Aeroelasticity Course – Exercise n^o3 – Academic Year 2015–2016

Consider the Lagrange equation of motion for a typical section (see Exercise n^o2) and assume the following formulas for the lift $\mathcal{L}$ and pitching moment $\mathcal{M}_A$ about the quarter-chord point (Theodorsen theory):[1]

$$\begin{aligned} \mathcal{L} &= \pi\rho b^2\left(\ddot{h}_0 + \frac{1}{2}b\ddot{\alpha} + U\dot{\alpha}\right) + 2\pi\rho U b C(k)\left(\dot{h}_0 + b\dot{\alpha} + U\alpha\right) , && (1) \\ \mathcal{M}_A &= -\pi\rho b^3\left(\frac{1}{2}\ddot{h}_0 + \frac{3}{8}b\ddot{\alpha} + U\dot{\alpha}\right) . && (2) \end{aligned}$$

Equations (1) and (2) define the unsteady aerodynamic operator for a thin and straight airfoil in the case $x_A = 0$ (origin of the frame of reference on the quarter-chord point). In Equation (1),

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + jH_0^{(2)}(k)} , \tag{3}$$

is Theodorsen complex function,[2] where $k = \omega\, b/U$ is the reduced frequency, $H_n^{(2)} = J_n - jY_n$ ($n = 0, 1$) is the Hänkel function of the second kind and order n, whereas J_n and Y_n are the Bessel functions of the first and second kind, respectively, and order n.

Once Eqs. (1) and (2) are inserted into the Lagrange equations of motion, these are made dimensionless as done in Exercise n^o2 but now introducing the following non-dimensional quantities (the non-dimensional time and Fourier/Laplace variables are defined in a different way):

$$\begin{aligned} &h = \frac{h_o}{b} , \quad \hat{t} = \frac{tU}{b} , \quad a = \frac{m}{\pi\rho b^2} , \quad \hat{U} = \frac{U}{b\omega_\alpha} , \\ &\xi_E = \frac{x_E}{b} , \quad \xi_G = \frac{x_G}{b} , \quad r_\alpha^2 = \frac{J_\alpha}{mb^2} , \quad \Omega^2 = \frac{\omega_h^2}{\omega_\alpha^2} . \end{aligned} \tag{4}$$

Hence, one obtains in Laplace domain (the non-dimensional Laplace variable is $p := sb/U$)

$$\begin{aligned} &\left\{ p^2\hat{U}^2 \begin{bmatrix} \left(1+\frac{1}{a}\right) & \left(\xi_G + \frac{1}{2a}\right) \\ \left(\xi_G + \frac{1}{2a}\right) & \left(r_\alpha^2 - \xi_E^2 + 2\xi_E\xi_G + \frac{3}{8a}\right) \end{bmatrix} + p\hat{U}^2 \begin{bmatrix} 0 & \frac{1}{a} \\ 0 & \frac{1}{a} \end{bmatrix} + \begin{bmatrix} \Omega^2 & \xi_E\Omega^2 \\ \xi_E\Omega^2 & \left(\xi_E^2\Omega^2 + r_\alpha^2\right) \end{bmatrix} + \right. \\ &\left. +\hat{U}^2 \begin{bmatrix} \frac{2}{a}\hat{C}(p)p & \frac{2}{a}\hat{C}(p) + \frac{2}{a}\hat{C}(p)p \\ 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} \tilde{h} \\ \tilde{\alpha} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} , \end{aligned} \tag{5}$$

where $\tilde{h}$ and $\tilde{\alpha}$ denote the Laplace transforms of h and α, respectively, and the function $\hat{C}(p)$ ($p := s\, b/U = \alpha + jk$) is the analytic continuation of Theodorsen complex function $C(k)$ in all the complex plane, obtained by setting $C(k) = \hat{C}(p)$ and using p in place of jk.

[1] If one takes $C(k) = 1$ in Theodorsen theory (this value approximates $C(k)$ for $k \to 0$) and neglects the contributions given by the second-order time-derivatives (aerodynamic added mass and moment of inertia), a quasi-steady aerodynamic model follows. Furthermore, if all the unsteady terms are neglected, one obtains the steady lift and pitching moment predicted by Glauert theory and discussed in Exercise n^o2.

[2] Note that part of Eqs. (1) and (2) is written in time domain, whereas Theodorsen complex function is in frequency domain. Although all terms should be written in frequency domain, classical textbooks on aeroelasticity (cfr. "Aeroelasticity", R. L. Bisplinghoff, H. Ashley, R. L. Halfman, Addison-Wesley publishing company inc., 1955) use such representation.

The system parameters are the same than in Exercise $n^o 2$:

$$a = 5\ , \quad \Omega = 0.5\ , \quad \xi_E = 0.3\ , \quad \xi_G = 0.45\ , \quad r_\alpha^2 = 0.25\ .$$

Answer to the following questions:

- Compare the exact Theodorsen complex function $C(k)$ written as a combination of Bessel functions (Eq. (3)) with the rational polynomial Padé approximation

$$\hat{C}(p) \ = \ \frac{1}{2} \frac{(p + 0.135)(p + 0.651)}{(p + 0.0965)(p + 0.4555)} \tag{6}$$

  restricted to the imaginary axis ($p = jk$).

- Consider characteristic equation of the system (5) written as

$$\hat{U}^2 = f(k)\ .$$

  To compute the stability margin, plot the real and imaginary parts of $f(k)$. The critical value of the reduced frequency k_F is such that the imaginary part of $f(k)$ vanishes. Then, the real part of $f(k)$ gives $\hat{U}_F^2$. Compare the stability margins obtained using the exact Theodorsen function (Eq. (3)) and Padé approximation (Eq. (6));

- Take the inverse Laplace transform of the system (5) using Eq. (6) for Theodorsen complex function and rewrite the system in first-order form (six first-order ODEs) by introducing an added aerodynamic state β.[3] Obtain the root locus and evaluate the flutter ($\hat{U}_F$) and divergence ($\hat{U}_D$) velocities;

- Obtain the root locus in the case of quasi-steady aerodynamics (set $C(k) = 1$ and neglect the second-order time derivatives in Eqs. (1) and (2)). Identify $\hat{U}_F$ and $\hat{U}_D$;

- Using the system written in time domain with the added aerodynamic state, compute the aeroelastic free response by means of the eigenvalues and eigenvectors of the state matrix for $\hat{U} = 0.8\,\hat{U}_F$ and $\hat{U} = 1.2\,\hat{U}_F$. The initial conditions are the same than in Exercise $n^o 2$ (consider $\beta_0 = \dot{\beta}_0 = 0$);

- Compare the present results with Exercise $n^o 2$. In particular, study the linear stability of the given general aviation aircraft now assuming unsteady aerodynamics.

[3] Develop the polynomial division in Eq. (6) and associate the added aerodynamic state only to the proper part of the rational function.