Rajnikant Sinha

Real and Complex Analysis

Volume 2

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Preface

The book is an introduction to real and complex analysis that will be useful to undergraduate students of mathematics and engineering. It is designed to equip the reader with tools that will help them to understand the concepts of real analysis and complex analysis. In addition, it contains the essential topics of analysis that are needed for the study of functional analysis. Its guiding principles help develop the necessary concepts rigorously with enough detail and with the minimum prerequisites. Further, I have developed the necessary tools to enhance the readability. This book contains complete solutions to almost all the problems discussed within. This will be beneficial to readers only if used correctly: readers are encouraged to look at the solution to a problem only after trying to solve the problem.

Certainly, at times, the reader may find the proofs excruciatingly detailed, but it is better to be detailed than concise. Furthermore, eliding over detailed calculation can sometimes be perplexing for the beginners. I have tried to make it a readable text that caters to a broad audience. This approach should certainly benefit beginners who have not yet tussled with the subject in a serious way.

This book contains several useful theorems and their proofs in the realm of real and complex analysis. Most of these theorems are the works of some the great mathematicians of the 19th and 20th centuries. In alphabetical order, these include: Arzela, Ascoli, Baire, Banach, Carathéodory, Cauchy, Dirichlet, Egoroff, Fatou, Fourier, Fubini, Hadamard, Jordan, Lebesgue, Liouville, Minkowski, Mittag-Leffler, Morera, Nikodym, Ostrowski, Parseval, Picard, Plancherel, Poisson, Radon, Riemann, Riesz, Runge, Schwarz, Taylor, Tietze, Urysohn, Weierstrass, and Young. I have spent several years providing their proofs in unprecedented detail.

There are plenty of superb texts on real and complex analysis, but there is a dearth of books that blend real analysis with complex analysis. Libraries already contain several excellent reference books on real and complex analysis, which interested students can consult for a deeper understanding. It was not my intention to replace such books. This book is written under the assumption that students already know the fundamentals of advanced calculus. The proofs of various named theorems should be considered to be at the core of the book by any reader who is serious about learning the subject.

This is the second volume of the two-volume book. Volume 2 contains four chapters: Holomorphic and Harmonic Functions, Conformal Mapping, Analytic Continuation, and Special Functions. In Chap. 1, we study the holomorphic functions and harmonic functions. Here, we have proved the fundamental theorem of algebra and global Cauchy theorem. The chapter ends with a discussion on the Schwarz reflection principle. The topics of discussion in Chap. 2 are infinite product and the Riemann mapping theorem. In Chap. 3, we have introduced analytic continuation and proved the monodromy theorem. A branch of logarithm function is also discussed in this chapter. In Chap. 4, we prove the prime number theorem. For this purpose, all the needed tools of analytic number theory are developed from scratches. Finally, we have supplied a beautiful proof of Picard's little theorem.

I am particularly indebted to Walter Rudin and Paul Richard Halmos for their letters discussing academic questions. By great good fortune, some colleagues of mine were able to join in with this enterprise a few years ago, some of whom have provided a meticulous reading of the manuscript from a user's viewpoint. I extend my great thanks to all of them for their expert services.

While studying this book, I hope that readers will experience the thrill of creative effort and the joy of achievement.

Varanasi, India Rajnikant Sinha

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About the Author

Rajnikant Sinha is Former Professor of Mathematics at Magadh University, Bodh Gaya, India. As a passionate mathematician, he has published numerous interesting research findings in international journals and books, including *Smooth Manifolds* (Springer) and the contributed book *Solutions to Weatherburn's Elementary Vector Analysis*. His research focuses on topological vector spaces, differential geometry and manifolds.

Chapter 1
Holomorphic and Harmonic Functions

1.1 Locally Path-Connected Function

Note 1.1 Let $a \in \mathbb{C}$, and let r be a positive real number.

Problem 1.2 $\{z : |z-a| < r\}^{-} = \{z : |z-a| \le r\}$.

(Solution

1. We first try to show that $\{z : |z-a| \le r\}$ is a closed set, that is $(\{z : r < |z-a|\} =)\{z : |z-a| \le r\}^c$ is an open set. For this purpose, let us fix any $z_0 \in \mathbb{C}$ such that $r < |z_0 - a|$. We have to show that z_0 is an interior point of $\{z : r < |z-a|\}$. Since $r < |z_0 - a|$, we have $0 < |z_0 - a| - r$. It suffices to show that

$$D(z_0; |z_0 - a| - r) \subset \{z : r < |z-a|\}.$$

For this purpose, let us take any $z \in D(z_0; |z_0 - a| - r)$. We have to show that $r < |z-a|$. Since $z \in D(z_0; |z_0 - a| - r)$, we have

$$\underbrace{|z - z_0| < (|z_0 - a| - r)}_{} \le (|z_0 - z| + |z - a|) - r,$$

and hence $r < |z-a|$.

2. Thus, $\{z : |z-a| \le r\}$ is a closed set containing $\{z : |z-a| < r\}$. It follows that

$$\{z : |z-a| < r\}^{-} \subset \{z : |z-a| \le r\}.$$

It remains to show that

$$\{z : |z-a| \le r\} \subset \{z : |z-a| < r\}^{-}.$$

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If not, otherwise, suppose that there exists $z_0 \in \{z : |z-a| \le r\}$ such that $z_0 \notin \{z : |z-a| < r\}^-(\supset \{z : |z-a| < r\} \ni a)$. We have to arrive at a contradiction. Here, we have $|z_0 - a| \le r$, and $|z_0 - a| \not< r$, so $|z_0 - a| = r$ and $z_0 \ne a$. Now, since for every positive integer n,

$$\left|\left(\left(1-\frac{1}{n}\right)z_0 + \frac{1}{n}a\right) - a\right| = \left(1-\frac{1}{n}\right)|z_0 - a| = \left(1-\frac{1}{n}\right)r < r,$$

we have, for every positive integer n,

$$\left|\left(\left(1-\frac{1}{n}\right)z_0 + \frac{1}{n}a\right) - a\right| < r,$$

and hence $\left\{\left(1-\frac{1}{n}\right)z_0 + \frac{1}{n}a\right\}$ is a sequence in $\{z : |z-a| < r\}$. Now, since

$$\lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)z_0 + \frac{1}{n}a\right) = z_0,$$

we have $z_0 \in \{z : |z-a| < r\}^-$. This is a contradiction ∎)

Conclusion 1.3 Let $a \in \mathbb{C}$, and let r be a positive real number. Then $\{z : |z-a| < r\}^- = \{z : |z-a| \le r\}$.

Here, $D(a;r)$ is called the ***open disk*** *with center at a and radius r*, and $D[a;r]$ is called the ***closed disk*** *with center at a and radius r*.

By ***punctured disk*** *with center at a and radius r* we mean $\{z : 0 < |z-a| < r\}$, and this is denoted by $D'(a;r)$.

Thus, $\overline{D(a;r)} = D[a;r]$, where $D(a;r) \equiv \{z : |z-a| < r\}$, and $D[a;r] \equiv \{z : |z-a| \le r\}$.

Note 1.4 Let X be a topological space. Let S be a connected subset of X.

Problem 1.5 The closure S^- of S is also connected.

(Solution If not, otherwise, let S^- be not connected. We have to arrive at a contradiction.

Since S^- is not connected, there exist open subsets G_1, G_2 of X such that $G_1 \cap (S^-)$ is nonempty, $G_2 \cap (S^-)$ is nonempty, $G_1 \cap G_2 \cap (S^-)$ is empty, and $S^- \subset G_1 \cup G_2$. Since $G_1 \cap (S^-)$ is nonempty, there exists $a \in G_1 \cap (S^-)$, and hence $a \in G_1$. Since $a \in G_1$, and G_1 is open, G_1 is an open neighborhood of a. Since $a \in G_1 \cap (S^-)$, we have $a \in S^-$. Since $a \in S^-$, and G_1 is an open neighborhood of a, $G_1 \cap S$ is nonempty. Similarly, $G_2 \cap S$ is nonempty. Since $S \subset S^-$, and $G_1 \cap G_2 \cap (S^-)$ is empty, $G_1 \cap G_2 \cap S$ is empty. Since $S \subset S^- \subset G_1 \cup G_2$, we have $S \subset G_1 \cup G_2$. Since G_1, G_2 are open subsets of X, $G_1 \cap S$ is nonempty, $G_2 \cap S$ is nonempty, $G_1 \cap G_2 \cap S$ is empty, and $S \subset G_1 \cup G_2$, S is not connected. This is a contradiction. ∎)

Conclusion 1.6 Let X be a topological space. Let S be a connected subset of X. Then the closure S^- of S is also connected.

Note 1.7 Let X be a topological space. Let S be a nonempty subset of X.

For every x, y in S, by $x \sim y$ we shall mean that there exists a connected subset C of X such that $C \subset S$, $x \in C$, and $y \in C$.

Problem 1.8 $\sim$ is an equivalence relation over S, that is

1. for every x in S, $x \sim x$,
2. if $x \sim y$ then $y \sim x$,
3. if $x \sim y$, and $y \sim z$ then $x \sim z$.

(**Solution** For 1: Let us take any x in S. Since the singleton set $\{x\}$ is a connected subset of X, and $x \in \{x\} \subset S$, by the definition of $\sim$, $x \sim x$.

For 2: Let $x \sim y$. Now, by the definition of $\sim$, there exists a connected subset C of X such that $C \subset S$, $x \in C$, and $y \in C$. Thus, C is a connected subset of X such that $C \subset S$, $y \in C$, and $x \in C$, and hence, by the definition of $\sim$, $y \sim x$.

For 3: Let $x \sim y$, and $y \sim z$. Since $x \sim y$, by the definition of $\sim$, there exists a connected subset C of X such that $C \subset S$, $x \in C$, and $y \in C$. Similarly, there exists a connected subset C_1 of X such that $C_1 \subset S$, $y \in C_1$, and $z \in C_1$. Since C is a connected subset of X, C_1 is a connected subset of X, and $y \in C \cap C_1$, $C \cup C_1$ is a connected subset of X. Also, $x \in C \subset C \cup C_1$, and $z \in C_1 \subset C \cup C_1$. Since $C \subset S$, and $C_1 \subset S$, we have $C \cup C_1 \subset S$. It follows, by the definition of $\sim$, that $x \sim z$. ∎)

Since $\sim$ is an equivalence relation over S, S is partitioned into equivalence classes. Here, each equivalence class is called a ***component*** *of S*.

Problem 1.9 Each component of S is a connected subset of X.

(**Solution** Let us take any component $[a]$ of S, where a is in S, and $[a] \equiv \{x : x \in S, \text{and } x \sim a\}$. We have to show that $[a]$ is connected.

If not, otherwise let $[a]$ be not connected. That is, there exist open subsets G_1, G_2 of X such that $G_1 \cap [a]$ is nonempty, $G_2 \cap [a]$ is nonempty, $G_1 \cap G_2 \cap [a]$ is empty, and $[a] \subset G_1 \cup G_2$. We have to arrive at a contradiction.

Since $G_1 \cap [a]$ is nonempty, there exists b in $G_1 \cap [a]$. Similarly, there exists c in $G_2 \cap [a]$. Here b and c are in $[a]$, and $[a]$ is an equivalence class, so $b \sim c$, and hence by the definition of $\sim$, there exists a connected subset C of X such that $C \subset S$, $b \in C$, and $c \in C$. Thus, b is in $G_1 \cap C$, and hence $G_1 \cap C$ is nonempty. Similarly, $G_2 \cap C$ is nonempty.

Problem 1.10 C is contained in $[a]$.

(**Solution** If not, otherwise there exists $t \in C$ such that $t \notin [a]$. Since C is a connected subset of X, $C \subset S$, $b \in C$, and $t \in C$, $b \sim t$. Since b is in $G_1 \cap [a]$, we have $b \sim a$. Since $b \sim t$, and $b \sim a$, we have $t \sim a$, and hence $t \in [a]$. This is a contradiction. ∎)

Since C is contained in $[a]$, and $G_1 \cap G_2 \cap [a]$ is empty, $G_1 \cap G_2 \cap C$ is empty. Next, since C is contained in $[a]$, and $[a] \subset G_1 \cup G_2$, we have $C \subset G_1 \cup G_2$. Since G_1, G_2 are open subsets of X, $G_1 \cap C$ is nonempty, $G_2 \cap C$ is nonempty, $G_1 \cap G_2 \cap C$ is empty, and $C \subset G_1 \cup G_2$, C is not connected. This is a contradiction. ∎)

Problem 1.11 $[a]$ is a maximal connected subset of S.

(**Solution** If not, otherwise let $[a]$ be not a maximal connected subset of S. We have to arrive at a contradiction. Since $[a]$ is not a maximal connected subset of S, there exists a connected subset C of S such that $[a]$ is a proper subset of C. So, there exists t in C such that t is not in $[a]$. Since $a \in [a] \subset C \subset S$, t is in C, and C is a connected subset of X, we have $t \sim a$, and hence t is in $[a]$. This is a contradiction. ∎)

Problem 1.12 Every component of X is a closed set.

(**Solution** Let us take any component C of X. Since C is a component of X, C is connected, and hence C^- is connected. Since C is a component of X, C^- is connected and $C \subset C^- \subset X$, we have $C = C^-$, and hence C is closed. ∎)

Conclusion 1.13 Let X be a topological space. Let S be a nonempty subset of X. Then

i. S is partitioned into components,
ii. every component of S is connected,
iii. if C is a component of S, C_1 is connected, and $C \subset C_1 \subset S$ then $C = C_1$,
iv. every component of X is a closed set.

Note 1.14 Let X be a topological space. Let S be a nonempty subset of X. Let a, b be elements of S. Let α, β be any real numbers such that $\alpha < \beta$. Let f be a function from closed interval $[\alpha, \beta]$ to S. If

i. f is continuous,
ii. $f(\alpha) = a$ and $f(\beta) = b$, then we shall say that *f is a **curve** in S from a to b*.

For the sake of simplicity, we sometimes take 0 for α, and 1 for β.

Let X be a topological space. Let S be a nonempty subset of X.

If for every a, b in S, there exists a curve f in S from a to b then we say that *S is a **path-connected** subset of X*.

By *X is **path connected*** we mean: X is a path connected subset of X.

In other words, if a topological space *X is **path connected***, it means that for every x, y in X, there exists a continuous mapping $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

a. Let X be a topological space. Let S be a nonempty subset of X.

For every x, y in S, by $x \sim y$, we shall mean that the following statement is true: there exists a curve f in S from x to y.

Problem 1.15 $\sim$ is an equivalence relation over S.

(Solution By the definition of equivalence relation, we must prove:

1. $x \sim x$ for every x in S,
2. if $x \sim y$ then $y \sim x$,
3. if $x \sim y$ and $y \sim z$ then $x \sim z$.

For 1: Let us take any x in S. By the definition of $\sim$, we must find a curve f in S from x to x. For this purpose, consider the constant function $f : [0,1] \to S$ defined by $f(t) \equiv x$ for every t in $[0,1]$. Since every constant function is a continuous function, and f is a constant function, f is a continuous function. Further, by the definition of f, $f(0) = x$ and $f(1) = x$. Hence, by the definition of curve, f is a curve in S from x to x. This proves (1).

For 2: Let $x \sim y$. So, by the definition of $\sim$, there exists a curve f in S from x to y. Since f is a curve in S from x to y, by the definition of curve, f is a function from the closed interval $[0,1]$ to S such that

i. f is continuous,
ii. $f(0) = x$ and $f(1) = y$.

We have to prove $y \sim x$. So, by the definition of $\sim$, we must find a curve in S from y to x, and hence by the definition of curve, we must find a function $g : [0,1] \to S$ such that

(1′) g is continuous,
(2′) $g(0) = y$ and $g(1) = x$.

Let us define $g : [0,1] \to X$ as follows: $g(t) \equiv f(1-t)$ for every t in $[0,1]$. Clearly the range of g is equal to the range of f. Since the range of g is equal to the range of f, and $f : [0,1] \to S$, we have $g : [0,1] \to S$.

For (1′) Since $t \mapsto 1-t$ is a polynomial function, this is continuous. Since $t \mapsto 1-t$ is a continuous function and f is a continuous function, their composite function $t \mapsto f(1-t)(= g(t))$ is continuous, and hence g is continuous. This proves (1′).

For (2′) Here, by the definition of g, $g(0) = f(1-0) = f(1) = y$, and $g(1) = f(1-1) = f(0) = x$. This proves (2′).

Thus we have shown that if $x \sim y$ then $y \sim x$. This proves (2).

For 3: Given that $x \sim y$ and $y \sim z$. We have to prove: $x \sim z$, that is, by the definition of $\sim$, we must find a curve h in S from x to z, that is, we must find a function $h : [0,1] \to S$ such that

(1′) h is continuous,
(2′) $h(0) = x$ and $h(1) = z$.

Since $x \sim y$, by the definition of $\sim$, there exists a curve in S from x to y, and hence there exists a function $f : [0,1] \to S$ such that

(1″) f is continuous,
(2″) $f(0) = x$ and $f(1) = y$.

Since $y \sim z$ so, by the definition of $\sim$, there exists a curve in S from y to z, and hence there exists a function $g : [0, 1] \to S$ such that

(1‴) g is continuous,
(2‴) $g(0) = y$ and $g(1) = z$.

Let us define a function $h : [0, 1] \to X$ as follows:

$$h(t) \equiv \begin{cases} f(2t) & \text{if } 0 \le t < \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Clearly, the range of h is the union of the range of f, and the range of g. Further, since $f : [0, 1] \to S$, and $g : [0, 1] \to S$, the range of h is contained in S, and hence $h : [0, 1] \to S$.

For (1′): We have to prove that h is continuous over $[0, 1]$, that is h is continuous at all points of $[0, 1]$. By the definition of h over $0 \le t < \frac{1}{2}$, h is the composite function of two continuous functions, namely $t \mapsto 2t$ and f, so h is continuous at all points of $\left\{t : 0 \le t < \frac{1}{2}\right\}$. Similarly, h is continuous at all points of $\left\{t : \frac{1}{2} < t \le 1\right\}$. So, it remains to prove that h is continuous at $\frac{1}{2}$. For this purpose, we should calculate the left-hand limit of h at $\frac{1}{2}$, the right-hand limit of h at $\frac{1}{2}$ and $h\left(\frac{1}{2}\right)$. If all of these three values are equal then we can say that h is continuous at $\frac{1}{2}$. In short, we have to prove

$$\lim_{t \to \left(\frac{1}{2}\right)^-} h(t) = \lim_{t \to \left(\frac{1}{2}\right)^+} h(t) = h\left(\frac{1}{2}\right).$$

Here, $\lim_{t \to \left(\frac{1}{2}\right)^-} h(t) = \lim_{t \to \frac{1}{2}} f(2t)$, by the definition of h. Since $t \mapsto f(2t)$ is a continuous function over $\left[0, \frac{1}{2}\right]$, we have $\lim_{t \to \frac{1}{2}} f(2t) = f\left(2 \times \frac{1}{2}\right) = f(1) = y$. Hence, $\lim_{t \to \left(\frac{1}{2}\right)^-} h(t) = y$. Similarly,

$$\lim_{t \to \left(\frac{1}{2}\right)^+} h(t) = \lim_{t \to \frac{1}{2}} g(2t-1) = g\left(2\left(\frac{1}{2}\right) - 1\right) = g(0) = y.$$

Hence, $\lim_{t \to \left(\frac{1}{2}\right)^-} h(t) = \lim_{t \to \left(\frac{1}{2}\right)^+} h(t) = y$. Further, by the definition of h, $h\left(\frac{1}{2}\right) = g\left(2\left(\frac{1}{2}\right) - 1\right) = g(0) = y$. Hence $\lim_{t \to \left(\frac{1}{2}\right)^-} h(t) = \lim_{t \to \left(\frac{1}{2}\right)^+} h(t) = h\left(\frac{1}{2}\right)$. Thus, we have shown that h is continuous over $[0, 1]$. This proves (1′).

For (2′): Here, $h(0) = f(2(0)) = f(0) = x$, and $h(1) = g(2(1) - 1) = g(1) = z$. This proves (2′).

Thus, we have shown if $x \sim y$ and $y \sim z$ then $x \sim z$. This proves (3). ∎)

Since $\sim$ is an equivalence relation over S, S is partitioned into equivalence classes.

Here, each equivalence class is called a ***path component*** *of* S.

Problem 1.16

1. Each path component of S is a path connected subset of X.
2. Let $[a]$ be a path component of S. Let C be a path component subset of X satisfying $[a] \subset C \subset S$. Then $[a] = C$.

(Solution

1. For this purpose, let us take any path component $[a]$ of S, where $a \in S, [a] \equiv \{x : x \in S, \text{and } x \sim a\}$, and $x \sim a$ stands for: there exists a curve in S from x to a. We have to show that $[a]$ is a path connected subset of X.

For this purpose, let us take any x, y in $[a]$. We have to find a curve in $[a]$ from x to y.

Since $[a]$ is a path component of S, $[a]$ is an equivalence class. Since $[a]$ is an equivalence class, and x is in $[a]$, we have $x \sim a$, and hence there exists a function $f : [0, 1] \to S$ such that

(1′) $f : [0, 1] \to S$ is continuous,
(2′) $f(0) = x$ and $f(1) = a$.

Problem 1.17 The range of f is contained in $[a]$.

(Solution Let us take any $b \equiv f(t_0) \in \text{ran} f$, where $t_0 \in (0, 1)$. We have to prove that $b \in [a]$, that is $a \sim b$. We must find a function $F : [0, 1] \to S$ such that

i. F is continuous,
ii. $F(0) = a$ and $F(1) = b$.

Let us define $F : [0, 1] \to S$ as follows: For every t in $[0, 1]$, $F(t) \equiv f(1 + (t_0 - 1)t)$. Clearly, the range of $[0, 1]$ under $t \mapsto (1 + (t_0 - 1)t)$ is $[t_0, 1]$, and hence the ran $F \subset f([t_0, 1]) \subset S$. Since ran $F \subset f([t_0, 1]) \subset S$, we have $F : [0, 1] \mapsto S$. Since F is the composite of two continuous functions, $t \mapsto (1 + (t_0 - 1)t)$, and f, F is continuous. Clearly, $F(0) = f(1 + (t_0 - 1)0) = f(1) = a$, and $F(1) = f(1 + (t_0 - 1)1) = f(t_0) = b$. ∎)

Since the range of f is contained in $[a]$, and $f : [0, 1] \to S$, we have $f : [0, 1] \to [a]$. Now, since $f : [0, 1] \to S$ is continuous, $f : [0, 1] \to [a]$ is continuous. Further, $f(0) = x$ and $f(1) = a$. Thus we get a curve in $[a]$ from x to a. Similarly, we get a curve in $[a]$ from a to y. Since there exists a curve in $[a]$ from x to a, and a curve in $[a]$ from a to y, there exists a curve in $[a]$ from x to y.

2. If not, otherwise let $[a]$ be a proper subset of C. We have to arrive at a contradiction. Since $[a]$ is a proper subset of C, there exists t in C such that t is not in $[a]$. Since $a \in [a] \subset C$, a is in C. Since a is in C, t is in C, and C is a path

connected subset of X, there exists a curve f in C from t to a. Since f is a curve in C, and $C \subset S$, f is a curve in S, and hence t is in $[a]$. This is a contradiction. ∎)

Conclusion 1.18 Let X be a topological space. Let S be a nonempty subset of X. Then

i. S is partitioned into path components,
ii. every path component of S is path connected,
iii. if C is a path component of S, C_1 is path connected and $C \subset C_1 \subset S$, then $C = C_1$.

Note 1.19 Let X be a topological space. Let S be a nonempty subset of X.

Problem 1.20 Each component of S is a union of path components of S.

(Solution Let us take any component $\{y : y \in S \,\text{and}\, y \sim a\}$ of S, where a is in S, and $y \sim a$ stands for: there exists a connected subset A of X such that $A \subset S$, and y, a are in A. It suffices to show that for every $a \in S$,

$$\{y : y \in S \,\text{and}\, y \sim' a\} \subset \{y : y \in S \,\text{and}\, y \sim a\}$$

where $y \sim' a$ stands for: there exists a curve in S from y to a.

For this purpose, let us fix any $a \in S$. Next, let us take any y in LHS. We have to prove that y is in RHS, that is $y \sim a$. By the definition of $\sim$, we should find a connected subset A of X such that $A \subset S$, and y, a are in A.

Since y is in LHS, we have $y \sim' a$. Since $y \sim' a$, by the definition of $\sim'$, there exists a curve f in S from y to a. Since f is a curve in S from y to a, f is a function from closed interval $[0, 1]$ to S satisfying

I. f is continuous,
II. $f(0) = y$ and $f(1) = a$.

Since $f : [0, 1] \to S$ is continuous, and $[0, 1]$ is connected, the f-image set $f([0, 1])$ of $[0, 1]$ is a connected subset of X. Since 0 is in $[0, 1]$, $f(0)$ is in $f([0, 1])$. Similarly, $f(1)$ is in $f([0, 1])$. Further, $f(0) = y$ and $f(1) = a$. Thus, $f([0, 1])$ is a connected subset of X such that $f([0, 1]) \subset S$, and y, a are in $f([0, 1])$. ∎)

Conclusion 1.21 Let X be a topological space. Let S be a nonempty subset of X. Then each component of S is a union of path components of S.

Note 1.22 Let X be a topological space. Let S be a nonempty subset of X. Let S be a path connected subset of X.

Problem 1.23 S is connected.

(Solution If not, otherwise let S be not connected. We have to arrive at a contradiction.

Since S is not connected, there exist open subsets G_1, G_2 of X such that $G_1 \cap S$ is nonempty, $G_2 \cap S$ is nonempty, $G_1 \cap G_2 \cap S$ is empty, and $S \subset G_1 \cup G_2$.

Since $G_1 \cap S$ is nonempty, there exists a in $G_1 \cap S$. Similarly, there exists b in $G_2 \cap S$. It follows that $a, b \in S$. Now, since S is a path connected subset of X, there exists a continuous function $f : [0,1] \to S$ such that $f(0) = a$, and $f(1) = b$. Since $f : [0,1] \to S$ is continuous, and $[0,1]$ is connected, the image set $f([0,1])(\subset S)$ is connected. Here, a is in $G_1 \cap S$, and $a = f(0) \in f([0,1])$, we have $a \in G_1 \cap f([0,1])$, and hence $G_1 \cap f([0,1])$ is nonempty. Similarly, $G_2 \cap f([0,1])$ is nonempty.

Next, since $G_1 \cap G_2 \cap S$ is empty, and $f([0,1]) \subset S$, $G_1 \cap G_2 \cap f([0,1])$ is empty. Since G_1, G_2 are open subsets of X such that $G_1 \cap f([0,1])$ is nonempty, $G_2 \cap f([0,1])$ is nonempty, $G_1 \cap G_2 \cap f([0,1])$ is empty, and $f([0,1]) \subset S \subset G_1 \cup G_2$, $f([0,1])$ is not connected. This is a contradiction. ∎)

Definition Let X be a topological space. If for every x in X, and for every open neighborhood U of x, there exists an open neighborhood V of x such that (i) $x \in V \subset U$, (ii) V is a connected subset of X, then we say that X is a ***locally connected space***.

Definition Let X be a topological space. If for every x in X, and for every open neighborhood U of x there exists an open neighborhood V of x such that (i) $x \in V \subset U$, (ii) V is a path connected subset of X, then we say that X is a ***locally path connected space***.

Problem 1.24 $\mathbb{C}$ is a locally path connected space.

(Solution For this purpose, let us take any $a \in \mathbb{C}$. Next, let U be an open neighborhood of a. It follows that there exists $r > 0$ such that $(a \in)D(a;r) \subset U$. Now, since $D(a;r)$ is open, it suffices to show that $D(a;r)$ is a path connected subset of $\mathbb{C}$. For this purpose, let us take any b, c in $D(a;r)$, where $b \neq c$. Now, since $D(a;r)$ is convex,

$$f : t \mapsto ((1-t)b + tc)$$

is a function from $[0,1]$ to $D(a;r)$. It is clear that $f : [0,1] \to D(a;r)$ is continuous. Also, $f(0) = (1-0)b + 0c = b$, and $f(1) = (1-1)b + 1c = c$. Thus, f is a curve in $D(a;r)$ from b to c, and hence $D(a;r)$ is a path connected subset of $\mathbb{C}$. ∎)

Conclusion 1.25 Every locally path connected space is a locally connected space.

Now, since $\mathbb{C}$ is a locally path connected space, $\mathbb{C}$ is a locally connected space.

Note 1.26 Let X be a locally connected space. Let G be a nonempty open subset of X.

Problem 1.27 Each component of G is open.

(Solution Let us take any component $([a] =)\{y : y \in G \text{ and } y \sim a\}$ of G, where a is in G, and $y \sim a$ stands for: there exists a connected subset A of X such that $A \subset G$, and y, a are in A. We have to show that $[a]$ is an open subset of X, that is, every point of $[a]$ is an interior point of $[a]$.

For this purpose, let us take any $x \in [a]$. We have to find an open neighborhood V of x such that $V \subset [a]$. Since $x \in [a]$ and $[a] \subset G$, we have $x \in G$. Since $x \in G$ and G is open in X, G is an open neighborhood of x. Since X is a locally connected space, and G is an open neighborhood of x, there exists an open neighborhood V of x such that

I. $x \in V \subset G$,
II. V is a connected subset of X.

It suffices to show that $V \subset [a]$. If not, otherwise let $V \not\subset [a]$. We have to arrive at a contradiction.

Since $V \not\subset [a]$, there exists b in V such that $b \nsim a$. Here $b \in V \subset G$, so $b \in G$. Further, since V is a connected subset of X, $V \subset G$, and x, b are in V, by the definition of $\sim$, $x \sim b$, and hence $[x] = [b]$. Since $x \in [a]$, $[x] = [a]$. Since $[x] = [b]$, and $[x] = [a]$, $[a] = [b]$ and hence $b \sim a$. This is a contradiction. ∎)

Conclusion 1.28 Let X be a locally connected space. Let G be a nonempty open subset of X. Then each component of G is open.

Note 1.29 Let X be a locally path connected space. Let G be a nonempty open subset of X.

Problem 1.30 Each path component of G is open.

(Solution Let us take any path component $([a] =)\{y : y \in G \text{ and } y \sim a\}$ of G, where a is in G, and $y \sim a$ stands for: there exists a curve in G from y to a. We have to show that $[a]$ is an open subset of X, that is every point of $[a]$ is an interior point of $[a]$. For this purpose, let us take any $x \in [a]$. We have to find an open neighborhood V of x such that $V \subset [a]$.

Since $x \in [a]$ and $[a] \subset G$, we have $x \in G$. Since $x \in G$ and G is open in X, G is an open neighborhood of x. Since X is a locally path connected space, and G is an open neighborhood of x, there exists an open neighborhood V of x such that

I. $x \in V \subset G$,
II. V is a path connected subset of X.

It suffices to show that $V \subset [a]$. If not, otherwise let $V \not\subset [a]$. We have to arrive at a contradiction.

Since $V \not\subset [a]$, there exists b in V such that $b \nsim a$. Here $b \in V \subset G$, so $b \in G$. Further, since V is a path connected subset of X, and x, b are in V, there exists a curve f in $V(\subset G)$ from x to b, and hence $x \sim b$. It follows that $[x] = [b]$. Since $x \in [a]$, $[x] = [a]$. Since $[x] = [b]$, and $[x] = [a]$, we have $[a] = [b]$, and hence $b \sim a$. This is a contradiction. ∎)

Conclusion 1.31 Let X be a locally path connected space. Let G be a nonempty open subset of X. Then each path component of G is open.

Lemma 1.32 Let Ω be a nonempty open subset of $\mathbb{C}$. Then, every component of Ω is open.

Proof By Conclusion 1.25, $\mathbb{C}$ is a path connected space, and hence by Conclusion 1.28 every component of Ω is open. ∎

Definition Let E be a nonempty subset of $\mathbb{C}$. By a ***region*** E, we mean that E is open and connected.

By Lemma 1.32 and Conclusion 1.13:

$\boxed{\text{every nonempty open subset of } \mathbb{C} \text{ is partitioned into regions.}}$

1.2 Representable by Power Series

Note 1.33

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$. If

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists, then this limit is denoted by $f'(a)$, and is called the ***derivative*** *of f at a*.

Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$. Suppose that $f'(a)$ exists.

For the present purpose, let us think of Ω as an open subset of $\mathbb{R}^2$, $\mathbb{C}$ as $\mathbb{R}^2$, and f as $(\mathrm{Re}(f), \mathrm{Im}(f))$. Since

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

that is

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} - f'(a) \right) = 0,$$

that is

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a) - (f'(a)) \cdot h}{h} \right) = 0,$$

that is

$$\lim_{h \to 0} \frac{1}{|h|} (f(a+h) - f(a) - (f'(a)) \cdot h) = 0,$$

that is

$$\lim_{(\mathrm{Re}(h)+i\mathrm{Im}(h))\to 0} \frac{1}{|(\mathrm{Re}(h)+i\mathrm{Im}(h))|}((\mathrm{Re}(f(a+h))+i\mathrm{Im}(f(a+h))) \\ -(\mathrm{Re}(f(a))+i\mathrm{Im}(f(a))) - (\mathrm{Re}(f'(a))+i\mathrm{Im}(f'(a)))\cdot(\mathrm{Re}(h)+i\mathrm{Im}(h))) = 0,$$

that is

$$\lim_{(\mathrm{Re}(h)+i\mathrm{Im}(h))\to 0} \frac{1}{|(\mathrm{Re}(h)+i\mathrm{Im}(h))|}((\mathrm{Re}(f(a+h))+i\mathrm{Im}(f(a+h))) \\ -(\mathrm{Re}(f(a))+i\mathrm{Im}(f(a))) - (((\mathrm{Re}(f'(a)))(\mathrm{Re}(h)) - (\mathrm{Im}(f'(a)))(\mathrm{Im}(h))) \\ + i((\mathrm{Im}(f'(a)))(\mathrm{Re}(h)) + (\mathrm{Re}(f'(a)))(\mathrm{Im}(h))))) = 0,$$

that is

$$\lim_{(h_1,h_2)\to(0,0)} \frac{1}{|(h_1,h_2)|}((\mathrm{Re}(f((\mathrm{Re}(a),\mathrm{Im}(a))+(h_1,h_2))), \\ \mathrm{Im}(f((\mathrm{Re}(a),\mathrm{Im}(a))+(h_1,h_2)))) - (\mathrm{Re}(f((\mathrm{Re}(a),\mathrm{Im}(a)))), \\ \mathrm{Im}(f((\mathrm{Re}(a),\mathrm{Im}(a))))) - ((\mathrm{Re}(f'(a)))h_1 - (\mathrm{Im}(f'(a)))h_2, \\ (\mathrm{Im}(f'(a)))h_1 + (\mathrm{Re}(f'(a)))h_2)) = (0,0),$$

that is

$$\lim_{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\to\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \frac{1}{\sqrt{(h_1)^2+(h_2)^2}} \left(\begin{bmatrix} (\mathrm{Re}(f))\left(\begin{bmatrix} \mathrm{Re}(a) \\ \mathrm{Im}(a) \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right) \\ (\mathrm{Im}(f))\left(\begin{bmatrix} \mathrm{Re}(a) \\ \mathrm{Im}(a) \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}\right) \end{bmatrix} - \begin{bmatrix} (\mathrm{Re}(f))\left(\begin{bmatrix} \mathrm{Re}(a) \\ \mathrm{Im}(a) \end{bmatrix}\right) \\ (\mathrm{Im}(f))\left(\begin{bmatrix} \mathrm{Re}(a) \\ \mathrm{Im}(a) \end{bmatrix}\right) \end{bmatrix} \right. \\ \left. - \begin{bmatrix} \mathrm{Re}(f'(a)) & -(\mathrm{Im}(f'(a))) \\ \mathrm{Im}(f'(a)) & \mathrm{Re}(f'(a)) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Bigg).$$

This shows that if

$$F : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} (\mathrm{Re}(f))\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ (\mathrm{Im}(f))\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \end{bmatrix}$$

is a function from $\Omega(\subset \mathbb{R}^2)$ to $\mathbb{R}^2$, then

$$F'\left(\begin{bmatrix}\mathrm{Re(a)}\\ \mathrm{Im(a)}\end{bmatrix}\right) : \begin{bmatrix}h_1\\ h_2\end{bmatrix} \mapsto \begin{bmatrix}\mathrm{Re}(f'(a)) & -(\mathrm{Im}(f'(a)))\\ \mathrm{Im}(f'(a)) & \mathrm{Re}(f'(a))\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix}$$
$$\left(= \begin{bmatrix}(\mathrm{Re}(f'(a)))h_1 - (\mathrm{Im}(f'(a)))h_2\\ (\mathrm{Im}(f'(a)))h_1 + (\mathrm{Re}(f'(a)))h_2\end{bmatrix}\right).$$

Conclusion 1.34 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$. Suppose that $f'(a)$ exists. Then $f'(a)$ can be thought of as the linear operator

$$\begin{bmatrix}h_1\\ h_2\end{bmatrix} \mapsto \begin{bmatrix}(\mathrm{Re}(f'(a)))h_1 - (\mathrm{Im}(f'(a)))h_2\\ (\mathrm{Im}(f'(a)))h_1 + (\mathrm{Re}(f'(a)))h_2\end{bmatrix}$$

from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Lemma 1.35 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let $a \in \Omega$. Suppose that $f'(a)$ exists. Then f is continuous at a.

Proof Since $f'(a)$ exists, $\lim_{z\to a}\frac{f(z)-f(a)}{z-a} = f'(a)$. Next, since $\lim_{z\to a}(z-a) = 0$, we have

$$\lim_{z\to a}(f(z) - f(a)) = \underbrace{\lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}\cdot(z-a)\right) = (f'(a)\cdot 0)}= 0,$$

and hence $\lim_{z\to a}(f(z) - f(a)) = 0$. This shows that f is continuous at a. ∎

Note 1.36 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$, and $g : \Omega \to \mathbb{C}$ be functions. Let $a \in \Omega$. Suppose that $f'(a)$ and $g'(a)$ exist.

Problem 1.37

I. $(f+g)'(a)$ exists and $(f+g)'(a) = f'(a) + g'(a)$,
II. $(f\cdot g)'(a)$ exists and $(f\cdot g)'(a) = f'(a)\cdot g(a) + f(a)\cdot g'(a)$.

(Solution

I. Since $f'(a)$ exists, $\lim_{z\to a}\frac{f(z)-f(a)}{z-a} = f'(a)$. Similarly, $\lim_{z\to a}\frac{g(z)-g(a)}{z-a} = g'(a)$. It follows that

$$\lim_{z\to a}\left(\frac{(f+g)(z)-(f+g)(a)}{z-a}\right) = \lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a} + \frac{g(z)-g(a)}{z-a}\right)$$
$$= (f'(a) + g'(a)).$$

Thus, $(f+g)'(a)$ exists and $(f+g)'(a) = f'(a) + g'(a)$.

II. Since $f'(a)$ exists, $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$. Similarly, $\lim_{z\to a} \frac{g(z)-g(a)}{z-a} = g'(a)$. We have to show that

$$\lim_{z\to a}\left(\frac{(f\cdot g)(z)-(f\cdot g)(a)}{z-a}\right) = f'(a)\cdot g(a) + f(a)\cdot g'(a).$$

Since $g'(a)$ exists, by Lemma 1.35 g is continuous at a, and hence $\lim\limits_{z\to a} g(z) = g(a)$. Since $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$, and $\lim_{z\to a} g(z) = g(a)$,

$$\lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}\cdot g(z)\right) = f'(a)\cdot g(a).$$

Similarly,

$$\lim_{z\to a}\left((f(a))\frac{g(z)-g(a)}{z-a}\right) = f(a)\cdot g'(a).$$

It follows that

$$\begin{aligned}
\text{LHS} &= \lim_{z\to a}\left(\frac{(f\cdot g)(z)-(f\cdot g)(a)}{z-a}\right) = \lim_{z\to a}\left(\frac{f(z)\cdot g(z)-f(a)\cdot g(a)}{z-a}\right)\\
&= \lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}\cdot g(z) + (f(a))\frac{g(z)-g(a)}{z-a}\right)\\
&= f'(a)\cdot g(a) + f(a)\cdot g'(a) = \text{RHS}.
\end{aligned}$$

∎)

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. If, for every $z \in \Omega$, $f'(z)$ exists, then we say that *f is a **holomorphic*** (or ***analytic***) *function in* Ω. The collection of all holomorphic functions in Ω is denoted by $H(\Omega)$.

From the above discussion, we get the following

Conclusion 1.38 Let Ω be a nonempty open subset of $\mathbb{C}$. Then $H(\Omega)$ is a commutative ring.

Note 1.39 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$. Suppose that $f'(a)$ exists. Let us define a function $\varphi : \Omega \to \mathbb{C}$ as follows: For every $z \in \Omega$,

$$\varphi(z) \equiv \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a. \end{cases}$$

Since

$$\lim_{z \to a} \varphi(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f'(a) = \varphi(a),$$

we have $\lim_{z \to a} \varphi(z) = \varphi(a)$, and hence $\varphi : \Omega \to \mathbb{C}$ is continuous at a. Also, for every $z \in \Omega$, $f(z) = f(a) + (\varphi(z))(z - a)$. Further, $\varphi(a) = f'(a)$.

Conclusion 1.40 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$. Suppose that $f'(a)$ exists. Then there exists a function $\varphi : \Omega \to \mathbb{C}$ such that

I. $\varphi : \Omega \to \mathbb{C}$ is continuous at a,
II. for every $z \in \Omega$, $f(z) = f(a) + (\varphi(z))(z - a)$,
III. $\varphi(a) = f'(a)$.

Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$. Let $\varphi : \Omega \to \mathbb{C}$ be a function such that

I. $\varphi : \Omega \to \mathbb{C}$ is continuous at a,
II. for every $z \in \Omega$, $f(z) = f(a) + (\varphi(z))(z - a)$.

Since $\varphi : \Omega \to \mathbb{C}$ is continuous at a,

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \underbrace{\lim_{z \to a} \varphi(z) = \varphi(a)},$$

and hence $\lim_{z \to a} \frac{f(z)-f(a)}{z-a} = \varphi(a)$. This shows that $f'(a)$ exists, and $f'(a) = \varphi(a)$.

Conclusion 1.41 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$. Let $\varphi : \Omega \to \mathbb{C}$ be a function such that

I. $\varphi : \Omega \to \mathbb{C}$ is continuous at a,
II. for every $z \in \Omega$, $f(z) = f(a) + (\varphi(z))(z - a)$.

Then $f'(a)$ exists, and $f'(a) = \varphi(a)$.

Note 1.42 Let Ω and Ω_1 be nonempty open subsets of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$, and $g : \Omega_1 \to \mathbb{C}$ be functions. Let $a \in \Omega$. Let $(f(a) \in) f(\Omega) \subset \Omega_1$. Suppose that $f'(a)$ exists, and $g'(f(a))$ exists.

Problem 1.43 $(g \circ f)'(a)$ exists, and $(g \circ f)'(a) = (g'(f(a)))(f'(a))$.

(Solution By Conclusion 1.41, it suffices to find a function $\varphi : \Omega \to \mathbb{C}$ such that

I. $\varphi : \Omega \to \mathbb{C}$ is continuous at a,
II. for every $z \in \Omega$, $(g \circ f)(z) = (g \circ f)(a) + (\varphi(z))(z - a)$,
III. $(g'(f(a)))(f'(a)) = \varphi(a)$.

Since $f'(a)$ exists, by Conclusion 1.40 there exists a function $\psi : \Omega \to \mathbb{C}$ such that

a. $\psi : \Omega \to \mathbb{C}$ is continuous at a,
b. for every $z \in \Omega$, $f(z) = f(a) + (\psi(z))(z - a)$,
c. $\psi(a) = f'(a)$.

Since $g'(f(a))$ exists, by Conclusion 1.40 there exists a function $\chi : \Omega_1 \to \mathbb{C}$ such that

a′. $\chi : \Omega_1 \to \mathbb{C}$ is continuous at $f(a)$,
b′. for every $z \in \Omega_1$, $g(z) = g(f(a)) + (\chi(z))(z - f(a))$,
c′. $\chi(f(a)) = g'(f(a))$.

Observe that for every $z \in \Omega$,

$$\begin{aligned}(g \circ f)(z) = g(f(z)) &= g(f(a)) + (\chi(f(z)))(f(z) - f(a))\\ &= (g \circ f)(a) + (\chi(f(z)))(f(z) - f(a))\\ &= (g \circ f)(a) + (\chi(f(z)))((f(a) + (\psi(z))(z - a)) - f(a))\\ &= (g \circ f)(a) + ((\chi(f(z)))(\psi(z)))(z - a)\\ &= (g \circ f)(a) + (((\chi \circ f)(z))(\psi(z)))(z - a)\\ &= (g \circ f)(a) + (\varphi(z))(z - a),\end{aligned}$$

where $\varphi : z \mapsto ((\chi \circ f)(z))(\psi(z))$ is a mapping from Ω to $\mathbb{C}$. Thus, for every $z \in \Omega$,

$$(g \circ f)(z) = (g \circ f)(a) + (\varphi(z))(z - a).$$

This proves II.

Since $f'(a)$ exists, by Lemma 1.35, f is continuous at a. Next, by (a′), $\chi : \Omega_1 \to \mathbb{C}$ is continuous at $f(a)$, so $(\chi \circ f)$ is continuous at a. Now, by (a), $\psi : \Omega \to \mathbb{C}$ is continuous at a, so the product function

$$\varphi : z \to ((\chi \circ f)(z))(\psi(z))$$

is continuous at a. This proves I. Here,

$$\begin{aligned}\varphi(a) &= ((\chi \circ f)(a))(\psi(a)) = ((\chi \circ f)(a))(f'(a)) = (\chi(f(a)))(f'(a))\\ &= (g'(f(a)))(f'(a)),\end{aligned}$$

so

$$(g'(f(a)))(f'(a)) = \varphi(a).$$

This proves III. ∎)

Conclusion 1.44 Let Ω and Ω_1 be nonempty open subsets of $\mathbb{C}$. Let $f \in H(\Omega)$, and $g \in H(\Omega_1)$. Let $f(\Omega) \subset \Omega_1$. Then $(g \circ f) \in H(\Omega)$. Also, for every $z \in \Omega$,

$$(g \circ f)'(z) = (g'(f(z)))(f'(z)).$$

This formula is known as the **chain rule**.

Note 1.45

Definition The members of $H(\mathbb{C})$ are called the ***entire*** *functions*.

Let Ω be a nonempty open subset of $\mathbb{C}$. If $f : \Omega \to \mathbb{C}$ is a constant function, then $f \in H(\Omega)$, and for every $z \in \Omega$, $f'(z) = 0$. If $f : z \mapsto z$ is a function from Ω to $\mathbb{C}$, then f is a holomorphic function in Ω, and, for every $z \in \Omega$, $f'(z) = 1$. It follows from Problem 1.37(II) that the mapping $z \mapsto (z \cdot z)(= z^2)$ from Ω to $\mathbb{C}$ is a member of $H(\Omega)$, and its derivative at any point $z \in \Omega$ is $2z$. Similarly, the mapping $z \mapsto z^3$ from Ω to $\mathbb{C}$ is a member of $H(\Omega)$, and its derivative at any point $z \in \Omega$ is $3z^2$, etc. Again, from Conclusion 1.38 we find that every polynomial from Ω to $\mathbb{C}$ is a member of $H(\Omega)$.

Thus, every polynomial from $\mathbb{C}$ to $\mathbb{C}$ is an entire function. By Conclusion 1.20, Vol. 1, exp is an entire function, and, for every $z \in \mathbb{C}$,

$$(\exp)'(\mathrm{z}) = \exp(z).$$

Conclusion 1.46 exp is an entire function, and, for every $z \in \mathbb{C}$, $(\exp)'(\mathrm{z}) = \exp(z)$.

Note 1.47 Let $c_0, c_1, c_2, c_3, \ldots$ be any complex numbers. It follows that $\left\{|c_1|^{\frac{1}{1}}, |c_2|^{\frac{1}{2}}, |c_3|^{\frac{1}{3}}, \ldots\right\}$ is a sequence of nonnegative real numbers. Hence $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in [0, \infty]$.

Case I: when $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in (0, \infty)$. In this case,

$$\frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)} \in (0, \infty).$$

For every $z \in D\left(0; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right)$, we have $|z| < \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}$, and hence

$$\limsup_{n\to\infty}\left(|c_nz^n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}|z|\left(|c_n|^{\frac{1}{n}}\right) = |z|\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) < 1.$$

Thus, for every $z \in D\left(0; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right)$, we have $\limsup_{n\to\infty}\left(|c_nz^n|^{\frac{1}{n}}\right) < 1$, and hence by the Cauchy root test, $\sum_{n=1}^{\infty} c_nz^n$ is an absolutely convergent series. It follows that for every $z \in D\left(0; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right)$,

$$\sum_{n=0}^{\infty} c_nz^n \left(\equiv c_0 + c_1z + c_2z^2 + \cdots\right)$$

is an absolutely convergent series. For every $z \notin D\left[0; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right]$, we have

$$\frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)} < |z|,$$

and hence

$$\limsup_{n\to\infty}\left(|c_nz^n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}|z|\left(|c_n|^{\frac{1}{n}}\right) = \underbrace{|z|\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) > 1}.$$

Thus, for every $z \notin D\left[0; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right]$, we have $\limsup_{n\to\infty}\left(|c_nz^n|^{\frac{1}{n}}\right) > 1$, and hence by the Cauchy root test, $\sum_{n=1}^{\infty} c_nz^n$ is a divergent series. Hence, for every $z \notin D\left[0; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right]$, $\sum_{n=0}^{\infty} c_nz^n$ is a divergent series.

Case II: when $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = 0$. For every $z \in \mathbb{C}$, we have

$$\limsup_{n\to\infty}\left(|c_nz^n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}|z|\left(|c_n|^{\frac{1}{n}}\right) = \underbrace{|z|\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right)} = 0(<1),$$

and hence

$$\limsup_{n\to\infty}\left(|c_nz^n|^{\frac{1}{n}}\right) < 1.$$

It follows, by the Cauchy root test, that for every $z \in \mathbb{C}$, $\sum_{n=1}^{\infty} c_n z^n$ is an absolutely convergent series, and hence for every $z \in \mathbb{C}$, $\sum_{n=0}^{\infty} c_n z^n$ is an absolutely an convergent series.

Case III: when $\left(\lim\ \sup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = \infty$. Let us take any nonzero complex number z. Thus, $|z| > 0$. Since

$$\left(\lim\ \sup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = \infty,$$

for every positive integer n,

$$\left\{|c_n|^{\frac{1}{n}}, |c_{n+1}|^{\frac{1}{n+1}}, |c_{n+2}|^{\frac{1}{n+2}}, \ldots\right\}$$

is not bounded above. Now, since $|z| > 0$, for every positive integer n,

$$\left\{|z|\left(|c_n|^{\frac{1}{n}}\right), |z|\left(|c_{n+1}|^{\frac{1}{n+1}}\right), |z|\left(|c_{n+2}|^{\frac{1}{n+2}}\right), \ldots\right\}$$

is not bounded above, and hence

$$\left(\limsup_{n\to\infty}\left(|c_n z^n|^{\frac{1}{n}}\right) = \right)\left(\limsup_{n\to\infty}|z|\left(|c_n|^{\frac{1}{n}}\right)\right) = \infty(>1).$$

Since

$$\limsup_{n\to\infty}\left(|c_n z^n|^{\frac{1}{n}}\right) > 1,$$

by the Cauchy root test, for every $z \notin \{0\}$, $\sum_{n=1}^{\infty} c_n z^n$ is a divergent series, and hence for every $z \notin \{0\}$, $\sum_{n=0}^{\infty} c_n z^n$ is a divergent series. Also, if $z \in \{0\}$, then

$$\sum_{n=0}^{\infty} c_n z^n (= c_0 + 0 + 0 + 0 + \cdots)$$

is a convergent series.

Conclusion 1.48 Suppose that each $c_n \in \mathbb{C}$.

I. If $\left(\lim\ \sup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in (0, \infty)$, then

   a. for every $z \in D\left(0; \frac{1}{\lim\ \sup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right)$, $\sum_{n=0}^{\infty} c_n z^n$ is an absolutely convergent series,

   b. for every $z \notin D\left[0; \frac{1}{\lim\ \sup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right]$, $\sum_{n=0}^{\infty} c_n z^n$ is a divergent series.

II. If $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = 0$, then for every $z \in \mathbb{C}$, $\sum_{n=0}^{\infty} c_n z^n$ is an absolutely convergent series.

III. If $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = \infty$, then $\left(\left(\sum_{n=0}^{\infty} c_n z^n \text{ is convergent}\right) \Leftrightarrow z = 0\right)$

Similarly, we get the following

Conclusion 1.49 Suppose that each $c_n \in \mathbb{C}$. Let $a \in \mathbb{C}$

I. If $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in (0, \infty)$, then

   a. for every $z \in D\left(a; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right)$, $\sum_{n=0}^{\infty} c_n (z-a)^n$ is an absolutely convergent series,

   b. for every $z \notin D\left[a; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right]$, $\sum_{n=0}^{\infty} c_n (z-a)^n$ is a divergent series.

II. If $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = 0$, then for every $z \in \mathbb{C}$, $\sum_{n=0}^{\infty} c_n (z-a)^n$ is an absolutely convergent series.

III. If $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = \infty$, then $\left(\left(\sum_{n=0}^{\infty} c_n (z-a)^n \text{ is convergent}\right) \Leftrightarrow z = a\right)$.

Here, it follows that whenever $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in (0, \infty)$, $f : z \mapsto \left(\sum_{n=0}^{\infty} c_n (z-a)^n\right)$ is a function from open disk $D\left(a; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right)$ to $\mathbb{C}$.

Further, if

$$\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = 0,$$

then

$$f : z \mapsto \left(\sum_{n=0}^{\infty} c_n (z-a)^n\right)$$

is a function from $\mathbb{C}$ to $\mathbb{C}$.

Note 1.50

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. If, for every $a \in \Omega$, and for every positive real number r satisfying $D(a; r) \subset \Omega$, there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that

1. for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent,
2. for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$,

then we say that f *is* ***representable by power series*** *in* Ω.

Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let f be representable by power series in Ω. Let $a \in \Omega$.

Since $a \in \Omega$, and Ω is open, there exists a positive real number r such that $D(a;r) \subset \Omega$. Now, since f is representable by power series in Ω, there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that

1. for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent,
2. for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$.

Problem 1.51 Let us first observe that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(**Solution** It suffices to show that $\lim_{n\to\infty}\left(n^{\frac{1}{n}} - 1\right) = 0$. Since for every positive integer n, $n^{\frac{1}{n}} \geq 1$, for every positive integer n, $\left(n^{\frac{1}{n}} - 1\right) \geq 0$. Now, for every positive integer $n \geq 2$, we have

$$n = \left(1 + \left(n^{\frac{1}{n}} - 1\right)\right)^n = \underbrace{1 + \binom{n}{1}\left(n^{\frac{1}{n}} - 1\right) + \binom{n}{2}\left(n^{\frac{1}{n}} - 1\right)^2 + \cdots}_{(n+1)\text{terms}}$$

$$\geq 1 + \binom{n}{2}\left(n^{\frac{1}{n}} - 1\right)^2 = 1 + \frac{n(n-1)}{2}\left(n^{\frac{1}{n}} - 1\right)^2.$$

Hence, for every positive integer $n \geq 2$,

$$0 \leq \left(n^{\frac{1}{n}} - 1\right)^2 \leq \frac{2}{n}.$$

Thus, for every positive integer $n \geq 2$,

$$0 \leq \left(n^{\frac{1}{n}} - 1\right) \leq \sqrt{2}\frac{1}{\sqrt{n}}.$$

Now, since $\lim_{n\to\infty} \sqrt{2}\frac{1}{\sqrt{n}} = 0$,

$$\lim_{n\to\infty}\left(n^{\frac{1}{n}} - 1\right) = 0.$$

∎)

Since for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent, by Conclusion 1.49(b) we have $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \neq \infty$, and hence $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in [0,\infty)$.

Problem 1.52 $\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)$.

(Solution By a theorem (e.g., see [5], Theorem 3.17), it suffices to show that

I. $\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)$ is a subsequential limit of $\left\{|c_n n|^{\frac{1}{n}}\right\}$,

II. if $x > \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)$, then there exists a positive integer N such that $\left(n \geq N \Rightarrow |c_n n|^{\frac{1}{n}} < x\right)$.

For I: Since $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in [0,\infty)$, there exist positive integers $n_1, n_2, n_3, \ldots$ such that $n_1 < n_2 < n_3 < \cdots$, and

$$\lim_{k\to\infty}\left(|c_{n_k}|^{\frac{1}{n_k}}\right) = \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right).$$

Since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, we have $\lim_{k\to\infty}(n_k)^{\frac{1}{n_k}} = 1$. Since $\lim_{k\to\infty}(n_k)^{\frac{1}{n_k}} = 1$, and

$$\lim_{k\to\infty}\left(|c_{n_k}|^{\frac{1}{n_k}}\right) = \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right),$$

we have

$$\lim_{k\to\infty}\left(|c_{n_k} n_k|^{\frac{1}{n_k}}\right) = \underbrace{\lim_{k\to\infty}\left(|c_{n_k}|^{\frac{1}{n_k}}\right)\left((n_k)^{\frac{1}{n_k}}\right) = \left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right)1}_{} = \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right),$$

and hence $\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)$ is a subsequential limit of $\left\{|c_n n|^{\frac{1}{n}}\right\}$.

For II: Let $x > \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)$. There exists a real number y such that $x > y > \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)$. It follows that there exists a positive integer N_1 such that $\left(n \geq N_1 \Rightarrow |c_n|^{\frac{1}{n}} < y\right)$, and hence $\left(n \geq N_1 \Rightarrow \left(|c_n|^{\frac{1}{n}}\right)\frac{x}{y} < x\right)$. Since $x > y > 0$, we have $\left(\lim_{n\to\infty} n^{\frac{1}{n}} =\right) 1 < \frac{x}{y}$, and hence there exists a positive integer N_2 such that

$$\left(n \geq N_2 \Rightarrow n^{\frac{1}{n}} < \frac{x}{y}\right),$$

again, and hence

$$\left(n \geq \max\{N_1, N_2\} \Rightarrow \left(|c_n|^{\frac{1}{n}}\right) n^{\frac{1}{n}} \leq \left(|c_n|^{\frac{1}{n}}\right) \frac{x}{y} < x\right).$$

Thus,

$$\left(n \geq \max\{N_1, N_2\} \Rightarrow |c_n n|^{\frac{1}{n}} < x\right).$$

■)

Similarly, $\limsup_{n\to\infty}\left(|(c_n n)n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right)$. Thus, $\limsup_{n\to\infty}$ $\left(\left|(c_n n^2)\right|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)$, etc.

Case I: when $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in (0, \infty)$. By Conclusion 1.49, for every

$$z \in D\left(a; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right),$$

$\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent, and for every

$$z \notin D\left[a; \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)}\right],$$

$\sum_{n=1}^{\infty} c_0(z-a)^n$ is divergent. Now, since for every $z \in D(a; r)$, the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent, we have

$$r \leq \frac{1}{\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)} \left(= \frac{1}{\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right)}\right).$$

Since

$$\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right), \text{and} \left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) \in (0, \infty),$$

we have

$$\left(\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right)\right) \in (0, \infty),$$

and hence, by Conclusion 1.49, for every

$$z \in D\left(a; \frac{1}{\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right)}\right),$$

$\sum_{n=0}^{\infty}(c_n n)(z-a)^n$ is absolutely convergent. Now, since

$$r \leq \frac{1}{\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right)},$$

for every $z \in D(a;r)$, $\sum_{n=0}^{\infty}(c_n n)(z-a)^n$ is an absolutely convergent.

Case II: when $\left(\limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right)\right) = 0$. Now, since

$$\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right) = \limsup_{n\to\infty}\left(|c_n|^{\frac{1}{n}}\right),$$

we have

$$\left(\limsup_{n\to\infty}\left(|c_n n|^{\frac{1}{n}}\right)\right) = 0,$$

and hence, by Conclusion 1.49, for every $z \in \mathbb{C}$, $\sum_{n=0}^{\infty} c_n n(z-a)^n$ is an absolutely convergent series. It follows that for every $z \in D(a;r)$, $\sum_{n=0}^{\infty}(c_n n)(z-a)^n$ is absolutely convergent.

Thus, in all cases, for every $z \in D(a;r)$,

$$\sum_{n=0}^{\infty}(c_n n)(z-a)^n$$

is absolutely convergent. It follows that, for every $z \in D'(a;r)$,

$$\sum_{n=1}^{\infty}\frac{1}{z-a}(c_n n)(z-a)^n\left(=\sum_{n=1}^{\infty}(c_n n)(z-a)^{n-1}\right)$$

is absolutely convergent, and hence for every $z \in D(a;r)$,

$$\sum_{n=1}^{\infty}(c_n n)(z-a)^{n-1}$$

is absolutely convergent. Similarly, for every $z \in D(a;r)$,

$$\sum_{n=2}^{\infty}\left(c_n n^2\right)(z-a)^{n-2}$$

is absolutely convergent,

$$\sum_{n=2}^{\infty}(c_n n(n-1))(z-a)^{n-2}$$

is absolutely convergent, etc.

Conclusion 1.53 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f:\Omega\to\mathbb{C}$. Let f be representable by power series in Ω. Let $D(a;r)\subset\Omega$. Suppose that for every $z\in D(a;r)$,

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n.$$

Then, for every $z\in D(a;r)$, $\sum_{n=1}^{\infty}c_n n(z-a)^{n-1}, \sum_{n=2}^{\infty}c_n n^2(z-a)^{n-2}, \sum_{n=2}^{\infty}(c_n n(n-1))(z-a)^{n-2}, \ldots$ are absolutely convergent. Also, for every $z\in D(a;r)$,

$$f'(z)=\sum_{n=1}^{\infty}c_n n(z-a)^{n-1}.$$

Proof of the remaining part For simplicity of calculation, let us take 0 for a. Now, let us fix any $b\in D(0;r)$. We have to show that

$$\lim_{z\to b}\left(\sum_{n=1}^{\infty}c_n\frac{1}{z-b}(z^n-b^n)\right)=\lim_{z\to b}\frac{1}{z-b}\left(\sum_{n=1}^{\infty}c_n(z^n-b^n)\right)$$
$$=\lim_{z\to b}\frac{1}{z-b}\left(\sum_{n=0}^{\infty}c_nz^n-\sum_{n=0}^{\infty}c_nb^n\right)=\underbrace{\lim_{z\to b}\frac{f(z)-f(b)}{z-b}=\sum_{n=1}^{\infty}c_nnb^{n-1}}\,,$$

that is

$$\lim_{z\to b}\left(\sum_{n=1}^{\infty}c_n\left(\frac{1}{z-b}(z^n-b^n)-nb^{n-1}\right)\right)$$
$$=\underbrace{\lim_{z\to b}\left(\sum_{n=1}^{\infty}c_n\frac{1}{z-b}(z^n-b^n)-\sum_{n=1}^{\infty}c_nnb^{n-1}\right)=0}\,,$$

that is

$$\lim_{z\to b}\left(\sum_{n=1}^{\infty}c_n\left(\frac{z^n-b^n}{z-b}-nb^{n-1}\right)\right)=0,$$

that is

$$\lim_{z \to b} \left(\sum_{n=2}^{\infty} c_n \left(\frac{z^n - b^n}{z - b} - nb^{n-1} \right) \right) = 0,$$

that is

$$\lim_{z \to b} \left| \sum_{n=2}^{\infty} c_n \left(\frac{z^n - b^n}{z - b} - nb^{n-1} \right) \right| = 0.$$

It suffices to show that

$$\lim_{z \to b} \sum_{n=2}^{\infty} \left| c_n \left(\frac{z^n - b^n}{z - b} - nb^{n-1} \right) \right| = 0,$$

that is

$$\lim_{z \to b} \sum_{n=2}^{\infty} |c_n| \left| \frac{z^n - b^n}{z - b} - nb^{n-1} \right| = 0.$$

Since $b \in D(0; r)$, $|b| < r$. There exists a positive real ρ such that $|b| < \rho < r$. Since for every positive integer $n \geq 2$, and for every $z \in (D(0; \rho) - \{b\})$,

$$\begin{aligned}
&\frac{z^n - b^n}{z - b} - nb^{n-1} = \underbrace{z^{n-1} + z^{n-2}b + z^{n-3}b^2 + \cdots + b^{n-1}}_{n\text{terms}} - nb^{n-1} \\
&= \underbrace{z^{n-1} + z^{n-2}b + z^{n-3}b^2 + \cdots + zb^{n-2}}_{(n-1)\text{terms}} - (n-1)b^{n-1} \\
&= \underbrace{\left(z^{n-1} - b^{n-1}\right) + \left(z^{n-2}b - b^{n-1}\right) + \left(z^{n-3}b^2 - b^{n-1}\right) + \cdots + \left(zb^{n-2} - b^{n-1}\right)}_{(n-1)\text{terms}} \\
&= \underbrace{\left(z^{n-1} - b^{n-1}\right) + \left(z^{n-2} - b^{n-2}\right)b + \left(z^{n-3} - b^{n-3}\right)b^2 + \cdots + (z-b)b^{n-2}}_{(n-1)\text{terms}} \\
&= (z-b)\left(z^{n-2} + z^{n-3}b + z^{n-4}b^2 + \cdots + b^{n-2}\right) \\
&\quad + (z-b)\left(z^{n-3} + z^{n-4}b + z^{n-5}b^2 + \cdots + b^{n-3}\right)b \\
&\quad + (z-b)\left(z^{n-4} + z^{n-5}b + z^{n-6}b^2 + \cdots + b^{n-4}\right)b^2 + \cdots + (z-b)b^{n-2} \\
&= (z-b)\left(z^{n-2} + z^{n-3}b + z^{n-4}b^2 + \cdots + b^{n-2}\right) + \left(z^{n-3}b + z^{n-4}b^2 + z^{n-5}b^3 + \cdots + b^{n-2}\right) \\
&\quad + \left(z^{n-4}b^2 + z^{n-5}b^3 + z^{n-6}b^4 + \cdots + b^{n-2}\right) + \cdots + (z-b)b^{n-2} \\
&= (z-b)\left(z^{n-2} + 2z^{n-3}b + 3z^{n-4}b^2 + \cdots + (n-1)b^{n-2}\right),
\end{aligned}$$

we have, for every positive integer $n \geq 2$, and for every $z \in (D(0;\rho) - \{b\})$,

$$|c_n|\left|\frac{z^n - b^n}{z - b} - nb^{n-1}\right| \leq |c_n||z - b|\Big(|z|^{n-2} + 2|z|^{n-3}|b|$$
$$+ 3|z|^{n-4}|b|^2 + \cdots + (n-1)|b|^{n-2}\Big) \leq |c_n||z - b|\big(\rho^{n-2} + 2\rho^{n-3}\rho$$
$$+ 3\rho^{n-4}\rho^2 + \cdots + (n-1)\rho^{n-2}\big) = |c_n||z - b|(1 + 2 + 3 + \cdots + (n-1))\rho^{n-2}$$
$$= |c_n||z - b|\frac{(n-1)n}{2}\rho^{n-2} \leq |z - b|\big(|c_n|n^2\rho^{n-2}\big),$$

and hence, for every positive integer $n \geq 2$, and for every $z \in (D(0;\rho) - \{b\})$,

$$\sum_{n=2}^{\infty} |c_n|\left|\frac{z^n - b^n}{z - b} - nb^{n-1}\right| \leq |z - b|\sum_{n=2}^{\infty}\big(|c_n|n^2\rho^{n-2}\big).$$

It suffices to show that $\sum_{n=2}^{\infty}(|c_n|n^2\rho^n)$ is convergent. Since for every $z \in D(0;r)$, the series $\sum_{n=2}^{\infty} c_n n^2 z^{n-1}$ is absolutely convergent, and $0 < \rho < r$, the series $\sum_{n=1}^{\infty} c_n n^2 \rho^{n-1}$ is absolutely convergent, and hence $\sum_{n=2}^{\infty}(|c_n|n^2\rho^n)$ is convergent. ∎

Note 1.54 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let f be representable by power series in Ω.

Problem 1.55

1. $f \in H(\Omega)$,
2. $f' : \Omega \to \mathbb{C}$ is representable by power series in Ω,
3. $f' \in H(\Omega)$,
4. $f'' : \Omega \to \mathbb{C}$ is representable by power series in Ω, etc.

(Solution For 1: Let us take any $a \in \Omega$. We have to show that $f'(a)$ exists. Since $a \in \Omega$, and Ω is open, there exists a positive real r such that $D(a;r) \subset \Omega$. Now, since f is representable by power series in Ω, there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that

a. for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent,
b. for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$.

By Conclusion 1.17, for every $z \in D(a;r)$, $\sum_{n=1}^{\infty}(c_n n)(z-a)^{n-1}$ is convergent, and for every $z \in D(a;r)$, $f'(z) = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1}$. This shows that $f'(a)(= c_1 1 = c_1)$ exists.

For 2: Since $f \in H(\Omega)$, we have $f' : \Omega \to \mathbb{C}$. Let us take any $D(a;r)$ such that $D(a;r) \subset \Omega$. Now, since f is representable by power series in Ω, there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that

a. for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is convergent,
b. for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$.

By Conclusion 1.53, for every $z \in D(a;r)$, $\sum_{n=1}^{\infty}(c_n n)(z-a)^{n-1}$ is convergent, and for every $z \in D(a;r)$, $f'(z) = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1}$. This shows that $f' : \Omega \to \mathbb{C}$ is representable by power series in Ω.

For 3: Now, by applying Conclusion 1.56, $f' \in H(\Omega)$.

For 4: Now, by applying Conclusion 1.56, $f'' : \Omega \to \mathbb{C}$ is representable by power series in Ω. ∎)

Conclusion 1.56 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let f be representable by power series in Ω. Then

1. $f \in H(\Omega)$,
2. $f' : \Omega \to \mathbb{C}$ is representable by power series in Ω,
3. $f' \in H(\Omega)$,
4. $f'' : \Omega \to \mathbb{C}$ is representable by power series in Ω, etc.

Note 1.57 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let f be representable by power series in Ω. Let $D(a;r) \subset \Omega$. Suppose that for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$.

Problem 1.58 For every $z \in D(a;r)$,

$$f'(z) = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1},$$

$$f''(z) = \sum_{n=2}^{\infty} c_n n(n-1)(z-a)^{n-2},$$

$$f'''(z) = \sum_{n=3}^{\infty} c_n n(n-1)(n-2)(z-a)^{n-3}, \text{etc.}$$

(Solution Since f is representable by power series in Ω, by Conclusion (1.53), for every $z \in D(a;r)$, $f'(z) = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1}$. By Conclusion (1.56), $f' : \Omega \to \mathbb{C}$ is representable by power series in Ω. Since $f' : \Omega \to \mathbb{C}$ is representable by power series in Ω, and for every $z \in D(a;r)$, $f'(z) = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1}$, by Conclusion (1.53), for every $z \in D(a;r)$,

$$f''(z) = \sum_{n=2}^{\infty} c_n n(n-1)(z-a)^{n-2}.$$

Similarly, for every $z \in D(a;r)$,

$$f'''(z) = \sum_{n=3}^{\infty} c_n n(n-1)(n-2)(z-a)^{n-3}, \text{etc.}$$

∎)

Conclusion 1.59 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let f be representable by power series in Ω. Let $D(a;r) \subset \Omega$. Suppose that for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$.

Then, for every $z \in D(a;r)$,

$$f'(z) = \sum_{n=1}^{\infty} c_n n (z-a)^{n-1},$$

$$f''(z) = \sum_{n=2}^{\infty} c_n n(n-1)(z-a)^{n-2},$$

$$f'''(z) = \sum_{n=3}^{\infty} c_n n(n-1)(n-2)(z-a)^{n-3}, \text{etc.}$$

And hence

$$f'(a) = c_1 1,\ \ f''(a) = c_2 2 \cdot 1,\ \ f'''(a) = c_3 3 \cdot 2 \cdot 1, \text{etc.}$$

Lemma 1.60 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let f be representable by power series in Ω. Let $D(a;r) \subset \Omega$. Then, there exist unique complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that

1. for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} c_n (z-a)^n$ is convergent,
2. for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$.

Furthermore, for every $z \in D(a;r)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n.$$

Proof It only remains to prove uniqueness part. Let $c_0, c_1, c_2, c_3, \ldots$, and $b_0, b_1, b_2, b_3, \ldots$ be complex numbers such that

1. for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} c_n (z-a)^n$ is convergent,
2. for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$,
3. for every $z \in D(a;r)$, the series $\sum_{n=0}^{\infty} b_n (z-a)^n$ is convergent,
4. for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$.

We have to show that, for every nonnegative integer n, $b_n = c_n$.

By Conclusion 1.59, for every nonnegative integer n, $f^{(n)}(a) = c_n(n!)$, and $f^{(n)}(a) = b_n(n!)$, and hence for every nonnegative integer n, $b_n = c_n$. Also, for every $z \in D(a;r)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

∎

1.3 Winding Number

Note 1.61 Let X be any nonempty set. Let $\mathcal{M}$ be a σ-algebra in X. Let $\mu : \mathcal{M} \to [0, \infty)$ be a measure on $\mathcal{M}$. Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\varphi : X \to \mathbb{C}$ be a member of $L^1(\mu)$. Let $\varphi(X) \subset \Omega^c$. Let $D(a;r) \subset \Omega$, where $r > 0$, and $a \in \Omega$. Let $b \in D(a;r)$.

Problem 1.62

$$\frac{1}{\varphi(\zeta)-a} + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right) + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^2 + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^3 + \cdots$$

converges uniformly to $\frac{1}{\varphi(\zeta)-a}$ on X.

(Solution Let us take any $\zeta \in X$. We first try to show that

$$\frac{1}{\varphi(\zeta)-a} + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right) + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^2 + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^3 + \cdots$$

is absolutely convergent, that is

$$\frac{1}{|\varphi(\zeta)-a|} + \left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right) + \left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)^2 + \left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)^3 + \cdots$$

is convergent. Since $b \in D(a;r)$, $|b-a| < r$. Since $\zeta \in X$, and $\varphi : X \to \Omega^c$, $\varphi(\zeta) \notin \Omega(\supset D(a;r))$, $\varphi(\zeta) \notin D(a;r)$, and hence $(0 \le |b-a| <)r \le |\varphi(\zeta)-a|$. This shows that $0 \le \frac{|b-a|}{|\varphi(\zeta)-a|} < 1$. That is, the common ratio of the geometric series

$$\frac{1}{|\varphi(\zeta)-a|}+\left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)+\left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)^2 \\ +\left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)^3+\cdots$$

is a member of $[0,1)$. It follows that

$$\frac{1}{|\varphi(\zeta)-a|}+\left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)+\left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)^2 \\ +\left(\frac{1}{|\varphi(\zeta)-a|}\right)\left(\frac{|b-a|}{|\varphi(\zeta)-a|}\right)^3+\cdots$$

is convergent. Since for every $\zeta \in X$,

$$\frac{1}{\varphi(\zeta)-a}+\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)+\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^2 \\ +\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^3+\cdots$$

is absolutely convergent, for every $\zeta \in X$,

$$\frac{1}{\varphi(\zeta)-a}+\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)+\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^2 \\ +\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^3+\cdots$$

is convergent, and its sum is

$$\lim_{n\to\infty}\left(\frac{\frac{1}{\varphi(\zeta)-a}}{1-\frac{b-a}{\varphi(\zeta)-a}}\left(1-\left(\frac{b-a}{\varphi(\zeta)-a}\right)^n\right)\right)=\frac{\frac{1}{\varphi(\zeta)-a}}{1-\frac{b-a}{\varphi(\zeta)-a}}\left(1-\lim_{n\to\infty}\left(\left(\frac{b-a}{\varphi(\zeta)-a}\right)^n\right)\right) \\ =\frac{\frac{1}{\varphi(\zeta)-a}}{1-\frac{b-a}{\varphi(\zeta)-a}}(1-0)=\frac{1}{\varphi(\zeta)-b}.$$

Thus,

$$\zeta\mapsto\left(\frac{1}{\varphi(\zeta)-a}+\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)+\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^2\right. \\ \left.+\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^3+\cdots\right)$$

is a function from X to $\mathbb{C}$. Since for every $\zeta \in X$, and for every positive integer n,

$$\left|\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^n\right| = |b-a|^n \frac{1}{|\varphi(\zeta)-a|^{n+1}} \leq |b-a|^n \frac{1}{r^{n+1}} = \frac{1}{r}\left(\frac{|b-a|}{r}\right)^n,$$

we have, for every $\zeta \in X$, and for every positive integer n,

$$\left|\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^n\right| \leq \frac{1}{r}\left(\frac{|b-a|}{r}\right)^n.$$

Since $0 \leq |b-a| < r$, we have $\frac{|b-a|}{r} \in [0,1)$, and hence the geometric series $\sum_{n=1}^{\infty} \frac{1}{r}\left(\frac{|b-a|}{r}\right)^n$ is convergent. Since for every $\zeta \in X$, and for every positive integer n,

$$\left|\left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^n\right| \leq \frac{1}{r}\left(\frac{|b-a|}{r}\right)^n,$$

and $\sum_{n=1}^{\infty} \frac{1}{r}\left(\frac{|b-a|}{r}\right)^n$ is convergent, by the Weierstrass M-test

$$\left(\frac{1}{\varphi(\zeta)-a} + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right) + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^2 + \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^3 + \cdots\right)$$

converges uniformly to $\frac{1}{\varphi(\zeta)-b}$ on X. ∎)

It follows that

$$\begin{aligned}
\int_X \frac{1}{\varphi(\zeta)-b}\,\mathrm{d}\mu(\zeta) = {} & \int_X \frac{1}{\varphi(\zeta)-a}\,\mathrm{d}\mu(\zeta) \\
& + \int_X \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)\mathrm{d}\mu(\zeta) \\
& + \int_X \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^2 \mathrm{d}\mu(\zeta) \\
& + \int_X \left(\frac{1}{\varphi(\zeta)-a}\right)\left(\frac{b-a}{\varphi(\zeta)-a}\right)^3 \mathrm{d}\mu(\zeta) + \cdots
\end{aligned}$$

$$= \left(\int_X \frac{1}{\varphi(\zeta) - a} \mathrm{d}\mu(\zeta) \right) + \left(\int_X \frac{1}{(\varphi(\zeta) - a)^2} \mathrm{d}\mu(\zeta) \right) (b - a)$$
$$+ \left(\int_X \frac{1}{(\varphi(\zeta) - a)^3} \mathrm{d}\mu(\zeta) \right) (b - a)^2 + \left(\int_X \frac{1}{(\varphi(\zeta) - a)^4} \mathrm{d}\mu(\zeta) \right) (b - a)^3$$
$$+ \cdots = c_0 + c_1(b - a) + c_2(b - a)^2 + c_3(b - a)^3 + \cdots,$$

where, for every nonnegative integer $n, c_n \equiv \int_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}\mu(\zeta)$.

Conclusion 1.63 Let X be any nonempty set. Let $\mathcal{M}$ be a σ-algebra in X. Let $\mu : \mathcal{M} \to [0, \infty)$ be a measure on $\mathcal{M}$. Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\varphi : X \to \mathbb{C}$ be a member of $L^1(\mu)$. Let $\varphi(X) \subset \Omega^c$. Let $D(a; r) \subset \Omega$, where $r > 0$, and $a \in \Omega$. Let $b \in D(a; r)$. Then

$$\int_X \frac{1}{\varphi(\zeta) - b} \mathrm{d}\mu(\zeta) = c_0 + c_1(b - a) + c_2(b - a)^2 + c_3(b - a)^3 + \cdots,$$

where, for every nonnegative integer $n, c_n \equiv \int_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}\mu(\zeta)$.

Note 1.64

Definition Let X be any nonempty set. Let $\mathcal{M}$ be a σ-algebra in X. Let $\mu : \mathcal{M} \to \mathbb{C}$ be a complex measure on $\mathcal{M}$.

Since $\mu : \mathcal{M} \to \mathbb{C}$ is a complex measure on $\mathcal{M}$, $\mathrm{Re}(\mu) : \mathcal{M} \to \mathbb{R}$, and $\mathrm{Im}(\mu) : \mathcal{M} \to \mathbb{R}$ are signed measures on $\mathcal{M}$. It follows that

$$\mathrm{Re}(\mu) = (\mathrm{Re}(\mu))^+ - (\mathrm{Re}(\mu))^-,$$

where $(\mathrm{Re}(\mu))^+, (\mathrm{Re}(\mu))^-$ are positive finite measures on $\mathcal{M}$. Also,

$$\mathrm{Im}(\mu) = (\mathrm{Im}(\mu))^+ - (\mathrm{Im}(\mu))^-,$$

where $(\mathrm{Im}(\mu))^+, (\mathrm{Im}(\mu))^-$ are positive finite measures on $\mathcal{M}$. Thus,

$$\mu = \left((\mathrm{Re}(\mu))^+ - (\mathrm{Re}(\mu))^-\right) + i\left((\mathrm{Im}(\mu))^+ - (\mathrm{Im}(\mu))^-\right).$$

Suppose that $\psi \in L^1(|\mu|)$. Since $\psi \in L^1(|\mu|)$, and

$$|\mu| \geq |\mathrm{Re}(\mu)| = (\mathrm{Re}(\mu))^+ + (\mathrm{Re}(\mu))^- \geq (\mathrm{Re}(\mu))^+,$$

we have $\psi \in L^1\left((\mathrm{Re}(\mu))^+\right)$, and hence $\int_X \psi \mathrm{d}\left((\mathrm{Re}(\mu))^+\right) \in \mathbb{C}$. Similarly, $\int_X \psi \mathrm{d}((\mathrm{Re}(\mu))^-) \in \mathbb{C}$, $\int_X \psi \mathrm{d}\left((\mathrm{Im}(\mu))^+\right) \in \mathbb{C}$, and $\int_X \psi \mathrm{d}((\mathrm{Im}(\mu))^-) \in \mathbb{C}$.

By $\int_X \psi \mathrm{d}\mu$ we mean

$$\left(\int\limits_X \psi \mathrm{d}\big((\mathrm{Re}(\mu))^+\big) - \int\limits_X \psi \mathrm{d}((\mathrm{Re}(\mu))^-)\right) + i\left(\int\limits_X \psi \mathrm{d}\big((\mathrm{Im}(\mu))^+\big) - \int\limits_X \psi \mathrm{d}((\mathrm{Im}(\mu))^-)\right).$$

$\int_X \psi \mathrm{d}\mu$ is also denoted by $\int_X \psi(\zeta)\mathrm{d}\mu(\zeta)$.

Let X be any nonempty set. Let $\mathcal{M}$ be a σ-algebra in X. Let $\mu : \mathcal{M} \to \mathbb{C}$ be a complex measure on $\mathcal{M}$. Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\varphi : X \to \mathbb{C}$ be a function such that $\varphi \in L^1(|\mu|)$. Let $\varphi(X) \subset \Omega^{\mathrm{c}}$. Let $D(a;r) \subset \Omega$, where $r > 0$, and $a \in \Omega$. Let $b \in D(a;r)$.

Problem 1.65

$$\int_X \frac{1}{\varphi(\zeta) - b} \mathrm{d}\mu(\zeta) = c_0 + c_1(b-a) + c_2(b-a)^2 + c_3(b-a)^3 + \cdots,$$

where, for every nonnegative integer $n, c_n \equiv \int_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}\mu(\zeta)$.

(Solution Since $\varphi \in L^1(|\mu|)$, we have $\varphi \in L^1\big((\mathrm{Re}(\mu))^+\big)$, and hence, by Conclusion 1.63,

$$\int\limits_X \frac{1}{\varphi(\zeta) - b} \mathrm{d}\big((\mathrm{Re}(\mu))^+\big)(\zeta) = \sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}\big((\mathrm{Re}(\mu))^+\big)(\zeta)\right)(b-a)^n.$$

Similarly,

$$\int\limits_X \frac{1}{\varphi(\zeta) - b} \mathrm{d}((\mathrm{Re}(\mu))^-)(\zeta) = \sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}((\mathrm{Re}(\mu))^-)(\zeta)\right)(b-a)^n,$$

$$\int\limits_X \frac{1}{\varphi(\zeta) - b} \mathrm{d}\big((\mathrm{Im}(\mu))^+\big)(\zeta) = \sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}\big((\mathrm{Im}(\mu))^+\big)(\zeta)\right)(b-a)^n,$$

$$\int\limits_X \frac{1}{\varphi(\zeta) - b} \mathrm{d}((\mathrm{Im}(\mu))^-)(\zeta) = \sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}((\mathrm{Im}(\mu))^-)(\zeta)\right)(b-a)^n.$$

Now

$$
\begin{aligned}
\text{LHS} &= \int\limits_X \frac{1}{\varphi(\zeta)-b}\,\mathrm{d}\mu(\zeta) \\
&= \left(\int\limits_X \frac{1}{\varphi(\zeta)-b}\mathrm{d}\big((\mathrm{Re}(\mu))^+\big)(\zeta) - \int\limits_X \frac{1}{\varphi(\zeta)-b}\mathrm{d}((\mathrm{Re}(\mu))^-)(\zeta)\right) \\
&\quad + i\left(\int\limits_X \frac{1}{\varphi(\zeta)-b}\mathrm{d}\big((\mathrm{Im}(\mu))^+\big)(\zeta) - \int\limits_X \frac{1}{\varphi(\zeta)-b}\mathrm{d}((\mathrm{Im}(\mu))^-)(\zeta)\right) \\
&= \left(\sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}\big((\mathrm{Re}(\mu))^+\big)(\zeta)\right)(b-a)^n\right. \\
&\quad \left. - \sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}((\mathrm{Re}(\mu))^-)(\zeta)\right)(b-a)^n\right) \\
&\quad + i\left(\sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}\big((\mathrm{Re}(\mu))^+\big)(\zeta)\right)(b-a)^n\right. \\
&\quad \left. - \sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}((\mathrm{Re}(\mu))^-)(\zeta)\right)(b-a)^n\right) \\
&= \sum_{n=0}^{\infty}\left(\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}\big((\mathrm{Re}(\mu))^+\big)(\zeta) - \int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}((\mathrm{Re}(\mu))^-)(\zeta)\right)\right. \\
&\quad + i\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}\big((\mathrm{Re}(\mu))^+\big)(\zeta)\right. \\
&\quad \left.\left. - \int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}((\mathrm{Re}(\mu))^-)(\zeta)\right)(b-a)^n\right) \\
&= \sum_{n=0}^{\infty}\left(\int\limits_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}\mu(\zeta)\right)(b-a)^n \\
&= c_0 + c_1(b-a) + c_2(b-a)^2 + c_3(b-a)^3 + \cdots = \text{RHS},
\end{aligned}
$$

where, for every nonnegative integer $n, c_n \equiv \int_X \frac{1}{(\varphi(\zeta)-a)^{n+1}}\mathrm{d}\mu(\zeta)$. ∎)

Conclusion 1.66 Let X be any nonempty set. Let $\mathcal{M}$ be a σ-algebra in X. Let $\mu : \mathcal{M} \to \mathbb{C}$ be a complex measure on $\mathcal{M}$. Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\varphi : X \to \mathbb{C}$ be a function such that $\varphi \in L^1(|\mu|)$. Let $\varphi(X) \subset \Omega^{\mathrm{c}}$.

a. If $D(a;r) \subset \Omega$, where $r > 0$, and $a \in \Omega$, and $b \in D(a;r)$, then

$$\int_X \frac{1}{\varphi(\zeta) - b} \mathrm{d}\mu(\zeta) = c_0 + c_1(b-a) + c_2(b-a)^2 + c_3(b-a)^3 + \cdots,$$

where, for every nonnegative integer $n, c_n \equiv \int_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}\mu(\zeta)$.

b. The function $f : z \mapsto \int_X \frac{1}{\varphi(\zeta)-z} \mathrm{d}\mu(\zeta)$ from Ω to $\mathbb{C}$ is representable by power series in Ω.

Proof of the remaining part (b) For this purpose, let us take any $a \in \Omega$, and a positive real number r such that $D(a;r) \subset \Omega$. By Conclusion 1.66(a), for every $z \in D(a;r)$,

$$(f(z) =) \int_X \frac{1}{\varphi(\zeta) - z} \mathrm{d}\mu(\zeta) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots,$$

where, for every nonnegative integer $n, c_n = \int_X \frac{1}{(\varphi(\zeta)-a)^{n+1}} \mathrm{d}\mu(\zeta)$. Thus, for every $z \in D(a;r), f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Hence, f is representable by power series in Ω. ∎

Lemma 1.67 Let a, b, c, d be any real numbers such that $a < b$, and $c < d$. Let $f : [a,b] \to [c,d]$ be a 1–1, onto, continuous function. Let $f(a) = c$, and $f(b) = d$. Then

a. $f([a,b]) = [f(a).f(b)]$,
b. $f : [a,b] \to [c,d]$ is strictly increasing.

Proof

a. Since $f : [a,b] \to [c,d]$, we have $f([a,b]) \subset [c,d](= [f(a),d] = [f(a),f(b)])$, and hence $f([a,b]) \subset [f(a),f(b)]$. It remains to show that $[f(a),f(b)] \subset f([a,b])$.

For this purpose, let us take any $y \in [f(a),f(b)]$. We have to show that $y \in f([a,b])$. Since $y \in [f(a),f(b)]$, we have $f(a) \leq y \leq f(b)$. Since $f : [a,b] \to [c,d]$ is continuous, $[a,b]$ is connected and $f(a) \leq y \leq f(b)$, there exists $t \in [a,b]$ such that $(f([a,b]) \ni) f(t) = y$, and hence $y \in f([a,b])$.

b. Let us take any $s, t \in [a,b]$, where $s < t$. We have to show that $f(s) < f(t)$. If not, otherwise, let $f(t) \leq f(s)$. We have to arrive at a contradiction.

Since $s < t$, we have $s \neq t$. Now, since $f : [a,b] \to [c,d]$ is 1–1, $f(s) \neq f(t)$. Since $f(s) \neq f(t)$, and $f(t) \leq f(s)$, we have $f(t) < f(s)$. Now, since $f(t), f(s) \in [c,d] = [f(a),f(b)]$, we have $f(a) \leq f(t) < f(s) \leq f(b)$.

Case I: when $s = a$. Now, since $f(t) < f(s)$, we have $f(t) < f(a)$. This is a contradiction.

Case II: when $s \in (a, b)$. It follows that

$$\underbrace{f(s) \in f((a,b))} = f([a,b] - \{a,b\}) = f([a,b]) - f(\{a,b\})$$

$$= [f(a), f(b)] - \{f(a), f(b)\} = (f(a), f(b)).$$

Thus, $f(s) < f(b)$. Since $f(t) < (f(s)) < f(b)$, $f|_{[t,b]} : [t,b] \to [c,d]$ is continuous, and $[t,b]$ is connected, there exists $u \in [t,b]$ such that $f(u) = (f(s))$. Now, since f is 1–1, we have $([t,b] \ni) u = s$, and hence $t \leq s$. This is a contradiction. ∎

Note 1.68

Definition Let X be a topological space. Let α, β be any real numbers such that $\alpha < \beta$. Let γ be a function from the closed interval $[\alpha, \beta]$ to X. Let γ be a curve in X, that is, $\gamma : [\alpha, \beta] \to X$ is continuous. Here $[\alpha, \beta]$ is called the ***parameter interval*** *of* γ, $\gamma(\alpha)$ is called the ***initial point*** *of* γ, and $\gamma(\beta)$ is called the ***end point*** *of* γ. If $\gamma(\alpha) = \gamma(\beta)$, then we say that γ *is a* ***closed curve***.

