17 February 2022 18:58

CORREZIONE SCRITTO 21/02/2022

ESERCIZIO 2:

(1)

$$Q(\bar{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_3 - 4x_2x_4$$

Deduciamo, se φ forma bilineare associata a Q, e $\mathcal{B}$ base canonica,

$$A := M^{\mathcal{B}}(\varphi)$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

Determiniamo segnatura

$$P_\lambda(A) = \begin{vmatrix} 1-\lambda & 0 & -1 & 0 \\ 0 & 1-\lambda & 0 & -2 \\ -1 & 0 & 1-\lambda & 0 \\ 0 & -2 & 0 & 1-\lambda \end{vmatrix} \leftarrow$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1-\lambda & -2 \\ -1 & 0 & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= \left[(1-\lambda)^2 - 1 \right] \left[(1-\lambda)^2 - 4 \right] = -\lambda (2-\lambda)(-1-\lambda)(3-\lambda)$$

autovalori $-1, 0, 2, 3$ segnatura $(2,1)$ INDEFINITA, DEGENERE

(2) $\ker \varphi = N \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} = \mathcal{L} \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$

Inoltre

$$V_{-1} = N \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix} = \mathcal{L} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$V_2 = N \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 \\ -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix} = \mathcal{L} \left(\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right)$$

$$V_3 = N \begin{pmatrix} -2 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 \\ -1 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix} = \mathcal{L} \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right)$$

$0 \quad -2 \quad 0 \quad -2$

Allora, nella base $\mathcal{B}' = \left\{ \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}_{\bar{v}_1}, \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}}_{\bar{v}_2}, \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}}_{\bar{v}_3}, \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}}_{\bar{v}_4} \right\}$

si ha

$$M^{\mathcal{B}'}(\varphi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

È UNA FORMA CANONICA

Allora $Q(\sqrt{2}\,\bar{v}_2 + \bar{v}_3) = 2\underbrace{Q(\bar{v}_2)}_{=-1} + \underbrace{Q(\bar{v}_3)}_{=2} + 2\sqrt{2}\,\underbrace{\varphi(\bar{v}_2, \bar{v}_3)}_{=0}$

$= 0$

$\Rightarrow \boxed{\sqrt{2}\,\bar{v}_2 + \bar{v}_3} \in I_\varphi$ ma non sta in $\ker\varphi = \mathcal{L}(\bar{v}_1)$

$$\sqrt{2}\,\bar{v}_2 + \bar{v}_3 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1\\-1/\sqrt{2}\\1\end{pmatrix}$$

③ Chiamiamo $P_k(\bar{x}) = Q(\bar{x}) + k x_1^2$

$$A_k = M^{\beta}(P_k) = \begin{bmatrix} 1+k & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

Se $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ autovettore di A relativo a -1, notiamo che

$$(0,1,0,1) A_k \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = (0,1,0,1) \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} = -2 < 0 \qquad \forall k$$

Quindi $\nexists$ k t.c. P_k semidefinita positiva

$\boxed{\text{Nota (fuori programma)}}$: Invece, se $Q_k(\bar{x}) = Q(\bar{x}) + k x_2^2$

Allora

$$C_k := M^{\beta}(Q_k) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1+k & 0 & -2 \\ -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$P_{Q_k}(\lambda) = (1-\lambda) \begin{vmatrix} 1+k-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1+k-\lambda & -2 \\ -1 & 0 & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 \begin{vmatrix} 1+k-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1+k-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix}$$

$$= \underbrace{\left[(1-\lambda)^2 - 1 \right]}_{\lambda(\lambda-2)} \underbrace{\left((1+k-\lambda)(1-\lambda) - 4 \right)}_{\lambda^2-(2+k)\lambda+k-3}$$

Per $\boxed{k=3}$

$$P_{C_u}(\lambda) = \lambda(\lambda-2)\left[\lambda^2 - 5\lambda\right] = \lambda^2(\lambda-2)(\lambda-5)$$ semidefinita positiva

