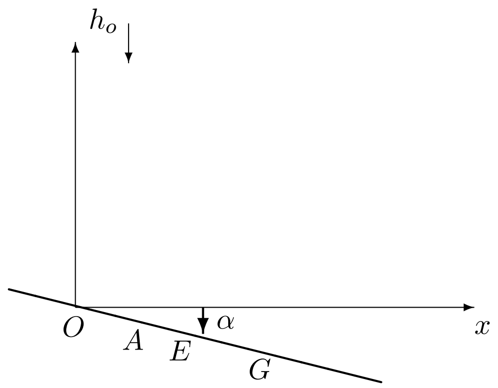
Aeroelasticity Course – Exercise $n^o 2$ – Academic Year 2015–2016

Consider the typical section model illustrated below, where

$$
\begin{aligned}
A &= \text{aerodynamic center (quarter-chord point)} \\
E &= \text{elastic center} \\
G &= \text{mass center} \\
O &= \text{origin of the frame of reference} \\
h_o &= \text{vertical displacement of the point O (downward positive)} \\
\alpha &= \text{angle of attack (clockwise positive)}
\end{aligned}
$$



The linearized Lagrange equations of motion are

$$
\begin{bmatrix} m & m x_G \\ m x_G & m x_G^2 + J_G \end{bmatrix} \begin{Bmatrix} \ddot{h}_o \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} k_h & k_h x_E \\ k_h x_E & k_\alpha + k_h x_E^2 \end{bmatrix} \begin{Bmatrix} h_o \\ \alpha \end{Bmatrix} = \begin{Bmatrix} -\mathcal{L} \\ \mathcal{M}_A - \mathcal{L} x_A \end{Bmatrix}, \tag{1}
$$

where

$$
\begin{aligned}
m &= \text{mass per unit span length} \\
J_G &= \text{principal moment of inertia with respect to G per unit span length} \\
k_h,\ k_\alpha &= \text{constants of the bending and torsional springs} \\
x_A,\ x_E,\ x_G &= \text{abscissas of the aerodynamic, elastic and mass centers} \\
\mathcal{L} &= \text{lift (upward positive)} \\
\mathcal{M}_A &= \text{aerodynamic moment with respect to A (clockwise positive)} \\
\rho &= \text{density of air} \\
b &= \text{half chord} \\
U &= \text{free stream velocity}
\end{aligned}
$$

For the sake of simplicity, assume that A is the origin of the frame of reference ($x_A = 0$). In order to obtain the non-dimensional equations of motion, divide the first equation of the system (1) by the force per unit span length $m b \omega_\alpha^2$ ($\omega_\alpha^2 = k_\alpha / J_\alpha$)[1] and the second equation by the moment per

[1] J_α is the moment of inertia with respect to E.

unit span length $mb^2\omega_\alpha^2$. Introduced the non-dimensional quantities

$$
\begin{aligned}
h &= \frac{h_o}{b}\ , \qquad \hat{t} = \omega_\alpha t\ , \qquad a = \frac{m}{\pi\rho b^2}\ , \qquad \hat{U} = \frac{U}{b\omega_\alpha}\ , \\
\xi_E &= \frac{x_E}{b}\ , \quad \xi_G = \frac{x_G}{b}\ , \quad r_\alpha^2 = \frac{J_\alpha}{mb^2}\ , \quad \Omega^2 = \frac{\omega_h^2}{\omega_\alpha^2}\ ,
\end{aligned}
\tag{2}
$$

and having assumed steady aerodynamics (according to Glauert theory for thin airfoils one has $\mathcal{L} = q\ 2b\ 2\pi\alpha$ and $\mathcal{M}_A = 0$, where $q = \rho U^2/2$), one obtains

$$
\begin{bmatrix} 1 & \xi_G \\ \xi_G & r_\alpha^2 - \xi_E^2 + 2\xi_E\xi_G \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \Omega^2 \begin{bmatrix} 1 & \xi_E \\ \xi_E & \xi_E^2 + r_\alpha^2/\Omega^2 \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} + \hat{U}^2 \begin{bmatrix} 0 & 2/a \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \mathbf{0}\ . \tag{3}
$$

The system parameters are:

$$
a = 5\ , \quad \Omega = 0.5\ , \quad \xi_E = 0.30\ , \quad \xi_G = 0.45\ , \quad r_\alpha^2 = 0.25\ .
$$

Answer to the following questions:

- Compute the non-dimensional divergence velocity $\hat{U}_D$ (static instability);
- Derive the non-dimensional equations of motion in the case of quasi-steady aerodynamics (assume $\mathcal{L} = q\ 2b\ 2\pi\hat{\alpha}$ and $\mathcal{M}_A = 0$, where $\hat{\alpha}$ is the dynamic incidence $\hat{\alpha} = \alpha + \dot{h}/U$). Introduce also a structural damping represented by the matrix $2\mu\Omega\mathbf{I}$ (take $\mu = 0.005$) applied on the vector $\{\dot{h}, \dot{\alpha}\}^{\mathrm{T}}$;
- Study the linear stability of the system with respect to the parameter $\hat{U}$ by means of the root locus in the cases of steady and quasi-steady aerodynamics. For both cases, identify the flutter velocity $\hat{U}_F$ and the associated reduced frequency $k_F = \omega_F/\omega_\alpha$;
- Compute the eigenvalues s_k and eigenvectors $\tilde{\mathbf{u}}^{(k)}$ of the system written in first-order form for $\hat{U} = 0.8\ \hat{U}_F$ and $\hat{U} = 1.2\ \hat{U}_F$ (use the flutter velocity estimated from the root locus in the case of quasi-steady aerodynamics). Assumed the initial conditions $h = 0.01,\ \alpha = 0,\ \dot{h} = 0,\ \dot{\alpha} = 0$, obtain the aeroelastic free response as:

  $$
  \mathbf{y}(t) = \sum_{k=1}^{4} c_k\ \tilde{\mathbf{u}}^{(k)}\ e^{s_k t}\ , \tag{4}
  $$

  where $\mathbf{y}(t) = \{h(t), \alpha(t), \dot{h}(t), \dot{\alpha}(t)\}^{\mathrm{T}}$ is the state vector;
- Assuming steady aerodynamics, study the linear stability of a general aviation aircraft in steady flight at sea level having the following characteristics: rectangular wing of chord $c = 1.5\,m$, half-wing span $l = 4\,m$, and half-wing mass $M = 70\,Kg$, and half-wing moment of inertia with respect to the line of the mass centers $J_g = 0.25\,Kgm^2$; mass centers at $c/3$ (from the leading edge) and elastic centers at half-chord; first uncoupled bending natural frequency $f_1 = 8\,Hz$ and first uncoupled torsional natural frequency $f_2 = 16\,Hz$.