Definition Let α, β be any real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a curve in $\mathbb{C}$, that is $\gamma : [\alpha, \beta] \to \mathbb{C}$ is continuous. If, for every $s \in (\alpha, \beta)$,

$$(\gamma'(s) \equiv) \lim_{h \to 0} \frac{\gamma(s+h) - \gamma(s)}{h}$$

exists,

$$(\gamma'(\alpha) \equiv) \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{\gamma(\alpha + \varepsilon) - \gamma(\alpha)}{\varepsilon}$$

exists, and

$$(\gamma'(\beta) \equiv) \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{\gamma(\beta - \varepsilon) - \gamma(\beta)}{-\varepsilon}$$

exists, then we say that $\gamma : [\alpha, \beta] \to \mathbb{C}$ *is* ***differentiable***. Thus, $\gamma' : [\alpha, \beta] \to \mathbb{C}$ is a function. Also, if $\gamma' : [\alpha, \beta] \to \mathbb{C}$ is continuous, then we say that $\gamma : [\alpha, \beta] \to \mathbb{C}$ *is* ***continuously differentiable***.

Definition Let α, β be any real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a curve in $\mathbb{C}$.

If, there exists a partition $\{s_0, s_1, \ldots, s_n\}$ of $[\alpha, \beta]$, where $\alpha = s_0 < s_1 < \cdots < s_n = \beta$, such that $\gamma|_{[s_0,s_1]}: [s_0, s_1] \to \mathbb{C}$ is continuously differentiable, $\gamma|_{[s_1,s_2]}: [s_1, s_2] \to \mathbb{C}$ is continuously differentiable,..., $\gamma|_{[s_{n-1},s_n]}: [s_{n-1}, s_n] \to \mathbb{C}$ is continuously differentiable, then we say that $\gamma : [\alpha, \beta] \to \mathbb{C}$ *is a **path***.

In short, $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path means $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a piecewise continuously differentiable curve in the plane.

Let $\alpha, \beta, \alpha_1, \beta_1$ be any real numbers such that $\alpha < \beta$, and $\alpha_1 < \beta_1$. Let $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ be a 1–1, onto, continuous function. Let $\varphi(\alpha_1) = \alpha$, and $\varphi(\beta_1) = \beta$. Let $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ be continuously differentiable. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a path.

Problem 1.69 $(\gamma \circ \varphi) : [\alpha_1, \beta_1] \to \mathbb{C}$ is a path.

(**Solution** Since $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a curve in $\mathbb{C}$, $\gamma : [\alpha, \beta] \to X$ is continuous. Next, since $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ is continuous, $(\gamma \circ \varphi) : [\alpha_1, \beta_1] \to \mathbb{C}$ is continuous. Thus, $(\gamma \circ \varphi) : [\alpha_1, \beta_1] \to \mathbb{C}$ is a curve.

Since $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path, there exists a partition $\{s_0, s_1, \ldots, s_n\}$ of $[\alpha, \beta]$, where $\alpha = s_0 < s_1 < \cdots < s_n = \beta$, such that $\gamma|_{[s_0,s_1]}: [s_0, s_1] \to \mathbb{C}$ is continuously differentiable, $\gamma|_{[s_1,s_2]}: [s_1, s_2] \to \mathbb{C}$ is continuously differentiable, $\cdots$, $\gamma|_{[s_{n-1},s_n]}: [s_{n-1}, s_n] \to \mathbb{C}$ is continuously differentiable. Since $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ is a 1–1, onto, continuous function, and $[\alpha_1, \beta_1]$ is compact, $\varphi^{-1} : [\alpha, \beta] \to [\alpha_1, \beta_1]$ is a 1–1, onto, continuous function. Now, since $\alpha = s_0 < s_1 < \cdots < s_n = \beta$, by Lemma 1.67 (b), $(\alpha_1 =)\varphi^{-1}(\alpha) = \varphi^{-1}(s_0) < \varphi^{-1}(s_1) < \cdots < \varphi^{-1}(s_n) = \varphi^{-1}(\beta)(= \beta_1)$, and hence

$$\left\{\varphi^{-1}(s_0), \varphi^{-1}(s_1), \ldots, \varphi^{-1}(s_n)\right\}$$

is a partition of $[\alpha_1, \beta_1]$.

Problem 1.70 $(\gamma \circ \varphi)|_{[\varphi^{-1}(s_0),\varphi^{-1}(s_1)]}: [\varphi^{-1}(s_0), \varphi^{-1}(s_1)] \to \mathbb{C}$ is continuously differentiable.

(**Solution** Let us fix any $t \in (\varphi^{-1}(s_0), \varphi^{-1}(s_1))$. We shall show that $(\gamma \circ \varphi)'(t)$ exists. By chain rule, it suffices to show that $\gamma'(\varphi(t))$ and $\varphi'(t)$ exist.

Since $t \in (\varphi^{-1}(s_0), \varphi^{-1}(s_1)), \varphi^{-1}(s_0) < t < \varphi^{-1}(s_1)$, and hence $s_0 < \varphi(t) < s_1$. Now, since $\varphi(t) \in (s_0, s_1)$, and $\gamma|_{[s_0,s_1]}: [s_0, s_1] \to \mathbb{C}$ is continuously differentiable, $\gamma'(\varphi(t))$ exists. Since $t \in (\varphi^{-1}(s_0), \varphi^{-1}(s_1))(\subset (\alpha_1, \beta_1))$, and $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ is continuously differentiable, $\varphi'(t)$ exists.

Now, we shall try to show that

$$(\gamma \circ \varphi)'(\alpha_1) = \underbrace{(\gamma \circ \varphi)'\left(\varphi^{-1}(s_0)\right) = \left(\gamma'\left(\varphi\left(\varphi^{-1}(s_0)\right)\right)\right)\left(\varphi'\left(\varphi^{-1}(s_0)\right)\right)}$$

$$= (\gamma'(s_0))(\varphi'(\alpha_1)) = (\gamma'(\alpha))(\varphi'(\alpha_1)),$$

that is

$$(\gamma \circ \varphi)'(\alpha_1) = (\gamma'(\alpha))(\varphi'(\alpha_1)).$$

Since $\varphi'(\alpha_1)$ exists, there exists a function $\psi : [\alpha_1, \beta_1] \to \mathbb{R}$ such that

a. $\psi : [\alpha_1, \beta_1] \to \mathbb{R}$ is continuous at α_1,
b. for every $s \in [\alpha_1, \beta_1], \varphi(s) = \varphi(\alpha_1) + (\psi(s))(s - \alpha_1)$,
c. $\psi(\alpha_1) = \varphi'(\alpha_1)$.

Since $\gamma'(\alpha)$ exists, there exists a function $\chi : [\alpha, \beta] \to \mathbb{C}$ such that

a′. $\chi : [\alpha, \beta] \to \mathbb{C}$ is continuous at α,
b′. for every $t \in [\alpha, \beta], \gamma(t) = \gamma(\alpha) + (\chi(t))(t - \alpha)$,
c′. $\chi(\alpha) = \gamma'(\alpha)$.

Observe that for every $s \in [\alpha_1, \beta_1]$,

$$\begin{aligned}(\gamma \circ \varphi)(s) = \gamma(\varphi(s)) &= \gamma(\alpha) + (\chi(\varphi(s)))(\varphi(s) - \alpha)\\ &= \gamma(\alpha) + (\chi(\varphi(s)))((\varphi(\alpha_1) + (\psi(s))(s - \alpha_1)) - \alpha)\\ &= \gamma(\alpha) + (\chi(\varphi(s)))((\alpha + (\psi(s))(s - \alpha_1)) - \alpha)\\ &= \gamma(\varphi(\alpha_1)) + ((\chi(\varphi(s)))(\psi(s)))(s - \alpha_1)\\ &= (\gamma \circ \varphi)(\alpha_1) + \mu(s)(s - \alpha_1),\end{aligned}$$

where $\mu : s \mapsto ((\chi(\varphi(s)))(\psi(s)))$ is a mapping from $[\alpha_1, \beta_1]$ to $\mathbb{C}$. Thus, for every $s \in [\alpha_1, \beta_1]$, $(\gamma \circ \varphi)(s) = (\gamma \circ \varphi)(\alpha_1) + \mu(s)(s - \alpha_1)$. Since $\chi : [\alpha, \beta] \to \mathbb{C}$ is continuous at $\alpha(= \varphi(\alpha_1))$, $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ is continuous at α_1, and ψ is continuous at α_1, the mapping $\mu : s \mapsto ((\chi(\varphi(s)))(\psi(s)))$ from $[\alpha_1, \beta_1]$ to $\mathbb{C}$ is continuous at α_1. It follows that

$$(\gamma \circ \varphi)'(\alpha_1) = (\chi(\varphi(\alpha_1)))(\psi(\alpha_1))(= (\chi(\alpha))(\psi(\alpha_1)) = (\gamma'(\alpha))(\varphi'(\alpha_1))),$$

and hence $(\gamma \circ \varphi)'(\alpha_1) = (\gamma'(\alpha))(\varphi'(\alpha_1))$. Similarly, $(\gamma \circ \varphi)'(\beta_1) = (\gamma'(\beta))(\varphi'(\beta_1))$.

Thus, for every $t \in [\varphi^{-1}(s_0), \varphi^{-1}(s_1)]$, $(\gamma \circ \varphi)'(t)$ exists, and $(\gamma \circ \varphi)'(t) = \gamma'(\varphi(t))\varphi'(t)$. Now, since γ', φ, and φ' are continuous on $[\varphi^{-1}(s_0), \varphi^{-1}(s_1)]$, $(\gamma \circ \varphi)'$ is continuous on $[\varphi^{-1}(s_0), \varphi^{-1}(s_1)]$. Thus,

$$(\gamma \circ \varphi)|_{[\varphi^{-1}(s_0), \varphi^{-1}(s_1)]} : \left[\varphi^{-1}(s_0), \varphi^{-1}(s_1)\right] \to \mathbb{C}$$

is continuously differentiable. ∎)

Similarly,

$$(\gamma \circ \varphi)|_{[\varphi^{-1}(s_1),\varphi^{-1}(s_2)]} : \left[\varphi^{-1}(s_1), \varphi^{-1}(s_2)\right] \to \mathbb{C}$$

is continuously differentiable, etc. Thus, $(\gamma \circ \varphi) : [\alpha_1, \beta_1] \to \mathbb{C}$ is a path. ∎)

Conclusion 1.71 Let $\alpha, \beta, \alpha_1, \beta_1$ be any real numbers such that $\alpha < \beta$, and $\alpha_1 < \beta_1$. Let $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ be a 1–1, onto, continuous function. Let $\varphi(\alpha_1) = \alpha$, and $\varphi(\beta_1) = \beta$. Let $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ be continuously differentiable. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a path. Then $(\gamma \circ \varphi) : [\alpha_1, \beta_1] \to \mathbb{C}$ is a path.

Definition Let $\alpha, \beta, \alpha_1, \beta_1$ be any real numbers such that $\alpha < \beta$, and $\alpha_1 < \beta_1$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$, and $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ be any paths. If there exists a function $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ such that

1. φ is 1–1, onto, continuous function,
2. $\varphi(\alpha_1) = \alpha$, and $\varphi(\beta_1) = \beta$,
3. $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ is continuously differentiable,
4. $(\gamma \circ \varphi) = \gamma_1$,

then we say that γ *is* ***equivalent*** *to* γ_1. Clearly, if path γ is equivalent to path γ_1, then $\text{ran}(\gamma) = \text{ran}(\gamma_1)$.

Note 1.72

Definition Let $\alpha, \beta, \alpha_1, \beta_1$ be any real numbers such that $\alpha < \beta$, and $\alpha_1 < \beta_1$. Let $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ be a 1–1, onto, continuous function. Let $\varphi(\alpha_1) = \alpha$, and $\varphi(\beta_1) = \beta$. Let $\varphi : [\alpha_1, \beta_1] \to [\alpha, \beta]$ be continuously differentiable. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a path. Let $f : \text{ran}(\gamma) \to \mathbb{C}$ be a continuous function.

Since $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path, γ is continuous, and γ' is continuous at all points of $[\alpha, \beta]$ except only at finite-many points. Next, since $f : \text{ran}(\gamma) \to \mathbb{C}$ is a continuous function, we get a continuous function $s \mapsto (f(\gamma(s)))(\gamma'(s))$ defined at all points of $[\alpha, \beta]$ except only at finite-many points. It follows that

$$\left(\int_{\alpha}^{\beta} (f(\gamma(s)))(\gamma'(s))\mathrm{d}s \right) \in \mathbb{C}.$$

By Conclusion 1.71, $(\gamma \circ \varphi) : [\alpha_1, \beta_1] \to \mathbb{C}$ is a path. Since $\text{ran}(\gamma) = \text{ran}(\gamma \circ \varphi)$, it follows, as above, that

$$\begin{aligned}
\int_{\alpha}^{\beta} (f(\gamma(s)))(\gamma'(s))\mathrm{d}s &= \int_{\varphi(\alpha_1)}^{\varphi(\beta_1)} ((f(\gamma(s)))(\gamma'(s)))\mathrm{d}s \\
&= \int_{\alpha_1}^{\beta_1} ((f(\gamma(\varphi(t))))(\gamma'(\varphi(t))))(\varphi'(t))\mathrm{d}t \\
&= \int_{\alpha_1}^{\beta_1} (f(\gamma(\varphi(t))))((\gamma'(\varphi(t)))(\varphi'(t)))\mathrm{d}t \\
&= \int_{\alpha_1}^{\beta_1} (f((\gamma\circ\varphi)(t)))((\gamma'(\varphi(t)))(\varphi'(t)))\mathrm{d}t \\
&= \underbrace{\left(\int_{\alpha_1}^{\beta_1} (f((\gamma\circ\varphi)(t)))\big((\gamma\circ\varphi)'(t)\big)\mathrm{d}t\right) \in \mathbb{C}\,.}
\end{aligned}$$

Thus,

$$\int_{\alpha_1}^{\beta_1} (f((\gamma\circ\varphi)(t)))\big((\gamma\circ\varphi)'(t)\big)\mathrm{d}t = \int_{\alpha}^{\beta} (f(\gamma(s)))(\gamma'(s))\mathrm{d}s.$$

In short,

$$\int_{\alpha}^{\beta} (f(\gamma(s)))(\gamma'(s))\mathrm{d}s$$

is independent of 'reparametrization' of $\gamma : [\alpha, \beta] \to \mathbb{C}$.

Here, $\int_{\alpha}^{\beta} (f(\gamma(s)))(\gamma'(s))\mathrm{d}s$ is denoted by $\int_{\gamma} f(z)\mathrm{d}z$.

Thus, if γ and γ_1 are equivalent paths, then $\int_{\gamma} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z$.

a. Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be any real numbers such that $\alpha_1 < \beta_1$, and $\alpha_2 < \beta_2$. Let $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$, and $\gamma_2 : [\alpha_2, \beta_2] \to \mathbb{C}$ be paths. Let $\gamma_1(\beta_1) = \gamma_2(\alpha_2)$. Let $f : \mathrm{ran}(\gamma_1) \cup \mathrm{ran}(\gamma_2) \to \mathbb{C}$ be a continuous function. It follows that $\int_{\gamma_1} f(z)\mathrm{d}z, \int_{\gamma_2} f(z)\mathrm{d}z \in \mathbb{C}$.

Case I: when $\beta_1 = \alpha_2$. Here, since $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$, and $\gamma_2 : [\alpha_2, \beta_2] \to \mathbb{C}$ are paths, $(\gamma_1 \cup \gamma_2) : [\alpha_1, \beta_2] \to \mathbb{C}$ is a path,

$$\operatorname{ran}(\gamma_1 \cup \gamma_2) = \operatorname{ran}(\gamma_1) \cup \operatorname{ran}(\gamma_2),$$

and

$$\int_{\gamma_1 \cup \gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z.$$

Case II: when $\beta_1 < \alpha_2$. Since $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ is a path, there exists a path $\gamma_3 : [\alpha_1, \alpha_2] \to \mathbb{C}$ such that γ_1 is equivalent to γ_3, and hence $\operatorname{ran}(\gamma_3) = \operatorname{ran}(\gamma_1)$, and $\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_3} f(z)\mathrm{d}z$. Now, by Case I, $(\gamma_3 \cup \gamma_2) : [\alpha_1, \beta_2] \to \mathbb{C}$ is a path, $\operatorname{ran}(\gamma_3 \cup \gamma_2) = \operatorname{ran}(\gamma_3) \cup \operatorname{ran}(\gamma_2) (= \operatorname{ran}(\gamma_1) \cup \operatorname{ran}(\gamma_2))$, and

$$\underbrace{\int_{\gamma_3 \cup \gamma_2} f(z)\mathrm{d}z = \int_{\gamma_3} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z}= \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z,$$

so $(\gamma_3 \cup \gamma_2) : [\alpha_1, \beta_2] \to \mathbb{C}$ is a path, $\operatorname{ran}(\gamma_3 \cup \gamma_2) = \operatorname{ran}(\gamma_1) \cup \operatorname{ran}(\gamma_2)$, and

$$\int_{\gamma_3 \cup \gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z.$$

Case III: when $\alpha_2 < \beta_1$. Since $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ is a path, there exists a path $\gamma_3 : [\alpha_3, \alpha_2] \to \mathbb{C}$ such that γ_1 is equivalent to γ_3, and hence $\operatorname{ran}(\gamma_3) = \operatorname{ran}(\gamma_1)$, and $\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_3} f(z)\mathrm{d}z$. Now, by Case I, $(\gamma_3 \cup \gamma_2) : [\alpha_3, \beta_2] \to \mathbb{C}$ is a path, $\operatorname{ran}(\gamma_3 \cup \gamma_2) = \operatorname{ran}(\gamma_3) \cup \operatorname{ran}(\gamma_2) (= \operatorname{ran}(\gamma_1) \cup \operatorname{ran}(\gamma_2))$, and

$$\underbrace{\int_{\gamma_3 \cup \gamma_2} f(z)\mathrm{d}z = \int_{\gamma_3} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z}= \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z,$$

so

$$(\gamma_3 \cup \gamma_2) : [\alpha_3, \beta_2] \to \mathbb{C} \text{ is a path, } \operatorname{ran}(\gamma_3 \cup \gamma_2) = \operatorname{ran}(\gamma_1) \cup \operatorname{ran}(\gamma_2),$$

and

$$\int_{\gamma_3 \cup \gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z.$$

So, in all cases, there exists a path γ such that

$$\int_{\gamma} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z,$$

and

$$\mathrm{ran}(\gamma) = \mathrm{ran}(\gamma_1) \cup \mathrm{ran}(\gamma_2).$$

Conclusion 1.73 Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be any real numbers such that $\alpha_1 < \beta_1$, and $\alpha_2 < \beta_2$. Let $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$, and $\gamma_2 : [\alpha_2, \beta_2] \to \mathbb{C}$ be paths. Let $\gamma_1(\beta_1) = \gamma_2(\alpha_2)$. Let $f : \mathrm{ran}(\gamma_1) \cup \mathrm{ran}(\gamma_2) \to \mathbb{C}$ be a continuous function. Then there exists a path γ such that $\mathrm{ran}(\gamma) = \mathrm{ran}(\gamma_1) \cup \mathrm{ran}(\gamma_2)$, and

$$\int_{\gamma} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z.$$

b. Let α, β be any real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be any path. For simplicity, let us take 0 for α, and 1 for β. Thus, $\gamma : [0, 1] \to \mathbb{C}$ is a piecewise continuously differentiable curve in the plane.

Let $\gamma_1 : t \mapsto \gamma(1-t)$ be a function from $[0, 1]$ to $\mathbb{C}$. Clearly, $\gamma_1 : [0, 1] \to \mathbb{C}$ is a piecewise continuously differentiable curve in the plane, that is $\gamma_1 : [0, 1] \to \mathbb{C}$ is a path. Also, if $(\gamma_1)'(t)$ exists, then $(\gamma_1)'(t) = -(\gamma'(1-t))$. Further, $\mathrm{ran}(\gamma_1) = \mathrm{ran}(\gamma)$.

Let $f : \mathrm{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Now, since $\mathrm{ran}(\gamma_1) = \mathrm{ran}(\gamma)$, $f : \mathrm{ran}(\gamma_1) \to \mathbb{C}$ is a continuous function, it follows that $\int_{\gamma} f(z)\mathrm{d}z, \int_{\gamma_1} f(z)\mathrm{d}z \in \mathbb{C}$.

Problem 1.74 $\int_{\gamma_1} f(z)\mathrm{d}z = -\int_{\gamma} f(z)\mathrm{d}z$.

(Solution

$$\begin{aligned} \mathrm{LHS} &= \int_{\gamma_1} f(z)\mathrm{d}z = \int_0^1 (f(\gamma_1(s)))\big((\gamma_1)'(s)\big)\mathrm{d}s = \int_0^1 (f(\gamma(1-s)))(-(\gamma'(1-s)))\mathrm{d}s \\ &= \int_1^0 (f(\gamma(1-s)))(\gamma'(1-s))\mathrm{d}s = \int_0^1 ((f(\gamma(t)))(\gamma'(t)))(-1)\mathrm{d}t \\ &= -\int_0^1 (f(\gamma(t)))(\gamma'(t))\mathrm{d}t = -\int_{\gamma} f(z)\mathrm{d}z = \mathrm{RHS}. \end{aligned}$$

∎)

Definition Here, $\gamma_1 : [0,1] \to \mathbb{C}$ is called the ***path opposite*** *to* γ.

Conclusion 1.75 Let α, β be any real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be any path, and $\gamma_1 : [0,1] \to \mathbb{C}$ be the path opposite to γ. Let $f : \operatorname{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Then, $\operatorname{ran}(\gamma_1) = \operatorname{ran}(\gamma)$, and $\int_{\gamma_1} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z$.

c. Let α, β be any real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be any path. Let $f : \operatorname{ran}(\gamma) \to \mathbb{C}$ be a continuous function.

Since $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path, $\gamma : [\alpha, \beta] \to \mathbb{C}$ is continuous. Now since $[\alpha, \beta]$ is compact, $\operatorname{ran}(\gamma)$ is a compact set. Since $f : \operatorname{ran}(\gamma) \to \mathbb{C}$ is a continuous function, $|f| : \operatorname{ran}(\gamma) \to [0, \infty)$ is a continuous function. Now, since $\operatorname{ran}(\gamma)$ is compact, there exists $z_0 \in \operatorname{ran}(\gamma)$ such that

$$[0,\infty) \ni \underbrace{|f|(z_0) = \max\{s : s \in |f|(\operatorname{ran}(\gamma))\}} = \max\{|f(z)| : z \in \operatorname{ran}(\gamma)\}.$$

We shall denote the nonnegative real number $\max\{|f(z)| : z \in \operatorname{ran}(\gamma)\}$ by $\|f\|_\infty$.

Problem 1.76 $\left|\int_\gamma f(z)\mathrm{d}z\right| \le \|f\|_\infty \int_\alpha^\beta |\gamma'(t)|\mathrm{d}t$.

(Solution Since

$$\left|\int_\gamma f(z)\mathrm{d}z\right| = \left|\int_\alpha^\beta (f(\gamma(t)))(\gamma'(t))\mathrm{d}t\right| \le \int_\alpha^\beta |(f(\gamma(t)))(\gamma'(t))|\mathrm{d}t$$
$$= \int_\alpha^\beta |f(\gamma(t))||\gamma'(t)|\mathrm{d}t \le \int_\alpha^\beta \|f\|_\infty |\gamma'(t)|\mathrm{d}t = \|f\|_\infty \int_\alpha^\beta |\gamma'(t)|\mathrm{d}t,$$

we have

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \le \|f\|_\infty \int_\alpha^\beta |\gamma'(t)|\mathrm{d}t.$$

∎)

Definition By the ***length*** *of path* γ, we mean $\int_\alpha^\beta |\gamma'(t)|\mathrm{d}t$.

Conclusion 1.77 Let α, β be any real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be any path. Let $f : \operatorname{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Then

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \le \|f\|_\infty (\text{length of } \gamma).$$

Note 1.78

Definition Let $a \in \mathbb{C}$, and let r be a positive real number. Let $\gamma : t \mapsto (a + re^{it})$ be a function from $[0, 2\pi]$ to $\mathbb{C}$. Clearly, γ is a closed path. Here γ is called the ***positively oriented circle*** *with* ***center*** *at a and* ***radius*** *r*.

I. Let γ be the positively oriented circle with center at a and radius r. Let $f : \operatorname{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Here

$$\int_\gamma f(z)\mathrm{d}z = \int_0^{2\pi} (f(\gamma(\theta)))(\gamma'(\theta))\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left(f\left(a + re^{i\theta}\right)\right)\left(0 + rie^{i\theta}\right)\mathrm{d}\theta = ri\int_0^{2\pi} \left(f\left(a + re^{i\theta}\right)\right)\left(e^{i\theta}\right)\mathrm{d}\theta,$$

and

$$\text{(length of the positively oriented circle with center at } a \text{ and radius } r) = \int_0^{2\pi} |\gamma'(\theta)|\mathrm{d}\theta$$

$$= \int_0^{2\pi} \left|0 + rie^{i\theta}\right|\mathrm{d}\theta = \int_0^{2\pi} r\mathrm{d}\theta = r(2\pi - 0) = 2\pi r.$$

Conclusion 1.79 Let γ be the positively oriented circle with center at a and radius r. Let $f : \operatorname{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Then (length of the positively oriented circle with center at a and radius r) $= 2\pi r$.

Definition Let $a, b \in \mathbb{C}$, where $a \neq b$. Let $\gamma : t \mapsto (a + t(b - a))$ be a function from $[0, 1]$ to $\mathbb{C}$. Clearly, γ is a path. Here, γ is called the ***oriented interval*** $[a, b]$. Here, a is called the ***initial point*** *of* $[a, b]$, and b is called the ***end point*** *of* $[a, b]$.

II. Let γ be the oriented interval $[a, b]$. Let $f : \operatorname{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Here

$$\int_\gamma f(z)\mathrm{d}z = \int_0^1 (f(\gamma(t)))(\gamma'(t))\mathrm{d}t = \int_0^1 (f(a + t(b - a)))(0 + 1(b - a))\mathrm{d}t$$

$$= (b - a)\int_0^1 f(a + t(b - a))\mathrm{d}t,$$

and

$$\text{(length of the oriented interval } [a, b]) = \int_0^1 |0 + 1(b - a)|\mathrm{d}t = |b - a|.$$

Next, let α, β be any real numbers such that $\alpha < \beta$. Let $\varphi : t \mapsto \frac{t-\alpha}{\beta-\alpha}$ be a function from $[\alpha, \beta]$ to $[0,1]$. Clearly, γ and $(\gamma \circ \varphi)$ are equivalent paths. Since

$$\underbrace{(\gamma \circ \varphi) : t \mapsto \gamma(\varphi(t))} = \gamma\left(\frac{t-\alpha}{\beta-\alpha}\right) = a + \frac{t-\alpha}{\beta-\alpha}(b-a) = \frac{\beta-t}{\beta-\alpha}a + \frac{t-\alpha}{\beta-\alpha}b,$$

we have

$$\begin{aligned}
\int\limits_{(\gamma\circ\varphi)} f(z)\mathrm{d}z &= \int\limits_{\alpha}^{\beta} \left(f\left(a + \frac{t-\alpha}{\beta-\alpha}(b-a)\right)\right)\left(0 + \frac{1-0}{\beta-\alpha}(b-a)\right)\mathrm{d}t \\
&= \frac{b-a}{\beta-\alpha}\int\limits_{\alpha}^{\beta} f\left(a + \frac{t-\alpha}{\beta-\alpha}(b-a)\right)\mathrm{d}t \\
&= \frac{b-a}{\beta-\alpha}\int\limits_{0}^{1} (f(a+s(b-a)))(\beta-\alpha)\mathrm{d}s \\
&= (b-a)\int\limits_{0}^{1} f(a+t(b-a))\mathrm{d}t = \int\limits_{\gamma} f(z)\mathrm{d}z,
\end{aligned}$$

and

$$(\text{length of the oriented interval } [a,b]) = \int\limits_{\alpha}^{\beta} \left|0 + \frac{1-0}{\beta-\alpha}(b-a)\right|\mathrm{d}t = |b-a|.$$

Conclusion 1.80 Let γ be the oriented interval $[a,b]$. Let $f : \mathrm{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Then (length of the oriented interval $[a,b]$) $= |b-a|$.

III. Let $a,b,c \in \mathbb{C}$, and let a,b,c be distinct. Let us denote the smallest convex set containing $\mathrm{ran}([a,b]) \cup \mathrm{ran}([b,c]) \cup \mathrm{ran}([c,a])$ by $\Delta(a,b,c)$. We call $\Delta(a,b,c)$ the ***triangle*** *with* ***vertices*** *at* a, b, *and* c.

Clearly, the boundary $\partial(\Delta(a,b,c))$ of $\Delta(a,b,c)$ is $\mathrm{ran}([a,b]) \cup \mathrm{ran}([b,c]) \cup \mathrm{ran}([c,a])$.

Let $f : \mathrm{ran}([a,b]) \cup \mathrm{ran}([b,c]) \cup \mathrm{ran}([c,a]) \to \mathbb{C}$ be a continuous function.

Clearly, $\int_{[a,b]} f(z)\mathrm{d}z, \int_{[b,c]} f(z)\mathrm{d}z, \int_{[c,a]} f(z)\mathrm{d}z \in \mathbb{C}$, and by Conclusion 1.73 there exists a path γ such that $\mathrm{ran}(\gamma) = \mathrm{ran}([a,b]) \cup \mathrm{ran}([b,c]) \cup \mathrm{ran}([c,a]) (= \partial(\Delta(a,b,c)))$, and

$$\int_\gamma f(z)\mathrm{d}z = \int_{[a,b]} f(z)\mathrm{d}z + \int_{[b,c]} f(z)\mathrm{d}z + \int_{[c,a]} f(z)\mathrm{d}z.$$

By $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z$, we mean $\int_\gamma f(z)\mathrm{d}z\Big(= \int_{[a,b]} f(z)\mathrm{d}z + \int_{[b,c]} f(z)\mathrm{d}z + \int_{[c,a]} f(z)\mathrm{d}z\Big)$.

Thus,

$$\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z \equiv \int_{[a,b]} f(z)\mathrm{d}z + \int_{[b,c]} f(z)\mathrm{d}z + \int_{[c,a]} f(z)\mathrm{d}z.$$

It follows that

$$\underbrace{\int_{\partial(\Delta(b,c,a))} f(z)\mathrm{d}z = \int_{[b,c]} f(z)\mathrm{d}z + \int_{[c,a]} f(z)\mathrm{d}z + \int_{[a,b]} f(z)\mathrm{d}z}$$
$$= \int_{[a,b]} f(z)\mathrm{d}z + \int_{[b,c]} f(z)\mathrm{d}z + \int_{[c,a]} f(z)\mathrm{d}z = \int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z,$$

and hence

$$\int_{\partial(\Delta(b,c,a))} f(z)\mathrm{d}z = \int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z.$$

Again,

$$\begin{aligned}\int_{\partial(\Delta(a,c,b))} f(z)\mathrm{d}z &= \int_{[a,c]} f(z)\mathrm{d}z + \int_{[c,b]} f(z)\mathrm{d}z + \int_{[b,a]} f(z)\mathrm{d}z \\ &= \left(-\int_{[c,a]} f(z)\mathrm{d}z\right) + \left(-\int_{[b,c]} f(z)\mathrm{d}z\right) + \left(-\int_{[a,b]} f(z)\mathrm{d}z\right) \\ &= -\left(\int_{[a,b]} f(z)\mathrm{d}z + \int_{[b,c]} f(z)\mathrm{d}z + \int_{[c,a]} f(z)\mathrm{d}z\right) \\ &= -\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z,\end{aligned}$$

so

$$\int_{\partial(\Delta(a,c,b))} f(z)\mathrm{d}z = -\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z, \text{etc.}$$

Conclusion 1.81 Let $a, b, c \in \mathbb{C}$, and let a, b, c be distinct. Let $f : \mathrm{ran}([a,b]) \cup \mathrm{ran}([b,c]) \cup \mathrm{ran}([c,a]) \to \mathbb{C}$ be a continuous function. Then, $\int_{\partial(\Delta(a,c,b))} f(z)\mathrm{d}z = -\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z$, etc.

Note 1.82 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path.

Since $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path, $\gamma : [\alpha, \beta] \to \mathbb{C}$ is continuous. Now since $[\alpha, \beta]$ is compact, $\mathrm{ran}(\gamma)$ is a compact subset of $\mathbb{C}$, and hence $\mathrm{ran}(\gamma)$ is a closed and bounded subset of $\mathbb{C}$. Since $\mathrm{ran}(\gamma)$ is a closed and bounded subset of $\mathbb{C}$, $(\mathrm{ran}(\gamma))^c$ is a nonempty open subset of $\mathbb{C}$.

Since $\mathrm{ran}(\gamma)$ is a bounded subset of $\mathbb{C}$, there exists a closed disk $D[a;r]$ such that $\mathrm{ran}(\gamma) \subset D[a;r]$, and hence

$$\{z : r < |z-a|\} = \underbrace{(D[a;r])^c \subset (\mathrm{ran}(\gamma))^c}.$$

Clearly, $\{z : r < |z-a|\}$ is path connected, so, by Conclusion 1.25, $\{z : r < |z-a|\}$ is a connected subset of $(\mathrm{ran}(\gamma))^c$. It follows that there exists a component C of $(\mathrm{ran}(\gamma))^c$ such that $((D[a;r])^c =)\{z : r < |z-a|\} \subset C$. Now, since components of $(\mathrm{ran}(\gamma))^c$ partitions $(\mathrm{ran}(\gamma))^c$, all components of $(\mathrm{ran}(\gamma))^c$ other than C are subsets of $C^c (\subset D[a;r])$. Now, since $D[a;r]$ is bounded, all components of $(\mathrm{ran}(\gamma))^c$ other than C are bounded. Since $\{z : r < |z-a|\} \subset C$ and $\{z : r < |z-a|\}$ is unbounded, C is unbounded.

Conclusion 1.83 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Then

1. $(\mathrm{ran}(\gamma))^c$ is nonempty open subset of $\mathbb{C}$,
2. there exists a unique component C of $(\mathrm{ran}(\gamma))^c$, such that C is unbounded, and C^c is bounded.

Note 1.84 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Let $z \in (\mathrm{ran}(\gamma))^c$. It follows that $z \notin \mathrm{ran}(\gamma)$, and hence, for every $\zeta \in \mathrm{ran}(\gamma)$, $z \neq \zeta$. This shows that the function $\zeta \mapsto \frac{1}{\zeta - z}$ from $\mathrm{ran}(\gamma)$ to $\mathbb{C}$ is continuous, and therefore $\left(\int_\gamma \frac{1}{\zeta - z}\mathrm{d}\zeta\right) \in \mathbb{C}$. Thus,

$$(\mathrm{Ind})_\gamma : z \mapsto \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\mathrm{d}\zeta \left(= \frac{1}{2\pi i}\int_\alpha^\beta \frac{1}{\gamma(s) - z}\gamma'(s)\mathrm{d}s \right)$$

is a function from nonempty open set $(\mathrm{ran}(\gamma))^c$ to $\mathbb{C}$.

Problem 1.85 For every $z \notin \mathrm{ran}(\gamma)$,

$$\frac{\int_\alpha^\beta \frac{1}{\gamma(s)-z}\gamma'(s)\mathrm{d}s}{2\pi i}$$

is an integer.

(Solution By Conclusion 1.53(b) Vol. 1, it suffices to show that

$$e^{\int_\alpha^\beta \frac{1}{\gamma(s)-z}\gamma'(s)\mathrm{d}s} = 1.$$

Let

$$\varphi : t \mapsto e^{\int_\alpha^t \frac{1}{\gamma(s)-z}\gamma'(s)\mathrm{d}s}$$

be a function from $[\alpha, \beta]$ to $\mathbb{C}$. We have to show that $\varphi(\beta) = 1$. It follows that there exists a finite subset S of $[\alpha, \beta]$ such that, for every $t \in ([\alpha, \beta] - S)$,

$$\varphi'(t) = \left(e^{\int_\alpha^t \frac{1}{\gamma(s)-z}\gamma'(s)\mathrm{d}s} \right) \left(\frac{1}{\gamma(t) - z}\gamma'(t) \right) \left(= \varphi(t)\frac{1}{\gamma(t) - z}\gamma'(t) \right),$$

and hence, for every $t \in ([\alpha, \beta] - S)$,

$$\frac{\varphi'(t)(\gamma(t) - z) - \varphi(t)(\gamma'(t) - 0)}{(\gamma(t) - z)^2} = 0.$$

∎)

Problem 1.86 The function

$$\chi : t \mapsto \frac{\varphi(t)}{\gamma(t) - z}$$

from $[\alpha, \beta]$ to $\mathbb{C}$ is continuous and, for every $t \in ([\alpha, \beta] - S)$, $\chi'(t) = 0$. Now, since S is a finite subset of $[\alpha, \beta]$, χ is a constant function, and hence for every $t \in [\alpha, \beta]$,

$$\frac{\varphi(t)}{\gamma(t)-z} = \chi(t) = \chi(\alpha) = \frac{\varphi(\alpha)}{\gamma(\alpha)-z} = \frac{e^{\int_\alpha^\alpha \frac{1}{\gamma(s)-z}\gamma'(s)\mathrm{d}s}}{\gamma(\alpha)-z} = \frac{e^0}{\gamma(\alpha)-z} = \frac{1}{\gamma(\alpha)-z}.$$

Thus, for every $t \in [\alpha, \beta]$, $\varphi(t) = \frac{\gamma(t)-z}{\gamma(\alpha)-z}$.

$$\text{LHS} = \varphi(\beta) = \frac{\gamma(\beta)-z}{\gamma(\alpha)-z} = \frac{\gamma(\alpha)-z}{\gamma(\alpha)-z} = 1 = \text{RHS}.$$

∎)

Conclusion 1.87 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Then

$$(\text{Ind})_\gamma \colon z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \mathrm{d}\zeta$$

is a function from nonempty open set $(\text{ran}(\gamma))^c$ to $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Note 1.88 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. By Conclusion 1.87,

$$(\text{Ind})_\gamma \colon z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \mathrm{d}\zeta$$

is a function from nonempty open set $(\text{ran}(\gamma))^c$ to $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. By Conclusion 1.66(b), the function

$$(\text{Ind})_\gamma \colon z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \mathrm{d}\zeta$$

from nonempty open set $(\text{ran}(\gamma))^c$ to $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is representable by power series in $(\text{ran}(\gamma))^c$, and hence, by Conclusion 1.56, $(\text{Ind})_\gamma \in H((\text{ran}(\gamma))^c)$. It follows that

$$(\text{Ind})_\gamma \colon (\text{ran}(\gamma))^c \to \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

is continuous. Let C_1 be any component of $(\text{ran}(\gamma))^c$. It follows that, C_1 is a connected subset of $(\text{ran}(\gamma))^c$. Now, since

$$(\text{Ind})_\gamma \colon (\text{ran}(\gamma))^c \to \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

is continuous, $(\text{Ind})_\gamma(C_1)$ is a connected subset of $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and hence $(\text{Ind})_\gamma(C_1)$ is a singleton set.

Conclusion 1.89 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Let C_1 be any component of $(\text{ran}(\gamma))^c$. Then there exists an integer n such that, for every $z \in C_1$, $(\text{Ind})_\gamma(z) = n$.

Note 1.90 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. By Conclusion 1.83, there exists a unique component C of $(\text{ran}(\gamma))^c$ such that C is unbounded. By Conclusion 1.89, there exists an integer n such that for every $z \in C$,

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\,\mathrm{d}\zeta = \underbrace{(\text{Ind})_\gamma(z) = n}.$$

Now, since C is unbounded, there exists $z_0 \in C$ such that

$$(|n| =)\left|\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\,\mathrm{d}\zeta\right| \in [0, 1).$$

It follows that $|n| \in [0, 1)$. Now, since $|n|$ is an integer, $|n| = 0$, and hence $n = 0$.

Conclusion 1.91 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Let C be the unbounded component of $(\text{ran}(\gamma))^c$. Then, for every $z \in C$, $(\text{Ind})_\gamma(z) = 0$.

Definition Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. By Conclusion 1.87, for every $z \in (\text{ran}(\gamma))^c$, $(\text{Ind})_\gamma(z)$ is an integer. This integer is called the ***index*** (or, ***winding number***) *of z with respect to γ*. Thus,

$$(\text{Ind})_\gamma : (\text{ran}(\gamma))^c \to \mathbb{Z}.$$

Lemma 1.92 Let $a \in \mathbb{C}$, and r be a positive real number. Let $\gamma : t \mapsto (a + re^{it})$ be the positively oriented circle with center at a and radius r. Then

1. for every $z \in \mathbb{C}$ satisfying $|z - a| > r$, $(\text{Ind})_\gamma(z) = 0$,
2. for every $z \in \mathbb{C}$ satisfying $|z - a| < r$, then $(\text{Ind})_\gamma(z) = 1$.

Proof Clearly, $\{z : |z - a| < r\}$, and $\{z : |z - a| > r\}$ are the only components of $(\text{ran}(\gamma))^c$, and

$$(\text{ran}(\gamma))^c = \{z : |z - a| = r\}^c = \{z : |z - a| \neq r\}.$$

Also, $\{z : |z-a| > r\}$ is the unbounded component of $(\mathrm{ran}(\gamma))^c$. Since $\{z : |z-a| > r\}$ is the unbounded component of $(\mathrm{ran}(\gamma))^c$, by Conclusion 1.91 for every $z \in \mathbb{C}$ satisfying $|z-a| > r$, $(\mathrm{Ind})_\gamma(z) = 0$. This proves (1).

Next, let z_0 be a complex number satisfying $|z_0 - a| < r$. We have to show that $(\mathrm{Ind})_\gamma(z_0) = 1$.

Since $z_0, a \in \{z : |z-a| < r\}$, and $\{z : |z-a| < r\}$ is a component of $(\mathrm{ran}(\gamma))^c$, by Conclusion 1.89,

$$\underbrace{(\mathrm{Ind})_\gamma(z_0) = (\mathrm{Ind})_\gamma(a)} = \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - a}\mathrm{d}\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{(a + re^{it}) - a}\left(0 + rie^{it}\right)\mathrm{d}t = 1.$$

Thus, $(\mathrm{Ind})_\gamma(z_0) = 1$. ∎

Note 1.93 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $F : \Omega \to \mathbb{C}$. Let $F \in H(\Omega)$. It follows that $F' : \Omega \to \mathbb{C}$ is a function. Let $F' : \Omega \to \mathbb{C}$ be continuous. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $\mathrm{ran}(\gamma) \subset \Omega$. It follows that $(F \circ \gamma) : [\alpha, \beta] \to \mathbb{C}$ is continuously differentiable at all but finite-many points of $[\alpha, \beta]$, and hence, by the fundamental theorem of calculus,

$$\begin{aligned}
\int_\gamma F'(z)\mathrm{d}z &= \int_\alpha^\beta ((F'(\gamma(t)))(\gamma'(t)))\mathrm{d}t \\
&= \underbrace{\int_\alpha^\beta (F \circ \gamma)'(t)\mathrm{d}t = (F \circ \gamma)(\beta) - (F \circ \gamma)(\alpha)} \\
&= F(\gamma(\beta)) - F(\gamma(\alpha)) = F(\gamma(\alpha)) - F(\gamma(\alpha)) = 0.
\end{aligned}$$

Thus, $\int_\gamma F'(z)\mathrm{d}z = 0$.

Conclusion 1.94 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $F : \Omega \to \mathbb{C}$. Let $F \in H(\Omega)$. Let $F' : \Omega \to \mathbb{C}$ be continuous. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $\mathrm{ran}(\gamma) \subset \Omega$. Then $\int_\gamma F'(z)\mathrm{d}z = 0$.

In short, if $F \in H(\Omega)$ such that $F' : \Omega \to \mathbb{C}$ is continuous, then for every closed path γ in Ω,

$$\int_\gamma F'(z)\mathrm{d}z = 0.$$

Note 1.95

a. Let n be a nonnegative integer. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Since $F : z \mapsto \frac{1}{n+1} z^{n+1}$ from $\mathbb{C}$ to $\mathbb{C}$ is a member of $H(\mathbb{C})$, and $F' : z \mapsto z^n$ from $\mathbb{C}$ to $\mathbb{C}$ is continuous, by Conclusion 1.94

$$\int_\gamma z^n \mathrm{d}z = \underbrace{\int_\gamma F'(z) \mathrm{d}z = 0}.$$

Conclusion 1.96 Let n be a nonnegative integer. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Then $\int_\gamma z^n \mathrm{d}z = 0$.

b. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $0 \notin \mathrm{ran}(\gamma)$. Since $F : z \mapsto \frac{-1}{z}$ from $\mathbb{C} - \{0\}$ to $\mathbb{C}$ is a member of $H(\mathbb{C} - \{0\})$, and $F' : z \mapsto \frac{1}{z^2}$ from $(\mathbb{C} - \{0\})$ to $\mathbb{C}$ is continuous, by Conclusion 1.94

$$\int_\gamma z^{-2} \mathrm{d}z = \int_\gamma \frac{1}{z^2} \mathrm{d}z = \underbrace{\int_\gamma F'(z) \mathrm{d}z = 0}.$$

Thus, $\int_\gamma z^{-2} \mathrm{d}z = 0$. Similarly, $\int_\gamma z^{-3} \mathrm{d}z = 0$, etc.

Conclusion 1.97 Let n be a negative integer different from -1. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $0 \notin \mathrm{ran}(\gamma)$. Then

$$\int_\gamma z^n \mathrm{d}z = 0.$$

c. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $0 \notin \mathrm{ran}(\gamma)$. It follows that $0 \in (\mathrm{ran}(\gamma))^c$. Now, by Conclusion 1.87,

$$\underbrace{(\mathrm{Ind})_\gamma(0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - 0} \mathrm{d}z} = \frac{1}{2\pi i} \int_\gamma z^{-1} \mathrm{d}z,$$

so,

$$\int\limits_{\gamma} z^{-1}\mathrm{d}z = \Big((\mathrm{Ind})_{\gamma}(0)\Big)2\pi i.$$

Conclusion 1.98 Let α, β be real numbers such that $\alpha<\beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $0 \notin \mathrm{ran}(\gamma)$. Then

$$\int\limits_{\gamma} z^{-1}\mathrm{d}z = \Big((\mathrm{Ind})_{\gamma}(0)\Big)2\pi i.$$

1.4 Cauchy's Theorem in a Triangle

Note 1.99 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$. It follows that $\Omega - \{p\}$ is an open subset of Ω. Let $f \in H(\Omega - \{p\})$ (more precisely, $f|_{(\Omega-\{p\})} \in H(\Omega - \{p\})$.

Let $a, b, c \in \mathbb{C}$, and let a, b, c be distinct. Here, $\Delta(a, b, c)$ denotes the triangle with vertices at a, b, and c, that is $\Delta(a, b, c)$ is the smallest convex set containing $\mathrm{ran}([a, b]) \cup \mathrm{ran}([b, c]) \cup \mathrm{ran}([c, a])$. Let $\Delta(a, b, c) \subset \Omega$, and $p \notin \Delta(a, b, c)$.

Problem 1.100 $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = 0$.

(Solution It follows, from Conclusion 1.81, that

$$\begin{aligned}\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = & \int\limits_{\partial\left(\Delta\left(a,\frac{a+b}{2},\frac{c+a}{2}\right)\right)} f(z)\mathrm{d}z + \int\limits_{\partial\left(\Delta\left(\frac{a+b}{2},b.\frac{b+c}{2}\right)\right)} f(z)\mathrm{d}z \\ & + \int\limits_{\partial\left(\Delta\left(\frac{b+c}{2},c,\frac{c+a}{2}\right)\right)} f(z)\mathrm{d}z + \int\limits_{\partial\left(\Delta\left(\frac{b+c}{2},\frac{c+a}{2},\frac{a+b}{2}\right)\right)} f(z)\mathrm{d}z,\end{aligned}$$

and hence

$$\frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| + \frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| + \frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right|$$
$$+ \frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| = \left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \le \left|\int_{\partial\left(\Delta\left(a,\frac{a+b}{2},\frac{c+a}{2}\right)\right)} f(z)\mathrm{d}z\right|$$
$$+\left|\int_{\partial\left(\Delta\left(\frac{a+b}{2},b.\frac{b+c}{2}\right)\right)} f(z)\mathrm{d}z\right| + \left|\int_{\partial\left(\Delta\left(\frac{b+c}{2},c,\frac{c+a}{2}\right)\right)} f(z)\mathrm{d}z\right| + \left|\int_{\partial\left(\Delta\left(\frac{b+c}{2},\frac{c+a}{2},\frac{a+b}{2}\right)\right)} f(z)\mathrm{d}z\right|.$$

This shows that

$$\frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \le \left|\int_{\partial\left(\Delta\left(a,\frac{a+b}{2},\frac{c+a}{2}\right)\right)} f(z)\mathrm{d}z\right|$$
$$\text{or } \frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \le \left|\int_{\partial\left(\Delta\left(\frac{a+b}{2},b.\frac{b+c}{2}\right)\right)} f(z)\mathrm{d}z\right|$$
$$\text{or } \frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \le \left|\int_{\partial\left(\Delta\left(\frac{b+c}{2},c,\frac{c+a}{2}\right)\right)} f(z)\mathrm{d}z\right|$$
$$\text{or } \frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \le \left|\int_{\partial\left(\Delta\left(\frac{b+c}{2},\frac{c+a}{2},\frac{a+b}{2}\right)\right)} f(z)\mathrm{d}z\right|.$$

Thus, there exists a triangle Δ_1 such that

1. $\Delta_1 \subset \Delta(a,b,c)$,
2. (length of $\partial(\Delta_1)$) $= \frac{1}{2}$(length of $\partial(\Delta(a,b,c))$),
3. $\frac{1}{4}\left|\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \le \left|\int_{\partial(\Delta_1)} f(z)\mathrm{d}z\right|$.

Similarly, there exists a triangle Δ_2 such that

1. $\Delta_2 \subset \Delta_1$,
2. (length of $\partial(\Delta_2)$) $= \frac{1}{2}$(length of $\partial(\Delta_1)$)$\left(= \frac{1}{2^2}(\text{length of } \partial(\Delta(a,b,c)))\right)$,
3. $\frac{1}{4}\left|\int_{\partial(\Delta_1)} f(z)\mathrm{d}z\right| \le \left|\int_{\partial(\Delta_2)} f(z)\mathrm{d}z\right|$, etc.

It follows that

$$\lim_{n\to\infty}(\text{length of } \partial(\Delta_n)) = 0.$$

Also,

$$\frac{1}{4^n}\left|\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \leq \left|\int\limits_{\partial(\Delta_n)} f(z)\mathrm{d}z\right|.$$

Since each Δ_n is a compact subset of $\mathbb{C}$, and $\ldots \subset \Delta_3 \subset \Delta_2 \subset \Delta_1 \subset \Delta(a,b,c)$, there exists $z_0 \in \Delta(a,b,c)(\subset \Omega)$ such that for every positive integer n, $z_0 \in \Delta_n$. Since $z_0 \in \Delta(a,b,c)$, and $p \notin \Delta(a,b,c)$, $z_0 \neq p$. Since $z_0 \neq p$, and $z_0 \in \Omega$, we have $z_0 \in (\Omega - \{p\})$. Now since $f \in H(\Omega - \{p\})$, we have $f'(z_0) \in \mathbb{C}$, and

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0).$$

Let us take any $\varepsilon > 0$.

Since

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0),$$

there exists $\delta > 0$ such that, for every $z \in D(z_0;\delta)$,

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon.$$

Since $\lim_{n\to\infty}(\text{length of } \partial(\Delta_n)) = 0$, and each Δ_n contains z_0, there exists a positive integer N such that

$$(n \geq N \Rightarrow \partial(\Delta_n) \subset D(z_0;\delta)).$$

Hence, for every positive integer $n \geq N$, and for every $z \in \partial(\Delta_n)$,

$$|f(z)-f(z_0)-(f'(z_0))(z-z_0)| \leq \varepsilon|z-z_0| \leq \varepsilon(\text{length of } \partial(\Delta_n)).$$

Thus, for every positive integer $n \geq N$, and for every $z \in \partial(\Delta_n)$,

$$|f(z)-f(z_0)-(f'(z_0))(z-z_0)| \leq \varepsilon(\text{length of } \partial(\Delta_n)).$$

It follows that, for every positive integer $n \geq N$,

$$\frac{1}{4^n}\left|\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \leq \left|\int\limits_{\partial(\Delta_n)} f(z)\mathrm{d}z\right| = \left|\int\limits_{\partial(\Delta_n)} f(z)\mathrm{d}z - (f(z_0) - (f'(z_0))z_0)0 - (f'(z_0))0\right|$$

$$= \left|\int\limits_{\partial(\Delta_n)} f(z)\mathrm{d}z - (f(z_0) - (f'(z_0))z_0)\int\limits_{\partial(\Delta_n)} z^0\mathrm{d}z - (f'(z_0))\int\limits_{\partial(\Delta_n)} z^1\mathrm{d}z\right|$$

$$= \underbrace{\left|\int\limits_{\partial(\Delta_n)} (f(z) - f(z_0) - (f'(z_0))(z - z_0))\mathrm{d}z\right| \leq (\varepsilon(\text{length of } \partial(\Delta_n)))(\text{length of } \partial(\Delta_n))}$$

$$= \varepsilon(\text{length of } \partial(\Delta_n))^2 = \varepsilon\left(\frac{1}{2^n}(\text{length of } \partial(\Delta(a,b,c)))\right)^2$$

$$= \frac{1}{4^n}\varepsilon(\text{length of } \partial(\Delta(a,b,c)))^2.$$

Hence, for every positive integer $n \geq N$,

$$\left|\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\right| \leq \varepsilon(\text{length of } \partial(\Delta(a,b,c)))^2.$$

Now, since ε is arbitrary, $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = 0$. ∎

Conclusion 1.101 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$. Let $f \in H(\Omega - \{p\})$. Let Δ be a triangle such that $\Delta \subset \Omega$, and $p \notin \Delta$. Then $\int_{\partial\Delta} f(z)\mathrm{d}z = 0$.

Note 1.102 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$. It follows that $\Omega - \{p\}$ is an open subset of Ω. Let $f \in H(\Omega - \{p\})$.

Let $a, b, c \in \mathbb{C}$, and let a, b, c be distinct. Here, $\Delta(a, b, c)$ denotes the triangle with vertices at a, b, and c, that is $\Delta(a, b, c)$ is the smallest convex set containing $\mathrm{ran}([a, b]) \cup \mathrm{ran}([b, c]) \cup \mathrm{ran}([c, a])$. Let $\Delta(a, b, c) \subset \Omega$, and $p \in \Delta(a, b, c)$.

Problem 1.103 $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = 0$.

(Solution Case I: when a, b, c are collinear. Here, for definiteness, suppose that b lies between a and c. Now, since f is continuous at all points of

$$\mathrm{ran}([a,b]) \cup \mathrm{ran}([b,c]) \cup \mathrm{ran}([c,a]),$$
$$\underbrace{\int\limits_{[a,b]} f(z)\mathrm{d}z + \int\limits_{[b,c]} f(z)\mathrm{d}z = \int\limits_{[a,c]} f(z)\mathrm{d}z} = -\int\limits_{[c,a]} f(z)\mathrm{d}z,$$

and hence

$$\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = \underbrace{\int\limits_{[a,b]} f(z)\mathrm{d}z + \int\limits_{[b,c]} f(z)\mathrm{d}z + \int\limits_{[c,a]} f(z)\mathrm{d}z = 0}$$

Thus, $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = 0$.

Case II: when a, b, c are not collinear.

Subcase I: when p is one of the vertices of $\Delta(a,b,c)$, say a. Let us take any x on $\mathrm{ran}([a,b])$ near to a. Let us take any y on $\mathrm{ran}([a,c])$ near to a. Observe that

$$\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = \int\limits_{\partial(\Delta(a,x,y))} f(z)\mathrm{d}z + \int\limits_{\partial(\Delta(x,b.y))} f(z)\mathrm{d}z + \int\limits_{\partial(\Delta(b,c,y))} f(z)\mathrm{d}z.$$

By Conclusion 1.101, $\int_{\partial(\Delta(b,c,y))} f(z)\mathrm{d}z = 0$, and $\int_{\partial(\Delta(x,b.y))} f(z)\mathrm{d}z = 0$. It follows that

$$\underbrace{\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = \int\limits_{\partial(\Delta(a,x,y))} f(z)\mathrm{d}z} = \int\limits_{[a,x]} f(z)\mathrm{d}z + \int\limits_{[x,y]} f(z)\mathrm{d}z + \int\limits_{[y,a]} f(z)\mathrm{d}z,$$

and hence

$$\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z\mathrm{d}z = \int\limits_{[a,x]} f(z)\mathrm{d}z + \int\limits_{[x,y]} f(z)\mathrm{d}z + \int\limits_{[y,a]} f(z)\mathrm{d}z.$$

Now since length of $[a,x]$, length of $[x,y]$, length of $[y,a]$ can be made arbitrarily short, and f is bounded on $\Delta(a,b,c)$, $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = 0$.

Subcase II: when p is any point of $\Delta(a,b,c)$. Observe that

$$\int\limits_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = \int\limits_{\partial(\Delta(b,p,a))} f(z)\mathrm{d}z + \int\limits_{\partial(\Delta(c,p.b))} f(z)\mathrm{d}z + \int\limits_{\partial(\Delta(a,p,c))} f(z)\mathrm{d}z.$$

By Case I, and by Case II (Subcase I), $\int_{\partial(\Delta(b,p,a))} f(z)\mathrm{d}z = 0$, $\int_{\partial(\Delta(c,p.b))} f(z)\mathrm{d}z = 0$, $\int_{\partial(\Delta(a,p,c))} f(z)\mathrm{d}z = 0$. This shows that $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = 0$.

Thus, in all cases, $\int_{\partial(\Delta(a,b,c))} f(z)\mathrm{d}z = 0$. ∎

Now, on using Conclusion 1.101, we get the following

Conclusion 1.104 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$. Let $f \in H(\Omega - \{p\})$. Let Δ be a triangle such that $\Delta \subset \Omega$. Then

$$\int_{\partial\Delta} f(z)\mathrm{d}z = 0.$$

This result, known as the **Cauchy's theorem for a triangle**, is due to A. L. Cauchy (21.08.1789–23.05.1857).

1.5 Cauchy's Theorem in a Convex Set

Note 1.105 Let Ω be a nonempty convex open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$. It follows that $\Omega - \{p\}$ is a nonempty open subset of Ω. Let $f \in H(\Omega - \{p\})$.

There exists $a \in (\Omega - \{p\})$.

Since Ω is convex, for every $z \in (\Omega - \{a\})$, $\mathrm{ran}([a,z]) \subset \Omega$. Now, since $f : \Omega \to \mathbb{C}$ is a continuous function, we have, for every $z \in (\Omega - \{a\})$, $\left(\int_{[a,z]} f(\zeta)\mathrm{d}\zeta\right) \in \mathbb{C}$. Let us define a function $F : \Omega \to \mathbb{C}$ as follows: For every $z \in \Omega$,

$$F(z) \equiv \begin{cases} \int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta & \text{if } z \neq a \\ 0 & \text{if } z = a. \end{cases}$$

Let us take any $z_0 \in (\Omega - \{a,p\})$.

Problem 1.106 $\lim_{z \to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0)$.

(Solution For this purpose, let us take any $\varepsilon > 0$.

Since $f : \Omega \to \mathbb{C}$ is continuous at $z_0 (\in (\Omega - \{a,p\}))$, there exists $\delta > 0$ such that $D(z_0;\delta) \subset (\Omega - \{a,p\})$, and for every $z \in D(z_0;\delta)$, $|f(z) - f(z_0)| < \varepsilon$. It follows that, for every $z \in D'(z_0;\delta)$, and for every $\zeta \in \mathrm{ran}([z_0,z]) (\subset D(z_0;\delta))$, $|f(\zeta) - f(z_0)| < \varepsilon$.

It suffices to show that for every $z \in D'(z_0;\delta)$,

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \le \varepsilon.$$

For this purpose, let us fix any $z \in D'(z_0;\delta)(\subset D(z_0;\delta) \subset (\Omega - \{a,p\}))$. It follows that z, z_0, a, p are distinct. Now, since Ω is convex, $\Delta(a,z,z_0) \subset \Omega$, and hence by Conclusion 1.104, $\int_{\partial(\Delta(a,z,z_0))} f(\zeta)\mathrm{d}\zeta = 0$. Since

$$\begin{aligned}
\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| &= \left|\frac{1}{z-z_0}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[a,z_0]} f(\zeta)\mathrm{d}\zeta\right) - f(z_0)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z_0,a]} f(\zeta)\mathrm{d}\zeta\right) - f(z_0)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z,z_0]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z_0,a]} f(\zeta)\mathrm{d}\zeta\right) + \left(\frac{-1}{z-z_0}\int\limits_{[z,z_0]} f(\zeta)\mathrm{d}\zeta - f(z_0)\right)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{\partial(\Delta(a,z,z_0))} f(\zeta)\mathrm{d}\zeta\right) + \left(\frac{1}{z-z_0}\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0)\right)\right| \\
&= \left|\frac{1}{z-z_0}(0) + \left(\frac{1}{z-z_0}\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0)\right)\right| = \left|\frac{1}{z-z_0}\left(\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0)(z-z_0)\right)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[z_0,z]} f(z_0)\mathrm{d}\zeta\right)\right| = \left|\frac{1}{z-z_0}\int\limits_{[z_0,z]} (f(\zeta)-f(z_0))\mathrm{d}\zeta\right| \\
&= \frac{1}{|z-z_0|}\left|\int\limits_{[z_0,z]} (f(\zeta)-f(z_0))\mathrm{d}\zeta\right| \le \frac{1}{|z-z_0|}(\varepsilon(\text{length of } [z_0,z])) \\
&= \frac{1}{|z-z_0|}(\varepsilon|z-z_0|) = \varepsilon,
\end{aligned}$$

we have $\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \le \varepsilon.$ ∎)

Thus, F' and f coincide on $(\Omega - \{a,p\})$.

Problem 1.107 $\lim_{z\to p}\frac{F(z)-F(p)}{z-p} = f(p)$.

(Solution For this purpose, let us take any $\varepsilon > 0$.

Since $f : \Omega \to \mathbb{C}$ is continuous at $p(\in (\Omega - \{a\}))$, there exists $\delta > 0$ such that $D(p;\delta) \subset (\Omega - \{a\})$, and for every $z \in D(p;\delta)$, $|f(z) - f(p)| < \varepsilon$. It follows that for every $z \in D'(p;\delta)$, and for every $\zeta \in \mathrm{ran}([p,z])(\subset D(p;\delta))$, $|f(\zeta) - f(p)| < \varepsilon$.

It suffices to show that for every $z \in D'(p;\delta)$,

$$\left|\frac{F(z)-F(p)}{z-p} - f(p)\right| \le \varepsilon.$$

For this purpose, let us fix any $z \in D'(p;\delta)(\subset D(p;\delta) \subset (\Omega - \{a\}))$. It follows that z, a, p are distinct. Now, since Ω is convex, $\Delta(a,z,p) \subset \Omega$, and hence, by Conclusion 1.104, $\int_{\partial(\Delta(a,z,p))} f(\zeta)\mathrm{d}\zeta = 0$. Since

$$\begin{aligned}
\left|\frac{F(z)-F(p)}{z-p} - f(p)\right| &= \left|\frac{1}{z-p}\left(\int_{[a,z]} f(\zeta)\mathrm{d}\zeta - \int_{[a,p]} f(\zeta)\mathrm{d}\zeta\right) - f(p)\right| \\
&= \left|\frac{1}{z-p}\left(\int_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int_{[z,p]} f(\zeta)\mathrm{d}\zeta + \int_{[p,a]} f(\zeta)\mathrm{d}\zeta\right) + \left(\frac{-1}{z-p}\int_{[z,p]} f(\zeta)\mathrm{d}\zeta - f(p)\right)\right| \\
&= \left|\frac{1}{z-p}\left(\int_{\partial(\Delta(a,z,p))} f(\zeta)\mathrm{d}\zeta\right) + \left(\frac{1}{z-p}\int_{[p,z]} f(\zeta)\mathrm{d}\zeta - f(p)\right)\right| \\
&= \left|\frac{1}{z-p}(0) + \left(\frac{1}{z-p}\int_{[p,z]} f(\zeta)\mathrm{d}\zeta - f(p)\right)\right| = \left|\frac{1}{z-p}\left(\int_{[p,z]} f(\zeta)\mathrm{d}\zeta - f(p)(z-p)\right)\right| \\
&= \left|\frac{1}{z-p}\left(\int_{[p,z]} f(\zeta)\mathrm{d}\zeta - \int_{[p,z]} f(p)\mathrm{d}\zeta\right)\right| = \left|\frac{1}{z-p}\int_{[p,z]} (f(\zeta) - f(p))\mathrm{d}\zeta\right| \\
&= \frac{1}{|z-p|}\left|\int_{[p,z]} (f(\zeta) - f(p))\mathrm{d}\zeta\right| \le \frac{1}{|z-p|}(\varepsilon(\text{length of } [p,z])) \\
&= \frac{1}{|z-p|}(\varepsilon|z-p|) = \varepsilon,
\end{aligned}$$

we have $\left|\frac{F(z)-F(p)}{z-p} - f(p)\right| \le \varepsilon$. ∎)

Thus, F' and f coincide on $(\Omega - \{a\})$.

Problem 108 $\lim_{z\to a}\frac{F(z)-F(a)}{z-a} = f(a)$.

(Solution For this purpose, let us take any $\varepsilon > 0$.

Since $f : \Omega \to \mathbb{C}$ is continuous at $a(\in (\Omega - \{p\}))$, there exists $\delta > 0$ such that $D(a;\delta) \subset (\Omega - \{p\})$, and for every $z \in D(a;\delta)$, $|f(z) - f(a)| < \varepsilon$. It follows that for every $z \in D'(a;\delta)$, and for every $\zeta \in \operatorname{ran}([a,z])(\subset D(a;\delta))$, $|f(\zeta) - f(a)| < \varepsilon$.

It suffices to show that for every $z \in D'(a;\delta)$,

$$\left|\frac{F(z) - F(a)}{z - a} - f(a)\right| \le \varepsilon.$$

For this purpose, let us fix any $z \in D'(a;\delta)(\subset D(a;\delta) \subset (\Omega - \{p\}))$. It follows that z, a, p are distinct. Now since Ω is convex, $\Delta(a,z,p) \subset \Omega$, and hence, by Conclusion 1.104, $\int_{\partial(\Delta(a,z,p))} f(\zeta)\mathrm{d}\zeta = 0$. Since

$$\begin{aligned}
\left|\frac{F(z) - F(a)}{z - a} - f(a)\right| &= \left|\frac{F(z) - 0}{z - a} - f(a)\right| = \left|\frac{1}{z-a}\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta - f(a)\right| \\
&= \left|\frac{1}{z-a}\int\limits_{[a,z]} (f(\zeta) - f(a))\mathrm{d}\zeta\right| = \frac{1}{|z-a|}\left|\int\limits_{[a,z]} (f(\zeta) - f(a))\mathrm{d}\zeta\right| \\
&\le \frac{1}{|z-a|}(\varepsilon(\text{length of}, [a,z])) = \frac{1}{|z-a|}(\varepsilon|z-a|) = \varepsilon,
\end{aligned}$$

we have $\left|\frac{F(z)-F(a)}{z-a} - f(a)\right| \le \varepsilon$. ∎)

Thus, F' and f coincide on Ω.

Conclusion 1.109 Let Ω be a nonempty convex open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$ Let $f \in H(\Omega - \{p\})$. There exists a function $F : \Omega \to \mathbb{C}$ such that $F \in H(\Omega)$, and $F' = f$.

Theorem 1.110 Let Ω be a nonempty convex open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$. Let $f \in H(\Omega - \{p\})$. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $\operatorname{ran}(\gamma) \subset \Omega$. Then

$$\int\limits_{\gamma} f(z)\mathrm{d}z = 0.$$

Proof By Conclusion 1.109, there exists a function $F : \Omega \to \mathbb{C}$ such that $F \in H(\Omega)$, and $F' = f$. Now since $f : \Omega \to \mathbb{C}$ is continuous, $F' : \Omega \to \mathbb{C}$ is continuous. By Conclusion 1.94,

$$\int_\gamma f(z)\mathrm{d}z = \underbrace{\int_\gamma F'(z)\mathrm{d}z = 0}_{},$$

and hence $\int_\gamma f(z)\mathrm{d}z = 0$. ∎

This theorem is known as the **Cauchy's theorem for a convex set**.

1.6 Cauchy Integral Formula in a Convex Set

Note 1.111 Let Ω be a nonempty convex open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $\mathrm{ran}(\gamma) \subset \Omega$. Let $a \in (\Omega - (\mathrm{ran}(\gamma)))(\subset (\mathrm{ran}(\gamma))^{\mathrm{c}})$.

Since

$$(\mathrm{Ind})_\gamma : (\mathrm{ran}(\gamma))^c \to \mathbb{Z},$$

and $a \in (\mathrm{ran}(\gamma))^{\mathrm{c}}$, $(\mathrm{Ind})_\gamma(a)$ is an integer. Clearly,

$$\zeta \mapsto \frac{1}{\zeta - a}$$

is a continuous function from $(\mathbb{C} - \{a\})$ to $\mathbb{C}$. Since $a \in (\mathrm{ran}(\gamma))^{\mathrm{c}}$, $\mathrm{ran}(\gamma) \subset (\mathbb{C} - \{a\})$. Since $f \in H(\Omega)$, $f : \Omega \to \mathbb{C}$ is continuous. Now, since $\mathrm{ran}(\gamma) \subset \Omega$,

$$\zeta \mapsto \left(\frac{1}{\zeta - a}\right)(f(\zeta))$$

is a continuous function from $\mathrm{ran}(\gamma)$ to $\mathbb{C}$, and hence $\left(\int_\gamma \left(\left(\frac{1}{\zeta - a}\right)(f(\zeta))\right)\mathrm{d}\zeta\right) \in \mathbb{C}$. Since $a \in (\Omega - (\mathrm{ran}(\gamma)))(\subset \Omega)$, and $f : \Omega \to \mathbb{C}$, we have $f(a) \in \mathbb{C}$.

Problem 1.112 $\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - a}\mathrm{d}\zeta = (f(a))\left((\mathrm{Ind})_\gamma(a)\right)$.

(Solution Since $f \in H(\Omega)$, and $a \in \Omega$, $f'(a) \in \mathbb{C}$. Let us define a function $g : \Omega \to \mathbb{C}$ as follows: For every $\zeta \in \Omega$,

$$g(\zeta) \equiv \begin{cases} \frac{1}{\zeta - a}(f(\zeta) - f(a)) & \text{if } \zeta \neq a \\ f'(a) & \text{if } \zeta = a. \end{cases}$$

Since f is continuous at all points of Ω,

$$\zeta \mapsto \frac{1}{\zeta - a}(f(\zeta) - f(a))$$

is continuous on $(\Omega - \{a\})$, and hence $g : \Omega \to \mathbb{C}$ is continuous at all points of $(\Omega - \{a\})$. Since

$$(g(a) =)f'(a) = \lim_{\zeta \to a} \frac{1}{\zeta - a}(f(\zeta) - f(a)) \left(= \lim_{\zeta \to a} g(\zeta) \right),$$

we have $\lim_{\zeta \to a} g(\zeta) = g(a)$, and hence g is continuous at a. Thus, $g : \Omega \to \mathbb{C}$ is continuous. Clearly, for every $\zeta \in (\Omega - \{a\})$,

$$g'(\zeta) = \frac{(f'(\zeta) - 0)(\zeta - a) - (f(\zeta) - f(a))(1 - 0)}{(\zeta - a)^2},$$

so $g \in H(\Omega - \{a\})$. If follows, by Theorem 1.110, that

$$\begin{aligned} \int_\gamma \frac{f(\zeta)}{\zeta - a} \mathrm{d}\zeta - f(a)\Big(2\pi i\Big((\mathrm{Ind})_\gamma(a)\Big)\Big) &= \int_\gamma \frac{f(\zeta)}{\zeta - a} \mathrm{d}\zeta - f(a) \int_\gamma \frac{1}{\zeta - a} \mathrm{d}\zeta \\ &= \int_\gamma \frac{1}{\zeta - a}(f(\zeta) - f(a)) \mathrm{d}\zeta \\ &= \underbrace{\int_\gamma g(\zeta) \mathrm{d}\zeta = 0}, \end{aligned}$$

and hence

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} \mathrm{d}\zeta = (f(a))\Big((\mathrm{Ind})_\gamma(a)\Big).$$

∎)

Conclusion 1.113 Let Ω be a nonempty convex open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Suppose that $\operatorname{ran}(\gamma) \subset \Omega$. Let $a \in (\Omega - (\operatorname{ran}(\gamma)))$. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - a} \mathrm{d}\zeta = (f(a))\Big((\mathrm{Ind})_{\gamma}(a)\Big).$$

This result is known as the **Cauchy integral formula for a convex set**.

1.7 Morera's Theorem

Note 1.114 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$.

Problem 1.115 f is representable by power series in Ω.

(Solution For this purpose, let us take any $a \in \Omega$, and a positive real number R satisfying $D(a; R) \subset \Omega$. Let us take any $r \in (0, R)$. Here,

$$\gamma_r : t \mapsto \left(a + re^{it}\right)$$

from $[0, 2\pi]$ to $\mathbb{C}$ is a closed path satisfying $(a \notin)\operatorname{ran}(\gamma_r) \subset D(a; R)$. Also, $D(a; R)$ is a nonempty convex open subset of $\mathbb{C}$. So, by Conclusion 1.113, for every $z \in (D(a; R) - (\operatorname{ran}(\gamma_r)))$,

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = (f(z))\Big((\mathrm{Ind})_{\gamma_r}(z)\Big).$$

By Lemma 1.92(2), for every $z \in D(a; r)$, $(\mathrm{Ind})_{\gamma_r}(z) = 1$, so for every $z \in D(a; r)$,

$$\begin{aligned}
\frac{r}{2\pi} \int_0^{2\pi} \frac{1}{(a + re^{it}) - z} f\left(a + re^{it}\right) e^{it} \mathrm{d}t &= \frac{r}{2\pi} \int_0^{2\pi} \frac{f(a + re^{it})}{(a + re^{it}) - z} e^{it} \mathrm{d}t \\
&= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{it})}{(a + re^{it}) - z} \left(0 + rie^{it}\right) \mathrm{d}t \\
&= \underbrace{\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = f(z) \cdot 1}_{} = f(z).
\end{aligned}$$

Thus, for every $z \in D(a;r)$,

$$f(z) = \int\limits_{[0,2\pi]} \frac{1}{(a+re^{it}) - z} \mathrm{d}\mu(t),$$

where $\mathrm{d}\mu(t) \equiv \frac{r}{2\pi} f(a+re^{it})e^{it}\mathrm{d}t$. By Conclusion 1.66(a), there exist complex numbers $c_1, c_2, c_3, \ldots$ such that for every $w \in D(a;r)$,

$$\begin{aligned} f(w) &= \int\limits_{[0,2\pi]} \frac{1}{(a+re^{it}) - w} \mathrm{d}\mu(t) \\ &= c_0 + c_1(w-a) + c_2(w-a)^2 + c_3(w-a)^3 + \cdots. \end{aligned}$$

Thus, for every $w \in D(a;r)$,

$$f(w) = c_0 + c_1(w-a) + c_2(w-a)^2 + c_3(w-a)^3 + \cdots.$$

This shows that f is representable by power series in $D(a;r)$. Now, let us take any $z \in (D(a;R) - D(a;r))$. It suffices to show that

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots.$$

There exists $r_1 \in (r,R)$ such that $z \in D(a;r_1)$. As above, there exist complex numbers $d_0, d_1, d_2, d_3, \ldots$ such that for every $w \in D(a;r_1)(\supset D(a;r))$,

$$f(w) = d_0 + d_1(w-a) + d_2(w-a)^2 + d_3(w-a)^3 + \cdots,$$

and hence for every $w \in D(a;r)$,

$$\begin{aligned} c_0 + c_1(w-a) + c_2(w-a)^2 + c_3(w-a)^3 + \cdots &= f(w) \\ &= d_0 + d_1(w-a) + d_2(w-a)^2 + d_3(w-a)^3 + \cdots. \end{aligned}$$

Now, by Lemma 1.60, $c_0 = d_0$, $c_1 = d_1$, $c_2 = d_2$, $\cdots$. It follows that for every $w \in D(a;r_1)$,

$$f(w) = c_0 + c_1(w-a) + c_2(w-a)^2 + c_3(w-a)^3 + \cdots.$$

Now, since $z \in D(a;r_1)$,
$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots.$ ∎)

Conclusion 1.116 Let Ω be a nonempty open subset of $\mathbb{C}$. Then every member of $H(\Omega)$ is representable by power series in Ω.

Lemma 1.117 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Then $f' \in H(\Omega)$.

Proof By Conclusion 1.116, f is representable by power series in Ω, and hence, by Conclusion 1.56, $f' \in H(\Omega)$. ∎

Note 1.118 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Suppose that, for every triangle Δ satisfying $\Delta \subset \Omega$, $\int_{\partial\Delta} f(z)\mathrm{d}z = 0$.

Problem 1.119 $f \in H(\Omega)$.

(**Solution** For this purpose, let us take any $p \in \Omega$. We have to show that $f'(p)$ exists. Now, since Ω is open, there exists $r > 0$ such that $D(p;r) \subset \Omega$. Also, $D(p;r)$ is a nonempty open convex set. There exists $a \in (D(p;r) - \{p\})$. Since $D(p;r)$ is convex, for every $z \in (D(p;r) - \{a\})$, $\mathrm{ran}([a,z]) \subset D(p;r)$. Now, since f is continuous on $D(p;r)$, we have for every $z \in (D(p;r) - \{a\})$,

$$\left(\int_{[a,z]} f(\zeta)\mathrm{d}\zeta \right) \in \mathbb{C}.$$

Let us define a function $F : D(p;r) \to \mathbb{C}$ as follows: For every $z \in D(p;r)$,

$$F(z) \equiv \begin{cases} \int_{[a,z]} f(\zeta)\mathrm{d}\zeta & \text{if } z \neq a \\ 0 & \text{if } z = a. \end{cases}$$

Let us take any $z_0 \in (D(p;r) - \{a,p\})$.

Problem 1.120 $\lim_{z\to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0)$.

(**Solution** For this purpose, let us take any $\varepsilon > 0$. Since f is continuous at $z_0(\in (D(p;r) - \{a,p\}))$, there exists $\delta > 0$ such that $D(z_0;\delta) \subset (D(p;r) - \{a,p\})$, and for every $z \in D(z_0;\delta)$, $|f(z) - f(z_0)| < \varepsilon$. It follows that for every $z \in D'(z_0;\delta)$, and for every $\zeta \in \mathrm{ran}([z_0,z])(\subset D(z_0;\delta))$, $|f(\zeta) - f(z_0)| < \varepsilon$. It suffices to show that for every $z \in D'(z_0;\delta)$,

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \varepsilon.$$

For this purpose, let us fix any $z \in D'(z_0;\delta)(\subset D(z_0;\delta) \subset (D(p;r) - \{a,p\}))$. It follows that z, z_0, a, p are distinct. Now, since $D(p;r)$ is convex, $\Delta(a,z,z_0) \subset D(p;r)(\subset \Omega)$, and hence, by assumption, $\int_{\partial(\Delta(a,z,z_0))} f(\zeta)\mathrm{d}\zeta = 0$. Since

$$\begin{aligned}
\left|\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)\right| &= \left|\frac{1}{z-z_0}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[a,z_0]} f(\zeta)\mathrm{d}\zeta\right)-f(z_0)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z_0,a]} f(\zeta)\mathrm{d}\zeta\right)-f(z_0)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z,z_0]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z_0,a]} f(\zeta)\mathrm{d}\zeta\right)+\left(\frac{-1}{z-z_0}\int\limits_{[z,z_0]} f(\zeta)\mathrm{d}\zeta - f(z_0)\right)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{\partial(\Delta(a,z,z_0))} f(\zeta)\mathrm{d}\zeta\right)+\left(\frac{1}{z-z_0}\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0)\right)\right| \\
&= \left|\frac{1}{z-z_0}(0)+\left(\frac{1}{z-z_0}\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0)\right)\right| = \left|\frac{1}{z-z_0}\left(\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0)(z-z_0)\right)\right| \\
&= \left|\frac{1}{z-z_0}\left(\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[z_0,z]} f(z_0)\mathrm{d}\zeta\right)\right| = \left|\frac{1}{z-z_0}\int\limits_{[z_0,z]} (f(\zeta)-f(z_0))\mathrm{d}\zeta\right| \\
&= \frac{1}{|z-z_0|}\left|\int\limits_{[z_0,z]} (f(\zeta)-f(z_0))\mathrm{d}\zeta\right| \le \frac{1}{|z-z_0|}(\varepsilon(\text{length of } [z_0,z])) = \frac{1}{|z-z_0|}(\varepsilon|z-z_0|) = \varepsilon,
\end{aligned}$$

we have $\left|\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)\right| \le \varepsilon$. ∎)

Thus, F' and f coincide on $(D(p;r)-\{a,p\})$.

Problem 1.121 $\lim_{z\to p}\frac{F(z)-F(p)}{z-p}=f(p)$.

(Solution For this purpose, let us take any $\varepsilon>0$.

Since f is continuous at $p(\in (D(p;r)-\{a\}))$, there exists $\delta>0$ such that $D(p;\delta)\subset(D(p;r)-\{a\})$, and for every $z\in D(p;\delta)$, $|f(z)-f(p)|<\varepsilon$. It follows that for every $z\in D'(p;\delta)$, and for every $\zeta\in \operatorname{ran}([p,z])(\subset D(p;\delta))$, $|f(\zeta)-f(p)|<\varepsilon$. It suffices to show that for every $z\in D'(p;\delta)$,

$$\left|\frac{F(z)-F(p)}{z-p}-f(p)\right|\le\varepsilon.$$

For this purpose, let us fix any $z\in D'(p;\delta)(\subset D(p;\delta)\subset(D(p;r)-\{a\}))$. It follows that z,a,p are distinct. Now, since $D(p;r)$ is convex, $\Delta(a,z,p)\subset D(p;r)(\subset\Omega)$, and hence, by assumption, $\int_{\partial(\Delta(a,z,p))}f(\zeta)\mathrm{d}\zeta=0$. Since

$$\begin{aligned}
\left|\frac{F(z)-F(p)}{z-p}-f(p)\right| &= \left|\frac{1}{z-p}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[a,p]} f(\zeta)\mathrm{d}\zeta\right) - f(p)\right| \\
&= \left|\frac{1}{z-p}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[p,a]} f(\zeta)\mathrm{d}\zeta\right) - f(p)\right| \\
&= \left|\frac{1}{z-p}\left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z,p]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[p,a]} f(\zeta)\mathrm{d}\zeta\right) + \left(\frac{-1}{z-p}\int\limits_{[z,p]} f(\zeta)\mathrm{d}\zeta - f(p)\right)\right| \\
&= \left|\frac{1}{z-p}\left(\int\limits_{\partial(\Delta(a,z,p))} f(\zeta)\mathrm{d}\zeta\right) + \left(\frac{1}{z-p}\int\limits_{[p,z]} f(\zeta)\mathrm{d}\zeta - f(p)\right)\right| \\
&= \left|\frac{1}{z-p}(0) + \left(\frac{1}{z-p}\int\limits_{[p,z]} f(\zeta)\mathrm{d}\zeta - f(p)\right)\right| = \left|\frac{1}{z-p}\left(\int\limits_{[p,z]} f(\zeta)\mathrm{d}\zeta - f(p)(z-p)\right)\right| \\
&= \left|\frac{1}{z-p}\left(\int\limits_{[p,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[p,z]} f(p)\mathrm{d}\zeta\right)\right| = \left|\frac{1}{z-p}\int\limits_{[p,z]} (f(\zeta)-f(p))\mathrm{d}\zeta\right| \\
&= \frac{1}{|z-p|}\left|\int\limits_{[p,z]} (f(\zeta)-f(p))\mathrm{d}\zeta\right| \le \frac{1}{|z-p|}(\varepsilon(\text{length of } [p,z])) = \frac{1}{|z-p|}(\varepsilon|z-p|) = \varepsilon,
\end{aligned}$$

we have $\left|\frac{F(z)-F(p)}{z-p}-f(p)\right| \le \varepsilon$. ∎)

Thus, F' and f coincide on $(D(p;r)-\{a\})$.

Problem 1.122 $\lim\limits_{z\to a}\frac{F(z)-F(a)}{z-a} = f(a)$.

(**Solution** For this purpose, let us take any $\varepsilon > 0$. Since f is continuous at $a(\in (D(p;r)-\{p\}))$, there exists $\delta > 0$ such that $D(a;\delta) \subset (D(p;r)-\{p\})$, and for every $z \in D(a;\delta)$, $|f(z)-f(a)| < \varepsilon$. It follows that for every $z \in D'(a;\delta)$, and for every $\zeta \in \operatorname{ran}([a,z])(\subset D(a;\delta))$, $|f(\zeta)-f(a)| < \varepsilon$. It suffices to show that for every $z \in D'(a;\delta)$,

$$\left|\frac{F(z)-F(a)}{z-a}-f(a)\right| \le \varepsilon.$$

For this purpose, let us fix any $z \in D'(a;\delta)(\subset D(a;\delta) \subset (D(p;r)-\{p\}))$. It follows that z,a,p are distinct. Now, since Ω is convex, $\Delta(a,z,p) \subset D(p;r)(\subset \Omega)$, and hence, by assumption, $\int_{\partial(\Delta(a,z,p))} f(\zeta)\mathrm{d}\zeta = 0$. Since

$$\left|\frac{F(z)-F(a)}{z-a}-f(a)\right| = \left|\frac{F(z)-0}{z-a}-f(a)\right| = \left|\frac{1}{z-a}\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta - f(a)\right|$$

$$= \left|\frac{1}{z-a}\int\limits_{[a,z]} (f(\zeta)-f(a))\mathrm{d}\zeta\right| = \frac{1}{|z-a|}\left|\int\limits_{[a,z]} (f(\zeta)-f(a))\mathrm{d}\zeta\right|$$

$$\le \frac{1}{|z-a|}(\varepsilon(\text{length of } [a,z])) = \frac{1}{|z-a|}(\varepsilon|z-a|) = \varepsilon,$$

we have $\left|\frac{F(z)-F(a)}{z-a}-f(a)\right| \le \varepsilon$. ∎)

Thus, F' and f coincide on $D(p;r)$. It follows that $F \in H(D(p;r))$, and hence by Lemma 1.117, $\left(f|_{D(p;r)}=\right)F' \in H(D(p;r))$. Since $f|_{D(p;r)} \in H(D(p;r))$, $f'(p)$ exists. ∎)

Conclusion 1.123 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Suppose that, for every triangle Δ satisfying $\Delta \subset \Omega$, $\int_{\partial\Delta} f(z)\mathrm{d}z = 0$. Then $f \in H(\Omega)$.

This result, known as the **Morera's theorem**, is due to G. Morera (18.07.1856 – 08.02.1907).

1.8 Parseval's Formula

Note 1.124 Let Ω be a region. That is, Ω is a nonempty open, connected subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Let $a \in f^{-1}(0)$. Suppose that a is a limit point of $(\Omega - (f^{-1}(0)))$.

There exists a $r > 0$ such that $D(a;r) \subset \Omega$. Since $f \in H(\Omega)$, by Conclusion 1.116, f is representable by power series in Ω. Now, since $D(a;r) \subset \Omega$, there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in D(a;r)$,

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots.$$

Since

$$c_0 = c_0 + c_1(a-a) + c_2(a-a)^2 + c_3(a-a)^3 + \cdots = f\underbrace{(a) = 0}_{},$$

we have, for every $z \in D(a;r)$,

$$f(z) = c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots.$$

Problem 1.125 There exists a positive integer m such that $c_m \neq 0$, and, for every $z \in D(a;r)$,

$$f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + c_{m+2}(z-a)^{m+2} + \cdots.$$

(Solution If not, otherwise, suppose that, for every positive integer m, either $c_m = 0$, or $\left(\text{for every } z \in D(a;r), f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + c_{m+2}(z-a)^{m+2} + \cdots\right)$ is false. We have to arrive at a contradiction. Since for every $z \in D(a;r)$,

$$f(z) = c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots$$

is true, we have $c_1 = 0$. Thus, for every $z \in D(a;r)$,

$$f(z) = c_2(z-a)^2 + c_3(z-a)^3 + \cdots$$

is true, and hence $c_2 = 0$. Similarly, $c_3 = 0$, etc. Thus, for every $z \in D(a;r)(\subset \Omega)$,

$$f(z) = 0(z-a) + 0(z-a)^2 + 0(z-a)^3 + \cdots (= 0).$$

Thus, $D(a;r) \subset f^{-1}(0)$, and hence a is not a limit point of $(\Omega - (f^{-1}(0)))$. This contradicts the assumption. ∎)

It follows that there exists the smallest positive integer m such that $c_m \neq 0$, and, for every $z \in D(a;r)$,

$$f(z) = c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + c_{m+2}(z-a)^{m+2} + \cdots.$$

Let us define a function $g : \Omega \to \mathbb{C}$ as follows: For every $z \in \Omega$,

$$g(z) \equiv \begin{cases} \frac{f(z)}{(z-a)^m} \text{ if } z \neq a \\ c_m \text{ if } z = a. \end{cases}$$

Since $f : \Omega \to \mathbb{C}$ is a holomorphic function,

$$z \mapsto \frac{f(z)}{(z-a)^m}$$

is a member of $H(\Omega - (a))$, and hence $g \in H(\Omega - (a))$. It follows that for every $z \in D'(a;r)$,

$$\underbrace{g(z) = \frac{1}{(z-a)^m} f(z)} = \frac{1}{(z-a)^m}\left(c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + c_{m+2}(z-a)^{m+2} + \cdots\right)$$

$$= c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + c_{m+3}(z-a)^3 + \cdots,$$

and

$$\underbrace{g(a) = c_m} = c_m + c_{m+1}(a-a) + c_{m+2}(a-a)^2 + c_{m+3}(a-a)^3 + \cdots,$$

and hence for every $z \in D(a;r)$,

$$g(z) = c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + c_{m+3}(z-a)^3 + \cdots.$$

Thus, g is representable by power series in $D(a;r)$. Now, by Conclusion 1.56, $g \in H(D(a;r))$. Since $g \in H(D(a;r))$, and $g \in H(\Omega - (a))$, we have $g \in H(\Omega)$. Since $(g(a) =)c_m \neq 0$, we have $g(a) \neq 0$. Since $g \in H(D(a;r))$, g is continuous at a. Now, since $g(a) \neq 0$, there exists $r_1 \in (0,r)$ such that for every $z \in D(a;r_1)$, $g(z) \neq 0$, and hence for every $z \in D'(a;r_1)$, $\frac{f(z)}{(z-a)^m} \neq 0$. Thus, for every $z \in D'(a;r_1)$, $f(z) \neq 0$, and hence

$$D'(a;r_1) \subset \left(f^{-1}(0)\right)^c.$$

Thus, a is an isolated point of $f^{-1}(0)$.

Conclusion 1.126 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Let $a \in f^{-1}(0)$. Suppose that a is a limit point of $(\Omega - (f^{-1}(0)))$. Then

1. every point of $f^{-1}(0)$ is an isolated point of $f^{-1}(0)$,
2. there exist $r > 0$, a positive integer m, and $g \in H(\Omega)$ such that for every $z \in D(a;r)$, $f(z) = (z-a)^m g(z)$, and $g(a) \neq 0$.

Note 1.127 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$.

Problem 1.128 $(f^{-1}(0))^0 = \emptyset$ or $f = 0$.
(**Solution**:

Problem 1.129 It suffices to show that $\left(\Omega - (f^{-1}(0))^0\right)$ is open.

(**Solution** Suppose that $\left(\Omega - (f^{-1}(0))^0\right)$ is open. We have to show that $(f^{-1}(0))^0 = \emptyset$ or $f^{-1}(0) = \Omega$.