(iv) Nella base $\mathcal{B} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4\}$ abbiamo

$$M^{\mathcal{B}}(\varphi) = \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}$$

Sia $W = \mathcal{L}(\bar{v}_2, \bar{v}_3)$

$$W^{\perp_\varphi} = \left\{ \bar{x} = x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 + x_4\bar{v}_4 : \begin{array}{l} (x_1\, x_2\, x_3\, x_4) \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 0 \\ (x_1\, x_2\, x_3\, x_4) \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0 \end{array} \right\}$$

$= $ Soluzioni di $\; -x_2 = 2x_3 = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$$W^{\perp_q} = \mathcal{L}(\bar{v}_1, \bar{v}_4)$$

$$(W^{\perp_q})^{\perp_q} = \left\{ \bar{x} = x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 + x_4\bar{v}_4 : \overbrace{\begin{matrix} (x_1,x_2,x_3,x_4) \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \\ (x_1,x_2,x_3,x_4) \begin{bmatrix} 0 & & & \\ & -1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \end{matrix}}^{\text{sempre}} \right\}$$

$$\{\bar{x} : x_4 = 0\} = \mathcal{L}(\bar{v}_1, \bar{v}_2, \bar{v}_3) \neq W$$

(5)

$$C = \{(x,y) : x^2 + x^2 + y^2 + y^2 - 2xy - 4xy + 7\sqrt{2}\,x - 3\sqrt{2}\,y = 0\}$$

Equazione matriciale

$$(x,y) \underbrace{\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} + \sqrt{2}\,(7,-3) \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$0 = \begin{vmatrix} 2-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 9 = (2-\lambda-3)(2-\lambda+3) = (-1-\lambda)(5-\lambda)$$

Autovalori $\lambda = -1$ e $\lambda = 5$

$$V_{-1} = \mathcal{L}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$V_5 = \mathcal{L}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \qquad \Rightarrow \qquad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$V_5 = \mathcal{L}\left(\begin{pmatrix}1\\-1\end{pmatrix}\right)$ $\qquad$ V_2 $|-1 \ 1)$

Da cui $\quad \Delta = {}^tPAP = \begin{bmatrix}5 & 0\\0 & -1\end{bmatrix}$

Nuove coordinate $\begin{pmatrix}X\\Y\end{pmatrix}$ date da $\begin{pmatrix}x\\y\end{pmatrix} = P\begin{pmatrix}X\\Y\end{pmatrix}$

