follows from (12). In turn, the fourth equality is a result of the fact that $\Pr(\widetilde{\Phi}(x) \leqslant \bullet)$ is continuous.

Combining expressions (13) and (14), it follows that:

$$\lim_{n\to\infty} Q_{1-\alpha}(L_n) \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X)).$$

Repeating the steps from expression (13), it follows that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\lim_{n\to\infty} \Pr(L_n \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X)) - \varepsilon) = (1-\alpha) - \delta.$$

Thus,

$$Q_{1-\alpha}(\widetilde{\Phi}(X)) - \varepsilon \leqslant \lim_{n\to\infty} Q_{1-\alpha}(L_n) \leqslant Q_{1-\alpha}(\widetilde{\Phi}(X)).$$

Since ε can be arbitrarily close to 0, the proof is complete. □

Appendix B. Parameterization of PDFs

This appendix outlines the parameterization of the densities in (6) and (7).

The estimates of the asset-return correlation and default probability are assumed to be delivered by *minimum-variance unbiased* estimators. Moreover, given candidate parameter values, ρ^c and PD^c, the standard deviations of the noise in the point estimates, $\hat{\rho}$ and $\widehat{PD}(\rho^c)$, are set equal to the respective *Cramer-Rao lower bounds*, denoted by $\sigma_\rho(\rho^c)$ and $\sigma_{PD}(PD^c, \rho^c)$ and derived in Appendix C. Since the Fischer information inequality, which is behind the Cramer-Rao lower bounds, specifies only the first two moments of minimum-variance unbiased estimators, the rest of the densities of the correlation and PD estimators still remains to be parameterized. This is done in the following two paragraphs.

The posterior of correlation candidates, $h(\bullet|\hat{\rho})$ is as implied by (6) under a particular parameterization of the prior, $g(\bullet)$, and the conditional density of the correlation estimator $f(\bullet|\rho^c)$. Specifically, the prior is assumed to be uniform (or diffuse): $g(\rho^c) = 1$ for $\rho^c \in [0, 1]$. In turn, $f(\bullet|\rho^c)$ is parameterized as a beta PDF with a mean ρ^c and variance $\sigma^2_\rho(\rho^c) : f(\bullet|\rho^c) = beta(\bullet; A, B), A \equiv \rho^c/(1-\rho^c)B$ and $B \equiv (1-\rho^c)(\rho^c(1-\rho^c) - \sigma^2_\rho(\rho^c))/\sigma^2_\rho(\rho^c)$.

The conditional density of the PD estimator, $\tilde{f}(\bullet|PD^c, \rho^c)$, and the associated posterior, $\tilde{h}(\bullet|\widehat{PD}(\rho^c), \rho^c)$, satisfy simultaneously the following three criteria. First, the mean and variance implied by the density $\tilde{f}(\bullet|PD^c, \rho^c)$ equal PD^c and $\sigma^2_{PD}(PD^c, \rho^c)$, respectively. Second, the relationship between $\tilde{f}(\bullet|PD^c, \rho^c)$ and $\tilde{h}(\bullet|\widehat{PD}(\rho^c), \rho^c)$ is as implied by expression (7) under a uniform prior, $\tilde{g}(PD^c) = 1$ for $PD^c \in [0, 1]$. Third, the implied posterior distribution of candidate default points $\Phi^{-1}(PD^c)$ is normal.

Appendix C. Deriving Cramer-Rao lower bounds

Given that the data-generating process is as specified in (1) and (2) and the dataset is as described in Section 3.1, the log likelihood of asset returns on a particular date $t \in \{1, 2, \ldots, T\}$ is simply the log-likelihood of N jointly-normal random variables, with mean μ^*, standard deviation σ^* and correlation ρ^c. Denote this log-likelihood by $LL^{ar}(\{V_{it}\}_{i=1}^N; \theta^*)$, where $\theta^* \equiv \{\mu^*, \sigma^*, \rho^c\}$. Then, the Fischer information matrix equals:

$$I(\theta^*) = E\left(\frac{\partial^2 LL^{ar}\left(\{V_{it}\}_{i=1}^N; \theta\right)}{\partial\theta^2}\bigg|_{\theta=\theta^*}\right).$$

When asset returns are serially uncorrelated, the Cramer-Rao lower bound on the variance of the correlation estimator is the (3,3) element of $I^{-1}(\theta^*)/T$:

$$\sigma^2_\rho(\rho^c) = \frac{2(1-\rho^c)^2(1+(N-1)\rho^c)^2}{TN(N-1)}.$$

In turn, the date-t log-likelihood of defaults is :

$$LL^{dr}\left(\{d_{it}\}_{i=1}^N; PD^c, \rho^c\right) = \log \int \Phi\left(\frac{\Phi^{-1}(PD^c) - \sqrt{\rho^c}M}{\sqrt{1-\rho^c}}\right)^{D_t} \times \Phi\left(\frac{\sqrt{\rho^c}M - \Phi^{-1}(PD^c)}{\sqrt{1-\rho^c}}\right)^{N-D_t} \phi(M)dM, \tag{15}$$

where $d_i = 1$ if obligor i defaults and $d_i = 0$ otherwise; $D_t = \sum_{i=1}^N d_{it}$. Then, conditional on the correlation candidate ρ^c, the Cramer-Rao lower bound on the noise in the estimate of PD^c is given by

$$\sigma^2_{PD}(PD^c, \rho^c) = 1/T \cdot E\left(\frac{d^2 LL^{dr}\left(\{d_{it}\}_{i=1}^N; \pi, \rho^c\right)}{d\pi^2}\bigg|_{\pi=PD^c}\right).$$

The value of this lower bound has to be derived via numerical simulations.

The Cramer-Rao lower bounds $\sigma^2_\rho(\rho^c)$ and $\sigma^2_{PD}(PD^c, \rho^c)$ are affected neither by the values of μ^* and σ^* nor by the fact that μ^* has to be estimated. However, these lower bounds are inflated because σ^* has to be estimated.

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