Since $\left(\Omega - (f^{-1}(0))^0\right)$ is open, $\left(\Omega - (f^{-1}(0))^0\right) \subset \Omega$ and Ω is open, $\left(\Omega - (f^{-1}(0))^0\right)$ is open in Ω. Since $(f^{-1}(0))^0$ is open, $(f^{-1}(0))^0 \subset \Omega$ and Ω is open, $(f^{-1}(0))^0$ is open in Ω. Since $(f^{-1}(0))^0$ is open in Ω, $\left(\Omega - (f^{-1}(0))^0\right)$ is open in Ω,

$$(f^{-1}(0))^0 \cup \left(\Omega - (f^{-1}(0))^0\right) = \Omega, (f^{-1}(0))^0 \cap \left(\Omega - (f^{-1}(0))^0\right) = \emptyset,$$

and Ω is connected, we have $(f^{-1}(0))^0 = \emptyset$ or $\left(\Omega - (f^{-1}(0))^0\right) = \emptyset$.

Now, let us consider the case when $\left(\Omega - (f^{-1}(0))^0\right) = \emptyset$. In this case, $(f^{-1}(0) \subset)\Omega = (f^{-1}(0))^0 (\subset f^{-1}(0))$, and hence $f^{-1}(0) = \Omega$. This shows that $f = 0$.

Thus, $(f^{-1}(0))^0 = \emptyset$ or $f = 0$. ∎)

For this purpose, let us take any $a \in \left(\Omega - (f^{-1}(0))^0\right)$. It suffices to show that a is an interior point of $\left(\Omega - (f^{-1}(0))^0\right)$.

Case I: when $f(a) \neq 0$. Since $f : \Omega \to \mathbb{C}$ is continuous at a, there exists $r > 0$ such that $D(a; r) \subset \Omega$ and $(z \in D(a; r) \Rightarrow f(z) \neq 0)$.

Problem 1.130 $D(a; r) \cap \left((f^{-1}(0))^0\right) = \emptyset$.

(**Solution** If not, otherwise, suppose that there exists $z \in D(a; r)$ such that $z \in (f^{-1}(0))^0$. We have to arrive at a contradiction. Since $z \in (f^{-1}(0))^0$, there exists $r_1 > 0$ such that $(z \in)D(z; r_1) \subset f^{-1}(0)$, and hence $f(z) = 0$. Since $z \in D(a; r)$, we have $f(z) \neq 0$. This is a contradiction. ∎)

Since $D(a; r) \cap \left((f^{-1}(0))^0\right) = \emptyset$, we have $D(a; r) \subset \left((f^{-1}(0))^0\right)^c$. Now, since $D(a; r) \subset \Omega$, we have

$$D(a; r) \subset \left(\Omega \cap \left(\left((f^{-1}(0))^0\right)^c\right)\right)\left(= \left(\Omega - (f^{-1}(0))^0\right)\right),$$

and hence

$$D(a; r) \subset \left(\Omega - (f^{-1}(0))^0\right).$$

Thus, a is an interior point of $\left(\Omega - (f^{-1}(0))^0\right)$.

Case II: when $f(a) = 0$. It follows that $a \in f^{-1}(0)$, and hence $a \notin (f^{-1}(0))^c$. Since $a \in \left(\Omega - (f^{-1}(0))^0\right)$, we have

$\underbrace{a \in \left(\left(f^{-1}(0)\right)^0\right)^c} = \left(\left(f^{-1}(0)\right)^c\right)^-$, and hence $a \in \left(\left(f^{-1}(0)\right)^c\right)^-$. Now, since $a \notin \left(f^{-1}(0)\right)^c$, a is a limit point of $\left(f^{-1}(0)\right)^c$. Now, since $a \in f^{-1}(0)$, and Ω is open, a is a limit point of $(\Omega - (f^{-1}(0)))$. By Conclusion 1.126, every point of $f^{-1}(0)$ is an isolated point of $f^{-1}(0)$, and hence $\left(f^{-1}(0)\right)^0 = \emptyset$. Since Ω is open, and $a \in \Omega$, a is an interior point of $\Omega\left(= (\Omega - \emptyset) = \left(\Omega - \left(f^{-1}(0)\right)^0\right)\right)$, and hence a is an interior point of $\left(\Omega - \left(f^{-1}(0)\right)^0\right)$. Thus, in all cases, a is an interior point of $\left(\Omega - \left(f^{-1}(0)\right)^0\right)$. ∎)

Conclusion 1.131 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Then $\left(f^{-1}(0)\right)^0 = \emptyset$ or $f = 0$.

Note 1.132 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Suppose that Ω contains a limit point of $f^{-1}(0)$.

Problem 1.133 $f = 0$.

(Solution By Conclusion 1.131, $\left(f^{-1}(0)\right)^0 = \emptyset$ or $f = 0$, so it suffices to show that $\left(f^{-1}(0)\right)^0 \neq \emptyset$. If not, otherwise let $\left(f^{-1}(0)\right)^0 = \emptyset$. We have to arrive at a contradiction.

By assumption, there exists $a \in \Omega$ such that a is a limit point of $f^{-1}(0)$, and hence there exists a sequence $\{a_n\}$ in $f^{-1}(0)$ such that $\lim_{n\to\infty} a_n = a$. Now, since $f : \Omega \to \mathbb{C}$ is continuous, $(0 = \lim_{n\to\infty} 0 =) \lim_{n\to\infty} f(a_n) = f(a)$, and hence $a \in f^{-1}(0)$. Since $a \in f^{-1}(0)$, and $\left(f^{-1}(0)\right)^0 = \emptyset$, a is a limit point of $(\Omega - (f^{-1}(0)))$, and hence, by Conclusion 1.126, every point of $f^{-1}(0)$ is an isolated point of $f^{-1}(0)$. Now, since $a \in f^{-1}(0)$, a is an isolated point of $f^{-1}(0)$, and hence a is not a limit point of $f^{-1}(0)$. This is a contradiction. ∎)

Conclusion 1.134 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$. Let $f \in H(\Omega)$. Suppose that Ω contains a limit point of $f^{-1}(0)$. Then $f = 0$.

Theorem 1.135 Let Ω be a region. Let $f, g \in H(\Omega)$. Suppose that Ω contains a limit point of $\{z : f(z) = g(z)\}$. Then $f = g$.

Proof Since $f, g \in H(\Omega)$, $(f - g) \in H(\Omega)$. Since Ω contains a limit point of $\{z : f(z) = g(z)\}\left(= (f-g)^{-1}(0)\right)$, by Conclusion 1.134, $(f - g) = 0$ and hence $f = g$. ∎

Theorem 1.136 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$. Let f be a nonzero member of $H(\Omega)$. Then

I. $f^{-1}(0)$ has no limit point in Ω,
II. corresponding to each $a \in f^{-1}(0)$, there exist a unique positive integer m, and a unique function $g : \Omega \to \mathbb{C}$ such that

a. $g \in H(\Omega)$,
b. for every $z \in \Omega, f(z) = (z-a)^m(g(z))$,
c. $g(a) \neq 0$.

(**Definition** Here, m is called the ***order of zero*** (*or*, ***multiplicity of zero***) *which f has at the point a*. Here, the multiplicity of zero which f has at the point a is also denoted by $m(f;a)$. The set $f^{-1}(0)$ is called the ***zero set*** *of f*.)

Proof

I: By Conclusion 1.134, either (Ω does not contain a limit point of $f^{-1}(0)$) or $f = 0$. Since $f \neq 0$, Ω does not contain a limit point of $f^{-1}(0)$.

II: Existence part: Let us fix any $a \in f^{-1}(0)$. By I, a is not a limit point of $f^{-1}(0)(\subset \Omega)$. Now, since Ω is open, a is a limit point of $(\Omega - (f^{-1}(0)))$. By Conclusion 1.126, there exist $r > 0$, a positive integer m, and $g \in H(\Omega)$ such that for every $z \in D(a;r), f(z) = (z-a)^m g(z)$, and $g(a) \neq 0$.

It remains to show that for every $z \in \Omega, f(z) = (z-a)^m(g(z))$. Since $g \in H(\Omega)$, the function $h : z \mapsto (z-a)^m(g(z))$ is a member of $H(\Omega)$. We have to show that $f = h$. Here, $D(a;r) \subset \{z : f(z) = h(z)\}$, so $(\Omega \ni)a$ is a limit point of $\{z : f(z) = h(z)\}$, and hence Ω contains a limit point of $\{z : f(z) = h(z)\}$. Now, by Theorem 1.135 $f = h$.

Proof of uniqueness part Let m and n be integers, and $g : \Omega \to \mathbb{C}$, $h : \Omega \to \mathbb{C}$ be functions such that

1. $g \in H(\Omega)$,
2. for every $z \in \Omega, f(z) = (z-a)^m(g(z))$,
3. $g(a) \neq 0$.
4. $h \in H(\Omega)$,
5. for every $z \in \Omega, f(z) = (z-a)^n(h(z))$,
6. $h(a) \neq 0$.

We have to show: I. $m = n$, II. $g = h$.

For I: If not, otherwise, let $m \neq n$. We have to arrive at a contradiction. Since m and n are two distinct integers, either $m < n$ or $n < m$. For definiteness, let $m < n$. For simplicity, suppose that $n = m + 1$. Here, for every $z \in \Omega$,

$$(z-a)^m(g(z)) = \underbrace{f(z) = (z-a)^n(h(z))} = (z-a)^{m+1}(h(z)),$$

so for every $z \in (\Omega - \{a\})$,

$$g(z) = (z-a)h(z),$$

and hence

$$g(a) = \underbrace{\lim_{\substack{z \to a \\ z \neq a}} g(z) = \lim_{\substack{z \to a \\ z \neq a}}}((z-a)h(z)) = (a-a)h(a) = 0.$$

Thus, $g(a) = 0$. This contradicts (3).

For II: From (I), for every $z \in \Omega$,

$$(z-a)^m(g(z)) = f(z) = (z-a)^m(h(z)),$$

so for every $z \in (\Omega - \{a\})$, $g(z) = h(z)$, and hence

$$g(a) = \underbrace{\lim_{\substack{z \to a \\ z \neq a}} g(z) = \lim_{\substack{z \to a \\ z \neq a}}} h(z) = h(a).$$

Thus, for every $z \in \Omega$, $g(z) = h(z)$, and hence $g = h$. ∎

Note 1.137

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let $a \in \Omega$. It follows that $(\Omega - \{a\})$ is a nonempty open subset of $\mathbb{C}$. Let $f : (\Omega - \{a\}) \to \mathbb{C}$. If $f \in H(\Omega - \{a\})$, then we say that *f has an **isolated singularity** at the point a.*

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let $a \in \Omega$. Let $f : (\Omega - \{a\}) \to \mathbb{C}$. Suppose that f has an isolated singularity at the point a. If there exists a function $g : \Omega \to \mathbb{C}$ such that $g|_{(\Omega - \{a\})} = f$, and $g \in H(\Omega)$, then we say that *f has a **removable singularity** at the point a.*

Let Ω be a nonempty open subset of $\mathbb{C}$. Let $a \in \Omega$. Let $f : (\Omega - \{a\}) \to \mathbb{C}$. Suppose that f has an isolated singularity at a. Let r be a positive real number such that $D(a;r) \subset \Omega$, and $f(D'(a;r))$ is a bounded subset of $\mathbb{C}$. Let us define a function $h : \Omega \to \mathbb{C}$ as follows: For every $z \in \Omega$,

$$h(z) \equiv \begin{cases} (z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a. \end{cases}$$

Since $f(D'(a;r))$ is bounded, there exists $M > 0$ such that for every $z \in D'(a;r)$, $|f(z)| \leq M$. Since f has an isolated singularity at a, $f \in H(\Omega - \{a\})$, and hence for every $z \in (\Omega - \{a\})$, $f'(z)$ exists. Hence, for every $z \in (\Omega - \{a\})$, $h'(z)$ exists and for every $z \in (\Omega - \{a\})$, $h'(z) = 2(z-a)f(z) + (z-a)^2 f'(z)$. Next,

$$0 \le \lim_{z\to a}\left|\frac{h(z)-h(a)}{z-a}\right| = \lim_{z\to a}\left|\frac{(z-a)^2 f(z)-0}{z-a}\right| = \lim_{z\to a}\left|\frac{(z-a)^2 f(z)}{z-a}\right|$$
$$= \lim_{z\to a}|(z-a)f(z)| = \lim_{z\to a}|z-a||f(z)| \le \lim_{z\to a}|z-a|M = 0,$$

so

$$\lim_{z\to a}\left|\frac{h(z)-h(a)}{z-a}\right| = 0,$$

and hence

$$\lim_{z\to a}\frac{h(z)-h(a)}{z-a} = 0.$$

This shows that $h'(a) = 0$. Thus, $h \in H(\Omega)$. Now, by Conclusion 1.116, h is representable by power series in Ω, and hence there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in D(a;r)$,

$$h(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots.$$

Now, since $h(a) = 0$, we have $c_0 = 0$. By Conclusion 1.59, $c_1 = h'(a)(= 0)$. Thus, for every $z \in D(a;r)$, $h(z) = c_2(z-a)^2 + c_3(z-a)^3 + \cdots$. It follows that for every $z \in D'(a;r)$,

$$f(z) = c_2 + c_3(z-a) + c_4(z-a)^2 + \cdots.$$

Problem 1.138 f has a removable singularity at the point a.

(Solution Let us define a function $g : \Omega \to \mathbb{C}$ as follows: For every $z \in \Omega$,

$$g(z) \equiv \begin{cases} f(z) & \text{if } z \ne a \\ c_2 & \text{if } z = a. \end{cases}$$

Clearly, $g|_{(\Omega-\{a\})} = f$. Since $f \in H(\Omega - \{a\})$, we have $g|_{(\Omega-\{a\})} \in H(\Omega - \{a\})$. It suffices to show that $g'(a)$ exists. Since for every $z \in D'(a;r)$,

$$g(z) = f(z) = c_2 + c_3(z-a) + c_4(z-a)^2 + \cdots,$$

for every $z \in D'(a;r)$,

$$g(z) = c_2 + c_3(z-a) + c_4(z-a)^2 + \cdots.$$

Also, since

$$g(a) = c_2 = c_2 + c_3(a-a) + c_4(a-a)^2 + \cdots,$$

for every $z \in D(a;r)$,

$$g(z) = c_2 + c_3(z-a) + c_4(z-a)^2 + \cdots.$$

Thus, g is representable by power series in $D(a;r)$, and hence, by Conclusion 1.53, $g'(a)$ exists. ∎)

Conclusion 1.139 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $a \in \Omega$. Let $f : (\Omega - \{a\}) \to \mathbb{C}$. Suppose that f has an isolated singularity at a. Let r be a positive real number such that $D(a;r) \subset \Omega$, and $f(D'(a;r))$ is a bounded subset of $\mathbb{C}$. Then f has a removable singularity at a.

1.9 Casorati–Weierstrass Theorem

Note 1.140 Let Ω be a region. That is, Ω is a nonempty, open, connected subset of $\mathbb{C}$. Let $a \in \Omega$. Let $f : (\Omega - \{a\}) \to \mathbb{C}$. Suppose that f has an isolated singularity at a. Suppose that (for every $r > 0$ satisfying $D(a;r) \subset \Omega, f(D'(a;r))$ is dense in $\mathbb{C}$) is a false statement.

It follows that there exists $r > 0$ such that $D(a;r) \subset \Omega$, and $f(D'(a;r))$ is not dense in $\mathbb{C}$. (Now, recall the formula $A^{c-c} = A^0$ in 'topology'.) Since $f(D'(a;r))$ is not dense in $\mathbb{C}$,

$$(f(D'(a;r)))^{c0} = \underbrace{(f(D'(a;r)))^{-c}}$$

is nonempty, and hence $(f(D'(a;r)))^{c0}$ is nonempty. It follows that there exists $b \in (f(D'(a;r)))^c$,

and $\delta > 0$ such that $D(b;\delta) \subset (f(D'(a;r)))^c$, and hence

$$\underbrace{f(D'(a;r)) \subset (D(b;\delta))^c} = \{z : \delta \leq |z-b|\}.$$

Thus, $b \notin f(D'(a;r))$ and, for every $z \in D'(a;r)$, $\delta \leq |f(z) - b|$. It follows that for every $z \in D'(a;r)$,

$$\left|\frac{1}{f(z)-b}\right| \leq \frac{1}{\delta}.$$

Since $b \notin f(D'(a;r))$,

$$g : z \mapsto \frac{1}{f(z) - b}$$

is a function from $D'(a;r)$ to $(\mathbb{C} - \{0\})$.

Problem 1.141 $g \in H(D'(a;r))$.

(**Solution** Since $D(a;r) \subset \Omega$, we have $D'(a;r) \subset (\Omega - \{a\})$. Since $f : (\Omega - \{a\}) \to \mathbb{C}$ has an isolated singularity at a, $f \in H(\Omega - \{a\})$. Now, since $D'(a;r) \subset (\Omega - \{a\})$, the function $g : z \mapsto \frac{1}{f(z)-b}$ from $D'(a;r)$ to $(\mathbb{C} - \{0\})$, is a member of $H(D'(a;r))$. ∎)

Thus, g has an isolated singularity at a. Since for every $z \in D'(a;r)$,

$$|g(z)| = \underbrace{\left|\frac{1}{f(z) - b}\right| \leq \frac{1}{\delta}},$$

$g(D'(a;r))$ is a bounded subset of $\mathbb{C}$. By Conclusion 1.139, g has a removable singularity at a. It follows that there exists $h \in H(D(a;r))$ such that $h|_{D'(a;r)} = g$. It follows that for every $z \in D'(a;r)$,

$$\underbrace{h(z) = g(z)} = \frac{1}{f(z) - b},$$

and hence for every $z \in D'(a;r)$,

$$f(z) = \frac{1}{h(z)} + b.$$

Case I: when $h(a) \neq 0$. Since $h|_{D'(a;r)} = g$, $g : D'(a;r) \to (\mathbb{C} - \{0\})$ and $h(a) \neq 0$, h is nonzero on $D(a;r)$. Now, since $h : D(a;r) \to (\mathbb{C} - \{0\})$ is continuous at a,

$$\frac{1}{h} : D(a;r) \to (\mathbb{C} - \{0\})$$

is continuous at a.

Since

$$\frac{1}{h} : D(a;r) \to (\mathbb{C} - \{0\})$$

is continuous at a, there exists $\rho \in (0, r)$ such that

$$\left\{\frac{1}{h(z)} : z \in D'(a;\rho)\right\} \subset D\left(\frac{1}{h(a)};1\right).$$

This shows that $\left\{\frac{1}{h(z)} : z \in D'(a;\rho)\right\}$ is bounded, and hence

$$f(D'(a;\rho)) = \{f(z) : z \in D'(a;\rho)\} = \underbrace{\left\{\frac{1}{h(z)} + b : z \in D'(a;\rho)\right\}}$$

is bounded. Thus, $f(D'(a;\rho))$ is bounded. Now, by Conclusion 1.139, f has a removable singularity at a.

Case II: when $h(a) = 0$. Here, $a \in h^{-1}(0)$. Since $h|_{D'(a;r)} = g$, and $g : D'(a;r) \to (\mathbb{C} - \{0\})$, h is nonzero on $D'(a;r)$. Now, by Theorem 1.136, there exists a unique positive integer m, and a unique function $k : D(a;r) \to \mathbb{C}$ such that

a. $k \in H(D(a;r))$,
b. for every $z \in D(a;r)$, $h(z) = (z-a)^m(k(z))$,
c. $k(a) \neq 0$.

By Case II(b), for every $z \in D'(a;r)$,

$$\frac{1}{f(z) - b} = \underbrace{g(z) = (z-a)^m(k(z))},$$

and hence for every $z \in D'(a;r)$, $k(z) \neq 0$ and

$$f(z) = b + \frac{1}{(z-a)^m}\left(\frac{1}{k(z)}\right).$$

Since $k \in H(D(a;r))$ and, for every $z \in D(a;r)$, $k(z) \neq 0$, we have $\frac{1}{k} \in H(D(a;r))$. By Conclusion 1.116, $\frac{1}{k}$ is representable by power series in $H(D(a;r))$. Hence, there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in D(a;r)$,

$$\frac{1}{k(z)} = \sum_{n=0}^{\infty} c_n (z-a)^n.$$

It follows that for every $z \in D'(a;r)$,

$$(f(z) - b =)\frac{1}{(z-a)^m}\left(\frac{1}{k(z)}\right) = \frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)} + c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \cdots.$$

Hence, for every $z \in D'(a;r)$,

$$f(z) - \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)}\right)$$
$$= (c_m + b) + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \cdots.$$

Since

$$\frac{1}{k(a)} = c_0 + \sum_{n=1}^{\infty} c_n (a-a)^n (= c_0),$$

we have $c_0 \neq 0$. Since

$$f : (\Omega - \{a\}) \to \mathbb{C},$$

and f has an isolated singularity at a, it follows that the function

$$F : z \mapsto \left(f(z) - \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)}\right)\right)$$

from $(\Omega - \{a\})$ to $\mathbb{C}$ has an isolated singularity at a.

Problem 1.142 F has a removable singularity at the point a.

(Solution Let us define a function $G : \Omega \to \mathbb{C}$ as follows: For every $z \in \Omega$,

$$G(z) \equiv \begin{cases} F(z) & \text{if } z \neq a \\ c_m + b & \text{if } z = a. \end{cases}$$

Clearly, $G|_{(\Omega - \{a\})} = F$. Since $F \in H(\Omega - \{a\})$, we have $G|_{(\Omega - \{a\})} \in H(\Omega - \{a\})$. It suffices to show that $G'(a)$ exists.

Since for every $z \in D'(a;r)$,

$$G(z) = F(z) = f(z) - \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)}\right)$$
$$= (c_m + b) + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \cdots,$$

for every $z \in D'(a;r)$,

$$G(z) = (c_m + b) + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \cdots.$$

Also, since

$$G(a) = (c_m + b) = (c_m + b) + c_{m+1}(a-a) + c_{m+2}(a-a)^2 + \cdots,$$

for every $z \in D(a; r)$,

$$G(z) = (c_m + b) + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \cdots.$$

Thus G is representable by power series in $D(a; r)$, and hence, by Conclusion 1.53, $G'(a)$ exists. ∎)

Thus, the function

$$F : z \mapsto \left(f(z) - \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)} \right) \right)$$

from $(\Omega - \{a\})$ to $\mathbb{C}$ has a removable singularity at the point a.

Conclusion 1.143 Let Ω be a region. Let $a \in \Omega$. Let $f : (\Omega - \{a\}) \to \mathbb{C}$. Suppose that f has an isolated singularity at a. Then, one of the following statements holds:

1. for every $r > 0$ satisfying $D(a; r) \subset \Omega, f(D'(a; r))$ is dense in $\mathbb{C}$,
2. f has a removable singularity at a,
3. there exist a positive real number m, and complex numbers $c_0, c_1, \ldots, c_{m-1}$ such that $c_0 \neq 0$, and the function

$$F : z \mapsto \left(f(z) - \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)} \right) \right)$$

from $(\Omega - \{a\})$ to $\mathbb{C}$ has a removable singularity at a.

(Definition If the statement (1) holds, then we say the *f has an* ***essential singularity*** *at a*. If the statement (3) holds, then we say the *f has a* ***pole of order*** *m at a*, and the function $z \to \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)} \right)$ from $(\Omega - \{a\})$ to $\mathbb{C}$ is called the ***principal part*** *of f at a*. Observe that if the statement (3) holds, then $\lim_{z \to a} |f(z)| = \infty$.)

Conclusion 1.143 is known as the **Casorati–Weierstrass theorem**.

1.10 Liouville's Theorem

Note 1.144 Suppose that for every $z \in D(a;R), f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$. Let $r \in (0,R)$.

It follows that for every real θ,

$$\underbrace{f\left(a+re^{i\theta}\right) = c_0 + c_1\left(\left(a+re^{i\theta}\right)-a\right) + c_2\left(\left(a+re^{i\theta}\right)-a\right)^2 + \cdots}$$

$$= c_0 + c_1\left(re^{i\theta}\right) + c_2\left(re^{i\theta}\right)^2 + \cdots = c_0 + c_1re^{i\theta} + c_2r^2e^{2i\theta} + \cdots.$$

Since for every $z \in D(a;R)$,

$$c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

is convergent and $r \in (0,R)$, for every $\theta \in [-\pi,\pi]$,

$$c_0 + c_1re^{i\theta} + c_2r^2e^{2i\theta} + \cdots$$

is absolutely convergent and its sum is $f\left(a+re^{i\theta}\right)$. Thus,

$$|c_0| + |c_1|r + |c_2|r^2 + \cdots$$

converges absolutely. It follows, by Mertens' theorem, that the Cauchy product of

$$|c_0| + |c_1|r + |c_2|r^2 + \cdots$$

with itself is convergent, and hence

$$|c_0|^2 + 2|c_0||c_1|r + (|c_0||c_2| + |c_1||c_1| + |c_2||c_0|)r^2 + \cdots$$

is convergent. Thus, $\sum_{n=0}^{\infty}\left(r^n\sum_{k=0}^{\infty}|c_k|\left|c_{(n-k)}\right|\right)$ is convergent. Since for every $\theta \in [-\pi,\pi]$,

$$c_0 + c_1re^{i\theta} + c_2r^2e^{2i\theta} + \cdots$$

is absolutely convergent, and its sum is $f\left(a+re^{i\theta}\right)$, for every $\theta \in [-\pi,\pi]$,

$$\overline{c_0} + \overline{c_1}re^{-i\theta} + \overline{c_2}r^2e^{-2i\theta} + \cdots$$

is absolutely convergent, and its sum is $\overline{f(a+re^{i\theta})}$. Now, for every $\theta \in [-\pi,\pi]$, by Mertens' theorem, the Cauchy product of

$$c_0 + c_1 re^{i\theta} + c_2 r^2 e^{2i\theta} + \cdots$$

and

$$\overline{c_0} + \overline{c_1} re^{-i\theta} + \overline{c_2} r^2 e^{-2i\theta} + \cdots$$

is absolutely convergent, and its sum is

$$f\left(a + re^{i\theta}\right) \cdot \overline{f\left(a + re^{i\theta}\right)} \left(= \left|f\left(a + re^{i\theta}\right)\right|^2\right).$$

It follows that for every $\theta \in [-\pi, \pi]$,

$$\begin{aligned}&(c_0\overline{c_0}) + \left(c_0\overline{c_1} re^{-i\theta} + c_1 re^{i\theta}\overline{c_0}\right) + \left(c_0\overline{c_2} r^2 e^{-2i\theta} + c_1 re^{i\theta}\overline{c_1} re^{-i\theta} + c_2 r^2 e^{2i\theta}\overline{c_0}\right) + \cdots \\ &\quad = \left|f\left(a + re^{i\theta}\right)\right|^2,\end{aligned}$$

and hence for every $\theta \in [-\pi, \pi]$,

$$\left|f\left(a + re^{i\theta}\right)\right|^2 = |c_0|^2 + r\left(c_0\overline{c_1} e^{-i\theta} + c_1\overline{c_0} e^{i\theta}\right) + r^2\left(c_0\overline{c_2} e^{-2i\theta} + |c_1|^2 r^2 + c_2\overline{c_0} e^{2i\theta}\right) + \cdots.$$

Problem 1.145 $|c_0|^2 + \left(c_0\overline{c_1} e^{-i\theta} + c_1\overline{c_0} e^{i\theta}\right) r + \left(c_0\overline{c_2} e^{-2i\theta} + |c_1|^2 r^2 + c_2\overline{c_0} e^{2i\theta}\right) r^2 + \cdots$ converges uniformly on $[-\pi, \pi]$.

(Solution Since for every $\theta \in [-\pi, \pi]$, and for every nonnegative integer n,

$$\begin{aligned}\left|\sum_{k=0}^{n}\left(c_k r^k e^{ki\theta}\right)\left(\overline{c_{(n-k)}} r^{n-k} e^{-(n-k)i\theta}\right)\right| &= \left|\sum_{k=0}^{n}\left(c_k \overline{c_{(n-k)}} r^n e^{-(n-2k)i\theta}\right)\right| \\ &= \left|\sum_{k=0}^{n}\left(c_k \overline{c_{(n-k)}} e^{-(n-2k)i\theta}\right)\right| r^n \\ &\le r^n \sum_{k=0}^{n}\left|c_k \overline{c_{(n-k)}} e^{-(n-2k)i\theta}\right| = r^n \sum_{k=0}^{n}|c_k|\left|\overline{c_{(n-k)}}\right| \\ &= r^n \sum_{k=0}^{n}|c_k|\left|c_{(n-k)}\right|,\end{aligned}$$

we have, for every $\theta \in [-\pi, \pi]$ and for every nonnegative integer n,

$$\left|\sum_{k=0}^{n}\left(c_k r^k e^{ki\theta}\right)\left(\overline{c_{(n-k)}} r^{n-k} e^{-(n-k)i\theta}\right)\right| \le r^n \sum_{k=0}^{n}|c_k|\left|c_{(n-k)}\right|.$$

Since $\sum_{n=0}^{\infty}\left(r^n \sum_{k=0}^{n}|c_k|\left|c_{(n-k)}\right|\right)$ is convergent, by M-test,

$$|c_0|^2 + \left(c_0\overline{c_1}e^{-i\theta} + c_1\overline{c_0}e^{i\theta}\right)r + \left(c_0\overline{c_2}e^{-2i\theta} + |c_1|^2r^2 + c_2\overline{c_0}e^{2i\theta}\right)r^2 + \cdots$$

converges uniformly on $[-\pi, \pi]$. ∎)

Hence,

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left|f\left(a + re^{i\theta}\right)\right|^2 \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} |c_0|^2 \mathrm{d}\theta + \int_{-\pi}^{\pi} \left(c_0\overline{c_1}e^{-i\theta} + c_1\overline{c_0}e^{i\theta}\right) r\mathrm{d}\theta + \int_{-\pi}^{\pi} \left(c_0\overline{c_2}e^{-2i\theta} + |c_1|^2 + c_2\overline{c_0}e^{2i\theta}\right) r^2\mathrm{d}\theta + \cdots \\
&= |c_0|^2(2\pi) + \left(c_0\overline{c_1}0 + c_1\overline{c_0}0\right)r + \left(c_0\overline{c_2}0 + |c_1|^2(2\pi) + c_2\overline{c_0}0\right)r^2 + \cdots \\
&= 2\pi\left(|c_0|^2 + |c_1|^2r^2 + |c_2|^2r^4 + \cdots\right).
\end{aligned}$$

Thus,

$$|c_0|^2 + |c_1|^2r^2 + |c_2|^2r^4 + \cdots = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(a + re^{i\theta}\right)\right|^2 \mathrm{d}\theta.$$

Conclusion 1.146 Suppose that for every $z \in D(a; R)$,

$$f(z) = c_0 + c_1(z - a) + c_2(z - a)^2 + \cdots.$$

Let $r \in (0, R)$. Then

$$|c_0|^2 + |c_1|^2r^2 + |c_2|^2r^4 + \cdots = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(a + re^{i\theta}\right)\right|^2 \mathrm{d}\theta.$$

This formula is known as **Parseval's formula**.

Note 1.147 Let $f : \mathbb{C} \to \mathbb{C}$. Let $f \in H(\mathbb{C})$. (In short, f is an entire function). Let f be non-constant.

Problem 1.148 f is unbounded. That is, $f(\mathbb{C})$ is an unbounded subset of $\mathbb{C}$.

Observe that

$$\{f(0)\} \cup \left\{f\left(re^{i\theta}\right) : r \in (0, \infty), \text{and } \theta \in [-\pi, \pi]\right\} = \{f(z) : z \in \mathbb{C}\} = f(\mathbb{C}).$$

(Solution If not, suppose otherwise that there exists $M > 0$ such that for every $z \in \mathbb{C}$, $|f(z)| \leq M$. We have to arrive at a contradiction.

Since $f \in H(\mathbb{C})$, by Problem (1.115), f is representable by power series in $\mathbb{C}$. Hence there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in D(0;1), f(z) = \sum_{n=0}^{\infty} c_n z^n$.

Problem 1.149 For every $z \in \mathbb{C}, f(z) = \sum_{n=0}^{\infty} c_n z^n$.

(**Solution** Let us take fix any $w \in \mathbb{C}$. We have to show that $f(w) = \sum_{n=0}^{\infty} c_n w^n$. Since f is representable by power series in $\mathbb{C}$, there exist complex numbers $b_0, b_1, b_2, b_3, \ldots$ such that for every $z \in D(0;|w|+1), f(z) = \sum_{n=0}^{\infty} b_n z^n$. It follows that for every $w \in D(0;1)$,

$$\sum_{n=0}^{\infty} c_n w^n = \underbrace{f(w) = \sum_{n=0}^{\infty} b_n w^n}.$$

Now, by Lemma 1.60, for every nonnegative integer $n, b_n = c_n$, and hence for every $z \in D(0;|w|+1)$, $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Now, since $w \in D(0;|w|+1)$, $f(w) = \sum_{n=0}^{\infty} c_n w^n$. ∎)

It follows, by Conclusion 1.146, that for every $r > 0$,

$$|c_1|^2 r^2 \leq \underbrace{|c_0|^2 + |c_1|^2 r^2 + |c_2|^2 r^4 + \cdots = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(a + re^{i\theta}\right)\right|^2 \mathrm{d}\theta} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} M^2 \mathrm{d}\theta$$

$$= M^2,$$

and hence, for every $r > 0, |c_1|^2 r^2 \leq M^2$. This shows that $c_1 = 0$. Similarly, $c_2 = 0, c_3 = 0$, etc. Thus, for every $z \in \mathbb{C}, f(z) = c_0$. This contradicts the assumption that f is nonconstant. ∎)

Conclusion 1.150 Let f be a nonconstant entire function. Then f is unbounded.

This result, known as **Liouville's theorem**, is due to J. Liouville (24.03.1809–08.09.1882).

1.11 Maximum Modulus Theorem

Note 1.151 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f \in H(\Omega)$. Let $a \in \Omega$, and let $r > 0$. Let $D[a;r] \subset \Omega$.

Since $f : \Omega \to \mathbb{C}$ is continuous and

$$\{z : |z-a| = r\} = \left\{a + re^{i\theta} \ : \theta \in [-\pi, \pi]\right\} \subset D[a;r] \subset \Omega,$$

$\{z : |z-a| = r\}$ is a compact subset of Ω, $\max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}$, and $\min\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}$ exist.

Problem 1.152 $|f(a)| \le \max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}$.

(Solution Since $f \in H(\Omega)$, by Problem (1.115), f is representable by power series in Ω, and hence there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Thus, $f(a) = c_0$.

It follows, by Conclusion 1.146, that

$$|f(a)|^2 = |c_0|^2 \le \underbrace{|c_0|^2 + |c_1|^2 r^2 + |c_2|^2 r^4 + \cdots = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(a+re^{i\theta}\right)\right|^2 \mathrm{d}\theta}$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}\right)^2 \mathrm{d}\theta = \left(\max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}\right)^2,$$

and hence

$$|f(a)|^2 \le \left(\max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}\right)^2.$$

This shows that

$$|f(a)| \le \max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}.$$

∎)

Conclusion 1.153 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f \in H(\Omega)$. Let $a \in \Omega$, and $r > 0$. Let $D[a;r] \subset \Omega$. Then,

1. $|f(a)| \le \max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}$,
2. If Ω is a region, then $|f(a)| = \max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} \Leftrightarrow f$ is a constant function.

Proof of the remaining part (2) Let f be a constant function. Then

$$\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} = \{f(a)\},$$

and hence

$$\max\{|f(a+re^{i\theta})| : \theta \in [-\pi,\pi]\} = \max\{|f(a)|\} = |f(a)|.$$

Conversely, let Ω be a region and

$$|f(a)| = \max\{|f(a+re^{i\theta})| : \theta \in [-\pi,\pi]\}.$$

We have to show that f is a constant function. Since $f \in H(\Omega)$, by Problem (1.115), f is representable by power series in Ω and hence there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in D(a;r)$, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Thus, $f(a) = c_0$.

It follows, by Conclusion 1.146, that

$$|f(a)|^2 + |c_1|^2 r^2 + |c_2|^2 r^4 + \cdots = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(a+re^{i\theta})|^2 \mathrm{d}\theta$$
$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\max\{|f(a+re^{i\theta})| : \theta \in [-\pi,\pi]\}\right)^2 \mathrm{d}\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(a)|^2 \mathrm{d}\theta = |f(a)|^2,$$

and hence

$$|c_1|^2 r^2 + |c_2|^2 r^4 + \cdots = 0.$$

This shows that

$$0 = c_1 = c_2 = \cdots,$$

and hence for every $z \in D(a;r)$, $f(z) = f(a)$. Thus, f is constant on $D(a;r)$. Now, by Theorem 1.135, f is constant on Ω ■)

This result is known as the **maximum modulus theorem**.

Let Ω be a bounded region. Let $f : \bar{\Omega} \to \mathbb{C}$. Let f be continuous, and $f|_{\Omega}$ is holomorphic.

Problem 1.154 There exists $z_0 \in \partial\Omega$ such that for every $z \in \bar{\Omega}$, $|f(z)| \leq |f(z_0)|$.

(Solution Case I: when f is a constant function. The result is trivially true.

Case II: when f is not a constant function. Since Ω is bounded, $\bar{\Omega}$ is bounded. Now, since $\bar{\Omega}$ is closed, $\bar{\Omega}$ is compact. Since $f : \bar{\Omega} \to \mathbb{C}$ is continuous, $|f| : \bar{\Omega} \to [0,\infty)$ is continuous. Now, since $\bar{\Omega}$ is compact, there exists $z_0 \in \bar{\Omega}$ such that for every $z \in \bar{\Omega}$, $|f(z)| \leq |f(z_0)|$.

Problem 1.155 $z_0 \in \partial\Omega \left(= \bar{\Omega} - \Omega^0 = \bar{\Omega} - \bar{\Omega}\right)$.

(Solution If not, otherwise, let $z_0 \in \Omega$. We have to arrive at a contradiction.

Since Ω is a region, Ω is open, and hence there exists a positive real number r such that $D[z_0;r] \subset \Omega$. So, by Conclusion 1.153,

$$|f(z_0)| \le \max\left\{\left|f\left(z_0 + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}.$$

Since for every $z \in \bar{\Omega}(\supset \Omega \supset D[z_0; r])$, $|f(z)| \le |f(z_0)|$,

$$\max\left\{\left|f\left(z_0 + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} \le |f(z_0)|.$$

Thus,

$$|f(z_0)| = \max\left\{\left|f\left(z_0 + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}.$$

Now, by Conclusion 1.153, $f|_\Omega$ is a constant function. Now, since $f : \bar{\Omega} \to \mathbb{C}$ is continuous, f is constant on $\bar{\Omega}$ This is a contradiction. ∎)

Thus, in all cases, there exists $z_0 \in \partial\Omega$ such that for every $z \in \bar{\Omega}$, $|f(z)| \le |f(z_0)|$. ∎)

Conclusion 1.156 Let Ω be a bounded region. Let $f : \bar{\Omega} \to \mathbb{C}$. Let f be continuous, and $f|_\Omega$ is holomorphic. Then there exists $z_0 \in \partial\Omega$ such that for every $z \in \bar{\Omega}$, $|f(z)| \le |f(z_0)|$.

Note 1.157 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f \in H(\Omega)$. Let $a \in \Omega$, and let $r > 0$. Let $D[a; r] \subset \Omega$, and $D(a; r)$ contains no zero of f.

Problem 1.158 $\min\left\{\left|f\left(a + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} \le |f(a)|$.

(Solution Case I: when there exists $\theta \in [-\pi, \pi]$ such that $f\left(a + re^{i\theta}\right) = 0$. In this case,

$$\min\left\{\left|f\left(a + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} = 0(\le |f(a)|),$$

and hence

$$\min\left\{\left|f\left(a + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} \le |f(a)|.$$

Case II: when, for every $\theta \in [-\pi, \pi]$, $f\left(a + re^{i\theta}\right) \ne 0$. Here, since $D(a; r)$ contains no zero of f, the compact set $D[a; r]$ contains no zero of f. Now, since $\theta \mapsto \left|f\left(a + re^{i\theta}\right)\right|$ is continuous on the compact set $[-\pi, \pi]$, for every $\theta \in [-\pi, \pi]$, we have $0 < m \le \left|f\left(a + re^{i\theta}\right)\right| \le M$, where

$$m \equiv \min\left\{\left|f\left(a + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\},$$

and

$$M \equiv \max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}.$$

This shows that there exists $r_1 > r$ such that $D[a;r_1] \subset \Omega$, and $D(a;r_1)$ contains no zero of f.

It follows that $g : z \mapsto \frac{1}{f(z)}$ is a function from $D(a;r_1)$ to $(\mathbb{C} - \{0\})$. Clearly, $g \in H(D(a;r_1))$. Since $f : \Omega \to \mathbb{C}$ is continuous and

$$\{z : |z-a| = r\} = \left\{a+re^{i\theta} : \theta \in [-\pi,\pi]\right\},$$

$\{z : |z-a| = r\}$ is a compact subset of Ω,

$$\min\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\},$$

and

$$\max\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}$$

exist, and are positive. Since $r_1 > r > 0$, we have $D[a;r] \subset D(a;r_1)$, and hence, by Conclusion 1.156,

$$\frac{1}{|f(a)|} = \left|\frac{1}{f(a)}\right| = |g(a)| \le \max\left\{\left|g\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\}$$
$$= \max\left\{\left|\frac{1}{f(a+re^{i\theta})}\right| : \theta \in [-\pi,\pi]\right\} = \max\left\{\frac{1}{|f(a+re^{i\theta})|} : \theta \in [-\pi,\pi]\right\} = \frac{1}{\min\{|f(a+re^{i\theta})| : \theta \in [-\pi,\pi]\}}.$$

It follows that

$$\min\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} \le |f(a)|.$$

∎)

Conclusion 1.159 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f \in H(\Omega)$. Let $a \in \Omega$, and let $r > 0$. Let $D[a;r] \subset \Omega$, and $D(a;r)$ contains no zero of f. Then

$$\min\left\{\left|f\left(a+re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} \le |f(a)|.$$

1.12 Fundamental Theorem of Algebra

Theorem 1.160 Let n be a positive integer. Let $a_{n-1}, a_{n-2}, \ldots, a_0$ be any complex numbers. Let $P : z \mapsto (z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0)$ be a function from $\mathbb{C}$ to $\mathbb{C}$. (In short, P is a polynomial with leading coefficient 1.) Then P has a zero in $\mathbb{C}$.

Proof If $n = 1$, then $P(-a_0) = 0$, so it remains to show that P has a zero in $\mathbb{C}$, where $n \geq 2$.

If not, otherwise, suppose that P has no zero in $\mathbb{C}$, and $n \geq 2$. We have to arrive at a contradiction.

Put

$$r \equiv 1 + 2|a_0| + |a_1| + \cdots + |a_{n-1}|(\geq 1).$$

Problem 1.161 For every $\theta \in [-\pi, \pi]$, $\left|P\left(re^{i\theta}\right)\right| > |P(0)|(>0)$. $\quad (*)$

(**Solution** Let us fix any $\theta \in [-\pi, \pi]$. We have to show that $\left|P\left(re^{i\theta}\right)\right| > |P(0)|$. Since

$$\begin{aligned}
\left|P\left(re^{i\theta}\right)\right| &= \left|\left(re^{i\theta}\right)^n + a_{n-1}\left(re^{i\theta}\right)^{n-1} + a_{n-2}\left(re^{i\theta}\right)^{n-2} + \cdots + a_0\right| \\
&= \left|r^n e^{in\theta} - \left(-a_{n-1}r^{n-1}e^{i(n-1)\theta}\right) - \left(-a_{n-2}r^{n-2}e^{i(n-2)\theta}\right) - \cdots - (-a_0)\right| \\
&\geq \left|r^n e^{in\theta}\right| - \left|-a_{n-1}r^{n-1}e^{i(n-1)\theta}\right| - \left|-a_{n-2}r^{n-2}e^{i(n-2)\theta}\right| - \cdots - |-a_0| \\
&= r^n - |a_{n-1}|r^{n-1} - |a_{n-2}|r^{n-2} - \cdots - |a_0| \\
&= |a_0| + \left(r^n - |a_{n-1}|r^{n-1} - |a_{n-2}|r^{n-2} - \cdots - |a_1|r - 2|a_0|\right) \\
&\geq |a_0| + \left(r^n - |a_{n-1}|r^{n-1} - |a_{n-2}|r^{n-1} - \cdots - |a_1|r^{n-1} - 2|a_0|r^{n-1}\right) \\
&= |a_0| + (r - (2|a_0| + |a_1| + \cdots + |a_{n-1}|))r^{n-1} \\
&= |a_0| + ((1 + 2|a_0| + |a_1| + \cdots + |a_{n-1}|) - (2|a_0| + |a_1| + \cdots + |a_{n-1}|))r^{n-1} \\
&= |a_0| + r^{n-1} > |a_0| = |P(0)|,
\end{aligned}$$

we have

$$\left|P\left(re^{i\theta}\right)\right| > |P(0)|.$$

∎)

Since P has no zero in $\mathbb{C}$, and $P \in H(\mathbb{C})$, the function $f : z \mapsto \frac{1}{P(z)}$ from $\mathbb{C}$ to $(\mathbb{C} - \{0\})$ is a member of $H(\mathbb{C})$, and hence, by Conclusion 1.156,

$$\begin{aligned}
\frac{1}{|P(0)|} &= \underbrace{|f(0)| \leq \max\left\{\left|f\left(0 + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}} \\
&= \max\left\{\frac{1}{|P(re^{i\theta})|} : \theta \in [-\pi, \pi]\right\}.
\end{aligned}$$

Thus, there exists $\alpha \in [-\pi, \pi]$ such that

$$\max\left\{\frac{1}{|P(re^{i\theta})|} : \theta \in [-\pi, \pi]\right\} = \frac{1}{|P(re^{i\alpha})|}.$$

Since

$$\frac{1}{|P(0)|} \leq \max\left\{\frac{1}{|P(re^{i\theta})|} : \theta \in [-\pi, \pi]\right\} = \frac{1}{|P(re^{i\alpha})|},$$

we have

$$\frac{1}{|P(0)|} \leq \frac{1}{|P(re^{i\alpha})|},$$

and hence $|P(re^{i\alpha})| \leq |P(0)|$. This contradicts $(*)$. ∎)

Note 1.162 Let n be a positive integer. Let $a_{n-1}, a_{n-2}, \ldots, a_0$ be any complex numbers. Let $P : z \mapsto (z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0)$ be a function from $\mathbb{C}$ to $\mathbb{C}$. (In short, P is a polynomial with leading coefficient 1.)

Problem 1.163 P has exactly n zeros in $\mathbb{C}$.

(Solution By Theorem 1.160, there exists $\alpha_1 \in \mathbb{C}$ such that $P(\alpha_1) = 0$. It follows that there exists a polynomial

$$Q : z \mapsto \left(z^{n-1} + b_{n-2}z^{n-2} + b_{n-3}z^{n-3} + \cdots + b_0\right)$$

from $\mathbb{C}$ to $\mathbb{C}$ such that, for every $z \in \mathbb{C}$, $P(z) = (z - \alpha_1)Q(z)$. Repeating such arguments, we get exactly n zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ of P such that for every $z \in \mathbb{C}$,

$$P(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n).$$

∎)

Conclusion 1.164 Every nonconstant polynomial of degree n has exactly n zeros in $\mathbb{C}$.

This result is known as the **fundamental theorem of algebra**.

1.13 Cauchy's Estimates

Note 1.165 Let $a \in \mathbb{C}$, and $R > 0$. Let $f : D(a; R) \to \mathbb{C}$ be a function. Let $f \in H(D(a; R))$. Let $M > 0$. Suppose that for every $z \in D(a; R)$, $|f(z)| \leq M$.

By Conclusion 1.116, f is representable by power series in $D(a; R)$, and hence by Lemma 1.60, for every $z \in D(a; R)$,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots.$$

It follows, by Conclusion 1.146, that for every $r \in (0,R)$,

$$|f(a)|^2 + \left|\frac{f'(a)}{1!}\right|^2 r^2 + \left|\frac{f''(a)}{2!}\right|^2 r^4 + \cdots = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(a+re^{i\theta}\right)\right|^2 \mathrm{d}\theta \le \frac{1}{2\pi}\int_{-\pi}^{\pi} M^2 \mathrm{d}\theta = M^2.$$

It follows that for every positive integer n, and for every $r \in (0,R)$,

$$\left|\frac{f^{(n)}(a)}{n!}\right|^2 r^{2n} \le M^2,$$

and hence for every positive integer n, and for every $r \in (0,R)$,

$$\frac{1}{n!}\left|f^{(n)}(a)\right| r^n \le M.$$

Thus, for every positive integer n,

$$\frac{1}{n!}\left|f^{(n)}(a)\right| R^n = \underbrace{\sup\left\{\frac{1}{n!}\left|f^{(n)}(a)\right| r^n : r \in (0,R)\right\} \le M},$$

and hence for every positive integer n,

$$\left|f^{(n)}(a)\right| \le \frac{M(n!)}{R^n}.$$

Conclusion 1.166 Let $a \in \mathbb{C}$, and $R > 0$. Let $f : D(a;R) \to \mathbb{C}$ be a function. Let $f \in H(D(a;R))$. Let $M > 0$. Suppose that for every $z \in D(a;R)$, $|f(z)| \le M$. Then for every positive integer n,

$$\left|f^{(n)}(a)\right| \le \frac{M(n!)}{R^n}.$$

Hence,

$$R|f'(a)| \le \|f\|_{D(a;R)}.$$

This result is known as the **Cauchy's estimates**.

1.14 Open Mapping Theorem

Note 1.167

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : \Omega \to \mathbb{C}$ be a function. Let $f : \Omega \to \mathbb{C}$ be a function. If, for every nonempty compact subset K of Ω, and for every $\varepsilon > 0$, there exists a positive integer $N(\equiv N(K, \varepsilon))$ such that

$$(n \geq N, \text{ and } z \in K) \Rightarrow |f_n(z) - f(z)| < \varepsilon,$$

then we say that the sequence $\{f_n\}$ ***converges to f uniformly on compact subsets of*** Ω.

Let us take $D(0;1)$ for Ω. For every positive integer n, let $f_n : \mathrm{z} \mapsto z^n$. Let $f : \mathrm{z} \mapsto 0$.

Problem 1.168 $\{f_n\}$ converges to f uniformly on compact subsets of Ω.

(Solution Let us take any compact subset K of $\Omega(= D(0;1))$. Let us take any $\varepsilon > 0$.

Since K is compact and $z \to |z|$ is continuous, there exists $z_0 \in K(\subset \Omega = D(0;1))$ such that for every $z \in K$, $|z| \leq |z_0|(<1)$. Since $\lim_{n\to\infty}|z_0|^n = 0$, there exists a positive integer N such that every $n \geq N$ implies $|z_0|^n < \varepsilon$. Let us fix any $z \in K$, and $n \geq N$. It suffices to show that $|z|^n < \varepsilon$. Since $z \in K$, we have $|z| \leq |z_0|$, and hence $|z|^n \leq |z_0|^n (< \varepsilon)$. Thus, $|z|^n < \varepsilon$. ∎)

Let Ω be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : \Omega \to \mathbb{C}$ be a function. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that each $f_n \in H(\Omega)$. It follows that for every positive integer n, $(f_n)' : \Omega \to \mathbb{C}$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of Ω.

a. Problem 1.169 $f : \Omega \to \mathbb{C}$ is continuous.

(Solution For this purpose, let us take any $a \in \Omega$. We have to show that f is continuous at a.

Since $a \in \Omega$, and Ω is open, there exists $r > 0$ such that $\left(D\left[a; \frac{r}{2}\right] \subset\right) D(a;r) \subset \Omega$. Thus, $D\left[a; \frac{r}{2}\right]$ is a nonempty compact subset of Ω. Now, since $\{f_n\}$ converges to f uniformly on compact subsets of Ω, $\{f_n\}$ converges to f uniformly on $D\left[a; \frac{r}{2}\right]$. Since each $f_n \in H(\Omega)$, each $f_n : \Omega \to \mathbb{C}$ is continuous. Now, since $D\left[a; \frac{r}{2}\right] \subset \Omega$, each f_n is continuous on $D\left[a; \frac{r}{2}\right]$. Since each f_n is continuous on $D\left[a; \frac{r}{2}\right]$, and $\{f_n\}$ converges to f uniformly on $D\left[a; \frac{r}{2}\right]$, by a theorem (e.g., [5], Theorem 7.12), f is continuous on $D\left[a; \frac{r}{2}\right](\ni a)$, and hence f is continuous at a. ∎)

b. Problem 1.170 $f \in H(\Omega)$(and hence, $f' : \Omega \to \mathbb{C}$).

(Solution Let us take any triangle Δ satisfying $\Delta \subset \Omega$. By Conclusion 1.123, it suffices to show that $\int_{\partial\Delta} f(z)\mathrm{d}z = 0$.

Since Δ is compact, and $\{f_n\}$ converges to f uniformly on compact subsets of Ω, $\{f_n\}$ converges to f uniformly on Δ, and hence, by a theorem (e.g., [5], Theorem 7.16), $\int_{\partial\Delta} f(z)\mathrm{d}z = \lim_{n\to\infty}\left(\int_{\partial\Delta} f_n(z)\mathrm{d}z\right)$. Since each $f_n \in H(\Omega)$, each $f_n : \Omega \to \mathbb{C}$ is a continuous function, and hence by Conclusion 1.104, each $\int\limits_{\partial\Delta} f_n(z)\mathrm{d}z = 0$. It follows that

$$\int\limits_{\partial\Delta} f(z)\mathrm{d}z = \underbrace{\lim_{n\to\infty}\left(\int\limits_{\partial\Delta} f_n(z)\mathrm{d}z = 0\right)}.$$

Hence, $\int_{\partial\Delta} f(z)\mathrm{d}z = 0$. ∎)

c. Problem 1.171 $\{(f_n)'\}$ converges to f' uniformly on compact subsets of Ω.

(Solution For this purpose, let us take any nonempty compact subset K of Ω. We have to show that $\{(f_n)'\}$ converges to f' uniformly on K. There exists $\rho > 0$ such that for every $z \in K$, $D[z;\rho] \subset \Omega$. Let us take any $\varepsilon > 0$.

Since $\{f_n\}$ converges to f uniformly on compact subsets of Ω and, for every $z \in K$, $D[z;\rho]$ is a compact subset of Ω, for every $z \in K$, $\{f_n\}$ converges to f uniformly on $D[z;\rho]$. Hence, for every $z \in K$, there exists a positive integer N such that

$$\left(n \geq N \Rightarrow \rho\varepsilon > \|f_n - f\|_{D[z;\rho]}\left(\geq \|f_n - f\|_{D(z;\rho)}\right)\right).$$

Thus, for every $z \in K$,

$$\left(n \geq N \Rightarrow \|f_n - f\|_{D(z;\rho)} < \rho\varepsilon\right).$$

Since each $f_n \in H(\Omega)$, and $f \in H(\Omega)$, each $(f_n - f) \in H(\Omega)$, and hence by Conclusion 1.166, for every $z \in K$, and for every $n \geq N$,

$$\rho\left|(f_n)'(z) - f'(z)\right| = \underbrace{\rho\left|(f_n - f)'(z)\right| \leq \|f_n - f\|_{D(z;\rho)}} < \rho\varepsilon.$$

Thus, for every $z \in K$, and for every $n \geq N$,

$$\left|(f_n)'(z) - f'(z)\right| < \varepsilon.$$

∎)

Conclusion 1.172 Let Ω be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : \Omega \to \mathbb{C}$ be a function. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that each $f_n \in H(\Omega)$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Then

1. $f \in H(\Omega)$,
2. $\{(f_n)'\}$ converges to f' uniformly on compact subsets of Ω.

Note 1.173 Let Ω be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : \Omega \to \mathbb{C}$ be a function. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that each $f_n \in H(\Omega)$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Then, by Conclusion 1.172,

1. $f \in H(\Omega)$,
2. $\{(f_n)'\}$ converges to f' uniformly on compact subsets of Ω.

Since each $f_n \in H(\Omega)$, by Lemma 1.117, each $(f_n)' \in H(\Omega)$. Now, since $\{(f_n)'\}$ converges to f' uniformly on compact subsets of Ω, by Conclusion 1.172, $\left\{((f_n)')'\right\}$ converges to $(f')'$ uniformly on compact subsets of Ω. That is, $\{(f_n)''\}$ converges to f'' uniformly on compact subsets of Ω. Similarly, $\{(f_n)'''\}$ converges to f''' uniformly on compact subsets of Ω, etc.

Conclusion 1.174 Let Ω be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : \Omega \to \mathbb{C}$ be a function. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that each $f_n \in H(\Omega)$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Then,

1. $f \in H(\Omega)$,
2. for every positive integer m, $\left\{(f_n)^{(m)}\right\}$ converges to $f^{(m)}$ uniformly on compact subsets of Ω.

Note 1.175 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in \Omega \times \Omega$,

$$\varphi(z, w) \equiv \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Problem 1.176 $\varphi : \Omega \times \Omega \to \mathbb{C}$ is continuous.

(Solution Observe that $\{(a, a) : a \in \mathbb{C}\}$ is a closed subset of $\mathbb{C} \times \mathbb{C}$. Since Ω is an open subset of $\mathbb{C}$, $(\Omega \times \Omega)$ is an open subset of $\mathbb{C} \times \mathbb{C}$. It follows that $(\Omega \times \Omega) - \{(a, a) : a \in \mathbb{C}\}$ is an open subset of $\mathbb{C} \times \mathbb{C}$. Since $f \in H(\Omega)$, $f : \Omega \to \mathbb{C}$ is continuous. Now, since $(z, w) \mapsto z$ is continuous, $(z, w) \mapsto f(z)$ is continuous from $\Omega \times \Omega$ to $\mathbb{C}$. Similarly, $(z, w) \mapsto f(w)$ is continuous from $\Omega \times \Omega$ to $\mathbb{C}$. Now, since $w \mapsto -w$ is a continuous function from $\mathbb{C}$ to $\mathbb{C}$, $(z, w) \mapsto -(f(w))$ is continuous from $\Omega \times \Omega$ to $\mathbb{C}$. Since $(z, w) \mapsto f(z)$ is continuous, and $(z, w) \mapsto -f(w)$ is continuous, $(z, w) \mapsto (f(z), -f(w))$ is a continuous function from $\Omega \times \Omega$ to $\mathbb{C} \times \mathbb{C}$ Next, since $(z, w) \mapsto (z + w)$ is continuous from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$,

$$(z, w) \mapsto f(z) + (-f(w))(= f(z) - f(w))$$

is continuous from $(\Omega \times \Omega)$ to $\mathbb{C}$, and hence its restriction $(z, w) \mapsto f(z) - f(w)$ from $(\Omega \times \Omega) - \{(a, a) : a \in \mathbb{C}\}$ to $\mathbb{C}$ is continuous. Similarly, $(z, w) \mapsto z - w$ is continuous from $(\Omega \times \Omega) - \{(a, a) : a \in \mathbb{C}\}$ to $(\mathbb{C} - \{0\})$. Now, since $u \mapsto \frac{1}{u}$ is a continuous function from $(\mathbb{C} - \{0\})$ to $(\mathbb{C} - \{0\})$, $(z, w) \mapsto \frac{1}{z-w}$ is continuous from $(\Omega \times \Omega) - \{(a, a) : a \in \mathbb{C}\}$ to $(\mathbb{C} - \{0\})$. It follows that $(z, w) \mapsto \left(f(z) - f(w), \frac{1}{z-w}\right)$ is continuous from $(\Omega \times \Omega) - \{(a, a) : a \in \mathbb{C}\}$ to $\mathbb{C} \times \mathbb{C}$. Now, since $(z, w) \mapsto (z \cdot w)$ is continuous from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$,

$$(z, w) \mapsto (f(z) - f(w)) \cdot \frac{1}{z - w} \left(= \frac{f(z) - f(w)}{z - w} = \varphi(z, w) \right)$$

is continuous from $(\Omega \times \Omega) - \{(a, a) : a \in \mathbb{C}\}$ to $\mathbb{C}$, and hence φ is continuous from $(\Omega \times \Omega) - \{(a, a) : a \in \mathbb{C}\}$ to $\mathbb{C}$.

It remains to show that φ is continuous at all points of $\{(a, a) : a \in \Omega\}$. For this purpose, let us take any $a \in \Omega$. We have to show that φ is continuous at (a, a).

For this purpose, let us take any $\varepsilon > 0$. Since $f \in H(\Omega)$, by Lemma 1.117, $f' \in H(\Omega)$, and hence $f' : \Omega \to \mathbb{C}$ is continuous at a. It follows that there exists $\delta > 0$ such that $D(a; \delta) \subset \Omega$, and $(z \in D(a; \delta) \Rightarrow |f'(z) - f'(a)| < \varepsilon)$.

It suffices to show that $(z, w) \in ((D(a; \delta) \times D(a; \delta)) - \{(b, b) : b \in \mathbb{C}\}) \Rightarrow |\varphi(z, w) - \varphi(a, a)| < \varepsilon$, that is

$$(z, w) \in ((D(a; \delta) \times D(a; \delta)) - \{(b, b) : b \in \mathbb{C}\}) \Rightarrow \left| \frac{f(z) - f(w)}{z - w} - f'(a) \right| \leq \varepsilon.$$

For this purpose, let us fix any $(z_0, w_0) \in ((D(a; \delta) \times D(a; \delta)) - \{(b, b) : b \in \mathbb{C}\})$. We have to show that

$$\left| \frac{f(z_0) - f(w_0)}{z_0 - w_0} - f'(a) \right| \leq \varepsilon.$$

Here, $z_0, w_0 \in D(a; \delta)$, and $z_0 \neq w_0$. Let $\gamma : t \mapsto (w_0 + t(z_0 - w_0))$ be a function from $[0, 1]$ to $D(a; \delta)$. Clearly γ is a path, and $\text{ran}(\gamma) \subset D(a; \delta)$. Here,

$$\begin{aligned}
&\int_\gamma (f'(z) - f'(a))\mathrm{d}z = \int_0^1 (f'(\gamma(t)) - f'(a))(\gamma'(t))\mathrm{d}t \\
&= \int_0^1 (f'(\gamma(t))(\gamma'(t)) - (f'(a))(\gamma'(t)))\mathrm{d}t = \int_0^1 \big((f \circ \gamma)'(t) - (f'(a))(\gamma'(t))\big)\mathrm{d}t \\
&= \int_0^1 (f \circ \gamma)'(t)\mathrm{d}t - f'(a) \int_0^1 \gamma'(t)\mathrm{d}t = (f \circ \gamma)(t)|_0^1 - f'(a)\gamma(t)|_0^1 \\
&= ((f \circ \gamma)(1) - (f \circ \gamma)(0)) - (f'(a))(\gamma(1) - \gamma(0)) \\
&= (f(\gamma(1)) - f(\gamma(0))) - (f'(a))(\gamma(1) - \gamma(0)) = (f(z_0) - f(w_0)) - (f'(a))(z_0 - w_0) \\
&= (z_0 - w_0)\left(\frac{f(z_0) - f(w_0)}{z_0 - w_0} - f'(a)\right),
\end{aligned}$$

so

$$\underbrace{\begin{aligned}&|z_0 - w_0|\left|\frac{f(z_0) - f(w_0)}{z_0 - w_0} - f'(a)\right| \\ &= \left|(z_0 - w_0)\left(\frac{f(z_0) - f(w_0)}{z_0 - w_0} - f'(a)\right)\right| = \left|\int_\gamma (f'(z) - f'(a))\mathrm{d}z\right|\end{aligned}}$$

$$\leq \int_\gamma |f'(z) - f'(a)|\mathrm{d}z \leq \varepsilon(\text{length of } \gamma) = \varepsilon|z_0 - w_0|,$$

and hence

$$\left|\frac{f(z_0) - f(w_0)}{z_0 - w_0} - f'(a)\right| \leq \varepsilon.$$

∎)

Conclusion 1.177 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in \Omega \times \Omega$,

$$\varphi(z, w) \equiv \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Then $\varphi : \Omega \times \Omega \to \mathbb{C}$ is continuous.

Note 1.178 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let $z_0 \in \Omega$. Let $f'(z_0) \neq 0$.

Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in \Omega \times \Omega$,

$$\varphi(z, w) \equiv \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

By Conclusion 1.177, $\varphi : \Omega \times \Omega \to \mathbb{C}$ is continuous. Since $(\varphi(z_0, z_0) =)f'(z_0) \neq 0$, we have $|\varphi(z_0, z_0)| > 0$. Now, since $\varphi : \Omega \times \Omega \to \mathbb{C}$ is continuous at (z_0, z_0), there exists there exists an open neighborhood V_1 of z_0 in Ω such that

$$\left((z, w) \in V_1 \times V_1 \Rightarrow \frac{1}{2}|\varphi(z_0, z_0)| > |\varphi(z, w) - \varphi(z_0, z_0)|(\geq |\varphi(z_0, z_0)| - |\varphi(z, w)|)\right).$$

Hence,

$$\left((z,w) \in V_1 \times V_1 \Rightarrow |\varphi(z,w)| > \frac{1}{2}|\varphi(z_0,z_0)|\left(=\frac{1}{2}|f'(z_0)|\right)\right).$$

It follows that for every $z, w \in V_1$ satisfying $z \neq w$,

$$\frac{1}{2}|f'(z_0)| < |\varphi(z,w)|\left(=\left|\frac{f(z)-f(w)}{z-w}\right| = \frac{|f(z)-f(w)|}{|z-w|}\right),$$

and hence for every $z, w \in V_1$,

$$\frac{1}{2}|f'(z_0)||z-w| \leq |f(z)-f(w)|. \quad (*)$$

Since $f \in H(\Omega)$, $f' : \Omega \to \mathbb{C}$ is continuous. Now, since $f'(z_0) \neq 0$, there exists an open neighborhood V_2 of z_0 such that, for every $z \in V_2, f'(z) \neq 0$. Here, $V_1 \cap V_2$ is an open neighborhood of z_0 contained in Ω Also, for every $z \in V_1 \cap V_2$, $f'(z) \neq 0$.

a. Problem 1.179 f is 1–1 on $V_1 \cap V_2$.

(Solution Let $z, w \in V_1 \cap V_2$, and $f(z) = f(w)$. We have to show that $z = w$. Since $z, w \in V_1$, we have

$$0 \leq \underbrace{\frac{1}{2}|f'(z_0)||z-w| \leq |f(z)-f(w)|} = |f(w)-f(w)| = 0$$

and hence

$$\frac{1}{2}|f'(z_0)||z-w| = 0.$$

Since $z_0 \in V_2$, we have $f'(z_0) \neq 0$. Now, since $\frac{1}{2}|f'(z_0)||z-w| = 0$, we have $|z-w| = 0$, and hence $z = w$. c ∎)

b. Problem 1.180 $f(V_1 \cap V_2)$ is open in $\mathbb{C}$.

(Solution For this purpose, let us take any $a \in V_1 \cap V_2$. We have to show that $f(a)$ is an interior point of $f(V_1 \cap V_2)$. Since $a \in V_1 \cap V_2$, we have $a \in V_1$, and $a \in V_2$. Since $a \in V_2$, we have $f'(a) \neq 0$, and hence $|f'(a)| > 0$. Since $a \in V_1 \cap V_2$, and $V_1 \cap V_2$ is open, there exists $R > 0$ such that $D(a;R) \subset V_1 \cap V_2$. It follows that for every $\theta \in [-\pi, \pi]$, $\left(a + \frac{R}{2}e^{i\theta}\right) \in V_1 \cap V_2 (\subset V_1)$. Since for every $\theta \in [-\pi, \pi]$, $\left(a + \frac{R}{2}e^{i\theta}\right) \in V_1$ and $a \in V_1$, by $(*)$, for every $\theta \in [-\pi, \pi]$,

$$\frac{1}{2}|f'(z_0)|\frac{R}{2} = \underbrace{\frac{1}{2}|f'(z_0)|\left|\left(a+\frac{R}{2}e^{i\theta}\right)-a\right| \leq \left|f\left(a+\frac{R}{2}e^{i\theta}\right)-f(a)\right|}.$$

Hence, for every $\theta \in [-\pi, \pi]$,

$$2c \leq \left|f\left(a+\frac{R}{2}e^{i\theta}\right)-f(a)\right|,$$

where $c \equiv \frac{1}{8}|f'(z_0)|R$. Since $|f'(z_0)| > 0$, and $R > 0$, we have $c > 0$. It suffices to show that

$$D(f(a),\, c) \subset f(V_1 \cap V_2).$$

For this purpose, let us take any $\lambda \in D(f(a),\, c)$, that is$|f(a)-\lambda| < c$, that is $-c < -|\lambda - f(a)|$. We have to show that $\lambda \in f(V_1 \cap V_2)$, that is $(f-\lambda)$ has a zero in $V_1 \cap V_2$. If not, suppose otherwise that $(f-\lambda)$ has a no zero in $V_1 \cap V_2$. We have to arrive at a contradiction.

Since $f \in H(\Omega)$, we have $(f-\lambda) \in H(\Omega)$. Since $D\left[a; \frac{R}{2}\right] \subset D(a;R) \subset V_1 \cap V_2 \subset V_1 \subset \Omega$, and $(f-\lambda)$ has a no zero in $V_1 \cap V_2$, $(f-\lambda)$ has a no zero in $D\left(a; \frac{R}{2}\right)$. Now, by Conclusion 1.159,

$$\underbrace{\min\left\{\left|(f-\lambda)\left(a+\frac{R}{2}e^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} \leq |(f-\lambda)(a)|} = |f(a)-\lambda| < c,$$

so

$$\min\left\{\left|(f-\lambda)\left(a+\frac{R}{2}e^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} < c. \quad (**)$$

It follows that, for every $\theta \in [-\pi, \pi]$,

$$\begin{aligned} c = 2c + (-c) &< \left|f\left(a+\frac{R}{2}e^{i\theta}\right)-f(a)\right| + (-|\lambda - f(a)|) = \left|f\left(a+\frac{R}{2}e^{i\theta}\right)-f(a)\right| - |\lambda - f(a)| \\ &\leq \left|\left(f\left(a+\frac{R}{2}e^{i\theta}\right)-f(a)\right) - (\lambda - f(a))\right| = \left|f\left(a+\frac{R}{2}e^{i\theta}\right)-\lambda\right| = \left|(f-\lambda)\left(a+\frac{R}{2}e^{i\theta}\right)\right|, \end{aligned}$$

so for every $\theta \in [-\pi, \pi]$,

$$c < \left|(f-\lambda)\left(a+\frac{R}{2}e^{i\theta}\right)\right|.$$

This shows that

$$c < \min\left\{\left|(f-\lambda)\left(a+\frac{R}{2}e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}.$$

This contradicts $(**)$. ∎)

Conclusion 1.181 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let $z_0 \in \Omega$. Let $f'(z_0) \neq 0$. Then there exists an open neighborhood V of z_0 such that $V \subset \Omega$, and

1. for every $z \in V, f'(z) \neq 0$,
2. f is 1–1 on V,
3. $f(V)$ is open in $\mathbb{C}$,
4. $\left(f|_V\right)^{-1} \in H(f(V))$.

Proof of the remaining part (4) For this purpose, let us fix any $f(z_0) \in f(V)$, where $z_0 \in V$. We have to show that

$$\underbrace{\lim_{w \to f(z_0)} \frac{\left(f|_V\right)^{-1}(w) - \left(f|_V\right)^{-1}(f(z_0))}{w - f(z_0)}} = \lim_{f(z) \to f(z_0)} \frac{\left(f|_V\right)^{-1}(f(z)) - \left(f|_V\right)^{-1}(f(z_0))}{f(z) - f(z_0)}$$

$$= \lim_{f(z) \to f(z_0)} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\lim\limits_{f(z) \to f(z_0)} \frac{f(z) - f(z_0)}{z - z_0}}$$

exists. If $f(z) \to f(z_0)$, then

$$\frac{1}{2}|f'(z_0)||z - z_0| \leq \underbrace{|f(z) - f(z_0)| \to 0},$$

and hence $z \to z_0$. This shows that

$$\underbrace{\lim_{f(z) \to f(z_0)} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}} = f'(z_0) \neq 0.$$

Hence,

$$\lim_{w \to f(z_0)} \frac{\left(f|_V\right)^{-1}(w) - \left(f|_V\right)^{-1}(f(z_0))}{w - f(z_0)} = \frac{1}{f'(z_0)} (\in \mathbb{C}).$$

∎

Note 1.182

Definition Let m be a positive integer. The function $z \mapsto z^m$ from $\mathbb{C}$ to $\mathbb{C}$ will be denoted by π_m, and is called the m th ***power function***. Thus, for every positive integer m, and for every $z \in \mathbb{C}$, $\pi_m(z) = z^m$.

a. Let m be a positive integer. Let $w \in (\mathbb{C} - \{0\})$.

Problem 1.183 $(\pi_m)^{-1}(w)$ has exactly m members.

(Solution By Conclusion 1.164,

$$\left(\text{the number of elements in}(\pi_m)^{-1}(w)\right) \leq m.$$

Since $w \in (\mathbb{C} - \{0\})$, there exist $r > 0$, and a real α such that $w = re^{i\alpha}$. For every $k \in \{1, 2, \ldots, m\}$,

$$\pi_m\left(r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k)}\right) = \left(r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k)}\right)^m = re^{i(\alpha + 2\pi k)} = re^{i\alpha} e^{i2\pi k} = re^{i\alpha} 1 = w,$$

so for every $k \in \{1, 2, \ldots, m\}$,

$$r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k)} \in (\pi_m)^{-1}(w).$$

Thus,

$$\left\{r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k)} : k = 1, 2, \ldots, m\right\} \subset (\pi_m)^{-1}(w).$$

It suffices to show that $\left\{r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k)} : k = 1, 2, \ldots, m\right\}$ has exactly m distinct elements. If not, otherwise suppose that there exist $k_1, k_2 \in \{1, 2, \ldots, m\}$ such that $k_1 < k_2$, and $r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k_1)} \neq r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k_2)}$. We have to arrive at a contradiction.

Since $r > 0$, $r^{\frac{1}{m}} \neq 0$. Now, since

$$r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k_1)} \neq r^{\frac{1}{m}} e^{i\frac{1}{m}(\alpha + 2\pi k_2)},$$

we have

$$e^{i2\pi\frac{k_2 - k_1}{m}} = 1.$$

Since $k_1, k_2 \in \{1, 2, \ldots, m\}$, and $k_1 < k_2$, we have $0 < k_2 - k_1 < m$, and hence $0 < 2\pi \frac{k_2 - k_1}{m} < 2\pi$. This shows that

$$e^{i2\pi\frac{k_2 - k_1}{m}} \neq 1.$$

This is a contradiction. ∎)

Thus $\pi_m : (\mathbb{C} - \{0\}) \to (\mathbb{C} - \{0\})$ is an m-to-1 function. Also, $(\pi_m)^{-1}(w) \subset \left\{z : |z| = |w|^{\frac{1}{m}}\right\}$.

Conclusion 1.184 Let m be a positive integer. Then $\pi_m : (\mathbb{C} - \{0\}) \to (\mathbb{C} - \{0\})$ is an m-to-1 function. Also, for every $re^{i\alpha} \in (\mathbb{C} - \{0\})$, $\left\{r^{\frac{1}{m}}e^{i\frac{1}{m}(\alpha+2\pi k)} : k = 1, 2, \ldots, m\right\} = (\pi_m)^{-1}(re^{i\alpha}) \subset \left\{z : |z| = r^{\frac{1}{m}}\right\}$.

b. Let m be a positive integer.

Problem 1.185 $\pi_m : \mathbb{C} \to \mathbb{C}$ is an open function.

(Solution For this purpose, let us take an open nonempty subset V of $\mathbb{C}$. We have to show that $\pi_m(V)$ is open. For this purpose, let us take any $a \in V$.

We have to show that $\pi_m(a)$ is an interior point of $\pi_m(V)$.

Case I: when $a \neq 0$. It follows that $(\pi_m(a) =)a^m \neq 0$, and hence $\mathbb{C} - \{0\}$ is an open neighborhood of $\pi_m(a)$. Now, since $z \mapsto z^m$ is continuous at a, there exists $R > 0$ such that $D(a;R) \subset V$, and $(z \in D(a;R) \Rightarrow \pi_m(z) \in (\mathbb{C} - \{0\}))$.

Since $D(a;R)$ is a region, $\pi_m|_{D(a;R)} \in H(D(a;R))$, and $\left(\pi_m|_{D(a;R)}\right)'(a)(= ma^{m-1}) \neq 0$, by Conclusion 1.181, there exists an open neighborhood W of a such that $W \subset D(a;R)(\subset V)$, and $(\pi_m(a) \in)\left(\pi_m|_{D(a;R)}\right)(W)$ is open in $\mathbb{C}$. Thus, $\left(\pi_m|_{D(a;R)}\right)(W)$ is an open neighborhood of $\pi_m(a)$. Now, since $W \subset D(a;R) \subset V$, we have $\left(\pi_m|_{D(a;R)}\right)(W) \subset \pi_m(V)$, and hence $\pi_m(a)$ is an interior point of $\pi_m(V)$.

Case II: when $a = 0$. It follows that $\pi_m(a) = 0$. We have to show that $\pi_m(a)$ is an interior point of $\pi_m(V)$. That is, 0 is an interior point of $\pi_m(V)$.

Since $(0 =)a \in V$, there exists $R > 0$ such that $D(0;R) \subset V$.

Problem 1.186 $D(0;R^m) \subset \pi_m(D(0;R))$.

(Solution Let us take any $w \in D'(0;R^m)$, that is $0 < |w| < R^m$. We have to show that $w \in \pi_m(D(0;R))$. Since w is a nonzero complex number, by (a), there exists $z \in \mathbb{C}$ such that $(z^m =)\pi_m(z) = w$. Now, since $|w| < R^m$, we have $(|z|^m =)|z^m| < R^m$, and hence $|z| < R$. This shows that $z \in D(0;R)$, and hence $(w =)\pi_m(z) \in \pi_m(D(0;R))$. Thus, $w \in \pi_m(D(0;R))$. ∎)

Since $D(0;R) \subset V$, we have $(D(0;R^m) \subset)\pi_m(D(0;R)) \subset \pi_m(V)$, and hence $D(0;R^m) \subset \pi_m(V)$. This shows that 0 is an interior point of $\pi_m(V)$.

So, in all cases $\pi_m(a)$ is an interior point of $\pi_m(V)$. ∎)

Conclusion 1.187 Let m be a positive integer. Then $\pi_m : \mathbb{C} \to \mathbb{C}$ is an open function.

c. Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a nonconstant function. Let $f \in H(\Omega)$. Let $z_0 \in \Omega$.

Since $f \in H(\Omega)$, we have $(f - f(z_0)) \in H(\Omega)$. Clearly, z_0 is a zero of $(f - f(z_0))$, that is $z_0 \in (f - f(z_0))^{-1}(0)$. Since $f : \Omega \to \mathbb{C}$ is a nonconstant function, $(f - f(z_0)) : \Omega \to \mathbb{C}$ is a nonzero function. Now, by Theorem 1.136,

I. $(f - f(z_0))^{-1}(0)$ has no limit point in Ω,

II. corresponding to each $a \in (f - f(z_0))^{-1}(0)$, there exists a unique positive integer m, and a unique function $g : \Omega \to \mathbb{C}$ such that a. $g \in H(\Omega)$, b. for every $z \in \Omega$, $(f - f(z_0))(z) = (z - a)^m(g(z))$, c. $g(a) \neq 0$.

From (I), z_0 is an isolated point of $(f - f(z_0))^{-1}(0)$, and hence there exists $R > 0$ such that $D'(z_0; R) \subset \Omega$, and $D'(z_0; R)$ contains no point of $(f - f(z_0))^{-1}(0)$. Since $z_0 \in (f - f(z_0))^{-1}(0)$, by (II), there exists a unique positive integer m, and a unique function $g : \Omega \to \mathbb{C}$ such that

a. $g \in H(\Omega)$,
b. for every $z \in \Omega$, $(f - f(z_0))(z) = (z - z_0)^m(g(z))$,
c. $g(z_0) \neq 0$.

Since $D'(z_0; R)$ contains no point of $(f - f(z_0))^{-1}(0)$, by (b), g has no zero in $D'(z_0; R)$. Now, since $g(z_0) \neq 0$, g has no zero in $D(z_0; R)$. Since g has no zero in $D(z_0; R)$, and $g \in H(\Omega)$, $\frac{1}{g|_{D(z_0;R)}} \in H(D(z_0; R))$. Since $g \in H(\Omega)$, we have $g|_{D(z_0;R)} \in H(D(z_0; R))$, and hence by Lemma 1.117, $\left((g')|_{D(z_0;R)} = \right)\left(g|_{D(z_0;R)}\right)' \in H(D(z_0; R))$. Thus, $(g')|_{D(z_0;R)} \in H(D(z_0; R))$. Now, since $\frac{1}{g|_{D(z_0;R)}} \in H(D(z_0; R))$, the product

$$\frac{(g')|_{D(z_0;R)}}{g|_{D(z_0;R)}} = \underbrace{\left((g')|_{D(z_0;R)} \cdot \frac{1}{g|_{D(z_0;R)}}\right) \in H(D(z_0; R)).}$$

Since $D(z_0; R)$ is a nonempty, convex, open subset of $\mathbb{C}$, and

$$\frac{(g')|_{D(z_0;R)}}{g|_{D(z_0;R)}} \in H(D(z_0; R)),$$

by Conclusion 1.109, there exists a function $h : D(z_0; R) \to \mathbb{C}$ such that $h \in H(D(z_0; R))$, and

$$h' = \frac{(g')|_{D(z_0;R)}}{g|_{D(z_0;R)}}.$$

Thus, for every $z \in D(z_0; R)$,

$$\begin{aligned}\left(g|_{D(z_0;R)} \cdot e^{-h}\right)'(z) &= \left(g|_{D(z_0;R)}\right)'(z)e^{-h(z)} + \left(g|_{D(z_0;R)}\right)(z)\left(e^{-h(z)}(-h'(z))\right) \\ &= e^{-h(z)}\left(\left(g|_{D(z_0;R)}\right)'(z) - \left(g|_{D(z_0;R)}\right)(z)h'(z)\right) = e^{-h(z)} \cdot 0 = 0,\end{aligned}$$

and hence for every $z \in D(z_0; R)$,

$$\left(g|_{D(z_0;R)} \cdot e^{-h}\right)'(z) = 0.$$

It follows that for every $z \in D(z_0; R)$,

$$\left(g|_{D(z_0;R)} \cdot e^{-h}\right)''(z) = 0, \left(g|_{D(z_0;R)} \cdot e^{-h}\right)'''(z) = 0, \text{ etc.}$$

Since $h \in H(D(z_0; R))$, and $z \mapsto e^z$ is a member of $H(\mathbb{C})$, $e^{-h} \in H(D(z_0; R))$. Now, since $g|_{D(z_0;R)} \in H(D(z_0; R))$, the product $\left(g|_{D(z_0;R)} \cdot e^{-h}\right) \in H(D(z_0; R))$, and hence, by Conclusion 1.116, $\left(g|_{D(z_0;R)} \cdot e^{-h}\right)$ is representable by power series in $D(z_0; R)$. Next, by Lemma 1.60, for every $z \in D(z_0; R)$,

$$\begin{aligned}g(z)e^{-h(z)} &= \left(g|_{D(z_0;R)} \cdot e^{-h}\right)(z) \\ &= \left(g|_{D(z_0;R)} \cdot e^{-h}\right)(z_0) + \sum_{n=1}^{\infty} \frac{\left(g|_{D(z_0;R)} \cdot e^{-h}\right)^{(n)}(z_0)}{n!}(z - z_0)^n \\ &= \left(g|_{D(z_0;R)} \cdot e^{-h}\right)(z_0) + \sum_{n=1}^{\infty} \frac{0}{n!}(z - z_0)^n \\ &= g|_{D(z_0;R)}(z_0) \cdot e^{-h(z_0)} = g(z_0)e^{-h(z_0)},\end{aligned}$$

and hence for every $z \in D(z_0; R)$, $g(z)e^{-h(z)} = c$, where $c \equiv g(z_0)e^{-h(z_0)}$. Now, since $g(z_0) \neq 0$, and $e^{-h(z_0)} \neq 0$, we have $c \neq 0$. There exists $\alpha \in (\mathbb{C} - \{0\})$ such that $\alpha^m = c$. It follows that for every $z \in D(z_0; R)$, $g(z) = \alpha^m e^{h(z)}$.

Let $\varphi : z \rightarrow (z - z_0)\alpha e^{\frac{h(z)}{m}}$ be a function from $D(z_0; R)$ to $\mathbb{C}$.

It follows that for every $z \in D(z_0; R)$,

$$(\varphi(z))^m = (z - z_0)^m\left(\alpha^m e^{h(z)}\right) = (z - z_0)^m g(z) = (f - f(z_0))(z) = f(z) - f(z_0).$$

Thus, for every $z \in D(z_0; R)$, $f(z) = f(z_0) + (\varphi(z))^m$.

Observe that $D(z_0; R)$ is a region. Since $h \in H(D(z_0; R))$, by the definition of φ, we have $\varphi \in H(D(z_0; R))$. For every $z \in D(z_0; R)$,

$$\varphi'(z) = 1\left(\alpha e^{\frac{h(z)}{m}}\right) + (z - z_0)\alpha e^{\frac{h(z)}{m}}\left(\frac{1}{m}h'(z)\right),$$

so

$$\varphi'(z_0) = \alpha e^{\frac{h(z_0)}{m}}.$$

Since $\alpha \neq 0$, and $e^{\frac{h(z_0)}{m}} \neq 0$, we have $(\varphi'(z_0) =)\alpha e^{\frac{h(z_0)}{m}} \neq 0$, and hence $\varphi'(z_0) \neq 0$.

Now, by Conclusion 1.181, there exists an open neighborhood V of z_0 such that $V \subset D(z_0; R)$ and

1. for every $z \in V$, $\varphi'(z) \neq 0$,
2. φ is 1–1 on V,
3. $\varphi(V)$ is open in $\mathbb{C}$,
4. $\left(\varphi|_V\right)^{-1} \in H(\varphi(V))$.

Conclusion 1.188 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a nonconstant function. Let $f \in H(\Omega)$. Let $z_0 \in \Omega$. Let m be the order of the zero that the function $f - f(z_0)$ has at z_0. Then there exists an open neighborhood V of z_0 in Ω and there exists $\psi \in H(V)$ such that

1. for every $z \in V, f(z) = f(z_0) + (\psi(z))^m$ (that is, $(f - f(z_0))|_V = (\pi_m \circ \psi)$,
2. for every $z \in V$, $\psi'(z) \neq 0$,
3. ψ is 1–1 on V,
4. $\psi(V)$ is open in $\mathbb{C}$,
5. $\psi^{-1} \in H(\psi(V))$.

From Conclusion 1.188(5), ψ^{-1} is continuous, so ψ is an open function. By conclusion (b), π_m is an open function. Since ψ and π_m are open functions, their composite $\left(f|_V - f(z_0) = (f - f(z_0))|_V = \right)(\pi_m \circ \psi)$ is an open function, and hence $f|_V$ is an open function.

This shows that $f : \Omega \to \mathbb{C}$ is an open mapping.

Conclusion 1.189 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a non-constant function. Let $f \in H(\Omega)$. Then $f : \Omega \to \mathbb{C}$ is an open mapping.

This result is known as the **open mapping theorem**.

Since every holomorphic function is continuous, and continuous image of a connected set is connected, by the conclusion (1.189), we get the following

Conclusion 1.190 Let Ω be a region. Let $f \in H(\Omega)$. Then $f(\Omega)$ is a region or a singleton.

Theorem 1.191 Let Ω be a region. Let $f \in H(\Omega)$. Let $f : \Omega \to \mathbb{C}$ be 1–1 (and hence, f is nonconstant). Then,

1. for every $z \in \Omega$, $f'(z) \neq 0$,
2. $f^{-1} \in H(f(\Omega))$.

Proof

1: If not, otherwise suppose that there exists $z_0 \in \Omega$ such that $f'(z_0) = 0$. We have to arrive at a contradiction.

Let m be the order of the zero that the function $f - f(z_0)$ has at z_0. By Conclusion 1.188, there exists an open neighborhood V of z_0 in Ω, and there exists $\psi \in H(V)$ such that

1. $(f - f(z_0))|_V = (\pi_m \circ \psi)$,
2. for every $z \in V$, $\psi'(z) \neq 0$,
3. ψ is 1–1 on V,
4. $\psi(V)$ is open in $\mathbb{C}$.
5. $\psi^{-1} \in H(\psi(V))$.

Problem 1.192 $m \neq 1$.

(**Solution** If not, otherwise let $m = 1$. We have to arrive at a contradiction. From (1), $(0 =)f'(z_0) = 0 + \psi'(z_0)$, so $0 = \psi'(z_0)$. This contradicts (2). ∎)

Thus, $m \in \{2, 3, 4, \ldots\}$.

By Conclusion 1.184, $\pi_m : (\mathbb{C} - \{0\}) \to (\mathbb{C} - \{0\})$ is an m-to-1 function, and ψ is 1–1 on $(V - \{z_0\})$, so

$$f|_{(V-\{z_0\})} - f(z_0) = (f - f(z_0))|_{(V-\{z_0\})} = \underbrace{(\pi_m \circ \psi)}$$

is an m-to-1 function on $(V - \{z_0\})$, and hence $f|_{(V-\{z_0\})}$ is an m-to-1 function on $(V - \{z_0\})$. Now, since $m \in \{2, 3, 4, \ldots\}$, f is not 1–1. This is a contradiction.

2: Since for every $z \in \Omega$, $(f^{-1})'(f(z)) = \frac{1}{f'(z)} \in \Omega$. This shows that $f^{-1} \in H(f(\Omega))$. ∎)

1.15 Global Cauchy Theorem

Note 1.193

Definition Let $\gamma_1, \ldots, \gamma_n$ be any paths.

It follows that $\text{ran}(\gamma_1), \ldots, \text{ran}(\gamma_n)$ are compact subsets of $\mathbb{C}$, and hence $\text{ran}(\gamma_1) \cup \cdots \cup \text{ran}(\gamma_n)$ is a compact subset of $\mathbb{C}$. Recall that $C(\text{ran}(\gamma_1) \cup \cdots \cup \text{ran}(\gamma_n))$ is the collection of all continuous functions from $\text{ran}(\gamma_1) \cup \cdots \cup \text{ran}(\gamma_n)$ to $\mathbb{C}$. We know that $C(\text{ran}(\gamma_1) \cup \cdots \cup \text{ran}(\gamma_n))$ is a complex linear space under pointwise addition, and pointwise scalar multiplication. Let $\tilde{\gamma}_1 : f \mapsto \int_{\gamma_1} f(z)\mathrm{d}z$ be a function from $C(\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n))$ to $\mathbb{C}$. Clearly,

$$\tilde{\gamma}_1 : C(\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n)) \to \mathbb{C}$$

is linear. Thus, $\tilde{\gamma}_1$ is a linear functional on $C(\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n))$. Similarly,

$$\tilde{\gamma}_2 : f \mapsto \int_{\gamma_2} f(z)\mathrm{d}z$$

is a linear functional on $C(\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n))$, etc. It follows that

$$\underbrace{(\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n) : f \mapsto (\tilde{\gamma}_1(f) + \cdots + \tilde{\gamma}_n(f))} = \int_{\gamma_1} f(z)\mathrm{d}z + \cdots + \int_{\gamma_n} f(z)\mathrm{d}z$$

is a linear functional on $C(\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n))$. Here, $(\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n)(f)$ is denoted by

$$\int_{\gamma_1 \dot{+} \cdots \dot{+} \gamma_n} f(z)\mathrm{d}z,$$

and the symbol $\gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ is called a ***chain***. If each γ_i is a closed path, then the chain $\gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ is called a ***cycle***. If each $\mathrm{ran}(\gamma_i)$ is contained in some open set Ω, then we say that $\gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ *is a **chain** in* Ω. Here, $\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n)$ is denoted by $(\gamma_1 \dot{+} \cdots \dot{+} \gamma_n)^*$.