L'equazione diventa

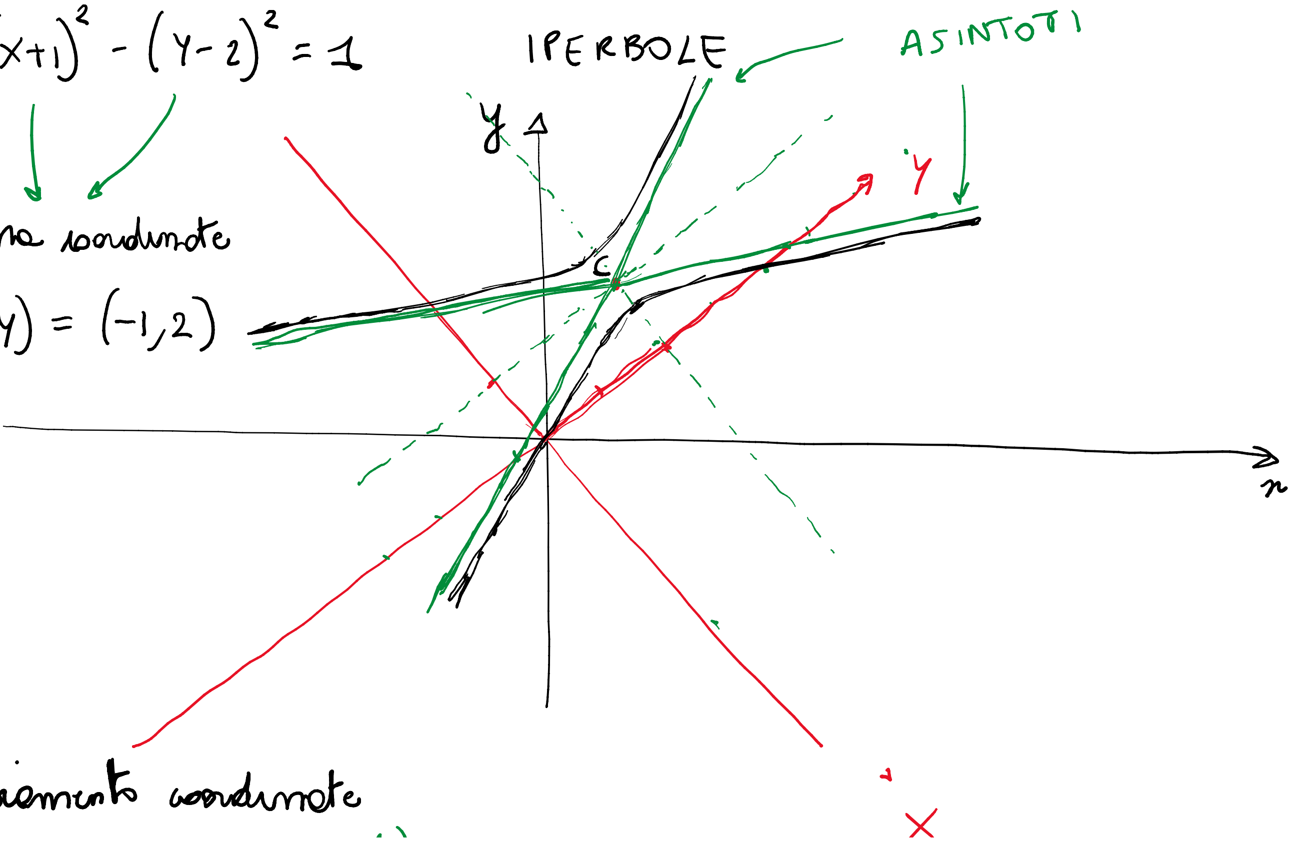
$$5X^2 - Y^2 + \cancel{\sqrt{2}}\,(7,-3)\cdot\frac{1}{\cancel{\sqrt{2}}}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix} = 0$$

$$5X^2 - Y^2 + 10X + 4Y = 0$$

$$5\left[(X+1)^2 - 1\right] - \left[(Y-2)^2 - 4\right] = 0 \quad \Rightarrow \text{Nuove coordinate } \begin{cases} X' = X+1 \\ Y' = Y-2 \end{cases} \quad (\otimes)$$

$$5(X+1)^2 - (Y-2)^2 = 1$$

C ha coordinate $(X,Y) = (-1,2)$



Cambiamento coordinate

Cambiamento coordinate

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \overset{(0)}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \times$$

ESERCIZIO 1:

$$f(\bar v) = \Big[\bar v \wedge \overbrace{(\bar\imath + \bar\jmath)}^{\bar a} \cdot \overbrace{(\bar\imath + \bar k)}^{\bar b}\Big]\bar\imath + \Big[\bar v \wedge \overbrace{(\bar\imath + \bar k)}^{=\bar b} \cdot \overbrace{(\bar\imath + \bar\jmath)}^{\bar a}\Big]\bar\jmath$$

$$+ \Big[\bar v \cdot (-\bar\imath + \bar\jmath - 2\bar k)\Big]\bar k$$

$$\boxed{\begin{matrix} \bar v, \bar w \in V_3 \\ \lambda, \mu \in \mathbb{R} \end{matrix}}$$

