Aeroelasticity Course – Exercise $n^o 1$ – Academic Year 2015–2016

Consider the governing equation of a simply-supported infinite plate in supersonic flow:

$$EI\left(1+\zeta\frac{\partial}{\partial t}\right)w^{IV}+Nw^{II}+\lambda w^{I}+kw=-\bar{\rho}\ddot{w}-\mu\dot{w}+p(x,t)\;, \tag{1}$$

where EI is the bending stiffness, ζ is the modal structural damping, N is the axial load, λ is an aerodynamic coefficient given by Ackeret theory, μ is an aerodynamic damping, k is the constant associated with the elastic floor, $\bar{\rho}$ is the linear mass density, and $p(x,t)$ is an external load.

The continuous model is discretized in space using the orthogonal eigenfunctions of the structural operator $L(\bullet):=EI\partial^4(\bullet)/\partial x^4+N\partial^2(\bullet)/\partial x^2+k(\bullet)$. These are defined as the functions $\phi_n(x)$, satisfying the given boundary conditions, such that $L\phi_n(x)=\lambda_n\phi_n$, where λ_n are the corresponding eigenvalues and $\omega_n^2:=\lambda_n/\bar{\rho}$ are the squares of the natural (angular) frequencies of vibration. In the present case, one has $\phi_n(x)=\sin(n\pi x/l)$, where l is the panel length in the flow direction.

Thus, the analytic solution of the problem can be written as:

$$w(x,t)=\sum_{n=1}^{\infty}w_n(t)\phi_n(x)\;, \tag{2}$$

where the components $w_n(t)$ are unequivocally determined. Applying Galerkin method (*i.e.*, projecting Eq. (1) on the generic eigenfunction $\phi_m(x)$), one obtains: [1]

$$\begin{aligned}\sum_{n=1}^{\infty}\left\{\lambda_n w_n(t)+\left[\zeta EI\left(\frac{n\pi}{l}\right)^4+\mu\right]\dot{w}_n(t)+\bar{\rho}\ddot{w}_n(t)\right\}\frac{l}{2}\,\delta_{nm}+\lambda w_n(t)\left\langle\phi_n^I(x),\phi_m(x)\right\rangle &= \\ =\langle p(x,t),\phi_m(x)\rangle\;,&\end{aligned} \tag{3}$$

where $\lambda_n=EI\,(n\pi/l)^4-N\,(n\pi/l)^2+k$. Furthermore, if the summation is extended up to a finite number of terms N and a skew-symmetric matrix of elements a_{nm} is introduced such that:

$$a_{nm}:=\frac{\left\langle\phi_n^I(x),\phi_m(x)\right\rangle}{l/2}=\frac{2}{l}\int_0^l\frac{n\pi}{l}\sin\frac{m\pi x}{l}\cos\frac{n\pi x}{l}dx\equiv-a_{mn}\;, \tag{4}$$

one obtains for the case $N=2$ (two modes):

$$\left\{\begin{array}{c}\ddot{w}_1\\ \ddot{w}_2\end{array}\right\}+\left[\begin{array}{cc}\zeta_1&0\\0&\zeta_2\end{array}\right]\left\{\begin{array}{c}\dot{w}_1\\ \dot{w}_2\end{array}\right\}+\left(\left[\begin{array}{cc}\omega_1^2&0\\0&\omega_2^2\end{array}\right]+\Lambda\left[\begin{array}{cc}0&1\\-1&0\end{array}\right]\right)\left\{\begin{array}{c}w_1\\ w_2\end{array}\right\}=\left\{\begin{array}{c}p_1\\ p_2\end{array}\right\}\;, \tag{5}$$

where

$$\begin{aligned}\zeta_n&:=\left[\zeta EI(n\pi/l)^4+\mu\right]/\bar{\rho}\;,\\ \omega_n^2&:=\left[EI(n\pi/l)^4-N(n\pi/l)^2+k\right]/\bar{\rho}\;,\\ \Lambda&:=\lambda a_{12}/\bar{\rho}\;,\\ p_m&:=\langle p(x,t),\phi_m(x)\rangle/(\bar{\rho}l/2)\;.\end{aligned} \tag{6}$$

The following data are given:

$$f_1=15\,Hz\;,\qquad f_2=30\,Hz\;,\qquad \zeta_1=0.03\,s^{-1}\;,\qquad \zeta_2=0.01\,s^{-1}\;.$$

[1] Defined the internal product between functions as $\langle a(x),b(x)\rangle:=\int_0^l a(x)\,b(x)\,dx$, one has, because of orthogonality, $\langle\phi_n(x),\phi_m(x)\rangle=l/2\,\delta_{nm}$.

Answer to the following questions:

- Study the linear stability ($p(x,t) = 0$) of the system with respect to the parameter Λ by means of the root locus in the following cases:
  - No damping ($\zeta_1 = \zeta_2 = 0$);
  - Given damping parameters ($\zeta_1 > \zeta_2$);
  - Varying ζ_2 (consider the case $\zeta_2 > \zeta_1$, *e.g.*, $\zeta_2 = 0.05\, s^{-1}$, and $\zeta_2 = \zeta_1 = 0.03\, s^{-1}$).
- The components of the free response in the case of complex conjugates eigenvalues s_n, $s_n^\star$ are written as

$$\begin{Bmatrix} w_1(t) \\ w_2(t) \end{Bmatrix} = \sum_{n=1}^{2} c_n \begin{Bmatrix} \tilde{w}_1^{(n)} \\ \tilde{w}_2^{(n)} \end{Bmatrix} e^{s_n t} + C.C. \,,$$

  where $N = 2$ is the number of eigenvalue pairs. The quantities $\tilde{w}_1^{(n)}$ and $\tilde{w}_2^{(n)}$ are the components of the eigenvector associated with s_n, whereas c_n are complex constants depending on the prescribed initial conditions.

  Evaluate the free response of the system ($p(x,t) = 0$) when $\Lambda < \Lambda_F$ (*e.g.*, $\Lambda = 0.9\Lambda_F$), where Λ_F is the stability margin. Assume the following initial conditions:

$$\begin{aligned} w(x,0) &= \sin\left(\frac{\pi x}{l}\right) \,, \\ \dot{w}(x,0) &= 0 \,. \end{aligned}$$

  Compare the results with those that would be obtained (with the same initial conditions) with air-free and damping-free vibrations ($\Lambda = \zeta_1 = \zeta_2 = 0$).
- Discuss the undamped free response ($\zeta_1 = \zeta_2 = 0$) for $\Lambda \to \Lambda_F$.
- **Optional I:** Study the driven response (initial conditions equal to zero) to the impulsive gust load ($P_0 = 1$):

$$\frac{p(x,t)}{\bar{\rho} l/2} = P_0\, \delta(t) \,.$$

- **Optional II:** Is it possible to find suitable initial conditions such that the free response does not diverge when $\Lambda > \Lambda_F$ (*e.g.*, $\Lambda = 1.1\Lambda_F$)? If it exists, describe the set of such initial conditions. If these initial conditions exist, can be still considered the system a *unstable* system as well?