Let $\gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ be a cycle, that is each γ_i is a closed path. Let $\alpha \in \left((\gamma_1 \dot{+} \cdots \dot{+} \gamma_n)^*\right)^c$, that is

$$\underbrace{\alpha \in (\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n))^c} = (\mathrm{ran}(\gamma_1))^c \cap \cdots \cap (\mathrm{ran}(\gamma_n))^c.$$

It follows that

$$\alpha \in (\mathrm{ran}(\gamma_1))^c \cap \cdots \cap (\mathrm{ran}(\gamma_n))^c (\subset (\mathrm{ran}(\gamma_1))^c),$$

and hence $(\mathrm{Ind})_{\gamma_1}(\alpha) \in \mathbb{C}$. Similarly, $(\mathrm{Ind})_{\gamma_2}(\alpha) \in \mathbb{C}$, etc.

The complex number

$$(\mathrm{Ind})_{\gamma_1}(\alpha) + \cdots + (\mathrm{Ind})_{\gamma_n}(\alpha)$$

is denoted by

$$(\mathrm{Ind})_{(\gamma_1 \dot{+} \cdots \dot{+} \gamma_n)}(\alpha),$$

and is called the ***index*** *of* α *with respect to cycle* $(\gamma_1 \dot{+} \cdots \dot{+} \gamma_n)$. Let us denote $\gamma_1 \dot{+} \cdots \dot{+} \gamma_n$ by Γ.

Now, for every

$$\underbrace{f \in C(\operatorname{ran}(\gamma_1) \cup \cdots \cup \operatorname{ran}(\gamma_n))} = C\big((\gamma_1 \dot{+} \cdots \dot{+} \gamma_n)^*\big) = C(\Gamma^*),$$

$$\int_\Gamma f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \cdots + \int_{\gamma_n} f(z)\mathrm{d}z.$$

Since

$$\Gamma^* = (\gamma_1 \dot{+} \cdots \dot{+} \gamma_n)^* = \underbrace{\operatorname{ran}(\gamma_1) \cup \cdots \cup \operatorname{ran}(\gamma_n)}$$

is a compact subset of $\mathbb{C}$, Γ^* is a compact subset of $\mathbb{C}$.

Also, if

$$\underbrace{\alpha \in (\operatorname{ran}(\gamma_1) \cup \cdots \cup \operatorname{ran}(\gamma_n))^c} = \big((\gamma_1 \dot{+} \cdots \dot{+} \gamma_n)^*\big)^c = (\Gamma^*)^c,$$

then the function

$$z \mapsto \frac{1}{z - \alpha}$$

from $(\operatorname{ran}(\gamma_1) \cup \cdots \cup \operatorname{ran}(\gamma_n))$ to $\mathbb{C}$ is continuous, and hence this is a member of $C(\operatorname{ran}(\gamma_1) \cup \cdots \cup \operatorname{ran}(\gamma_n))$. Thus, if $\alpha \in (\Gamma^*)^c$, then

$$\begin{aligned}
(\mathrm{Ind})_\Gamma(\alpha) &= (\mathrm{Ind})_{\gamma_1}(\alpha) + \cdots + (\mathrm{Ind})_{\gamma_n}(\alpha) \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z-\alpha}\mathrm{d}z + \cdots + \frac{1}{2\pi i}\int_{\gamma_n} \frac{1}{z-\alpha}\mathrm{d}z \\
&= \frac{1}{2\pi i}\left(\int_{\gamma_1} \frac{1}{z-\alpha}\mathrm{d}z + \cdots + \int_{\gamma_n} \frac{1}{z-\alpha}\mathrm{d}z\right) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-\alpha}\mathrm{d}z.
\end{aligned}$$

Hence, for every $z \in (\Gamma^*)^c$, $(\mathrm{Ind})_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\xi - z}\mathrm{d}\xi$.

If each γ_i is replaced by its opposite path, the new chain will be denoted by $-\Gamma$. Clearly, for every $f \in C(\Gamma^*)$, $\int_{-\Gamma} f(z)\mathrm{d}z = -\int_\Gamma f(z)\mathrm{d}z$.

If Γ is a cycle, then for every $z \in (\Gamma^*)^c$, $(\mathrm{Ind})_{-\Gamma}(z) = \frac{1}{2\pi i}\int_{-\Gamma} \frac{1}{\xi - z} \mathrm{d}\xi \left(= -\frac{1}{2\pi i}\int_{\Gamma} \frac{1}{\xi - z}\mathrm{d}\xi\right) = -\left((\mathrm{Ind})_{\Gamma}(z)\right)$.

Let $\gamma_1 \dot{+} \cdots \dot{+} \gamma_n$, and $\delta_1 \dot{+} \cdots \dot{+} \delta_k$ be chains, where $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_k$ are paths. If for every $f \in C(\mathrm{ran}(\gamma_1) \cup \cdots \cup \mathrm{ran}(\gamma_n) \cup \mathrm{ran}(\delta_1) \cup \cdots \cup \mathrm{ran}(\delta_k))$, $\int\limits_{\gamma_1} f(z)\mathrm{d}z + \cdots + \int\limits_{\gamma_n} f(z)\mathrm{d}z = \int\limits_{\delta_1} f(z)\mathrm{d}z + \cdots + \int\limits_{\delta_k} f(z)\mathrm{d}z$, then we write $\gamma_1 \dot{+} \cdots \dot{+} \gamma_n = \delta_1 \dot{+} \cdots \dot{+} \delta_k$.

Let Γ_1 and Γ_2 be chains. Clearly, $\Gamma_1 \dot{+} \Gamma_2$ is a chain. Also, for every $f \in C((\Gamma_1)^* \cup (\Gamma_2)^*)$,

$$\int\limits_{\Gamma_1 \dot{+} \Gamma_2} f(z)\mathrm{d}z = \int\limits_{\Gamma_1} f(z)\mathrm{d}z + \int\limits_{\Gamma_2} f(z)\mathrm{d}z.$$

Conclusion 1.194 Let Γ_1 and Γ_2 be chains. Then for every $f \in C((\Gamma_1)^* \cup (\Gamma_2)^*)$, $\int_{\Gamma_1 \dot{+} \Gamma_2} f(z)\mathrm{d}z = \int_{\Gamma_1} f(z)\mathrm{d}z + \int_{\Gamma_2} f(z)\mathrm{d}z$.

Note 1.195 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ be a cycle in Ω.

It follows that Γ^* is a compact subset of Ω. Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in \Omega \times \Omega$,

$$\varphi(z, w) \equiv \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

By Conclusion 1.177, $\varphi : \Omega \times \Omega \to \mathbb{C}$ is continuous. It follows that for every $z \in \Omega$, the function $\varphi_z : w \mapsto \varphi(z, w)$ from Ω to $\mathbb{C}$ is continuous. Since Γ is a chain in Ω and, for every $z \in \Omega$, the function $\varphi_z : w \mapsto \varphi(z, w)$ from Ω to $\mathbb{C}$ is continuous, for every $z \in \Omega$,

$$\int\limits_{\Gamma} \varphi(z, \zeta)\mathrm{d}\zeta = \underbrace{\int\limits_{\Gamma} \varphi_z(\zeta)\mathrm{d}\zeta \in \mathbb{C}}.$$

Let $h : z \mapsto \int_{\Gamma} \varphi(z, \zeta)\mathrm{d}\zeta$ be a function from Ω to $\mathbb{C}$.

Problem 1.196 $h : \Omega \to \mathbb{C}$ is continuous.

(Solution Let us take any $z \in \Omega$. We have to show that h is continuous at z.

Since Γ^* is a compact subset of Ω, and $\{z\}$ is a compact subset of Ω, $\Gamma^* \cup \{z\}$ is a compact subset of Ω. By Lemma 1.163, Vol. 1, there exists an open set V such that

$$\Gamma^* \cup \{z\} \subset \underbrace{V \subset \overline{V}} \subset \Omega,$$

and $\overline{V}$ is compact. Let us take any convergent sequence $\{z_n\}$ in V such that $\lim\limits_{n \to \infty} z_n = z$. We have to show that

$$\lim_{n\to\infty}\int_\Gamma \varphi(z_n,\zeta)\mathrm{d}\zeta = \underbrace{\lim_{n\to\infty} h(z_n) = h(z)} = \int_\Gamma \varphi(z,\zeta)\mathrm{d}\zeta,$$

that is

$$\lim_{n\to\infty}\int_\Gamma \varphi(z_n,\zeta)\mathrm{d}\zeta = \int_\Gamma \varphi(z,\zeta)\mathrm{d}\zeta.$$

Observe that the function $\psi:\zeta\mapsto(z,\zeta)$ from V to $V\times V$ is continuous. Similarly, for every positive integer n, the function $\psi_n:\zeta\mapsto(z_n,\zeta)$ from V to $V\times V$ is continuous.

Problem 1.197 $\{\psi_n\}$ converges uniformly on V, and its uniform limit is ψ.

(Solution For this purpose, let us take any $\varepsilon>0$. Since $\lim_{n\to\infty} z_n = z$, there exists a positive integer N such that $(n\geq N \Rightarrow |z_n - z| < \varepsilon)$. It follows that, for every $\zeta\in V$, and for every $n\geq N$,

$$\|\psi_n(\zeta)-\psi(\zeta)\| = \|(z_n,\zeta)-(z,\zeta)\| = \sqrt{|z_n-z|^2+|\zeta-\zeta|^2} = |z_n-z| < \varepsilon,$$

and hence $\{\psi_n\}$ converges uniformly on V, and its uniform limit is ψ. ∎)

We have to show that $\lim_{n\to\infty}\int_\Gamma \varphi(\psi_n(\zeta))\mathrm{d}\zeta = \int_\Gamma \varphi(\psi(\zeta))\mathrm{d}\zeta$, that is $\lim_{n\to\infty}\int_\Gamma(\varphi\circ\psi_n)(\zeta)\mathrm{d}\zeta = \int_\Gamma(\varphi\circ\psi)(\zeta)\mathrm{d}\zeta$. By a theorem (e.g., see [5], Theorem 7.16), it suffices to show that

1. For every $\zeta\in\Gamma^*$, $\lim_{n\to\infty}(\varphi\circ\psi_n)(\zeta) = (\varphi\circ\psi)(\zeta)$,
2. $\{\varphi\circ\psi_n\}$ is uniformly convergent on the compact set Γ^*.

For 1: Let us take any $\zeta\in\Gamma^*$. We have to show that

$$\lim_{n\to\infty}\varphi(z_n,\zeta) = \lim_{n\to\infty}\varphi(\psi_n(\zeta)) = \underbrace{\lim_{n\to\infty}(\varphi\circ\psi_n)(\zeta) = (\varphi\circ\psi)(\zeta)} = \varphi(\psi(\zeta))$$
$$= \varphi(z,\zeta),$$

that is $\lim_{n\to\infty}\varphi(z_n,\zeta) = \varphi(z,\zeta)$. Since $\varphi:\Omega\times\Omega\to\mathbb{C}$ is continuous, and $\lim_{n\to\infty} z_n = z$, $\lim_{n\to\infty}\varphi(z_n,\zeta) = \varphi(z,\zeta)$.

For 2: Let us take any $\varepsilon>0$. Since $\varphi:\Omega\times\Omega\to\mathbb{C}$ is continuous, it is uniformly continuous on the compact set $\bar{V}\times\bar{V}$, and hence there exists $\delta>0$ such that for every $u,v,w,z\in\bar{V}$ satisfying $\|(u,v)-(w,z)\|<\delta$, we have $|\varphi(u,v)-\varphi(w,z)|<\varepsilon$. Since $\{\psi_n\}$ converges uniformly on V, there exists a positive integer N such that for every $m,n\geq N$, and for every $\zeta\in V$, $\|\psi_m(\zeta)-\psi_n(\zeta)\|<\delta$. Thus, for every $\zeta\in V$, and for every $m, n \geq N$, we have $\|(z_m, \zeta) - (z_n, \zeta)\| < \delta$. It follows that for every $\zeta \in V(\supset \Gamma^* \cup \{z\})$, and for every $m, n \geq N$,

$$|(\varphi \circ \psi_m)(\zeta) - (\varphi \circ \psi_n)(\zeta)| = |\varphi(\psi_m(\zeta)) - \varphi(\psi_n(\zeta))| = \underbrace{|\varphi(z_m, \zeta) - \varphi(z_n, \zeta)| < \varepsilon}.$$

Hence, for every $\zeta \in \Gamma^*$, and for every $m, n \geq N$, $|(\varphi \circ \psi_m)(\zeta) - (\varphi \circ \psi_n)(\zeta)| < \varepsilon$. This shows that $\{\varphi \circ \psi_n\}$ is uniformly convergent on the compact set Γ^*. ∎)

Conclusion 1.198 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ be a cycle in Ω. Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in \Omega \times \Omega$,

$$\varphi(z, w) \equiv \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Then the function

$$h : z \mapsto \frac{1}{2\pi i} \int_{\Gamma} \varphi(z, \zeta) \mathrm{d}\zeta$$

from Ω to $\mathbb{C}$ is continuous.

Note 1.199 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ be a cycle in Ω. Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in \Omega \times \Omega$,

$$\varphi(z, w) \equiv \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Let $h : z \mapsto \frac{1}{2\pi i} \int_{\Gamma} \varphi(z, \zeta) \mathrm{d}\zeta$ be the function from Ω to $\mathbb{C}$.

Problem 1.200 $h \in H(\Omega)$.

(Solution By Conclusion 1.198, $h : \Omega \to \mathbb{C}$ is a continuous function. By Conclusion 1.123, it suffices to show that for every triangle Δ satisfying $\Delta \subset \Omega$, $\int_{\partial\Delta} h(z) \mathrm{d}z = 0$.

For this purpose, let us take any triangle Δ satisfying $\Delta \subset \Omega$. We have to show that $\int_{\partial\Delta} h(z) \mathrm{d}z = 0$. Here,

$$\int_{\partial\Delta} h(z)\mathrm{d}z = \int_{\partial\Delta}\left(\frac{1}{2\pi i}\int_{\Gamma}\varphi(z,\zeta)\mathrm{d}\zeta\right)\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\Delta}\left(\int_{\Gamma}\varphi(z,\zeta)\mathrm{d}\zeta\right)\mathrm{d}z$$
$$= \frac{1}{2\pi i}\int_{\Gamma}\left(\int_{\partial\Delta}\varphi(z,\zeta)\mathrm{d}z\right)\mathrm{d}\zeta,$$

so

$$\int_{\partial\Delta} h(z)\mathrm{d}z = \frac{1}{2\pi i}\int_{\Gamma}\left(\int_{\partial\Delta}\varphi(z,\zeta)\mathrm{d}z\right)\mathrm{d}\zeta.$$

It suffices to show that, for every $\zeta \in \Gamma^*(\subset \Omega)$,

$$\int_{\partial\Delta}\varphi(z,\zeta)\mathrm{d}z = 0.$$

By Conclusion 1.177, $\varphi : \Omega \times \Omega \to \mathbb{C}$ is continuous, so, for every $\zeta \in \Gamma^*$, the function $\psi_\zeta : z \mapsto \varphi(z,\zeta)$ from Ω to $\mathbb{C}$ is continuous, and hence by Conclusion 1.104, it suffices to show that, for every $\zeta \in \Gamma^*$, the function $\psi_\zeta : z \mapsto \varphi(z,\zeta)$ from Ω to $\mathbb{C}$ is holomorphic.

For this purpose, let us fix any $\zeta \in \Gamma^*$. Since

$$\psi_\zeta : z \mapsto \varphi(z,\zeta)\left(= \begin{cases} \frac{f(z)-f(\zeta)}{z-\zeta} & \text{if } z \neq \zeta \\ f'(\zeta) & \text{if } z = \zeta. \end{cases}\right),$$

and $f \in H(\Omega)$, we have $\psi_\zeta \in H(\Omega)$. ∎)

Conclusion 1.201 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ be a cycle in Ω. Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z,w) \in \Omega \times \Omega$,

$$\varphi(z,w) \equiv \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Let

$$h : z \mapsto \frac{1}{2\pi i}\int_{\Gamma}\varphi(z,\zeta)\mathrm{d}\zeta$$

be the function from Ω to $\mathbb{C}$. Then $h \in H(\Omega)$.

Note 1.202 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ be a cycle in Ω. It follows that Γ^* is a compact subset of Ω, and hence $\Omega^c \subset (\Gamma^*)^c$. Suppose that for every $z \in \Omega^c$,

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta = \underbrace{(\mathrm{Ind})_{\Gamma}(z) = 0}.$$

By Conclusions 1.89 and 1.91,

$$\left\{z : z \in (\Gamma^*)^c, \ \text{ and } \ (\mathrm{Ind})_{\Gamma}(z) = 0\right\}$$

is the unbounded component of $(\Gamma^*)^c$, and hence by Lemma 1.32,

$$\left\{z : z \in (\Gamma^*)^c, \ \text{ and } \ \left(\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta =\right)(\mathrm{Ind})_{\Gamma}(z) = 0\right\}$$

is an open set. Let $h_1 : z \mapsto \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$ be a function from open set

$$\left\{z : z \in (\Gamma^*)^c, \ \text{ and } \ \left(\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta =\right)(\mathrm{Ind})_{\Gamma}(z) = 0\right\}(\supset \Omega^c)$$

to $\mathbb{C}$. Clearly, $h_1 \in H\big(\{z : z \in (\Gamma^*)^c \ \text{ and } \ (\mathrm{Ind})_{\Gamma}(z) = 0\}\big)$. It follows that

$$\left\{z : z \in (\Gamma^*)^c, \ \text{ and } \ \left(\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta =\right)(\mathrm{Ind})_{\Gamma}(z) = 0\right\} \cup \Omega = \mathbb{C}.$$

Let $\varphi : \Omega \times \Omega \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in \Omega \times \Omega$,

$$\varphi(z, w) \equiv \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w. \end{cases}$$

Let $h : z \mapsto \frac{1}{2\pi i}\int_{\Gamma}\varphi(z, \zeta)\mathrm{d}\zeta$ be the function from Ω to $\mathbb{C}$. By Conclusion 1.201, $h \in H(\Omega)$.

Problem 1.203 h and h_1 coincide on $\left\{z : z \in (\Gamma^*)^c, \text{ and } \left(\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta =\right)\right.$ $\left.(\mathrm{Ind})_{\Gamma}(z) = 0\right\} \cap \Omega$.

(Solution For this purpose, let us take any

$$w \in \left\{z : z \in (\Gamma^*)^c, \ \text{ and } \ \left(\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta =\right)(\mathrm{Ind})_{\Gamma}(z) = 0\right\} \cap \Omega.$$

We have to show that $h(w) = h_1(w)$.

$$\begin{aligned}
\text{LHS} = h(w) &= \frac{1}{2\pi i}\int_\Gamma \varphi(w,\zeta)\mathrm{d}\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)-f(\zeta)}{w-\zeta}\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta-w}\mathrm{d}\zeta - f(w)\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta-w}\mathrm{d}\zeta = h_1(w) - f(w)\left(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta-w}\mathrm{d}\zeta\right) \\
&= h_1(w) - f(w)\big((\mathrm{Ind})_\Gamma(w)\big) = h_1(w) - f(w)(0) = h_1(w) = \text{RHS}.
\end{aligned}$$

∎)

Since $h_1 \in H\big(\{z : z \in (\Gamma^*)^c, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\}\big)$, $h \in H(\Omega)$, and h and h_1 coincide on $\{z : z \in (\Gamma^*)^c, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\} \cap \Omega$, there exists

$$\psi \in H\big(\{z : z \in (\Gamma^*)^c, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\} \cup \Omega\big)(= H(\mathbb{C}))$$

such that ψ is an extension of h_1, and ψ is an extension of h. Next, since

$$\left\{z : z \in (\Gamma^*)^c, \text{ and } \left(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta-z}\mathrm{d}\zeta =\right)(\mathrm{Ind})_\Gamma(z) = 0\right\}$$

is the unbounded component of $(\Gamma^*)^c$,

$$\lim_{|z|\to\infty}\psi(z) = \lim_{|z|\to\infty} h_1(z) = \lim_{|z|\to\infty}\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta = 0,$$

and hence $\lim_{|z|\to\infty}\psi(z) = 0$. Since $\psi \in H(\mathbb{C})$, and $\lim_{|z|\to\infty}\psi(z) = 0$, $\psi : \mathbb{C} \to \mathbb{C}$ is bounded, and hence by Conclusion 1.150, $\psi : \mathbb{C} \to \mathbb{C}$ is a constant function. Now, since $\lim_{|z|\to\infty}\psi(z) = 0$, we have $\psi = 0$. Since $\psi = 0$, and ψ is an extension of $h : \Omega \to \mathbb{C}$, for every $z \in (\Omega - \Gamma^*)$, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta - f(z)\cdot(\mathrm{Ind})_\Gamma(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta - f(z)\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta-z}\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)-f(z)}{\zeta-z}\mathrm{d}\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(\zeta)}{z-\zeta}\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_\Gamma \varphi(z,\zeta)\mathrm{d}\zeta = \underbrace{h(z) = 0},
\end{aligned}$$

and hence for every $z \in (\Omega - \Gamma^*)$, we have

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = f(z) \cdot (\mathrm{Ind})_{\Gamma}(z).$$

Conclusion 1.204 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ be a cycle in Ω. Suppose that, for every $z \in \Omega^c$, $\left(\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta =\right)(\mathrm{Ind})_{\Gamma}(z) = 0$. Then for every $z \in (\Omega - \Gamma^*)$,

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = f(z) \cdot (\mathrm{Ind})_{\Gamma}(z).$$

Theorem 1.205 Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ be a cycle in Ω. Suppose that for every $z \in \Omega^c$, $\left(\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\zeta - z}\mathrm{d}\zeta =\right)(\mathrm{Ind})_{\Gamma}(z) = 0$. Then $\int_{\Gamma} f(\zeta)\mathrm{d}\zeta = 0$.

Proof Let us take any $a \in (\Omega - \Gamma^*)$. Since $f \in H(\Omega)$, the function $z \mapsto (z - a)f(z)$ from Ω to $\mathbb{C}$ is a member of $H(\Omega)$. Now, by Conclusion 1.204,

$$\frac{1}{2\pi i}\int_{\Gamma} f(\zeta)\mathrm{d}\zeta = \underbrace{\frac{1}{2\pi i}\int_{\Gamma} \frac{(\zeta - a)f(\zeta)}{\zeta - a}\mathrm{d}\zeta = ((a - a)f(a)) \cdot (\mathrm{Ind})_{\Gamma}(a)} = 0.$$

Thus, $\int_{\Gamma} f(\zeta)\mathrm{d}\zeta = 0$. ∎)

Theorem 1.206 Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Let Γ_1, Γ_2 be cycles in Ω. Suppose that for every $z \in \Omega^c$, $(\mathrm{Ind})_{\Gamma_1}(z) = (\mathrm{Ind})_{\Gamma_2}(z)$. Then

$$\int_{\Gamma_1} f(\zeta)\mathrm{d}\zeta = \int_{\Gamma_2} f(\zeta)\mathrm{d}\zeta.$$

Proof Since Γ_1, Γ_2 are cycles in Ω, $\Gamma_1 \dot{+} (-\Gamma_2)$ is a cycle in Ω. For every $z \in \Omega^c$,

$$\begin{aligned}(\mathrm{Ind})_{\Gamma_1 \dot{+} (-\Gamma_2)}(z) &= (\mathrm{Ind})_{\Gamma_1}(z) + (\mathrm{Ind})_{(-\Gamma_2)}(z) = (\mathrm{Ind})_{\Gamma_1}(z) - (\mathrm{Ind})_{\Gamma_2}(z)\\ &= (\mathrm{Ind})_{\Gamma_1}(z) - (\mathrm{Ind})_{\Gamma_1}(z) = 0,\end{aligned}$$

so for every $z \in \Omega^c$, $(\mathrm{Ind})_{\Gamma_1 \dot{+} (-\Gamma_2)}(z) = 0$. Now, by Theorem 1.66,

$$\int_{\Gamma_1} f(\zeta)\mathrm{d}\zeta - \int_{\Gamma_2} f(\zeta)\mathrm{d}\zeta = \int_{\Gamma_1} f(\zeta)\mathrm{d}\zeta + \int_{-\Gamma_2} f(\zeta)\mathrm{d}\zeta = \underbrace{\int_{\Gamma_1 \dot{+} (-\Gamma_2)} f(\zeta)\mathrm{d}\zeta = 0}_{},$$

so

$$\int_{\Gamma_1} f(\zeta)\mathrm{d}\zeta = \int_{\Gamma_2} f(\zeta)\mathrm{d}\zeta.$$

■

Theorems 1.205 and 1.206 together are known as the **global Cauchy theorem**.

Note 1.207 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Let u, v be real numbers satisfying $\alpha < u < v < \beta$. Let $\gamma(u) \neq \gamma(v)$. Put $a \equiv \frac{\gamma(u)+\gamma(v)}{2}$, and $b \equiv \frac{\gamma(v)-\gamma(u)}{2}$. Suppose that

1. $\gamma^{-1}(D(a; |b|)) = (u, v)$,
2. $\gamma^{-1}(\{z : |z-a| = |b|\}) = \{u, v\}$.

Here $\mathrm{ran}(\gamma)$ is a compact subset of $\mathbb{C}$, and hence $(D(a; |b|) - \mathrm{ran}(\gamma))$ is an open subset of $\mathbb{C}$. By Lemma 1.32, $(D(a; |b|) - \mathrm{ran}(\gamma))$ is partitioned into regions.

Suppose that $(D(a; |b|) - \mathrm{ran}(\gamma))$ is partitioned into two regions D_+, and D_- such that $(a+ib) \in (D_+)^-$, and $(a-ib) \in (D_-)^-$. Let $w_1 \in D_-$, and $w_2 \in D_+$. It follows that

$$w_1, w_2 \in (D(a; |b|) - \mathrm{ran}(\gamma))(\subset (\mathrm{ran}(\gamma))^c),$$

and hence $w_1, w_2 \in (\mathrm{ran}(\gamma))^c$. Also, $w_1, w_2 \in D(a; |b|)$.

Problem 1.208 $(\mathrm{Ind})_\gamma(w_2) - (\mathrm{Ind})_\gamma(w_1) = 1$, that is $\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - w_2}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - w_1}\mathrm{d}\zeta = 1$.

(Solution For simplicity, let $u = 0$, and $v = \pi$. Let

$$C : t \mapsto \left(a - be^{it}\right)$$

be the function from $[0, 2\pi]$ to $\mathbb{C}$. Since $C(0) = C(2\pi), C : [0, 2\pi] \to \mathbb{C}$ is a closed path. Also,

$$C(0) = a - be^{i0} = a - b = \frac{\gamma(u)+\gamma(v)}{2} - \frac{\gamma(v)-\gamma(u)}{2} = \gamma(u) = \gamma(0),$$

so $C(0)=\gamma(0)$. Similarly, $C(\pi)=\gamma(\pi)$. Let

$$C_1 : t \mapsto \begin{cases} C(t) & \text{if } t \in [0,\pi] \\ \gamma(2\pi - t) & \text{if } t \in [\pi, 2\pi], \end{cases}$$

be the function from $[0, 2\pi]$ to $\mathbb{C}$. Since

$$C_1(0) = C(0) = \gamma(0) = \gamma(2\pi - 2\pi) = C_1(2\pi),$$

we have $C_1(0) = C_1(2\pi)$. Next, since $C(\pi)=\gamma(\pi)$, C_1 is a closed path. Let

$$C_2 : t \mapsto \begin{cases} \gamma(t) & \text{if } t \in [0,\pi] \\ C(t) & \text{if } t \in [\pi, 2\pi], \end{cases}$$

be the function from $[0, 2\pi]$ to $\mathbb{C}$. Since

$$C_2(0) = \gamma(0) = C(0) = C(2\pi) = C_2(2\pi),$$

we have $C_2(0) = C_2(2\pi)$. Next, since $C(\pi)=\gamma(\pi)$, C_2 is a closed path.

Let

$$C_3 : t \mapsto \begin{cases} \gamma(t) & \text{if } t \in [\alpha,0] \cup [\pi,\beta] \\ C(t) & \text{if } t \in [0, \pi], \end{cases}$$

be the function from $[\alpha, \beta]$ to $\mathbb{C}$. Since

$$C_3(\alpha) = \gamma(\alpha) = \gamma(\beta) = C_3(\beta),$$

we have $C_3(\alpha) = C_3(\beta)$. Next, since $C(0)=\gamma(0)$, and $C(\pi)=\gamma(\pi)$, C_3 is a closed path. By the repeated application of Theorem 1.110,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - w_2}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - w_1}\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\left(\int_{\gamma \dot{+} C_1} \frac{1}{\zeta - w_2}\,\mathrm{d}\zeta - \int_{C_1} \frac{1}{\zeta - w_2}\,\mathrm{d}\zeta\right) - \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - w_1}\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\left(\int_{C_3} \frac{1}{\zeta - w_2}\,\mathrm{d}\zeta - \int_{C_1} \frac{1}{\zeta - w_2}\,\mathrm{d}\zeta\right) - \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - w_1}\,\mathrm{d}\zeta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\left(\int_{C_3}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - 0\right) - \frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta - w_1}\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\left(\int_{C_3}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \int_{\gamma}\frac{1}{\zeta - w_1}\mathrm{d}\zeta\right) \\
&= \frac{1}{2\pi i}\left(\int_{C_3}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \left(\int_{\gamma \dot{+} C}\frac{1}{\zeta - w_1}\mathrm{d}\zeta - \int_{C}\frac{1}{\zeta - w_1}\mathrm{d}\zeta\right)\right) \\
&= \frac{1}{2\pi i}\left(\int_{C_3}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \left(\int_{C_2 \dot{+} C_3}\frac{1}{\zeta - w_1}\mathrm{d}\zeta - \int_{C}\frac{1}{\zeta - w_1}\mathrm{d}\zeta\right)\right) \\
&= \frac{1}{2\pi i}\left(\int_{C_3}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \left(\left(\int_{C_2}\frac{1}{\zeta - w_1}\mathrm{d}\zeta + \int_{C_3}\frac{1}{\zeta - w_1}\mathrm{d}\zeta\right) - \int_{C}\frac{1}{\zeta - w_1}\mathrm{d}\zeta\right)\right) \\
&= \frac{1}{2\pi i}\left(\int_{C_3}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \left(\left(0 + \int_{C_3}\frac{1}{\zeta - w_1}\mathrm{d}\zeta\right) - \int_{C}\frac{1}{\zeta - w_1}\mathrm{d}\zeta\right)\right) \\
&= \frac{1}{2\pi i}\int_{C_3}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{C_3}\frac{1}{\zeta - w_1}\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{C}\frac{1}{\zeta - w_1}\mathrm{d}\zeta \\
&= (\mathrm{Ind})_{C_3}(w_2) - (\mathrm{Ind})_{C_3}(w_1) + (\mathrm{Ind})_{C}(w_1),
\end{aligned}
$$

so

$$
\frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta - w_1}\mathrm{d}\zeta = (\mathrm{Ind})_{C_3}(w_2) - (\mathrm{Ind})_{C_3}(w_1) + (\mathrm{Ind})_{C}(w_1).
$$

By Lemma 1.92, $(\mathrm{Ind})_C(w_1) = 1$. Since w_1 and w_2 are the members of the same component of $(\mathrm{ran}(C_3))^c$, by Conclusion 1.89, $(\mathrm{Ind})_{C_3}(w_2) = (\mathrm{Ind})_{C_3}(w_1)$. Thus,

$$
\frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta - w_2}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{\gamma}\frac{1}{\zeta - w_1}\mathrm{d}\zeta = 1.
$$

∎)

Conclusion 1.209 Let α, β be real numbers such that $\alpha < \beta$. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed path. Let u, v be real numbers satisfying $\alpha < u < v < \beta$. Let $\gamma(u) \neq \gamma(v)$. Put $a \equiv \frac{\gamma(u)+\gamma(v)}{2}$, and $b \equiv \frac{\gamma(v)-\gamma(u)}{2}$. Suppose that

1. $\gamma^{-1}(D(a; |b|)) = (u, v)$,
2. $\gamma^{-1}(\{z : |z-a| = |b|\}) = \{u, v\}$.

Suppose that $(D(a; |b|) - \text{ran}(\gamma))$ is partitioned into two regions D_+, and D_- such that $(a+ib) \in (D_+)^-$, and $(a-ib) \in (D_-)^-$. Let $w_1 \in D_-$ and $w_2 \in D_+$. Then,

$$(\text{Ind})_\gamma(w_2) - (\text{Ind})_\gamma(w_1) = 1.$$

Note 1.210

a. Let $\gamma_0 : [0,1] \to \mathbb{C}$, and $\gamma_1 : [0,1] \to \mathbb{C}$ be closed paths. Let $\alpha \in \mathbb{C}$. Suppose that, for every $s \in [0,1]$,

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| (*).$$

(This shows that, for every $s \in [0,1], \alpha \neq \gamma_0(s)$ and hence $\alpha \in (\text{ran}(\gamma_0))^c$. It follows that $(\text{Ind})_{\gamma_0}(\alpha) \in \mathbb{C}$. From $(*)$, for every $s \in [0,1], \alpha \neq \gamma_1(s)$ and hence $\alpha \in (\text{ran}(\gamma_1))^c$. It follows that $(\text{Ind})_{\gamma_1}(\alpha) \in \mathbb{C}$.)

Problem 1.211 $(\text{Ind})_{\gamma_0}(\alpha) = (\text{Ind})_{\gamma_1}(\alpha)$.

(Solution Since $\gamma_0 : [0,1] \to \mathbb{C}$, and $\gamma_1 : [0,1] \to \mathbb{C}$ are closed paths, the function

$$\gamma : s \mapsto \frac{\gamma_1(s) - \alpha}{\gamma_0(s) - \alpha}$$

from $[0,1]$ to $(\mathbb{C} - \{0\})$ is a closed path. Also, $0 \in (\text{ran}(\gamma))^c$. So, $(\text{Ind})_\gamma(0) \in \mathbb{C}$. It follows that

$$\gamma' = \frac{(\gamma_1)'(\gamma_0 - \alpha) - (\gamma_1 - \alpha)(\gamma_0)'}{(\gamma_0 - \alpha)^2},$$

and hence

$$\frac{\gamma'}{\gamma} = \frac{1}{\gamma}\left(\frac{(\gamma_1)'(\gamma_0 - \alpha) - (\gamma_1 - \alpha)(\gamma_0)'}{(\gamma_0 - \alpha)^2}\right) = \left(\frac{\gamma_0 - \alpha}{\gamma_1 - \alpha}\right)\left(\frac{(\gamma_1)'(\gamma_0 - \alpha) - (\gamma_1 - \alpha)(\gamma_0)'}{(\gamma_0 - \alpha)^2}\right)$$
$$= \frac{(\gamma_1)'(\gamma_0 - \alpha) - (\gamma_1 - \alpha)(\gamma_0)'}{(\gamma_0 - \alpha)(\gamma_1 - \alpha)} = \frac{(\gamma_1)'}{\gamma_1 - \alpha} - \frac{(\gamma_0)'}{\gamma_0 - \alpha}.$$

Thus,

$$\int_0^1 \frac{\gamma'(s)}{\gamma(s)}\mathrm{d}s = \int_0^1 \left(\frac{(\gamma_1)'(s)}{\gamma_1(s)-\alpha} - \frac{(\gamma_0)'(s)}{\gamma_0(s)-\alpha}\right)\mathrm{d}s = \int_0^1 \frac{1}{\gamma_1(s)-\alpha}(\gamma_1)'(s)\mathrm{d}s - \int_0^1 \frac{1}{\gamma_0(s)-\alpha}(\gamma_0)'(s)\mathrm{d}s$$
$$= \int_{\gamma_1} \frac{1}{\zeta-\alpha}\mathrm{d}\zeta - \int_{\gamma_0} \frac{1}{\zeta-\alpha}\mathrm{d}\zeta = (\mathrm{Ind})_{\gamma_1}(\alpha) - (\mathrm{Ind})_{\gamma_0}(\alpha).$$

It suffices to show that

$$\int_0^1 \frac{\gamma'(s)}{\gamma(s)}\mathrm{d}s = 0.$$

From $(*)$,

$$|1-\gamma| = \left|1 - \frac{\gamma_1-\alpha}{\gamma_0-\alpha}\right| = \left|\frac{\gamma_0-\gamma_1}{\gamma_0-\alpha}\right| = \underbrace{\frac{|\gamma_1-\gamma_0|}{|\alpha-\gamma_0|} < 1},$$

so $\mathrm{ran}(\gamma) \subset D(1;1)$, it follows that 0 is a member of the unbounded component of $(\mathrm{ran}(\gamma))^c$, and hence by Conclusion 1.91,

$$\int_0^1 \frac{\gamma'(s)}{\gamma(s)}\mathrm{d}s = \int_0^1 \frac{1}{\gamma(s)}\gamma'(s)\mathrm{d}s = \int_\gamma \frac{1}{\zeta}\mathrm{d}\zeta = \int_\gamma \frac{1}{\zeta-0}\mathrm{d}\zeta = \underbrace{(\mathrm{Ind})_\gamma(0) = 0}.$$

Thus,

$$\int_0^1 \frac{\gamma'(s)}{\gamma(s)}\mathrm{d}s = 0.$$

∎)

Conclusion 1.212 Let $\gamma_0 : [0,1] \to \mathbb{C}$, and $\gamma_1 : [0,1] \to \mathbb{C}$ be closed paths. Let $\alpha \in \mathbb{C}$. Suppose that for every $s \in [0,1]$,

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)|.$$

Then,

$$(\mathrm{Ind})_{\gamma_0}(\alpha) = (\mathrm{Ind})_{\gamma_1}(\alpha).$$

Definition Let X be a topological space. Let $\gamma_0 : [0,1] \to X$, and $\gamma_1 : [0,1] \to X$ be closed curves in X (that is, $\gamma_0 : [0,1] \to X$, and $\gamma_1 : [0,1] \to X$ are continuous, $\gamma_0(0) = \gamma_0(1)$, and $\gamma_1(0) = \gamma_1(1)$). If there exists a mapping $H : [0,1] \times [0,1] \to X$ such that

1. H is continuous,
2. for every $s \in [0,1], H(s,0) = \gamma_0(s)$, and $H(s,1) = \gamma_1(s)$,
3. for every $t \in [0,1], H(0,t) = H(1,t)$,

then we say that γ_0 *and* γ_1 *are* X-***homotopic***. Here, for every $t \in [0,1]$, the mapping $\gamma_t : s \mapsto H(s,t)$ from $[0,1]$ to X is a closed curve in X. We say that $\{\gamma_t\}_{t\in[0,1]}$ *is a one-parameter family of closed curves in* X, *which connects* γ_0 *and* γ_1. Intuitively, γ_0 can be continuously deformed to γ_1, within X. Here, t resembles 'time'.

Let Ω be a region, that is, Ω is a nonempty open, connected subset of $\mathbb{C}$. Let $\Gamma_0 : [0,1] \to \Omega$, and $\Gamma_1 : [0,1] \to \Omega$ be closed paths in Ω. Suppose that Γ_0 and Γ_1 are Ω-homotopic. Let $\alpha \in \Omega^c$.

(Since $\Gamma_0 : [0,1] \to \Omega$, we have $\mathrm{ran}(\Gamma_0) \subset \Omega$, and hence $(\alpha \in)\Omega^c \subset (\mathrm{ran}(\Gamma_0))^c$. This shows that $(\mathrm{Ind})_{\Gamma_0}(\alpha) \in \mathbb{C}$. Similarly, $\mathrm{ran}(\Gamma_1) \subset \Omega$, and $(\mathrm{Ind})_{\Gamma_1}(\alpha) \in \mathbb{C}$.)

Problem 1.213 $(\mathrm{Ind})_{\Gamma_0}(\alpha) = (\mathrm{Ind})_{\Gamma_1}(\alpha)$.

(Solution Since Γ_0 and Γ_1 are Ω-homotopic, there exists a mapping $H : [0,1] \times [0,1] \to \Omega$ such that

1. H is continuous,
2. for every $s \in [0,1], H(s,0) = \Gamma_0(s)$, and $H(s,1) = \Gamma_1(s)$,
3. for every $t \in [0,1], H(0,t) = H(1,t)$.

Since $[0,1]$ is compact, $[0,1] \times [0,1]$ is compact. Since $H : [0,1] \times [0,1] \to \Omega$ is continuous, and $\alpha \in \Omega^c$, the mapping

$$(s,t) \mapsto |\alpha - H(s,t)|$$

from $[0,1] \times [0,1]$ to $(0,\infty)$ is continuous. Now, since $[0,1] \times [0,1]$ is compact,

$$(\min\{|\alpha - H(s,t)| : (s,t) \in [0,1] \times [0,1]\}) \in (0,\infty).$$

Put $\varepsilon \equiv \frac{1}{2}(\min\{|\alpha - H(s,t)| : (s,t) \in [0,1] \times [0,1]\})(> 0)$.

Since $H : [0,1] \times [0,1] \to \Omega$ is continuous, and $[0,1] \times [0,1]$ is compact, $H : [0,1] \times [0,1] \to \Omega$ is uniformly continuous and hence there exists a positive integer n such that

$$\left((s,t),(s',t') \in [0,1] \times [0,1], \text{and} \left(\sqrt{(s-s')^2 + (t-t')^2} = \right)|(s,t) - (s',t')| \le \frac{1}{n}\right)$$
$$\Rightarrow |H(s,t) - H(s',t')| < \varepsilon.$$

For the sake of simplicity of discussion, let us take 3 for n. Thus,

$$\left((s,t),(s',t') \in [0,1]\times[0,1], \text{and } \sqrt{(s-s')^2+(t-t')^2} \le \frac{1}{3}\right)$$
$$\Rightarrow |H(s,t)-H(s',t')| < \varepsilon.$$

$$\text{Let } \gamma_0 : s \mapsto \begin{cases} (1-3s)\left(H\left(\frac{0}{3},\frac{0}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{0}{3}\right)\right) & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] (\text{that is}, 3s\in[0,1]) \\ (1-(3s-1))\left(H\left(\frac{1}{3},\frac{0}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{0}{3}\right)\right) & \text{if } s\in\left[\frac{1}{3},\frac{2}{3}\right] (\text{that is}, (3s-1)\in[0,1]) \\ (1-(3s-2))\left(H\left(\frac{2}{3},\frac{0}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{0}{3}\right)\right) & \text{if } s\in\left[\frac{2}{3},\frac{3}{3}\right] (\text{that is}, (3s-2)\in[0,1]) \end{cases}$$

be a function from $[0,1]$ to $\mathbb{C}$. Clearly, γ_0 is a polygonal closed path of 3-sides.

$$\text{Let } \gamma_1 : s \mapsto \begin{cases} (1-3s)\left(H\left(\frac{0}{3},\frac{1}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{1}{3}\right)\right) & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] (\text{that is}, 3s\in[0,1]) \\ (1-(3s-1))\left(H\left(\frac{1}{3},\frac{1}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{1}{3}\right)\right) & \text{if } s\in\left[\frac{1}{3},\frac{2}{3}\right] (\text{that is}, (3s-1)\in[0,1]) \\ (1-(3s-2))\left(H\left(\frac{2}{3},\frac{1}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{1}{3}\right)\right) & \text{if } s\in\left[\frac{2}{3},\frac{3}{3}\right] (\text{that is}, (3s-2)\in[0,1]) \end{cases}$$

be a function from $[0,1]$ to $\mathbb{C}$. Clearly, γ_1 is a polygonal closed path of 3-sides.

$$\text{Let } \gamma_2 : s \mapsto \begin{cases} (1-3s)\left(H\left(\frac{0}{3},\frac{2}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{2}{3}\right)\right) & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] (\text{that is}, 3s\in[0,1]) \\ (1-(3s-1))\left(H\left(\frac{1}{3},\frac{2}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{2}{3}\right)\right) & \text{if } s\in\left[\frac{1}{3},\frac{2}{3}\right] (\text{that is}, (3s-1)\in[0,1]) \\ (1-(3s-2))\left(H\left(\frac{2}{3},\frac{2}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{2}{3}\right)\right) & \text{if } s\in\left[\frac{2}{3},\frac{3}{3}\right] (\text{that is}, (3s-2)\in[0,1]) \end{cases}$$

be a function from $[0,1]$ to $\mathbb{C}$. Clearly, γ_2 is a polygonal closed path of 3-sides.

$$\text{Let } \gamma_3 : s \mapsto \begin{cases} (1-3s)\left(H\left(\frac{0}{3},\frac{3}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{3}{3}\right)\right) & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] (\text{that is}, 3s\in[0,1]) \\ (1-(3s-1))\left(H\left(\frac{1}{3},\frac{3}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{3}{3}\right)\right) & \text{if } s\in\left[\frac{1}{3},\frac{2}{3}\right] (\text{that is}, (3s-1)\in[0,1]) \\ (1-(3s-2))\left(H\left(\frac{2}{3},\frac{3}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{3}{3}\right)\right) & \text{if } s\in\left[\frac{2}{3},\frac{3}{3}\right] (\text{that is}, (3s-2)\in[0,1]) \end{cases}$$

be a function from $[0,1]$ to $\mathbb{C}$. Clearly, γ_3 is a polygonal closed path of 3-sides.

Here, for every $s \in [0, 1]$,

$$\begin{aligned}
&\left|\gamma_0(s) - H\left(s, \frac{0}{3}\right)\right| \\
&= \begin{cases}
\left|\left((1-3s)\left(H\left(\frac{0}{3},\frac{0}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{0}{3}\right)\right)\right)-H\left(s,\frac{0}{3}\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\left|\left((1-(3s-1))\left(H\left(\frac{1}{3},\frac{0}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{0}{3}\right)\right)\right)-H\left(s,\frac{0}{3}\right)\right| \text{ if } s\in\left[\frac{1}{3},\frac{2}{3}\right] \\
\left|\left((1-(3s-2))\left(H\left(\frac{2}{3},\frac{0}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{0}{3}\right)\right)\right)-H\left(s,\frac{0}{3}\right)\right| \text{ if } s\in\left[\frac{2}{3},\frac{3}{3}\right]
\end{cases} \\
&= \begin{cases}
\left|\left((1-3s)\left(H\left(\frac{0}{3},\frac{0}{3}\right)-H\left(s,\frac{0}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{0}{3}\right)-H\left(s,\frac{0}{3}\right)\right)\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\left|\left((1-(3s-1))\left(H\left(\frac{1}{3},\frac{0}{3}\right)-H\left(s,\frac{0}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{0}{3}\right)-H\left(s,\frac{0}{3}\right)\right)\right)\right| \text{ if } s\in\left[\frac{1}{3},\frac{2}{3}\right] \\
\left|\left((1-(3s-2))\left(H\left(\frac{2}{3},\frac{0}{3}\right)-H\left(s,\frac{0}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{0}{3}\right)-H\left(s,\frac{0}{3}\right)\right)\right)\right| \text{ if } s\in\left[\frac{2}{3},\frac{3}{3}\right]
\end{cases} \\
&< \begin{cases}
(1-3s)\varepsilon+(3s)\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-(3s-1))\varepsilon+(3s-1)\varepsilon \text{ if } s\in\left[\frac{1}{3},\frac{2}{3}\right] \\
(1-(3s-2))\varepsilon+(3s-2)\varepsilon \text{ if } s\in\left[\frac{2}{3},\frac{3}{3}\right]
\end{cases} \\
&= \begin{cases}
\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\varepsilon \text{ if } s\in\left[\frac{1}{3},\frac{2}{3}\right] \\
\varepsilon \text{ if } s\in\left[\frac{2}{3},\frac{3}{3}\right].
\end{cases}
\end{aligned}$$

Thus, for every $s \in [0, 1]$, we have

$$|\gamma_0(s) - \Gamma_0(s)| = |\gamma_0(s) - H(s, 0)| = \underbrace{\left|\gamma_0(s) - H\left(s, \frac{0}{3}\right)\right| < \varepsilon}.$$

Similarly, for every $s \in [0, 1]$, we have $\left|\gamma_1(s) - H\left(s, \frac{1}{3}\right)\right| < \varepsilon$; for every $s \in [0, 1]$, we have $\left|\gamma_2(s) - H\left(s, \frac{2}{3}\right)\right| < \varepsilon$; and for every $s \in [0, 1]$, we have

$$|\gamma_3(s) - \Gamma_1(s)| = |\gamma_3(s) - H(s,1)| = \underbrace{\left|\gamma_3(s) - H\left(s, \frac{3}{3}\right)\right| < \varepsilon}.$$

Since for every $s \in [0,1], |\gamma_0(s) - \Gamma_0(s)| < \varepsilon$, and

$$2\varepsilon = (\min\{|\alpha - H(s,t)| : (s,t) \in [0,1] \times [0,1]\}),$$

for every $s \in [0,1]$,

$$\underbrace{|\gamma_0(s) - \Gamma_0(s)| < \frac{1}{2}|\alpha - H(s,0)|} = \frac{1}{2}|\alpha - \Gamma_0(s)| \leq |\alpha - \Gamma_0(s)|.$$

Now, by Conclusion 1.212, $(\mathrm{Ind})_{\gamma_0}(\alpha) = (\mathrm{Ind})_{\Gamma_0}(\alpha)$.

Observe that for every $s \in [0,1]$,

$$\begin{aligned} |\alpha - \gamma_1(s)| &= \left|\left(\alpha - H\left(s, \frac{1}{3}\right)\right) - \left(\gamma_1(s) - H\left(s, \frac{1}{3}\right)\right)\right| \\ &\geq \left|\alpha - H\left(s, \frac{1}{3}\right)\right| - \left|\gamma_1(s) - H\left(s, \frac{1}{3}\right)\right| \geq 2\varepsilon - \left|\gamma_1(s) - H\left(s, \frac{1}{3}\right)\right| > 2\varepsilon - \varepsilon = \varepsilon, \end{aligned}$$

so for every $s \in [0,1], \varepsilon < |\alpha - \gamma_1(s)|$.

Observe that for every $s \in [0,1]$,

$$\begin{aligned} |\alpha - \gamma_2(s)| &= \left|\left(\alpha - H\left(s, \frac{2}{3}\right)\right) - \left(\gamma_2(s) - H\left(s, \frac{2}{3}\right)\right)\right| \\ &\geq \left|\alpha - H\left(s, \frac{2}{3}\right)\right| - \left|\gamma_2(s) - H\left(s, \frac{2}{3}\right)\right| \geq 2\varepsilon - \left|\gamma_2(s) - H\left(s, \frac{2}{3}\right)\right| > 2\varepsilon - \varepsilon = \varepsilon, \end{aligned}$$

so for every $s \in [0,1], \varepsilon < |\alpha - \gamma_2(s)|$. Since for every $s \in [0,1], |\gamma_3(s) - \Gamma_1(s)| < \varepsilon$, and $2\varepsilon = (\min\{|\alpha - H(s,t)| : (s,t) \in [0,1] \times [0,1]\})$, for every $s \in [0,1]$,

$$\underbrace{|\gamma_3(s) - \Gamma_1(s)| < \frac{1}{2}|\alpha - H(s,1)|} = \frac{1}{2}|\alpha - \Gamma_1(s)| \leq |\alpha - \Gamma_1(s)|.$$

Now, by Conclusion 1.212, $(\mathrm{Ind})_{\gamma_3}(\alpha) = (\mathrm{Ind})_{\Gamma_1}(\alpha)$. Observe that for every $s \in [0,1]$,

$$
\begin{aligned}
&|\gamma_0(s)-\gamma_1(s)| \\
&=\begin{cases}
\left|\left((1-3s)\left(H\left(\frac{0}{3},\frac{0}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{0}{3}\right)\right)\right)-\left((1-3s)\left(H\left(\frac{0}{3},\frac{1}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{1}{3}\right)\right)\right)\right| & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\left|\left((1-3s)\left(H\left(\frac{0}{3},\frac{0}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{0}{3}\right)\right)\right)-\left((1-3s)\left(H\left(\frac{0}{3},\frac{1}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{1}{3}\right)\right)\right)\right| & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\left|\left((1-3s)\left(H\left(\frac{0}{3},\frac{0}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{0}{3}\right)\right)\right)-\left((1-3s)\left(H\left(\frac{0}{3},\frac{1}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{1}{3}\right)\right)\right)\right| & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right]
\end{cases} \\
&\le\begin{cases}
(1-3s)\left|H\left(\frac{0}{3},\frac{0}{3}\right)-H\left(\frac{0}{3},\frac{1}{3}\right)\right|+(3s)\left|H\left(\frac{1}{3},\frac{0}{3}\right)-H\left(\frac{1}{3},\frac{1}{3}\right)\right| & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-3s)\left|H\left(\frac{0}{3},\frac{0}{3}\right)-H\left(\frac{0}{3},\frac{1}{3}\right)\right|+(3s)\left|H\left(\frac{1}{3},\frac{0}{3}\right)-H\left(\frac{1}{3},\frac{1}{3}\right)\right| & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-3s)\left|H\left(\frac{0}{3},\frac{0}{3}\right)-H\left(\frac{0}{3},\frac{1}{3}\right)\right|+(3s)\left|H\left(\frac{1}{3},\frac{0}{3}\right)-H\left(\frac{1}{3},\frac{1}{3}\right)\right| & \text{if } s\in\left[\frac{0}{3},\frac{1}{3}\right]
\end{cases} \\
&<\begin{cases}
(1-3s)\varepsilon+(3s)\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-(3s-1))\varepsilon+(3s-1)\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-(3s-2))\varepsilon+(3s-2)\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right]
\end{cases} \\
&=\begin{cases}
\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right].
\end{cases}
\end{aligned}
$$

Thus, for every $s \in [0,1]$, we have $|\gamma_0(s) - \gamma_1(s)| < \varepsilon$. Now, since for every $s \in [0,1]$, $\varepsilon < |\alpha - \gamma_1(s)|$, for every $s \in [0,1]$, $|\gamma_0(s) - \gamma_1(s)| < |\alpha - \gamma_1(s)|$. Now, by Conclusion 1.212, $(\mathrm{Ind})_{\gamma_0}(\alpha) = (\mathrm{Ind})_{\gamma_1}(\alpha)$.

Observe that for every $s \in [0, 1]$,

$$
\begin{aligned}
&|\gamma_1(s)-\gamma_2(s)| \\
&=\begin{cases}
\left|\left((1-3s)\left(H\left(\frac{0}{3},\frac{1}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{1}{3}\right)\right)\right)-\left((1-3s)\left(H\left(\frac{0}{3},\frac{2}{3}\right)\right)+(3s)\left(H\left(\frac{1}{3},\frac{2}{3}\right)\right)\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\left|\left((1-(3s-1))\left(H\left(\frac{1}{3},\frac{1}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{1}{3}\right)\right)\right)-\left((1-(3s-1))\left(H\left(\frac{1}{3},\frac{2}{3}\right)\right)+(3s-1)\left(H\left(\frac{2}{3},\frac{2}{3}\right)\right)\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\left|\left((1-(3s-2))\left(H\left(\frac{2}{3},\frac{1}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{1}{3}\right)\right)\right)-\left((1-(3s-2))\left(H\left(\frac{2}{3},\frac{2}{3}\right)\right)+(3s-2)\left(H\left(\frac{3}{3},\frac{2}{3}\right)\right)\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right]
\end{cases} \\
&\le\begin{cases}
(1-3s)\left|H\left(\frac{0}{3},\frac{1}{3}\right)-H\left(\frac{0}{3},\frac{2}{3}\right)\right|+(3s)\left|H\left(\frac{1}{3},\frac{1}{3}\right)-H\left(\frac{1}{3},\frac{2}{3}\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-(3s-1))\left|H\left(\frac{1}{3},\frac{1}{3}\right)-H\left(\frac{1}{3},\frac{2}{3}\right)\right|+(3s-1)\left|H\left(\frac{2}{3},\frac{1}{3}\right)-H\left(\frac{2}{3},\frac{2}{3}\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-(3s-2))\left|H\left(\frac{2}{3},\frac{1}{3}\right)-H\left(\frac{2}{3},\frac{2}{3}\right)\right|+(3s-2)\left|H\left(\frac{3}{3},\frac{1}{3}\right)-H\left(\frac{3}{3},\frac{2}{3}\right)\right| \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right]
\end{cases} \\
&<\begin{cases}
(1-3s)\varepsilon+(3s)\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-(3s-1))\varepsilon+(3s-1)\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
(1-(3s-2))\varepsilon+(3s-2)\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right]
\end{cases} \\
&=\begin{cases}
\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right] \\
\varepsilon \text{ if } s\in\left[\frac{0}{3},\frac{1}{3}\right].
\end{cases}
\end{aligned}
$$

Thus, for every $s \in [0, 1]$, we have $|\gamma_1(s) - \gamma_2(s)| < \varepsilon$. Now, since for every $s \in [0, 1]$, $\varepsilon < |\alpha - \gamma_2(s)|$, for every $s \in [0, 1]$, $|\gamma_1(s) - \gamma_2(s)| < |\alpha - \gamma_2(s)|$. Now, by Conclusion 1.212, $(\mathrm{Ind})_{\gamma_1}(\alpha) = (\mathrm{Ind})_{\gamma_2}(\alpha)$.

Observe that for every $s \in [0,1]$,

$$
\begin{aligned}
&|\gamma_2(s) - \gamma_3(s)| \\
&= \begin{cases} \left|\left((1-3s)\left(H\left(\frac{0}{3},\frac{2}{3}\right)\right) + (3s)\left(H\left(\frac{1}{3},\frac{2}{3}\right)\right)\right) - \left((1-3s)\left(H\left(\frac{0}{3},\frac{3}{3}\right)\right) + (3s)\left(H\left(\frac{1}{3},\frac{3}{3}\right)\right)\right)\right| \text{if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ \left|\left((1-(3s-1))\left(H\left(\frac{1}{3},\frac{2}{3}\right)\right) + (3s)\left(H\left(\frac{2}{3},\frac{2}{3}\right)\right)\right) - \left((1-3s)\left(H\left(\frac{1}{3},\frac{3}{3}\right)\right) + (3s)\left(H\left(\frac{2}{3},\frac{3}{3}\right)\right)\right)\right| \text{if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ \left|\left((1-(3s-2))\left(H\left(\frac{2}{3},\frac{2}{3}\right)\right) + (3s)\left(H\left(\frac{3}{3},\frac{2}{3}\right)\right)\right) - \left((1-3s)\left(H\left(\frac{2}{3},\frac{3}{3}\right)\right) + (3s)\left(H\left(\frac{3}{3},\frac{3}{3}\right)\right)\right)\right| \text{if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \end{cases} \\
&\le \begin{cases} (1-3s)\left|H\left(\frac{0}{3},\frac{2}{3}\right) - H\left(\frac{0}{3},\frac{3}{3}\right)\right| + (3s)\left|H\left(\frac{1}{3},\frac{2}{3}\right) - H\left(\frac{1}{3},\frac{3}{3}\right)\right| \text{if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ (1-(3s-1))\left|H\left(\frac{1}{3},\frac{2}{3}\right) - H\left(\frac{1}{3},\frac{3}{3}\right)\right| + (3s)\left|H\left(\frac{2}{3},\frac{2}{3}\right) - H\left(\frac{2}{3},\frac{3}{3}\right)\right| \text{if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ (1-(3s-2))\left|H\left(\frac{2}{3},\frac{2}{3}\right) - H\left(\frac{2}{3},\frac{3}{3}\right)\right| + (3s)\left|H\left(\frac{3}{3},\frac{2}{3}\right) - H\left(\frac{3}{3},\frac{3}{3}\right)\right| \text{if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \end{cases} \\
&< \begin{cases} (1-3s)\varepsilon + (3s)\varepsilon \text{ if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ (1-(3s-1))\varepsilon + (3s-1)\varepsilon \text{ if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ (1-(3s-2))\varepsilon + (3s-2)\varepsilon \text{ if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \end{cases} \\
&= \begin{cases} \varepsilon \text{ if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ \varepsilon \text{ if } s \in \left[\frac{0}{3},\frac{1}{3}\right] \\ \varepsilon \text{ if } s \in \left[\frac{0}{3},\frac{1}{3}\right]. \end{cases}
\end{aligned}
$$

Thus, for every $s \in [0,1]$, we have $|\gamma_3(s) - \gamma_2(s)| < \varepsilon$. Now, since for every $s \in [0,1]$, $\varepsilon < |\alpha - \gamma_2(s)|$, for every $s \in [0,1]$, $|\gamma_3(s) - \gamma_2(s)| < |\alpha - \gamma_2(s)|$. Now, by Conclusion 1.212, $(\mathrm{Ind})_{\gamma_3}(\alpha) = (\mathrm{Ind})_{\gamma_2}(\alpha)$.

Since $(\mathrm{Ind})_{\gamma_3}(\alpha) = (\mathrm{Ind})_{\gamma_2}(\alpha)$, $(\mathrm{Ind})_{\gamma_1}(\alpha) = (\mathrm{Ind})_{\gamma_2}(\alpha)$, $(\mathrm{Ind})_{\gamma_0}(\alpha) = (\mathrm{Ind})_{\gamma_1}(\alpha)$, $(\mathrm{Ind})_{\gamma_3}(\alpha) = (\mathrm{Ind})_{\Gamma_1}(\alpha)$ and $(\mathrm{Ind})_{\gamma_0}(\alpha) = (\mathrm{Ind})_{\Gamma_0}(\alpha)$, we have $(\mathrm{Ind})_{\Gamma_0}(\alpha) = (\mathrm{Ind})_{\Gamma_1}(\alpha)$. ■)

Conclusion 1.214 Let Ω be a region. Let $\Gamma_0 : [0,1] \to \Omega$ and $\Gamma_1 : [0,1] \to \Omega$ be closed paths in Ω. Suppose that Γ_0 and Γ_1 are Ω-homotopic. Let $\alpha \in \Omega^c$. Then $(\mathrm{Ind})_{\Gamma_0}(\alpha) = (\mathrm{Ind})_{\Gamma_1}(\alpha)$.

1.16 Meromorphic Functions

Note 1.215

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let f be a function. If there exists a subset A of Ω such that

1. A has no limit point in Ω (that is, for every $z \in \Omega$, there exists $r > 0$ such that $D'(z;r) \cap A = \emptyset$),
2. for every $z \in (\Omega - A)$, $f'(z)$ exists,

3. for every $a \in A, f$ has a pole at a (that is, there exist a positive integer m, and complex numbers $b_{-1}, \ldots, b_{-m}$ such that the function $z \mapsto \left(f(z) - \left(\frac{b_{-1}}{(z-a)^1} + \cdots + \frac{b_{-m}}{(z-a)^m}\right)\right)$ from $(\Omega - A)$ has a removable singularity at a),

then we say that $f : \Omega \to \mathbb{C}$ *is a* ***meromorphic function*** *in* Ω.

Problem 1.216 If $f \in H(\Omega)$, then $f : \Omega \to \mathbb{C}$ is a meromorphic function in Ω.

(Solution: Let us take $\emptyset$ for A. We must show:

1. $\emptyset$ has no limit point in Ω,
2. for every $z \in (\Omega - \emptyset), f'(z)$ exists,
3. for every $a \in \emptyset, f$ has a pole at a.

For 1: This is trivially true.
For 2: This is trivial, because $\Omega - \emptyset = \Omega$, and $f \in H(\Omega)$.
For 3: Since $a \in \emptyset$ is false, the statement is vacuously true. ∎)

Problem 1.217 In the above definition, A is countable.

(Solution Since Ω is open, there exists compact sets $K_1, K_2, \ldots$ such that $\Omega = K_1 \cup K_2 \cup \cdots$. It follows that $(A =)\Omega \cap A = (K_1 \cup K_2 \cup \cdots) \cap A (= (K_1 \cap A) \cup (K_2 \cap A) \cup \cdots)$.

Problem 1.218 $K_1 \cap A$ is finite.

(Solution If not, otherwise, let $K_1 \cap A$ be infinite. We have to arrive at a contradiction. Since K_1 is compact, $((K_1 \cap A) \subset) K_1$ is closed and bounded, and hence $(K_1 \cap A)$ is bounded. Now, since $K_1 \cap A$ is infinite, by Weierstrass theorem, there exists $\alpha \in \mathbb{C}$ such that α is a limit point of $K_1 \cap A$. It follows that $\alpha \in \overline{K_1 \cap A}\left(\subset \overline{K_1} = K_1 \subset \Omega\right)$, and hence $\alpha \in \Omega$. Since α is a limit point of $K_1 \cap A (\subset A)$, α is a limit point of A. Now, since $\alpha \in \Omega$, A has a limit point in Ω. This contradicts 1. ∎)

Similarly, $K_2 \cap A$ is finite, etc. It follows that $(A =)(K_1 \cap A) \cup (K_2 \cap A) \cup \cdots$ is countable and hence A is countable. ∎)

In the above definition, for every cycle Γ satisfying $a \in (\mathrm{ran}(\Gamma))^{\mathrm{c}}$,

$$\int_{\Gamma}\left(\frac{b_{-1}}{(z-a)^1}+\cdots+\frac{b_{-m}}{(z-a)^m}\right)\mathrm{d}z$$

$$=b_{-1}\int_{\Gamma}\frac{1}{z-a}\mathrm{d}z+\underbrace{b_{-2}\int_{\Gamma}\frac{1}{(z-a)^2}\mathrm{d}z+\cdots+b_{-m}\int_{\Gamma}\frac{1}{(z-a)^m}\mathrm{d}z}_{(m-1)\text{terms}}$$

$$=b_{-1}\int_{\Gamma}\frac{1}{z-a}\mathrm{d}z+\underbrace{b_{-2}\int_{\Gamma}\left(\zeta\mapsto\frac{-1}{(\zeta-a)}\right)'(z)\mathrm{d}z+\cdots}_{(m-1)\text{terms}}$$

$$=b_{-1}\int_{\Gamma}\frac{1}{z-a}\mathrm{d}z+\underbrace{b_{-2}\cdot 0+\cdots}_{(m-1)\text{terms}}=b_{-1}(\mathrm{Ind})_{\Gamma}(a).$$

Definition Here, b_{-1} is called the ***residue*** *of f at a*, and is denoted by $\mathrm{Res}(f;a)$.
Thus, for every cycle Γ satisfying $a\in(\mathrm{ran}(\Gamma))^{\mathrm{c}}$,

$$\int_{\Gamma}\left(\frac{b_{-1}}{(z-a)^1}+\cdots+\frac{b_{-m}}{(z-a)^m}\right)\mathrm{d}z=(\mathrm{Res}(f;a))\big((\mathrm{Ind})_{\Gamma}(a)\big).$$

Conclusion 1.219 If $f\in H(\Omega)$, then

1. $f:\Omega\to\mathbb{C}$ is a meromorphic function in Ω,
2. from the above definition, A is countable.

1.17 Residue Theorem

Note 1.220 Let Ω be a region. Let $f:\Omega\to\mathbb{C}$ be a meromorphic function in Ω. Let A be the set of all points of Ω at which f has pole. Suppose that

1. A has no limit point in Ω,
2. for every $z\in(\Omega-A)$, $f'(z)$ exists.

Let Γ be a cycle in $(\Omega-A)$. Suppose that for every $\alpha\in\Omega^{\mathrm{c}}$, $(\mathrm{Ind})_{\Gamma}(\alpha)=0$.

Problem 1.221 $\{z:z\in A,\ \text{and}\ (\mathrm{Ind})_{\Gamma}(z)\neq 0\}$ is finite.

(Solution If not, otherwise let $\{z:z\in A,\ \text{and}\ (\mathrm{Ind})_{\Gamma}(z)\neq 0\}$ be infinite. We have to arrive at a contradiction.

Let V be the unbounded component of $(\mathrm{ran}(\Gamma))^{\mathrm{c}}$. Since for every $\alpha\in\Omega^{\mathrm{c}}$, $(\mathrm{Ind})_{\Gamma}(\alpha)=0$, by Conclusions 1.91, 1.89, and 1.83, $\Omega^{\mathrm{c}}\subset V$ and hence

$\left(\left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\} \subset\right) V^c \subset \Omega$. Since V is the unbounded component of $(\mathrm{ran}(\Gamma))^{\mathrm{c}}$, V^c is bounded, and closed, and therefore V^c is compact. Since $\left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}$ is an infinite set contained in the bounded set V^c, there exists a limit point β of $\left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\} (\subset A)$ in $(V^c)^-(= V^c \subset \Omega)$. It follows that β is a limit point of A, and $\beta \in \Omega$. This contradicts 1. ∎)

Thus, $\sum_{a \in \left\{z : z \in A, \text{ and} (\mathrm{Ind})_\Gamma(z) \neq 0\right\}} (\mathrm{Res}(f;a))\left((\mathrm{Ind})_\Gamma(a)\right)$ is actually a finite sum.

Problem 1.222 $\underbrace{\Omega - \left(A - \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}\right)} = \Omega - (\{z : z \in A, \text{and}$ $(\mathrm{Ind})_\Gamma(z) = 0\})$ is open.

(Solution Here,

$$\begin{aligned}
&\Omega - \left(A - \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}\right) \\
&= \Omega - \left(A \cap \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}^c\right) \\
&= \Omega \cap \left(A^c \cup \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}\right) \\
&= (\Omega \cap A^c) \cup \left(\Omega \cap \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}\right) \\
&= (\Omega \cap A^c) \cup \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\} \\
&= (\Omega - A) \cup \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}.
\end{aligned}$$

We have to show that each point of $\Omega - \left(\left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\right\}\right)$ is an interior point of $\Omega - \left(\left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\right\}\right)$. For this purpose, let us take any $a \in \left(\Omega - \left(\left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\right\}\right)\right)$ $(= (\Omega - A) \cup \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\})$, that is, $a \in (\Omega - A)$ or $a \in \{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\}$. We have to show that a is an interior point of $(\Omega - A) \cup \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}$.

Case I: when $a \in (\Omega - A)$. By assumption 1, a is not a limit point of A. Also, Ω is open, $a \in \Omega$ and $a \notin A (\subset \Omega)$. It follows that there exists an open neighborhood W of a such that $W \subset \Omega$ and $W \cap A = \emptyset$. It follows that $W \subset (\Omega - A) (\subset (\Omega - A) \cup \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\})$. Thus, a is an interior point of $(\Omega - A) \cup \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}$.

Case II: when $a \in \left\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\right\}$. It follows that $a \in A (\subset \Omega)$ and $(\mathrm{Ind})_\Gamma(a) \neq 0$. It follows that $a \in \cup \left\{\left((\mathrm{Ind})_\Gamma\right)^{-1}(m) : m \text{ is a nonzero integer}\right\}$. Since each $\left((\mathrm{Ind})_\Gamma\right)^{-1}(m)$ is open, $\cup \left\{\left((\mathrm{Ind})_\Gamma\right)^{-1}(m) : m \text{ is a nonzero integer}\right\}$ is open. Now, since Ω is open, $a \in \Omega$, and $a \in \cup \left\{\left((\mathrm{Ind})_\Gamma\right)^{-1}(m) : m \text{ is a nonzero integer}\right\}$, there exists an open neighborhood W of a such that $W \subset \Omega$ and $W \subset \cup \left\{\left((\mathrm{Ind})_\Gamma\right)^{-1}(m) : m \text{ is a nonzero integer}\right\}$.

Problem 1.223 $W \subset (\Omega - A) \cup \{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\}$.

(Solution If not, otherwise suppose that there exists $z_0 \in W(\subset \Omega)$ such that $z_0 \notin (\Omega - A)$, and $z_0 \notin \{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\}$. We have to arrive at a contradiction.

Since $z_0 \notin (\Omega - A)$, and $z_0 \in \Omega$, we have $z_0 \in A$. Now, since $z_0 \notin \{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\}$, we have $(\mathrm{Ind})_\Gamma(z_0) = 0$. Since $z_0 \in W \subset \cup\left\{\left((\mathrm{Ind})_\Gamma\right)^{-1}(m) : m \text{ is a nonzero integer}\right\}$, $(\mathrm{Ind})_\Gamma(z_0)$ is a nonzero integer. This is a contradiction. ∎)

Now, since W is an open neighborhood of a, a is an interior point of $(\Omega - A) \cup \{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\}$. ∎)

For the simplicity of discussion, suppose that $\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\}$ has exactly two distinct elements, say a and b.

Problem 1.224 $\frac{1}{2\pi i}\int_\Gamma f(z)\mathrm{d}z = (\mathrm{Res}(f;a))\left((\mathrm{Ind})_\Gamma(a)\right) + (\mathrm{Res}(f;b))\left((\mathrm{Ind})_\Gamma(b)\right)$.

(Solution Since $\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\} = \{a, b\}$, we have $a \in A, b \in A$, $(\mathrm{Ind})_\Gamma(a) \neq 0$, and $(\mathrm{Ind})_\Gamma(b) \neq 0$. Since $a \in A$, by assumption, f has a pole at a, and hence there exist a positive integer m and complex numbers $a_{-1}, \ldots, a_{-m}$ such that the function

$$Q_1 : z \mapsto \left(f(z) - \left(\frac{a_{-1}}{(z-a)^1} + \cdots + \frac{a_{-m}}{(z-a)^m}\right)\right)$$

has a removable singularity at a. Similarly, there exist a positive integer n, and complex numbers $b_{-1}, \ldots, b_{-n}$ such that the function

$$Q_2 : z \mapsto \left(f(z) - \left(\frac{b_{-1}}{(z-b)^1} + \cdots + \frac{b_{-n}}{(z-b)^n}\right)\right)$$

has a removable singularity at b. Here, $\mathrm{Res}(f;a) = a_{-1}$, and $\mathrm{Res}(f;b) = b_{-1}$. Since

$$Q_1 : z \mapsto \left(f(z) - \left(\frac{a_{-1}}{(z-a)^1} + \cdots + \frac{a_{-m}}{(z-a)^m}\right)\right)$$

has a removable singularity at a, and $a \neq b$, the function

$$g : z \mapsto \left(f(z) - \left(\frac{a_{-1}}{(z-a)^1} + \cdots + \frac{a_{-m}}{(z-a)^m}\right) - \left(\frac{b_{-1}}{(z-b)^1} + \cdots + \frac{b_{-n}}{(z-b)^n}\right)\right)$$

has a removable singularity at a. Similarly, the function

$$g : z \mapsto \left(f(z) - \left(\frac{a_{-1}}{(z-a)^1} + \cdots + \frac{a_{-m}}{(z-a)^m} \right) - \left(\frac{b_{-1}}{(z-b)^1} + \cdots + \frac{b_{-n}}{(z-b)^n} \right) \right)$$

has a removable singularity at b.

Now, since for every $z \in (\Omega - A)$, $f'(z)$ exists, we have

$$\underbrace{g \in H((\Omega - A) \cup \{a, b\})} = H\big((\Omega - A) \cup \{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\}\big)$$

$$= H\big(\Omega - (\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\})\big).$$

Since for every $\alpha \in \big(\Omega - (\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\})\big)^c$ $(= \Omega^c \cup \{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\})$, we have, by the given assumption, $(\mathrm{Ind})_\Gamma(\alpha) = 0$.

Since $\Omega - (\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\})$ is open, $g \in H(\Omega - (\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\}))$, and for every $\alpha \in \big(\Omega - (\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) = 0\})\big)^c$, $(\mathrm{Ind})_\Gamma(\alpha) = 0$, we have, by Theorem 1.205,

$$\begin{aligned}
&\int_\Gamma f(\zeta)\mathrm{d}\zeta - \mathrm{Res}(f; a) \cdot 2\pi i (\mathrm{Ind})_\Gamma(a) - \mathrm{Res}(f; b) \cdot 2\pi i (\mathrm{Ind})_\Gamma(b) \\
&= \int_\Gamma f(\zeta)\mathrm{d}\zeta - a_{-1} 2\pi i (\mathrm{Ind})_\Gamma(a) - b_{-1} 2\pi i (\mathrm{Ind})_\Gamma(b) \\
&= \int_\Gamma f(\zeta)\mathrm{d}\zeta - a_{-1} \int_\Gamma \frac{1}{\zeta - a}\mathrm{d}\zeta - b_{-1} \int_\Gamma \frac{1}{\zeta - b}\mathrm{d}\zeta \\
&= \int_\Gamma \left(f(\zeta) - \left(\frac{a_{-1}}{(\zeta-a)^1} + \cdots + \frac{a_{-m}}{(\zeta-a)^m} \right) - \left(\frac{b_{-1}}{(\zeta-b)^1} + \cdots + \frac{b_{-n}}{(\zeta-b)^n} \right) \right)\mathrm{d}\zeta \\
&= \int_\Gamma g(\zeta)\mathrm{d}\zeta = \underbrace{\int_\Gamma g(\zeta)\mathrm{d}\zeta = 0},
\end{aligned}$$

and hence

$$\frac{1}{2\pi i} \int_\Gamma f(z)\mathrm{d}z = (\mathrm{Res}(f; a))\big((\mathrm{Ind})_\Gamma(a)\big) + (\mathrm{Res}(f; b))\big((\mathrm{Ind})_\Gamma(b)\big).$$

∎)

Conclusion 1.225 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a meromorphic function in Ω. Let A be the set of all points of Ω at which f has pole. Suppose that

1. A has no limit point in Ω,
2. for every $z \in (\Omega - A)$, $f'(z)$ exists.

Let Γ be a cycle in $(\Omega - A)$. Suppose that for every $\alpha \in \Omega^{c}$, $(\mathrm{Ind})_{\Gamma}(\alpha) = 0$. Then,

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)\mathrm{d}z = \sum_{a\in\{z:z\in A,\text{ and }(\mathrm{Ind})_{\Gamma}(z)\neq 0\}} (\mathrm{Res}(f;a))\big((\mathrm{Ind})_{\Gamma}(a)\big).$$

This result is known as the **residue theorem**.

1.18 Rouché's Theorem and Hurwitz's Theorem

Note 1.226 Let Ω be a region. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed path in Ω. Suppose that for every $\alpha \in \Omega^{c}$, $(\mathrm{Ind})_{\gamma}(\alpha) = 0$. Suppose that for every $\alpha \in (\Omega - \mathrm{ran}(\gamma))$, $(\mathrm{Ind})_{\gamma}(\alpha) = 0$ or 1. Let $f : \Omega \to \mathbb{C}$, and $f \in H(\Omega)$. Suppose that $(f^{-1}(0)) \cap (\mathrm{ran}(\gamma)) = \emptyset$.

It follows that γ is a closed path in $(\Omega - f^{-1}(0))$. Since $f \in H(\Omega)$, by Lemma 1.117, $f' \in H(\Omega)$.

Problem 1.227 $\varphi : z \mapsto \frac{f'(z)}{f(z)}$ is a meromorphic function in Ω.

(Solution We must show:

1. $f^{-1}(0)$ has no limit point in Ω,
2. for every $z \in (\Omega - f^{-1}(0))$, $\varphi'(z)$ exists,
3. for every $a \in f^{-1}(0)$, φ has a pole at a,
4. if φ has a pole at a, then $f(a) = 0$.

For 1: Since $(f^{-1}(0)) \cap (\mathrm{ran}(\gamma)) = \emptyset$, $\mathrm{ran}(\gamma)$ is nonempty, $\mathrm{ran}(\gamma) \subset \Omega$, and $f : \Omega \to \mathbb{C}$, we have $f \neq 0$. Now, by Theorem 1.136, $f^{-1}(0)$ has no limit point in Ω,

For 2: Since $f \in H(\Omega)$, and $\Omega - f^{-1}(0)$ is a nonempty open subset of Ω, for every $z \in (\Omega - f^{-1}(0))$, $f'(z)$ exists. Similarly, for every $z \in (\Omega - f^{-1}(0))$, $f''(z)$ exists. Thus, for every $z \in (\Omega - f^{-1}(0))$,

$$\varphi'(z) = \frac{f''(z)f(z) - f'(z)f'(z)}{(f(z))^2}.$$

Hence, for every $z \in (\Omega - f^{-1}(0))$, $\varphi'(z)$ exists.

For 3: Let us take any $a \in f^{-1}(0)$. We have to show that φ has a pole at a.

Since $f \neq 0$, by Theorem 1.136, there exists a unique positive integer $m(a)$, and a unique function $g : \Omega \to \mathbb{C}$ such that

. $$g \in H(\Omega),$$

b. for every $z \in \Omega$, $f(z) = (z-a)^{m(a)}(g(z))$,

. $$g(a) \neq 0.$$

Since $g \in H(\Omega)$, and $g(a) \neq 0$, there exists an open neighborhood V of a such that $V \subset \Omega$, and $g, \frac{1}{g}$ are holomorphic in V. Now, for every $z \in (V - \{a\})$,

$$\varphi(z) = \frac{m(a)(z-a)^{m(a)-1}(g(z)) + (z-a)^{m(a)}(g'(z))}{(z-a)^{m(a)}(g(z))} = \frac{m(a)}{z-a} + \frac{1}{g(z)}g'(z).$$

Since $g, \frac{1}{g}$ are holomorphic in V, $z \mapsto \frac{1}{g(z)}g'(z)$ is holomorphic in V. Next, since V is an open neighborhood of a, $z \mapsto \frac{1}{g(z)}g'(z)$ has a removable singularity at a. Thus, φ has a pole at a. Also, $\mathrm{Res}(\varphi; a) = m(a)$.

For 4: If not, otherwise suppose that there exists $a \in \Omega$ such that φ has a pole at a, and $f(a) \neq 0$. We have to arrive at a contradiction. Since $f(a) \neq 0$, and $f, f' \in H(\Omega)$, $\left(\lim_{z\to a}|\varphi(z)| =\right) \lim_{z\to a}\left|\frac{f'(z)}{f(z)}\right| \neq \infty$, and hence, φ has no pole at a. This is a contradiction. ∎)

By Conclusion 1.225,

$$\begin{aligned}
(\mathrm{Ind})_{f\circ\gamma}(0) &= \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{\zeta - 0}\mathrm{d}\zeta = \frac{1}{2\pi i}\int_0^1 \frac{1}{(f\circ\gamma)(t) - 0}\left((f\circ\gamma)'(t)\right)\mathrm{d}t \\
&= \frac{1}{2\pi i}\int_0^1 \frac{1}{(f\circ\gamma)(t)}\left((f\circ\gamma)'(t)\right)\mathrm{d}t = \frac{1}{2\pi i}\int_0^1 \frac{1}{f(\gamma(t))}(f'(\gamma(t))\gamma'(t))\mathrm{d}t \\
&= \frac{1}{2\pi i}\int_0^1 \left(\frac{1}{f(\gamma(t))}f'(\gamma(t))\right)\gamma'(t)\mathrm{d}t = \frac{1}{2\pi i}\int_\gamma \frac{1}{f(z)}f'(z)\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z
\end{aligned}$$

$$= \underbrace{\frac{1}{2\pi i}\int_{\gamma} \varphi(z)\mathrm{d}z = \sum_{a \in \left\{z : z \in f^{-1}(0),\text{ and }(\mathrm{Ind})_{\gamma}(z)=1\right\}} (\mathrm{Res}(\varphi;a))\Big((\mathrm{Ind})_{\gamma}(a)\Big)}$$

$$= \sum_{a \in \left\{z : z \in f^{-1}(0),\text{ and }(\mathrm{Ind})_{\gamma}(z)=1\right\}} (\mathrm{Res}(\varphi;a))(1) = \sum_{a \in \left\{z : z \in f^{-1}(0),\text{ and }(\mathrm{Ind})_{\gamma}(z)=1\right\}} (m(a))$$

$$= \left(\text{number of zeros of } f \text{ with their multiplicity in} \left\{z : (\mathrm{Ind})_{\gamma}(z) = 1\right\}\right),$$

so

$$\left(\text{number of zeros of } f \text{ with their multiplicity in} \left\{z : (\mathrm{Ind})_{\gamma}(z) = 1\right\}\right)$$
$$= (\mathrm{Ind})_{f \circ \gamma}(0) = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\mathrm{d}z.$$

Conclusion 1.228 Let Ω be a region. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed path in Ω. Suppose that for every $\alpha \in \Omega^{c}$, $(\mathrm{Ind})_{\gamma}(\alpha) = 0$. Suppose that for every $\alpha \in (\Omega - \mathrm{ran}(\gamma))$, $(\mathrm{Ind})_{\gamma}(\alpha) = 0$ or 1. Let $f : \Omega \to \mathbb{C}$ and $f \in H(\Omega)$. Suppose that $(f^{-1}(0)) \cap (\mathrm{ran}(\gamma)) = \emptyset$. Then

$$\left(\text{number of zeros of } f \text{ with their multiplicity in} \left\{z : (\mathrm{Ind})_{\gamma}(z) = 1\right\}\right)(\mathrm{Ind})_{f \circ \gamma}(0)$$
$$= \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\mathrm{d}z.$$

Theorem 1.229 Let Ω be a region. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed path in Ω. Suppose that for every $\alpha \in \Omega^{c}$, $(\mathrm{Ind})_{\gamma}(\alpha) = 0$. Suppose that for every $\alpha \in (\Omega - \mathrm{ran}(\gamma))$, $(\mathrm{Ind})_{\gamma}(\alpha) = 0$ or 1. Let $f : \Omega \to \mathbb{C}$, and $f \in H(\Omega)$. Suppose that $(f^{-1}(0)) \cap (\mathrm{ran}(\gamma)) = \emptyset$. Let $g : \Omega \to \mathbb{C}$ and $g \in H(\Omega)$. Suppose that for every $z \in \mathrm{ran}(\gamma)$,

$$|f(z) - g(z)| < |f(z)|(*).$$

Then

$$\left(\text{number of zeros of } f \text{ with their multiplicity in} \left\{z : (\mathrm{Ind})_{\gamma}(z) = 1\right\}\right)$$
$$= \left(\text{number of zeros of } g \text{ with their multiplicity in} \left\{z : (\mathrm{Ind})_{\gamma}(z) = 1\right\}\right).$$

Proof By $(*)$, for every $z \in \mathrm{ran}(\gamma)$, $g(z) \neq 0$. It follows that $(g^{-1}(0)) \cap (\mathrm{ran}(\gamma)) = \emptyset$. It follows from Conclusion 1.228, that

$$\left(\text{number of zeros of } g \text{ with their multiplicity in} \left\{ z : (\mathrm{Ind})_\gamma(z) = 1 \right\}\right)$$
$$= (\mathrm{Ind})_{g\circ\gamma}(0),$$

and

$$\left(\text{number of zeros of} f \text{ with their multiplicity in} \left\{ z : (\mathrm{Ind})_\gamma(z) = 1 \right\}\right)$$
$$= (\mathrm{Ind})_{f\circ\gamma}(0).$$

Now, it suffices to show that $(\mathrm{Ind})_{f\circ\gamma}(0) = (\mathrm{Ind})_{g\circ\gamma}(0)$.

Since $\gamma : [0,1] \to \mathbb{C}$ is a closed path in Ω, and $f \in H(\Omega)$, $(f \circ \gamma) : [0,1] \to \mathbb{C}$ is a closed path in Ω. Similarly, $(g \circ \gamma) : [0,1] \to \mathbb{C}$ is a closed path in Ω. For every $s \in [0,1]$, $\gamma(s) \in \mathrm{ran}(\gamma)$, and hence, by $(*)$, for every $s \in [0,1]$,

$$|(f \circ \gamma)(s) - (g \circ \gamma)(s)| = \underbrace{|f(\gamma(s)) - g(\gamma(s))| < |f(\gamma(s))|} = |0 - (f \circ \gamma)(s)|.$$

Thus, for every $s \in [0,1]$,

$$|(f \circ \gamma)(s) - (g \circ \gamma)(s)| < |0 - (f \circ \gamma)(s)|.$$

Now, by Conclusion 1.212, $(\mathrm{Ind})_{f\circ\gamma}(0) = (\mathrm{Ind})_{g\circ\gamma}(0)$. ∎

Theorem 1.229, known as **Rouche's theorem**, is due to E. Rouche (18.08.1832 – 19.08.1910).

Corollary 1.230 Let Ω be a region. For every positive integer n, let $f_n : \Omega \to \mathbb{C}$ be a function. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that each $f_n \in H(\Omega)$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Let $f \neq 0$. Let $D[a;R] \subset \Omega$ and, for every z satisfying $|z-a| = R, f(z) \neq 0$. Then there exists a positive integer N such that, for every $n \geq N$, f and f_n have the same number of zeros in $D(a;R)$.

Proof By Conclusion 1.172, we have $f \in H(\Omega)$, and hence $f : \Omega \to \mathbb{C}$ is continuous. Since $f : \Omega \to \mathbb{C}$ is a continuous function, and $\{z : |z-a| = R\}$ is a compact set, $\min\{|f(z)| : |z-a| = R\}$ exists. Now, since for every z satisfying $|z-a| = R$, $f(z) \neq 0$, $\min\{|f(z)| : |z-a| = R\}$ is a positive real number. Since $\{f_n\}$ converges to f uniformly on compact subsets of Ω, and $\{z : |z-a| = R\}$ is a compact subset of Ω, there exists a positive integer N such that for every $n \geq N$ and for every z satisfying $|z-a| = R$, we have $|f_n(z) - f(z)| < \frac{1}{2}\min\{|f(\zeta)| : |\zeta - a| = R\}(< |f(z)|)$. Thus, for every $n \geq N$ and for every z satisfying $|z-a| = R$, we have $|f_n(z) - f(z)| < |f(z)|$. Now, since $f \neq 0$, by Theorem 1.229, for every $n \geq N$,

$$\Big(\text{number of zeros of } f \text{ with their multiplicity in} \Big\{z : (\text{Ind})_\gamma(z) = 1\Big\}(= D(a;R))\Big)$$
$$= \Big(\text{number of zeros of } f_n \text{ with their multiplicity in} \Big\{z : (\text{Ind})_\gamma(z) = 1\Big\}(= D(a;R))\Big).$$

Thus, for every $n \geq N$, f and f_n have the same number of zeros in $D(a;R)$. c ∎

This result, known as the **Hurwitz's theorem**, is due to A. Hurwitz (26.03.1859 – 18.11.1919).

Corollary 1.231 Let Ω be a region. For every positive integer n let $f_n : \Omega \to \mathbb{C}$ be a function. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that each $f_n \in H(\Omega)$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Suppose that each f_n has no zero on Ω. Then $f = 0$ or f has no zero on Ω.

Proof If not, otherwise let $f \neq 0$, and f has a zero on Ω. We have to arrive at a contradiction.

Since f has a zero on Ω, there exists $a \in \Omega$ such that $f(a) = 0$. Since $a \in \Omega$, and Ω is open, there exists $R > 0$ such that $(a \in)D[a;R] \subset \Omega$. Also, since $f(a) = 0$, the number of zeros of f in $D(a;R)$ is nonzero.

By Conclusion 1.172, we have $f \in H(\Omega)$, and hence $f : \Omega \to \mathbb{C}$ is continuous. Since $f : \Omega \to \mathbb{C}$ is a continuous function, and $\{z : |z - a| = R\}$ is a compact set, $\min\{|f(z)| : |z - a| = R\}$ exists.

Now, since for every z satisfying $|z - a| = R$, $f(z) \neq 0$, $\min\{|f(z)| : |z - a| = R\}$ is a positive real number and hence f has no zero in $\{z : |z - a| = R\}$. Now, by Corollary 1.230, there exists a positive integer N such that f and f_N have the same number of zeros in $D(a;R)$. Now, since the number of zeros of f in $D(a;R)$ is nonzero, the number of zeros of f_N in $D(a;R)$ is nonzero. This contradicts the assumption. ∎

1.19 Cauchy–Riemann Equations

Note 1.232 For the present purpose, for every real number x, y, we shall identify the complex number $x + iy$ with the ordered pair (x, y).

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $a \in \Omega$.