(1) $$f(\lambda \bar v + \mu \bar w) = \Big[(\lambda \bar v + \mu \bar w) \wedge \bar a \cdot \bar b\Big]\bar\imath$$

$$+ \Big[(\lambda \bar v + \mu \bar w) \wedge \bar b \cdot \bar a\Big]\bar\jmath$$

$$+ \Big[(\lambda \bar v + \mu \bar w) \cdot (-\bar\imath + \bar\jmath - 2\bar k)\Big]\bar k$$

$$= \Big[\lambda\, \bar v \wedge \bar a \cdot \bar b + \mu\, \bar w \wedge \bar a \cdot \bar b\Big]\bar\imath$$

$$+ \Big[\lambda\, \bar v \wedge \bar b \cdot \bar a + \mu\, \bar w \wedge \bar b \cdot \bar a\Big]\bar\jmath$$

$$+ \Big[\lambda\, \bar v \cdot (-\bar\imath + \bar\jmath - 2\bar k) + \mu\, \bar w \cdot (-\bar\imath + \bar\jmath - 2\bar k)\Big]\bar k$$

$$= \lambda\Big(\big[\bar v \wedge \bar a \cdot \bar b\big]\bar\imath + \big[\bar v \wedge \bar b \cdot \bar a\big]\bar\jmath + \big[\bar v \cdot (-\bar\imath + \bar\jmath - 2\bar k)\big]\bar k\Big)$$

$$+ \mu\Big(\big[\bar w \wedge \bar a \cdot \bar b\big]\bar\imath + \big[\bar w \wedge \bar b \cdot \bar a\big]\bar\jmath + \big[\bar w \cdot (-\bar\imath + \bar\jmath - 2\bar k)\big]\bar k\Big)$$

$$= \lambda f(\bar v) + \mu f(\bar w)$$

Da cui f è lineare

(2)

$$f(\bar\imath) = \Big[\overbrace{\bar\imath \wedge (\bar\imath+\bar\jmath)}^{\bar k} \cdot (\bar\imath+\bar k)\Big]\bar\imath + \Big[\overbrace{\bar\imath \wedge (\bar\imath+\bar k)}^{-\bar\jmath} \cdot (\bar\imath+\bar\jmath)\Big]\bar\jmath + \Big[\bar\imath \cdot (-\bar\imath+\bar\jmath-2\bar k)\Big]\bar k$$

$$= \bar\imath - \bar\jmath - \bar k \qquad [f(\bar\imath)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$f(\bar\jmath) = \Big[\overbrace{\bar\jmath \wedge (\bar\imath+\bar\jmath)}^{-\bar k} \cdot (\bar\imath+\bar k)\Big]\bar\imath + \Big[\overbrace{\bar\jmath \wedge (\bar\imath+\bar k)}^{-\bar k+\bar\imath} \cdot (\bar\imath+\bar\jmath)\Big]\bar\jmath + \Big[\bar\jmath \cdot (-\bar\imath+\bar\jmath-2\bar k)\Big]\bar k$$

$$= -\bar\imath + \bar\jmath + \bar k \qquad [f(\bar\jmath)]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$f(\bar k) = \Big[\overbrace{\bar k \wedge (\bar\imath+\bar\jmath)}^{\bar\jmath-\bar\imath} \cdot (\bar\imath+\bar k)\Big]\bar\imath + \Big[\overbrace{\bar k \wedge (\bar\imath+\bar k)}^{\bar\jmath} \cdot (\bar\imath+\bar\jmath)\Big]\bar\jmath + \Big[\bar k \cdot (-\bar\imath+\bar\jmath-2\bar k)\Big]\bar k$$

$$= -\bar\imath + \bar\jmath - 2\bar k \qquad [f(\bar k)]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

