ESERCIZIO 2

$v = x\bar{i} + y\bar{j} + z\bar{k}$ $\qquad\qquad$ $\mathcal{B} = \{\bar{i}, \bar{j}, \bar{k}\}$

Determiniamo prima $M^{\mathcal{B}}(f) := A$

$$v \wedge (\bar{i}+\bar{j}) = x\,\bar{i}\wedge(\bar{i}+\bar{j}) + y\,\bar{j}\wedge(\bar{i}+\bar{j}) + z\,\bar{k}\wedge(\bar{i}+\bar{j})$$

$$= x\bar{k} - y\bar{k} + z\bar{j} - z\bar{i}$$

$$= -z\bar{i} + z\bar{j} + (x-y)\bar{k}$$

$$\Rightarrow f(\bar{v}) = (x-z)\bar{i} + (y+z)\bar{j} + (x+y)\bar{k}$$

$f(\bar{i}) = \bar{i} + \bar{k}$

$f(\bar{j}) = \bar{j} + \bar{k}$

$f(\bar{k}) = -\bar{i} + \bar{j}$

$$M^{\mathcal{B}}(f) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

In componenti

$$\varphi(\bar{v}, \bar{w}) = {}^t\!\left[A\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] \cdot \left[A\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right] = (x, y, z)\, {}^t\!A \cdot A \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

se $[\bar{v}]_{\mathcal{B}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad$ $[\bar{w}]_{\mathcal{B}} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$= M^{\mathcal{B}}(\varphi)$

(i)

$$ {}^tAA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = M^B(\varphi) $$

(ii) Notiamo che $\varphi(v,v) = \|f(v)\|^2 \geq 0 \Rightarrow \varphi$ semidefinita

e inoltre, $\ker f = N(A) = \mathcal{L}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\mathrm{rk}\, {}^tAA = 2$

e $\ker f \equiv \ker \varphi \Rightarrow$ segnatura $(2,0)$

Essendo φ semidefinita, $B_\varphi \equiv \ker \varphi$ è un sottospazio isotropo

(iii) Trovo autovettori/autovalori

$$ \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 2-\lambda \\ -1 & 1 \end{vmatrix} $$

$$ = (2-\lambda)\left[(2-\lambda)^2-1\right] + (\lambda-3) - (-1+2-\lambda) $$

$$ = (2-\lambda)(3-\lambda)(1-\lambda) - 2(3-\lambda) $$

$$ = (3-\lambda)\left[\lambda^2-3\lambda+\cancel{2}-\cancel{2}\right] = \lambda^2(\lambda-3) $$

Autovalori $\lambda = \overset{\text{doppio}}{3}$, $\lambda = 0$

$$V_3 = N\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} = \mathcal{L}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$V_0 = N\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \mathcal{L}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

Base ortonormale: $v_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad v_2' = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\|^2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$v_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \sqrt{\frac{2}{3}}\begin{pmatrix} -1 \\ -1/2 \\ 1/2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{Base } \mathcal{B}' = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\-1\\1\end{pmatrix} \right\}$$

$$M^{\mathcal{B}'}(\varphi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Base t.c. φ in forma normale:

$$\begin{cases} v_0' = v_0 = \dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix} \\[2ex] v_1' = \dfrac{v_1}{\sqrt{3}} = \dfrac{1}{\sqrt{6}}\begin{pmatrix}0\\1\\1\end{pmatrix} \\[2ex] v_2' = \dfrac{v_2}{\sqrt{3}} = \dfrac{1}{3\sqrt{2}}\begin{pmatrix}-2\\-1\\1\end{pmatrix} \end{cases}$$

(iv) $v \in f^{-1}\left((\operatorname{Im} f)^{\perp}\right) \Longleftrightarrow f(v) \in (\operatorname{Im} f)^{\perp}$

$$\Updownarrow$$

$\varphi(v,w) = 0 \quad \forall w \in V_3 \Longleftrightarrow f(v)\cdot f(w) = 0 \quad \forall w \in V_3$

$v \in \ker \varphi \Longleftrightarrow \varphi(v,w)=0 \quad \forall w \in V_3$

Scelgo quindi $v \in \ker \varphi = \mathcal{L}\left(\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)$

(iv) Posso scegliere $W = \mathcal{L}(v_1', v_2')$, per cui

$\varphi|_{W\times W}$ has matrix $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$

(v) $Q(x,y,y) - \varphi(x,+y,-y)$

$$= \cancel{2x^2} + \cancel{2y^2} + \cancel{2y^2} + \cancel{2xy} - \cancel{2xy} + 2y^2$$
$$-\left(\cancel{2x^2} + \cancel{2y^2} + \cancel{2y^2} + 2xy + 2xy - 2y^2\right)$$

$$= -4xy + 4y^2$$

$\Rightarrow$ Conica $\mathcal{C}$ $-4xy + 4y^2 = 1 - 4x^2$

$\Rightarrow \mathcal{C}:\ 4x^2 - 4xy + 4y^2 = 1$

$$(x, y)\begin{pmatrix}4 & -2\\-2 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 1$$

$$\begin{vmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 4 = (6-\lambda)(2-\lambda)$$

$$V_2 = N\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \mathcal{L}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$V_6 = N\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} = \mathcal{L}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \Delta = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$6(x')^2 + 2(y')^2 = 1$ ellisse di semiassi $\frac{1}{\sqrt{6}}$ e $\frac{1}{\sqrt{2}}$