We shall not distinguish

$$\big\{(x,y) : (x,y) \in \mathbb{R}^2, \text{ and } (x + iy) \in \Omega\big\}$$

with Ω,

$$(\mathrm{Re}(f), \mathrm{Im}(f))$$

with

$$(\mathrm{Re}(f) + i(\mathrm{Im}(f))),$$

and the function

$$(x,y) \mapsto ((\mathrm{Re}(f))(x,y), (\mathrm{Im}(f))(x,y))$$

from

$$\{(x,y) : (x,y) \in \mathbb{R}^2, \text{ and } (x+iy) \in \Omega\}$$

to $\mathbb{R}^2$ with f.

If the linear transformation

$$f'(\mathrm{Re}(a), \mathrm{Im}(a)) : \mathbb{R}^2 \to \mathbb{R}^2$$

exists, then we say that *f has a* ***differential*** *at a*.

Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let us denote $\mathrm{Re}(f)$ by u, and $\mathrm{Im}(f)$ by v. Suppose that for every $a \in \Omega$, f has a differential at a. Let $f \in H(\Omega)$.

Let us take any $a \in \Omega$. Since $f \in H(\Omega)$, $\lim\limits_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists, and hence by Conclusion 1.34, f has a differential at a It follows that

$$\begin{aligned}\lim_{(x,y)\to(\mathrm{Re}(a),\mathrm{Im}(a))} \frac{1}{|(x,y)-(\mathrm{Re}(a),\mathrm{Im}(a))|} \big|(u(x,y),\ v(x,y)) - (u(\mathrm{Re}(a),\mathrm{Im}(a)),\ v(\mathrm{Re}(a),\mathrm{Im}(a)))\\ -(u_x(\mathrm{Re}(a),\mathrm{Im}(a)))(x-\mathrm{Re}(a))\\ + \big(u_y(\mathrm{Re}(a),\mathrm{Im}(a))\big)(y-\mathrm{Im}(a)), (v_x(\mathrm{Re}(a),\mathrm{Im}(a)))(x-\mathrm{Re}(a))\\ + \big(v_y(\mathrm{Re}(a),\mathrm{Im}(a))\big)(y-\mathrm{Im}(a))\big| = 0,\end{aligned}$$

and hence

$$\begin{aligned}&\lim_{z\to a} \frac{1}{|z-a|} \Big|f(z) - f(a)\\ &\quad - \left((u_x(a))\left(\frac{1}{2}((z-a)+(z-a)^-)\right) + (u_y(a))\left(\frac{1}{2i}((z-a)-(z-a)^-)\right)\right)\\ &\quad + i\left((v_x(a))\left(\frac{1}{2}((z-a)+(z-a)^-)\right) + (v_y(a))\left(\frac{1}{2i}((z-a)-(z-a)^-)\right)\right)\Big| = 0,\end{aligned}$$

Thus,

$$\lim_{z\to a}\frac{1}{|z-a|}\left|f(z)-f(a)\right.$$
$$-\left((z-a)\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)\right.$$
$$\left.\left.+(z-a)^-\left(\left(\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}\right)\right)\right)\right|=0,$$

that is

$$\lim_{z\to a}\left|\frac{f(z)-f(a)}{z-a}\right.$$
$$-\left(\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)\right.$$
$$\left.\left.+\frac{(z-a)^-}{z-a}\left(\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}\right)\right)\right|=0,$$

that is

$$\lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}-\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)\right.$$
$$\left.-\left(\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}\right)\frac{(z-a)^-}{z-a}\right)=0.$$

Now, since

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a}=f'(a),$$

we have

$$\left(\lim_{z\to a}\left(\left(\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}\right)=-\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)\frac{(z-a)^-}{z-a}\right)\right.$$
$$=\Bigg)\lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}\right.$$
$$-\left(\frac{f(z)-f(a)}{z-a}-\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)\right.$$
$$\left.\left.-\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)\frac{(z-a)^-}{z-a}\right)\right)=f'(a)-0(=f'(a)),$$

and hence

$$\lim_{z\to a}\left(\left(\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}\right)\frac{(z-a)^-}{z-a}\right)$$
$$=-\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)+f'(a).$$

Now, since

$$\lim_{z\to a}\frac{(z-a)^-}{z-a}$$

does not exist,

$$\left(\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}\right)=0,\ \text{and}\, f'(a)$$
$$=\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}.$$

Next,

$$\left(\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f\right)(a)\equiv\frac{1}{2}\left(\left(\frac{\partial u}{\partial x}(a)+i\frac{\partial v}{\partial x}(a)\right)+i\left(\frac{\partial u}{\partial y}(a)+i\frac{\partial v}{\partial y}(a)\right)\right)$$
$$=\frac{1}{2}\left((u_x(a)+iv_x(a))+i\big(u_y(a)+iv_y(a)\big)\right)=\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}=0,$$

so

$$\left(\bar{\partial}f\right)(a)=0,$$

where $\bar{\partial}$ stands for the operator $\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$. Also,

$$\left(\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)f\right)(a)\equiv\frac{1}{2}\left(\left(\frac{\partial u}{\partial x}(a)+i\frac{\partial v}{\partial x}(a)\right)-i\left(\frac{\partial u}{\partial y}(a)+i\frac{\partial v}{\partial y}(a)\right)\right)$$
$$=\frac{1}{2}\left((u_x(a)+iv_x(a))-i\big(u_y(a)+iv_y(a)\big)\right)=\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}=f'(a),$$

so

$$(\partial f)(a)=f'(a),$$

where ∂ stands for the operator $\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$.

Conclusion 1.233 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let us denote $\mathrm{Re}(f)$ by u, and $\mathrm{Im}(f)$ by v. Let $f \in H(\Omega)$. Then, for every $z \in \Omega$,

$$\left(\bar{\partial} f\right)(z) = 0, \text{ and } (\partial f)(z) = f'(z),$$

where $\bar{\partial} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, and $\partial \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$.

Note 1.234 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let us denote $\mathrm{Re}(f)$ by u, and $\mathrm{Im}(f)$ by v. Suppose that for every $a \in \Omega$, f has a differential at a. Suppose that for every $z \in \Omega$, $\left(\bar{\partial} f\right)(z) = 0$, where $\bar{\partial} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Problem 1.235 $f \in H(\Omega)$.

(Solution For this purpose, let us take any $a \in \Omega$. We have to show that

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists. By assumption,

$$\frac{1}{2}\left(\left(u_x(a) - v_y(a)\right) + i\left(v_x(a) + u_y(a)\right)\right) = \frac{1}{2}\left(\left(u_x(a) + iv_x(a)\right) + i\left(u_y(a) + iv_y(a)\right)\right)$$
$$= \left(\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f\right)(a) = \underbrace{\left(\bar{\partial} f\right)(a) = 0},$$

so

$$u_x(a) - v_y(a) = 0, \text{ and } v_x(a) + u_y(a) = 0.$$

Thus,

$$u_x(a) = v_y(a), \text{ and } u_y(a) = -v_x(a).$$

Since f has a differential at a, we have

$$\lim_{(x,y) \to (\mathrm{Re}(a), \mathrm{Im}(a))} \frac{1}{|(x,y) - (\mathrm{Re}(a), \mathrm{Im}(a))|} \left|(u(x,y), v(x,y))\right.$$
$$- (u(\mathrm{Re}(a), \mathrm{Im}(a)), v(\mathrm{Re}(a), \mathrm{Im}(a))) - ((u_x(\mathrm{Re}(a), \mathrm{Im}(a)))(x - \mathrm{Re}(a))$$
$$+ \left(u_y(\mathrm{Re}(a), \mathrm{Im}(a))\right)(y - \mathrm{Im}(a)), (v_x(\mathrm{Re}(a), \mathrm{Im}(a)))(x - \mathrm{Re}(a))$$
$$\left. + \left(v_y(\mathrm{Re}(a), \mathrm{Im}(a))\right)(y - \mathrm{Im}(a))\right)\Big| = 0,$$

and hence

$$\lim_{z\to a}\frac{1}{|z-a|}\Big|f(z)-f(a)-$$
$$\left(\left((u_x(a))\left(\frac{1}{2}((z-a)+(z-a)^-)\right)+\left(u_y(a)\right)\left(\frac{1}{2i}((z-a)-(z-a)^-)\right)\right)\right.$$
$$\left.+i\left((v_x(a))\left(\frac{1}{2}((z-a)+(z-a)^-)\right)+\left(v_y(a)\right)\left(\frac{1}{2i}((z-a)-(z-a)^-)\right)\right)\right)\Big|=0.$$

Thus,

$$\lim_{z\to a}\frac{1}{|z-a|}\Big|f(z)-f(a)$$
$$-\left((z-a)\left(\frac{u_x(a)+v_y(a)}{2}+i\frac{-u_y(a)+v_x(a)}{2}\right)\right.$$
$$\left.+(z-a)^-\left(\left(\frac{u_x(a)-v_y(a)}{2}+i\frac{u_y(a)+v_x(a)}{2}\right)\right)\right)\Big|=0,$$

that is

$$\lim_{z\to a}\Big|\frac{f(z)-f(a)}{z-a}-\left(\left(\frac{v_y(a)+v_y(a)}{2}+i\frac{-(-v_x(a))+v_x(a)}{2}\right)\right.$$
$$\left.+\frac{(z-a)^-}{z-a}\left(\frac{v_y(a)-v_y(a)}{2}+i\frac{(-v_x(a))+v_x(a)}{2}\right)\right)\Big|=0,$$

that is

$$\left(\lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}-(u_x(a)+iv_x(a))\right)=\right)\lim_{z\to a}\left(\frac{f(z)-f(a)}{z-a}-\left(v_y(a)+iv_x(a)\right)\right)$$
$$=0.$$

This shows that $\lim_{z\to a}\frac{f(z)-f(a)}{z-a}$ exists, and $f'(a)=u_x(a)+iv_x(a)$. ∎

Conclusion 1.236 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f:\Omega\to\mathbb{C}$ be a function. Let us denote $\mathrm{Re}(f)$ by u, and $\mathrm{Im}(f)$ by v. Suppose that for every $a\in\Omega$, f has a differential at a Suppose that for every $z\in\Omega$,

$$\left(\bar{\partial}f\right)(z)=0,$$

where $\bar{\partial}\equiv\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$. Then $f\in H(\Omega)$, and for every $z\in\Omega$,

$$f'(z) = u_x(z) + iv_x(z).$$

The equation $\bar{\partial}f = 0$, that is, the simultaneous equations

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\},$$

are known as the **Cauchy–Riemann equations**.

Note 1.237 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let us denote $\mathrm{Re}(f)$ by u and $\mathrm{Im}(f)$ by v. Suppose that for every $a \in \Omega$,

$$\underbrace{f_{xx}(a)} = u_{xx}(a) + iv_{xx}(a), \text{ and } \underbrace{f_{yy}(a)} = u_{yy}(a) + iv_{yy}(a)$$

exist. Thus, $f_{xx} : \Omega \to \mathbb{C}$, and $f_{yy} : \Omega \to \mathbb{C}$ are functions.

Definition The function $(f_{xx} + f_{yy}) : \Omega \to \mathbb{C}$ is denoted by Δf, and is called the ***Laplacian*** *of* f.

Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let us denote $\mathrm{Re}(f)$ by u and $\mathrm{Im}(f)$ by v. Let $f \in H(\Omega)$. It follows that $f : \Omega \to \mathbb{C}$ is continuous.

Problem 1.238 $\Delta f = 4\partial(\bar{\partial}f) = 4\bar{\partial}(\partial f)$.

(Solution Let us take any $a \in \Omega$. We have to show that $(\Delta f)(a) = (4\partial(\bar{\partial}f))(a)$.

$$\begin{aligned} \text{LHS} = (\Delta f)(a) = (f_{xx} + f_{yy})(a) = f_{xx}(a) + f_{yy}(a) &= (u_{xx}(a) + iv_{xx}(a)) + (u_{yy}(a) + iv_{yy}(a)) \\ &= (u_{xx}(a) + u_{yy}(a)) + i(v_{xx}(a) + v_{yy}(a)), \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= (4\partial(\bar{\partial}f))(a) = 4\left(\partial\left(\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f)\right)\right)(a) = 2(\partial((u_x + iv_x) + i(u_y + iv_y)))(a) \\ &= 2(\partial((u_x - v_y) + i(v_x + u_y)))(a) = 2\left(\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)((u_x - v_y) + i(v_x + u_y))\right)(a) \\ &= ((u_{xx}(a) - v_{yx}(a)) + i(v_{xx}(a) + u_{yx}(a))) - i((u_{xy}(a) - v_{yy}(a)) + i(v_{xy}(a) + u_{yy}(a))) \\ &= ((u_{xx}(a) - v_{xy}(a)) + i(v_{xx}(a) + u_{xy}(a))) - i((u_{xy}(a) - v_{yy}(a)) + i(v_{xy}(a) + u_{yy}(a))) \\ &= (u_{xx}(a) - v_{xy}(a)) + i(v_{xx}(a) + u_{xy}(a)) - i(u_{xy}(a) - v_{yy}(a)) + (v_{xy}(a) + u_{yy}(a)) \\ &= (u_{xx}(a) + u_{yy}(a)) + i(v_{xx}(a) + v_{yy}(a)), \end{aligned}$$

so LHS = RHS. Similarly, $\Delta f = 4\bar{\partial}(\partial f)$. ∎)

Since $f \in H(\Omega)$, by Conclusion 1.239, $\bar{\partial} f = 0$, and hence $\Delta f = \underbrace{4\partial\left(\bar{\partial} f\right) = 0}$.

Thus, $\Delta f = 0$.

Conclusion 1.239 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. Let $f \in H(\Omega)$. Then $f : \Omega \to \mathbb{C}$ is continuous, and $\Delta f = 0$.

Definition Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a function. If $f : \Omega \to \mathbb{C}$ is continuous, and $\Delta f = 0$, then we say that *f is* ***harmonic*** *in* Ω.

Now, the above conclusion can be stated as:

$\boxed{\text{every holomorphic function is harmonic.}}$

Further, it is clear that the linear combination of harmonic functions is a harmonic function. Also, f is harmonic in Ω if and only if $\mathrm{Re}(f)$ is harmonic in Ω, and $\mathrm{Im}(f)$is harmonic in Ω.

1.20 Dual Space of a Banach Space

Note 1.240 Let X be a normed linear space. Let X^* be the collection of all bounded linear functional on X. Under pointwise vector addition and scalar multiplication, X^* is a linear space. For every $f \in X^*$, we define

$$\|f\| \equiv \sup\{|f(x)| : x \in X, \text{and} \|x\| \leq 1\}.$$

Problem 1.241 Under this norm, X^* is a normed linear space.

(Solution We must prove

1. for every $f \in X^*$, $\|f\| \geq 0$,
2. $\|0\| = 0$; if $\|f\| = 0$, then $f = 0$,
3. for every $f \in X^*$, and for every $\alpha \in \mathbb{C}$, $\|\alpha f\| = |\alpha|\|f\|$,
4. for every $f, g \in X^*$, $\|f + g\| \leq \|f\| + \|g\|$.

For 1: This is clear.

For 2: $\|0\| = 0$ is clear. Suppose that $(\sup\{|f(x)| : x \in X, \text{and } \|x\| \leq 1\} =)$ $\|f\| = 0$. It follows that for every $x \in X$ satisfying $\|x\| \leq 1$, $|f(x)| = 0$, that is for every $x \in X$ satisfying $\|x\| \leq 1$, $f(x) = 0$. Now, let us take any $y \in X$ satisfying $1 < \|y\|$. We have to show that $f(y) = 0$. Here, $\left\|\frac{1}{\|y\|} y\right\| = 1$, so

$$\left(\frac{1}{\|y\|} f(y) = \right) f\left(\frac{1}{\|y\|} y\right) = 0,$$

and hence $f(y) = 0$.

For 3: Let us take any $f \in X^*$, and a nonzero complex number α. We have to show that $\|\alpha f\| = |\alpha|\|f\|$.

$$\begin{aligned}\text{LHS} &= \|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in X, \text{ and } \|x\| \le 1\}\\ &= \sup\{|\alpha(f(x))| : x \in X, \text{ and } \|x\| \le 1\}\\ &= \sup\{|\alpha||f(x)| : x \in X, \text{ and } \|x\| \le 1\}\\ &= |\alpha|(\sup\{|f(x)| : x \in X, \text{ and } \|x\| \le 1\}) = |\alpha|\|f\| = \text{RHS}.\end{aligned}$$

For 4: Let us take any $f, g \in X^*$. We have to show that $\|f+g\| \le \|f\| + \|g\|$. Here,

$$\begin{aligned}\|f+g\| &= \sup\{|(f+g)(x)| : x \in X, \text{ and } \|x\| \le 1\}\\ &= \sup\{|f(x)+g(x)| : x \in X, \text{ and } \|x\| \le 1\}\sup\{|f(x)|+|g(x)| : x \in X, \text{ and } \|x\| \le 1\}\\ &\le \sup\{|f(x)| : x \in X, \text{ and } \|x\| \le 1\}\\ &\quad + \sup\{|g(x)| : x \in X, \text{ and } \|x\| \le 1\} = \|f\| + \|g\|,\end{aligned}$$

so

$$\|f+g\| \le \|f\| + \|g\|.$$

∎)

Problem 1.242 X^* is a Banach space.

(Solution For this purpose, let us take any sequence $\{f_n\}$ in X^*. Let $\{f_n\}$ be Cauchy. We have to show that $\{f_n\}$ is convergent.

Problem 1.243 For every $x \in X$, $\{f_n(x)\}$ is convergent.

(Solution Let us fix any nonzero a in X. We have to show that $\{f_n(a)\}$ is convergent. It suffices to show that $\{f_n(a)\}$ is Cauchy. For this purpose, let us take any $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, there exists a positive integer N such that

$$\begin{aligned}\underbrace{m,n \ge N \Rightarrow \frac{\varepsilon}{\|a\|} > \|f_n - f_m\|} &= \sup\{|(f_n - f_m)(x)| : x \in X, \text{ and } \|x\| \le 1\}\\ &= \sup\{|f_n(x) - f_m(x)| : x \in X, \text{ and } \|x\| \le 1\}\\ &\ge \left|f_n\left(\frac{1}{\|a\|}a\right) - f_m\left(\frac{1}{\|a\|}a\right)\right| = \frac{1}{\|a\|}|f_n(a) - f_m(a)|,\end{aligned}$$

and hence

$$m,n \ge N \Rightarrow |f_n(a) - f_m(a)| < \varepsilon.$$

This shows that $\{f_n(a)\}$ is Cauchy. ∎)

For every $x \in X$, $\{f_n(x)\}$ is convergent, so for every $x \in X$, there exists a complex number $f(x)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$.

Problem 1.244 $f : X \to \mathbb{C}$ is linear.

(Solution Let $x, y \in X$, and $\alpha, \beta \in \mathbb{C}$. We have to show that $f(\alpha x + \beta y) = \alpha(f(x)) + \beta(f(y))$.

$$\begin{aligned}\text{LHS} &= f(\alpha x + \beta y) = \lim_{n\to\infty} f_n(\alpha x + \beta y) = \lim_{n\to\infty} (\alpha(f_n(x)) + \beta(f_n(y))) \\ &= \alpha\Big(\lim_{n\to\infty} f_n(x)\Big) + \beta\Big(\lim_{n\to\infty} f_n(y)\Big) = \alpha(f(x)) + \beta(f(y)) = \text{RHS}.\end{aligned}$$

∎)

Problem 1.245 $f \in X^*$.

(Solution Since $\{f_n\}$ is Cauchy in X^*, there exists a positive real number M such that for every $n \in N$, $\|f_n\| \le M$. Let us take any $x \in X$ satisfying $\|x\| \le 1$. It suffices to show that

$$\left(\lim_{n\to\infty} |f_n(x)| = \left|\lim_{n\to\infty} f_n(x)\right| = \right) |f(x)| \le M,$$

that is $\lim_{n\to\infty} |f_n(x)| \le M$. Since

$$\lim_{n\to\infty} |f_n(x)| \le \lim_{n\to\infty} \|f_n\| \|x\| \le \lim_{n\to\infty} \|f_n\| \cdot 1 \le M,$$

we have

$$\lim_{n\to\infty} |f_n(x)| \le M.$$

∎)

It suffices to show that $\lim_{n\to\infty} \|f_n - f\| = 0$.

For this purpose, let us take any $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, there exists a positive integer N such that for every $m, n \ge N$, $\|f_n - f_m\| < \varepsilon$. Here, for every $m, n \ge N$,

$$\begin{aligned}&\sup\{|f_n(x) - f_m(x)| : x \in X, \text{ and } \|x\| \le 1\} \\ &\quad = \sup\{|(f_n - f_m)(x)| : x \in X, \text{ and } \|x\| \le 1\} = \underbrace{\|f_n - f_m\| < \varepsilon}_{},\end{aligned}$$

so for every $x \in X$ satisfying $\|x\| \le 1$, and for every $m, n \ge N$, $|f_n(x) - f_m(x)| < \varepsilon$. Hence, for every $x \in X$ satisfying $\|x\| \le 1$, and for every $n \ge N$,

$$
\begin{aligned}
|(f_n - f)(x)| &= |f_n(x) - f(x)| = \left| f_n(x) - \lim_{m\to\infty} f_m(x) \right| \\
&= \left| \lim_{m\to\infty} (f_n(x) - f_m(x)) \right| = \underbrace{\lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon}.
\end{aligned}
$$

Thus, for every $x \in X$ satisfying $\|x\| \leq 1$, and for every $n \geq N$, $|(f_n - f)(x)| \leq \varepsilon$. It follows that for every $n \geq N$,

$$\|f_n - f\| = \sup\{|(f_n - f)(x)| : x \in X, \text{and } \|x\| \leq 1\} \leq \varepsilon.$$

Thus,

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

∎)

Conclusion 1.246 Let X be a normed linear space. Let X^* be the collection of all bounded linear functionals on X. Then X^* is a Banach space. Also, for every $a \in X$,

$$\sup\{|f(a)| : f \in X^*, \text{ and } \|f\| = 1\} = \|a\|.$$

(**Definition** Here, the Banach space X^* is called the ***dual space*** *of* X.)

Proof of the remaining part Let us fix any nonzero a in X. Next, let us take any $f \in X^*$ satisfying $\|f\| = 1$. Since $|f(a)| \leq \|f\|\|a\| = 1 \cdot \|a\| = \|a\|$, we have $\sup\{|f(a)| : f \in X^*, \text{ and } \|f\| = 1\} \leq a$. It suffices to show that

$$\|a\| \in \{|f(a)| : f \in X^*, \text{ and } \|f\| = 1\}.$$

By Conclusion 2.221, Vol. 1, there exists $g \in X^*$ such that $\|g\| = 1$, and $g(a) = \|a\|$. It follows that

$$(\|a\| =)|g(a)| \in \{|f(a)| : f \in X^*, \text{and} \|f\| = 1\}.$$

Thus,

$$\|a\| \in \{|f(a)| : f \in X^*, \text{and} \|f\| = 1\}.$$

∎)

Note 1.247 Let K be a topological space. Let K be Hausdorff and compact. Let H be a nonempty compact subset of K. Let $C(K)$ be the collection of all continuous functions $f : K \to \mathbb{C}$.

Problem 1.248 $C(K) = C_c(K)$.

(Solution By the definition of $C_c(K)$, we have $C_c(K) \subset C(K)$. It remains to show that $C(K) \subset C_c(K)$. For this purpose, let us take any $f \in C(K)$. We have to show that $f \in C_c(K)$. It suffices to show that

$$\underbrace{\left(f^{-1}(\mathbb{C} - \{0\})\right)^{-} = \text{supp}(f)}$$

is compact. Since supp(f) is a closed subset of the compact space K, supp(f) is compact. ∎)

Since $C_c(K)$ is a complex linear space, and $C(K) = C_c(K)$, $C(K)$ is a complex linear space. For every $f \in C(K)$, $f(K)$ is a nonempty bounded subset of $\mathbb{C}$. It follows that for every $f \in C(K)$, $\sup\{|f(x)| : x \in K\}$ is a nonnegative real number. For every $f \in C(K)$, we define

$$\|f\| \equiv \sup\{|f(x)| : x \in K\}.$$

Problem 1.249 Under this norm, $C(K)$ is a normed linear space.

(Solution We must prove

1. for every $f \in C(K)$, $\|f\| \geq 0$,
2. $\|0\| = 0$; if $\|f\| = 0$, then $f = 0$,
3. for every $f \in C(K)$, and for every $\alpha \in \mathbb{C}$, $\|\alpha f\| = |\alpha|\|f\|$,
4. for every $f, g \in C(K)$, $\|f + g\| \leq \|f\| + \|g\|$.

For 1: This is clear.

For 2: $\|0\| = 0$ is clear. Suppose that $(\sup\{|f(x)| : x \in K\} =)\|f\| = 0$. It follows that, for every $x \in K$, $|f(x)| = 0$, that is for every $x \in K$, $f(x) = 0$. Thus, $f = 0$.

For 3: Let us take any $f \in C(K)$, and a nonzero complex number α. We have to show that $\|\alpha f\| = |\alpha|\|f\|$.

$$\begin{aligned}\text{LHS} = \|\alpha f\| &= \sup\{|(\alpha f)(x)| : x \in K\} = \sup\{|\alpha(f(x))| : x \in K\} \\ &= \sup\{|\alpha||f(x)| : x \in K\} = |\alpha|(\sup\{|f(x)| : x \in K\}) = |\alpha|\|f\| = \text{RHS}.\end{aligned}$$

For 4: Let us take any $f, g \in C(K)$. We have to show that $\|f + g\| \leq \|f\| + \|g\|$. Here,

$$\begin{gathered}\|f + g\| = \sup\{|(f + g)(x)| : x \in K\} = \sup\{|f(x) + g(x)| : x \in K\} \\ \leq \sup\{|f(x)| + |g(x)| : x \in K\} \leq \sup\{|f(x)| : x \in K\} + \sup\{|g(x)| : x \in K\} = \|f\| + \|g\|,\end{gathered}$$

so

$$\|f+g\| \le \|f\| + \|g\|.$$

∎)

Problem 1.250 $C(K)$ is a Banach space.

(Solution For this purpose, let us take any sequence $\{f_n\}$ in $C(K)$. Let $\{f_n\}$ be Cauchy. We have to show that $\{f_n\}$ is convergent.

Problem 1.251 For every $x \in K$, $\{f_n(x)\}$ is convergent.

(Solution Let us fix any a in K. We have to show that $\{f_n(a)\}$ is convergent. It suffices to show that $\{f_n(a)\}$ is Cauchy. For this purpose, let us take any $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, there exists a positive integer N such that

$$\begin{aligned}\underbrace{m,n \ge N \Rightarrow \varepsilon > \|f_n - f_m\|} &= \sup\{|(f_n - f_m)(x)| : x \in K\} \\ &= \sup\{|f_n(x) - f_m(x)| : x \in K\} \ge |f_n(a) - f_m(a)|,\end{aligned}$$

and hence

$$m,n \ge N \Rightarrow |f_n(a) - f_m(a)| < \varepsilon.$$

This shows that $\{f_n(a)\}$ is Cauchy. ∎)

It follows that for every $x \in K$, there exists a complex number $f(x)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$. Thus,

$$f : K \to \mathbb{C}.$$

Problem 1.252 $f \in C(K)$.

(Solution Let us take any $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy in $C(K)$, there exists a positive integer N such that for every $m,n \ge N$,

$$\{|f_n(x) - f_m(x)| : x \in K\} = \{|(f_n - f_m)(x)| : x \in K\} = \underbrace{\|f_n - f_m\| < \varepsilon}.$$

Thus, for every $x \in K$, and for every $m,n \ge N$, $|f_n(x) - f_m(x)| < \varepsilon$. It follows that $\{f_n\}$ converges uniformly on K. Now, since $\{f_n\}$ is a sequence of continuous functions on K, and for every $x \in K$, $\lim_{n\to\infty} f_n(x) = f(x)$, $f : K \to \mathbb{C}$ is continuous, we have $f \in C(K)$. ∎)

It suffices to show that $\lim_{n\to\infty} \|f_n - f\| = 0$.

For this purpose, let us take any $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, there exists a positive integer N such that for every

$$m, n \geq N, \ \|f_n - f_m\| < \varepsilon.$$

Here, for every $m, n \geq N$,

$$\sup\{|f_n(x) - f_m(x)| : x \in K\} = \sup\{|(f_n - f_m)(x)| : x \in K\} = \underbrace{\|f_n - f_m\| < \varepsilon},$$

so for every $x \in K$ and for every $m, n \geq N$,

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Hence, for every $x \in K$ and for every $n \geq N$,

$$\begin{aligned} |(f_n - f)(x)| &= |f_n(x) - f(x)| = \left| f_n(x) - \lim_{m \to \infty} f_m(x) \right| \\ &= |\lim_{m \to \infty}(f_n(x) - f_m(x))| = \underbrace{\lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \varepsilon}. \end{aligned}$$

Thus, for every $x \in K$ and for every $n \geq N$, $|(f_n - f)(x)| \leq \varepsilon$. It follows that for every $n \geq N$,

$$\|f_n - f\| = \underbrace{\sup\{|(f_n - f)(x)| : x \in K\} \leq \varepsilon}.$$

Thus,

$$\lim_{n \to \infty} \|f_n - f\| = 0.$$

∎)

Let A be a linear subspace of $C(K)$. Clearly, the constant function 1 on K is a member of $C(K)$. Suppose that $1 \in A$. Suppose that for every $f \in A$,

$$\sup\{|f(x)| : x \in K\} = \sup\{|f(x)| : x \in H\}.$$

In short, we shall write: For every $f \in A$, $\|f\|_K = \|f\|_H$.

Problem 1.253 $\{h|_H : h \in A\}$ is a linear subspace of $C(H)$.

(Solution For this purpose, let us take any $f, g \in A$, and $\alpha, \beta \in \mathbb{C}$. We have to show that $(\alpha(f|_H) + \beta(g|_H)) \in \{h|_H : h \in A\}$. Since $f, g \in A$, $\alpha, \beta \in \mathbb{C}$, and A is a linear subspace of $C(K)$, we have $(\alpha f + \beta g) \in A$ and hence

$$\alpha(f|_H) + \beta(g|_H) = \underbrace{(\alpha f + \beta g)|_H \in \{h|_H : h \in A\}}.$$

Thus,

$$\left(\alpha\left(f|_H\right)+\beta\left(g|_H\right)\right)\in\left\{h|_H\colon h\in A\right\}.$$

∎)

Problem 1.254 Each member of $\left\{h|_H\colon h\in A\right\}$ has a unique extension to a member of A.

(Solution Existence: Let $f\in\left\{h|_H\colon h\in A\right\}$. So there exists $h\in A$ such that $h|_H=f$. Thus, f has an extension h, where $h\in A$.

Uniqueness: Let $f\in\left\{k|_H\colon k\in A\right\}$. Let $g\in A$ such that $g|_H=f$. Let $h\in A$ such that $h|_H=f$. We have to show that $g=h$.

Since $g,h\in A$, and A is a linear subspace of $C(K)$, we have $(g-h)\in A$, and hence

$$0=(f-f)=\left(g|_H-h|_H\right)=\underbrace{(g-h)|_H\in\left\{k|_H\colon k\in A\right\}}.$$

Since $(g-h)\in A$, by the assumption,

$$\begin{aligned}\|g-h\|_K&=\underbrace{\sup\{|(g-h)(x)|:x\in K\}=\sup\{|(g-h)(x)|:x\in H\}}\\&=\sup\{|0|:x\in H\}=0,\end{aligned}$$

and hence $\|g-h\|_K=0$. This shows that $g=h$. ∎)

Let us denote $\left\{h|_H\colon h\in A\right\}$ by M.

Clearly, $h\mapsto h|_H$ is a mapping from A to M. Since each member of M has a unique extension to a member of A, the mapping $h\mapsto h|_H$ from A to M is 1–1, onto. Since for every $f\in A$, $\|f\|_K=\|f\|_H$, the mapping $h\mapsto h|_H$ from A to M is norm preserving.

In light of the above discussion, we shall not distinguish between $h|_H$ and h, where $h\in A$.

Problem 1.255 For every $a\in K$, the function

$$F_a:f\mapsto f(a)$$

from M to $\mathbb{C}$ is a bounded linear functional on M of norm 1.

(Solution Linearity: Let $f,g\in M$, and $\alpha,\beta\in\mathbb{C}$. We have to show that $(\alpha f+\beta g)(a)=\alpha(f(a))+\beta(g(a))$. This is clearly true.

Boundedness: We have to show that $\left\{|f(a)|:f\in M,\text{ and }\|f\|_H\le 1\right\}$ is bounded.

For this purpose, let us take any $f \in M$ satisfying

$$|f(a)| \leq \sup\{|f(x)| : x \in K\} = \|f\|_K = \underbrace{\|f\|_H \leq 1}.$$

It follows that 1 is an upper bound of $\{|f(a)| : f \in M, \text{ and } \|f\|_H \leq 1\}$. Thus, $\{|f(a)| : f \in M, \text{ and } \|f\|_H \leq 1\}$ is bounded. Also,

$$1 = |1| = |1(a)| = |F_a(1)| \leq \sup\{|F_a(f)| : f \in M, \text{ and } \|f\|_H \leq 1\} = \underbrace{\|F_a\| \leq 1},$$

so $\|F_a\| = 1$. ∎)

Since M is a linear subspace of the Banach space $C(H)$, and for every $a \in K$, $F_a : M \to \mathbb{C}$ is a bounded linear functional on M satisfying $\|F_a\| = 1$, by Theorem 2.214, for every $a \in K$, there exists a function $\Lambda_a : C(H) \to \mathbb{C}$ such that

1. $\Lambda_a : C(H) \to \mathbb{C}$ is an extension of $F_a : M \to \mathbb{C}$, (that is, for every $f \in M$, $\Lambda_a(f) = F_a(f) = f(a)$.)
2. $\Lambda_a : C(H) \to \mathbb{C}$ is a bounded real-linear functional,
3. $\|\Lambda_a\| = \|F_a\|(= 1)$.

Problem 1.256 For every $a \in K$, the linear functional $\Lambda_a : C(H) \to \mathbb{C}$ is positive.

(Solution Let us take any $a \in K$. Next, let us take any nonzero $f \in C(H)$ satisfying $f \geq 0$. It follows that $\|f\|_H > 0$. We have to show that $\Lambda_a(f) \geq 0$.

Since $f \in C(H)$, and $\|f\|_H > 0$, we have

$$\left(\frac{1}{\|f\|_H}\right) f \in C(H), \text{ and } \left\{\sup\left|\left(\frac{1}{\|f\|_H}\right) f(x)\right| : x \in H\right\} = \underbrace{\left\|\left(\frac{1}{\|f\|_H}\right) f\right\|_H = 1}.$$

From 1, $\Lambda_a(1) = F_a(1) = 1(a) = 1$, so $\Lambda_a(1) = 1$. From 3, $\|\Lambda_a\| = 1$.

Put $g \equiv \left(\frac{1}{\|f\|_H}\right) f$. Thus, $g \in C(H)$ and $0 \leq g \leq 1$. It follows that $-1 \leq 2g - 1 \leq 1$, and hence for every $x \in K$, $|(2g-1)(x)| \leq 1$. This shows that $\|2g - 1\|_H \leq 1$. It suffices to show that $\Lambda_a(g)$ is a nonnegative real number. For every real r,

$$\begin{aligned}
&(\mathrm{Im}(\Lambda_a(2g-1)))^2+2(\mathrm{Im}(\Lambda_a(2g-1)))r+r^2=(\mathrm{Im}(\Lambda_a(2g-1))+r)^2\\
&\leq|\mathrm{Re}(\Lambda_a(2g-1))+i(\mathrm{Im}(\Lambda_a(2g-1))+r)|^2=|\Lambda_a(2g-1)+ir\cdot 1|^2\\
&=|\Lambda_a(2g-1)+ir\cdot(\Lambda_a(1))|^2=|\Lambda_a((2g-1)+(ir)1)|^2\leq\left(\|\Lambda_a\|\|(2g-1)+(ir)1\|_H\right)^2\\
&=\left(1\|(2g-1)+(ir)1\|_H\right)^2=\left(\|(2g-1)+ir1\|_H\right)^2=(\sup\{|(2g-1)(x)+ir|:x\in H\})^2\\
&=\sup\left\{|(2g-1)(x)+ir|^2:x\in H\right\}=\sup\left\{|(2g-1)(x)|^2+r^2:x\in H\right\}\\
&=\left(\sup\left\{|(2g-1)(x)|^2:x\in H\right\}\right)+r^2=(\sup\{|(2g-1)(x)|:x\in H\})^2+r^2\\
&=\left(\|2g-1\|_H\right)^2+r^2\leq(1)^2+r^2=1+r^2
\end{aligned}$$

so for every real r,

$$(\mathrm{Im}(\Lambda_a(2g-1)))^2+2(\mathrm{Im}(\Lambda_a(2g-1)))r\leq 1.$$

Problem 1.257 $\mathrm{Im}(\Lambda_a(2g-1))=0$.

(Solution If not, otherwise, let $\mathrm{Im}(\Lambda_a(2g-1))\neq 0$. We have to arrive at a contradiction.

Case I: when $\mathrm{Im}(\Lambda_a(2g-1))$ is positive. Now, since for every real r,

$$(\mathrm{Im}(\Lambda_a(2g-1)))^2+2(\mathrm{Im}(\Lambda_a(2g-1)))r\leq 1,$$

we have

$$\mathbb{R}\subset\left[\frac{1-(\mathrm{Im}(\Lambda_a(2g-1)))^2}{2(\mathrm{Im}(\Lambda_a(2g-1)))},\infty\right).$$

This is a contradiction.

Case II: when $\mathrm{Im}(\Lambda_a(2g-1))$ is negative. Now, since for every real r,

$$(\mathrm{Im}(\Lambda_a(2g-1)))^2+2(\mathrm{Im}(\Lambda_a(2g-1)))r\leq 1,$$

we have

$$\mathbb{R}\subset\left(-\infty,\frac{1-(\mathrm{Im}(\Lambda_a(2g-1)))^2}{2(\mathrm{Im}(\Lambda_a(2g-1)))}\right].$$

This is a contradiction. ∎)

Thus, $\Lambda_a(2g-1)$ is a real number. Since

$$\Lambda_a(g) = \Lambda_a\left(\frac{1}{2}(2g-1) + \frac{1}{2}1\right) = \frac{1}{2}\Lambda_a(2g-1) + \frac{1}{2}\Lambda_a(1)$$
$$= \frac{1}{2}\Lambda_a(2g-1) + \frac{1}{2}\cdot 1 = \frac{1}{2}(\Lambda_a(2g-1)+1),$$

we have $\Lambda_a(g) = \frac{1}{2}(\Lambda_a(2g-1)+1)$. It suffices to show that $-1 \leq \Lambda_a(2g-1)$. Since for every real r,

$$|\Lambda_a(2g-1) + ir\cdot 1|^2 \leq 1 + r^2,$$

we have

$$|\Lambda_a(2g-1) + i0\cdot 1|^2 \leq 1 + 0^2,$$

so

$$-1 \leq \Lambda_a(2g-1).$$

∎)

By Lemma 1.231, Vol. 1, for every $a \in K$, there exists a σ-algebra $\mathcal{M}$ in H that contains all Borel sets in H, and there exists a regular, positive, Borel measure μ_a on $\mathcal{M}$ such that for every $f \in C(H)$,

$$\Lambda_a(f) = \int\limits_H f \mathrm{d}\mu_a.$$

Next, for every $f \in A$,

$$f(a) = \Lambda_a(f) = \int\limits_H f \mathrm{d}\mu_a.$$

Conclusion 1.258 Let K be a Hausdorff, compact space. Let H be a nonempty compact subset of K Let A be a linear subspace of $C(K)$. Suppose that $1 \in A$, and for every $f \in A$, $\|f\|_K = \|f\|_H$.

Then for every $a \in K$, there exists a σ-algebra $\mathcal{M}$ in H that contains all Borel sets in H, and there exists a regular, positive, Borel measure μ_a on $\mathcal{M}$ such that, for every $f \in A$,

$$f(a) = \int\limits_H f \mathrm{d}\mu_a.$$

1.21 Poisson's Integral

Note 1.259 Let $a_0, \ldots, a_N$ be any complex numbers. Let $f : z \mapsto (a_0 + \cdots + a_N z^N)$ be any polynomial function.

Problem 1.260 $\sup\{|f(z)| : z \in D[0; 1]\} = \sup\{|f(z)| : z \in D(0; 1)\}$.

(Solution Since $D(0; 1) \subset D[0; 1]$, we have $\sup\{|f(z)| : z \in D(0; 1)\} \leq \sup\{|f(z)| : z \in D[0; 1]\}$. It remains to show that $\sup\{|f(z)| : z \in D[0; 1]\} \leq \sup\{|f(z)| : z \in D(0; 1)\}$.

If not, otherwise suppose that $\sup\{|f(z)| : z \in D(0; 1)\} < \sup\{|f(z)| : z \in D[0; 1]\}$. We have to arrive at a contradiction.

There exists a real number r such that $\sup\{|f(z)| : z \in D(0; 1)\} < r < \sup\{|f(z)| : z \in D[0; 1]\}$. It follows that there exists $z_0 \in \mathbb{C}$ such that $|z_0| = 1$, and $r < |f(z_0)|$. Since $|z_0| = 1$, there exists a sequence $\{a_n\}$ in $D(0; 1)$ such that $\lim_{n\to\infty} a_n = z_0$. Now, since $|f|$ is a continuous function, $\lim_{n\to\infty}|f(a_n)| = |f(z_0)|$. Since $\{a_n\}$ is a sequence in $D(0; 1)$, $\{|f(a_n)|\}$ is a sequence in $\{|f(z)| : z \in D(0; 1)\}(\subset [0, r))$, and hence $(|f(z_0)| =) \lim_{n\to\infty}|f(a_n)| \leq r$. Thus, $|f(z_0)| \leq r$. This is a contradiction. ∎)

Problem 1.261 $\sup\{|f(z)| : z \in D[0; 1]\} = \sup\{|f(z)| : z \in \{w : |w| = 1\}\}$.

(Solution Since $\{w : |w| = 1\} \subset D[0; 1]$, we have $\sup\{|f(z)| : z \in \{w : |w| = 1\}\} \leq \sup\{|f(z)| : z \in D[0; 1]\}$. It suffices to show that, for nonconstant polynomial f,

$$\sup\{|f(z)| : z \in D[0; 1]\} \leq \sup\{|f(z)| : z \in \{w : |w| = 1\}\}.$$

Suppose that f is a nonconstant polynomial.

Since $D[0; 1]$ is compact, and $|f|$ is continuous, there exists $z_0 \in D[0; 1]$ such that for every $z \in D[0; 1]$, $|f(z)| \leq |f(z_0)|$. Hence, $\sup\{|f(z)| : z \in D[0; 1]\} \leq |f(z_0)|$.

It suffices to show that $|z_0| = 1$.

If not, otherwise let $z_0 \in D(0; 1)$. We have to arrive at a contradiction.

It follows that $|z_0| \in [0, 1)$, and hence $(1 - |z_0|) \in (0, 1]$. There exist complex numbers $b_0, \ldots, b_N$ such that for every $z \in \mathbb{C}$,

$$f(z) = \underbrace{b_0 + b_1(z - z_0)^1 + \cdots}_{(N+1)\text{terms}}.$$

Now, for every $r \in (0, 1 - |z_0|)$, we have $\left\{z_0 + re^{i\theta} : \theta \in [-\pi, \pi]\right\} \subset D(0; 1)$, and

$$
\begin{aligned}
|b_0|^2 2\pi &= \Bigg|\underbrace{b_0 + b_1(z_0 - z_0)^1 + \cdots}_{(N+1)\text{terms}}\Bigg|^2 2\pi = |f(z_0)|^2 2\pi = \int_{-\pi}^{\pi} |f(z_0)|^2 \mathrm{d}\theta \\
&\geq \int_{-\pi}^{\pi} \left|f\left(z_0 + re^{i\theta}\right)\right|^2 \mathrm{d}\theta = \int_{-\pi}^{\pi} \Bigg|\underbrace{b_0 + b_1\left(\left(z_0 + re^{i\theta}\right) - z_0\right)^1 + \cdots}_{(N+1)\text{terms}}\Bigg|^2 \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \Bigg|\underbrace{b_0 + b_1\left(re^{i\theta}\right)^1 + \cdots}_{(N+1)\text{terms}}\Bigg|^2 \mathrm{d}\theta = \int_{-\pi}^{\pi} \Bigg(\underbrace{b_0 + b_1\left(re^{i\theta}\right)^1 + \cdots}_{(N+1)\text{terms}}\Bigg)\Bigg(\underbrace{\overline{b_0} + \overline{b_1}\left(re^{-i\theta}\right)^1 + \cdots}_{(N+1)\text{terms}}\Bigg) \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \Bigg(\Bigg(\underbrace{|b_0|^2 + |b_1|^2 r^2 + \cdots}_{(N+1)\text{terms}}\Bigg) + \left(b_0\overline{b_1}r\left(e^{-i\theta}\right)^1 + \cdots\right)\Bigg) \mathrm{d}\theta \\
&= \Bigg(\underbrace{|b_0|^2 + |b_1|^2 r^2 + \cdots}_{(N+1)\text{terms}}\Bigg) 2\pi + \left(b_0\overline{b_1}r \int_{-\pi}^{\pi} \left(e^{-i\theta}\right)^1 \mathrm{d}\theta + \cdots\right) \\
&= \Bigg(\underbrace{|b_0|^2 + |b_1|^2 r^2 + \cdots}_{(N+1)\text{terms}}\Bigg) 2\pi + \left(b_0\overline{b_1}r \cdot 0 + \cdots\right) = \Bigg(\underbrace{|b_0|^2 + |b_1|^2 r^2 + \cdots}_{(N+1)\text{terms}}\Bigg) 2\pi.
\end{aligned}
$$

Hence for every $r \in (0, 1 - |z_0|)$, we have

$$0 \geq \underbrace{|b_1|^2 + |b_2|^2 r^2 + \cdots}_{N\text{ terms}}\left(\geq |b_1|^2 \geq 0\right).$$

This shows that $|b_1|^2 = 0$ and hence $b_1 = 0$. Similarly, $b_2 = 0$, etc. Thus, for every $z \in \mathbb{C}$, $f(z) = b_0$, and hence f is a constant polynomial. This is a contradiction. ∎)

Conclusion 1.262 Let $a_0, \ldots, a_N$ be any complex numbers. Let $f : z \mapsto (a_0 + \cdots + a_N z^N)$ be any polynomial function. Then

$$\sup\{|f(z)| : z \in D(0;1)\} = \sup\{|f(z)| : z \in \{w : |w| = 1\}\}.$$

Note 1.263 Let A be a linear subspace of $C(D[0;1])$. Suppose that A contains all polynomials and, for every $f \in A$, $\|f\|_{D(0;1)} = \|f\|_{\{z:|z|=1\}}$.

Clearly, for every $f \in A$,

$$\sup\{|f(z)| : z \in D[0;1]\} = \sup\{|f(z)| : z \in D(0;1)\},$$

so for every $f \in A$,

$$\|f\|_{D[0;1]} = \|f\|_{\{z:|z|=1\}}.$$

Now, by Conclusion 1.258, for every $a \in D[0;1]$, there exists a σ-algebra $\mathcal{M}$ in $\{z : |z| = 1\}$ that contains all Borel sets in $\{z : |z| = 1\}$, and there exists a regular, positive, Borel measure μ_a on $\mathcal{M}$ such that for every $f \in A$,

$$f(a) = \int\limits_{\{z:|z|=1\}} f \, \mathrm{d}\mu_a.$$

Let $r \in [0,1)$, and let θ be any real number.

It follows that $re^{i\theta} \in D[0;1]$. For every integer n, let $u_n : w \mapsto w^n$ be a function. Hence, for every nonnegative integer n, $u_n \in A$. Thus, there exists a regular, positive, Borel measure $\mu_{re^{i\theta}}$ such that for every nonnegative integer n,

$$r^n e^{i\theta n} = \left(re^{i\theta}\right)^n = \underbrace{u_n\left(re^{i\theta}\right) = \int\limits_{\{z:|z|=1\}} u_n \mathrm{d}\mu_{re^{i\theta}}}.$$

Observe that for every $w \in \{z : |z| = 1\}$, we have

$$\overline{u_n}(w) = \overline{u_n(w)} = \overline{w^n} = (\overline{w})^n = \left(\frac{|w|^2}{w}\right)^n = \left(\frac{1^2}{w}\right)^n = w^{-n} = u_{-n}(w),$$

so, $\overline{u_n} = u_{-n}$ on $\{z : |z| = 1\}$. Since

$$r^1 e^{i\theta 1} = \int\limits_{\{z:|z|=1\}} u_1 \mathrm{d}\mu_{re^{i\theta}},$$

we have

$$r^{|-1|} e^{i\theta(-1)} = r^1 e^{i\theta(-1)} = \underbrace{\overline{r^1 e^{i\theta 1}} = \int\limits_{\{z:|z|=1\}} \overline{u_1} \mathrm{d}\mu_{re^{i\theta}}} = \int\limits_{\{z:|z|=1\}} u_{-1} \mathrm{d}\mu_{re^{i\theta}},$$

and hence

$$\int\limits_{\{z:|z|=1\}} u_{-1} \mathrm{d}\mu_{re^{i\theta}} = r^{|-1|} e^{i\theta(-1)}.$$

Similarly,

$$\int\limits_{\{z:|z|=1\}} u_{-2}\mathrm{d}\mu_{re^{i\theta}} = r^{|-2|}e^{i\theta(-2)}, \text{etc.}$$

Thus, for every integer n,

$$\int\limits_{\{z:|z|=1\}} u_n\mathrm{d}\mu_{re^{i\theta}} = r^{|n|}e^{i\theta n}.$$

Definition For every real t, we denote

$$1+\left(r^{|1|}e^{i(\theta-t)1}+r^{|-1|}e^{i(\theta-t)(-1)}\right)+\left(r^{|2|}e^{i(\theta-t)2}+r^{|-2|}e^{i(\theta-t)(-2)}\right)+\cdots$$

by

$$P_r(\theta-t), \text{ or } \sum_{n=-\infty}^{\infty} r^{|n|}e^{i(\theta-t)n}.$$

We observe that $\sum_{n=-\infty}^{\infty} r^{|n|}e^{i(\theta-t)n}$ is dominated by the convergent series $\sum_{n=-\infty}^{\infty} r^{|n|}$, so $\sum_{n=-\infty}^{\infty} r^{|n|}e^{i(\theta-t)n}$ converges uniformly to $P_r(\theta-t)$, and hence

$$\begin{aligned}
&\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_r(\theta-t)e^{i2t}\mathrm{d}t\\
&=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(1+\left(r^{|1|}e^{i(\theta-t)1}+r^{|-1|}e^{i(\theta-t)(-1)}\right)+\left(r^{|2|}e^{i(\theta-t)2}+r^{|-2|}e^{i(\theta-t)(-2)}\right)+\cdots\right)e^{i2t}\mathrm{d}t\\
&=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(e^{i2t}+\left(r^{|1|}e^{i\theta 1}e^{it(-1+2)}+r^{|-1|}e^{i\theta(-1)}e^{it(1+2)}\right)+\left(r^{|2|}e^{i\theta 2}e^{it(-2+2)}+r^{|-2|}e^{i\theta(-2)}e^{it(2+2)}\right)+\cdots\right)\mathrm{d}t\\
&=\frac{1}{2\pi}\left(\int\limits_{-\pi}^{\pi}e^{i2t}\mathrm{d}t+\left(r^{|1|}e^{i\theta 1}\int\limits_{-\pi}^{\pi}e^{it(-1+2)}\mathrm{d}t+r^{|-1|}e^{i\theta(-1)}\int\limits_{-\pi}^{\pi}e^{it(1+2)}\mathrm{d}t\right)\right.\\
&\quad\left.+\left(r^{|2|}e^{i\theta 2}\int\limits_{-\pi}^{\pi}1\mathrm{d}t+r^{|-2|}e^{i\theta(-2)}\int\limits_{-\pi}^{\pi}e^{it(2+2)}\mathrm{d}t\right)+\cdots\right)\\
&=\frac{1}{2\pi}\left(0+\left(r^{|1|}e^{i\theta 1}0+r^{|-1|}e^{i\theta(-1)}0\right)+\left(r^{|2|}e^{i\theta 2}2\pi+r^{|-2|}e^{i\theta(-2)}0\right)+0+\cdots\right)=r^{|2|}e^{i\theta 2}.
\end{aligned}$$

Thus,

$$\frac{1}{2\pi}\pi\int_{-\pi}^{\pi} P_r(\theta - t)e^{i2t}\mathrm{d}t = r^{|2|}e^{i\theta 2}.$$

Similarly, for every integer n,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)u_n\left(e^{it}\right)\mathrm{d}t = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)e^{int}\mathrm{d}t = r^{|n|}e^{i\theta n}} = \int_{\{z:|z|=1\}} u_n \mathrm{d}\mu_{re^{i\theta}}.$$

This shows that for every trigonometric polynomial function

$$f : t \mapsto \left(c_0 + \left(c_1 u_1\left(e^{it}\right) + c_{-1}u_{-1}\left(e^{it}\right)\right) + \cdots + \left(c_N u_N\left(e^{it}\right) + c_{-N}u_{-N}\left(e^{it}\right)\right)\right),$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f\left(e^{it}\right)\mathrm{d}t = \int_{\{z:|z|=1\}} f\mathrm{d}\mu_{re^{i\theta}}.$$

Now, by Conclusion 2.156, Vol. 1, for every $g \in C(\{z : |z| = 1\})$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)g\left(e^{it}\right)\mathrm{d}t = \int_{\{z:|z|=1\}} g\mathrm{d}\mu_{re^{i\theta}}.$$

This shows that for every $h \in A(\subset C(D[0;1]))$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)h\left(e^{it}\right)\mathrm{d}t = \int_{\{z:|z|=1\}} h\mathrm{d}\mu_{re^{i\theta}}\left(= h\left(re^{i\theta}\right)\right)(*).$$

Now, since $1 \in A$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\mathrm{d}t = 1(*\prime).$$

Here,

$$\begin{aligned}
P_r(\theta-t) &= 1+\left(r^{|1|}e^{i(\theta-t)1}+r^{|-1|}e^{i(\theta-t)(-1)}\right)+\left(r^{|2|}e^{i(\theta-t)2}+r^{|-2|}e^{i(\theta-t)(-2)}\right)+\cdots \\
&= 1+\left(r^1(2\cos(\theta-t))\right)+\left(r^2(2\cos 2(\theta-t))\right)+\cdots \\
&= 1+2\left(r\cos(\theta-t)+r^2\cos 2(\theta-t)+\cdots\right) \\
&= \mathrm{Re}\left(1+2\left(\left(re^{i\theta}e^{-it}\right)^1+\left(re^{i\theta}e^{-it}\right)^2+\cdots\right)\right) \\
&= \mathrm{Re}\left(1+2\left(\frac{re^{i\theta}e^{-it}}{1-re^{i\theta}e^{-it}}\right)\right)=\mathrm{Re}\left(\frac{1+re^{i\theta}e^{-it}}{1-re^{i\theta}e^{-it}}\right) \\
&= \mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right)=\mathrm{Re}\left(\frac{\left(e^{it}+re^{i\theta}\right)\left(e^{-it}-re^{-i\theta}\right)}{\left|e^{it}-re^{i\theta}\right|^2}\right) \\
&= \mathrm{Re}\left(\frac{1-r^2+r\left(e^{i(\theta-t)}-e^{-i(\theta-t)}\right)}{(\cos t-r\cos\theta)^2+(\sin t-r\sin\theta)^2}\right) \\
&= \frac{1-r^2}{(\cos t-r\cos\theta)^2+(\sin t-r\sin\theta)^2}=\frac{1-r^2}{1+r^2-2r\cos(\theta-t)},
\end{aligned}$$

so for every real θ, t, and $r\in[0,1)$,

$$P_r(\theta-t)=\frac{1-r^2}{1-2r\,\cos(\theta-t)+r^2}\,(**).$$

Here, for every $h\in A$, for every $r\in[0,1)$, and for every real θ,

$$h\left(re^{i\theta}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\,\cos(\theta-t)+r^2}h\left(e^{it}\right)\mathrm{d}t.$$

Conclusion 1.264 Let A be a linear subspace of $C(D[0;1])$. Suppose that A contains all polynomials and, for every $f\in A$, $\|f\|_{D(0;1)}=\|f\|_{\{z:|z|=1\}}$. Then for every $f\in A$, for every $r\in[0,1)$, and for every real θ,

$$f\left(re^{i\theta}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\,\cos(\theta-t)+r^2}f\left(e^{it}\right)\mathrm{d}t.$$

Note 1.265 Since for every $r\in[0,1)$, and for every real t,

$$\begin{aligned}
&1+\left(r^{|1|}e^{it1}+r^{|-1|}e^{it(-1)}\right)+\left(r^{|2|}e^{it2}+r^{|-2|}e^{it(-2)}\right)+\cdots \\
&\quad = 1+2r\,\cos t+2r^2\cos 2t+\cdots,\ |2r^n\cos nt|\le 2r^n,
\end{aligned}$$

and $1+2r+2r^2+\cdots$ is convergent,

$$1+\left(r^{|1|}e^{it1}+r^{|-1|}e^{it(-1)}\right)+\left(r^{|2|}e^{it2}+r^{|-2|}e^{it(-2)}\right)+\cdots$$

converges uniformly on $\mathbb{R}$. It follows that

$$P:(r,t)\mapsto\left(1+\left(r^{|1|}e^{it1}+r^{|-1|}e^{it(-1)}\right)+\left(r^{|2|}e^{it2}+r^{|-2|}e^{it(-2)}\right)+\cdots\right)\left(\equiv\sum_{n=-\infty}^{\infty}r^{|n|}e^{itn}\right)$$

is a function from $[0,1)\times\mathbb{R}$ to $\mathbb{R}$.

Definition Here P is known as the **Poisson's kernel**. It has been named after S. D. Poisson (21.06.1781 – 25.04.1840).

For every $(r,t)\in[0,1)\times\mathbb{R}$, $P(r,t)$ is also denoted by $P_r(t)$. By $(**)$ of Note 1.263,

$$\begin{aligned}P_r(-t)=P_r(0-t)&=\frac{1-r^2}{1-2r\cos(0-t)+r^2}=\frac{1-r^2}{1-2r\cos(-t)+r^2}\\&=\frac{1-r^2}{1-2r\cos t+r^2},\end{aligned}$$

so

$$P_r(-t)=\frac{1-r^2}{1-2r\cos t+r^2}.$$

It follows that

$$\underbrace{P_r(t)=\frac{1-r^2}{1-2r\cos(-t)+r^2}}=\frac{1-r^2}{1-2r\cos t+r^2}=\frac{1-r^2}{(r-\cos t)^2+\sin^2 t}>0.$$

Conclusion 1.266 For every $(r,t)\in[0,1)\times\mathbb{R}$, $P_r(-t)=P_r(t)$, and $P_r(t)>0$. Further, $P:[0,1)\times\mathbb{R}\to(0,\infty)$, and for every $(r,t)\in[0,1)\times\mathbb{R}$,

$$P_r(t)=\frac{1-r^2}{1-2r\cos r\cos t+r^2}.$$

Clearly, for every $r\in[0,1)$, P_r is 2π-periodic. Let $r\in(0,1)$. Let $t,\delta\in(0,\pi]$. Let $t<\delta$.

It follows that $\cos t>\cos\delta$, and hence $-2r\cos t<-2r\cos\delta$. Thus,

$$0<1-2r\cos t+r^2<1-2r\cos r\cos\delta+r^2,$$

and hence

$$\frac{1}{1-2r\cos r\cos t+r^2}>\frac{1}{1-2r\cos\delta+r^2}.$$

It follows that

$$P_r(t) = \frac{1 - r^2}{1 - 2r \cos r \cos t + r^2} > \frac{1 - r^2}{1 - 2r \cos \delta + r^2} = P_r(\delta),$$

and hence

$$P_r(\delta) < P_r(t) = P_r(-t).$$

Conclusion 1.267 Let $r \in (0, 1)$. Let $|t|, \delta \in (0, \pi]$. Let $|t| < \delta$. Then,

$$P_r(\delta) < P_r(|t|).$$

Problem 1.268 For every $\delta \in (0, \pi]$, $\lim\limits_{r \to 1^-} P_r(\delta) = 0$.

(Solution Let us fix any $\delta \in (0, \pi]$.

$$\begin{aligned}
\text{LHS} &= \lim_{r \to 1^-} P_r(\delta) = \lim_{r \to 1^-} \frac{1 - r^2}{1 - 2r \cos \delta + r^2} \\
&= \lim_{r \to 1^-} \left(\left(\frac{1 + r}{1 - 2r \cos r \cos \delta + r^2} \right) (1 - r) \right) \\
&= \lim_{r \to 1^-} \left(\frac{1 + r}{1 - 2r \cos \delta + r^2} \right) \cdot \lim_{r \to 1^-} (1 - r) \\
&= \frac{2}{2(1 - \cos \delta)} \cdot 0 = 0 = \text{RHS}.
\end{aligned}$$

■)

Definition Let $f : \{z : |z| = 1\} \to \mathbb{C}$. Let $f \in L^1(\{z : |z| = 1\})$. The function

$$\underbrace{P[f] : re^{i\theta} \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f\left(e^{it}\right) \mathrm{d}t}= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} f\left(e^{it}\right) \mathrm{d}t$$

from $D(0; 1)$ to $\mathbb{C}$ is called the ***Poisson's integral*** *of* f.

If $f, g \in L^1(\{z : |z| = 1\})$, and $\alpha, \beta \in \mathbb{C}$, then clearly

$$P[\alpha f + \beta g] = \alpha(P[f]) + \beta(P[g]).$$

Problem 1.269 $P[f] : D(0;1) \to \mathbb{C}$ is a harmonic function in $D(0;1)$.

(Solution Since for every $re^{i\theta} \in D(0;1)$,

$$\begin{aligned}
(P[f])\left(re^{i\theta}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)f\left(e^{it}\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\left((\mathrm{Re}(f))\left(e^{it}\right)+i(\mathrm{Im}(f))\left(e^{it}\right)\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t + i\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)(\mathrm{Im}(f))\left(e^{it}\right)\mathrm{d}t \\
&= (P[\mathrm{Re}(f)])\left(re^{i\theta}\right) + i\left((P[\mathrm{Im}(f)])\left(re^{i\theta}\right)\right) \\
&= (P[\mathrm{Re}(f)] + i(P[\mathrm{Im}(f)]))\left(re^{i\theta}\right),
\end{aligned}$$

we have $P[f] = P[\mathrm{Re}(f)] + i(P[\mathrm{Im}(f)])$. It suffices to show that

1. $P[\mathrm{Re}(f)] : D(0;1) \to \mathbb{R}$ is a harmonic function in $D(0;1)$,
2. $P[\mathrm{Im}(f)] : D(0;1) \to \mathbb{R}$ is a harmonic function in $D(0;1)$.

For 1: Since for every $re^{i\theta} \in D(0;1)$,

$$\begin{aligned}
(P[\mathrm{Re}(f)])\left(re^{i\theta}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1+\left(r^{|1|}e^{i(\theta-t)1}+r^{|-1|}e^{i(\theta-t)(-1)}\right)\right. \\
&\quad \left. + \left(r^{|2|}e^{i(\theta-t)2}+r^{|-2|}e^{i(\theta-t)(-2)}\right)+\cdots\right)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1+\left(r^1(2\cos(\theta-t))\right)+\left(r^2(2\cos 2(\theta-t))\right)+\cdots\right)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathrm{Re}\left(1+2\left(\left(re^{i\theta}e^{-it}\right)^1+\left(re^{i\theta}e^{-it}\right)^2+\cdots\right)\right)\right)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathrm{Re}\left(1+2\left(\frac{re^{i\theta}e^{-it}}{1-re^{i\theta}e^{-it}}\right)\right)\right)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(\frac{1+re^{i\theta}e^{-it}}{1-re^{i\theta}e^{-it}}\right)\right)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right)\right)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}(\mathrm{Re}(f))\left(e^{it}\right)\right)\mathrm{d}t$$

$$= \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t\right)$$

$$= \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{2e^{it}}{e^{it}-re^{i\theta}}-1\right)(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t\right)$$

$$= \mathrm{Re}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{e^{it}-re^{i\theta}}e^{it}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t\right)-\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t,$$

we have

$$(P[\mathrm{Re}(f)]) : z \mapsto \left(\mathrm{Re}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{e^{it}-z}e^{it}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t-\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t\right)\right)$$

from $D(0;1)$ to $\mathbb{R}$. Since $f \in L^1(\{z : |z| = 1\})$, by Conclusion 1.66(b), the function

$$z \mapsto \int_{-\pi}^{\pi}\frac{1}{e^{it}-z}e^{it}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t$$

from $D(0;1)$ to $\mathbb{R}$ is representable by power series of z in $D(0;1)$, and hence

$$z \mapsto \left(\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{e^{it}-z}e^{it}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t-\frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathrm{Re}(f))\left(e^{it}\right)\mathrm{d}t\right)\right)$$

from $D(0;1)$ to $\mathbb{R}$ is holomorphic. Now, by Conclusion 1.239,

$$P[\mathrm{Re}(f)] : D(0;1) \to \mathbb{R}$$

is a harmonic function in $D(0;1)$.

For 2: This is similar to 1. ∎)

Conclusion 1.270 Let $f \in L^1(\{z : |z| = 1\})$. Then

$$P[f] : D(0;1) \to \mathbb{C}$$

is a harmonic function in $D(0;1)$.

Note 1.271 Let $f : \{z : |z| = 1\} \to \mathbb{C}$. Let $f \in C(\{z : |z| = 1\})(\subset L^1(\{z : |z| = 1\}))$.

a. Problem 1.272 For every $(r, \theta) \in [0, 1) \times \mathbb{R}$,

$$\left|(P[f])\left(re^{i\theta}\right)\right| \leq \|f\|_{\{z:|z|=1\}}.$$

(Solution Let us take any $(r, \theta) \in [0, 1) \times \mathbb{R}$. Now,

$$\begin{aligned}
\left|(P[f])\left(re^{i\theta}\right)\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f\left(e^{it}\right)\mathrm{d}t\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} P_r(\theta - t)f\left(e^{it}\right)\mathrm{d}t\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|P_r(\theta - t)f\left(e^{it}\right)\right|\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\theta - t)|\left|f\left(e^{it}\right)\right|\mathrm{d}t \\
&\leq \sup\left\{\left|f\left(e^{it}\right)\right| : t \in [-\pi, \pi]\right\} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\theta - t)|\mathrm{d}t \\
&= \|f\|_{\{z:|z|=1\}} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\theta - t)|\mathrm{d}t \\
&= \|f\|_{\{z:|z|=1\}} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\mathrm{d}t \\
&= \|f\|_{\{z:|z|=1\}} \cdot 1 = \|f\|_{\{z:|z|=1\}},
\end{aligned}$$

so

$$\left|(P[f])\left(re^{i\theta}\right)\right| \leq \|f\|_{\{z:|z|=1\}}.$$

∎)

b. Let $(Hf) : D[0;1] \to \mathbb{C}$ be the function defined as follows: For every $(r,\theta) \in [0,1] \times \mathbb{R}$,

$$(Hf)\left(re^{i\theta}\right) \equiv \begin{cases} f\left(e^{i\theta}\right) & \text{if } r = 1 \\ (P[f])\left(re^{i\theta}\right) & \text{if } r \in [0,1). \end{cases}$$

Problem 1.273 If $f : e^{i\theta} \mapsto \sum\limits_{n=-N}^{N} c_n e^{in\theta}$, where $c_0, c_1, \ldots, c_N$ are complex numbers, then $f \in C(\{z : |z| = 1\})$, and $(Hf) : D[0;1] \to \mathbb{C}$ is continuous.

(**Solution** By Conclusion 1.270, Hf is continuous at all points of $D(0;1)$. It remains to show that Hf is continuous at all points of $\{z : |z| = 1\}$. For every $(r,\theta) \in [0,1] \times \mathbb{R}$,

$$\begin{aligned}
(Hf)\left(re^{i\theta}\right) &= \begin{cases} f\left(e^{i\theta}\right) & \text{if } r = 1, \\ \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_r(\theta - t) f\left(e^{it}\right)\mathrm{d}t & \text{if } r \in [0,1) \end{cases} \\
&= \begin{cases} \sum\limits_{n=-N}^{N} c_n e^{in\theta} & \text{if } r = 1, \\ \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_r(\theta - t)\left(\sum\limits_{n=-N}^{N} c_n e^{int}\right)\mathrm{d}t & \text{if } r \in [0,1) \end{cases} \\
&= \begin{cases} \sum\limits_{n=-N}^{N} c_n e^{in\theta} & \text{if } r = 1, \\ \sum\limits_{n=-N}^{N} c_n \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_r(\theta - t) e^{int}\mathrm{d}t & \text{if } r \in [0,1) \end{cases} \\
&= \begin{cases} \sum\limits_{n=-N}^{N} c_n e^{in\theta} & \text{if } r = 1, \\ \sum\limits_{n=-N}^{N} c_n r^{|n|} e^{i\theta n} & \text{if } r \in [0,1) \end{cases} \\
&= \begin{cases} \underbrace{c_0 + 2c_1 \cos\theta + \cdots + 2c_N \cos N\theta}_{(N+1)\text{terms}} & \text{if } r = 1, \\ \underbrace{c_0 + 2c_1 r \cos\theta + \cdots + 2c_N r^N \cos N\theta}_{(N+1)\text{terms}} & \text{if } r \in [0,1), \end{cases}
\end{aligned}$$

so for every $(r,\theta) \in [0,1] \times \mathbb{R}$,

$$(Hf)\left(re^{i\theta}\right) = \begin{cases} \underbrace{c_0 + 2c_1 \cos\theta + \cdots + 2c_N \cos N\theta}_{(N+1)\text{terms}} & \text{if } r = 1, \\ \underbrace{c_0 + 2c_1 r\ \cos\theta + \cdots + 2c_N r^N \cos N\theta}_{(N+1)\text{terms}} & \text{if } r \in [0,1), \end{cases}$$

This shows that $(Hf) : D[0;1] \to \mathbb{C}$ is continuous. ∎)

c. Let $(Hf) : D[0;1] \to \mathbb{C}$ be the function defined as follows: For every $(r,\theta) \in [0,1] \times \mathbb{R}$,

$$(Hf)\left(re^{i\theta}\right) \equiv \begin{cases} f\left(e^{i\theta}\right) & \text{if } r = 1 \\ (P[f])\left(re^{i\theta}\right) & \text{if } r \in [0,1). \end{cases}$$

Problem 1.274 $\|Hf\|_{D[0;1]} = \|f\|_{\{z:|z|=1\}}$.

(Solution By a, for every $(r,\theta) \in [0,1) \times \mathbb{R}$,

$$\left|(Hf)\left(re^{i\theta}\right)\right| = \underbrace{\left|(P[f])\left(re^{i\theta}\right)\right| \le \|f\|_{\{z:|z|=1\}}} = \sup\left\{\left|f\left(e^{i\theta}\right)\right| : \theta \in \mathbb{R}\right\} = \|f\|_{\{z:|z|=1\}},$$

So for every $(r,\theta) \in [0,1] \times \mathbb{R}$,

$$\left|(Hf)\left(re^{i\theta}\right)\right| \le \sup\left\{\left|f\left(e^{i\theta}\right)\right| : \theta \in \mathbb{R}\right\}\left(\le \|f\|_{\{z:|z|=1\}}\right).$$

This shows that

$$\|Hf\|_{D[0;1]} \le \|f\|_{\{z:|z|=1\}}.$$

It is clear that

$$\|f\|_{\{z:|z|=1\}} = \underbrace{\|Hf\|_{\{z:|z|=1\}} \le \|Hf\|_{D[0;1]}}.$$

Thus,

$$\|Hf\|_{D[0;1]} = \|f\|_{\{z:|z|=1\}}.$$

∎)

d. Let $(Hf) : D[0;1] \to \mathbb{C}$ be the function defined as follows: For every $(r,\theta) \in [0,1] \times \mathbb{R}$,

$$(Hf)\left(re^{i\theta}\right) \equiv \begin{cases} f\left(e^{i\theta}\right) \text{ if } r = 1 \\ (P[f])\left(re^{i\theta}\right) \text{ if } r \in [0,1). \end{cases}$$

Clearly, $(Hf) : D[0;1] \to \mathbb{C}$ is continuous. Also, $(Hf) : D[0;1] \to \mathbb{C}$ is an extension of $f : \{z : |z| = 1\} \to \mathbb{C}$.

Problem 1.275 The restriction of Hf on $D(0;1)$ is a harmonic function in $D(0;1)$.

(Solution By Conclusion 2.156, Vol. 1, there exists a sequence $\{P_n\}$ of trigonometric polynomials

$$P_n : \{z : |z| = 1\} \to \mathbb{C}$$

such that

$$\lim_{n\to\infty} \|P_n - f\|_{\{z:|z|=1\}} = 0.$$

By Conclusion 1.273, each $(HP_n) : D[0;1] \to \mathbb{C}$ is continuous. It suffices to show that $\{HP_n\}$ converges uniformly to Hf on $D[0;1]$, that is

$$\lim_{n\to\infty} \|HP_n - Hf\|_{D[0;1]} = 0.$$

$$\begin{aligned}\text{LHS} &= \lim_{n\to\infty} \|HP_n - Hf\|_{D[0;1]} = \lim_{n\to\infty} \|H(P_n - f)\|)_{D[0;1]} = \lim_{n\to\infty} \|P_n - f\|_{\{z:|z|=1\}} \\ &= 0 = \text{RHS}.\end{aligned}$$

■)

Conclusion 1.276 Let $f : \{z : |z| = 1\} \to \mathbb{C}$. Let $f \in C(\{z : |z| = 1\})$. Then there exists $g \in C(D[0;1])$ such that g is an extension of f, and the restriction of g to $D(0;1)$ is a harmonic function in $D(0;1)$.

Note 1.277 Let $u : D[0;1] \to \mathbb{R}$ be a continuous function. Suppose that the restriction of u to $D(0;1)$ is a harmonic function in $D(0;1)$.

Problem 1.278 The function $f : z \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u(e^{it})\mathrm{d}t$ from $D(0;1)$ to $\mathbb{C}$ is holomorphic.

(Solution Since for every $z \in D(0;1)$,

$$\begin{aligned}f(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u\left(e^{it}\right)\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{2e^{it}}{e^{it}-z} - 1\right) u\left(e^{it}\right)\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{e^{it}-z} e^{it} u\left(e^{it}\right)\mathrm{d}t - \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{it}\right)\mathrm{d}t,\end{aligned}$$

we have

$$f : z \mapsto \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{e^{it}-z} e^{it} u\left(e^{it}\right)\mathrm{d}t - \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{it}\right)\mathrm{d}t\right)$$

from $D(0;1)$ to $\mathbb{C}$. Since $u : D[0;1] \to \mathbb{R}$ is a continuous function, $u \in L^1(\{z : |z| = 1\})$, and hence by Conclusion 1.66, the function

$$z \mapsto \int_{-\pi}^{\pi} \frac{1}{e^{it} - z} e^{it} u\left(e^{it}\right) \mathrm{d}t$$

from $D(0;1)$ to $\mathbb{C}$ is representable by power series in $D(0;1)$. Thus,

$$f : z \mapsto \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{e^{it} - z} e^{it} u\left(e^{it}\right) \mathrm{d}t - \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(e^{it}\right) \mathrm{d}t \right)$$

from $D(0;1)$ to $\mathbb{C}$ is holomorphic. ∎)

Problem 1.279 $P\left[u|_{\{z:|z|=1\}}\right] : re^{i\theta} \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u|_{\{z:|z|=1\}}(e^{it}) \mathrm{d}t$ from $D(0;1)$ to $\mathbb{C}$ is $\mathrm{Re}(f)$.

(Solution Let us take any $re^{i\theta} \in D(0;1)$. We have to show that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u\left(e^{it}\right) \mathrm{d}t &= (\mathrm{Re}(f))\left(re^{i\theta}\right) = \mathrm{Re}\left(f\left(re^{i\theta}\right)\right) \\ &= \mathrm{Re}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} u\left(e^{it}\right) \mathrm{d}t \right), \end{aligned}$$

that is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u\left(e^{it}\right) \mathrm{d}t = \mathrm{Re}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} u\left(e^{it}\right) \mathrm{d}t \right).$$

$$\begin{aligned} \mathrm{LHS} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u\left(e^{it}\right) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 + \left(r^{|1|} e^{i(\theta - t)1} + r^{|-1|} e^{i(\theta - t)(-1)}\right)\right. \\ &\quad \left. + \left(r^{|2|} e^{i(\theta - t)2} + r^{|-2|} e^{i(\theta - t)(-2)}\right) + \cdots \right) u\left(e^{it}\right) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 + \left(r^1 (2\cos(\theta - t))\right) + \left(r^2 (2\cos 2(\theta - t))\right) + \cdots \right) u\left(e^{it}\right) \mathrm{d}t \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(1+2\left(\left(re^{i\theta}e^{-it}\right)^1+\left(re^{i\theta}e^{-it}\right)^2+\cdots\right)\right)\right)u\left(e^{it}\right)\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(1+2\left(\frac{re^{i\theta}e^{-it}}{1-re^{i\theta}e^{-it}}\right)\right)\right)u\left(e^{it}\right)\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(\frac{1+re^{i\theta}e^{-it}}{1-re^{i\theta}e^{-it}}\right)\right)u\left(e^{it}\right)\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right)\right)u\left(e^{it}\right)\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Re}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}u\left(e^{it}\right)\right)\mathrm{d}t = \mathrm{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}u\left(e^{it}\right)\mathrm{d}t\right) = \mathrm{RHS}.$$

∎)

Let us define a function $u_1 : D[0;1] \to \mathbb{R}$ as follows: For every $re^{i\theta} \in D[0;1]$,

$$u_1\left(re^{i\theta}\right) \equiv \begin{cases} u\left(e^{i\theta}\right) \text{ if} r=1, \\ (\mathrm{Re}(f))\left(re^{i\theta}\right) \text{ if} r \in (0,1), \end{cases}$$

that is for every $re^{i\theta} \in D[0;1]$,

$$u_1\left(re^{i\theta}\right) \equiv \begin{cases} u|_{\{z:|z|=1\}}\left(e^{i\theta}\right) \text{ if} r=1, \\ \left(P\left[u|_{\{z:|z|=1\}}\right]\right)\left(re^{i\theta}\right) \text{ if} r \in (0,1). \end{cases}$$

By Conclusion (d) of Note 1.271, $u_1 : D[0;1] \to \mathbb{R}$ is continuous.

Since $u_1 : D[0;1] \to \mathbb{R}$ is continuous, and $u : D[0;1] \to \mathbb{R}$ is continuous, $(u-u_1) : D[0;1] \to \mathbb{R}$ is a continuous function. Clearly, $u_1 = u$ on $\{z : |z| = 1\}$.

Problem 1.280 $(u-u_1)|_{D(0;1)}$ is harmonic in $D(0;1)$.

(Solution By assumption, $u|_{D(0;1)}$ is harmonic in $D(0;1)$, so it suffices to show that $P\left[u|_{\{z:|z|=1\}}\right]$ is harmonic in $D(0;1)$. Since $u : D[0;1] \to \mathbb{R}$ is continuous,

$$u|_{\{z:|z|=1\}} \in C(\{z : |z| = 1\})\left(\subset L^1(\{z : |z| = 1\})\right),$$

and hence by Note 1.265, $P\left[u|_{\{z:|z|=1\}}\right]$ is harmonic in $D(0;1)$. ∎)

Problem 1.281 For every $z \in D(0;1)$, $u(z) = u_1(z)$.

(**Solution** If not, otherwise suppose that there exists $z_0 \in D(0;1)$ such that $u(z_0) \neq u_1(z_0)$. We have to arrive at a contradiction.

Case I: when $u(z_0) > u_1(z_0)$. It follows that $(u(z_0) - u_1(z_0)) > 0$.

Let

$$g : z \mapsto \left((u(z) - u_1(z)) + \frac{1}{2}(u(z_0) - u_1(z_0))|z|^2 \right)$$

be a function from $D[0;1]$ to $\mathbb{R}$. Since u, u_1 are continuous, $g : D[0;1] \to \mathbb{R}$ is continuous on the compact set $D[0;1]$, and hence there exists $z_1 \in D[0;1]$ such that $g(z_1) = \max\{g(z) : z \in D[0;1]\}$.

For every $w \in \{z : |z| = 1\}$,

$$\begin{aligned} g(w) &= (u(w) - u_1(w)) + \frac{1}{2}(u(z_0) - u_1(z_0))|w|^2 = (u(w) - u_1(w)) + \frac{1}{2}(u(z_0) - u_1(z_0))1^2 \\ &= 0 + \frac{1}{2}(u(z_0) - u_1(z_0))1^2 = \frac{1}{2}(u(z_0) - u_1(z_0)) < u(z_0) - u_1(z_0) \\ &\leq (u(z_0) - u_1(z_0)) + \frac{1}{2}(u(z_0) - u_1(z_0))|z_0|^2 = g(z_0) \leq g(z_1), \end{aligned}$$

so for every $w \in \{z : |z| = 1\}$, $g(w) < g(z_1)$. This shows that $z_1 \notin \{z : |z| = 1\}$, and hence $z_1 \in D(0;1)$.

Since $u_1 = \mathrm{Re}(f)$ on $D(0;1)$, and $f : D(0;1) \to \mathbb{C}$ is holomorphic, $u_1|_{D(0;1)} : D(0;1) \to \mathbb{C}$ is holomorphic, and hence by Conclusion 1.239, $u_1|_{D(0;1)} : D(0;1) \to \mathbb{C}$ is harmonic. Now, since the restriction of u to $D(0;1)$ is a harmonic function in $D(0;1)$, $g|_{D(0;1)}$ is harmonic in $D(0;1)$, and hence over $D(0;1)$,

$$\begin{aligned} g_{xx} + g_{yy} = \Delta g &= (\Delta u - \Delta u_1) + \frac{1}{2}(u(z_0) - u_1(z_0))(2+2) \\ &= (0-0) + \frac{1}{2}(u(z_0) - u_1(z_0))(2+2) = 2(u(z_0) - u_1(z_0)) > 0. \end{aligned}$$

Now, since $z_1 \in D(0;1)$, we have $g_{xx}(z_1) + g_{yy}(z_1) > 0$. Since

$$g(z_1) = \max\{g(z) : z \in D[0;1]\},$$

and $z_1 \in D(0;1)$, we have $g_{xx}(z_1) \leq 0$ and $g_{yy}(z_1) \leq 0$. Thus, $g_{xx}(z_1) + g_{yy}(z_1) \leq 0$. This is a contradiction.

Case II: when $u(z_0) < u_1(z_0)$. This case is similar to case I.

Thus, in all cases we get a contradiction. ∎)

Since $u_1 = u$ on $\{z : |z| = 1\}$, and for every $z \in D(0;1)$, $u(z) = u_1(z)$, we have $u = u_1$.

Since for every $re^{i\theta} \in D[0;1]$,

$$u\left(re^{i\theta}\right) = u_1\left(re^{i\theta}\right) = \begin{cases} u\left(e^{i\theta}\right) \text{ if} r = 1, \\ (\mathrm{Re}(f))\left(re^{i\theta}\right) \text{ if} r \in (0,1), \end{cases}$$

we have $u|_{D(0;1)} = \mathrm{Re}(f)$.

Conclusion 1.282 Let $u : D[0;1] \to \mathbb{R}$ be a continuous function. Suppose that the restriction of u to $D(0;1)$ is a harmonic function in $D(0;1)$. Then,

1. the function $f : z \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u(e^{it})\mathrm{d}t$ from $D(0;1)$ to $\mathbb{C}$ is holomorphic,
2. $u|_{D(0;1)} = \mathrm{Re}(f) = P\left[u|_{\{z:|z|=1\}}\right]$.

Note 1.283 Let $u : \{z : |z| = 1\} \to \mathbb{R}$ be a continuous function.

It follows from Note 1.265, that $P[u] : D(0;1) \to \mathbb{R}$ is a harmonic function in $D(0;1)$.

Let us define a function $h : D[0;1] \to \mathbb{R}$ as follows: For every $re^{i\theta} \in D[0;1]$,

$$h\left(re^{i\theta}\right) \equiv \begin{cases} u\left(e^{i\theta}\right) \text{ if} r = 1, \\ (P[u])\left(re^{i\theta}\right) \text{ if} r \in (0,1). \end{cases}$$

By Conclusion 1.275, $h : D[0;1] \to \mathbb{R}$ is continuous, and hence by the conclusion of Note 1.83, $h|_{D(0;1)} = P\left[h|_{\{z:|z|=1\}}\right](= P[u])$, and $h|_{D(0;1)} = \mathrm{Re}(f)$, where

$$f : z \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u\left(e^{it}\right)\mathrm{d}t$$

from $D(0;1)$ to $\mathbb{C}$ is a holomorphic function. Now, since $P[u] : D(0;1) \to \mathbb{R}$ is a harmonic function in $D(0;1)$, $h|_{D(0;1)} : D(0;1) \to \mathbb{R}$ is a harmonic function in $D(0;1)$. Also, h coincides with u on $\{z : |z| = 1\}$.

Conclusion 1.284 Let $u : \{z : |z| = 1\} \to \mathbb{R}$ be a continuous function. Then,

1. $f : z \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} u(e^{it})\mathrm{d}t$ from $D(0;1)$ to $\mathbb{C}$ is a holomorphic function,
2. there exists a continuous function $h : D[0;1] \to \mathbb{R}$ such that
   a. h coincides with u on $\{z : |z| = 1\}$,
   b. $h|_{D(0;1)} = \mathrm{Re}(f)$,
   c. $h|_{D(0;1)} : D(0;1) \to \mathbb{R}$ is a harmonic function in $D(0;1)$,
   d. for every $re^{i\theta} \in D(0;1)$, $h\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} u(e^{it})\mathrm{d}t$.

Note 1.285 Let $a \in \mathbb{C}$, and R be a positive real number. Let $u : \{z : |z - a| = R\} \to \mathbb{R}$ be a continuous function. Then, as in Note 1.283,

1. $f : z \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Re^{it} + (z-a)}{Re^{it} - (z-a)} u(a + Re^{it}) \mathrm{d}t$ from $D(a;R)$ to $\mathbb{C}$ is a holomorphic function,
2. there exists a continuous function $h : D[a;R] \to \mathbb{R}$ such that
a. h coincides with u on $\{z : |z - a| = R\}$,
b. $h|_{D(a;R)} = \mathrm{Re}(f)$,
c. $h|_{D(a;R)} : D(a;R) \to \mathbb{R}$ is a harmonic function in $D(a;R)$,
d. for every $\left(a + re^{i\theta}\right) \in D(a;R)$ satisfying $r \in [0, R)$,

$$h\left(a + re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u\left(a + Re^{it}\right) \mathrm{d}t.$$

(Reason for (d): Let us take any $\left(a + re^{i\theta}\right) \in D(a;R)$, where $r \in [0, R)$. We have to show that

$$h\left(a + re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u\left(a + Re^{it}\right) \mathrm{d}t.$$

$$\begin{aligned}
\text{LHS} &= h\left(a + re^{i\theta}\right) = (\mathrm{Re}(f))\left(a + re^{i\theta}\right) = \mathrm{Re}\left(f\left(a + re^{i\theta}\right)\right) \\
&= \mathrm{Re}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Re^{it} + \left(\left(a + re^{i\theta}\right) - a\right)}{Re^{it} - \left(\left(a + re^{i\theta}\right) - a\right)} u\left(a + Re^{it}\right) \mathrm{d}t\right) \\
&= \mathrm{Re}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{Re^{it} + re^{i\theta}}{Re^{it} - re^{i\theta}} u\left(a + Re^{it}\right) \mathrm{d}t\right) \\
&= \mathrm{Re}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left(Re^{it} + re^{i\theta}\right)\left(Re^{-it} - re^{-i\theta}\right)}{\left(Re^{it} - re^{i\theta}\right)\left(Re^{-it} - re^{-i\theta}\right)} u\left(a + Re^{it}\right) \mathrm{d}t\right) \\
&= \mathrm{Re}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left(R^2 - r^2\right) + Rr\left(e^{i(\theta - t)} - e^{-i(\theta - t)}\right)}{R^2 - Rr\left(e^{i(t-\theta)} + e^{-i(t-\theta)}\right) + r^2} u\left(a + Re^{it}\right) \mathrm{d}t\right) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{Re}\left(\frac{\left(R^2 - r^2\right) + Rr\left(e^{i(\theta - t)} - e^{-i(\theta - t)}\right)}{R^2 - Rr\left(e^{i(t-\theta)} + e^{-i(t-\theta)}\right) + r^2} u\left(a + Re^{it}\right)\right) \mathrm{d}t
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(\frac{(R^2-r^2)+Rr\left(e^{i(\theta-t)}-e^{-i(\theta-t)}\right)}{R^2-Rr(e^{i(t-\theta)}+e^{-i(t-\theta)})+r^2}\right)\right)u\left(a+Re^{it}\right)\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\mathrm{Re}\left(\frac{(R^2-r^2)+Rr(2i\,\sin(\theta-t))}{R^2-Rr(2\cos(t-\theta))+r^2}\right)\right)u\left(a+Re^{it}\right)\mathrm{d}t$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-r^2}{R^2-2Rr\,\cos(\theta-t)+r^2}u\left(a+Re^{it}\right)\mathrm{d}t = \mathrm{RHS.})$$

Conclusion 1.286 Let $a\in\mathbb{C}$, and R be a positive real number. Let $u:\{z:|z-a|=R\}\to\mathbb{R}$ be a continuous function. Then,

1. $f:z\mapsto\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Re^{it}+(z-a)}{Re^{it}-(z-a)}u(a+Re^{it})\mathrm{d}t$ from $D(a;R)$ to $\mathbb{C}$ is a holomorphic function,
2. there exists a continuous function $h:D[a;R]\to\mathbb{R}$ such that
a. h coincides with u on $\{z:|z-a|=R\}$,
b. $h|_{D(a;R)}=\mathrm{Re}(f)$,
c. $h|_{D(a;R)}:D(a;R)\to\mathbb{R}$ is a harmonic function in $D(a;R)$,
d. for every $\left(a+re^{i\theta}\right)\in D(a;R)$ satisfying $r\in[0,R)$, $h\left(a+re^{i\theta}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-r^2}{R^2-2Rr\,\cos(\theta-t)+r^2}u(a+Re^{it})\mathrm{d}t$.

1.22 Mean Value Property

Note 1.287

a. Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u:\Omega\to\mathbb{R}$ be a function harmonic in Ω. Let $a\in\Omega$, and R be a positive real number such that $D[a;R]\subset\Omega$.

Since $u:\Omega\to\mathbb{R}$ is harmonic in Ω, $u:\Omega\to\mathbb{R}$ is continuous, and hence u is continuous at every point of $D[a;R]$. Also, the restriction of u to $D(a;R)$ is a harmonic function in $D(a;R)$. Then, as in Note 1.277,

1. the function $f:z\mapsto\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{Re^{it}+(z-a)}{Re^{it}-(z-a)}u(a+Re^{it})\mathrm{d}t$ from $D(a;R)$ to $\mathbb{C}$ is holomorphic,
2. $u|_{D(a;R)}=\mathrm{Re}(f)$,
3. for every $\left(a+re^{i\theta}\right)\in D(a;R)$ satisfying $r\in[0,R)$, $u\left(a+re^{i\theta}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-r^2}{R^2-2R\,r\,\cos(\theta-t)+r^2}u(a+Re^{it})\mathrm{d}t$.

Suppose that $g : D(a;R) \to \mathbb{C}$ is a holomorphic function such that $(\mathrm{Re}(f) =)u|_{D(a;R)} = \mathrm{Re}(g)$. It follows that $(g-f) : D(a;R) \to \mathbb{C}$ is a holomorphic function such that $0 = \mathrm{Re}(g-f)$.

Problem 1.288 $\mathrm{Im}(g-f) = ki$ for some real k.