Da cui

$$A := M^{\mathcal{B}}(f) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

(3)

$$\ker f = \{\bar v \text{ t.c. } [\bar v]_{\mathcal{B}} \in N(A)\}$$

$$N(A) = \{\text{soluzioni di } \overset{X\in\mathbb{R}^3}{AX=0}\}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{Risolvo} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{L}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \mathcal{L}\Big(\underbrace{\tfrac{1}{\sqrt2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\text{Norma } 1}\Big)$$

$$\Rightarrow \ker f = \mathcal{L}(\bar\imath+\bar\jmath) \qquad \text{base ortonormale: } \frac{\bar\imath+\bar\jmath}{\sqrt2}$$

$$\operatorname{Im} f = \{ \bar v \in V_3 : [\bar v]_{\mathcal{B}} \in \mathcal{C}(A) \}$$

$$\mathcal{C}(A) = \mathcal{L}\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right)$$

$C_3 \to C_3/-3$

$$\xrightarrow{\substack{C_2 \to C_2 + C_1 \\ C_3 \to C_3 + C_1}} = \mathcal{L}\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \right) = \mathcal{L}\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Base ortonormale

$\bar v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ha già norma 1

$$\bar v_2' = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\|^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\bar v_2 = \frac{\bar v_2'}{\|\bar v_2'\|} = \frac{1}{\sqrt 2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

base ortonormale $= \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt 2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

Cioè base ortonormale per $\operatorname{Im} f$ è $\left\{ \bar k, \frac{\bar i - \bar j}{\sqrt 2} \right\}$

(4)
$$f^{-1}(\bar k) = \{ \bar v \in V_3 : f(\bar v) = \bar k \}$$

$$= \left\{ \bar v \in V_3 : X = [\bar v]_{\mathcal{B}} \text{ risolve } AX = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{= [\bar k]_{\mathcal{B}}} \right\}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3+R_1]{R_2 \mapsto R_2+R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

$$z = -\frac{1}{3} \qquad x = z+y = -\frac{1}{3}+y$$

$$\text{Soluzioni} = \left\{ \begin{pmatrix} -\frac{1}{3}+y \\ y \\ -\frac{1}{3} \end{pmatrix} : y \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -1/3 \\ 0 \\ -1/3 \end{pmatrix} + \underbrace{y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{N(A)} : y \in \mathbb{R} \right\}$$

$$\Rightarrow f^{-1}(\bar{k}) = \left\{ -\frac{1}{3}(\bar{\imath}+\bar{k}) + \bar{w} : \bar{w} \in \ker f \right\}$$

(NON È UN SOTTOSPAZIO VETTORIALE)

Invece

$f^{-1}(\mathcal{L}(\bar{k}))$ è un sottospazio, e

$$f^{-1}(\mathcal{L}(\bar{k})) = \{ \bar{v} \in V_3 : f(\bar{v}) = \lambda \bar{k} \quad \text{per un } \lambda \in \mathbb{R} \}$$

$$\text{Ma } f^{-1}(\lambda \bar{k}) = \left\{ -\frac{1}{3}\lambda(\bar{\imath}+\bar{k}) + \bar{w} : \bar{w} \in \ker f \right\}$$

$$\Rightarrow f^{-1}(\mathcal{L}(\bar{k})) = \left\{ \bar{v} \in V_3 : \bar{v} = -\frac{1}{3}\lambda(\bar{\imath}+\bar{k}) + \bar{w} \text{ per un } \lambda \in \mathbb{R}, \bar{w} \in \ker f \right\}$$

$$= \mathcal{L}(\bar{\imath}+\bar{k}) + \ker f$$

$$= \mathcal{L}(\bar{\imath}+\bar{k}, \bar{\imath}+\bar{\jmath})$$

$$f^{-1}(\operatorname{Im} f) = V_3 \quad \text{(direttamente dalla definizione di } \operatorname{Im} f)$$

$f^{-1}(\mathrm{Im} f) = V_3$ (direttamente dalla definizione di $\mathrm{Im} f$)