(Solution By Conclusion 1.190,

$$\begin{aligned}\underbrace{(g-f)(D(a;R))} &= \{\mathrm{Re}(g-f)(z) + i\mathrm{Im}(g-f)(z) : z \in D(a;R)\} \\ &= \{0 + i\mathrm{Im}(g-f)(z) : z \in D(a;R)\} \\ &= \{i\mathrm{Im}(g-f)(z) : z \in D(a;R)\} \subset i\mathbb{R}\end{aligned}$$

is a region or a singleton. Now, since $(g-f)(D(a;R)) \subset i\mathbb{R}$, $(g-f)(D(a;R))$ is not open, and hence $(g-f)(D(a;R))$ is not a region, it follows that $(\{i\mathrm{Im}(g-f)(z) : z \in D(a;R)\} =)(g-f)(D(a;R))$ is a singleton. Hence, there exists a real number k such that for every $z \in D(a;R)$, $(g-f)(z) = ki$. ∎)

This shows that for every $z \in D(a;R)$, $g(z) = f(z) + ki$.

Conclusion 1.289 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{R}$ be a function harmonic in Ω. Let $a \in \Omega$, and R be a positive real number such that $D[a;R] \subset \Omega$. Then,

1. the function $f : z \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{Re^{it}+(z-a)}{Re^{it}-(z-a)} u(a+Re^{it})\mathrm{d}t$ from $D(a;R)$ to $\mathbb{C}$ is holomorphic,
2. $u|_{D(a;R)} = \mathrm{Re}(f)$,
3. for every $(a+re^{i\theta}) \in D(a;R)$ satisfying $r \in [0,R)$, $u(a+re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2} u(a+Re^{it})\mathrm{d}t$,
4. if $g : D(a;R) \to \mathbb{C}$ is a holomorphic function such that $u|_{D(a;R)} = \mathrm{Re}(g)$, then there exists a real k such that for every $z \in D(a;R)$, $g(z) = f(z) + ki$.

b. Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{R}$ be a function harmonic in Ω.

Let $a \in \Omega$, and R be a positive real number such that $D[a;R] \subset \Omega$. Then by Conclusion 1.289, there exists a holomorphic function $f : D(a;R) \to \mathbb{C}$ such that $u|_{D(a;R)} = \mathrm{Re}(f)$. Since $f \in H(D(a;R))$, by Lemma 1.117,

$$u_x + i(\mathrm{Im}(f))_x = (\mathrm{Re}(f))_x + i(\mathrm{Im}(f))_x = \underbrace{f' \in H(D(a;R))}.$$

Again, by Lemma 1.117,

$$u_{xx} + i(\mathrm{Im}(f))_{xx} = (u_x)_x + i\big((\mathrm{Im}(f))_x\big)_x = \underbrace{\big(u_x + i(\mathrm{Im}(f))_x\big)' \in H(D(a;R))}.$$

Also,

$$u_{xy} + i(\mathrm{Im}(f))_{xy} = (u_x)_y + i\big((\mathrm{Im}(f))_x\big)_y = \underbrace{\big(u_x + i(\mathrm{Im}(f))_x\big)' \in H(D(a;R))}, \text{etc.}$$

Thus, u has continuous partial derivatives of all orders in Ω.

Conclusion 1.290 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{C}$ be a function harmonic in Ω. Then u has continuous partial derivatives of all orders in Ω.

c. Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{R}$ be a function harmonic in Ω. Let $a \in \Omega$, and R be a positive real number such that $D[a;R] \subset \Omega$.

Problem 1.291 For every $r \in [0,R)$, $u(a) = \frac{\int_{-\pi}^{\pi} u(a+re^{i\theta})\mathrm{d}\theta}{\int_{-\pi}^{\pi} 1\mathrm{d}\theta}$.

(Solution Let us fix any $r \in [0,R)$. We have to show that

$$u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\big(a+re^{i\theta}\big)\mathrm{d}\theta.$$

On using Conclusion 1.289,

$$\begin{aligned} u(a) = u\big(a+0e^{i0}\big) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-0^2}{R^2-2R0\cos(0-t)+0^2}u\big(a+Re^{it}\big)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u\big(a+Re^{it}\big)\mathrm{d}t, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} u\big(a+re^{i\theta}\big)\mathrm{d}\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2}u\big(a+Re^{it}\big)\mathrm{d}t\right)\mathrm{d}\theta \\ &= \frac{(R^2-r^2)}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{1}{R^2-2Rr\cos(\theta-t)+r^2}u\big(a+Re^{it}\big)\mathrm{d}t\right)\mathrm{d}\theta \\ &= \frac{(R^2-r^2)}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{1}{R^2-2Rr\cos(\theta-t)+r^2}u\big(a+Re^{it}\big)\mathrm{d}\theta\right)\mathrm{d}t \\ &= \frac{(R^2-r^2)}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}\frac{1}{b-c\cos(\theta-t)}\mathrm{d}\theta\right)u\big(a+Re^{it}\big)\mathrm{d}t, \end{aligned}$$

where $b \equiv R^2 + r^2$, and $c \equiv 2Rr$. Now, since

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{1}{b - c\ \cos(\theta - t)} \mathrm{d}\theta &= \int_{-\pi-t}^{\pi-t} \frac{1}{b - c\ \cos\varphi} \mathrm{d}\varphi = \int_{-\pi}^{\pi} \frac{1}{b - c\ \cos\varphi} \mathrm{d}\varphi \\
&= \int_{T=-\infty}^{T=\infty} \frac{1}{b - c\frac{1-T^2}{1+T^2}} \mathrm{d}\left(2\tan^{-1} T\right) = \int_{-\infty}^{\infty} \frac{1}{b(1+T^2) - c(1-T^2)} 2\mathrm{d}T \\
&= 4\int_{0}^{\infty} \frac{1}{b(1+T^2) - c(1-T^2)} \mathrm{d}T = 4\int_{0}^{\infty} \frac{1}{(b-c) + (b+c)T^2} \mathrm{d}T \\
&= 4\int_{0}^{\infty} \frac{1}{(R-r)^2 + (R+r)^2 T^2} \mathrm{d}T = 4\frac{1}{(R-r)}\frac{1}{(R+r)} \tan^{-1} \frac{(R+r)T}{R-r}\bigg|_0^\infty \\
&= \frac{4}{R^2 - r^2}\left(\frac{\pi}{2} - 0\right) = \frac{2\pi}{R^2 - r^2},
\end{aligned}$$

we have

$$\begin{aligned}
\mathrm{RHS} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(a + re^{i\theta}\right)\mathrm{d}\theta = \frac{(R^2 - r^2)}{4\pi^2}\int_{-\pi}^{\pi}\left(\frac{2\pi}{R^2 - r^2}\right) u\left(a + Re^{it}\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(a + Re^{it}\right)\mathrm{d}t = u(a) = \mathrm{LHS}.
\end{aligned}$$

∎)

Definition Let Ω be a nonempty subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{C}$ be a continuous function. By *u has the* ***mean value property***, we mean: For every $a \in \Omega$, there exists a sequence $\{r_n\}$ of positive real numbers such that $\lim_{n\to\infty} r_n = 0$, and for every positive integer n,

$$u(a) = \frac{\int_{-\pi}^{\pi} u\left(a + r_n e^{i\theta}\right)\mathrm{d}\theta}{\int_{-\pi}^{\pi} 1\mathrm{d}\theta}.$$

Conclusion 1.292 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{C}$ be a function harmonic in Ω.

Problem 1.293 u has a ‘stronger’ mean value property.

(Solution Since $u : \Omega \to \mathbb{C}$ is a function harmonic in Ω, $u : \Omega \to \mathbb{C}$ is a continuous function.

Let us take any $a \in \Omega$. Now, since Ω is open, there exists $R > 0$ such that $D[a; R] \subset \Omega$. Since $u : \Omega \to \mathbb{C}$ is a function harmonic in Ω, $\mathrm{Re}(u) : \Omega \to \mathbb{R}$ is a function harmonic in Ω, and $\mathrm{Im}(u) : \Omega \to \mathbb{R}$ is a function harmonic in Ω. From the foregoing discussion, for every $r \in [0, R)$,

$$(\mathrm{Re}(u))(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathrm{Re}(u))\left(a + re^{i\theta}\right)\mathrm{d}\theta, \text{ and } (\mathrm{Im}(u))(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathrm{Im}(u))\left(a + re^{i\theta}\right)\mathrm{d}\theta.$$

It follows that for every $r \in [0, R)$,

$$\begin{aligned} u(a) &= (\mathrm{Re}(u))(a) + i((\mathrm{Im}(u))(a)) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathrm{Re}(u))\left(a + re^{i\theta}\right)\mathrm{d}\theta + i\frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathrm{Im}(u))\left(a + re^{i\theta}\right)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left((\mathrm{Re}(u))\left(a + re^{i\theta}\right) + i\left((\mathrm{Im}(u))\left(a + re^{i\theta}\right)\right)\right)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(a + re^{i\theta}\right)\mathrm{d}\theta. \end{aligned}$$

Since for every $r \in [0, R)$,

$$u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(a + re^{i\theta}\right)\mathrm{d}\theta,$$

u has the 'stronger' mean value property, and hence u has the mean value property. ∎)

Conclusion 1.294 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{C}$ be a function harmonic in Ω. Then for every $r \in [0, R), u(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(a + re^{i\theta}\right)\mathrm{d}\theta$.

Note 1.295 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{R}$ be a continuous function. Suppose that u has the mean value property.

Problem 1.296 $u : \Omega \to \mathbb{R}$ is harmonic in Ω.

(Solution For this purpose, let us take any $a \in \Omega$. We have to show that $u_{xx}(a) + u_{yy}(a) = 0$.

Since $a \in \Omega$, and Ω is open, there exists $R > 0$ such that $D[a; R] \subset \Omega$. Now, since $u : \Omega \to \mathbb{R}$ is a continuous function, the restriction of u to $\{z : |z - a| = R\}$ is continuous, and hence by Note 1.277 there exists a continuous function $h : D[a; R] \to \mathbb{R}$ such that

a. h coincides with u on $\{z : |z-a| = R\}$,
b. $h|_{D(a;R)} \colon D(a;R) \to \mathbb{R}$ is a harmonic function in $D(a;R)$,

Since $u|_{D[a;R]}$ is continuous, and h is continuous, $\left(u|_{D[a;R]} - h\right)$ is continuous on the compact set $D[a;R]$, and hence there exists $z_0 \in D[a;R]$ such that for every $z \in D[a;R]$, $\left(u|_{D[a;R]} - h\right)(z) \leq \left(u|_{D[a;R]} - h\right)(z_0)(= u(z_0) - h(z_0))$.

Since the singleton set $\left\{\left(u|_{D[a;R]} - h\right)(z_0)\right\}$ is a closed set, and $\left(u|_{D[a;R]} - h\right)$ is continuous on $D[a;R]$, $\left(u|_{D[a;R]} - h\right)^{-1}\left(\left(u|_{D[a;R]} - h\right)(z_0)\right)$ is a closed subset of the compact set $D[a;R]$, and hence

$$(z_0 \ni)\left(u|_{D[a;R]} - h\right)^{-1}\left(\left(u|_{D[a;R]} - h\right)(z_0)\right)$$

is a compact subset of $D[a;R]$. By b, and Conclusion 1.292, h has a stronger mean value property. Now, since u has the mean value property, $\left(u|_{D[a;R]} - h\right)$ has the mean value property.

Problem 1.297 $u(z_0) \leq h(z_0)$.

(Solution If not, otherwise let $u(z_0) > h(z_0)$. We have to arrive at a contradiction.

Since $u(z_0) > h(z_0)$, by a, $z_0 \in D(a;R)$. Since $\left(u|_{D[a;R]} - h\right)^{-1}\left(\left(u|_{D[a;R]} - h\right)(z_0)\right)$ is a compact subset of $D[a;R]$, and $z \mapsto |z-a|$ is continuous, there exists

$$z_1 \in \left(u|_{D[a;R]} - h\right)^{-1}\left(\left(u|_{D[a;R]} - h\right)(z_0)\right)$$

such that for every $z \in \left(u|_{D[a;R]} - h\right)^{-1}\left(\left(u|_{D[a;R]} - h\right)(z_0)\right)$, we have $|z-a| \leq |z_1 - a|$.

It follows that for all sufficiently small positive r, at least half of the circle with center z_1 and radius r lies outside $\left(u|_{D[a;R]} - h\right)^{-1}\left(\left(u|_{D[a;R]} - h\right)(z_0)\right)$, and hence the corresponding mean value of $\left(u|_{D[a;R]} - h\right)$ are less than $\left(u|_{D[a;R]} - h\right)(z_0)$. This contradicts the fact that $\left(u|_{D[a;R]} - h\right)$ has the mean value property. ∎)

Since for every $z \in D[a;R]$, $\left(u|_{D[a;R]} - h\right)(z) \leq u(z_0) - h(z_0)$, and $u(z_0) \leq h(z_0)$, for every $z \in D[a;R]$, $u(z) \leq h(z)$.

Similarly, for every $z \in D[a;R]$, $h(z) \le u(z)$. Hence, for every $z \in D[a;R]$, $h(z) = u(z)$. Thus, $u|_{D(a;R)} = h|_{D(a;R)}$. Now, by b, $u|_{D(a;R)} : D(a;R) \to \mathbb{R}$ is a harmonic function in $D(a;R)$, and hence $u_{xx}(a) + u_{yy}(a) = 0$. ∎)

Conclusion 1.298 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{C}$ be a continuous function. Suppose that u has the mean value property. Then $u : \Omega \to \mathbb{C}$ is harmonic in Ω.

Proof of the remaining part It suffices to show that $\mathrm{Re}(u) : \Omega \to \mathbb{R}$ is harmonic in Ω, and $\mathrm{Im}(u) : \Omega \to \mathbb{R}$ is harmonic in Ω.
Since $u : \Omega \to \mathbb{C}$ is continuous, $\mathrm{Re}(u) : \Omega \to \mathbb{R}$ is continuous. Since u has the mean value property, $\mathrm{Re}(u)$ has the mean value property, and hence by 1.296, $\mathrm{Re}(u) : \Omega \to \mathbb{R}$ is harmonic in Ω. Similarly, $\mathrm{Im}(u) : \Omega \to \mathbb{R}$ is harmonic in Ω. ∎

1.23 Schwarz Reflection Principle

Note 1.299 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a harmonic function. Let $\Omega^* \equiv \{\bar{z} : z \in \Omega\}$. Thus, $z \in \Omega \Leftrightarrow \bar{z} \in \Omega^*$. Clearly, Ω^* is a nonempty open subset of $\mathbb{C}$.

Let us define a function $g : \Omega^* \to \mathbb{C}$ as follows: For every $z \in \Omega^*$(and hence $\bar{z} \in \Omega$, and $f(\bar{z}) \in \mathbb{C}$), $g(z) \equiv f(\bar{z})$. In short, $g : z \mapsto f(\bar{z})$ is a function from Ω^* to $\mathbb{C}$.

Problem 1.300 $g : \Omega^* \to \mathbb{C}$ is a harmonic function.

(Solution Since $f : \Omega \to \mathbb{C}$ is a harmonic function, $f : \Omega \to \mathbb{C}$ is a continuous function. Now, since $z \mapsto \bar{z}$ is a continuous function, $g : z \mapsto f(\bar{z})$ is a continuous function from Ω^* to $\mathbb{C}$. Next, let us take any $((a_1, a_2) = a_1 + ia_2 \equiv) a \in \Omega^*$. It suffices to show that

$$g_{xx}(a_1, a_2) + g_{yy}(a_1, a_2) = 0.$$

From the definition of g, for every real x, y satisfying $(x + iy) \in \Omega^*$, $g(x, y) = f(x, -y)$. It follows that

$$\begin{aligned} g_y(a_1, a_2) &= f_y(a_1, -a_2)(-1), \\ g_{yy}(a_1, a_2) &= f_{yy}(a_1, -a_2)(-1)(-1)\big(= f_{yy}(a_1, -a_2)\big), \end{aligned}$$

and

$$g_{xx}(a_1, a_2) = f_{xx}(a_1, -a_2).$$

Since $f : \Omega \to \mathbb{C}$ is a harmonic function, and $(a_1 - ia_2) \in \Omega$,

$$\left(g_{xx}(a_1,a_2)+g_{yy}(a_1,a_2)=\right)f_{xx}(a_1,-a_2)+f_{yy}(a_1,-a_2)=0,$$

and we have

$$g_{xx}(a_1,a_2)+g_{yy}(a_1,a_2)=0.$$

∎)

I. In short, if f is harmonic, then $z \mapsto f(\bar{z})$ is harmonic.

Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f:\Omega\to\mathbb{C}$ be a holomorphic function.

Let $\Omega^* \equiv \{\bar{z} : z\in\Omega\}$. Thus, $z\in\Omega \Leftrightarrow \bar{z}\in\Omega^*$. Clearly, Ω^* is a nonempty open subset of $\mathbb{C}$.

Let us define a function $g:\Omega^*\to\mathbb{C}$ as follows: For every $z\in\Omega^*$(and hence $\bar{z}\in\Omega$, and $f(\bar{z})\in\mathbb{C}$), $g(z)\equiv\overline{f(\bar{z})}$. In short, $g: z\mapsto\overline{f(\bar{z})}$ is a function from Ω^* to $\mathbb{C}$.

Problem 1.301 $g:\Omega^*\to\mathbb{C}$ is a holomorphic function.

(Solution Let us take any $((a_1,a_2)=a_1+ia_2\equiv)a\in\Omega^*$. It suffices to show that $\lim\limits_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exists, that is $\lim\limits_{h\to 0}\frac{\overline{f\left(\overline{a+h}\right)}-\overline{f(\bar{a})}}{h}$ exists, that is $\lim\limits_{h\to 0}\left(\frac{f\left(\overline{a+h}\right)-f(\bar{a})}{\bar{h}}\right)^{-}$ exists, that is $\lim\limits_{h\to 0}\frac{f\left(\overline{a+h}\right)-f(\bar{a})}{\bar{h}}$ exists, that is $\lim\limits_{h\to 0}\frac{f(\bar{a}+\bar{h})-f(\bar{a})}{\bar{h}}$ exists, that is $\lim\limits_{\bar{h}\to 0}\frac{f(\bar{a}+\bar{h})-f(\bar{a})}{\bar{h}}$ exists, that is $\lim\limits_{h\to 0}\frac{f(\bar{a}+h)-f(\bar{a})}{h}$ exists. Since $a\in\Omega^*$, we have $\bar{a}\in\Omega$. Now, since $f:\Omega\to\mathbb{C}$ is a holomorphic function, $\lim\limits_{h\to 0}\frac{f(\bar{a}+h)-f(\bar{a})}{h}$ exists. ∎)

II. Conclusion 1.302 If f is holomorphic, then $z\mapsto\overline{f(\bar{z})}$ is holomorphic.

Note 1.303 Let L be an open segment of the real axis in $\mathbb{C}$. Let Ω^+ be a region in the 'upper half plane' $\{x+iy : x\in\mathbb{R}, \text{ and } y\in(0,\infty)\}$. Put $\Omega^-\equiv\left\{z:\bar{z}\in\Omega^+\right\}$.

Definition We shall call Ω^- the ***reflection*** *of* Ω^+.

Suppose that for every $t\in L$, D_t is an open disk with center t such that $D_t\subset\Omega^+\cup L\cup\Omega^-$. Let $f\in H\left(\Omega^+\right)$. Suppose that for every convergent sequence $\{z_n\}$ in Ω^+ satisfying $\left(\lim\limits_{n\to\infty}z_n\right)\in L$, $\lim\limits_{n\to\infty}(\mathrm{Im}(f))(z_n)=0$.

Observe that Ω^+, L, Ω^- are pairwise disjoint subsets of $\mathbb{C}$, and $\left(\Omega^+\cup L\cup\Omega^-\right)$ is symmetric about the real axis. Also, $(\mathrm{Im}(f)):\Omega^+\to\mathbb{R}$.

Problem 1.304 $\left(\Omega^+\cup L\cup\Omega^-\right)$ is an open subset of $\mathbb{C}$.

(Solution Since Ω^+,Ω^- are open sets, it suffices to show that every point of L is an interior point of $\left(\Omega^+\cup L\cup\Omega^-\right)$. For this purpose, let us take any $t\in L$. We have to show that t is an interior point of $\left(\Omega^+\cup L\cup\Omega^-\right)$.

By assumption, D_t is an open disk with center t, and $D_t \subset \Omega^+ \cup L \cup \Omega^-$. It follows that t is an interior point of $(\Omega^+ \cup L \cup \Omega^-)$. ∎)

Let us define a function $v : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{R}$ as follows: For every $z \in (\Omega^+ \cup L \cup \Omega^-)$,

$$v(z) \equiv \begin{cases} (\mathrm{Im}(f))(z) & \text{if } z \in \Omega^+ \\ -(\mathrm{Im}(f))(\bar{z}) & \text{if } z \in \Omega^- \\ 0 & \text{if } z \in L. \end{cases}$$

Clearly,

$$v : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{R}$$

is an extension of $(\mathrm{Im}(f)) : \Omega^+ \to \mathbb{R}$.

Problem 1.305 $v : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{R}$ is harmonic in $(\Omega^+ \cup L \cup \Omega^-)$.

(**Solution** Since $f \in H(\Omega^+)$, $\mathrm{Im}(f) : \Omega^+ \to \mathbb{R}$ is continuous. Similarly, $z \mapsto -(\mathrm{Im}(f))(\bar{z})$ from Ω^- to $\mathbb{R}$ is continuous. Now, since for every convergent sequence $\{z_n\}$ in Ω^+ satisfying $\left(\lim_{n\to\infty} z_n\right) \in L$, $\lim_{n\to\infty} (\mathrm{Im}(f))(z_n) = 0$, by the definition of v, $v : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{R}$ is continuous.

Since $f \in H(\Omega^+)$, $f : \Omega^+ \to \mathbb{C}$ is harmonic, and hence $(\mathrm{Im}(f)) : \Omega^+ \to \mathbb{R}$ is harmonic. Now, by Problem 1.301, $z \mapsto (\mathrm{Im}(f))(\bar{z})$ is harmonic, and hence $z \mapsto -(\mathrm{Im}(f))(\bar{z})$ is harmonic. Since $(\mathrm{Im}(f)) : \Omega^+ \to \mathbb{R}$ is harmonic, and $z \mapsto -(\mathrm{Im}(f))(\bar{z})$ is harmonic, by the definition of v, $v : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{R}$ has the mean value property, and hence by Conclusion 1.298, $v : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{R}$ is harmonic in $(\Omega^+ \cup L \cup \Omega^-)$. ∎)

Let us take any $t \in L$.

Here, D_t is an open disk with center t, and $D_t \subset \Omega^+ \cup L \cup \Omega^-$. Now, by Conclusion 1.289, there exists a holomorphic function $-ig_t \in H(D_t)$ such that $v|_{D_t} = \mathrm{Re}(-ig_t)(= \mathrm{Im}(g_t))$.

Thus,

1. $g_t \in H(D_t)$,
2. $v|_{D_t} = \mathrm{Im}(g_t)$,

From 2, for every $z \in D_t \cap \Omega^+$, $((\mathrm{Im}(f))(z) =)v(z) = (\mathrm{Im}(g_t))(z)$.

By Conclusion 1.190,

$$\begin{aligned}(g_t - f)(D_t \cap \Omega^+) &= \{\mathrm{Re}(g_t - f)(z) + i\mathrm{Im}(g_t - f)(z) : z \in D_t \cap \Omega^+\} \\ &= \{\mathrm{Re}(g_t - f)(z) + i0 : z \in D_t \cap \Omega^+\} \\ &= \{\mathrm{Re}(g_t - f)(z) : z \in D_t \cap \Omega^+\} \subset \mathbb{R}\end{aligned}$$

is a region or a singleton. Now, since $(g_t - f)(D_t \cap \Omega^+) \subset \mathbb{R}$, $(g_t - f)(D_t \cap \Omega^+)$ is not open, and hence $(g_t - f)(D_t \cap \Omega^+)$ is not a region, it follows that

$$\left\{\operatorname{Re}(g_t - f)(z) : z \in D_t \cap \Omega^+\right\} = \underbrace{(g_t - f)(D_t \cap \Omega^+)}$$

is a singleton. Hence, there exists a real number k such that for every $z \in D_t \cap \Omega^+$, $(g_t - f)(z) = k$.

This shows that for every $z \in D_t \cap \Omega^+$, $g_t(z) = f(z) + k$.

Since $D_t \cap \Omega^+$ is nonempty, there exists $z_0 \in D_t \cap \Omega^+$. It follows that $(\operatorname{Im}(f))(z_0) = (\operatorname{Im}(g_t))(z_0)$, and $f(z_0) - g_t(z_0)$ is a real number.

Put $h_t \equiv g_t + (f(z_0) - g_t(z_0))$. From 1,

3. $h_t \in H(D_t)$, and from 2,
4. $v|_{D_t} = \operatorname{Im}(h_t)$.

From 4, for every $z \in D_t \cap \Omega^+$,

$$(\operatorname{Im}(f))(z) = \underbrace{v(z) = (\operatorname{Im}(h_t))(z)}.$$

Also,

$$h_t(z_0) = g_t(z_0) + (f(z_0) - g_t(z_0)) = f(z_0),$$

so $h_t(z_0) = f(z_0)$. By Conclusion 1.190,

$$\begin{aligned}
\underbrace{(h_t - f)(D_t \cap \Omega^+)} &= \left\{\operatorname{Re}(h_t - f)(z) + i\operatorname{Im}(h_t - f)(z) : z \in D_t \cap \Omega^+\right\} \\
&= \left\{\operatorname{Re}(h_t - f)(z) + i0 : z \in D_t \cap \Omega^+\right\} \\
&= \left\{\operatorname{Re}(h_t - f)(z) : z \in D_t \cap \Omega^+\right\} \subset \mathbb{R}
\end{aligned}$$

is a region or a singleton. Now, since $(h_t - f)(D_t \cap \Omega^+) \subset \mathbb{R}$, $(h_t - f)(D_t \cap \Omega^+)$ is not open, and hence $(h_t - f)(D_t \cap \Omega^+)$ is not a region, it follows that

$$\left\{\operatorname{Re}(h_t - f)(z) : z \in D_t \cap \Omega^+\right\} = \underbrace{(h_t - f)(D_t \cap \Omega^+)}$$

is a singleton. Hence, there exists a real number l such that for every $z \in D_t \cap \Omega^+$, $(h_t - f)(z) = l$. Now, since $z_0 \in D_t \cap \Omega^+$,

$$0 = f(z_0) - f(z_0) = h_t(z_0) - f(z_0) = \underbrace{(h_t - f)(z_0) = l},$$

and hence

5. $h_t = f$ on $D_t \cap \Omega^+$.

By the definition of v, $v = 0$ on L, so by 4, $\mathrm{Im}(h_t) = 0$ on $L \cap D_t$. This shows that the derivatives of all orders of h_t are real numbers at t, and hence the power series expansion of h_t in powers of $(z - t)$ has only real coefficients. It follows that,

6. for every $z \in D_t$, $\overline{h_t(\bar{z})} = h_t(z)$.

7. Problem 1.306 For every $z \in D_t \cap \Omega^-$, $h_t(z) = \overline{f(\bar{z})}$.

(Solution Let us take any $z \in D_t \cap \Omega^-$. We have to show that $h_t(z) = \overline{f(\bar{z})}$. Since $z \in D_t \cap \Omega^-$, we have $\bar{z} \in D_t \cap \Omega^+$. Now, by 5, $h_t(\bar{z}) = f(\bar{z})$, and hence

$$h_t(z) = \underbrace{\overline{h_t(\bar{z})} = \overline{f(\bar{z})}}.$$

Thus,

$$h_t(z) = \overline{f(\bar{z})}.$$

∎)

8. Suppose that $z \in D_s \cap D_t$, where $s, t \in L$.

Problem 1.307 $h_s(z) = h_t(z)$.

(Solution Case I: when $z \in \Omega^-$. Since $z \in D_s \cap D_t$, we have $\bar{z} \in D_s \cap D_t$. Since $z \in \Omega^-$, we have $\bar{z} \in \Omega^+$, and hence from 6, $h_t(z) = \overline{h_t(\bar{z})}\left(= \overline{f(\bar{z})}\right)$. Thus, $h_t(z) = \overline{f(\bar{z})}$. Similarly,

$$\underbrace{h_s(z) = \overline{f(\bar{z})}} = h_t(z).$$

Hence, $h_s(z) = h_t(z)$.

Case II: when $z \in \Omega^+$. From 5, $h_t = f = h_s$ on $D_s \cap D_t \cap \Omega^+$. It follows that $h_s(z) = h_t(z)$.

Case III: when $z \in L$. Here, $z \in D_s \cap D_t \cap L$. There exists a convergent sequence $\{z_n\}$ in $D_s \cap D_t \cap \Omega^+$ such that $\lim_{n\to\infty} z_n = z$. Now, since $h_t \in H(D_t)$, $\lim_{n\to\infty} h_t(z_n) = h_t(z)$. Similarly, $\lim_{n\to\infty} h_s(z_n) = h_s(z)$. Since each $z_n \in D_s \cap D_t \cap \Omega^+$, by Case II, for every positive integer n, $h_s(z_n) = h_t(z_n)$, and hence

$$h_s(z) = \underbrace{\lim_{n\to\infty} h_s(z_n) = \lim_{n\to\infty} h_t(z_n)} = h_t(z).$$

It follows that $h_s(z) = h_t(z)$.

So, in all cases, $h_s(z) = h_t(z)$. ∎)

From 5, 7, 8, it follows that

$$F : z \mapsto \begin{cases} f(z) & \text{if} z \in \Omega^+ \\ h_t(z) & \text{if} z \in D_t \\ \overline{f(\bar{z})} & \text{if} z \in \Omega^- \end{cases}$$

is a well-defined function from $\Omega^+ \cup L \cup \Omega^-$ to $\mathbb{C}$.

Clearly, F is an extension of f. Since $f \in H(\Omega^+)$, and $h_t \in H(D_t)$, by Conclusion 1.302,

$$F : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{C}$$

is a holomorphic function.

Problem 1.308 For every $z \in (\Omega^+ \cup L \cup \Omega^-)$, $F(\bar{z}) = \overline{F(z)}$.

(**Solution** Case I: when $z \in \Omega^+$. It follows that $\bar{z} \in \Omega^-$, and hence $F(\bar{z}) = \overline{f(\bar{\bar{z}})}\left(= \overline{f(z)}\right)$. Since $z \in \Omega^+$, $F(z) = f(z)$, and hence $\overline{F(z)} = \overline{f(z)}$. Thus, $F(\bar{z}) = \overline{F(z)}$.

Case II: when $z \in \Omega^-$. It follows that $\bar{z} \in \Omega^+$, and hence $F(\bar{z}) = f(\bar{z})$. Since $z \in \Omega^-$, $F(z) = \overline{f(\bar{z})}$, and hence $\overline{F(z)} = f(\bar{z})$. Thus, $F(\bar{z}) = \overline{F(z)}$.

Case III: when $z \in L$. It follows that $\bar{z} = z$. Now, we have to show that $F(z) = \overline{F(z)}$. Thus, it suffices to show that for every $t \in L$, $(h_t(t) =)F(t)$ is real, that is for every $t \in L$, $h_t(t)$ is real. Since for every $t \in L$, $t \in D_t$, we have, from 6, $\left(\overline{h_t(t)} =\right)\overline{h_t(\bar{t})} = h_t(t)$. Thus, for every $t \in L$, $h_t(t)$ is real. ∎)

Conclusion 1.309 Let L be an open segment of the real axis in $\mathbb{C}$. Let Ω^+ be a region in the 'upper half plane' $\{x + iy : x \in \mathbb{R}, \text{and}\, y \in (0, \infty)\}$. Put $\Omega^- \equiv \{z : \bar{z} \in \Omega^+\}$. Suppose that for every $t \in L$, D_t is an open disk with center t such that $D_t \subset \Omega^+ \cup L \cup \Omega^-$. Let $f \in H(\Omega^+)$. Suppose that, for every convergent sequence $\{z_n\}$ in Ω^+ satisfying $\left(\lim_{n\to\infty} z_n\right) \in L$, $\lim_{n\to\infty} (\mathrm{Im}(f))(z_n) = 0$.

Then there exists a function $F : (\Omega^+ \cup L \cup \Omega^-) \to \mathbb{C}$ such that

1. F is an extension of f,
2. F is holomorphic,
3. for every $z \in (\Omega^+ \cup L \cup \Omega^-)$, $F(\bar{z}) = \overline{F(z)}$.

This result is known as the **Schwarz reflection principle**.

Exercises

1.1 Let Ω be a nonempty open subset of $\mathbb{C}$. Show that there exists a family $\{C_i\}_{i\in I}$ of sets such that

a. $\cup_{i\in I} C_i = \Omega$,
b. $i, j \in I$, and $i \neq j \Rightarrow C_i \cap C_j = \emptyset$,
c. each C_i is open,
d. each C_i is connected.

1.2 Let $f : D(0; r) \to \mathbb{C}$ be representable by power series in $D(0; r)$. Show that for every $z \in D(0; r)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

1.3 Let Ω be a nonempty convex open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let $p \in \Omega$. Let $f \in H(\Omega - \{p\})$. Let $\gamma : [0, 1] \to \mathbb{C}$ be a closed path. Suppose that $\operatorname{ran}(\gamma) \subset \Omega$. Show that

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

1.4 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Suppose that for every triangle Δ satisfying $\Delta \subset \Omega$,

$$\int_{\partial\Delta} (f(z))^2 \mathrm{d}z = 0.$$

Show that $(f)^2 \in H(\Omega)$.

1.5 Suppose that $f : D'(0; 2) \to \mathbb{C}$ has an isolated singularity at 0, and $f(D'(0; 1))$ is a bounded subset of $\mathbb{C}$. Show that f has a removable singularity at 1.

1.6 Let $f : D(0; 1) \to \mathbb{C}$ be a function. Let $f \in H(D(0; 1))$. Let $M > 0$. Suppose that for every $z \in D(0; 1)$, $|f(z)| \leq M$. Show that for every positive integer n,

$$\left|f^{(n)}(0)\right| \leq M(n!).$$

1.7 Let $f \in H(D(0; 1))$. Let $\varphi : D(0; 1) \times D(0; 1) \to \mathbb{C}$ be a function defined as follows: For every $(z, w) \in D(0; 1) \times D(0; 1)$,

$$\varphi(z,w) \equiv \begin{cases} \frac{f(z)-f(w)}{z-w} \text{ if} z \neq w \\ f'(w) \text{ if} z = w. \end{cases}$$

Show that $\varphi : D(0;1) \times D(0;1) \to \mathbb{C}$ is continuous.

1.8 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a meromorphic function in Ω. Let A be the set of all points of Ω at which f has pole. Suppose that

a. A has no limit point in Ω,
b. for every $z \in (\Omega - A)$, $f'(z)$ exists.

Let Γ be a cycle in $(\Omega - A)$. Suppose that for every $\alpha \in \Omega^c$, $(\mathrm{Ind})_\Gamma(\alpha) = 0$. Suppose that

$$\{z : z \in A, \text{ and } (\mathrm{Ind})_\Gamma(z) \neq 0\} = \{0, 1\}.$$

Show that

$$\frac{1}{2\pi i}\int_\Gamma f(z)\mathrm{d}z = (\mathrm{Res}(f;0))\big((\mathrm{Ind})_\Gamma(0)\big) + (\mathrm{Res}(f;1))\big((\mathrm{Ind})_\Gamma(1)\big).$$

1.9 For every positive integer n, let $f_n : D(0;1) \to \mathbb{C}$ be a function. Let $f : D(0;1) \to \mathbb{C}$ be a nonzero function. Suppose that each $f_n \in H(D(0;1))$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of $D(0;1)$. Suppose that each f_n has no zero on $D(0;1)$. Show that f has no zero on $D(0;1)$.

1.10 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a harmonic function. Let $\Omega^* \equiv \{\bar{z} : z \in \Omega\}$. Let $g : z \mapsto f(\bar{z})$ be a function from Ω^* to $\mathbb{C}$. Show that $g : \Omega^* \to \mathbb{C}$ is a harmonic function.

Chapter 2
Conformal Mapping

The topics of discussion in this chapter are infinite product and the Riemann mapping theorem. We also prove Weierstrass factorization theorem, Montel's theorem and the Mittag-Leffler theorem.

2.1 Schwarz Lemma

Note 2.1 The collection

$$\{f : f \in H(D(0;1)), \text{and} f \text{ is bounded}\}$$

is denoted by H^∞. Clearly, H^∞ is a complex linear space under pointwise addition and scalar multiplication.

For every $f \in H^\infty$, f is bounded, and hence

$$\{|f(z)| : z \in D(0;1)\}$$

is a nonempty bounded above set of nonnegative real numbers. It follows that

$$\sup\{|f(z)| : z \in D(0;1)\}$$

is a nonnegative real number. We denote

$$\sup\{|f(z)| : z \in D(0;1)\} \text{ by } \|f\|_\infty.$$

Problem 2.2 Under the norm $\|\ \|_\infty$, H^∞ is a normed linear space.

(Solution We must prove

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1. $\|0\|_\infty = 0$,
2. if $\|f\|_\infty = 0$ then $f = 0$,
3. for every $\alpha \in \mathbb{C}$, and for every $f \in H^\infty$, $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$,
4. for every $f, g \in H^\infty$, $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

For 1: This is clear.

For 2: Let $f \in H^\infty$, and $\|f\|_\infty = 0$. We have to show that for every $z \in D(0;1)$, $f(z) = 0$. For this purpose, let us take any $z \in D(0;1)$. We have to show that $f(z) = 0$, that is $|f(z)| = 0$. Since

$$0 \leq |f(z)| \leq \sup\{|f(w)| : w \in D(0;1)\} = \underbrace{\|f\|_\infty = 0},$$

we have $|f(z)| = 0$.

For 3: Let $\alpha \in \mathbb{C}$, and $f \in H^\infty$. We have to show that $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$.

$$\begin{aligned} \text{LHS} = \|\alpha f\|_\infty &= \sup\{|\alpha f(w)| : w \in D(0;1)\} \\ &= \sup\{|\alpha(f(w))| : w \in D(0;1)\} \\ &= \sup\{|\alpha||f(w)| : w \in D(0;1)\} \\ &= |\alpha| \cdot \sup\{|f(w)| : w \in D(0;1)\} \\ &= |\alpha|\|f\|_\infty = \text{RHS}. \end{aligned}$$

For 4: Let $f, g \in H^\infty$. We have to show that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Here,

$$\begin{aligned} \|f + g\|_\infty &= \sup\{|(f + g)(w)| : w \in D(0;1)\} \\ &= \sup\{|f(w) + g(w)| : w \in D(0;1)\} \\ &\leq \sup\{|f(w)| + |g(w)| : w \in D(0;1)\} \\ &\leq \sup\{|f(w)| : w \in D(0;1)\} \\ &\quad + \sup\{|g(w)| : w \in D(0;1)\} \\ &= \|f\|_\infty + \|g\|_\infty, \end{aligned}$$

so

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

∎)

Let $f \in H^\infty$, $f(0) = 0$, and $\sup\{|f(z)| : z \in D(0;1)\} = \|f\|_\infty \in [0, 1]$.

Problem 2.3 The holomorphic function $g : z \mapsto \frac{f(z)}{z}$ from $D'(0;1)$ to $\mathbb{C}$ has a removable singularity at the point 0.

(Solution Let us define a function $h : D(0;1) \to \mathbb{C}$ as follows: For every $z \in D(0;1)$,

$$h(z) \equiv \begin{cases} g(z) & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Clearly, $h|_{D'(0;1)} = g$. Since $\left(h|_{D'(0;1)} =\right) g \in H(D'(0;1))$, it suffices to show that $h'(0)$ exists.

Since $f \in H(D(0;1))$, by Conclusion 1.116, f is representable by power series in $D(0;1)$, and hence there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in D(0;1)$,

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

Now, since $f(0) = 0$, we have $c_0 = 0$. By Conclusion 1.59, $c_1 = f'(0)(= h(0))$. For every $z \in D(0;1)$,

$$f(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

So it follows that for every $z \in D'(0;1)$,

$$h(z) = c_1 + c_2 z + c_3 z^2 + \cdots.$$

Now, since $c_1 = h(0)$, for every $z \in D(0;1)$, $h(z) = c_1 + c_2 z + c_3 z^2 + \cdots$. Thus, h is representable by power series in $D(0;1)$, and hence by Conclusion 1.53, $h'(0)$ exists. ∎)

Thus, for every $z \in D(0;1)$,

$$h(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0. \end{cases}$$

For every $r \in (0,1)$, and for every $z \in D[0;r](\subset D(0;1))$, by Conclusion 1.156,

$$\begin{aligned} |h(z)| &\leq \max\left\{\left|h\left(0 + re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} = \max\left\{\left|h\left(re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} \\ &= \max\left\{\left|\frac{f\left(re^{i\theta}\right)}{re^{i\theta}}\right| : \theta \in [-\pi,\pi]\right\} = \max\left\{\frac{1}{r}\left|f\left(re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} \\ &= \frac{1}{r}\max\left\{\left|f\left(re^{i\theta}\right)\right| : \theta \in [-\pi,\pi]\right\} \leq \frac{1}{r}\cdot 1 = \frac{1}{r} \end{aligned}$$

so for every $r \in (0,1)$, and for every $z \in D[0;r]$, $|h(z)| \leq \frac{1}{r}$. On letting $r \to 1^-$, we get

$$\frac{|f(z)|}{|z|} = \left|\frac{f(z)}{z}\right| = |h(z)| \leq 1$$

for every $z \in D'(0;1)$. Now, since $f(0) = 0$, for every $z \in D(0;1)$, $|f(z)| \leq |z|$.

Conclusion 2.4 Let $f \in H^\infty$, $f(0) = 0$, and $(\sup\{|f(z)| : z \in D(0;1)\} =)\|f\|_\infty \in [0,1]$. Then,

1. for every $z \in D(0;1)$, $|f(z)| \le |z|$,
2. $|f'(0)| \le 1$,
3. if there exists $z_0 \in D'(0;1)$ such that $|f(z_0)| = |z_0|$, then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$, $f(z) = \lambda z$,
4. if $|f'(0)| = 1$, then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and, for every $z \in D(0;1)$, $f(z) = \lambda z$.

Proof of the remaining part

2. Since

$$f'(0) = \lim_{z\to 0} \frac{f(0+z) - f(0)}{z} = \lim_{z\to 0} \frac{f(z) - 0}{z} = \lim_{z\to 0} \frac{f(z)}{z},$$

we have

$$|f'(0)| = \left|\lim_{z\to 0} \frac{f(z)}{z}\right| = \lim_{z\to 0} \left|\frac{f(z)}{z}\right| = \lim_{z\to 0} \frac{|f(z)|}{|z|} \le 1,$$

and hence $|f'(0)| \le 1$.

3. Suppose that there exists $z_0 \in D'(0;1)$ such that $|f(z_0)| = |z_0|$. Since $z_0 \in D'(0;1)$, there exists $r > 0$ such that $D[z_0;r] \subset D'(0;1)$, and $0 \notin D[z_0;r]$. For every $z \in D'(0;1)(\supset D[z_0;r])$, we have

$$\underbrace{|h(z)| \le 1}_{} = \frac{|f(z_0)|}{|z_0|} = |h(z_0)|,$$

so

$$|h(z_0)| \ge \max\left\{\left|h\left(z_0 + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}.$$

Again, by Conclusion 1.153,

$$|h(z_0)| \le \max\left\{\left|h\left(z_0 + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}.$$

It follows that

$$|h(z_0)| = \max\left\{\left|h\left(z_0 + re^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\},$$

and hence, by Conclusion 1.153, h is a constant function, say λ. It follows that $h(z_0) = \lambda$, and hence $|\lambda| = |h(z_0)| = 1$. Thus, $|\lambda| = 1$. Also, for every $z \in D'(0;1)$,

$$\frac{f(z)}{z} = h(z) = \lambda.$$

Now, since $f(0) = 0$, for every $z \in D(0;1), f(z) = \lambda z$.

4. Let $|f'(0)| = 1$. We have to show that there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$,

$$f(z) = \lambda z.$$

If not, otherwise, suppose that there does not exist $\lambda \in \mathbb{C}$ satisfying $|\lambda| = 1$, and for every $z \in D(0;1), f(z) = \lambda z$. We have to arrive at a contradiction.

By Conclusions 2.4(3), and 2.4(1), for every $z \in D'(0;1)$, $|f(z)| < |z|$, and hence for every $z \in D'(0;1)$, $|h(z)| < 1$. Since for every $z \in D(0;1)$,

$$h(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0, \end{cases}$$

for every $z \in D(0;1)$,

$$|h(z)| = \begin{cases} \left|\frac{f(z)}{z}\right| & \text{if } z \neq 0 \\ |f'(0)| & \text{if } z = 0, \end{cases}$$

and hence for every $z \in D(0;1)$,

$$|h(z)| = \begin{cases} \frac{|f(z)|}{|z|}\,(<1) & \text{if } z \neq 0 \\ |f'(0)|(=1) & \text{if } z = 0. \end{cases}$$

Now, by Conclusion 1.153,

$$\begin{aligned} |h(0)| &\leq \max\left\{\left|h\left(0 + \frac{1}{2}e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} \\ &= \max\left\{\left|h\left(\frac{1}{2}e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} < 1 = |h(0)|, \end{aligned}$$

so $|h(0)| < |h(0)|$. This is a contradiction. ∎

This result is known as the **Schwarz lemma**.

2.2 Radial Limits

Note 2.5 Definition Let $\alpha \in D'(0;1)$. It follows that $\frac{1}{\bar{\alpha}} \notin D[0;1]$. The mapping

$$z \mapsto \frac{z-\alpha}{1-\bar{\alpha}z}$$

from $\left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right)$ to $\mathbb{C}$ will be denoted by φ_α. By φ_0 we shall mean the identity function from $\mathbb{C}$ onto $\mathbb{C}$.

Problem 2.6 φ_α is 1-1.

(Solution Let $\varphi_\alpha(z) = \varphi_\alpha(w)$, where $z, w \in \left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right)$. We have to show that $z = w$. Since

$$\frac{z-\alpha}{1-\bar{\alpha}z} = \varphi_\alpha(z) = \varphi_\alpha(w) = \frac{w-\alpha}{1-\bar{\alpha}w},$$

we have

$(z-\alpha)(1-\bar{\alpha}w) = (w-\alpha)(1-\bar{\alpha}z)$, that is $z - \bar{\alpha}zw - \alpha + \alpha\bar{\alpha}w = w - \bar{\alpha}zw - \alpha + \alpha\bar{\alpha}z$, that is $z + \alpha\bar{\alpha}w = w + \alpha\bar{\alpha}z$, that is $(z-w)\left(1-|\alpha|^2\right) = 0$. Now, since $\alpha \in D(0;1)$, $z = w$. ∎)

Problem 2.7 $\varphi_\alpha|_{D[0;1]} \colon D[0;1] \to D[0;1]$.

(Solution Let us take any $z \in D[0;1]$. We have to show that $\left(\frac{z-\alpha}{1-\bar{\alpha}z} =\right)\varphi_\alpha(z) \in D[0;1]$, that is

$$\frac{|z-\alpha|}{|1-\bar{\alpha}z|} = \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right| \leq 1,$$

that is

$$|z-\alpha| \leq |1-\bar{\alpha}z|,$$

that is

$$\begin{aligned} |z|^2 + |\alpha|^2 - 2\mathrm{Re}(\bar{\alpha}z) &= |z-\alpha|^2 \leq |1-\bar{\alpha}z|^2 = 1 + |\bar{\alpha}z|^2 - 2\mathrm{Re}(\bar{\alpha}z) \\ &= 1 + |\alpha|^2|z|^2 - 2\mathrm{Re}(\bar{\alpha}z), \end{aligned}$$

that is

$$0 \leq 1 + |\alpha|^2|z|^2 - |z|^2 - |\alpha|^2,$$

that is

$$0 \le \left(1 - |z|^2\right)\left(1 - |\alpha|^2\right),$$

that is

$$0 \le \left(1 - |z|^2\right),$$

that is $|z| \le 1$. Now, since $z \in D[0;1]$, we have $|z| \le 1$, and hence $\varphi_\alpha(z) \in D[0;1]$. ∎)

Problem 2.8 $\varphi_\alpha|_{D[0;1]}: D[0;1] \to D[0;1]$ is onto.

(Solution Let us take any $w \in D[0;1]$. Since $\alpha \in D(0;1)$, we have $(-\alpha) \in D(0;1)$, and hence $\varphi_{-\alpha}|_{D[0;1]}: D[0;1] \to D[0;1]$. It follows that $\varphi_{-\alpha}(w) \in D[0;1]$. It suffices to show that

$$\varphi_\alpha(\varphi_{-\alpha}(w)) = w.$$

$$\begin{aligned}
\text{LHS} = \varphi_\alpha(\varphi_{-\alpha}(w)) &= \frac{\varphi_{-\alpha}(w) - \alpha}{1 - \bar{\alpha}\varphi_{-\alpha}(w)} \\
&= \frac{\frac{w+\alpha}{1+\bar{\alpha}w} - \alpha}{1 - \bar{\alpha}\frac{w+\alpha}{1+\bar{\alpha}w}} = \frac{w + \alpha - \alpha(1+\bar{\alpha}w)}{(1+\bar{\alpha}w) - \bar{\alpha}(w+\alpha)} \\
&= \frac{w - \alpha\bar{\alpha}w}{1 - \bar{\alpha}\alpha} = \frac{\left(1 - |\alpha|^2\right)w}{1 - |\alpha|^2} \\
&= w = \text{RHS}.
\end{aligned}$$

∎)

Problem 2.9 $\varphi_\alpha|_{\{z:|z|=1\}}: \{z : |z| = 1\} \to \{z : |z| = 1\}$ is 1-1, onto.

(Solution Let us take any $w \in \{z : |z| = 1\}$. Hence $|w| = 1$. We shall show that $\varphi_\alpha(w) \in \{z : |z| = 1\}$, that is $|\varphi_\alpha(w)| = 1$, that is $|\varphi_\alpha(w)|^2 = 1$.

$$\begin{aligned}
\text{LHS} = |\varphi_\alpha(w)|^2 = \left|\frac{w - \alpha}{1 - \bar{\alpha}w}\right|^2 = \frac{|w-\alpha|^2}{|1 - \bar{\alpha}w|^2} = \frac{|w|^2 + |\alpha|^2 - 2\mathrm{Re}(\bar{\alpha}w)}{1 + |\bar{\alpha}w|^2 - 2\mathrm{Re}(\bar{\alpha}w)} \\
= \frac{|w|^2 + |\alpha|^2 - 2\mathrm{Re}(\bar{\alpha}w)}{1 + |\alpha|^2|w|^2 - 2\mathrm{Re}(\bar{\alpha}w)} = \frac{1 + |\alpha|^2 - 2\mathrm{Re}(\bar{\alpha}w)}{1 + |\alpha|^2 1 - 2\mathrm{Re}(\bar{\alpha}w)} = 1 = \text{RHS}.
\end{aligned}$$

Thus

$$\varphi_\alpha|_{\{z:|z|=1\}}: \{z : |z| = 1\} \to \{z : |z| = 1\}.$$

Now, since φ_α is 1-1, $\varphi_\alpha|_{\{z:|z|=1\}}$ is 1-1. It remains to show that

$$\varphi_\alpha|_{\{z:|z|=1\}}\colon \{z : |z| = 1\} \to \{z : |z| = 1\}$$

is onto. For this purpose, let us take any $w \in \{z : |z| = 1\}$. Hence, $|w| = 1$. Since $\alpha \in D(0;1)$, we have $(-\alpha) \in D(0;1)$, and hence

$$\varphi_{-\alpha}|_{\{z:|z|=1\}}\colon \{z : |z| = 1\} \to \{z : |z| = 1\}.$$

Now, since $w \in \{z : |z| = 1\}$, $\varphi_{-\alpha}(w) \in \{z : |z| = 1\}$. It suffices to show that $\varphi_\alpha(\varphi_{-\alpha}(w)) = w$.

$$\text{LHS} = \varphi_\alpha(\varphi_{-\alpha}(w)) = \frac{\varphi_{-\alpha}(w) - \alpha}{1 - \bar{\alpha}\varphi_{-\alpha}(w)} = \frac{\frac{w+\alpha}{1+\bar{\alpha}w} - \alpha}{1 - \bar{\alpha}\frac{w+\alpha}{1+\bar{\alpha}w}} = \frac{w + \alpha - \alpha(1 + \bar{\alpha}w)}{(1 + \bar{\alpha}w) - \bar{\alpha}(w + \alpha)}$$
$$= \frac{w - \alpha\bar{\alpha}w}{1 - \bar{\alpha}\alpha} = \frac{\left(1 - |\alpha|^2\right)w}{1 - |\alpha|^2} = w = \text{RHS}.$$

∎)

Problem 2.10 1. $(\varphi_\alpha)^{-1} = \varphi_{-\alpha}$, 2. $\varphi_\alpha(\alpha) = 0$.

(Solution

1. For this purpose, let us take any $z \in \left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right)$. It suffices to show that $\varphi_{-\alpha}(\varphi_\alpha(z)) = z$.

$$\text{LHS} = \varphi_{-\alpha}(\varphi_\alpha(z)) = \frac{\varphi_\alpha(z) + \alpha}{1 + \bar{\alpha}\varphi_\alpha(z)} = \frac{\frac{z-\alpha}{1-\bar{\alpha}z} + \alpha}{1 + \bar{\alpha}\frac{z-\alpha}{1-\bar{\alpha}z}} = \frac{z - \alpha + \alpha(1 - \bar{\alpha}z)}{(1 - \bar{\alpha}z) + \bar{\alpha}(z - \alpha)}$$
$$= \frac{z - \alpha\bar{\alpha}z}{1 - \bar{\alpha}\alpha} = \frac{\left(1 - |\alpha|^2\right)z}{1 - |\alpha|^2} = z = \text{RHS}.$$

2. Since $\varphi_0(0) = 0$, we consider the case when $\alpha \in D'(0;1)$. Since $(\alpha\bar{\alpha} =)|\alpha|^2 \neq 1$, we have $\alpha \in \left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right)$. Now,

$$\varphi_\alpha(\alpha) = \frac{\alpha - \alpha}{1 - \bar{\alpha}\alpha} = \frac{\alpha - \alpha}{1 - |\alpha|^2} = 0,$$

so $\varphi_\alpha(\alpha) = 0$. ∎)

For every $\alpha \in D'(0;1)$, $\varphi_\alpha : \left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right) \to \left(\mathbb{C} - \left\{-\frac{1}{\bar{\alpha}}\right\}\right)$ is holomorphic, and for every $z \in \left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right)$,

$$(\varphi_\alpha)'(z) = \frac{(1-0)(1-\bar{\alpha}z)-(z-\alpha)(0-\bar{\alpha}1)}{(1-\bar{\alpha}z)^2} = \frac{1-\bar{\alpha}z+z\bar{\alpha}-|\alpha|^2}{(1-\bar{\alpha}z)^2} = \frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2},$$

so

$$(\varphi_\alpha)'(\alpha) = \frac{1-|\alpha|^2}{(1-\bar{\alpha}\alpha)^2} = \frac{1}{1-|\alpha|^2}.$$

Thus,

$$(\varphi_\alpha)'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Also,

$$(\varphi_\alpha)'(0) = \frac{1-|\alpha|^2}{(1-\bar{\alpha}0)^2} = 1-|\alpha|^2.$$

Next, since

$$(\varphi_0)'(0) = 1 = \frac{1}{1-|0|^2} = 1-|0|^2,$$

for every $\alpha \in D(0;1)$,

$$(\varphi_\alpha)'(\alpha) = \frac{1}{1-|\alpha|^2}, \text{and } (\varphi_\alpha)'(0) = 1-|\alpha|^2.$$

Conclusion 2.11

1. For every $\alpha \in D'(0;1)$, $\varphi_\alpha : \left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right) \to \left(\mathbb{C} - \left\{-\frac{1}{\bar{\alpha}}\right\}\right)$ is a holomorphic, 1-1 function,
2. $\varphi_\alpha|_{D[0;1]}: D[0;1] \to D[0;1]$ is 1-1, onto,
3. $\varphi_\alpha|_{\{z:|z|=1\}}: \{z : |z| = 1\} \to \{z : |z| = 1\}$ is 1-1, onto,
4. $(\varphi_\alpha)^{-1} = \varphi_{-\alpha}$,
5. $(\varphi_\alpha)'(\alpha) = \frac{1}{1-|\alpha|^2}$, and $(\varphi_\alpha)'(0) = 1-|\alpha|^2$,
6. For every $\alpha \in D'(0;1)$, $\varphi_\alpha : \left(\mathbb{C} - \left\{\frac{1}{\bar{\alpha}}\right\}\right) \to \left(\mathbb{C} - \left\{-\frac{1}{\bar{\alpha}}\right\}\right)$ has a pole of order 1 at $\frac{1}{\bar{\alpha}}$.

Proof of the remaining part

6. Let us take any $\alpha \in D'(0;1)$. Since

$$\underbrace{\varphi_\alpha : z \mapsto \frac{z-\alpha}{1-\bar{\alpha}z}}=\frac{\frac{-1}{\bar{\alpha}}(1-\bar{\alpha}z)+\left(\frac{1}{\bar{\alpha}}-\alpha\right)}{1-\bar{\alpha}z}=\frac{-1}{\bar{\alpha}}+\frac{\left(\frac{1}{\bar{\alpha}}-\alpha\right)}{-\bar{\alpha}\left(z-\frac{1}{\bar{\alpha}}\right)}$$
$$=\frac{-1}{\bar{\alpha}}+\frac{\frac{-1}{(\bar{\alpha})^2}\left(1-|\alpha|^2\right)}{\left(z-\frac{1}{\bar{\alpha}}\right)},$$

and

$$\frac{-1}{(\bar{\alpha})^2}\left(1-|\alpha|^2\right)\neq 0,$$

$$\varphi_\alpha : \left(\mathbb{C}-\left\{\frac{1}{\bar{\alpha}}\right\}\right)\to\left(\mathbb{C}-\left\{-\frac{1}{\bar{\alpha}}\right\}\right)$$

has a pole of order 1 at $\frac{1}{\bar{\alpha}}$. ∎

Note 2.12 Let $\alpha, \beta \in D'(0;1)$. Let $f \in H^\infty$, $f(\alpha)=\beta$, and

$$\sup\{|f(z)| : z \in D(0;1)\} = \underbrace{\|f\|_\infty \in [0,1]}.$$

Since $f \in H^\infty$, $f : D(0;1) \to \mathbb{C}$ is a holomorphic function. Since

$$\sup\{|f(z)| : z \in D(0;1)\} = \underbrace{f_\infty \in [0,1]},$$

for every $z \in D(0;1)$, $|f(z)| \le 1$, and hence $f : D(0;1) \to D[0;1]$. Since $\beta \in D(0;1)$,

$$\varphi_\beta\big|_{D[0;1]} : D[0;1] \to D[0;1]$$

is 1-1, onto. Similarly,

$$\varphi_{-\alpha}|_{D(0;1)} : D(0;1) \to D(0;1)$$

is 1-1, onto. It follows that

$$\left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right) : D(0;1) \to D[0;1].$$

Since $\beta \in D'(0;1)$, we have $\frac{1}{\beta} \notin D[0;1]$, and hence $D[0;1] \subset \left(\mathbb{C} - \left\{\frac{1}{\beta}\right\}\right)$. Now, since

$$\varphi_\beta : \left(\mathbb{C} - \left\{\frac{1}{\overline{\beta}}\right\}\right) \to \left(\mathbb{C} - \left\{-\frac{1}{\overline{\beta}}\right\}\right)$$

is holomorphic,

$$\varphi_{-\alpha}|_{D(0;1)} : D(0;1) \to D(0;1)$$

is holomorphic and $f : D(0;1) \to D[0;1]$ is holomorphic,

$$\left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right) : D(0;1) \to D[0;1]$$

is holomorphic. It follows that

$$\left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right) \in H^\infty, \left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right)(0) = 0,$$

and

$$\left\|\left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right)\right\|_\infty = \underbrace{\sup\left\{\left|\left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right)(z)\right| : z \in D(0;1)\right\} \in [0,1]}.$$

Now, by Conclusion 2.4,

$$\left|\left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right)'(0)\right| \le 1,$$

so

$$|f'(\alpha)| \le \frac{1 - |\beta|^2}{1 - |\alpha|^2}.$$

Further,

$$\left(|f'(\alpha)| = \frac{1 - |\beta|^2}{1 - |\alpha|^2}\right) \Leftrightarrow \left(\left|\left(\varphi_\beta \circ f \circ \left(\varphi_{-\alpha}|_{D(0;1)}\right)\right)'(0)\right| = 1\right)$$

$\Rightarrow$ (there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$, $\left(\varphi_\beta \circ f \circ \left(\left(\varphi_{-\alpha}|_{D(0;1)}\right)\right)(z) = \lambda z\right)$
$\Rightarrow$ (there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$, $(\varphi_\beta \circ f)(z) = \lambda\varphi_\alpha(z)$)
$\Rightarrow$ (there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$, $f(z) = \varphi_{-\beta}(\lambda\varphi_\alpha(z))$).

Conclusion 2.13 Let $\alpha, \beta \in D'(0;1)$. Let $f \in H^\infty$, $f(\alpha) = \beta$, and $(\sup\{|f(z)| : z \in D(0;1)\} =)\|f\|_\infty \in [0,1]$. Then,

1. $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$,
2. if $|f'(\alpha)| = \frac{1-|\beta|^2}{1-|\alpha|^2}$, then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$, $f(z) = \varphi_{-\beta}(\lambda\varphi_\alpha(z))$.

Similarly,

Conclusion 2.14 Let $\alpha, \beta \in D(0;1)$. Let $f \in H^\infty$, $f(\alpha) = \beta$, and $(\sup\{|f(z)| : z \in D(0;1)\} =)\|f\|_\infty \in [0,1]$. Then,

1. $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$,
2. if $|f'(\alpha)| = \frac{1-|\beta|^2}{1-|\alpha|^2}$, then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$,

$$\underbrace{f(z) = \varphi_{-\beta}(\lambda\varphi_\alpha(z))} = \varphi_{-\beta}\left(\lambda\frac{z-\alpha}{1-\bar{\alpha}z}\right) = \frac{\lambda\frac{z-\alpha}{1-\bar{\alpha}z} + \beta}{1 + \bar{\beta}\lambda\frac{z-\alpha}{1-\bar{\alpha}z}}$$
$$= \frac{\lambda(z-\alpha) + \beta(1-\bar{\alpha}z)}{(1-\bar{\alpha}z) + \bar{\beta}\lambda(z-\alpha)} = \frac{(\lambda - \bar{\alpha}\beta)z + (-\lambda\alpha + \beta)}{(\lambda\bar{\beta} - \bar{\alpha})z + (1 - \lambda\alpha\bar{\beta})}.$$

Note 2.15 Let $f : D(0;1) \to D(0;1)$ be 1-1, onto. Let $f \in H(D(0;1))$. Let $\alpha \in D(0;1)$, and $f(\alpha) = 0$.

Since $f : D(0;1) \to D(0;1)$ is 1-1, onto, and $f \in H(D(0;1))$, by Theorem 1.191,

1. for every $z \in D(0;1)$, $f'(z) \neq 0$,
2. $f^{-1} \in H(D(0;1))$,
3. for every $z \in D(0;1)$, $(f^{-1})'(f(z)) \cdot f'(z) = 1$.

Now, by Conclusion 2.14,

$$|f'(\alpha)| \leq \frac{1-|0|^2}{1-|\alpha|^2}\left(= \frac{1}{1-|\alpha|^2}\right).$$

Since $f^{-1} : D(0;1) \to D(0;1)$, $f^{-1}(0) = \alpha$, and $f^{-1} \in H(D(0;1))$, by Conclusion 2.14,

$$\left|(f^{-1})'(0)\right| \leq \frac{1-|\alpha|^2}{1-|0|^2} = 1 - |\alpha|^2.$$

Now, by 3,

$$1 = \left(1 - |\alpha|^2\right)\frac{1}{1 - |\alpha|^2} \geq \left(1 - |\alpha|^2\right)|f'(\alpha)| \geq \left|\left(f^{-1}\right)'(0)\right||f'(\alpha)|$$
$$= \left|\left(f^{-1}\right)'(0) \cdot f'(\alpha)\right| = \underbrace{\left|\left(f^{-1}\right)'(f(\alpha)) \cdot f'(\alpha)\right| = 1},$$

so

$$\left(1 - |\alpha|^2\right)|f'(\alpha)| = 1,$$

and hence

$$|f'(\alpha)| = \frac{1 - |0|^2}{1 - |\alpha|^2}.$$

Now, by Conclusion 2.14, there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$,

$$f(z) = \varphi_{-0}(\lambda\varphi_\alpha(z)) = \lambda\varphi_\alpha(z).$$

Conclusion 2.16 Let $f : D(0;1) \to D(0;1)$ be 1-1, onto. Let $f \in H(D(0;1))$. Let $\alpha \in D(0;1)$, and $f(\alpha) = 0$. Then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and for every $z \in D(0;1)$,

$$\underbrace{f(z) = \lambda\varphi_\alpha(z)} = \lambda\frac{z - \alpha}{1 - \bar{\alpha}z}.$$

Note 2.17 Definition Let $u : D(0;1) \to \mathbb{C}$ be any function. Let $r \in [0,1)$. It follows that for every real θ, $re^{i\theta} \in D(0;1)$, and hence for every real θ, $u\left(re^{i\theta}\right) \in \mathbb{C}$. The function

$$e^{i\theta} \mapsto u\left(re^{i\theta}\right)$$

from $\{z : |z| = 1\}$ to $\mathbb{C}$ will be denoted by u_r. Thus,

$$u_r : \{z : |z| = 1\} \to \mathbb{C}.$$

Problem 2.18 For a fixed $r \in [0,1)$, if $u : D(0;1) \to \mathbb{C}$ is continuous, then $u_r : \{z : |z| = 1\} \to \mathbb{C}$ is continuous.

(**Solution** Since u_r is the composite of continuous function $e^{i\theta} \mapsto re^{i\theta}$ from $\{z : |z| = 1\}$ to $D(0;1)$, and continuous function $u : D(0;1) \to \mathbb{C}$, $u_r : \{z : |z| = 1\} \to \mathbb{C}$ is continuous. ∎)

Let

$$f:\{z:|z|=1\}\to\mathbb{C}.\,\mathrm{Let} f\in C(\{z:|z|=1\}).$$

It follows that $P[f]:D(0;1)\to\mathbb{C}$, and hence for every $r\in[0,1)$,

$$(P[f])_r\colon \{z:|z|=1\}\to\mathbb{C}.$$

Since $P[f]:D(0;1)\to\mathbb{C}$ is a harmonic function, $P[f]:D(0;1)\to\mathbb{C}$ is a continuous function, and hence for every $r\in[0,1)$,

$$(P[f])_r\colon \{z:|z|=1\}\to\mathbb{C}$$

is continuous. It follows that for every $r\in[0,1)$,

$$\left((P[f])_r - f\right):\{z:|z|=1\}\to\mathbb{C}$$

is continuous. Now, since $\{z:|z|=1\}$ is compact, for every $r\in[0,1)$,

$$\sup\left\{\left|\left((P[f])_r - f\right)(z)\right|:|z|=1\right\}$$

is a nonnegative real number.

Problem 2.19 $\lim_{r\to1^-}\left(\sup\left\{\left|\left((P[f])_r - f\right)(z)\right|:|z|=1\right\}\right)=0.$

(Solution By Problem 1.275, $(Hf):D[0;1]\to\mathbb{C}$ is continuous, where for every $(r,\theta)\in[0,1]\times\mathbb{R}$,

$$(Hf)\left(re^{i\theta}\right)\equiv\begin{cases} f\left(e^{i\theta}\right) & \text{if } r=1\\ (P[f])\left(re^{i\theta}\right) & \text{if } r\in[0,1).\end{cases}$$

Now, since $D[0;1]$ is compact, $(Hf):D[0;1]\to\mathbb{C}$ is uniformly continuous.
Let us take any $\varepsilon>0$. Since

$$(Hf):D[0;1]\to\mathbb{C}$$

is uniformly continuous, there exists $\delta\in(0,1)$ such that every $z,w\in D[0;1]$ satisfying $|z-w|<\delta$ implies

$$|(Hf)(z)-(Hf)(w)|<\varepsilon,$$

and hence every $z\in D[0;1]$ and every real θ satisfying $\left|z-e^{i\theta}\right|<\delta$ implies

$$\left|(Hf)(z)-(Hf)\left(e^{i\theta}\right)\right|<\varepsilon.$$

It follows that for every real θ, and, for every $r\in(1-\delta,1)$, we have $\left|re^{i\theta}-e^{i\theta}\right|<\delta$, and hence,

$$\begin{aligned}\left|\left((P[f])_r - f\right)\left(e^{i\theta}\right)\right| &= \left|(P[f])_r\left(e^{i\theta}\right) - f\left(e^{i\theta}\right)\right| \\ &= \left|(P[f])\left(re^{i\theta}\right) - f\left(e^{i\theta}\right)\right| \\ &= \left|(Hf)\left(re^{i\theta}\right) - f\left(e^{i\theta}\right)\right| < \varepsilon \\ &= \underbrace{\left|(Hf)\left(re^{i\theta}\right) - (Hf)\left(e^{i\theta}\right)\right| < \varepsilon}.\end{aligned}$$

Since for every real θ, and for every $r \in (1-\delta, 1)$, $\left|\left((P[f])_r - f\right)\left(e^{i\theta}\right)\right| < \varepsilon$, for every $r \in (1-\delta, 1)$,

$$\sup\left\{\left|\left((P[f])_r - f\right)(z)\right| : |z| = 1\right\} \leq \varepsilon,$$

and hence

$$\lim_{r \to 1^-}\left(\sup\left\{\left|\left(((P[f])_r\right) - f\right)(z)\right| : |z| = 1\right\}\right) = 0.$$

∎)

In short, we write

$$\lim_{r \to 1^-}\left\|\left((P[f])_r\right) - f\right\|_\infty = 0.$$

Further, for every real θ,

$$\lim_{r \to 1^-}\left((P[f])_r\right)\left(e^{i\theta}\right) = f\left(e^{i\theta}\right).$$

Such limits are known as the '***radial limits***'.

Conclusion 2.20 Let $f : \{z : |z| = 1\} \to \mathbb{C}$. Let $f \in C(\{z : |z| = 1\})$. Then

1. for every real θ, $\lim_{r \to 1}\left((P[f])_r\right)\left(e^{i\theta}\right) = f\left(e^{i\theta}\right)$,
2. $\lim_{r \to 1}\left\|\left((P[f])_r\right) - f\right\|_\infty = 0$.

2.3 Infinite Product

Note 2.21 Definition Let $\{u_n\}$ be any sequence of complex numbers. Put

$$p_1 \equiv (1+u_1), p_2 \equiv (1+u_1)(1+u_2),\ p_3 \equiv (1+u_1)(1+u_2)(1+u_3), \ldots.$$

If $\{p_n\}$ is convergent, then we denote $\lim_{n\to\infty} p_n$ by

$$(1+u_1)(1+u_2)(1+u_3)\ldots, \quad \text{or} \quad \prod_{n=1}^{\infty}(1+u_n),$$

and we say that p_n is the n-th ***partial product*** *of the* ***infinite product*** $\prod_{n=1}^{\infty}(1+u_n)$. We also say that $\prod_{n=1}^{\infty}(1+u_n)$ *is* ***convergent***.

Let $\{u_n\}$ be any sequence of complex numbers. Put

$$\begin{aligned} p_1 &\equiv (1+u_1), p_2 \equiv (1+u_1)(1+u_2), p_3 \equiv (1+u_1)(1+u_2)(1+u_3), \ldots, \\ (p_1)^* &\equiv (1+|u_1|), (p_2)^* \equiv (1+|u_1|)(1+|u_2|), \\ (p_3)^* &\equiv (1+|u_1|)(1+|u_2|)(1+|u_3|), \ldots. \end{aligned}$$

Problem 2.22 For every positive integer N, $1 \le (p_N)^* \le e^{|u_1|+\cdots+|u_N|}$.

(Solution Let us fix any positive integer N. Since for every nonnegative real x, $(1+x) \le e^x$, we have

$$(1 \le (p_N)^* =)(1+|u_1|)\ldots(1+|u_N|) \le e^{|u_1|}\ldots e^{|u_N|}\left(= e^{|u_1|+\cdots+|u_N|}\right),$$

and hence

$$1 \le (p_N)^* \le e^{|u_1|+\cdots+|u_N|}.$$

∎)

Problem 2.23 For every positive integer N, $|p_N - 1| \le (p_N)^* - 1$.

(Solution We must prove

$$|p_1 - 1| \le (p_1)^* - 1, |p_2 - 1| \le (p_2)^* - 1, |p_3 - 1| \le (p_3)^* - 1, \text{ etc.}$$

Since

$$|p_1 - 1| = |(1+u_1) - 1| = |u_1| \le ((1+|u_1|) - 1) = (p_1)^* - 1,$$

we have $|p_1 - 1| \le (p_1)^* - 1$. Since

$$\begin{aligned} |p_2 - 1| &= |p_1(1+u_2) - 1| \\ &= |(p_1 - 1)(1+u_2) + u_2| \\ &\le |(p_1 - 1)(1+u_2)| + |u_2| \le |p_1 - 1||1+u_2| + |u_2| \\ &\le ((p_1)^* - 1)|1+u_2| + |u_2| \le ((p_1)^* - 1)(1+|u_2|) + |u_2| \\ &= (p_1)^*(1+|u_2|) - 1 \\ &= (p_2)^* - 1, \end{aligned}$$

we have $|p_2 - 1| \le (p_2)^* - 1$. Since

$$\begin{aligned}|p_3 - 1| = |p_2(1+u_3) - 1| \\ &= |(p_2-1)(1+u_3)+u_3| \le |(p_2-1)(1+u_3)| + |u_3| \\ &\le |p_2-1||1+u_3| + |u_3| \le ((p_2)^* - 1)|1+u_3| + |u_3| \\ &\le ((p_2)^* - 1)(1+|u_3|) + |u_3| \\ &= (p_2)^*(1+|u_3|) - 1 = (p_3)^* - 1,\end{aligned}$$

we have $|p_3 - 1| \le (p_3)^* - 1$, etc. ∎)

Conclusion 2.24 Let $\{u_n\}$ be any sequence of complex numbers. Put

$$p_1 \equiv (1+u_1), p_2 \equiv (1+u_1)(1+u_2), p_3 \equiv (1+u_1)(1+u_2)(1+u_3), \ldots,$$
$$(p_1)^* \equiv (1+|u_1|), (p_2)^* \equiv (1+|u_1|)(1+|u_2|),$$
$$(p_3)^* \equiv (1+|u_1|)(1+|u_2|)(1+|u_3|), \ldots.$$

Then

1. for every positive integer N, $1 \le (p_N)^* \le e^{|u_1| + \cdots + |u_N|}$,
2. for every positive integer N, $|p_N - 1| \le (p_N)^* - 1$.

Note 2.25 Let S be a nonempty set. Suppose that for every positive integer n, $u_n : S \to \mathbb{C}$ is a bounded function. It follows that for every positive integer n, $|u_n| : S \to [0, \infty)$ is a bounded function. Let $u : S \to [0, \infty)$ be a function. Suppose that $\sum_{n=1}^{\infty} |u_n|$ converges to u uniformly on S.

Problem 2.26 $u : S \to [0, \infty)$ is a bounded function.

(Solution Since $\sum_{n=1}^{\infty} |u_n|$ converges to u uniformly on S, there exists a positive integer N such that for every $m > n \ge N$, $\sum_{j=n+1}^{m} |u_j| < 1$ and hence

$$|u_{N+1}| + |u_{N+2}| + |u_{N+3}| + \cdots \le 1.$$

Since each $|u_n| : S \to [0, \infty)$ is a bounded function, for every positive integer n, there exists a positive real number M_n such that for every positive integer n, $|u_n| \le M_n$. Since

$$|u_1| \le M_1, \ldots, |u_N| \le M_N, \text{and } |u_{N+1}| + |u_{N+2}| + |u_{N+3}| + \cdots \le 1,$$

we have

$$\begin{aligned}\left(u = \sum_{n=1}^{\infty} |u_n|\right) &= (|u_1| + \cdots + |u_N|) + |u_{N+1}| + |u_{N+2}| \\ &\quad + |u_{N+3}| + \cdots \le (M_1 + \cdots + M_N) + 1,\end{aligned}$$

and hence $u : S \to [0, \infty)$ is a bounded function. ∎)

For every positive integer N, put $p_N \equiv (1+u_1)\dots(1+u_N)$. Thus, for every positive integer N, $p_N : S \to \mathbb{C}$.

Problem 2.27 There exists a positive real number C such that for every positive integer N, $|p_N| \le C$.

(Solution Let us fix a positive integer N. We have to show that $|p_N| \le C$.

By Conclusion 2.24(2),

$$(|p_N| - 1 \le)\, |p_N - 1| \le (1+|u_1|)\dots(1+|u_N|) - 1,$$

so

$$|p_N| \le (1+|u_1|)\dots(1+|u_N|).$$

By Conclusion 2.24(1),

$$|p_N| \le \underbrace{(1+|u_1|)\dots(1+|u_N|) \le e^{|u_1|+\dots+|u_N|}} \le e^{|u_1|+|u_2|+\cdots} = e^u,$$

so $|p_N| \le e^u$. Since $u : S \to [0,\infty)$ is a bounded function, there exists a positive real number M such that $u \le M$, and hence $|p_N| \le e^u \le e^M$. Thus, for every positive integer N, $|p_N| \le C$, where $C \equiv e^M$. ∎)

Problem 2.28 $\{p_N\}$ converges uniformly on S. In other words, $\prod_{n=1}^{\infty}(1+u_n)$ converges uniformly on S.

(Solution Let us take any $\varepsilon \in \left(0, \frac{1}{2}\right)$. Since $\sum_{n=1}^{\infty}|u_n|$ converges to u uniformly on S, there exists a positive integer N such that for every positive integer m, n satisfying $m > n \ge N$, $\sum_{j=n+1}^{m}\left|u_j\right| < \varepsilon$, and hence

$$|u_{N+1}| + |u_{N+2}| + |u_{N+3}| + \cdots \le \varepsilon.$$

Now, let us take any positive integers m, n such that $m > n \ge N$. It suffices to show that

$$\begin{aligned}|p_n||(1+u_{n+1})\dots(1+u_m) - 1| &= |p_n((1+u_{n+1})\dots(1+u_m) - 1)| \\ &= \underbrace{|p_m - p_n| \le 2C\varepsilon},\end{aligned}$$

that is

$$|p_n||(1+u_{n+1})\dots(1+u_m) - 1| \le 2C\varepsilon.$$

By Conclusion 2.24(2),

$$|(1+u_{n+1})\ldots(1+u_m)-1| \leq (1+|u_{n+1}|)\ldots(1+|u_m|)-1,$$

so

$$|p_n||(1+u_{n+1})\ldots(1+u_m)-1| \leq |p_n|((1+|u_{n+1}|)\ldots(1+|u_m|)-1).$$

By Conclusion 2.24(1),

$$(1+|u_{n+1}|)\ldots(1+|u_m|) \leq e^{|u_{n+1}|+\cdots+|u_m|} = e^{\sum_{j=n+1}^{m}|u_j|} \leq e^{\varepsilon},$$

so

$$\begin{aligned}|p_n||(1+u_{n+1})\ldots(1+u_m)-1| &\leq |p_n|(e^{\varepsilon}-1)\\ &= |p_n|\left(\varepsilon+\frac{1}{2!}\varepsilon^2+\frac{1}{3!}\varepsilon^3+\cdots\right)\\ &\leq |p_n|\left(\varepsilon+\varepsilon^2+\varepsilon^3+\cdots\right)\\ &= |p_n|\left(\varepsilon+\frac{\varepsilon^2}{1-\varepsilon}\right) \leq |p_n|\varepsilon\left(1+\frac{\varepsilon}{1-\varepsilon}\right)\\ &\leq |p_n|\varepsilon(1+1) = 2|p_n|\varepsilon \leq 2C\varepsilon,\end{aligned}$$

and hence

$$|p_n||(1+u_{n+1})\ldots(1+u_m)-1| \leq 2C\varepsilon.$$

Also,

$$\begin{aligned}|p_N|-|p_m| \leq |p_m-p_N| &= |p_N(1+u_{N+1})\ldots(1+u_m)-p_N|\\ &= |p_N||(1+u_{N+1})\ldots(1+u_m)-1|\\ &\leq 2|p_N|\varepsilon,\end{aligned}$$

so

$$(|p_N|(1-2\varepsilon)=)|p_N|-2|p_N|\varepsilon \leq |p_m|,$$

and hence

$$0 \leq |p_N|(1-2\varepsilon) \leq \lim_{m\to\infty}|p_m| = |\lim_{m\to\infty} p_m| = \left|\prod_{n=1}^{\infty}(1+u_n)\right|.$$

If, for some $s_0 \in S$, $\prod_{n=1}^{\infty}(1+u_n(s_0)) = 0$, then on using

$$0 \le |p_N(s_0)|(1-2\varepsilon) \le \left|\prod_{n=1}^{\infty}(1+u_n(s_0))\right|,$$

we get $|p_N(s_0)|(1-2\varepsilon) = 0$, and hence

$$(1+u_1(s_0))\dots(1+u_N(s_0)) = \underbrace{p_N(s_0) = 0}.$$

It follows that there exists a positive integer N_0 such that $u_{N_0}(s_0) = -1$. ∎)

Conclusion 2.29 Let S be a nonempty set. Suppose that for every positive integer n, $u_n : S \to \mathbb{C}$ is a bounded function. Suppose that $\sum_{n=1}^{\infty}|u_n|$ converges uniformly on S. Then

1. $\prod_{n=1}^{\infty}(1+u_n)$ converges uniformly on S.
2. If, for some $s_0 \in S$, $\prod_{n=1}^{\infty}(1+u_n(s_0)) = 0$, then there exists a positive integer N_0 such that $u_{N_0}(s_0) = -1$.
3. If $\{n_1, n_2, n_3, \dots\}$ is a permutation of $\{1,2,3,\dots\}$, then $\prod_{n=1}^{\infty}(1+u_n) = \prod_{k=1}^{\infty}(1+u_{n_k})$.

Proof of the remaining part

3. Let $\{n_1, n_2, n_3, \dots\}$ be a permutation of $\{1,2,3,\dots\}$. Put

$$q_1 \equiv (1+u_{k_1}), q_2 \equiv (1+u_{n_1})(1+u_{n_2}), q_3 \equiv (1+u_{n_1})(1+u_{n_2})(1+u_{n_3}), \dots.$$

Since $\sum_{n=1}^{\infty}|u_n|$ converges uniformly on S, and $\{n_1, n_2, n_3, \dots\}$ is a permutation of $\{1,2,3,\dots\}$, $\sum_{k=1}^{\infty}|u_{n_k}|$ converges uniformly on S, and hence, by 1, $\prod_{k=1}^{\infty}(1+u_{n_k})$ converges uniformly on S. It follows that $\prod_{k=1}^{\infty}(1+u_{n_k}) = \lim_{k\to\infty} q_k$. Also, $\prod_{n=1}^{\infty}(1+u_n) = \lim_{k\to\infty} p_k$. We have to show that $\lim_{k\to\infty} p_k = \lim_{k\to\infty} q_k$.

It not, otherwise let $\lim_{k\to\infty} p_k \neq \lim_{k\to\infty} q_k$, that is $|(\lim_{k\to\infty} p_k) - (\lim_{k\to\infty} q_k)| > 0$. We have to arrive at a contradiction.

Put $\varepsilon \equiv \min\{\frac{1}{4C}|(\lim_{k\to\infty} p_k) - (\lim_{k\to\infty} q_k)|, \frac{1}{2}\}(>0)$.

Since $\sum_{n=1}^{\infty}|u_n|$ converges uniformly on S, there exists a positive integer N_0 such that for every positive integer m,n satisfying $m > n \ge N_0$, $\sum_{j=n+1}^{m}|u_j| < \varepsilon$, and hence

$$|u_{N_0+1}| + |u_{N_0+2}| + |u_{N_0+3}| + \cdots \le \varepsilon.$$

Let us take any positive integer N such that $N \ge N_0$. Since $\{n_1, n_2, n_3, \dots\}$ is a permutation of $\{1,2,3,\dots\}$, there exists a positive integer M such that $\{1,2,\dots,N\} \subset \{n_1, n_2, \dots, n_M\}$. It follows that for every positive integer m satisfying $m \ge M$, we have

$$
\begin{aligned}
|q_m - p_N| &= \left| \prod_{j \in \{n_1, n_2, \dots, n_m\}} (1+u_j) - p_N \right| \\
&= \left| \prod_{j \in \{1,2,\dots,N\} \cup (\{n_1,n_2,\dots,n_m\} - \{1,2,\dots,N\})} (1+u_j) - p_N \right| \\
&= \left| \prod_{j \in \{1,2,\dots,N\}} (1+u_j) \cdot \prod_{j \in (\{n_1,n_2,\dots,n_m\} - \{1,2,\dots,N\})} (1+u_j) - p_N \right| \\
&= \left| p_N \cdot \prod_{j \in (\{n_1,n_2,\dots,n_m\} - \{1,2,\dots,N\})} (1+u_j) - p_N \right| \\
&= \left| p_N \left(\prod_{j \in (\{n_1,n_2,\dots,n_m\} - \{1,2,\dots,N\})} (1+u_j) - 1 \right) \right| \\
&= |p_N| \left| \prod_{j \in (\{n_1,n_2,\dots,n_m\} - \{1,2,\dots,N\})} (1+u_j) - 1 \right| \\
&\le |p_N| \left(\prod_{j \in (\{n_1,n_2,\dots,n_m\} - \{1,2,\dots,N\})} (1+|u_j|) - 1 \right) \\
&\le |p_N| \left(e^{\sum_{j \in (\{n_1,n_2,\dots,n_m\} - \{1,2,\dots,N\})} |u_j|} - 1 \right) \le |p_N| \left(e^{\sum_{j \in \{N+1, N+2, \dots\}} |u_j|} - 1 \right) \\
&\le |p_N| \left(e^{\sum_{j \in \{N_0+1, N_0+2, \dots\}} |u_j|} - 1 \right) \le |p_N| (e^{\varepsilon} - 1) = |p_N| \left(\varepsilon + \frac{1}{2!}\varepsilon^2 + \frac{1}{3!}\varepsilon^3 + \cdots \right) \\
&\le |p_N| (\varepsilon + \varepsilon^2 + \varepsilon^3 + \cdots) = |p_N| \left(\varepsilon + \frac{\varepsilon^2}{1-\varepsilon} \right) \le |p_N| \varepsilon \left(1 + \frac{\varepsilon}{1-\varepsilon} \right) \\
&\le |p_N| \varepsilon (1+1) = 2|p_N|\varepsilon \le 2C\varepsilon,
\end{aligned}
$$

and hence

$$(N \ge N_0 \text{ and } m \ge M \Rightarrow |q_m - p_N| \le 2C\varepsilon).$$

It follows that

$$\left(N \ge N_0 \Rightarrow 2C\varepsilon \ge \lim_{m\to\infty} |q_m - p_N| \left(= \left| \left(\lim_{m\to\infty} q_m \right) - p_N \right| \right) \right),$$

and hence

$$\left(N \geq N_0 \Rightarrow \left|p_N - \left(\lim_{m\to\infty} q_m\right)\right| \leq 2C\varepsilon\right).$$

It follows that

$$2C\varepsilon < 4C\varepsilon \leq \left|\left(\lim_{N\to\infty} p_N\right) - \left(\lim_{m\to\infty} q_m\right)\right| = \underbrace{\lim_{N\to\infty}\left|p_N - \left(\lim_{m\to\infty} q_m\right)\right| \leq 2C\varepsilon}.$$

This gives a contradiction. ∎

Note 2.30 Let $\{u_n\}$ be any sequence of real numbers in $[0,1)$. It follows that for every positive integer n, $(1-u_n) \in (0,1]$. Let $\sum_{n=1}^{\infty} u_n$ be convergent.

Problem 2.31 $\prod_{n=1}^{\infty}(1-u_n) > 0$.

(Solution If not, otherwise let $\prod_{n=1}^{\infty}(1-u_n) = 0$, that is $\lim_{N\to\infty} p_N = 0$, where for every positive integer N,

$$p_N \equiv (1-u_1)\ldots(1-u_N)(= (1+(-u_1))\ldots(1+(-u_N))).$$

We have to arrive at a contradiction.

For every positive integer n, let

$$\hat{u}_n : 1 \mapsto (-u_n)$$

be any function from $\{1\}$ to $\mathbb{C}$. Clearly, for every positive integer n, $\hat{u}_n : \{1\} \to \mathbb{C}$ is a bounded function. Since $\{u_n\}$ is a sequence of real numbers in $[0,1)$, and $\sum_{n=1}^{\infty} u_n \left(= \sum_{n=1}^{\infty}|-u_n| = \sum_{n=1}^{\infty}|\hat{u}_n|\right)$ is convergent, $\sum_{n=1}^{\infty}|\hat{u}_n|$ converges uniformly on $\{1\}$. Now, by Conclusion 2.29,

1. $\underbrace{\prod_{n=1}^{\infty}(1+\hat{u}_n)} = \prod_{n=1}^{\infty}(1+(-u_n)) = \prod_{n=1}^{\infty}(1-u_n)$ converges uniformly on $\{1\}$, that is $\prod_{n=1}^{\infty}(1-u_n)$ is convergent.
2. If $\prod_{n=1}^{\infty}(1+(-u_n)) = 0$, then there exists a positive integer N_0 such that $(-u_{N_0}) = -1$.
   Since $\prod_{n=1}^{\infty}(1-u_n) = 0$, by 2, there exists a positive integer N_0 such that $(-u_{N_0}) = -1$, and hence $u_{N_0} \notin [0,1)$. This contradicts the assumption. ∎)

Conclusion 2.32 Let $\{u_n\}$ be any sequence of real numbers in $[0,1)$. Then

1. $\sum_{n=1}^{\infty} u_n$ is convergent implies $\prod_{n=1}^{\infty}(1-u_n) > 0$,
2. $\prod_{n=1}^{\infty}(1-u_n) > 0$ implies that $\sum_{n=1}^{\infty} u_n$ is convergent.

Proof of the remaining part (2) Let $\prod_{n=1}^{\infty}(1-u_n)>0$. We have to show that $\sum_{n=1}^{\infty} u_n$ is convergent.

If not, otherwise suppose that $\sum_{n=1}^{\infty} u_n$ diverges to $+\infty$. We have to arrive at a contradiction.

Put

$$p_1 \equiv (1-u_1), p_2 \equiv (1-u_1)(1-u_2), p_3 \equiv (1-u_1)(1-u_2)(1-u_3), \ldots.$$

Since $\prod_{n=1}^{\infty}(1-u_n)>0$, we have $\lim_{n\to\infty} p_n>0$. Since for every positive integer n, $(1-u_n)\in(0,1]$,

$$(1-u_1)\geq(1-u_1)(1-u_2)\geq(1-u_1)(1-u_2)(1-u_3)\geq\cdots>0,$$

and hence

$$0<\cdots\leq p_3\leq p_2\leq p_1\leq 1.$$

It follows that for every positive integer N, $0<\lim_{n\to\infty} p_n\leq p_N\leq 1$.

Problem 2.33 For every $a\in[0,1)$, $0<(1-a)\leq e^{-a}$.

(Solution Let us take any $a\in[0,1)$. We have to show that $(1-a)\leq e^{-a}$. Since $a\in[0,1)$, we have

$$e^{-a}=1-a+\left(\frac{1}{2!}\left(1-\frac{a}{3}\right)a^2+\frac{1}{4!}\left(1-\frac{a}{5}\right)a^4+\cdots\right)\geq(1-a).$$

Thus,

$$0<(1-a)\leq e^{-a}.$$

∎)

Now, since each $u_n\in[0,1)$, for every positive integer N, $0<(1-u_n)\leq e^{-u_n}$. It follows that for every positive integer N,

$$0<p_N=\underbrace{(1-u_1)\ldots(1-u_N)\leq e^{-u_1}\ldots e^{-u_N}}=e^{-u_1-\ldots-u_N}=e^{-\sum\limits_{n=1}^{N}u_n},$$

and hence

$$\prod_{n=1}^{\infty}(1-u_n)=\underbrace{\lim_{N\to\infty}p_N=\lim_{N\to\infty}e^{-\sum\limits_{n=1}^{N}u_n}}=0.$$

Thus, $\prod_{n=1}^{\infty}(1-u_n)=0$. This contradicts the assumption. ∎

Note 2.34 Let Ω be a nonempty open subset of $\mathbb{C}$. (By Lemma 1.32, Ω is partitioned into regions.) For every positive integer n, let $f_n : \Omega \to \mathbb{C}$. Suppose that each $f_n \in H(\Omega)$. (It follows that each $(1 - f_n) \in H(\Omega)$.).

Suppose that no f_n is identically 0 on any region of Ω. Suppose that $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on compact subsets of Ω.

a. Problem 2.35 $\prod_{n=1}^{\infty} f_n$ converges uniformly on compact subsets of Ω.

(Solution Let us take any nonempty compact subset K of Ω. We have to show that $\prod_{n=1}^{\infty} f_n$ converges uniformly on K.

Since each $f_n \in H(\Omega)$, each f_n is continuous on $\Omega\,(\supset K)$. Now, since K is compact, each $(f_n - 1)|_K$ is a bounded function. By assumption, $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on K. Now, by Conclusion 2.29, $\left(\prod_{n=1}^{\infty} f_n =\right)$ $\prod_{n=1}^{\infty}(1 + (f_n - 1))$ converges uniformly on K. Thus, $\prod_{n=1}^{\infty} f_n$ converges uniformly on K. ∎)

b. Problem 2.36 $\left(\prod_{n=1}^{\infty} f_n\right) \in H(\Omega)$.

(Solution Put

$$p_1 \equiv (1 - f_1), p_2 \equiv (1 - f_1)(1 - f_2), p_3 \equiv (1 - f_1)(1 - f_2)(1 - f_3), \ldots.$$

We have to show that $(\lim_{N\to\infty} p_N) \in H(\Omega)$.

By a, $\{p_N\}$ converges uniformly on compact subsets of Ω. Since each $f_n \in H(\Omega)$, each $p_n \in H(\Omega)$, and hence by Conclusion 1.172, $(\lim_{N\to\infty} p_N) \in H(\Omega)$. ∎)

By Theorem 1.136, for every positive integer n, and for every $a \in (f_n)^{-1}(0)$, there exists a unique positive integer $m(f_n; a)$ and a unique function $g_n : \Omega \to \mathbb{C}$ such that

a. $g_n \in H(\Omega)$,
b. for every $z \in \Omega$, $f_n(z) = (z - a)^{m(f_n;a)}(g_n(z))$,
c. $g_n(a) \neq 0$.

For every positive integer n, if $a \notin (f_n)^{-1}(0)$, then $m(f_n; a)$ is defined to be 0.

Since $\left(\prod_{n=1}^{\infty} f_n\right) \in H(\Omega)$, for every $a \in \Omega$, $m\left(\left(\prod_{n=1}^{\infty} f_n\right); a\right)$ is a nonnegative integer.

Problem 2.37 For every $a \in \Omega$, $m\left(\left(\prod_{n=1}^{\infty} f_n\right); a\right) = \sum_{n=1}^{\infty} m(f_n; a)$. Also, only finite-many terms of $\sum_{n=1}^{\infty} m(f_n; a)$ are nonzero.

(Solution Let us fix any $a \in \Omega$. We have to show that $m\left(\left(\prod_{n=1}^{\infty} f_n\right); a\right) = \sum_{n=1}^{\infty} m(f_n; a)$.

Since $a \in \Omega$, and Ω is an open set, there exists $r > 0$ such that $D[a; r] \subset \Omega$. Now, since $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on compact subsets of Ω, and $D[a; r]$ is compact, each point of $(a \in)D[a; r]$ is a zero of only finite-many f_ns, and hence a is a zero of only finite-many f_ns, say, $f_{n_1}, \ldots, f_{n_k}$.

It follows that for every $z \in \Omega$,

$$\left(\prod_{n=1}^{\infty} f_n\right)(z) = (z-a)^{m(f_{n_1};a)}(g_{n_1}(z))\ldots(z-a)^{m(f_{n_k};a)}(g_{n_k}(z))\left(\prod_{n\notin\{n_1,\ldots,n_k\}} f_n(z)\right)$$
$$= (z-a)^{m(f_{n_1};a)+\cdots+m(f_{n_k};a)}\left(g_{n_1}(z)\ldots g_{n_k}(z)\cdot\prod_{n\notin\{n_1,\ldots,n_k\}} f_n(z)\right).$$

By Conclusion 2.29(2), for every $z \in \Omega$, $g_{n_1}(z)\ldots g_{n_k}(z)\cdot\prod_{n\notin\{n_1,\ldots,n_k\}} f_n(z) \neq 0$. It follows that

$$m\left(\left(\prod_{n=1}^{\infty} f_n\right);a\right) = m(f_{n_1};a) + \cdots + m(f_{n_k};a)\left(=\sum_{n=1}^{\infty} m(f_n;a)\right).$$

Thus,

$$m\left(\left(\prod_{n=1}^{\infty} f_n\right);a\right) = \sum_{n=1}^{\infty} m(f_n;a).$$

∎)

Conclusion 2.38 Let Ω be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : \Omega \to \mathbb{C}$. Suppose that each $f_n \in H(\Omega)$. Suppose that no f_n is identically 0 on any region of Ω. Suppose that $\sum_{n=1}^{\infty}|f_n - 1|$ converges uniformly on compact subsets of Ω. Then

1. $\prod_{n=1}^{\infty} f_n$ converges uniformly on compact subsets of Ω,
2. $\left(\prod_{n=1}^{\infty} f_n\right) \in H(\Omega)$,
3. for every $a \in \Omega$, $m\left(\left(\prod_{n=1}^{\infty} f_n\right);a\right) = \sum_{n=1}^{\infty} m(f_n;a)$. Also, only finite-many terms of $\sum_{n=1}^{\infty} m(f_n;a)$ are nonzero.

2.4 Elementary Factors

Note 2.39 Let $z \in D[0;1]$. Let p be a positive integer.

Problem 2.40 $\left|(1-z)e^{\sum_{n=1}^{p}\frac{1}{n}z^n} - 1\right| \leq |z|^{p+1}$.

(Solution Let

$$f : z \mapsto (1-z)e^{\sum_{n=1}^{p}\frac{1}{n}z^n}$$

be a function from $\mathbb{C}$ to $\mathbb{C}$. Clearly, $f \in H(\mathbb{C})$, and for every $z \in \mathbb{C}$,

$$f'(z) = (0-1)e^{\sum_{n=1}^{p}\frac{1}{n}z^n} + (1-z)\left(e^{\sum_{n=1}^{p}\frac{1}{n}z^n}\left(\sum_{n=1}^{p} z^{n-1}\right)\right)$$
$$= e^{\sum_{n=1}^{p}\frac{1}{n}z^n}\left(-1+(1-z)\left(\sum_{n=1}^{p} z^{n-1}\right)\right)$$
$$= e^{\sum_{n=1}^{p}\frac{1}{n}z^n}(-1+(1-z^p)) = -z^p e^{\sum_{n=1}^{p}\frac{1}{n}z^n}.$$

Thus, for every $z \in \mathbb{C}$,

$$f'(z) = -z^p e^{\sum_{n=1}^{p}\frac{1}{n}z^n}.$$

Also, $f(1) = 0$. Since $f \in H(\mathbb{C})$, there exists complex numbers $a_1, a_2, a_3, \ldots$ such that for every $z \in \mathbb{C}$, $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$. It follows that for every $z \in \mathbb{C}$,

$$-z^p\left(1+\left(z+\frac{1}{2}z^2+\cdots+\frac{1}{p}z^p\right)+\frac{1}{2!}\left(z+\frac{1}{2}z^2+\cdots+\frac{1}{p}z^p\right)^2+\cdots\right)$$
$$= -z^p e^{\sum_{n=1}^{p}\frac{1}{n}z^n} = \underbrace{f'(z) = \sum_{n=1}^{\infty} a_n n z^{n-1}}$$
$$= a_1 + a_2 2z + \cdots + a_p p z^{p-1} + a_{p+1}(p+1)z^p + \cdots,$$

and hence by Lemma 1.60,

1. $a_1 = a_2 = \cdots = a_p = 0$,
2. $a_{p+1}, a_{p+2}, a_{p+3}, \ldots$ are negative real numbers.

Here,

$$\left|(1-z)e^{\sum_{n=1}^{p}\frac{1}{n}z^n} - 1\right| = |f(z)-1| = \left|\left(1+\sum_{n=1}^{\infty} a_n z^n\right)-1\right| = \left|\sum_{n=1}^{\infty} a_n z^n\right|$$
$$= \left|\sum_{n=p+1}^{\infty} a_n z^n\right| = \left|z^{p+1}\sum_{n=p+1}^{\infty} a_n z^{n-(p+1)}\right| \le \left|z^{p+1}\right|\sum_{n=p+1}^{\infty}\left|a_n z^{n-(p+1)}\right|$$
$$= \left|z^{p+1}\right|\sum_{n=p+1}^{\infty}|a_n||z|^{n-(p+1)} \le \left|z^{p+1}\right|\sum_{n=p+1}^{\infty}|a_n|1^{n-(p+1)}$$
$$= \left|z^{p+1}\right|\sum_{n=p+1}^{\infty}|a_n| = \left|z^{p+1}\right|\sum_{n=p+1}^{\infty}(-a_n) = -\left|z^{p+1}\right|\sum_{n=1}^{\infty} a_n 1^n$$
$$= -\left|z^{p+1}\right|(f(1)-1) = -\left|z^{p+1}\right|(0-1) = |z|^{p+1}$$

so $\left|(1-z)e^{\sum_{n=1}^{p}\frac{1}{n}z^n}-1\right|\leq|z|^{p+1}$. ∎)

Conclusion 2.41 Let $z\in D[0;1]$. Let p be a positive integer. Then

$$\left|(1-z)e^{\sum\limits_{n=1}^{p}\frac{1}{n}z^n}-1\right|\leq|z|^{p+1}.$$

Definition By E_0, we mean the function $z\mapsto(1-z)$ from $\mathbb{C}$ to $\mathbb{C}$. By E_1, we mean the function $z\mapsto(1-z)e^z$ from $\mathbb{C}$ to $\mathbb{C}$. By E_2, we mean the function $z\mapsto(1-z)e^{z+\frac{1}{2}z^2}$ from $\mathbb{C}$ to $\mathbb{C}$. By E_3, we mean the function $z\mapsto(1-z)e^{z+\frac{1}{2}z^2+\frac{1}{3}z^3}$ from $\mathbb{C}$ to $\mathbb{C}$, etc. Here, each function E_n is called an ***elementary factor***. Clearly, for every nonnegative integer p, 1 is the only zero, with multiplicity 1, of E_p.
Now, from Conclusion 2.41, we get the following

Conclusion 2.42 Let $z\in D[0;1]$. Let p be a nonnegative integer. Then

$$\left|E_p(z)-1\right|\leq|z|^{p+1}.$$

Note 2.43 Let $\{a_n\}$ be any sequence of nonzero complex numbers. Let $\lim_{n\to\infty}|a_n|=\infty$. Let $r>0$.

It follows that there exists a positive integer N such that

$$(n\geq N\Rightarrow 2r<|a_n|),$$

and hence

$$\left(n\geq N\Rightarrow\frac{r}{|a_n|}<\frac{1}{2}\right).$$

Now,

$$\sum_{n=N}^{\infty}\left(\frac{r}{|a_n|}\right)^n\leq\sum_{n=N}^{\infty}\left(\frac{1}{2}\right)^n\leq\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1<\infty,$$

so

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+(n-1)}<\infty.$$

Thus, there exists an infinite sequence $\{p_n\}$ of nonnegative integers such that

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+p_n}<\infty.$$

Let $\{p_n\}$ be any sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+p_n}<\infty.$$

Let $z \in D[0;r]$. Since $\lim_{n\to\infty}|a_n| = \infty$, and each a_n is nonzero, there exists a positive integer N such that

$$(n \geq N \Rightarrow (|z| \leq)\; r < |a_n|),$$

and hence

$$\left(n \geq N \Rightarrow \left|\frac{z}{a_n}\right| < 1\right).$$

Since $\frac{z}{a_N}, \frac{z}{a_{N+1}}, \frac{z}{a_{N+2}}, \ldots$ are in $D[0;1]$, and each p_n is a nonnegative integer, by Conclusion 2.42,

$$\left|E_{p_N}\left(\tfrac{z}{a_N}\right) - 1\right| \leq \left|\tfrac{z}{a_N}\right|^{p_N+1}\left(= \left(\tfrac{|z|}{|a_N|}\right)^{p_N+1} \leq \left(\tfrac{r}{|a_N|}\right)^{p_N+1}\right),$$
$$\left|E_{p_{N+1}}\left(\tfrac{z}{a_{N+1}}\right) - 1\right| \leq \left(\tfrac{r}{|a_{N+1}|}\right)^{p_{N+1}+1}, \left|E_{p_{N+2}}\left(\tfrac{z}{a_{N+2}}\right) - 1\right| \leq \left(\tfrac{r}{|a_{N+2}|}\right)^{p_{N+2}+1}, \text{etc.}$$

It follows that

$$\infty > \sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1} \geq \underbrace{\sum_{n=N}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1} \geq \sum_{n=N}^{\infty}\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right|}_{},$$

and hence

$$\sum_{n=1}^{\infty}\left|E_{p_n}\left(\frac{z}{a_n}\right) - 1\right| < \infty.$$

Let $f_1 : z \mapsto E_{p_1}\left(\frac{z}{a_1}\right)$ be a function from $\mathbb{C}$ to $\mathbb{C}$, and $f_2 : z \mapsto E_{p_2}\left(\frac{z}{a_2}\right)$ be a function from $\mathbb{C}$ to $\mathbb{C}$, etc. Since $E_{p_1} \in H(\mathbb{C})$, we have $f_1 \in H(\mathbb{C})$. Similarly, $f_2 \in H(\mathbb{C})$, etc. Since each E_n has exactly one zero at 1, no f_n is identically 0 on $\mathbb{C}$. Since for every $r > 0$, and for every $z \in D[0;r]$, $\sum_{n=1}^{\infty}|f_n(z) - 1| < \infty$, and compact subsets of $\mathbb{C}$ are bounded, $\sum_{n=1}^{\infty}|f_n - 1|$ converges uniformly on compact subsets of $\mathbb{C}$. Hence, by Conclusion 2.38,

1. $\prod_{n=1}^{\infty} f_n$ converges uniformly on compact subsets of $\mathbb{C}$,
2. $\left(\prod_{n=1}^{\infty} f_n\right) \in H(\mathbb{C})$.

3. for every $a \in \mathbb{C}$, $m\left(\left(\prod_{n=1}^{\infty} f_n\right); a\right) = \sum_{n=1}^{\infty} m(f_n; a)$. Also, only finite-many terms of $\sum_{n=1}^{\infty} m(f_n; a)$ are nonzero.

From 2,

$$P : z \mapsto \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

is an entire function. Since for every nonnegative integer p, 1 is the only zero of E_p, and $a_1, a_2, a_3, \ldots$ are the only zeros of P.

Conclusion 2.44 Let $\{a_n\}$ be any sequence of nonzero complex numbers. Let $\lim_{n\to\infty}|a_n| = \infty$. Then there exists an infinite sequence $\{p_n\}$ of nonnegative integers such that

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+p_n} < \infty.$$

Let $\{p_n\}$ be any sequence of nonnegative integers such that, for every $r > 0$,

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+p_n} < \infty.$$

Then,

1. $P : z \mapsto \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ is an entire function,
2. $a_1, a_2, a_3, \ldots$ are the only zeros of P,
3. if a occurs in the sequence $\{a_1, a_2, a_3, \ldots\}$ exactly m times, then P has a zero at $z = a$ of multiplicity m.

2.5 Weierstrass Factorization Theorem

Note 2.45 Let $f \in H(\mathbb{C})$ (that is, $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function). Suppose that $f(0) \neq 0$.

Since $f(0) \neq 0$, we have $f \neq 0$. It follows, by Conclusion 1.134, that $f^{-1}(0)$ has no limit point.

Let $\{a_1, a_2, a_3, \ldots\}$ be an infinite sequence of the zeros of f, listed according to their multiplicities.

(**Example** Suppose that $f^{-1}(0) = \{i, -2i, 3i, -4i, \ldots\}$. Next, let $m(f; i) = 2$, $m(f; -2i) = 3$, $m(f; 3i) = 3$, $m(f; -4i) = 5, \ldots$. In this case, $\underbrace{a_1 = i, a_2 = i}_{2}$, $\underbrace{a_3 = -2i, a_4 = -2i, a_5 = -2i}_{3}$, $\underbrace{a_6 = 3i, a_7 = 3i, a_8 = 3i}_{3}$, $\underbrace{a_9 = -4i, a_{10} = -4i, a_{11} = -4i, a_{12} = -4i, a_{13} = -4i}_{5}, \ldots$.)

Problem 2.46 $\lim_{n\to\infty}|a_n| = \infty$.

(Solution If not, otherwise there exists a positive real M such that for every positive integer n, there exists a positive integer $k_n \geq n$ such that $|a_{k_n}| \leq M$. Now, since $\{a_1, a_2, a_3, \ldots\}$ is an infinite sequence of the zeros of f, listed according to their multiplicities, $\{a_{k_1}, a_{k_2}, a_{k_3}, \ldots\}$ is an infinite bounded subset of $f^{-1}(0)$, and hence $f^{-1}(0)$ has a limit point. This is a contradiction. ∎)

It follows, by Conclusion 2.48, that there exists a sequence $\{p_n\}$ of nonnegative integers such that

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+p_n} < \infty.$$

Also,

1. $P : z \mapsto \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ is an entire function,
2. f and P have the same zeros with the same multiplicities.

Since $(a_1 \in)f^{-1}(0)$ has no limit point, a_1 is not a limit point of $f^{-1}(0)$, and hence there exists $r_1 > 0$ such that $D'(a_1; r_1)$ contains no point of $f^{-1}(0)$. It follows that for every $z \in D'(a_1; r_1)$, $f(z) \neq 0$, and hence, by 2, for every $z \in D'(a_1; r_1), P(z) \neq 0$. Now, since f and P are holomorphic functions,$z \mapsto \frac{f(z)}{P(z)}$ from $D'(a_1; r_1)$ to $\mathbb{C}$ is holomorphic, and hence, by 2,

$$z \mapsto \frac{f(z)}{P(z)}$$

has a removable singularity at a_1. Similarly,

$$z \mapsto \frac{f(z)}{P(z)}$$

has a removable singularity at each a_n. Next, by 2, there exists a holomorphic function $h : \mathbb{C} \to \mathbb{C}$ such that for every $z \notin f^{-1}(0)$,

$$h(z) = \frac{f(z)}{P(z)}, \text{and}\, h^{-1}(0) = \emptyset.$$

Now, by 2, for every $z \in \mathbb{C}$,

$$\underbrace{f(z) = h(z)P(z)} = h(z)\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right),$$

so for every $z \in \mathbb{C}$,

$$f(z) = h(z) \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Problem 2.47 There exists a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $h = e^g$.

(Solution Since $h \in H(\mathbb{C})$, by Lemma 1.117, $h' \in H(\mathbb{C})$. Now, since $h^{-1}(0) = \emptyset$, the mapping $z \mapsto \frac{h'(z)}{h(z)}$ from $\mathbb{C}$ to $\mathbb{C}$ is holomorphic. Thus, by Conclusion 1.116, $z \mapsto \frac{h'(z)}{h(z)}$ from $\mathbb{C}$ to $\mathbb{C}$ is representable by power series in $\mathbb{C}$. It follows that there exist complex numbers $c_0, c_1, c_2, c_3, \ldots$ such that for every $z \in \mathbb{C}$,

$$\frac{h'(z)}{h(z)} = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

and

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 0.$$

Now, since

$$0 \le \limsup_{n\to\infty} \left|\frac{c_n}{n+1}\right|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \left(\frac{1}{(n+1)^{\frac{1}{n+1}}}\right)^{1+\frac{1}{n}}$$
$$\le \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} 1 = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 0,$$

we have

$$\limsup_{n\to\infty} \left|\frac{c_n}{n+1}\right|^{\frac{1}{n}} = 0,$$

and hence

$$g : z \mapsto \left(c_0 z + \frac{c_1}{2} z^2 + \frac{c_2}{3} z^3 + \cdots\right)$$

from $\mathbb{C}$ to $\mathbb{C}$ is holomorphic. Also, for every $z \in \mathbb{C}$,

$$\underbrace{g'(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}_{} = \frac{h'(z)}{h(z)},$$

and hence for every $z \in \mathbb{C}$,

$$\frac{(e^g)'(z)}{(e^g)(z)} = g'(z) = \frac{h'(z)}{h(z)}.$$

Thus, for every $z \in \mathbb{C}$,

$$\left(e^{g(z)}\right)^2 \left(\frac{h}{e^g}\right)'(z) = \underbrace{(e^g)(z)h'(z) - (e^g)'(z)h(z) = 0},$$

and hence for every $z \in \mathbb{C}$, $\left(\frac{h}{e^g}\right)'(z) = 0$. Now, since $g \in H(\mathbb{C})$, we have $e^g \in H(\mathbb{C})$. Since $e^g, h \in H(\mathbb{C})$, and for every $z \in \mathbb{C}$, $e^{g(z)} \neq 0$, we have $\frac{h}{e^g} \in H(\mathbb{C})$. Now, since for every $z \in \mathbb{C}$, $\left(\frac{h}{e^g}\right)'(z) = 0$, we have for every $z \in \mathbb{C}$,

$$\frac{h(z)}{e^{g(z)}} = \left(\frac{h}{e^g}\right)(z) = \left(\frac{h}{e^g}\right)(0) = \frac{h(0)}{e^{g(0)}}.$$

Since $h^{-1}(0) = \emptyset$, we have $\frac{h(0)}{e^{g(0)}} = e^a$, for some $a \in \mathbb{C}$. Thus, for every $z \in \mathbb{C}$, $h(z) = e^{g(z)+a}$, where

$$z \mapsto (g(z) + a)$$

is a holomorphic function from $\mathbb{C}$ to $\mathbb{C}$. ∎)

Conclusion 2.48 Let $f \in H(\mathbb{C})$. Suppose that $f(0) \neq 0$, and let $a_1, a_2, a_3, \ldots$ be the zeros of f, listed according to their multiplicities. Then there exists $g \in H(\mathbb{C})$, and a sequence $\{p_n\}$ of nonnegative integers such that for every $z \in \mathbb{C}$,

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

This result, known as the **Weierstrass factorization theorem**, is due to K. T. W. Weierstrass (31.10.1815–19.02.1897).

2.6 Conformal Mapping

Note 2.49 Let r be a positive real number. By $D'(\infty; r)$ we mean the set $(\mathbb{C} - D[0; r])\ (= \{z : r < |z|\})$, and by $D(\infty; r)$ we mean the set $(\{\infty\} \cup D'(\infty; r))$.

Let $\mathcal{O}$ be the collection of those subsets of $\left(\mathbb{R}^2 \cup \{\infty\}\right)$ (or equivalently, $(\mathbb{C} \cup \{\infty\})$), which are either $\emptyset$ or $\left(\mathbb{R}^2 \cup \{\infty\}\right)$ or can be expressed as an arbitrary union of sets of the form $D(a; r)$, where $a \in \left(\mathbb{R}^2 \cup \{\infty\}\right)$, and $r \in (0, \infty)$.

It is easy to see that

1. $\mathcal{O}$ contains all open subsets of $\mathbb{R}^2$,
2. each member of $\mathcal{O}$ is either $D(\infty; r)$ for some $r > 0$, or an open subset of $\mathbb{R}^2$, or $D(\infty; r) \cup (\text{an open subset of } \mathbb{R}^2)$ for some $r > 0$.
3. $\mathcal{O}$ is a Hausdorff topology over $(\mathbb{R}^2 \cup \{\infty\})$,
4. the subspace topology of $\mathcal{O}$ over $\mathbb{R}^2$ is the usual topology of $\mathbb{R}^2$.

Let

$$\varphi : (\mathbb{R}^2 \cup \{\infty\}) \to \left\{ (x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}, \text{and} \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = 1 \right\}$$

be the function defined as follows: $\varphi(\infty) = (0, 0, 1), \varphi(0) = (0, 0, -1)$, and, for every $(r, \theta) \in (0, \infty) \times \mathbb{R}$,

$$\varphi\left(re^{i\theta}\right) = \left(\frac{2r \cos\theta}{r^2 + 1}, \frac{2r \sin\theta}{r^2 + 1}, \frac{r^2 - 1}{r^2 + 1} \right).$$

It is clear that

$$\varphi : (\mathbb{R}^2 \cup \{\infty\}) \to \left\{ (x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}, \text{and} \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = 1 \right\}$$

is a 1-1, onto mapping. Further,

$$\varphi : (\mathbb{R}^2 \cup \{\infty\}) \to \left\{ (x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}, \text{and} \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = 1 \right\}$$

is a homeomorphism. That is why $(\mathbb{R}^2 \cup \{\infty\})$ (or say, $(\mathbb{C} \cup \{\infty\})$) is also denoted by $\mathbb{S}^2$.

Here, $\mathbb{S}^2$ is called the ***Riemann sphere***.

Observe that for every $r > 0, D'(\infty; r)$ is an open subset of $\mathbb{C}$.

Definition Let $r > 0$. Let $f : D'(\infty; r) \to \mathbb{C}$. If $f \in H(D'(\infty; r))$, then we say that f *has an* ***isolated singularity*** *at* ∞.

Here, for every $z \in D'\left(0; \frac{1}{r}\right)$, we have $0 < |z| < \frac{1}{r}$, and hence $0 < r < \left|\frac{1}{z}\right|$. Thus, for every $z \in D'\left(0; \frac{1}{r}\right)$, we have $\frac{1}{z} \in D'(\infty; r)$. Now, since $f : D'(\infty; r) \to \mathbb{C}$, we have, for every $z \in D'\left(0; \frac{1}{r}\right), f\left(\frac{1}{z}\right) \in \mathbb{C}$.

We denote the function $z \mapsto f\left(\frac{1}{z}\right)$ from $D'\left(0; \frac{1}{r}\right)$ to $\mathbb{C}$ by $\tilde{f}$, and we define that the nature of singularity of f at ∞ is the same as the nature of singularity of $\tilde{f}$ at 0.

Let Ω be a nonempty open subset of $\mathbb{C}$.

Problem 2.50 $\left(\mathbb{C} - \left(D'(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right) = \right)\mathbb{S}^2 - \left(D(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)$ is a compact subset of Ω, that is $\mathbb{C} - \left(D'(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)$ is a compact subset of Ω.

(**Solution** Since $D'(\infty;2)$ is open in $\mathbb{C}$, and for every $a\in(\mathbb{C}-\Omega), D\left(a;\frac{1}{2}\right)$ in open in $\mathbb{C}$,

$$D'(\infty;2) \cup \left(\bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)$$

is open in $\mathbb{C}$, and hence

$$\mathbb{C} - \left(D'(\infty;2) \cup \left(\bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)$$

is a closed subset of $\mathbb{C}$. It is clear that

$$\mathbb{C} - \left(D'(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right) \left(\subset \{z : |z| \leq 2\}\right)$$

is bounded. Now, by the Heine–Borel theorem,

$$\mathbb{C} - \left(D'(\infty;2) \cup \left(\bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)$$

is a compact subset of $\mathbb{C}$. Since

$$(\mathbb{C}-\Omega) \subset \bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right),$$

we have

$$\mathbb{C} - \left(D'(\infty;2) \cup \left(\bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right) \subset \mathbb{C} - (\mathbb{C}-\Omega) = \Omega.$$

Thus,

$$\mathbb{C} - \left(D'(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)$$

is a compact subset of Ω. ∎)

Similarly,

$$\mathbb{S}^2 - \left(D(\infty;1) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right)$$

is a compact subset of Ω, and

$$\mathbb{S}^2 - \left(D(\infty;3) \cup \left(\bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{3}\right)\right)\right)$$

is a compact subset of Ω, etc. It follows that

$$\begin{aligned}&\left(\mathbb{S}^2 - \left(D(\infty;1) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right)\right)\\ &\quad \cup \left(\mathbb{S}^2 - \left(D(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)\right)\\ &\quad \cup \left(\mathbb{S}^2 - \left(D(\infty;3) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{3}\right)\right)\right)\right) \cup \cdots \subset \Omega.\end{aligned}$$

Problem 2.51

$$\begin{aligned}&\left(\mathbb{S}^2 - \left(D(\infty;1) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right)\right) \cup \left(\mathbb{S}^2 - \left(D(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)\right)\\ &\quad \cup \left(\mathbb{S}^2 - \left(D(\infty;3) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{3}\right)\right)\right)\right) \cup \cdots = \Omega.\end{aligned}$$

(Solution It suffices to show that

$$\begin{aligned}&\underbrace{\Omega \subset \cup_{n=1}^{\infty}\left(\mathbb{C} - \left(D'(\infty;n) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{n}\right)\right)\right)\right)}\\ &\quad = \cup_{n=1}^{\infty}\left((\mathbb{C} - D'(\infty;n)) \cap \left(\mathbb{C} - \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{n}\right)\right)\right)\right)\\ &\quad = \cup_{n=1}^{\infty}\left((\mathbb{C} - D'(\infty;n)) \cap \left(\cap_{a\in(\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{n}\right)\right)\right)\right),\end{aligned}$$

that is

$$\Omega \subset \bigcup_{n=1}^{\infty}\left((\mathbb{C} - D'(\infty;n)) \cap \left(\bigcap_{a\in(\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{n}\right)\right)\right)\right).$$

For this purpose, let us take any $z \in \Omega$. We have to show that

$$z \in \cup_{n=1}^{\infty}\left((\mathbb{C} - D'(\infty;n)) \cap \left(\cap_{a\in(\mathbb{C}-\Omega)}\left(\mathbb{C} - D\left(a;\frac{1}{n}\right)\right)\right)\right).$$

Since $z \in \Omega$, there exists a positive real r such that $D(z;r) \subset \Omega$. There exists a positive integer N_1 such that

$$\left(n \geq N_1 \Rightarrow \frac{1}{n} < r\right).$$

There exists a positive integer N_2 such that

$$(n \geq N_2 \Rightarrow |z| \leq n),$$

and hence

$$(n \geq N_2 \Rightarrow z \notin D'(\infty;n)).$$

Put $N \equiv \max\{N_1, N_2\}$.
It follows that

$$(n \geq N \Rightarrow z \in (\mathbb{C} - D'(\infty;n))),$$

and hence

$$z \in (\mathbb{C} - D'(\infty;N)).$$

Also, $\frac{1}{N} < r$. Since $D(z;r) \subset \Omega$, for every $a \in (\mathbb{C} - \Omega)$, we have $z \notin D(a;r)$, and hence

$$\underbrace{z \in \bigcap_{a\in(\mathbb{C}-\Omega)} (\mathbb{C} - D(a;r))} \subset \bigcap_{a\in(\mathbb{C}-\Omega)}\left(\mathbb{C} - D\left(a;\frac{1}{N}\right)\right).$$

Now, since $z \in (\mathbb{C} - D'(\infty;N))$, and $z \in \bigcap_{a\in(\mathbb{C}-\Omega)}\left(\mathbb{C} - D\left(a;\frac{1}{N}\right)\right)$, we have

$$z \in \left((\mathbb{C} - D'(\infty;N)) \cap \left(\bigcap_{a\in(\mathbb{C}-\Omega)}\left(\mathbb{C} - D\left(a;\frac{1}{N}\right)\right)\right)\right).$$

∎)

For every positive integer n, let us put

$$K_n \equiv \mathbb{S}^2 - \left(D(\infty;n) \cup \left(\bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{n}\right)\right)\right).$$

Since

$$D(\infty;1) \supset D(\infty;2), \text{and} \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right) \supset \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right),$$

we have

$$\left(D(\infty;1) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right) \supset \left(D(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right),$$

and hence

$$K_1 = \underbrace{\left(\mathbb{S}^2 - \left(D(\infty;1) \cup \left(\bigcup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right)\right) \subset \left(\mathbb{S}^2 - \left(D(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)\right)} = K_2.$$

Thus, $K_1 \subset K_2$. Similarly, $K_2 \subset K_3$. Thus, we have seen that each K_n is a compact subset of Ω, $K_1 \subset K_2 \subset K_3 \subset \cdots \subset \Omega$, and

$$\Omega = \cup_{n=1}^{\infty} K_n.$$

Problem 2.52 $K_1 \subset (K_2)^0$, that is

$$\mathbb{S}^2 - \left(D(\infty;1) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right)$$
$$\subset \left(\mathbb{S}^2 - \left(D(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)\right)^0,$$

that is

$$\begin{aligned}
&(\mathbb{C} - D'(\infty;1)) \cap \left(\cap_{a\in(\mathbb{C}-\Omega)}\left(\mathbb{C} - D\left(a;\frac{1}{1}\right)\right)\right) \\
&= (\mathbb{C} - D'(\infty;1)) \cap \left(\mathbb{C} - \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right) \\
&= \underbrace{\mathbb{C} - \left(D'(\infty;1) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{1}\right)\right)\right) \subset \left(\mathbb{C} - \left(D'(\infty;2) \cup \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)\right)^0} \\
&= \left((\mathbb{C} - D'(\infty;2)) \cap \left(\mathbb{C} - \left(\cup_{a\in(\mathbb{C}-\Omega)} D\left(a;\frac{1}{2}\right)\right)\right)\right)^0 \\
&= \left((\mathbb{C} - D'(\infty;2)) \cap \left(\cap_{a\in(\mathbb{C}-\Omega)}\left(\mathbb{C} - D\left(a;\frac{1}{2}\right)\right)\right)\right)^0.
\end{aligned}$$

That is, every point of

$$(\mathbb{C} - D'(\infty;1)) \cap \left(\bigcap_{a \in (\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{1}\right)\right)\right)$$

is an interior point of

$$(\mathbb{C} - D'(\infty;2)) \cap \left(\bigcap_{a \in (\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{2}\right)\right)\right).$$

(**Solution** For this purpose, let us take any

$$z \in \left((\mathbb{C} - D'(\infty;1)) \cap \left(\bigcap_{a \in (\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{1}\right)\right)\right)\right).$$

It follows that $z \in (\mathbb{C} - D'(\infty;1))$, and $z \in \left(\cap_{a\in(\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{1}\right)\right)\right)$, and hence $|z| \le 1$, and, for every $a \in (\mathbb{C} - \Omega)$, we have $\frac{1}{1} \le |z-a|$. It suffices to show that

$$D\left(z;\frac{1}{1}-\frac{1}{2}\right) \subset (\mathbb{C} - D'(\infty;2)) \cap \left(\cap_{a \in (\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{2}\right)\right)\right).$$

For this purpose, let us take any $w \in D\left(z;\frac{1}{1}-\frac{1}{2}\right)$. We have to show that

1. $w \in (\mathbb{C} - D'(\infty;2))$, that is $|w| \le 2$,
2. $w \in \cap_{a\in(\mathbb{C}-\Omega)} \left(\mathbb{C} - D\left(a;\frac{1}{2}\right)\right)$, that is for every $a \in (\mathbb{C} - \Omega)$, $\frac{1}{2} \le |w-a|$.

For 1: Since $w \in D\left(z;\frac{1}{1}-\frac{1}{2}\right)$, we have $|w-z| < \left(\frac{1}{1}-\frac{1}{2}\right)$. Now,

$$|w| = |(w-z)+z| \le |w-z| + |z| < \left(\frac{1}{1}-\frac{1}{2}\right) + |z| \le \left(\frac{1}{1}-\frac{1}{2}\right) + 1 < (1) + 1 = 2,$$

so $|w| \le 2$.

For 2: Since $w \in D\left(z;\frac{1}{1}-\frac{1}{2}\right)$, we have $|w-z| < \left(\frac{1}{1}-\frac{1}{2}\right)$, and hence $-\left(\frac{1}{1}-\frac{1}{2}\right) < -|w-z|$. Now, since $\frac{1}{1} \le |a-z|$, we have

$$\frac{1}{2} = \underbrace{\frac{1}{1} + \left(-\left(\frac{1}{1}-\frac{1}{2}\right)\right) \le |a-z| + (-|w-z|)}$$

$$= |a-z| - |w-z| \le |(a-z)-(w-z)| = |a-w| = |w-a|,$$

and hence $\frac{1}{2} \le |w-a|$. ∎)

Similarly, $K_2 \subset (K_3)^0$, $K_3 \subset (K_4)^0$, etc. Thus, $(K_1)^0 \subset K_1 \subset (K_2)^0 \subset K_2 \subset (K_3)^0 \subset K_3 \subset \cdots \subset \Omega$.

Suppose that K is a compact set contained in Ω. Since

$$(K_1)^0 \subset K_1 \subset (K_2)^0 \subset K_2 \subset (K_3)^0 \subset K_3 \subset \cdots \subset \Omega,$$

and $\Omega = \cup_{n=1}^{\infty} K_n$, we have $(K \subset)\Omega = \cup_{n=1}^{\infty} (K_n)^0$, and hence

$$\left\{(K_1)^0, (K_2)^0, (K_3)^0, \ldots\right\}$$

is an open cover of the compact set K. Now, since

$$(K_1)^0 \subset (K_2)^0 \subset (K_3)^0 \subset \ldots,$$

there exists a positive integer N such that

$$K \subset (K_1)^0 \cup (K_2)^0 \cup \cdots \cup (K_N)^0 \Big(= (K_N)^0 \subset K_N\Big).$$

Thus, $K \subset K_N$.

Conclusion 2.53 Let Ω be a nonempty open subset of $\mathbb{C}$. Then, there exists a sequence $\{K_1, K_2, K_3, \ldots\}$ of compact subsets of Ω satisfying

1. $(K_1)^0 \subset K_1 \subset (K_2)^0 \subset K_2 \subset (K_3)^0 \subset K_3 \subset \cdots \subset \Omega$,
2. $\Omega = \cup_{n=1}^{\infty} K_n$,
3. for every compact set K contained in Ω, there exists a positive integer N such that $K \subset K_N$,
4. for every positive integer n, and for every $z \in K_n$, $D\left(z; \frac{1}{n} - \frac{1}{n+1}\right) \subset K_{n+1}$,
5. for every positive integer n, if C is a component of $\left(\mathbb{S}^2 - K_n\right)$, then there exists a component D of $\left(\mathbb{S}^2 - \Omega\right)$ such that $D \subset C$.

Proof of the remaining part

5. Let us fix any positive integer n. Let C be a component of $\left(\mathbb{S}^2 - K_n\right)$.

Observe that

$$\begin{aligned}\left(\mathbb{S}^2 - K_n\right) &= \mathbb{S}^2 - \left(\mathbb{S}^2 - \left(D(\infty; n) \cup \left(\cup_{a \in (\mathbb{C} - \Omega)} D\left(a; \frac{1}{n}\right)\right)\right)\right) \\ &= D(\infty; n) \cup \left(\cup_{a \in (\mathbb{C} - \Omega)} D\left(a; \frac{1}{n}\right)\right).\end{aligned}$$

Thus, C is a connected subset of $\left(\mathbb{S}^2 - K_n\right)$.

Since $\infty \in D(\infty; n)$, and $\infty \in (\mathbb{S}^2 - \Omega)$, $D(\infty; n)$ intersects $(\mathbb{S}^2 - \Omega)$. For every $a \in (\mathbb{C} - \Omega)(\subset (\mathbb{S}^2 - \Omega))$, $a \in D(a; \frac{1}{n})$, and $a \in (\mathbb{S}^2 - \Omega)$. Thus, for every $a \in (\mathbb{C} - \Omega)$, $D(a; \frac{1}{n})$ intersects $(\mathbb{S}^2 - \Omega)$. Hence, $D(\infty; n)$, and each $D(a; \frac{1}{n})$ intersect $(\mathbb{S}^2 - \Omega)$.

Since $D'(\infty; n)$, and each $D(a; \frac{1}{n})$ are connected sets in $\mathbb{C}$, $D(\infty; n)$, and each $D(a; \frac{1}{n})$ are connected sets in $\mathbb{S}^2$.

Since C is a component of $D(\infty; n) \cup \left(\cup_{a \in (\mathbb{C} - \Omega)} D(a; \frac{1}{n}) \right)$; and $D(\infty; n)$, each $D(a; \frac{1}{n})$ are connected subsets of $D(\infty; n) \cup \left(\cup_{a \in (\mathbb{C} - \Omega)} D(a; \frac{1}{n}) \right)$, either $D(\infty; n)$ is contained in C, or some $D(a; \frac{1}{n})$ is contained in C. Now, since $D(\infty; n)$, and each $D(a; \frac{1}{n})$ intersect $(\mathbb{S}^2 - \Omega)$, C intersects $(\mathbb{S}^2 - \Omega)$. Since C intersects $(\mathbb{S}^2 - \Omega)$, and the components of $(\mathbb{S}^2 - \Omega)$ partitions $(\mathbb{S}^2 - \Omega)$, there exists a component D of $(\mathbb{S}^2 - \Omega)$ such that C intersects D. Since D is a component of $(\mathbb{S}^2 - \Omega)$, D is a connected subset of $(\mathbb{S}^2 - \Omega)$.

Since $K_n \subset \Omega$, we have $(D \subset)(\mathbb{S}^2 - \Omega) \subset (\mathbb{S}^2 - K_n)$. Since C is connected, D is connected and C intersects D, $C \cup D$ is connected. Now, since $D \subset (\mathbb{S}^2 - K_n)$, and $C \subset (\mathbb{S}^2 - K_n)$, $C \cup D$ is a connected subset of $(\mathbb{S}^2 - K_n)$. Since $C \cup D$ is a connected subset of $(\mathbb{S}^2 - K_n)$, C is a component of $(\mathbb{S}^2 - K_n)$, and $C \subset C \cup D$, we have $C = C \cup D$, and hence $D \subset C$. ∎

Note 2.54 Definition Let z be a nonzero complex number. By $A[z]$ we mean $\frac{1}{|z|} z$, and we say that $A[z]$ *determines a* ***direction*** *of z from the origin.*

Definition Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a mapping. Let $z_0 \in \Omega$. Suppose that there exists $\rho > 0$ such that $D'(z_0; \rho) \subset \Omega$, and for every $z \in D'(z_0; \rho)$, $f(z) \neq f(z_0)$.

(It follows that for every $r \in (0, \rho)$, $D'[z_0; r] \subset \Omega$, and for every $z \in D'[z_0; r]$, $f(z) \neq f(z_0)$. Thus, for every $r \in (0, \rho)$, and for every real θ, $f(z_0 + re^{i\theta}) \neq f(z_0)$, and hence $f(z_0 + re^{i\theta}) - f(z_0)$ is a nonzero complex number. Hence, for every $r \in (0, \rho)$, and for every real θ, $A[f(z_0 + re^{i\theta}) - f(z_0)]$ is a complex number.)

By *f* ***preserves angle*** *at z_0*, we mean that for every real θ, $\lim_{r \to 0^+} e^{-i\theta} A[f(z_0 + re^{i\theta}) - f(z_0)]$ exists and is independent of θ.

Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a mapping. Let $z_0 \in \Omega$. Suppose that there exists $\rho > 0$ such that $D'(z_0; \rho) \subset \Omega$, and for every $z \in D'(z_0; \rho)$, $f(z) \neq f(z_0)$. Suppose that $f'(z_0)$ exists and $f'(z_0) \neq 0$. Now, for every real θ,

$$\begin{aligned}
&\lim_{r\to 0^+} e^{-i\theta}A\left[f\left(z_0+re^{i\theta}\right)-f(z_0)\right]\\
&\quad=\lim_{r\to 0^+} e^{-i\theta}\frac{1}{\left|f\left(z_0+re^{i\theta}\right)-f(z_0)\right|}\left(f\left(z_0+re^{i\theta}\right)-f(z_0)\right)\\
&\quad=\lim_{r\to 0^+}\frac{1}{\left|\frac{f\left(z_0+re^{i\theta}\right)-f(z_0)}{\left(z_0+re^{i\theta}\right)-z_0}\right|}\frac{f\left(z_0+re^{i\theta}\right)-f(z_0)}{\left(z_0+re^{i\theta}\right)-z_0}\\
&\quad=\frac{1}{|f'(z_0)|}f'(z_0)
\end{aligned}$$

so for every real θ, $\lim_{r\to 0^+} e^{-i\theta}A\left[f\left(z_0+re^{i\theta}\right)-f(z_0)\right]$ exists and is independent of θ.

Let Ω be a region. Let $f:\Omega\to\mathbb{C}$ be a mapping. Let $z_0\in\Omega$. Suppose that there exists $\rho>0$ such that $D'(z_0;\rho)\subset\Omega$, and for every $z\in D'(z_0;\rho)$, $f(z)\neq f(z_0)$.

(It follows that for every $r\in(0,\rho)$, $D'[z_0;r]\subset\Omega$, and for every $z\in D'[z_0;r]$, $f(z)\neq f(z_0)$. Thus, for every $r\in(0,\rho)$, and for every real θ, $f\left(z_0+re^{i\theta}\right)\neq f(z_0)$, and hence $f\left(z_0+re^{i\theta}\right)-f(z_0)$ is a nonzero complex number. Hence, for every $r\in(0,\rho)$, and for every real θ, $A\left[f\left(z_0+re^{i\theta}\right)-f(z_0)\right]$ is a complex number.)

Suppose that $f'(z_0)$ exists, and $f'(z_0)\neq 0$. Now, for every real θ,

$$\begin{aligned}
&\lim_{r\to 0^+} e^{-i\theta}A\left[f\left(z_0+re^{i\theta}\right)-f(z_0)\right]\\
&\quad=\lim_{r\to 0^+} e^{-i\theta}\frac{1}{\left|f\left(z_0+re^{i\theta}\right)-f(z_0)\right|}\left(f\left(z_0+re^{i\theta}\right)-f(z_0)\right)\\
&\quad=\lim_{r\to 0^+}\frac{1}{\left|\frac{f\left(z_0+re^{i\theta}\right)-f(z_0)}{\left(z_0+re^{i\theta}\right)-z_0}\right|}\frac{f\left(z_0+re^{i\theta}\right)-f(z_0)}{\left(z_0+re^{i\theta}\right)-z_0}\\
&\quad=\frac{1}{|f'(z_0)|}f'(z_0)
\end{aligned}$$

so for every real θ,

$$\lim_{r\to 0^+} e^{-i\theta}A\left[f\left(z_0+re^{i\theta}\right)-f(z_0)\right]$$

exists and is independent of θ.

Conclusion 2.55 Let Ω be a region. Let $f:\Omega\to\mathbb{C}$ be a mapping. Let $z_0\in\Omega$. Suppose that there exists $\rho>0$ such that $D'(z_0;\rho)\subset\Omega$, and for every $z\in D'(z_0;\rho)$, $f(z)\neq f(z_0)$. Suppose that $f'(z_0)$ exists, and $f'(z_0)\neq 0$. Then, for every real θ, $\lim\limits_{r\to 0^+} e^{-i\theta}A\left[f\left(z_0+re^{i\theta}\right)-f(z_0)\right]$ exists and is independent of θ.

Definition Let Ω be a region. Let $f:\Omega\to\mathbb{C}$ be a mapping. Suppose that for every $z\in\Omega$, $f'(z)$ exists, and $f'(z)\neq 0$, then we say that *f is a* ***conformal mapping***.

Conclusion 2.56 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a mapping. Suppose that for every $z \in \Omega$, there exists $\rho > 0$ such that $D'(z;\rho) \subset \Omega$, and for every $w \in D'(z;\rho)$, $f(w) \neq f(z)$. Let f be a conformal mapping. Then f preserves angle at every point of Ω.

Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a mapping. Let $a \in \Omega$. Suppose that there exists $\rho > 0$ such that $D'(a;\rho) \subset \Omega$, and for every $z \in D'(a;\rho)$, $f(z) \neq f(a)$. Suppose that f preserves angle at a. Suppose that the differential of f at a exists and is nonzero.

(That is, there exists a complex-valued function η, and there exist real numbers $A_{11}, A_{12}, A_{21}, A_{22}$ such that

1. not all $A_{11}, A_{12}, A_{21}, A_{22}$ are zero,
2. $\lim_{z\to 0} \eta(z) = 0$,
3. for every $z \in D(0;\rho)$,

$$\begin{aligned}
&\underbrace{f(a+z) = f(a) + ((A_{11}\mathrm{Re}(z) + A_{12}\mathrm{Im}(z)) + i(A_{21}\mathrm{Re}(z) + A_{22}\mathrm{Im}(z))) + |z|\eta(z)}\\
&\quad = f(a) + (((A_{11} + iA_{21})\mathrm{Re}(z) + (A_{12} + iA_{22})\mathrm{Im}(z))) + |z|\eta(z)\\
&\quad = f(a) + \left((A_{11} + iA_{21})\frac{1}{2}(z + \bar{z}) + (A_{12} + iA_{22})\frac{-i}{2}(z - \bar{z})\right) + |z|\eta(z)\\
&\quad = f(a) + \left(\left(\frac{1}{2}(A_{11} + iA_{21}) - \frac{i}{2}(A_{12} + iA_{22})\right)z\right.\\
&\qquad \left. + \left(\frac{1}{2}(A_{11} + iA_{21}) + \frac{i}{2}(A_{12} + iA_{22})\right)\bar{z}\right) + |z|\eta(z)\\
&\quad = f(a) + (\alpha z + \beta\bar{z}) + |z|\eta(z),
\end{aligned}$$

where $\alpha \equiv \frac{1}{2}((A_{11} + A_{22}) + i(A_{21} - A_{12}))$, and $\beta \equiv \frac{1}{2}((A_{11} - A_{22}) + i(A_{21} + A_{12}))$.

Thus, for every $z \in D(0;\rho)$,

$$f(a+z) = f(a) + (\alpha z + \beta\bar{z}) + |z|\eta(z),$$

where $\alpha \equiv \frac{1}{2}((A_{11} + A_{22}) + i(A_{21} - A_{12}))$, and $\beta \equiv \frac{1}{2}((A_{11} - A_{22}) + i(A_{21} + A_{12}))$. Since not all $A_{11}, A_{12}, A_{21}, A_{22}$ are zero, α and β both cannot be 0 simultaneously. ■)

Problem 2.57 $f'(a)$ exists, and $f'(a) \neq 0$.

(Solution Since for every $z \in D(0;\rho)$,

$$f(a+z) = f(a) + (\alpha z + \beta\bar{z}) + |z|\eta(z),$$

and $\lim_{z\to 0}\eta(z) = 0$, it suffices to show that $\beta = 0$. If not, otherwise, let $\beta \neq 0$, that is $|\beta| > 0$. We have to arrive at a contradiction.

Since f preserves angle at a, for every real θ, $\lim_{r\to 0^+} e^{-i\theta}A\left[f\left(a+re^{i\theta}\right) - f(a)\right]$ exists and is independent of θ. Here, for every real θ,

$$\begin{aligned}
&\lim_{r\to 0^+} e^{-i\theta}A\left[f\left(a+re^{i\theta}\right) - f(a)\right] \\
&\quad = \lim_{r\to 0^+} e^{-i\theta}A\left[\left(f(a) + \left(\alpha\left(re^{i\theta}\right) + \beta\overline{\left(re^{i\theta}\right)}\right) + \left|re^{i\theta}\right|\eta\left(re^{i\theta}\right)\right) - f(a)\right] \\
&\quad = \lim_{r\to 0^+} e^{-i\theta}A\left[\left(\alpha\left(re^{i\theta}\right) + \beta\overline{\left(re^{i\theta}\right)}\right) + \left|re^{i\theta}\right|\eta\left(re^{i\theta}\right)\right] \\
&\quad = \lim_{r\to 0^+} e^{-i\theta}A\left[\left(\alpha\left(re^{i\theta}\right) + \beta\left(re^{-i\theta}\right)\right) + r\eta\left(re^{i\theta}\right)\right] \\
&\quad = \lim_{r\to 0^+} e^{-i\theta}\frac{\left(\alpha\left(re^{i\theta}\right) + \beta\left(re^{-i\theta}\right)\right) + r\eta\left(re^{i\theta}\right)}{\left|\left(\alpha\left(re^{i\theta}\right) + \beta\left(re^{-i\theta}\right)\right) + r\eta\left(re^{i\theta}\right)\right|} \\
&\quad = \lim_{r\to 0^+} e^{-i\theta}\frac{\left(\alpha\left(e^{i\theta}\right) + \beta\left(e^{-i\theta}\right)\right) + \eta\left(re^{i\theta}\right)}{\left|\left(\alpha\left(e^{i\theta}\right) + \beta\left(e^{-i\theta}\right)\right) + \eta\left(re^{i\theta}\right)\right|} \\
&\quad = e^{-i\theta}\frac{\left(\alpha\left(e^{i\theta}\right) + \beta\left(e^{-i\theta}\right)\right) + 0}{\left|\left(\alpha\left(e^{i\theta}\right) + \beta\left(e^{-i\theta}\right)\right) + 0\right|} = \frac{\alpha + \left(e^{-i2\theta}\right)}{\left|\left(\alpha\left(e^{i\theta}\right) + \beta\left(e^{-i\theta}\right)\right)\right|} \\
&\quad = \frac{\alpha + \beta\left(e^{-i2\theta}\right)}{\left|e^{i\theta}\left(\alpha + \beta\left(e^{-i2\theta}\right)\right)\right|} = \frac{\alpha + \beta\left(e^{-i2\theta}\right)}{\left|e^{i\theta}\right|\left|\alpha + \beta\left(e^{-i2\theta}\right)\right|} = \frac{\alpha + \beta\left(e^{-i2\theta}\right)}{\left|\alpha + \beta\left(e^{-i2\theta}\right)\right|},
\end{aligned}$$

so $\frac{\alpha+\beta\left(e^{-i2\theta}\right)}{\left|\alpha+\beta\left(e^{-i2\theta}\right)\right|}$ is a complex number, which is independent of θ, and hence

$$\left\{\frac{\alpha + \beta\left(e^{-i2\theta}\right)}{\left|\alpha + \beta\left(e^{-i2\theta}\right)\right|} : \theta \in \mathbb{R}\right\}$$

contains exactly one point. Since $\left\{\alpha + \beta\left(e^{-i2\theta}\right) : \theta \in \mathbb{R}\right\}$ is a circle of radius $|\beta|$,

$$\left\{\frac{\alpha + \beta\left(e^{-i2\theta}\right)}{\left|\alpha + \beta\left(e^{-i2\theta}\right)\right|} : \theta \in \mathbb{R}\right\}$$

contains infinite-many points. This is a contradiction. ∎)

Conclusion 2.58 Let Ω be a region. Let $f : \Omega \to \mathbb{C}$ be a mapping. Let $a \in \Omega$. Suppose that there exists $\rho > 0$ such that $D'(a;\rho) \subset \Omega$, and for every $z \in D'(a;\rho)$, $f(z) \neq f(a)$. Suppose that f preserves angle at a. Suppose that the differential of f at a exists and is nonzero. Then $f'(a)$ exists and $f'(a) \neq 0$.

2.7 Linear Fractional Transformations

Note 2.59 Let a, b, c, d be any complex numbers satisfying $ad - bc \neq 0$. Let us define a mapping $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:

Case I: when $c \neq 0$. It follows that $\frac{-d}{c}$ is a complex number, and for every $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$, $cz + d$ is a nonzero complex number. Thus, for every $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$, $\frac{az+b}{cz+d}$ is a complex number. For every $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$, we define

$$\varphi(z) \equiv \frac{az+b}{cz+d}\left(\in \mathbb{C} \subset \mathbb{S}^2\right).$$

Here, observe that

$$(az+b)|_{z=\frac{-d}{c}} = a\left(\frac{-d}{c}\right) + b = \frac{ad-bc}{-c} \in (\mathbb{C} - \{0\}), \text{and}$$
$$(cz+d)|_{z=\frac{-d}{c}} = c\left(\frac{-d}{c}\right) + d = 0.$$

Naturally, we define $\varphi\left(\frac{-d}{c}\right) \equiv \infty\left(\in \mathbb{S}^2\right)$. It remains to define $\varphi(\infty)$. Since $c \neq 0$, $\frac{a}{c}$ is a complex number. Observe that

$$\lim_{n\to\infty} \frac{an+b}{cn+d} = \lim_{n\to\infty} \frac{a + b\frac{1}{n}}{c + d\frac{1}{n}} = \frac{a+b0}{c+d0} = \frac{a}{c}.$$

Naturally, we define $\varphi(\infty) \equiv \frac{a}{c}\left(\in \mathbb{C} \subset ^{\not\subset}\right)$. In short, for every $z \in \mathbb{S}^2\left(= \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \cup \left\{\frac{-d}{c}\right\} \cup \{\infty\}\right)$,

$$\varphi(z) \equiv \begin{cases} \frac{az+b}{cz+d}(\neq \infty) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{a}{c}(\neq \infty) & \text{if } z = \infty. \end{cases}$$

Case II: when $c = 0$. Here, $(ad = ad - b0 =)\, ad - bc \neq 0$, so a and d both are nonzero complex numbers. For every $z \in \mathbb{C}$, we define $\varphi(z) \equiv \frac{az+b}{d}\left(\in \mathbb{C} \subset \mathbb{S}^2\right)$. It remains to define $\varphi(\infty)$. Naturally, we define $\varphi(\infty) \equiv \infty\left(\in \mathbb{S}^2\right)$. In short, for every $z \in \mathbb{S}^2(= \mathbb{C} \cup \{\infty\})$,

$$\varphi(z) \equiv \begin{cases} \frac{az+b}{d}(\neq \infty) & \text{if } z \in C \\ \infty & \text{if } z = \infty. \end{cases}$$

Problem 2.60 $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is 1-1.

(Solution For this purpose, suppose that $\varphi(z) = \varphi(w)$, where $z, w \in \mathbb{S}^2$. We have to show that $z = w$.

Case I: when $c \neq 0$. Here,

$$\varphi(u) = \begin{cases} \frac{au+b}{cu+d} (\neq \infty) & \text{if } u \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } u = \frac{-d}{c} \\ \frac{a}{c} (\neq \infty) & \text{if } u = \infty. \end{cases}$$

Subcase I: when $\varphi(z) = \infty$. Here, by the definition of φ, $z = \frac{-d}{c}$. Since $(\varphi(w) =)\, \varphi(z) = \infty$, we have $\varphi(w) = \infty$, and hence by the definition of φ, $w = \frac{-d}{c}\ (= z)$. Thus, $z = w$.

Subcase II: when $\varphi(z) \neq \infty$. Here, by the definition of φ, $z \neq \frac{-d}{c}$, and hence $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \cup \{\infty\}$. Similarly, $w \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \cup \{\infty\}$. Also,

$$\varphi(z) = \begin{cases} \frac{az+b}{cz+d} (\neq \infty) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \frac{a}{c} (\neq \infty) & \text{if } z = \infty, \end{cases}$$

and

$$\varphi(w) = \begin{cases} \frac{aw+b}{cw+d} (\neq \infty) & \text{if } w \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \frac{a}{c} (\neq \infty) & \text{if } w = \infty. \end{cases}$$

Situation I: when $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$. It follows that

$$\left(\frac{a}{c} \text{or} \frac{aw+b}{cw+d} = \varphi(w) = \right) \varphi(z) = \frac{az+b}{cz+d}.$$

Thus,

$$\left(\frac{a}{c} = \frac{az+b}{cz+d}\right) \text{or} \left(\frac{aw+b}{cw+d} = \frac{az+b}{cz+d}\right).$$

If $\frac{a}{c} = \frac{az+b}{cz+d}$, then we get $ad - bc = 0$. This is false, so $\frac{aw+b}{cw+d} = \frac{az+b}{cz+d}$, and hence

$$aczw + bcz + adw + bd = \underbrace{(cz+d)(aw+b) = (az+b)(cw+d)}$$
$$= aczw + adz + bcw + bd.$$

Thus, $bc(z-w) + ad(w-z) = 0$, that is $(ad-bc)(w-z) = 0$. Now, since $ad - bc \neq 0$, $z = w$.

Situation II: when $z = \infty$. It follows that $(\varphi(w) =)\, \varphi(z) = \frac{a}{c}$, and hence $\varphi(w) = \frac{a}{c}$.

Problem 2.61 $w \notin \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$.

(Solution If not, otherwise, let $w \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$. We have to arrive at a contradiction. Since $w \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$, we have $\left(\frac{a}{c} =\right) \varphi(w) = \frac{aw+b}{cw+d}$, and hence $a(cw+d) = c(aw+b)$, that is $ad - bc = 0$. This is a contradiction. ∎)

It follows that $w = \infty$. Since $w = \infty$, and $z = \infty$, $z = w$.

So, in all situations, $z = w$. Thus, in all subcases of Case I, $z = w$.

Case II: when $c = 0$. Here, $ad = ad - b0 = \underbrace{ad - bc \neq 0}$, so a and d both are nonzero complex numbers. Also,

$$\varphi(u) \equiv \begin{cases} \frac{au+b}{d} \ (\neq \infty) & \text{if } u \in \mathbb{C} \\ \infty & \text{if } u = \infty. \end{cases}$$

Subcase I: when $\varphi(z) = \infty$. Here, by the definition of φ, $z = \infty$. Since $(\varphi(w) =) \, \varphi(z) = \infty$, we have $\varphi(w) = \infty$, and hence by the definition of φ, $w = \infty \, (= z)$. Thus, $z = w$.

Subcase II: when $\varphi(z) \neq \infty$. Here, by the definition of φ, $z \in \mathbb{C}$, and $\varphi(z) = \frac{az+b}{d}$. Similarly, $w \in \mathbb{C}$, and $\frac{az+b}{d} = \varphi(z) = \underbrace{\varphi(w) = \frac{aw+b}{d}}$, and hence $az + b = aw + b$. Thus, $a(z - w) = 0$. Now, since a is a nonzero complex number, $z = w$.

Thus, in all subcases of Case II, $z = w$.

We have shown that, in all cases, $z = w$. ∎)

Problem 2.62 $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is onto.

(Solution For this purpose, let us take any $w \in \mathbb{S}^2$. We have to find $z \in \mathbb{S}^2$ such that $\varphi(z) = w$.

Case I: when $c \neq 0$. Here,

$$\varphi(u) = \begin{cases} \frac{au+b}{cu+d} \ (\neq \infty) & \text{if } u \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } u = \frac{-d}{c} \\ \frac{a}{c} \ (\neq \infty) & \text{if } u = \infty. \end{cases}$$

Subcase I: when $w = \infty$. Here, by the definition of φ, $\varphi\left(\frac{-d}{c}\right) = \infty \, (= w)$, and $\frac{-d}{c} \in \mathbb{C} \left(\subset \mathbb{S}^2\right)$.

Subcase II: when $w \neq \infty$.

Situation I: when $w = \frac{a}{c}$. Here, by the definition of φ, $\varphi(\infty) = \frac{a}{c} \, (= w)$, and $\infty \in \mathbb{S}^2$.

Situation II: when $w \neq \frac{a}{c}$. Here, w is a complex number different from $\frac{a}{c}$, so $(-c)w + a$ is a nonzero complex number, and $dw + (-b)$ is a complex number. It follows that $\frac{dw+(-b)}{(-c)w+a} \in \mathbb{C} \left(\subset \mathbb{S}^2\right)$.

Problem 2.63 $\frac{dw+(-b)}{(-c)w+a} \neq \frac{-d}{c}$.

(**Solution** If not, otherwise, let

$$\frac{dw+(-b)}{(-c)w+a} = \frac{-d}{c}.$$

We have to arrive at a contradiction. Since

$$\frac{dw+(-b)}{(-c)w+a} = \frac{-d}{c},$$

we have

$$c(dw+(-b)) = -d((-c)w+a),$$

that is

$$ad - bc = 0.$$

This is a contradiction. ∎)

It follows that

$$\frac{dw+(-b)}{(-c)w+a} \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)(\subset \mathbb{S}^2),$$

and hence by the definition of φ,

$$\begin{aligned}\varphi\left(\frac{dw+(-b)}{(-c)w+a}\right) &= \frac{a\left(\frac{dw+(-b)}{(-c)w+a}\right)+b}{c\left(\frac{dw+(-b)}{(-c)w+a}\right)+d} = \frac{a(dw-b)+b(-cw+a)}{c(dw-b)+d(-cw+a)} \\ &= \frac{(ad-bc)w}{(ad-bc)} = w.\end{aligned}$$

Thus,

$$\frac{dw+(-b)}{(-c)w+a} \in \mathbb{S}^2, \text{and } \varphi\left(\frac{dw+(-b)}{(-c)w+a}\right) = w.$$

Case II: when $c = 0$. Here, $(ad = ad - b0 =)\, ad - bc \neq 0$, so a and d are both nonzero complex numbers. Also,

$$\varphi(u) \equiv \left\{ \begin{array}{ll} \frac{au+b}{d}\,(\neq \infty) & \text{if}\, u \in \mathbb{C} \\ \infty & \text{if}\, u = \infty \end{array} \right\}$$

Subcase I: when $w = \infty$. Here, by the definition of φ, $\varphi(\infty) = \infty\,(= w)$, and $\infty \in \mathbb{S}^2$.

Subcase II: when $w \neq \infty$. Here, w is a complex number, and a is nonzero complex number, we have $\frac{dw+(-b)}{a} \in \mathbb{C}\left(\subset \mathbb{S}^2\right)$, and hence by the definition of φ,

$$\varphi\left(\frac{dw+(-b)}{a}\right) = \frac{a\left(\frac{dw+(-b)}{a}\right)+b}{d} = \frac{dw+(-b)+b}{d} = w.$$

Thus,

$$\frac{dw+(-b)}{a} \in \mathbb{S}^2, \text{and } \varphi\left(\frac{dw+(-b)}{a}\right) = w.$$

So, in all cases, there exists $z \in \mathbb{S}^2$ such that $\varphi(z) = w$. ∎)

Definition Let a, b, c, d be any complex numbers satisfying $ad - bc \neq 0$. Let us define a mapping $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ as follows: when $c \neq 0$,

$$\varphi(z) \equiv \begin{cases} \frac{az+b}{cz+d}\ (\neq \infty) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{a}{c}\ (\neq \infty) & \text{if } z = \infty, \end{cases}$$

and when $c = 0$,

$$\varphi(z) \equiv \begin{cases} \frac{az+b}{d}\ (\neq \infty) & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$$

Here, $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is called a ***linear fractional transformation***, and φ is also denoted by $\varphi_{(a,b,c,d)}$.

Conclusion 2.64 Let $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ be a linear fractional transformation. Then $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is a 1-1, onto mapping.

Observe that when $c \neq 0$, for every $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$,

$$\varphi'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2}\left(= \frac{ad-bc}{(cz+d)^2} \neq 0\right).$$

Also, φ is 1-1. So, $\varphi|_{\mathbb{C}-\left\{\frac{-d}{c}\right\}}$ is a conformal mapping.

Next, when $c = 0$, for every $z \in \mathbb{C}$, $\varphi'(z) = \frac{a}{d} \neq 0$. Also, φ is 1-1. So, $\varphi|_{\mathbb{C}}$ is a conformal mapping.

In short, every linear fractional transformation is a conformal mapping.

Note 2.65 Definition For every complex number b, the linear fractional transformation

$$\varphi_{(1,b,0,1)} : z \mapsto \begin{cases} \frac{1z+b}{0z+1}\ (= z+b) & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases}$$

is called a ***translation***.

For every real θ, the linear fractional transformation

$$\varphi_{(e^{i\theta},0,0,1)} : z \mapsto \begin{cases} \frac{e^{i\theta}z+0}{0z+1}\ \left(= e^{i\theta}z\right) & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases}$$

is called a ***rotation***. For every positive real r, the linear fractional transformation

$$\varphi_{(r,0,0,1)} : z \mapsto \begin{cases} \frac{rz+0}{0z+1}\ (= rz) & \text{if } z \in C \\ \infty & \text{if } z = \infty \end{cases}$$

is called a ***homothety***. The linear fractional transformation

$$\varphi_{(0,1,1,0)} : z \mapsto \begin{cases} \frac{0z+1}{1z+0}\ \left(= \frac{1}{z}\right) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-0}{1}\right\}\right) (= \mathbb{C} - \{0\}) \\ \infty & \text{if } z = \frac{-0}{1}\ (= 0) \\ \frac{0}{1} (= 0) & \text{if } z = \infty, \end{cases}$$

that is

$$\varphi_{(0,1,1,0)} : z \mapsto \begin{cases} \frac{1}{z} & \text{if } z \in (\mathbb{C} - \{0\}) \\ \infty & \text{if } z = 0 \\ 0 & \text{if } z = \infty, \end{cases}$$

is called an ***inversion***.

Let a, b, c, d be any complex numbers satisfying $ad - bc \neq 0$.

Case I: when $c = 0$. Here, a and d are nonzero. There exists a real number θ such that $\frac{a}{d} = \left|\frac{a}{d}\right| e^{i\theta}$. Here $\left|\frac{a}{d}\right| > 0$, and

$$\varphi_{(a,b,c,d)}(z) \equiv \begin{cases} \frac{az+b}{d} \left(\begin{aligned} &= \frac{a}{d}\left(z + \frac{b}{a}\right) = \left|\frac{a}{d}\right| \left(e^{i\theta} \left(z + \frac{b}{a} \right) \right) \\ &= \left(\varphi_{\left(\left|\frac{a}{d}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{\left(1,\frac{b}{a},0,1\right)} \right)(z) \end{aligned} \right) & \text{if } z \in \mathbb{C} \\ = \left(\varphi_{\left(\left|\frac{a}{d}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{\left(1,\frac{b}{a},0,1\right)} \right)(z) & \text{if } z = \infty. \end{cases}$$

This shows that

$$\varphi_{(a,b,c,d)} = \varphi_{\left(\left|\frac{a}{d}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{\left(1,\frac{b}{a},0,1\right)}.$$

Thus, $\varphi_{(a,b,c,d)}$ is a composition of translation $\varphi_{\left(1,\frac{b}{a},0,1\right)}$, rotation $\varphi_{(e^{i\theta},0,0,1)}$, and homothety $\varphi_{\left(\left|\frac{a}{d}\right|,0,0,1\right)}$.

Case II: when $c \neq 0$. Here, $ad - bc$ and c^2 are nonzero complex numbers, so there exists a real number θ such that $\frac{ad-bc}{-c^2} = \left|\frac{ad-bc}{c^2}\right| e^{i\theta}$. Here $\left|\frac{ad-bc}{c^2}\right| > 0$, and

$$\varphi_{(a,b,c,d)}(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$$

Observe that for every $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$,

$$\begin{aligned} \varphi_{(a,b,c,d)}(z) &= \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) + \left(b - \frac{ad}{c}\right)}{cz+d} \\ &= \frac{a}{c} + \frac{ad-bc}{-c^2}\left(\frac{1}{z+\frac{d}{c}}\right) = \left|\frac{ad-bc}{c^2}\right|\left(e^{i\theta}\left(\frac{1}{z+\frac{d}{c}}\right)\right) + \frac{a}{c} \\ &= \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)} \circ \varphi_{\left(1,\frac{d}{c},0,1\right)}\right)(z), \end{aligned}$$

so for every $z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right)$,

$$\varphi_{(a,b,c,d)}(z) = \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)} \circ \varphi_{\left(1,\frac{d}{c},0,1\right)}\right)(z).$$

Next,

$$\begin{aligned} &\left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)} \circ \varphi_{\left(1,\frac{d}{c},0,1\right)}\right)\left(\frac{-d}{c}\right) \\ &\quad = \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)}\right)(0) \\ &\left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)}\right)(\infty) = \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)}\right)(\infty) \\ &\quad = \varphi_{\left(1,\frac{a}{c},0,1\right)}(\infty) = \infty = \varphi_{(a,b,c,d)}\left(\frac{-d}{c}\right), \end{aligned}$$

and

$$\begin{aligned}
&\left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)} \circ \varphi_{\left(1,\frac{d}{c},0,1\right)}\right)(\infty) \\
&= \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)}\right)(\infty) \\
&= \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)}\right)(0) \\
&= \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)}\right)(0) = \varphi_{\left(1,\frac{a}{c},0,1\right)}(0) = \frac{a}{c} = \varphi_{(a,b,c,d)}(\infty),
\end{aligned}$$

so for every $z \in \mathbb{S}^2$,

$$\varphi_{(a,b,c,d)}(z) = \left(\varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)} \circ \varphi_{\left(1,\frac{d}{c},0,1\right)}\right)(z),$$

and hence

$$\varphi_{(a,b,c,d)} = \varphi_{\left(1,\frac{a}{c},0,1\right)} \circ \varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)} \circ \varphi_{(e^{i\theta},0,0,1)} \circ \varphi_{(0,1,1,0)} \circ \varphi_{\left(1,\frac{d}{c},0,1\right)}.$$

Thus, $\varphi_{(a,b,c,d)}$ is the composition of translation $\varphi_{\left(1,\frac{d}{c},0,1\right)}$, inversion $\varphi_{(0,1,1,0)}$, rotation $\varphi_{(e^{i\theta},0,0,1)}$, homothety $\varphi_{\left(\left|\frac{ad-bc}{c^2}\right|,0,0,1\right)}$, and translation $\varphi_{\left(1,\frac{a}{c},0,1\right)}$.

Conclusion 2.66 Every linear fractional transformation can be expressed as a composition of translations, rotation, homothety and inversion.

Note 2.67 Observe that the equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

where g, f, c are real numbers such that $g^2 + f^2 > c$. Thus, the equation of a circle is

$$1\left(x^2 + y^2\right) + 2gx + 2fy + c = 0,$$

where g, f, c are real numbers such that $1c < g^2 + f^2$. In other words, the equation of a circle in complex variable z is $1(z\bar{z}) + 2\mathrm{Re}((g - if)z) + c = 0$, where g, f, c are real numbers such that

$$1c < (g - if)\overline{(g - if)}.$$

Observe that the equation of a straight line is $ax + by + c = 0$, where a, b, c are real numbers such that a and b both cannot be zero. Thus, the equation of a straight line is $0(x^2 + y^2) + 2\left(\frac{a}{2}\right)x + 2\left(\frac{b}{2}\right)y + c = 0$, where a, b, c are real numbers such that

$(0c =)\, 0 < \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$. In other words, the equation of a straight line in complex variable z is

$$0(z\bar{z}) + 2\mathrm{Re}\left(\left(\frac{a}{2} - i\frac{b}{2}\right)z\right) + c = 0,$$

where a, b, c are real numbers such that

$$(0c =)\, 0 < \left(\frac{a}{2} - i\frac{b}{2}\right)\overline{\left(\frac{a}{2} - i\frac{b}{2}\right)}.$$

Thus, the equation of every circle and the equation of every straight line, in complex variable z, are of the form

$$\alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0,$$

where $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$, and $\alpha\gamma < \beta\bar{\beta}$.

Let us denote the collection of all straight lines and all circles by $\mathcal{F}$.

Problem 2.68

$\mathcal{F} = \left\{\{z : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0\} : \alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C}, \text{and } \alpha\gamma < \beta\bar{\beta}\right\}.$

(Solution From the above discussion,

$$\mathcal{F} \subset \left\{\{z : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0\} : \alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C}, \text{and } \alpha\gamma < \beta\bar{\beta}\right\}.$$

It remains to show that

$$\left\{\{z : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0\} : \alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C}, \text{and } \alpha\gamma < \beta\bar{\beta}\right\} \subset \mathcal{F}.$$

For this purpose, let us take any $\alpha, \gamma \in \mathbb{R}$, and $\beta \in \mathbb{C}$ such that $\alpha\gamma < \beta\bar{\beta}$. Let $\beta \equiv \beta_1 + i\beta_2$, where $\beta_1, \beta_2 \in \mathbb{R}$. We have to show that $\alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0$ is an equation of a straight line or a circle.

On putting $z \equiv x + iy$, where $x, y \in \mathbb{R}$, our equation

$$\left(\alpha\left(x^2 + y^2\right) + 2(\beta_1 x - \beta_2 y) + \gamma =\right) \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0$$

becomes

$$\begin{cases} 2\beta_1 x - 2\beta_2 y + \gamma & \text{if } \alpha = 0\left(\text{and hence } \alpha\gamma\beta\bar{\beta} \text{ becomes } 0(\beta_1)^2 + (\beta_2)^2\right) \\ x^2 + y^2 + 2\frac{\beta_1}{\alpha}x - 2\frac{\beta_2}{\alpha}y + \frac{\gamma}{\alpha} = 0 & \text{if } \alpha \neq 0\left(\text{and hence } \alpha\gamma; \beta\bar{\beta} \text{ becomes } \frac{\gamma}{\alpha}\left(\frac{\beta_1}{\alpha}\right)^2 + \left(\frac{-\beta_2}{\alpha}\right)^2\right) \end{cases}$$

Thus, $\alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0$ is an equation of a straight line or a circle. ∎)

a. Problem 2.69 Every translation sends straight line to straight line.

(Solution Let $\varphi_{(1,b,0,1)}$ be a translation, where $b \in \mathbb{C}$. Let $\{a + td : t \in \mathbb{R}\}$ be a straight line where $a, d \in \mathbb{C}$ and $d \neq 0$. We have to show that $\left\{\varphi_{(1,b,0,1)}(a + td) : t \in \mathbb{R}\right\}$ is a straight line. Here,

$$\left\{\varphi_{(1,b,0,1)}(a + td) : t \in \mathbb{R}\right\} = \{(a + td) + b : t \in \mathbb{R}\} = \{(a + b) + td : t \in \mathbb{R}\}.$$

Now, since $\{(a + b) + td : t \in \mathbb{R}\}$ is a straight line, $\left\{\varphi_{(1,b,0,1)}(a + td) : t \in \mathbb{R}\right\}$ is a straight line. ∎)

b. Problem 2.70 Every translation sends circle to circle.

(Solution Let $\varphi_{(1,b,0,1)}$ be a translation where $b \in \mathbb{C}$. Let $\{z : |z - c| = R\}$ be a circle where $c \in \mathbb{C}$ and $R > 0$. We have to show that

$$\left\{\varphi_{(1,b,0,1)}(z) : |z - c| = R\right\}$$

is a circle. Here

$$\begin{aligned}\left\{\varphi_{(1,b,0,1)}(z) : |z - c| = R\right\} &= \{z + b : |z - c| = R\} \\ &= \{z + b : |(z + b) - (c + b)| = R\} \\ &= \{w : |w - (c + b)| = R\}.\end{aligned}$$

Now, since $\{w : |w - (c + b)| = R\}$ is a circle, $\left\{\varphi_{(1,b,0,1)}(z) : |z - c| = R\right\}$ is a circle. ∎)

c. Problem 2.71 Every rotation sends straight line to straight line.

(Solution Let $\varphi_{(e^{i\theta},0,0,1)}$ be a rotation where $\theta \in \mathbb{R}$. Let $\{a + td : t \in \mathbb{R}\}$ be a straight line where $a, d \in \mathbb{C}$ and $d \neq 0$. We have to show that

$$\left\{\varphi_{(e^{i\theta},0,0,1)}(a + td) : t \in \mathbb{R}\right\}$$

is a straight line. Here

$$\left\{\varphi_{(e^{i\theta},0,0,1)}(a + td) : t \in \mathbb{R}\right\} = \left\{e^{i\theta}(a + td) : t \in \mathbb{R}\right\} = \left\{\left(e^{i\theta}a\right) + t\left(e^{i\theta}d\right) : t \in \mathbb{R}\right\}.$$

Now, since

$$\left\{\left(e^{i\theta}a\right) + t\left(e^{i\theta}d\right) : t \in \mathbb{R}\right\}$$

is a straight line,

$$\left\{\varphi_{(e^{i\theta},0,0,1)}(a+td) : t \in \mathbb{R}\right\}$$

is a straight line. ∎)

d. Problem 2.72 Every rotation sends circle to circle.

(Solution Let $\varphi_{(e^{i\theta},0,0,1)}$ be a rotation where $\theta \in \mathbb{R}$. Let $\{z : |z-c| = R\}$ be a circle where $c \in \mathbb{C}$ and $R > 0$. We have to show that

$$\left\{\varphi_{(e^{i\theta},0,0,1)}(z) : |z-c| = R\right\}$$

is a circle. Here

$$\begin{aligned}\left\{\varphi_{(e^{i\theta},0,0,1)}(z) : |z-c| = R\right\} &= \left\{e^{i\theta}z : |z-c| = R\right\} = \left\{e^{i\theta}z : \left|e^{i\theta}z - e^{i\theta}c\right| = R\right\} \\ &= \left\{w : \left|w - e^{i\theta}c\right| = R\right\}.\end{aligned}$$

Now, since $\left\{w : \left|w - e^{i\theta}c\right| = R\right\}$ is a circle, $\left\{\varphi_{(e^{i\theta},0,0,1)}(z) : |z-c| = R\right\}$ is a circle. ∎)

e. Problem 2.73 Every homothety sends straight line to straight line.

(Solution Let $\varphi_{(r,0,0,1)}$ be a rotation where $r > 0$. Let $\{a + td : t \in \mathbb{R}\}$ be a straight line, where $a, d \in \mathbb{C}$ and $d \neq 0$. We have to show that

$$\left\{\varphi_{(r,0,0,1)}(a+td) : t \in \mathbb{R}\right\}$$

is a straight line. Here,

$$\left\{\varphi_{(r,0,0,1)}(a+td) : t \in \mathbb{R}\right\} = \{r(a+td) + b : t \in \mathbb{R}\} = \{(ra) + t(rd) : t \in \mathbb{R}\}.$$

Now, since $\{(ra) + t(rd) : t \in \mathbb{R}\}$ is a straight line, $\left\{\varphi_{(r,0,0,1)}(a+td) : t \in \mathbb{R}\right\}$ is a straight line. ∎)

f. Problem 2.74 Every homothety sends circle to circle.

(Solution Let $\varphi_{(r,0,0,1)}$ be a rotation where $r > 0$. Let $\{z : |z-c| = R\}$ be a circle where $c \in \mathbb{C}$ and $R > 0$. We have to show that

$$\left\{\varphi_{(r,0,0,1)}(z) : |z-c| = R\right\}$$

is a circle. Here,

$$\left\{\varphi_{(r,0,0,1)}(z) : |z-c| = R\right\} = \{rz : |z-c| = R\} = \{rz : |rz - rc| = rR\}$$
$$= \{w : |w - rc| = rR\}.$$

Now, since $\{w : |w - rc| = rR\}$ is a circle, $\left\{\varphi_{(r,0,0,1)}(z) : |z-c| = R\right\}$ is a circle. ∎)

g. Problem 2.75 Inverse sends (straight line or circle) to a (straight line or circle).

(Solution Let

$$\{z : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0, \text{and}\, z \neq 0\}$$

be a (straight line or circle), where $\alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C}$ and $\alpha\gamma < \beta\bar{\beta}$. We have to show that

$$\left\{\varphi_{(0,1,1,0)}(z) : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0, \text{and}\, z \neq 0\right\}$$

is a (straight line or circle). Here,

$$\begin{aligned}
&\left\{\varphi_{(0,1,1,0)}(z) : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0, \text{and } z \neq 0\right\} \\
&= \left\{\frac{1}{z} : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0, \text{and } z \neq 0\right\} \\
&= \left\{\frac{1}{z} : \alpha(z\bar{z}) + \left(\beta z + \bar{\beta}\bar{z}\right) + \gamma = 0, \text{and } z \neq 0\right\} \\
&= \left\{w : \alpha\left(\frac{1}{w}\frac{1}{\bar{w}}\right) + \left(\beta\frac{1}{w} + \bar{\beta}\frac{1}{\bar{w}}\right) + \gamma = 0, \text{and } w \neq 0\right\} \\
&= \left\{w : \gamma w\bar{w} + \left(\bar{\beta}w + \bar{\bar{\beta}}\bar{w}\right) + \alpha = 0, \text{and } w \neq 0\right\}.
\end{aligned}$$

Since $\alpha\gamma < \beta\bar{\beta}$, we have $\gamma\alpha < \bar{\beta}\bar{\bar{\beta}}$, and hence

$$\left\{w : \gamma w\bar{w} + \left(\bar{\beta}w + \bar{\bar{\beta}}\bar{w}\right) + \alpha = 0, \text{and}\, w \neq 0\right\}$$

is a (straight line or circle). Thus,

$$\left\{\varphi_{(0,1,1,0)}(z) : \alpha(z\bar{z}) + 2\mathrm{Re}(\beta z) + \gamma = 0, \text{and}\, z \neq 0\right\}$$

is a (straight line or circle). ∎)

In short, we get the following

Conclusion 2.76 Every linear fractional transformation sends members of $\mathcal{F}$ to members of $\mathcal{F}$.

Note 2.77

I. Let a, b, c be distinct complex numbers. It follows that $b-c, c-a, a-b$ are nonzero complex numbers. Let us define a mapping $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:

$$\varphi(z) \equiv \begin{cases} \frac{(b-c)z-a(b-c)}{(b-a)z-c(b-a)} \left(= \frac{(b-c)(z-a)}{(b-a)(z-c)}\right) & \text{if}\, z \in \left(\mathbb{C} - \left\{\frac{c(b-a)}{(b-a)}\right\}\right) (= \mathbb{C} - \{c\}) \\ \infty & \text{if}\, z = \frac{c(b-a)}{(b-a)} \,(= c) \\ \frac{(b-c)}{(b-a)} & \text{if}\, z = \infty. \end{cases}$$

Since

$$\begin{aligned} (b-c)(-c(b-a)) - (-a(b-c))(b-a) &= (b-c)(b-a)(a-c) \\ &= (b-c)(c-a)(a-b) \neq 0, \end{aligned}$$

φ is a linear fractional transformation. Here,

$$\left.\begin{aligned} \varphi(c) &= \infty \\ \varphi(\infty) &= \frac{(b-c)}{(b-a)} \end{aligned}\right\},$$

and for every $z \in \mathbb{C} - \{c\}$,

$$\varphi(z) = \frac{(b-c)(z-a)}{(b-a)(z-c)}.$$

Next,

$$\varphi(a) = \frac{(b-c)(a-a)}{(b-a)(a-c)} = 0, \;\; \varphi(b) = \frac{(b-c)(b-a)}{(b-a)(b-c)} = 1, \text{and } \varphi(c) = \infty.$$

In short, the linear fractional transformation φ maps ordered triplet (a, b, c) into ordered triplet $(0, 1, \infty)$.

II. Let b, c be distinct complex numbers. It follows that $b-c$ is a nonzero complex number. Let us define a mapping $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:

$$\varphi(z) \equiv \begin{cases} \frac{0z+(b-c)}{1z-c} \left(= \frac{b-c}{z-c}\right) & \text{if}\, z \in (\mathbb{C} - \{c\}) \\ \infty & \text{if}\, z = c \\ \frac{0}{1} (= 0) & \text{if}\, z = \infty. \end{cases}$$

Since $0(-c) - (b-c)1 = -(b-c) \neq 0$, φ is a linear fractional transformation. Here, φ maps ordered triplet (∞, b, c) into ordered triplet $(0, 1, \infty)$.

III. Let a, c be distinct complex numbers. It follows that $a - c$ is a nonzero complex number. Let us define a mapping $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:

$$\varphi(z) \equiv \begin{cases} \frac{1z-a}{1z-c} \left(= \frac{z-a}{z-c}\right) & \text{if } z \in (\mathbb{C} - \{c\}) \\ \infty & \text{if } z = c \\ \frac{1}{1}(= 1) & \text{if } z = \infty. \end{cases}$$

Since $1(-c) - (-a)1 = (a - c) \neq 0$, φ is a linear fractional transformation. Here, φ maps ordered triplet (a, ∞, c) into ordered triplet $(0, 1, \infty)$.

IV. Let a, b be distinct complex numbers. It follows that $a - b$ is a nonzero complex number. Let us define a mapping $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:

$$\varphi(z) \equiv \begin{cases} \frac{1z-a}{0z+(b-a)} \left(= \frac{z-a}{b-a}\right) & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$$

Since $1(b - a) - (-a)0 = (b - a) \neq 0$, φ is a linear fractional transformation. Here, φ maps ordered triplet (a, b, ∞) into ordered triplet $(0, 1, \infty)$.

Conclusion 2.78 Let a, b, c be distinct members of $\mathbb{S}^2$. There exists a linear fractional transformation $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$, which maps ordered triplet (a, b, c) into ordered triplet $(0, 1, \infty)$.

Note 2.79 Let $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$, and $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ be linear fractional transformations.

Problem 2.80 $(\psi \circ \varphi) : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation.

(Solution Since $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, there exist complex numbers a, b, c, d such that $ad - bc \neq 0$, and

$$c \neq 0, \quad \varphi(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty, \end{cases}$$

or,

$$c = 0, \quad \varphi(z) = \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$$

Since $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, by Conclusion 2.66, ψ can be expressed as a composition of translations, rotation, homothety and inversion. Hence, it suffices to show that

1. $(\psi \circ \varphi) : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, if ψ is a translation,
2. $(\psi \circ \varphi) : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, if ψ is a rotation,

3. $(\psi \circ \varphi) : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, if ψ is a homothety,
4. $(\psi \circ \varphi) : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, if ψ is the inversion.

For 1: Let $\psi = \varphi_{(1,\beta,0,1)}$, where $\beta \in \mathbb{C}$. We have to show that $\varphi_{(1,\beta,0,1)} \circ \varphi$ is a linear fractional transformation.

Case 1a: when $c \neq 0$, $\quad \varphi(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$

Here, for every $z \in \mathbb{S}^2$,

$$\left(\varphi_{(1,\beta,0,1)}^{\circ}\varphi\right)(z) = \begin{cases} \varphi_{(1,\beta,0,1)}\left(\frac{az+b}{cz+d}\right) & \text{if} z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \varphi_{(1,\beta,0,1)}(\infty) & \text{if} z = -\frac{d}{c} \\ \varphi_{(1,\beta,0,1)}\left(\frac{a}{c}\right) & \text{if} z = \infty \end{cases}$$
$$= \begin{cases} \frac{az+b}{cz+d} + \beta & \text{if} z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if} z = -\frac{d}{c} \\ \frac{a}{c} + \beta & \text{if} z = \infty \end{cases}$$
$$= \begin{cases} \frac{(a+c\beta)z + (b+d\beta)}{cz+d} & \text{if} z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if} z = -\frac{d}{c} \\ \frac{a+c\beta}{c} & \text{if} z = \infty \end{cases}$$
$$= \varphi_{(a+c\beta,b+d\beta,c,d)}(z),$$

so $\varphi_{(1,\beta,0,1)} \circ \varphi = \varphi_{(a+c\beta,b+d\beta,c,d)}$. Now, since $\varphi_{(a+c\beta,b+d\beta,c,d)}$ is a linear fractional transformation, $\varphi_{(1,\beta,0,1)} \circ \varphi$ is a linear fractional transformation.

Case 1b: when $c = 0$, $\quad \varphi(z) = \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$

Here, for every $z \in \mathbb{S}^2$,

$$\left(\varphi_{(1,\beta,0,1)}^{\circ}\varphi\right)(z) = \begin{cases} \varphi_{(1,\beta,0,1)}\left(\frac{az+b}{d}\right) & \text{if} z \in \mathbb{C} \\ \varphi_{(1,\beta,0,1)}(\infty) & \text{if} z = \infty \end{cases}$$
$$= \begin{cases} \frac{az+b}{d} + \beta & \text{if} z \in \mathbb{C} \\ \infty & \text{if} z = \infty \end{cases}$$
$$= \begin{cases} \frac{az+(b+d\beta)}{d} & \text{if} z \in \mathbb{C} \\ \infty & \text{if} z = \infty \end{cases} = \varphi_{(a,b+d\beta,0,d)}(z),$$

so $\varphi_{(1,\beta,0,1)} \circ \varphi = \varphi_{(a,b+d\beta,0,d)}$. Now, since $\varphi_{(a,b+d\beta,0,d)}$ is a linear fractional transformation, $\varphi_{(1,\beta,0,1)} \circ \varphi$ is a linear fractional transformation.