(5)

A simmetrica $\Rightarrow$ f autoaggiunto $\Rightarrow$ f diagonalizzabile

(e $\mathcal{B} = \{\vec{\imath}, \vec{\jmath}, \vec{k}\}$ ortonormale)

$$P_A(\lambda) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & 1 \\ -1 & 1 & -2-\lambda \end{vmatrix}$$

$$\overset{R_3 \to R_3 - R_2}{=} \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & 1 \\ 0 & \lambda & -3-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1 \end{vmatrix} - (3+\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= -\lambda(\cancel{1} - \lambda \cancel{-1}) - (3+\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= \lambda^2 - (3+\lambda)\lambda(\lambda - 2) = \lambda\left[\lambda - (3+\lambda)(\lambda-2)\right]$$

$$= \lambda\left[\cancel{\lambda} - \lambda^2 \cancel{-\lambda} + 6\right] = -\lambda(\lambda - \sqrt{6})(\lambda + \sqrt{6})$$

autovalori $\lambda_1 = 0$ $\quad m_a(0) = 1$

$\lambda_2 = \sqrt{6}$ $\quad m_a(\sqrt{6}) = 1$

$\lambda_3 = -\sqrt{6}$ $\quad m_a(-\sqrt{6}) = 1$

Dalla teoria sappiamo allora che le molteplicità geometriche

sono $m_g(0) = 1$

$m_g(\sqrt{6}) = 1$

siccome

$1 \leq m_g(\lambda) \leq m_a(\lambda)$

$m_g(\sqrt{6}) = 1$ ricorda

$1 \leq m_g(\lambda) \leq m_a(\lambda)$

$m_g(-\sqrt{6}) = 1$

$\forall \lambda$ autovalore

⑥ Ricordiamo che il rango di $f|_{Z_i}$ è la dimensione di $Im(f|_{Z_i})$

$f|_{Z_0} : Z_0 \to V_3$ deve avere rango 0

L'unica applicazione con rango 0 è l'applicazione nulla.

Da cui deve essere $Z_0 \leq \ker f = \mathcal{L}(\bar{\imath}+\bar{\jmath})$

Posso prendere $Z_0 = \{\bar{0}\}$ (caso banale)

oppure $Z_0 = \mathcal{L}(\bar{\imath}+\bar{\jmath})$

$f|_{Z_1} : Z_1 \to V_3$ deve avere rango 1

Basta prendere come Z_1 un qualsiasi sottospazio di dimensione 1 e diverso da $\ker f$

Ad esempio $Z_1 = \mathcal{L}(\bar{k})$

$f|_{Z_2} : Z_2 \to V_3$ deve avere rango 2

posso prendere $Z_2 = V_3$ siccome $\dim(Im f) = 2$

Se voglio un sottospazio Z_2 di dimensione 2, posso prendere

$Z_2 = (\ker f)^{\perp}$ (perchè $f : (\ker f)^{\perp} \to V_3$ ha rango 2 ??)

$$= \{ \bar{v} \in V_3 : [\bar{v}]_B \in N(A)^{\perp} \}$$

$$= \{ \bar{v} \in V_3 : [\bar{v}]_B \in R(A) \} \qquad R(A) = \mathcal{L}\left(\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right)$$

$$= \mathcal{L}\left(\bar{\imath} - \bar{\jmath} - \bar{k},\ \cancel{-\bar{\imath} + \bar{\jmath} + \bar{k}},\ -\bar{\imath} + \bar{\jmath} - 2\bar{k} \right)$$

$$= \mathcal{L}\left(\bar{\imath} - \bar{\jmath} - \bar{k},\ -\bar{\imath} + \bar{\jmath} - 2\bar{k} \right)$$

(Note che $R(A) = C(A)$ perché A simmetrica, quindi $(\ker f)^{\perp} = \operatorname{Im} f$)