For 2: Let $\psi = \varphi_{(e^{i\theta},0,0,1)}$, where $\theta \in \mathbb{R}$. We have to show that $\varphi_{(e^{i\theta},0,0,1)} \circ \varphi$ is a linear fractional transformation.

Case 2a: when $c \neq 0$, $\varphi(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$

Here, for every $z \in \mathbb{S}^2$,

$$\left(\varphi_{(e^{i\theta},0,0,1)} \circ \varphi\right)(z) = \begin{cases} \varphi_{(e^{i\theta},0,0,1)}\left(\frac{az+b}{cz+d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \varphi_{(e^{i\theta},0,0,1)}(\infty) & \text{if } z = -\frac{d}{c} \\ \varphi_{(e^{i\theta},0,0,1)}\left(\frac{a}{c}\right) & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} e^{i\theta}\frac{az+b}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if } z = -\frac{d}{c} \\ e^{i\theta}\frac{a}{c} & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \frac{\left(e^{i\theta}a\right)z + \left(e^{i\theta}b\right)}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{e^{i\theta}a}{c} & \text{if } z = \infty \end{cases} = \varphi_{(e^{i\theta}a,e^{i\theta}b,c,d)}(z),$$

so $\varphi_{(e^{i\theta},0,0,1)} \circ \varphi = \varphi_{(e^{i\theta}a,e^{i\theta}b,c,d)}$. Now, since $\varphi_{(e^{i\theta}a,e^{i\theta}b,c,d)}$ is a linear fractional transformation, $\varphi_{(e^{i\theta},0,0,1)} \circ \varphi$ is a linear fractional transformation.

Case 2b: when $c = 0$, $\quad \varphi(z) = \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$

Here, for every $z \in \mathbb{S}^2$,

$$\left(\varphi^{\circ}_{(e^{i\theta},0,0,1)}\varphi\right)(z) = \begin{cases} \varphi_{(e^{i\theta},0,0,1)}\left(\frac{az+b}{d}\right) & \text{if } z \in \mathbb{C} \\ \varphi_{(e^{i\theta},0,0,1)}(\infty) & \text{if} z = \infty \end{cases}$$
$$= \begin{cases} e^{i\theta}\frac{az+b}{d} & \text{if} z \in \mathbb{C} \\ \infty & \text{if} z = \infty \end{cases}$$
$$= \begin{cases} \frac{\left(e^{i\theta}a\right)z + \left(e^{i\theta}b\right)}{d} & \text{if} z \in \mathbb{C} \\ \infty & \text{if} z = \infty \end{cases}$$
$$= \varphi_{(e^{i\theta}a,e^{i\theta}b,0,d)}(z),$$

so $\varphi_{(e^{i\theta},0,0,1)} \circ \varphi = \varphi_{(e^{i\theta}a,e^{i\theta}b,0,d)}$. Now, since $\varphi_{(e^{i\theta}a,e^{i\theta}b,0,d)}$ is a linear fractional transformation, $\varphi_{(e^{i\theta},0,0,1)} \circ \varphi$ is a linear fractional transformation.

For 3: Let $\psi = \varphi_{(r,0,0,1)}$, where $r > 0$. We have to show that $\varphi_{(r,0,0,1)} \circ \varphi$ is a linear fractional transformation.

Case 3a: when $c \neq 0$, $\varphi(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$

Here, for every $z \in \mathbb{S}^2$,

$$
\begin{aligned}
\left(\varphi^{\circ}_{(r,0,0,1)}\varphi\right)(z) &= \begin{cases} \varphi_{(r,0,0,1)}\left(\frac{az+b}{cz+d}\right) & \text{if}\, z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \varphi_{(r,0,0,1)}(\infty) & \text{if}\, z = -\frac{d}{c} \\ \varphi_{(r,0,0,1)}\left(\frac{a}{c}\right) & \text{if}\, z = \infty \end{cases} \\
&= \begin{cases} r\frac{az+b}{cz+d} & \text{if}\, z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if}\, z = -\frac{d}{c} \\ r\frac{a}{c} & \text{if}\, z = \infty \end{cases} \\
&= \begin{cases} \frac{(ra)z+(rb)}{cz+d} & \text{if}\, z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if}\, z = -\frac{d}{c} \\ \frac{ra}{c} & \text{if}\, z = \infty \end{cases} \\
&= \varphi_{(ra,rb,c,d)}(z),
\end{aligned}
$$

so $\varphi_{(r,0,0,1)} \circ \varphi = \varphi_{(ra,rb,c,d)}$. Now, since $\varphi_{(ra,rb,c,d)}$ is a linear fractional transformation, $\varphi_{(r,0,0,1)} \circ \varphi$ is a linear fractional transformation.

Case 3b: when $c = 0$, $\quad \varphi(z) = \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$

Here, for every $z \in \mathbb{S}^2$,

$$
\begin{aligned}
\left(\varphi_{(r,0,0,1)} \circ \varphi\right)(z) &= \begin{cases} \varphi_{(r,0,0,1)}\left(\frac{az+b}{d}\right) & \text{if } z \in \mathbb{C} \\ \varphi_{(r,0,0,1)}(\infty) & \text{if } z = \infty \end{cases} = \begin{cases} r\frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases} \\
&= \begin{cases} \frac{(ra)z+(rb)}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases} = \varphi_{(ra,rb,0,d)}(z),
\end{aligned}
$$

so $\varphi_{(r,0,0,1)} \circ \varphi = \varphi_{(ra,rb,0,d)}$. Now, since $\varphi_{(ra,rb,0,d)}$ is a linear fractional transformation, $\varphi_{(r,0,0,1)} \circ \varphi$ is a linear fractional transformation.

For 4: Let $\psi = \varphi_{(0,1,1,0)}$. We have to show that $\varphi_{(0,1,1,0)} \circ \varphi$ is a linear fractional transformation.

Case 4a: when $c \neq 0$, $\varphi(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \infty & \text{if } z = -\frac{d}{c} \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$

Subcase 4aI: when $a \neq 0$.

Here, for every $z \in \mathbb{S}^2$,

$$\left(\varphi_{(0,1,1,0)} \circ \varphi\right)(z) = \begin{cases} \varphi_{(0,1,1,0)}\left(\frac{az+b}{cz+d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \varphi_{(0,1,1,0)}(\infty) & \text{if } z = -\frac{d}{c} \\ \varphi_{(0,1,1,0)}\left(\frac{a}{c}\right) & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \varphi_{(0,1,1,0)}\left(\frac{az+b}{cz+d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ 0 & \text{if } z = -\frac{d}{c} \\ \frac{c}{a} & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \frac{cz+d}{az+b} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}, -\frac{b}{a}\right\}\right) \\ \infty & \text{if } z = -\frac{b}{a} \\ 0 & \text{if } z = -\frac{d}{c} \\ \frac{c}{a} & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \frac{cz+d}{az+b} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{b}{a}\right\}\right) \\ \infty & \text{if } z = -\frac{b}{a} \\ \frac{c}{a} & \text{if } z = \infty \end{cases} = \varphi_{(c,d,a,b)}(z)$$

so $\varphi_{(0,1,1,0)} \circ \varphi = \varphi_{(c,d,a,b)}$. Now, since $\varphi_{(c,d,a,b)}$ is a linear fractional transformation, $\varphi_{(0,1,1,0)} \circ \varphi$ is a linear fractional transformation.

Subcase 4aII: when $a = 0$.

Here, for every $z \in \mathbb{S}^2$,

$$\left(\varphi_{(0,1,1,0)} \circ \varphi\right)(z) = \begin{cases} \varphi_{(0,1,1,0)}\left(\frac{b}{cz+d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ \varphi_{(0,1,1,0)}(\infty) & \text{if } z = -\frac{d}{c} \\ \varphi_{(0,1,1,0)}\left(\frac{0}{c}\right) & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \varphi_{(0,1,1,0)}\left(\frac{b}{cz+d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ 0 & \text{if } z = -\frac{d}{c} \\ \infty & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \frac{cz+d}{b} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{d}{c}\right\}\right) \\ 0 & \text{if } z = -\frac{d}{c} \\ \infty & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \frac{cz+d}{b} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases} = \varphi_{(c,d,0,b)}(z),$$

so $\varphi_{(0,1,1,0)} \circ \varphi = \varphi_{(c,d,0,b)}$. Now, since $\varphi_{(c,d,0,b)}$ is a linear fractional transformation, $\varphi_{(0,1,1,0)} \circ \varphi$ is a linear fractional transformation.

Case 4b: when $c = 0$, $\varphi(z) = \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$

Clearly, a and d are nonzero complex numbers. Here, for every $z \in \mathbb{S}^2$,

$$\left(\varphi_{(0,1,1,0)} \circ \varphi\right)(z) = \begin{cases} \varphi_{(0,1,1,0)}\left(\frac{az+b}{d}\right) & \text{if } z \in \mathbb{C} \\ \varphi_{(0,1,1,0)}(\infty) & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \varphi_{(0,1,1,0)}\left(\frac{az+b}{d}\right) & \text{if } z \in \mathbb{C} \\ 0 & \text{if } z = \infty. \end{cases}$$
$$= \begin{cases} \varphi_{(0,1,1,0)}\left(\frac{az+b}{d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{b}{a}\right\}\right) \\ \varphi_{(0,1,1,0)}(0) & \text{if } z = -\frac{b}{a} \\ 0 & \text{if } z = \infty \end{cases}$$
$$= \begin{cases} \frac{d}{az+b} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{b}{a}\right\}\right) \\ \varphi_{(0,1,1,0)}(0) & \text{if } z = -\frac{b}{a} \\ 0 & \text{if} z = \infty \end{cases}$$
$$= \begin{cases} \frac{d}{az+b} & \text{if } z \in \left(\mathbb{C} - \left\{-\frac{b}{a}\right\}\right) \\ \infty & \text{if } z = -\frac{b}{a} \\ \frac{0}{a} & \text{if } z = \infty \end{cases} = \varphi_{(0,d,a,b)}(z),$$

so $\varphi_{(0,1,1,0)} \circ \varphi = \varphi_{(0,d,a,b)}$. Now, since $\varphi_{(0,d,a,b)}$ is a linear fractional transformation, $\varphi_{(0,1,1,0)} \circ \varphi$ is a linear fractional transformation. ∎)

Conclusion 2.81 Composition of two linear fractional transformations is a linear fractional transformation.

Theorem 2.82 The collection of all linear fractional transformations is a group, under composition of mappings as the binary operation.

Proof

1. Closure law: By Conclusion 2.81, closure law is satisfied.
2. Associative law: It is well known that composition of mappings is associative.
3. Existence of identity element: Since $\varphi_{(1,0,0,1)}$ is a linear fractional transformation, and $\varphi_{(1,0,0,1)} : z \mapsto z$ from $\mathbb{S}^2$ to $\mathbb{S}^2$, $\varphi_{(1,0,0,1)}$ serves the purpose of identity element.
4. Existence of inverse element: For this purpose, let us take any linear fractional transformation φ. By Conclusions 2.66 and 2.81, it suffices to show that

   I. the inverse of a translation is a translation,
   II. the inverse of a rotation is a rotation,
   III. the inverse of a homothety is a homothety,
   IV. the inverse of inversion is inversion.

For I: For every complex number b, $\left(\varphi_{(1,b,0,1)}\right)^{-1} = \left(\varphi_{(1,-b,0,1)}\right)$, so the inverse of a translation is a translation.

For II: For every real number θ, $\left(\varphi_{(e^{i\theta},0,0,1)}\right)^{-1} = \left(\varphi_{(e^{i(-\theta)},0,0,1)}\right)$, so the inverse of a rotation is a rotation.

For III: For every $r > 0$, $\left(\varphi_{(r,0,0,1)}\right)^{-1} = \left(\varphi_{(\frac{1}{r},0,0,1)}\right)$, so the inverse of a homothety is a homothety.

For IV: Since $\left(\varphi_{(0,1,1,0)}\right)^{-1} = \varphi_{(0,1,1,0)}$, the inverse of inversion is inversion.

Thus, the collection of all linear fractional transformations is a group, under composition of mappings as the binary operation. ∎

Note 2.83 Let $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ be a linear fractional transformation, which maps ordered triplet $(0, 1, \infty)$ into ordered triplet $(0, 1, \infty)$.

a. Problem 2.84 $\varphi = I$, where I denotes the identity mapping from $\mathbb{S}^2$ to $\mathbb{S}^2$.

(**Solution** Since $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, which maps ordered triplet $(0, 1, \infty)$ into ordered triplet $(0, 1, \infty)$, $\varphi(\infty) = \infty$, hence there exist complex numbers a, b, d such that

1. a, d are nonzero,
2. 2. $\varphi(z) = \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$

Now, since $\left(\frac{b}{d} = \frac{a0+b}{d} =\right)\varphi(0) = 0$, we have $b = 0$. Thus, $\varphi(z) = \begin{cases} \frac{a}{d}z & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$ Since $\left(\frac{a}{d} = \frac{a}{d}1 =\right)\varphi(1) = 1$, we have $\frac{a}{d} = 1$. Thus, for every $z \in \mathbb{S}^2$, $\varphi(z) = \begin{cases} 1z & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases} (= I(z))$, and hence $\varphi = I$. ∎)

b. Let a, b, c be distinct points of $\mathbb{S}^2$. Let α, β, γ be distinct points of $\mathbb{S}^2$.

Problem 2.85 There exists a unique linear fractional transformation φ, which maps ordered triplet (a, b, c) into (α, β, γ).

(**Solution** Existence: By Conclusion 2.78, there exists a linear fractional transformation $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$, which maps ordered triplet (a, b, c) into ordered triplet $(0, 1, \infty)$. Similarly, there exists a linear fractional transformation $\psi : \mathbb{S}^2 \to \mathbb{S}^2$, which maps ordered triplet (α, β, γ) into ordered triplet $(0, 1, \infty)$. Hence, by Theorem 2.82, $\psi^{-1} : \mathbb{S}^2 \to \mathbb{S}^2$ is a linear fractional transformation, which maps ordered triplet $(0, 1, \infty)$ into ordered triplet (α, β, γ). Now, by Theorem 2.82,

$\left(\left(\psi^{-1}\right)\circ\varphi\right):\mathbb{S}^2\to\mathbb{S}^2$ is a linear fractional transformation, which maps ordered triplet (a,b,c) into ordered triplet (α,β,γ).

Uniqueness: Let $\varphi:\mathbb{S}^2\to\mathbb{S}^2$ be a linear fractional transformation, which maps ordered triplet (a,b,c) into (α,β,γ). Let $\psi:\mathbb{S}^2\to\mathbb{S}^2$ be a linear fractional transformation, which maps ordered triplet (a,b,c) into (α,β,γ). We have to show that $\varphi=\psi$.

By Conclusion 2.78, there exists a linear fractional transformation $\chi_1:\mathbb{S}^2\to\mathbb{S}^2$, which maps ordered triplet (a,b,c) into ordered triplet $(0,1,\infty)$. Similarly, there exists a linear fractional transformation $\chi_2:\mathbb{S}^2\to\mathbb{S}^2$, which maps ordered triplet (α,β,γ) into ordered triplet $(0,1,\infty)$. It follows that $\chi_2\circ\varphi\circ(\chi_1)^{-1}$ is a linear fractional transformation, which maps ordered triplet $(0,1,\infty)$ into ordered triplet $(0,1,\infty)$. Similarly, $\chi_1\circ\psi^{-1}\circ(\chi_2)^{-1}$ is a linear fractional transformation, which maps ordered triplet $(0,1,\infty)$ into ordered triplet $(0,1,\infty)$. It follows, by a, that $\chi_1\circ\psi^{-1}\circ(\chi_2)^{-1}=I$, and $\chi_2\circ\varphi\circ(\chi_1)^{-1}=I$. Since $\chi_2\circ\varphi\circ(\chi_1)^{-1}=I$, we have $\varphi=(\chi_2)^{-1}\circ\chi_1$. Since $\chi_1\circ\psi^{-1}\circ(\chi_2)^{-1}=I$, we have $\psi=(\chi_2)^{-1}\circ\chi_1$. Thus $\varphi=\psi$. ∎)

Conclusion 2.86 Let a,b,c be distinct points of $\mathbb{S}^2$. Let α,β,γ be distinct points of $\mathbb{S}^2$. Then there exists a unique linear fractional transformation φ, which maps ordered triplet (a,b,c) into (α,β,γ).

2.8 Arzela–Ascoli Theorem

Note 2.87 Definition Let (X,ρ) be a metric space. Let $\mathcal{F}$ be a collection of functions $f:X\to\mathbb{C}$. If for every $\varepsilon>0$, there exists $\delta>0$ such that for every $f\in\mathcal{F}$, and for every x,y in X satisfying $\rho(x,y)<\delta$, we have $|f(x)-f(y)|<\varepsilon$, then we say that $\mathcal{F}$ *is* ***equicontinuous***.

Definition Let (X,ρ) be a metric space. Let $\mathcal{F}$ be a collection of functions $f:X\to\mathbb{C}$. If, for every $x\in X$, there exists a positive real number $M(x)$ such that, for every $f\in\mathcal{F}$, $|f(x)|\le M(x)$, then we say that $\mathcal{F}$ *is* ***pointwise bounded***.
Let (X,ρ) be a metric space. Let $\mathcal{F}$ be a collection of functions $f:X\to\mathbb{C}$. Let $\mathcal{F}$ be equicontinuous and pointwise bounded. Let E be a countable dense subset of X. Let $\{f_n\}$ be a sequence in $\mathcal{F}$.

Since E is countable, we can write: $(X\supset)E=\{x_1,x_2,x_3,\ldots\}$.

Since $\mathcal{F}$ is pointwise bounded, and $x_1\in X$, there exists a positive real number $M(x_1)$ such that for every $f\in\mathcal{F}$, $|f(x_1)|\le M(x_1)$, and hence for every positive integer n, $|f_n(x_1)|\le M(x_1)$. Thus, $\{f_1(x_1),f_2(x_1),f_3(x_1),\ldots\}$ is a bounded sequence of complex numbers, and hence there exist positive integers $k_{11},k_{12},k_{13},\ldots$ such that

1. $\{k_{11},k_{12},k_{13},\ldots\}\subset\{1,2,3,\ldots\}$,
2. $k_{11}<k_{12}<k_{13}<\cdots$,
3. $\lim_{n\to\infty}f_{k_{1n}}(x_1)$ exists.

Let us put

$$S_0 \equiv \{1, 2, 3, \ldots\},$$

and

$$S_1 \equiv \{k_{11}, k_{12}, k_{13}, \ldots\}.$$

Thus, $S_1 \subset S_0$. Since $\lim_{n\to\infty} f_{k_{1n}}(x_1)$ exists, $S_1 = \{k_{11}, k_{12}, k_{13}, \ldots\}$, and $k_{11} < k_{12} < k_{13} < \cdots$, $\lim_{\substack{n \to \infty \\ n \in S_1}} f_n(x_1)$ exists. Since $\mathcal{F}$ is pointwise bounded, and $x_2 \in X$, there exists a positive real number $M(x_2)$ such that for every $f \in \mathcal{F}$, $|f(x_2)| \leq M(x_2)$, and hence for every positive integer n, $|f_n(x_2)| \leq M(x_2)$. Thus, $\{f_{k_{11}}(x_2), f_{k_{12}}(x_2), f_{k_{13}}(x_2), \ldots\}$ is a bounded sequence of complex numbers, and hence there exist positive integers $k_{21}, k_{22}, k_{23}, \ldots$ such that

1. $\{k_{21}, k_{22}, k_{23}, \ldots\} \subset \{k_{11}, k_{12}, k_{13}, \ldots\}$
2. $k_{21} < k_{22} < k_{23} < \ldots$,
3. $\lim_{n\to\infty} f_{k_{2n}}(x_2)$ exists.

Let us put $S_2 \equiv \{k_{21}, k_{22}, k_{23}, \cdots\}$. Clearly, $S_2 \subset S_1 \subset S_0$. Since $\lim_{n\to\infty} f_{k_{1n}}(x_1)$ exists, and $\{f_{k_{2n}}(x_1)\}$ is a subsequence of $\{f_{k_{1n}}(x_1)\}$, $\lim_{n\to\infty} f_{k_{2n}}(x_1)$ exists, and hence $\lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_1)$ exists. Since $\lim_{n\to\infty} f_{k_{2n}}(x_2)$ exists, $S_2 = \{k_{21}, k_{22}, k_{23}, \ldots\}$, and $k_{21} < k_{22} < k_{23} < \cdots$, $\lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_2)$ exists.

Thus, $\lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_1), \lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_2)$ exist.

Since $\{k_{21}, k_{22}, k_{23}, \ldots\} \subset \{k_{11}, k_{12}, k_{13}, \ldots\}$, $k_{11} < k_{12} < k_{13} < \cdots$, and $k_{21} < k_{22} < k_{23} < \cdots$, we have $k_{11} < k_{22}$.

Since $\mathcal{F}$ is pointwise bounded, and $x_3 \in X$, there exists a positive real number $M(x_3)$ such that for every $f \in \mathcal{F}$, $|f(x_3)| \leq M(x_3)$, and hence for every positive integer n, $|f_n(x_3)| \leq M(x_3)$. Thus,

$$\{f_{k_{21}}(x_3), f_{k_{22}}(x_3), f_{k_{23}}(x_3), \ldots\}$$

is a bounded sequence of complex numbers, and hence there exist positive integers $k_{31}, k_{32}, k_{33}, \ldots$ such that

1. $\{k_{31}, k_{32}, k_{33}, \ldots\} \subset \{k_{21}, k_{22}, k_{23}, \ldots\}$
2. $k_{31} < k_{32} < k_{33} < \ldots$,
3. $\lim_{n\to\infty} f_{k_{3n}}(x_3)$ exists.

Let us put $S_3 \equiv \{k_{31}, k_{32}, k_{33}, \ldots\}$. Clearly, $S_3 \subset S_2 \subset S_1 \subset S_0$. Since $\lim_{n\to\infty} f_{k_{1n}}(x_1)$ exists, and $\{f_{k_{3n}}(x_1)\}$ is a subsequence of $\{f_{k_{1n}}(x_1)\}$, $\lim_{n\to\infty} f_{k_{3n}}(x_1)$ exists, and hence $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_1)$ exists. Since $\lim_{n\to\infty} f_{k_{2n}}(x_2)$ exists, and $\{f_{k_{3n}}(x_2)\}$ is a subsequence of $\{f_{k_{2n}}(x_2)\}$, $\lim_{n\to\infty} f_{k_{3n}}(x_2)$ exists and hence $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_2)$ exists. Since $\lim_{n\to\infty} f_{k_{3n}}(x_3)$ exists, $S_3 = \{k_{31}, k_{32}, k_{33}, \ldots\}$, and $k_{31} < k_{32} < k_{33} < \cdots$, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_3)$ exists.

Thus, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_1)$, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_2)$, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_3)$ exist.

Since $\{k_{31}, k_{32}, k_{33}, \ldots\} \subset \{k_{21}, k_{22}, k_{23}, \ldots\}$, $k_{21} < k_{22} < k_{23} < \cdots$, and $k_{31} < k_{32} < k_{33} < \cdots$, we have $k_{22} < k_{33}$, etc.

Since $k_{11} < k_{22} < k_{33} < \cdots$, $\{f_{k_{11}}, f_{k_{22}}, f_{k_{33}}, \ldots\}$ is a subsequence of $\{f_n\}$.

Observe that all but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2.

(**Reason** Here, $k_{22} \in S_2$. Since $k_{33} \in S_3 \subset S_2$, $k_{33} \in S_2$. Since $k_{44} \in S_4 \subset S_2$, $k_{44} \in S_2$, etc. Thus, $\{k_{22}, k_{33}, \ldots\} \subset S_2$, and hence, all but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2.)

Since all but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2, and $\lim_{\substack{n\to\infty \\ n\in S_2}} f_n(x_1)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_1)$ exists. Since all but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2, and $\lim_{\substack{n\to\infty \\ n\in S_2}} f_n(x_2)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_2)$ exists.

Similarly, $\lim_{n\to\infty} f_{k_{nn}}(x_3)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_4)$ exists, etc.

Thus, for every $a \in E$, $\lim_{n\to\infty} f_{k_{nn}}(a)$ exists.

Problem 2.88 $\{f_{k_{11}}, f_{k_{22}}, f_{k_{33}}, \ldots\}$ converges uniformly on every compact subset of X.

(**Solution** For this purpose, let us take any compact subset K of Ω. Next, let us take any $\varepsilon > 0$. It suffices to find a positive integer N such that for every $m, n \geq N$, and for every $x \in K$, $|f_{k_{mm}}(x) - f_{k_{nn}}(x)| < 5\varepsilon$.

Since $\mathcal{F}$ is equicontinuous, and $\varepsilon > 0$, there exists $\delta > 0$ such that for every $f \in \mathcal{F}$, and for every x, y in X satisfying $\rho(x, y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$. It follows that for every positive integer n, and for every x, y in X satisfying $\rho(x, y) < \delta$, we have $|f_{k_{nn}}(x) - f_{k_{nn}}(y)| < \varepsilon$.

Since K is compact, and $\left\{B_{\frac{\delta}{2}}(a) : a \in K\right\}$ is an open cover of K, there exist finite-many $a_1, \ldots, a_M$ in K such that $K \subset B_{\frac{\delta}{2}}(a_1) \cup \cdots \cup B_{\frac{\delta}{2}}(a_M)$. Since $\{x_1, x_2, x_3, \ldots\}$ is a dense subset of X, and $B_{\frac{\delta}{2}}(a_1)$ is an open neighborhood of a_1, there exists a positive integer k_1 such that $\rho(x_{k_1}, a_1) < \frac{\delta}{2}$. Similarly, there exists a positive integer k_2 such that $\rho(x_{k_2}, a_2) < \frac{\delta}{2}$, …, and there exists a positive integer k_M such that $\rho(x_{k_M}, a_M) < \frac{\delta}{2}$. Since $x_{k_1} \in E$, and for every $a \in E$, $\lim_{n\to\infty} f_{k_{nn}}(a)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_{k_1})$ exists, and hence $\{f_{k_{nn}}(x_{k_1})\}$ is a Cauchy sequence. Similarly, $\{f_{k_{nn}}(x_{k_2})\}$ is a Cauchy sequence, …, $\{f_{k_{nn}}(x_{k_M})\}$ is a Cauchy sequence.

It follows that there exists a positive integer N such that for every $m, n \geq N$, we have $|f_{k_{mm}}(x_{k_1}) - f_{k_{nn}}(x_{k_1})| < \varepsilon, \ldots, |f_{k_{mm}}(x_{k_M}) - f_{k_{nn}}(x_{k_M})| < \varepsilon$.

Let us fix any positive integers $m, n \geq N$. Let us fix any $a \in K$. It suffices to show that $|f_{k_{mm}}(a) - f_{k_{nn}}(a)| < 5\varepsilon$.

Since $a \in K\left(\subset B_{\frac{\delta}{2}}(a_1) \cup \cdots \cup B_{\frac{\delta}{2}}(a_M)\right)$, there exists $j \in \{1, \ldots, M\}$ such that $a \in B_{\frac{\delta}{2}}(a_j)$, and hence $\left|f_{k_{nn}}(a) - f_{k_{nn}}(a_j)\right| < \varepsilon$. Since $\rho\left(x_{k_j}, a_j\right) < \frac{\delta}{2}$, we have $\left|f_{k_{nn}}(a_j) - f_{k_{nn}}(x_{k_j})\right| < \varepsilon$. Thus $\left|f_{k_{nn}}(a) - f_{k_{nn}}(x_{k_j})\right| < 2\varepsilon$. Similarly, $|f_{k_{mm}}(a) - f_{k_{mm}}(x_{k_j})| < 2\varepsilon$.

Now, since

$|f_{k_{mm}}(a) - f_{k_{mm}}(x_{k_j})| < 2\varepsilon$, $\left|f_{k_{mm}}\left(x_{x_{k_j}}\right) - f_{k_{nn}}\left(x_{x_{k_j}}\right)\right| < \varepsilon$, and $\left|f_{k_{nn}}(a) - f_{k_{nn}}(x_{k_j})\right| < 2\varepsilon$, we have

$$|f_{k_{mm}}(a) - f_{k_{nn}}(a)| < 5\varepsilon.$$

■)

Conclusion 2.89 Let (X, ρ) be a metric space. Let $\mathcal{F}$ be a collection of functions $f : X \to \mathbb{C}$. Let $\mathcal{F}$ be equicontinuous and pointwise bounded. Suppose that there exists a countable dense subset of X. Then every sequence in $\mathcal{F}$ has a subsequence, which converges uniformly on every compact subset of X.

This result, known as the **Arzela–Ascoli theorem**, is due to C. Arzela (06.03.1847–15.03.1912) and G. Ascoli (12.12.1887–10.05.1957).

2.9 Montel's Theorem

Note 2.90 Let Ω be a region. Let $\mathcal{F}$ be a nonempty subset of $H(\Omega)$. Suppose that for every nonempty compact subset K of Ω, there exists a positive real number $M(K)$ such that for every $f \in \mathcal{F}$, and for every $z \in K$, $|f(z)| \leq M(K)$. In short, we say that $\mathcal{F}$ is uniformly bounded on each compact subset of Ω.

Now, since for every $z \in \Omega$, $\{z\}$ is a compact subset of Ω, $\mathcal{F}$ is pointwise bounded. Observe that $\{z : z \in \Omega, \text{and}, \operatorname{Re}(z), \operatorname{Im}(z) \text{are rational numbers}\}$ is a countable dense subset of Ω. Put $E \equiv \{z : z \in \Omega, \text{and}, \operatorname{Re}(z), \operatorname{Im}(z)\text{are rational numbers}\}$. Thus, E is a countable dense subset of Ω.

Problem 2.91 Every sequence in $\mathcal{F}$ has a subsequence, which converges uniformly on every compact subset of Ω.

(Solution For this purpose, let us take any sequence $\{f_n\}$ in $\mathcal{F}$.

Since E is countable, we can write: $(\Omega \supset)E = \{x_1, x_2, x_3, \ldots\}$. Since $\mathcal{F}$ is pointwise bounded, and $x_1 \in \Omega$, there exists a positive real number $M(x_1)$ such that for every $f \in \mathcal{F}, |f(x_1)| \leq M(x_1)$, and hence for every positive integer n, $|f_n(x_1)| \leq M(x_1)$. Thus, $\{f_1(x_1), f_2(x_1), f_3(x_1), \ldots\}$ is a bounded sequence of complex numbers, and hence there exist positive integers $k_{11}, k_{12}, k_{13}, \ldots$ such that

1. $\{k_{11}, k_{12}, k_{13}, \ldots\} \subset \{1, 2, 3, \ldots\}$,
2. $k_{11} < k_{12} < k_{13} < \cdots$,
3. $\lim_{n\to\infty} f_{k_{1n}}(x_1)$ exists.

Let us put $S_0 \equiv \{1, 2, 3, \ldots\}$, and $S_1 \equiv \{k_{11}, k_{12}, k_{13}, \ldots\}$. Thus, $S_1 \subset S_0$. Since $\lim_{n\to\infty} f_{k_{1n}}(x_1)$ exists, $S_1 = \{k_{11}, k_{12}, k_{13}, \ldots\}$, and $k_{11} < k_{12} < k_{13} < \cdots$, $\lim_{\substack{n \to \infty \\ n \in S_1}} f_n(x_1)$ exists.

Since $\mathcal{F}$ is pointwise bounded, and $x_2 \in \Omega$, there exists a positive real number $M(x_2)$ such that for every $f \in \mathcal{F}, |f(x_2)| \leq M(x_2)$, and hence for every positive integer n, $|f_n(x_2)| \leq M(x_2)$. Thus, $\{f_{k_{11}}(x_2), f_{k_{12}}(x_2), f_{k_{13}}(x_2), \ldots\}$ is a bounded sequence of complex numbers, and hence there exist positive integers $k_{21}, k_{22}, k_{23}, \ldots$ such that

1. $\{k_{21}, k_{22}, k_{23}, \ldots\} \subset \{k_{11}, k_{12}, k_{13}, \ldots\}$
2. $k_{21} < k_{22} < k_{23} < \cdots$,
3. $\lim_{n\to\infty} f_{k_{2n}}(x_2)$ exists.

Let us put $S_2 \equiv \{k_{21}, k_{22}, k_{23}, \ldots\}$. Clearly, $S_2 \subset S_1 \subset S_0$. Since $\lim_{n\to\infty} f_{k_{1n}}(x_1)$ exists, and $\{f_{k_{2n}}(x_1)\}$ is a subsequence of $\{f_{k_{1n}}(x_1)\}$, $\lim_{n\to\infty} f_{k_{2n}}(x_1)$ exists, and hence $\lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_1)$ exists. Since $\lim_{n\to\infty} f_{k_{2n}}(x_2)$ exists, $S_2 = \{k_{21}, k_{22}, k_{23}, \ldots\}$, and $k_{21} < k_{22} < k_{23} < \cdots$, $\lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_2)$ exists.

Thus, $\lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_1), \lim_{\substack{n \to \infty \\ n \in S_2}} f_n(x_2)$ exist.

Since $\{k_{21}, k_{22}, k_{23}, \ldots\} \subset \{k_{11}, k_{12}, k_{13}, \ldots\}$, $k_{11} < k_{12} < k_{13} < \cdots$, and $k_{21} < k_{22} < k_{23} < \cdots$, we have $k_{11} < k_{22}$.

Since $\mathcal{F}$ is pointwise bounded, and $x_3 \in \Omega$, there exists a positive real number $M(x_3)$ such that for every $f \in \mathcal{F}, |f(x_3)| \leq M(x_3)$, and hence for every positive integer n, $|f_n(x_3)| \leq M(x_3)$. Thus, $\{f_{k_{21}}(x_3), f_{k_{22}}(x_3), f_{k_{23}}(x_3), \ldots\}$ is a bounded sequence of complex numbers, and hence there exist positive integers $k_{31}, k_{32}, k_{33}, \ldots$ such that

1. $\{k_{31}, k_{32}, k_{33}, \ldots\} \subset \{k_{21}, k_{22}, k_{23}, \ldots\}$
2. $k_{31} < k_{32} < k_{33} < \cdots$,
3. $\lim_{n\to\infty} f_{k_{3n}}(x_3)$ exists.

Let us put $S_3 \equiv \{k_{31}, k_{32}, k_{33}, \ldots\}$. Clearly, $S_3 \subset S_2 \subset S_1 \subset S_0$. Since $\lim_{n\to\infty} f_{k_{1n}}(x_1)$ exists, and $\{f_{k_{3n}}(x_1)\}$ is a subsequence of $\{f_{k_{1n}}(x_1)\}$, $\lim_{n\to\infty} f_{k_{3n}}(x_1)$ exists, and hence $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_1)$ exists. Since $\lim_{n\to\infty} f_{k_{2n}}(x_2)$ exists, and $\{f_{k_{3n}}(x_2)\}$ is a subsequence of $\{f_{k_{2n}}(x_2)\}$, $\lim_{n\to\infty} f_{k_{3n}}(x_2)$ exists, and hence $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_2)$ exists. Since $\lim_{n\to\infty} f_{k_{3n}}(x_3)$ exists, $S_3 = \{k_{31}, k_{32}, k_{33}, \ldots\}$, and $k_{31} < k_{32} < k_{33} < \cdots$, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_3)$ exists.

Thus, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_1)$, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_2)$, $\lim_{\substack{n\to\infty \\ n\in S_3}} f_n(x_3)$ exist.

Since $\{k_{31}, k_{32}, k_{33}, \ldots\} \subset \{k_{21}, k_{22}, k_{23}, \ldots\}$, $k_{21} < k_{22} < k_{23} < \cdots$, and $k_{31} < k_{32} < k_{33} < \cdots$, we have $k_{22} < k_{33}$, etc.

Since $k_{11} < k_{22} < k_{33} < \cdots$, $\{f_{k_{11}}, f_{k_{22}}, f_{k_{33}}, \ldots\}$ is a subsequence of $\{f_n\}$.

Problem 2.92 All but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2.

(Solution Here, $k_{22} \in S_2$. Since $k_{33} \in S_3 \subset S_2$, $k_{33} \in S_2$. Since $k_{44} \in S_4 \subset S_2$, $k_{44} \in S_2$, etc. Thus, $\{k_{22}, k_{33}, \ldots\} \subset S_2$, and hence, all but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2. ∎)

Since all but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2, and $\lim_{\substack{n\to\infty \\ n\in S_2}} f_n(x_1)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_1)$ exists. Since all but finite-many elements of $\{k_{11}, k_{22}, k_{33}, \ldots\}$ are in S_2, and $\lim_{\substack{n\to\infty \\ n\in S_2}} f_n(x_2)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_2)$ exists.

Similarly, $\lim_{n\to\infty} f_{k_{nn}}(x_3)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_4)$ exists, etc.

Thus, for every $a \in E$, $\lim_{n\to\infty} f_{k_{nn}}(a)$ exists.

Problem 2.93 $\{f_{k_{11}}, f_{k_{22}}, f_{k_{33}}, \ldots\}$ converges uniformly on every compact subset of Ω.

(Solution For this purpose, let us take any compact subset K of Ω. Next, let us take any $\varepsilon > 0$. It suffices to find a positive integer N such that, for every $m, n \geq N$, and for every $z \in K$, $|f_{k_{mm}}(z) - f_{k_{nn}}(z)| < 5\varepsilon$.

By Conclusion 2.53, there exists a sequence $\{K_1, K_2, K_3, \ldots\}$ of compact subsets of Ω satisfying

1. $(K_1)^0 \subset K_1 \subset (K_2)^0 \subset K_2 \subset (K_3)^0 \subset K_3 \subset \cdots \subset \Omega$,
2. $\Omega = \cup_{n=1}^{\infty} K_n$,
3. there exists a positive integer n_0 such that $K \subset K_{n_0}$,
4. for every $z \in K_{n_0}$, $D\left(z; \frac{1}{n_0} - \frac{1}{n_0+1}\right) \subset K_{n_0+1}$.

It suffices to find a positive integer N such that for every $m, n \geq N$, and for every $z \in K_{n_0}$, $|f_{k_{mm}}(z) - f_{k_{nn}}(z)| < 5\varepsilon$.

Let us take any $z', z'' \in K_{n_0}$ such that $|z' - z''| < \frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)$. Let γ be the positively oriented circle with center at z', and radius $\frac{1}{2}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)$. Since $z' \in K_{n_0}$, we have $(\operatorname{ran}(\gamma) \subset) D\left(z'; \frac{1}{n_0} - \frac{1}{n_0+1}\right) \subset K_{n_0+1}$, and hence for every $\zeta \in \operatorname{ran}(\gamma)$, and for every $f \in \mathcal{F}$, $|f(\zeta)| \leq M(K_{n_0+1})$.

Now, by Conclusion 1.116, for every $f \in \mathcal{F}$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z'}\mathrm{d}\zeta = (f(z'))\left((\mathrm{Ind})_\gamma(z')\right)(= (f(z'))(1) = f(z')),$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z''}\mathrm{d}\zeta = (f(z''))\left((\mathrm{Ind})_\gamma(z'')\right)(= (f(z''))(1) = f(z'')).$$

So, for every $f \in \mathcal{F}$,

$$\begin{aligned}|f(z') - f(z'')| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z'}\mathrm{d}\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z''}\mathrm{d}\zeta\right| \\ &= \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{\zeta - z'} - \frac{1}{\zeta - z''}\right) f(\zeta)\mathrm{d}\zeta\right| \\ &= \frac{1}{2\pi}\left|\int_\gamma \left(\frac{z' - z''}{(\zeta - z')(\zeta - z'')}\right) f(\zeta)\mathrm{d}\zeta\right|,\end{aligned}$$

and hence

$$|f(z') - f(z'')| = \frac{1}{2\pi}\left|\int_\gamma \left(\frac{z' - z''}{(\zeta - z')(\zeta - z'')}\right) f(\zeta)\mathrm{d}\zeta\right|.$$

Since for every $\zeta \in \operatorname{ran}(\gamma)$,

$$\begin{aligned}\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right) &= \frac{1}{2}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right) - \frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right) \leq \frac{1}{2}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right) \\ &\quad - |z'' - z'| \\ &= |\zeta - z'| - |z'' - z'| \leq |\zeta - z''|,\end{aligned}$$

we have for every $\zeta \in \operatorname{ran}(\gamma)$,

$$\frac{1}{|\zeta - z''|} \le \frac{1}{\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)}.$$

Hence,

$$\begin{aligned}(2\pi|f(z') - f(z'')|) &= \left|\int_\gamma \left(\frac{(z'-z'')}{(\zeta - z')(\zeta - z'')}\right) f(\zeta)\mathrm{d}\zeta\right| \\ &\le \frac{1}{\frac{1}{2}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)} \frac{1}{\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)} |z' - z''| \cdot M(K_{n_0+1}) \cdot 2\pi \frac{1}{2}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right) \\ &= \frac{M(K_{n_0+1}) \cdot 2\pi}{\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)} |z' - z''|.\end{aligned}$$

Thus, for every $f \in \mathcal{F}$, and, for every $z', z'' \in K_{n_0}$,

$$|f(z') - f(z'')| \le \frac{M(K_{n_0+1})}{\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)} |z' - z''|.$$

Problem 2.94 $\left\{f|_{K_{n_0}} : f \in \mathcal{F}\right\}$ is an equicontinuous family.

(Solution For this purpose, let us take any $\varepsilon_1 > 0$. Put $\delta \equiv \frac{\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)\varepsilon_1}{M\left(K_{n_0+1}\right)} (>0)$. Let us take any $f \in \mathcal{F}$. Let us take any $z', z'' \in K_{n_0}$ satisfying $|z' - z''| < \delta$. We have to show that $|f(z') - f(z'')| < \varepsilon_1$.

Since $|z' - z''| < \delta \left(= \frac{\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)\varepsilon_1}{M\left(K_{n_0+1}\right)}\right)$, we have

$$|f(z') - f(z'')| \le \underbrace{\frac{M(K_{n_0+1})}{\frac{1}{4}\left(\frac{1}{n_0} - \frac{1}{n_0+1}\right)} |z' - z''| < \varepsilon_1},$$

and hence $|f(z') - f(z'')| < \varepsilon_1$. ∎)

Now, since $K \subset K_{n_0}$, $\left\{f|_K : f \in \mathcal{F}\right\}$ is an equicontinuous family.

It follows that there exists $\delta > 0$ such that for every $f \in \mathcal{F}$, and for every z', z'' in K satisfying $|z' - z''| < \delta$, we have $|f(z') - f(z'')| < \varepsilon$.

Since K_{n_0} is compact, and $\left\{B_{\frac{\delta}{2}}(a) : a \in K_{n_0}\right\}$ is an open cover of K_{n_0}, there exist finite-many $a_1, \ldots, a_M$ in K_{n_0} such that $K_{n_0} \subset B_{\frac{\delta}{2}}(a_1) \cup \cdots \cup B_{\frac{\delta}{2}}(a_M)$. Since $\{x_1, x_2, x_3, \ldots\}$ is a dense subset of Ω, and $B_{\frac{\delta}{2}}(a_1)$ is an open neighborhood of a_1, there exists a positive integer k_1 such that $|x_{k_1} - a_1| < \frac{\delta}{2}$. Similarly, there exists a positive integer k_2 such that $|x_{k_2} - a_2| < \frac{\delta}{2}$, ..., and there exists a positive integer k_M such that $|x_{k_M} - a_M| < \frac{\delta}{2}$. Since $x_{k_1} \in E$ and, for every $a \in E$, $\lim_{n\to\infty} f_{k_{nn}}(a)$ exists, $\lim_{n\to\infty} f_{k_{nn}}(x_{k_1})$ exists, and hence $\{f_{k_{nn}}(x_{k_1})\}$ is a Cauchy sequence. Similarly, $\{f_{k_{nn}}(x_{k_2})\}$ is a Cauchy sequence, ..., $\{f_{k_{nn}}(x_{k_M})\}$ is a Cauchy sequence.

It follows that there exists a positive integer N such that for every $m, n \geq N$, we have $|f_{k_{mm}}(x_{k_1}) - f_{k_{nn}}(x_{k_1})| < \varepsilon, \ldots, |f_{k_{mm}}(x_{k_M}) - f_{k_{nn}}(x_{k_M})| < \varepsilon$.

Let us fix any positive integers $m, n \geq N$. Let us fix any $a \in K_{n_0}$. It suffices to show that $|f_{k_{mm}}(a) - f_{k_{nn}}(a)| < 5\varepsilon$.

Since

$$a \in K_{n_0}\left(\subset B_{\frac{\delta}{2}}(a_1) \cup \cdots \cup B_{\frac{\delta}{2}}(a_M)\right),$$

there exists $j \in \{1, \ldots, M\}$ such that $a \in B_{\frac{\delta}{2}}(a_j)$, and hence $\left|f_{k_{nn}}(a) - f_{k_{nn}}(a_j)\right| < \varepsilon$. Since $\left|x_{k_j} - a_j\right| < \frac{\delta}{2}$, we have $\left|f_{k_{nn}}(a_j) - f_{k_{nn}}(x_{k_j})\right| < \varepsilon$. Thus, $\left|f_{k_{nn}}(a) - f_{k_{nn}}(x_{k_j})\right| < 2\varepsilon$. Similarly, $\left|f_{k_{mm}}(a) - f_{k_{mm}}(x_{k_j})\right| < 2\varepsilon$.

Now, since

$$\left|f_{k_{mm}}(a) - f_{k_{mm}}(x_{k_j})\right| < 2\varepsilon, \left|f_{k_{mm}}\left(x_{x_{k_j}}\right) - f_{k_{nn}}\left(x_{x_{k_j}}\right)\right| < \varepsilon, \text{ and } \left|f_{k_{nn}}(a) - f_{k_{nn}}(x_{k_j})\right| < 2\varepsilon,$$

we have

$$|f_{k_{mm}}(a) - f_{k_{nn}}(a)| < 5\varepsilon.$$

∎)

Conclusion 2.95 Let Ω be a region. Let $\mathcal{F}$ be a nonempty subset of $H(\Omega)$. Suppose that $\mathcal{F}$ is uniformly bounded on each compact subset of Ω. Then every sequence in $\mathcal{F}$ has a subsequence, which converges uniformly on every compact subset of Ω.

Definition Let Ω be a region. Let $\mathcal{F}$ be a nonempty subset of $H(\Omega)$. If every sequence in $\mathcal{F}$ has a subsequence, which converges uniformly on every compact subset of Ω, then we say that $\mathcal{F}$ *is a* ***normal family***.
Now, the above conclusion can be stated as follows:

Conclusion 2.96 Let Ω be a region. Let $\mathcal{F}$ be a nonempty subset of $H(\Omega)$. Suppose that $\mathcal{F}$ is uniformly bounded on each compact subset of Ω. Then, $\mathcal{F}$ is a normal family.

This result, known as **Montel's theorem**, is due to P. Montel (29.04.1876–22.01.1975).

2.10 Riesz Representation Theorem for Bounded Functionals

Note 2.97 Let X be a locally compact Hausdorff space.

(By Conclusion 2.248, Vol. 1, $C_0(X)$ is a Banach space, whose norm is defined by $\|f\| \equiv \sup\{|f(x)| : x \in X\}$. By Conclusion 2.237, Vol. 1, $C_c(X) \subset C_0(X)$, and, by Lemma 1.171, Vol. 1, $C_c(X)$ is a complex linear space so $C_c(X)$ is a linear subspace of $C_0(X)$. Now, since $C_0(X)$ is a Banach space under the supremum norm, $C_c(X)$ is a normed linear space under the supremum norm. Further, by Conclusion 2.245, Vol. 1, $C_c(X)$ is a dense subset of $C_0(X)$.)

Let $\Phi : C_0(X) \to \mathbb{C}$ be a bounded linear functional on $C_0(X)$. Let $\|\Phi\| = 1$.

Let $C_c^+(X)$ be the collection of those $f \in C_c(X)$ for which $\mathrm{ran}(f) \subset [0, \infty)$.

For every $f \in C_c^+(X)$, $\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \le f\}$ is nonempty, because $0 \in C_c(X), \text{and } |0| \le f$. Thus, $\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \le f\}$ is a nonempty set of nonnegative real numbers.

Problem 2.98 For every $f \in C_c^+(X)$,

$$\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \le f\}$$

exists, and

$$(\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \le f\}) \le \|f\|.$$

(Solution Let us fix any $f \in C_c^+(X)$. For every $h \in C_c(X)$ satisfying $|h| \le f$, we have

$$\begin{aligned}|\Phi(h)| \le \|\Phi\|\|h\| = 1\|h\| = \|h\| = \sup\{|h(x)| : x \in X\} \le \sup\{f(x) : x \in X\} \\ = \sup\{|f(x)| : x \in X\} = \|f\| \in [0, \infty),\end{aligned}$$

so

$$\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \le f\}$$

is bounded above, and f is an upper bound of

$$\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \le f\}.$$

It follows that $\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \le f\}$ exists, and

$$(\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \le f\}) \le \|f\|.$$

∎)

For every $f \in C_c^+(X)$, let us denote

$$(\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq f\})$$

by $\Lambda(f)$. Thus, for every $f \in C_c^+(X)$, $0 \leq \Lambda(f) \leq \|f\|$.

For every $f \in C_c^+(X)(\subset C_c(X))$, we have $|f| = f \leq f$, and hence $|\Phi(f)| \in \{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq f\}$. It follows that for every $f \in C_c^+(X)$, $|\Phi(f)| \leq \sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq f\} (= \Lambda(f) \leq \|f\|)$, and hence for every $f \in C_c^+(X)$, we have $|\Phi(f)| \leq \Lambda(f) \leq \|f\|$. Now, since for every $f \in C_c^+(X)$, $|f| = f$, we have

a. for every $f \in C_c^+(X)$, $|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|$. Hence, $\Lambda(0) = 0$.
b. Let $f_1, f_2 \in C_c^+(X)$. Let $f_1 \leq f_2$.

Problem 2.99 $\Lambda(f_1) \leq \Lambda(f_2)$.

(Solution Since $f_1, f_2 \in C_c^+(X)$, and $f_1 \leq f_2$, we have

$$\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq f_1\} \subset \{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq f_2\},$$

and hence

$$\Lambda(f_1) = \underbrace{\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq f_1\} \leq \sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq f_2\}} = \Lambda(f_2).$$

Thus, $\Lambda(f_1) \leq \Lambda(f_2)$. ∎)

c. Let $f \in C_c^+(X)$. Let c be a positive real number. It follows that $(cf) \in C_c^+(X)$. Clearly, $\Lambda(cf) = c(\Lambda(f))$.

(Reason Since

$$\begin{aligned}\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq cf\} &= \{|\Phi(ck)| : k \in C_c(X), \text{and } |ck| \leq cf\} \\ &= \{|c\Phi(k)| : k \in C_c(X), \text{and } c|k| \leq cf\} \\ &= \{|c\Phi(k)| : k \in C_c(X), \text{and } |k| \leq f\} \\ &= \{c|\Phi(k)| : k \in C_c(X), \text{and } |k| \leq f\},\end{aligned}$$

we have

$$\begin{aligned}(\Lambda(cf)) &= \sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq cf\} = \sup\{c|\Phi(k)| : k \in C_c(X), \text{and } |k| \leq f\} \\ &= c(\sup\{|\Phi(k)| : k \in C_c(X), \text{and } |k| \leq f\}) = c(\Lambda(f)),\end{aligned}$$

and hence $\Lambda(cf) = c(\Lambda(f))$.)

d. Let $f, g \in C_c^+(X)$. It follows that $(f+g) \in C_c^+(X)$.

Problem 2.100 $\Lambda(f) + \Lambda(g) \leq \Lambda(f+g)$, that is

$$\sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq f\} + \sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq g\}$$
$$\leq \sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq (f+g)\}.$$

(Solution Let us take any $h_1 \in C_c(X)$ such that $|h_1| \leq f$. Let us take any $h_2 \in C_c(X)$ such that $|h_2| \leq g$. It suffices to show that

$$(|\Phi(h_1)| + |\Phi(h_2)|) \in \{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq (f+g)\}.$$

There exists a complex number α_1 such that $|\alpha_1| = 1$, and $|\Phi(h_1)| = \alpha_1(\Phi(h_1))(= \Phi(\alpha_1 h_1))$. Similarly, there exists a complex number α_2 such that $|\alpha_2| = 1$, and $|\Phi(h_2)| = \Phi(\alpha_2 h_2)$. Thus, $|\Phi(h_1)| + |\Phi(h_2)| = \Phi(\alpha_1 h_1) + \Phi(\alpha_2 h_2) = \Phi(\alpha_1 h_1 + \alpha_2 h_2)$, and hence

$$|\Phi(\alpha_1 h_1 + \alpha_2 h_2)| = |\Phi(h_1)| + |\Phi(h_2)|.$$

Now, we have to show that $|\Phi(\alpha_1 h_1 + \alpha_2 h_2)| \in \{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq (f+g)\}$. Since $h_1, h_2 \in C_c(X)$, and $C_c(X)$ is a linear space, we have $(\alpha_1 h_1 + \alpha_2 h_2) \in C_c(X)$. Hence, it suffices to show that

$$|\alpha_1 h_1 + \alpha_2 h_2| \leq (f+g).$$

Since

$$|\alpha_1 h_1 + \alpha_2 h_2| \leq |\alpha_1||h_1| + |\alpha_2||h_2| = 1|h_1| + 1|h_2| = |h_1| + |h_2| \leq f + |h_2| \leq (f+g),$$

we have $|\alpha_1 h_1 + \alpha_2 h_2| \leq (f+g)$. ∎)

e. Let $f, g \in C_c^+(X)$. It follows that $(f+g) \in C_c^+(X)$.

Problem 2.101 $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$.

(Solution From d, it suffices to show that

$$\sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq (f+g)\} \leq \sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq f\}$$
$$+ \sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq g\}.$$

For this purpose, let us take any $h \in C_c(X)$ such that $|h| \leq (f+g)$. It is enough to show that

$|\Phi(h)| \leq \sup\{|\Phi(k)| : k \in C_c(X), \text{and} |k| \leq f\} + \sup\{|\Phi(k)| : k \in C_c(X), \text{and} |k| \leq g\}.$

Since f, g, h are continuous on X, the restriction of $h \cdot \frac{f}{f+g}$ to the open set $(f^{-1}(0) \cap g^{-1}(0))^c$ is continuous.

Problem 2.102 The function

$$h_1 : x \mapsto \begin{cases} h(x) \cdot \frac{f(x)}{f(x)+g(x)} & \text{if } x \in (f^{-1}(0) \cap g^{-1}(0))^c \\ 0 & \text{if } x \in (f^{-1}(0) \cap g^{-1}(0)) \end{cases}$$

from X to $\mathbb{C}$ is continuous.

(Solution Here,

$$h_1 : x \mapsto \begin{cases} h(x) \cdot \frac{f(x)}{f(x)+g(x)} & \text{if } x \in (f^{-1}(0) \cap g^{-1}(0))^c \\ 0 & \text{if } x \in (f^{-1}(0) \cap g^{-1}(0)) \end{cases}$$

is a function from X to $\mathbb{C}$, and the restriction of $h \cdot \frac{f}{f+g}$ to the open set $(f^{-1}(0) \cap g^{-1}(0))^c$ is continuous, h_1 is continuous at every point of $(f^{-1}(0) \cap g^{-1}(0))^c$. It remains to show that h_1 is continuous at every point of $(f^{-1}(0) \cap g^{-1}(0))$. For this purpose, let us take any $a \in (f^{-1}(0) \cap g^{-1}(0))$. We have to show that h_1 is continuous at a.

For this purpose, let us take any $\varepsilon > 0$. Since h is continuous at a, there exists an open neighborhood V of a such that for every $x \in V$, $|h(x) - h(a)| < \varepsilon$. Since $a \in (f^{-1}(0) \cap g^{-1}(0))$, we have $f(a) = 0$, and $g(a) = 0$. Now, since $|h| \leq (f+g)$, we have $(0 \leq)|h(a)| \leq (f(a)+g(a))(= 0+0 = 0)$, and hence $h(a) = 0$. Also, by the definition of h_1, $h_1(a) = 0$. Since $f, g \in C_c^+(X)$, we have $\text{ran}(f) \subset [0, \infty)$, and $\text{ran}(g) \subset [0, \infty)$, and hence by the definition of h_1, $|h_1| \leq |h|$. Since for every $x \in V$,

$$|h_1(x) - h_1(a)| = |h_1(x) - 0| = |h_1(x)| \leq |h(x)| = |h(x) - 0| = \underbrace{|h(x) - h(a)| < \varepsilon},$$

h_1 is continuous at a. ∎)

Similarly, the function

$$h_2 : x \mapsto \begin{cases} h(x) \cdot \frac{g(x)}{f(x)+g(x)} & \textit{if } x \in (f^{-1}(0) \cap g^{-1}(0))^c \\ 0 & \textit{if } x \in (f^{-1}(0) \cap g^{-1}(0)) \end{cases}$$

from X to $\mathbb{C}$ is continuous.

Problem 2.103 $h_1 \in C_c(X)$. Also, $(h_1)^{-1}(0) = f^{-1}(0)$.

(Solution Since $|h| \leq (f+g)$, we have $(f^{-1}(0) \cap g^{-1}(0)) \subset h^{-1}(0)$. Since

$$\begin{aligned}(h_1)^{-1}(\mathbb{C}-\{0\}) &= \left(h^{-1}(\mathbb{C}-\{0\})\right) \cap \left(f^{-1}(\mathbb{C}-\{0\})\right) \\ &= \left(h^{-1}(0)\right)^c \cap \left(f^{-1}(0)\right)^c \subset \left(f^{-1}(0) \cap g^{-1}(0)\right)^c \cap \left(f^{-1}(0)\right)^c \\ &= \left(f^{-1}(0)\right)^c = f^{-1}(\mathbb{C}-\{0\}),\end{aligned}$$

we have

$$(\mathrm{supp}(h_1) =)\Big((h_1)^{-1}(\mathbb{C}-\{0\})\Big)^- = \left(f^{-1}(\mathbb{C}-\{0\})\right)^- (= \mathrm{supp}(f)),$$

and hence $\mathrm{supp}(h_1) = \mathrm{supp}(f)$. Since $f \in C_c^+(X)$, we have $f \in C_c(X)$, and hence $(\mathrm{supp}(h_1) =)\mathrm{supp}(f)$ is compact. Since $h_1 : X \to \mathbb{C}$ is continuous and $\mathrm{supp}(h_1)$ is compact, we have $h_1 \in C_c(X)$. ∎)

Similarly, $h_2 \in C_c(X)$. Also, $(h_2)^{-1}(0) = g^{-1}(0)$.

It is clear that $h = h_1 + h_2$.

By the definition of h_1, and the fact that $|h| \leq (f+g)$, it is clear that $|h_1| \leq f$, and hence $|\Phi(h_1)| \in \{|\Phi(k)| : k \in C_c(X), \text{and } |k| \leq f\}$. It follows that

$$|\Phi(h_1)| \leq \sup\{|\Phi(k)| : k \in C_c(X), \text{and } |k| \leq f\}.$$

Similarly, $|\Phi(h_2)| \leq \sup\{|\Phi(k)| : k \in C_c(X), \text{and } |k| \leq g\}$. Now,

$$\begin{gathered}|\Phi(h)| = |\Phi(h_1+h_2)| = |\Phi(h_1)+\Phi(h_2)| \leq |\Phi(h_1)| + |\Phi(h_2)| \\ \leq \sup\{|\Phi(k)| : k \in C_c(X), \text{and } |k| \leq f\} + \sup\{|\Phi(k)| : k \in C_c(X), \text{and} |k| \leq g\},\end{gathered}$$

so

$$|\Phi(h)| \leq \sup\{|\Phi(k)| : k \in C_c(X), \text{and} |k| \leq f\} + \sup\{|\Phi(k)| : k \in C_c(X), \text{and} |k| \leq g\}.$$

∎)

Conclusion 2.104 Let X be a locally compact Hausdorff space. Let $\Phi : C_0(X) \to \mathbb{C}$ be a bounded linear functional on $C_0(X)$ satisfying $\Phi = 1$. Then there exists a function $\Lambda : C_c^+(X) \to [0, \infty)$ such that

1. for every $f \in C_c^+(X)$, $|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|$,
2. if $f_1, f_2 \in C_c^+(X)$, and $f_1 \leq f_2$, then $\Lambda(f_1) \leq \Lambda(f_2)$,
3. for every $f \in C_c^+(X)$, and for every positive real number c, $\Lambda(cf) = c(\Lambda(f))$,
4. for every $f, g \in C_c^+(X)$, $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$,
5. for every $f \in C_c^+(X)$, $\Lambda(f) = \sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq f\}$.

Note 2.105 Let X be a locally compact Hausdorff space.

(By Conclusion 2.248, Vol. 1, $C_0(X)$ is a Banach space, whose norm is defined by $\|f\| \equiv \sup\{|f(x)| : x \in X\}$. By Conclusion 2.237, Vol. 1, $C_c(X) \subset C_0(X)$, and by Lemma 1.171, Vol. 1, $C_c(X)$ is a complex linear space so $C_c(X)$ is a linear subspace of $C_0(X)$. Now, since $C_0(X)$ is a Banach space under the supremum norm, $C_c(X)$ is a normed linear space under the supremum norm. Further, by Conclusion 2.245, Vol. 1, $C_c(X)$ is a dense subset of $C_0(X)$.)

Let $\Phi : C_0(X) \to \mathbb{C}$ be a bounded linear functional on $C_0(X)$. Let $\|\Phi\| = 1$.

By Conclusion 2.104, there exists a function $\Lambda_1 : C_c^+(X) \to [0, \infty)$ such that

1. for every $f \in C_c^+(X)$, $|\Phi(f)| \le \Lambda_1(|f|) \le \|f\|$,
2. if $f_1, f_2 \in C_c^+(X)$, and $f_1 \le f_2$, then $\Lambda_1(f_1) \le \Lambda_1(f_2)$,
3. for every $f \in C_c^+(X)$, and for every positive real number c, $\Lambda_1(cf) = c(\Lambda_1(f))$.
4. for every $f, g \in C_c^+(X)$, $\Lambda_1(f+g) = \Lambda_1(f) + \Lambda_1(g)$,
5. for every $f \in C_c^+(X)$, $\Lambda_1(f) = \sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \le f\}$.

Let $C_c^{\mathbb{R}}(X)$ be the collection of those $f \in C_c(X)$ for which $\text{ran}(f) \subset \mathbb{R}$.

a. For every $f \in C_c^{\mathbb{R}}(X)$, we have $f = f^+ - f^-$.

Problem 2.106 $f^+, f^- \in C_c^+(X)$.

(Solution Since $f \in C_c^{\mathbb{R}}(X)$, we have $f \in C_c(X)$, and hence $f(= f^+ - f^-)$ is continuous. It follows that $|f|(= f^+ + f^-)$ is continuous, and hence $f + |f|(= 2f^+)$ is continuous. Hence, f^+ is continuous. Similarly, f^- is continuous. By the definition of f^+, $f^{-1}(0) \subset (f^+)^{-1}(0)$, so $(f^+)^{-1}(\mathbb{C} - \{0\}) \subset f^{-1}(\mathbb{C} - \{0\})$, and hence

$$\underbrace{\left((f^+)^{-1}(\mathbb{C} - \{0\})\right)^- \subset \left(f^{-1}(\mathbb{C} - \{0\})\right)^-} = \text{supp}(f).$$

Since $f \in C_c(X)$, $\text{supp}(f)$ is compact. Now, since $\left((f^+)^{-1}(\mathbb{C} - \{0\})\right)^-$ is a closed set, and $\left((f^+)^{-1}(\mathbb{C} - \{0\})\right)^- \subset \text{supp}(f)$, $(\text{supp}(f^+) =)\left((f^+)^{-1}(\mathbb{C} - \{0\})\right)^-$ is a compact set, and hence $\text{supp}(f^+)$ is a compact set. Since $\text{supp}(f^+)$ is a compact set, and f^+ is continuous, $f^+ \in C_c^+(X)$. Similarly, $f^- \in C_c^+(X)$. ∎)

Thus, for every $f \in C_c^{\mathbb{R}}(X)$, we have $\Lambda_1(f^+), \Lambda_1(f^-) \in [0, \infty)$, and hence $(\Lambda_1(f^+) - \Lambda_1(f^-)) \in \mathbb{R}$.

The function $\Lambda_2 : f \mapsto (\Lambda_1(f^+) - \Lambda_1(f^-))$ from $C_c^{\mathbb{R}}(X)$ to $\mathbb{R}$ is an extension of $\Lambda_1 : C_c^+(X) \to [0, \infty)$.

b. Let $f, g \in C_c^{\mathbb{R}}(X)$. Then, $f + g \in C_c^{\mathbb{R}}(X)$.

Problem 2.107 $\Lambda_2(f+g) = \Lambda_2(f) + \Lambda_2(g)$.

(Solution Here,

$$((f^+ - f^-) + (g^+ - g^-) =)(f+g) = (f+g)^+ - (f+g)^-,$$

so

$$(f+g)^- + f^+ + g^+ = (f+g)^+ + f^- + g^-.$$

Hence,

$$\Lambda_1((f+g)^-) + \Lambda_1(f^+) + \Lambda_1(g^+) = \underbrace{\Lambda_1((f+g)^- + f^+ + g^+) = \Lambda_1\big((f+g)^+ + f^- + g^-\big)}$$
$$= \Lambda_1\big((f+g)^+\big) + \Lambda_1(f^-) + \Lambda_1(g^-).$$

It follows that

$$\Lambda_2((f+g)) = \underbrace{\big(\Lambda_1(f+g)^+ - \Lambda_1((f+g)^-)\big) = (\Lambda_1(f^+) - \Lambda_1(f^-)) + (\Lambda_1(g^+) - \Lambda_1(g^-))} = \Lambda_2(f) + \Lambda_2(g),$$

and hence

$$\Lambda_2((f+g)) = \Lambda_2(f) + \Lambda_2(g).$$

∎)

c. Let $f \in C_c^{\mathbb{R}}(X)$. Then, $(-f) \in C_c^{\mathbb{R}}(X)$.

Problem 2.108 $\Lambda_2(-f) = -(\Lambda_2(f))$.

(Solution Here,

$$(-f)^+ - (-f)^- = -f = -(f^+ - f^-) = f^- - f^+,$$

so

$$(-f)^+ + f^+ = (-f)^- + f^-,$$

and hence

$$\Lambda_1\big((-f)^+\big) + \Lambda_1(f^+) = \underbrace{\Lambda_1\big((-f)^+ + f^+\big) = \Lambda_1((-f)^- + f^-)}$$
$$= \Lambda_1((-f)^-) + \Lambda_1(f^-)$$

It follows that

$$\Lambda_2(-f) = \underbrace{\Lambda_1\left((-f)^+\right) - \Lambda_1((-f)^-) = -(\Lambda_1(f^+) - \Lambda_1(f^-))} = -(\Lambda_2(f)),$$

and hence

$$\Lambda_2(-f) = -(\Lambda_2(f)).$$

∎)

d. Let $f \in C_c^{\mathbb{R}}(X)$, and $c \in \mathbb{R}$. Then, $(cf) \in C_c^{\mathbb{R}}(X)$.

Problem 2.109 $\Lambda_2(cf) = c(\Lambda_2(f))$.

(Solution

Case I: when $c = 0$. This is trivial.
Case II: when $c > 0$.

$$\begin{aligned}\text{LHS} &= \Lambda_2(cf) = \Lambda_1\left((cf)^+\right) - \Lambda_1((cf)^-) = \Lambda_1(c(f^+)) - \Lambda_1(c(f^-)) \\ &= c\Lambda_1(f^+) - c\Lambda_1(f^-) = c(\Lambda_1(f^+) - \Lambda_1(f^-)) = c(\Lambda_2(f)) = \text{RHS}.\end{aligned}$$

Case III: when $c<0$.

$$\begin{aligned}\text{LHS} &= \Lambda_2(cf) = \Lambda_2((-|c|)f) = \Lambda_2(|c|(-f)) = |c|(\Lambda_2(-f)) \\ &= |c|(-\Lambda_2(f)) = -|c|(\Lambda_2(f)) = c(\Lambda_2(f)) = \text{RHS}.\end{aligned}$$

So, in all cases, $\Lambda_2(cf) = c(\Lambda_2(f))$. ∎)

e. For every $f \in C_c(X)$, we have $f = \mathrm{Re}(f) + i\mathrm{Im}(f)$.

Problem 2.110 $\mathrm{Re}(f), \mathrm{Im}(f) \in C_c^{\mathbb{R}}(X)$.

(Solution Since $f \in C_c(X)$, we have $f \in C_c(X)$, and hence $f(= \mathrm{Re}(f) + i\mathrm{Im}(f))$ is continuous. It follows that $\mathrm{Re}(f), \mathrm{Im}(f)$ are continuous.

Since $f^{-1}(0) \subset (\mathrm{Re}(f))^{-1}(0)$, we have $(\mathrm{Re}(f))^{-1}(\mathbb{C} - \{0\}) \subset f^{-1}(\mathbb{C} - \{0\})$, and hence

$$\underbrace{\left((\mathrm{Re}(f))^{-1}(\mathbb{C} - \{0\})\right)^- \subset \left(f^{-1}(\mathbb{C} - \{0\})\right)^-} = \mathrm{supp}(f).$$

Since $f \in C_c(X)$, supp(f) is compact. Now, since $\left((\mathrm{Re}(f))^{-1}(\mathbb{C}-\{0\})\right)^{-}$ is a closed set, and

$$\left((\mathrm{Re}(f))^{-1}(\mathbb{C}-\{0\})\right)^{-} \subset \mathrm{supp}(f), (\mathrm{supp}(\mathrm{Re}(f)) =)\left((\mathrm{Re}(f))^{-1}(\mathbb{C}-\{0\})\right)^{-}$$

is a compact set, and hence supp(Re(f)) is a compact set. Since supp(Re(f)) is a compact set, and Re(f) is continuous, $\mathrm{Re}(f) \in C_c^{\mathbb{R}}(X)$. Similarly, $\mathrm{Im}(f) \in C_c^{\mathbb{R}}(X)$. ∎)

For every $f \in C_c(X)$, we have $\Lambda_2(\mathrm{Re}(f)), \Lambda_2(\mathrm{Im}(f)) \in \mathbb{R}$, and hence $(\Lambda_2(\mathrm{Re}(f)) + i\Lambda_2(\mathrm{Im}(f))) \in \mathbb{C}$.

The function

$$\begin{aligned}\Lambda : f \mapsto (\Lambda_2(\mathrm{Re}(f)) + i\Lambda_2(\mathrm{Im}(f)))\big(= \big(\Lambda_1\big((\mathrm{Re}(f))^{+}\big) - \Lambda_1((\mathrm{Re}(f))^{-})\big)\\ + i\big(\Lambda_1\big((\mathrm{Im}(f))^{+}\big) - \Lambda_1((\mathrm{Im}(f))^{-})\big)\big)\end{aligned}$$

from $C_c(X)$ to $\mathbb{C}$ is an extension of $\Lambda_2 : C_c^{\mathbb{R}}(X) \to \mathbb{R}$. Now, since Λ_2 is an extension of Λ_1, Λ is an extension of Λ_1.

f. Let $f, g \in C_c(X)$. Then, $f + g \in C_c(X)$.

Problem 2.111 $\Lambda(f + g) = \Lambda(f) + \Lambda(g)$.

(Solution Let us denote Re(f) by u_1, Im(f) by u_2, Re(g) by v_1, and Im(g) by v_2. It follows that $\mathrm{Re}(f+g) = u_1 + v_1$, $\mathrm{Im}(f+g) = u_2 + v_2$, and $u_1, v_1, u_2, v_2 \in C_c^{\mathbb{R}}(X)$. Here,

$$\begin{aligned}\mathrm{LHS} = \Lambda(f+g) &= (\Lambda_2(u_1+v_1) + i\Lambda_2(u_2+v_2))\\ &= (\Lambda_2(u_1) + \Lambda_2(v_1) + i(\Lambda_2(u_2) + \Lambda_2(v_2)))\\ &= (\Lambda_2(u_1) + i\Lambda_2(u_2)) + (\Lambda_2(v_1) + i\Lambda_2(v_2))\\ &= \Lambda(f) + \Lambda(g) = \mathrm{RHS}.\end{aligned}$$

∎)

g. Let $f \in C_c(X)$. Let $c \in \mathbb{C}$.

Problem 2.112 $\Lambda(cf) = c\Lambda(f)$.

(Solution Let us denote Re(f) by u, Im(f) by v, Re(c) by a, and Im(c) by b. It follows that $\mathrm{Re}(cf) = au - bv$, and $\mathrm{Im}(cf) = av + bu$. Here,

$$\begin{aligned}\text{LHS} = \Lambda(cf) &= \Lambda_2(au - bv) + i\Lambda_2(av + bu) \\ &= (a\Lambda_2(u) - b\Lambda_2(v)) + i(a\Lambda_2(v) + b\Lambda_2(u)) \\ &= (a + ib)(\Lambda_2(u) + i\Lambda_2(v)) \\ &= c\Lambda(f) = \text{RHS}.\end{aligned}$$

∎)

h. Let $f \in C_c(X)$.

Problem 2.113 $|f| \in C_c^+(X)$.

(Solution Since $f \in C_c(X)$, f is continuous, and hence $|f|$ is continuous. Clearly, $f^{-1}(0) = |f|^{-1}(0)$, so $\text{supp}(f) = \text{supp}(|f|)$. Since $f \in C_c(X)$, $(\text{supp}(|f|) =)\text{supp}(f)$ is compact, and hence $\text{supp}(|f|)$ is compact. Now, since $|f|$ is continuous, we have $|f| \in C_c(X)$. Next, since $\text{ran}(|f|) \subset [0, \infty)$, we have $|f| \in C_c^+(X)$. ∎)

Problem 2.114 $\underbrace{|\Phi(f)| \leq \Lambda(|f|)} = \Lambda_1(|f|) = \sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq |f|\}$.

(Solution Since $f \in C_c(X), \text{and} |f| \leq |f|$, we have

$$|\Phi(f)| \in \{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq |f|\},$$

and hence,

$$|\Phi(f)| \leq \sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \leq |f|\}.$$

∎)

Problem 2.115 $\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \leq |f|\} = \Lambda_1(|f|) = \underbrace{\Lambda(|f|) \leq \|f\|}$.

(Solution For this purpose, let us take any $h \in C_c(X)$ satisfying $|h| \leq |f|$. It suffices to show that $|\Phi(h)| \leq \|f\|$. Since $|h| \leq |f|$, we have

$$\|h\| = \sup\{|h(x)| : x \in X\} = \underbrace{\sup\{|h|(x) : x \in X\} \leq \sup\{|f|(x) : x \in X\}}$$

$$= \sup\{|f(x)| : x \in X\} = \|f\|,$$

and hence $\|h\| \leq \|f\|$. Since

$$|\Phi(h)| \leq \|\Phi\| \|h\| = 1\|h\| = \|h\| \leq \|f\|,$$

we have $|\Phi(h)| \leq f$. ∎)

Conclusion 2.116 Let X be a locally compact Hausdorff space. Let $\Phi : C_0(X) \to \mathbb{C}$ be a bounded linear functional on $C_0(X)$ satisfying $\|\Phi\| = 1$. Then there exists a positive linear functional $\Lambda : C_c(X) \to \mathbb{C}$ such that for every $f \in C_c(X)$,

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|.$$

Note 2.117 Let X be a locally compact Hausdorff space. Let $\Phi : C_0(X) \to \mathbb{C}$ be a bounded linear functional on $C_0(X)$. Let $\Phi = 1$.

(By Conclusion 2.248, Vol. 1, $C_0(X)$ is a Banach space whose norm is defined by $\|f\| \equiv \sup\{|f(x)| : x \in X\}$. By Conclusion 2.237, Vol. 1, $C_c(X) \subset C_0(X)$, and by Lemma 1.171, Vol. 1, $C_c(X)$ is a complex linear space so $C_c(X)$ is a linear subspace of $C_0(X)$. Now, since $C_0(X)$ is a Banach space under the supremum norm, $C_c(X)$ is a normed linear space under the supremum norm. Further, by Conclusion 2.245, Vol. 1, $C_c(X)$ is a dense subset of $C_0(X)$.)

By Conclusion 2.116, Vol. 1, there exists a positive linear functional $\Lambda : C_c(X) \to \mathbb{C}$ such that for every $f \in C_c(X)$, $|\Phi(f)| \leq \Lambda(|f|) \leq \|f\|$, that is for every $f \in C_c(X), \Lambda(|f|) \in [|\Phi(f)|, \|f\|]$. Let $\mathcal{B}$ be the σ-algebra of all Borel sets in X.

By Conclusion 1.219, Vol. 1, and Theorem 1.225, Vol. 1, there exists a positive Borel measure λ on $\mathcal{B}$ satisfying the following conditions:

1. for every compact subset K of X, $K \in \mathcal{B}$, and $\lambda(K) < \infty$,
2. for every $E \in \mathcal{B}$, $\lambda(E) = \inf\{\lambda(V) : E \subset V, \text{and } V \text{ is open}\}$,
3. for every open set V in X, $V \in \mathcal{B}$, and $\lambda(V) = \sup\{\lambda(K) : K \subset V, \text{and } K \text{ is a compact set}\}$,
4. for every $E \in \mathcal{B}$ satisfying $\lambda(E) < \infty$, $\lambda(E) = \sup\{\lambda(K) : K \subset E, \text{and } K \text{ is a compact set}\}$,
5. for every compact subset K of X, $\lambda(K) = \inf\{\Lambda(f) : K \prec f\}$,
6. for every open set V in X, $\lambda(V) = \sup\{\Lambda(f) : f \prec V\}$,
7. for every $f \in C_c(X)$, $\Lambda(f) = \int_X f \mathrm{d}\lambda$.

a. Problem 2.118 If $\lambda(X) < \infty$, then λ is regular (that is, for every $E \in \mathcal{B}$, $\inf\{\lambda(V) : E \subset V, \text{and } V \text{ is open}\} = \sup\{\lambda(K) : K \subset E, \text{and } K \text{ is a compact set}\}$).

(Solution Let $\lambda(X) < \infty$. We have to show that λ is regular. For this purpose, let us take any $E \in \mathcal{B}$. We have to show that

$$\inf\{\lambda(V) : E \subset V, \text{and } V \text{ is open}\} = \sup\{\lambda(K) : K \subset E, \text{and } K \text{ is a compact set}\}).$$

Since $\lambda(X) < \infty$, and $E \in \mathcal{B}$, we have $\lambda(E) < \infty$, and hence, by 4, $\lambda(E) = \sup\{\lambda(K) : K \subset E, \text{and } K \text{ is a compact set}\}$. Next, by 2,

$$\lambda(E) = \inf\{\lambda(V) : E \subset V, \text{and } V \text{ is open}\},$$

so

$$\inf\{\lambda(V) : E \subset V, \text{and } V \text{ is open}\} = \sup\{\lambda(K) : K \subset E, \text{and } K \text{ is a compact set}\}.$$

∎)

b. Problem 2.119 $\lambda(X) \le 1$.

(Solution By 6,

$$\underbrace{\lambda(X) = \sup\{\Lambda(f) : f \prec X\}}_{} = \sup\{\Lambda(f) : f \in C_c(X) \text{ and } \operatorname{ran}(f) \subset [0,1]\},$$

so,

$$\lambda(X) = \sup\{\Lambda(f) : C_c(X) \text{and } \operatorname{ran}(f) \subset [0,1]\}.$$

Let us take any $f \in C_c(X)$ satisfying $\operatorname{ran}(f) \subset [0,1]$. It suffices to show that

$$(\sup\{|\Phi(h)| : h \in C_c(X), \text{and } |h| \le f\}) = \underbrace{\Lambda(f) \le 1}_{},$$

that is

$$\sup\{|\Phi(h)| : h \in C_c(X), \text{and} |h| \le f\} \le 1.$$

For this purpose, let us take any $h \in C_c(X)$ satisfying $|h| \le f(\le 1)$. It is enough to show that $|\Phi(h)| \le 1$. Since $|h| \le 1$, we have

$$(\|h\| =)\sup\{|h(x)| : x \in X\} \le 1,$$

and hence $\|h\| \le 1$. Here,

$$|\Phi(h)| \le \|\Phi\|\|h\| = 1\|h\| = \|h\| \le 1,$$

so, $|\Phi(h)| \le 1$. ∎)

c. Problem 2.120 For every $f \in C_c(X)$, $|\Phi(f)| \le \|f\|_1 \le \|f\|$, where $\| \|_1$ denotes the norm of $L^1(\lambda)$ space.

(Solution Let us take any $f \in C_c(X)$. We have to show that $|\Phi(f)| \le \|f_1\|$, that is $|\Phi(f)| \le \int_X |f| d\lambda$. By 7,

$$([|\Phi(f)|, \|f\|] \ni)\Lambda(|f|) = \int\limits_X |f| d\lambda = \|f\|_1,$$

so

$$|\Phi(f)| \leq \|f\|_1 \leq \|f\|.$$

■)

By Conclusion 2.50, Vol. 1, $C_c(X)$ is a dense subspace of the normed linear space $L^1(\lambda)$. By c, for every $f \in C_c(X)$, $|\Phi(f)| \leq \|f\|_1$, so $\Phi|_{C_c(X)}$ is a linear functional on $C_c(X)$, whose norm $\left\|\Phi|_{C_c(X)}\right\|$ with respect to the $L^1(\lambda)$-norm on $C_c(X)$, is at most 1. Now, since $C_c(X)$ is a subspace of the normed linear space $L^1(\lambda)$, by Theorem 2.214, Vol. 1, there exists an extension function $\Psi : L^1(\lambda) \to \mathbb{C}$ of $\Phi|_{C_c(X)}$ such that

1. $\Psi : L^1(\lambda) \to \mathbb{C}$ is a bounded linear functional,
2. $\|\Psi\| = \left\|\Phi|_{C_c(X)}(\leq 1)\right\|$.

Now, by Theorem 3.84, Vol. 1, there exists a Borel function $g : X \to \mathbb{C}$ such that

1. $g \in L^\infty(\lambda)$,
2. for every $f \in L^1(\lambda)$, $\Psi(f) = \int_X (f \cdot g) \mathrm{d}\lambda$,
3. $(1 \geq)\|\Psi\| = \|g\|_\infty$.

Since $\|g\|_\infty \leq 1$, by Conclusion 2.18, Vol. 1, $|g(x)| \leq 1$ holds a.e. on X. Since $\Psi : L^1(\lambda) \to \mathbb{C}$ is an extension of $\Phi|_{C_c(X)}$, for every $f \in C_c(X)(\subset L^1(\lambda))$,

$$\Phi(f) = \underbrace{\left(\Phi|_{C_c(X)}\right)(f) = \Psi(f)} = \int_X (f \cdot g) \mathrm{d}\lambda.$$

Thus, for every $f \in C_c(X)$,

$$\Phi(f) = \int_X (f \cdot g) \mathrm{d}\lambda.$$

Problem 2.121 For every $f \in C_0(X)$, $\Phi(f) = \int_X (f \cdot g) \mathrm{d}\lambda$.

(Solution Let us fix any $f \in C_0(X)$. We have to show that $\Phi(f) = \int_X (f \cdot g) \mathrm{d}\lambda$.

Since $C_c(X)$ is a dense subset of $C_0(X)$, and $f \in C_0(X)$, there exists a sequence $\{f_n\}$ in $C_c(X)$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$. Since $\Phi : C_0(X) \to \mathbb{C}$ is a bounded linear functional on $C_0(X)$, $\Phi : C_0(X) \to \mathbb{C}$ is continuous. Now, since $\lim_{n\to\infty} \|f_n - f\| = 0, f \in C_0(X)$, and each $f_n \in C_c(X)(\subset C_0(X))$,

$$\begin{aligned}\int_X (f \cdot g)\mathrm{d}\lambda &= \int_X \left(\lim_{n\to\infty}(f_n \cdot g)\right)\mathrm{d}\lambda = \lim_{n\to\infty}\left(\int_X (f_n \cdot g)\mathrm{d}\lambda\right) \\ &= \underbrace{\lim_{n\to\infty} \Phi(f_n) = \Phi(f)}, \end{aligned}$$

and hence

$$\Phi(f) = \int_X (f \cdot g)\mathrm{d}\lambda.$$

■)

Thus, for every $f \in C_0(X)$, $\Phi(f) = \int_X f\mathrm{d}\mu$, where $\mathrm{d}\mu = g\mathrm{d}\lambda$, and μ is a complex Borel measure.

d. Problem 2.122 $\lambda(X) = 1$, $|g(x)| = 1$ a.e. on X, and $g \in L^1(\lambda)$.

(Solution Since $\|g\|_\infty \leq 1$, by Conclusion 2.18, $|g(x)| \leq 1$ holds a.e. on X. Since

$$\begin{aligned}1 = \Phi &= \sup\{|\Phi(f)| : f \in C_0(X) \text{and} f \leq 1\} \\ &= \sup\left\{\left|\int_X (f \cdot g)\mathrm{d}\lambda\right| : f \in C_0(X) \text{and} f \leq 1\right\} \\ &\leq \sup\left\{\int_X |f \cdot g|\mathrm{d}\lambda : f \in C_0(X) \text{and} f \leq 1\right\} \\ &= \sup\left\{\int_X |f||g|\mathrm{d}\lambda : f \in C_0(X) \text{and} f \leq 1\right\},\end{aligned}$$

we have

$$1 \leq \sup\left\{\int_X |f||g|\mathrm{d}\lambda : f \in C_0(X) \text{ and } \|f\| \leq 1\right\}.$$

Let us take any $f \in C_0(X)$ satisfying

$$\sup\{|f(x)| : x \in X\} = \underbrace{\|f\| \leq 1}.$$

It follows that $|f| \leq 1$. Next,

$$\int_X |f||g| \mathrm{d}\lambda \leq \int_X 1|g| \mathrm{d}\lambda = \int_X |g| \mathrm{d}\lambda,$$

so

$$(1 \leq) \sup \left\{ \int_X |f||g| \mathrm{d}\lambda : f \in C_0(X) \text{and} \|f\| \leq 1 \right\} \leq \int_X |g| \mathrm{d}\lambda,$$

and hence $1 \leq \int_X |g| \mathrm{d}\lambda$. Since $|g(x)| \leq 1$ holds a.e. on X, we have

$$(1 \leq) \int_X |g| \mathrm{d}\lambda \leq \lambda(X)(\leq 1),$$

and hence, $\int_X |g| \mathrm{d}\lambda = 1$, and $\lambda(X) = 1$. Since $\int_X |g| \mathrm{d}\lambda = 1$, $|g(x)| \leq 1$ holds a.e. on X, and $\lambda(X) = 1$, we have $|g(x)| = 1$ a.e. on X. Since

$$\int_X |g| \mathrm{d}\lambda = 1(<\infty),$$

we have

$$g \in L^1(\lambda).$$

∎)

Since $g \in L^1(\lambda)$, and $\mathrm{d}\mu = g\mathrm{d}\lambda$, by Conclusion 3.54, Vol. 1, $\mathrm{d}|\mu| = |g|\mathrm{d}\lambda (= 1\mathrm{d}\lambda \text{ a.e.})$. Thus, $|\mu|(X) = \lambda(X)(= 1 = \|\Phi\|)$, and hence $|\mu|(X) = \|\Phi\|$.

Conclusion 2.123 Let X be a locally compact Hausdorff space. Let $\Phi : C_0(X) \to \mathbb{C}$ be a bounded linear functional on $C_0(X)$ satisfying $\|\Phi\| = 1$. Then there exists a complex Borel measure μ such that

1. for every $f \in C_0(X)$, $\Phi(f) = \int_X f \mathrm{d}\mu$,
2. $|\mu|(X) = \|\Phi\|$.

Theorem 2.124 Let X be a locally compact Hausdorff space. Let $\Phi : C_0(X) \to \mathbb{C}$ be a bounded linear functional. Then there exists a complex Borel measure μ such that

1. for every $f \in C_0(X)$, $\Phi(f) = \int_X f \mathrm{d}\mu$,
2. $|\mu|(X) = \|\Phi\|$.

Proof

Case I: when $\Phi = 0$. It is trivial to take $\mu = 0$.
Case II: when $\Phi \neq 0$. It follows that $\|\Phi\| > 0$. Put $\Psi \equiv \frac{1}{\|\Phi\|}\Phi$. Clearly, $\Psi : C_0(X) \to \mathbb{C}$ is a bounded linear functional, and $\|\Psi\| = 1$. Now, by Conclusion 2.123, Vol. 1, there exists a complex Borel measure μ_1 such that

1. for every $f \in C_0(X)$, $\frac{1}{\|\Phi\|}(\Phi(f)) = \left(\frac{1}{\|\Phi\|}\Phi\right)(f) = \underbrace{\Psi(f) = \int_X f \mathrm{d}\mu_1}$,

2. 2. $|\mu_1|(X) = \|\Psi\|(= 1)$

Put $\mu \equiv \|\Phi\|\mu_1$. Since μ_1 is a Borel measure, $(\mu =)\|\Phi\|\mu_1$ is a Borel measure, and hence μ is a Borel measure. Since for every $f \in C_0(X)$, $\frac{1}{\|\Phi\|}(\Phi(f)) = \int_X f \mathrm{d}\mu_1$, and $\int_X f \mathrm{d}\mu = \|\Phi\| \int_X f \mathrm{d}\mu_1$, we have $\int_X f \mathrm{d}\mu = \Phi(f)$.

Since $\mu = \|\Phi\|\mu_1$, we have $|\mu|(X) = \|\Phi\|(|\mu_1|(X)) = \|\Phi\|1 = \|\Phi\|$, and hence $|\mu|(X) = \|\Phi\|$. ∎

Theorem 2.124 is known as the **Riesz representation theorem for bounded functionals**.

2.11 Rung's Theorem

Note 2.125 Definition Let Φ be a nonempty, finite collection of oriented intervals in $\mathbb{C}$. For every $p \in \mathbb{C}$, the number of members of Φ that have initial point p will be denoted by $m_I(p)$. For every $p \in \mathbb{C}$, the number of members of Φ that have end point p will be denoted by $m_E(p)$. If for every $p \in \mathbb{C}$, $m_I(p) = m_E(p)$, then we say that Φ *is* ***balanced***.

Suppose that Φ is balanced.

Let us take any oriented interval $[a_0, a_1] \in \Phi$. It follows that a_1 is an end point of a member of Φ, and hence $1 \leq m_E(a_2)(= m_I(a_2))$. Since $1 \leq m_I(a_1)$, there exists $a_2 \in \mathbb{C}$ such that $[a_1, a_2] \in \Phi$. Now, two cases arise: either $a_2 = a_0$ or $a_2 \neq a_0$.

In the case when $a_2 \neq a_0$, a_2 is an end point of a member of Φ, and hence $1 \leq m_E(a_2)(= m_I(a_2))$. Since $1 \leq m_I(a_2)$, there exists $a_3 \in \mathbb{C}$ such that $[a_2, a_3] \in \Phi$. Now, two cases arise: either $a_3 = a_0$ or $a_3 \neq a_0$, etc. Since Φ is finite, this process cannot go in an unending manner. Hence, there exists an integer $k \geq 2$ such that $[a_0, a_1], [a_2, a_3], \ldots, [a_{k-1}, a_k] \in \Phi$, and $a_k = a_0$. Thus, $[a_0, a_1], [a_2, a_3], \ldots, [a_{k-1}, a_k]$ constitute a closed path.

Put $\Phi_1 \equiv \Phi - \{[a_0, a_1], [a_2, a_3], \ldots, [a_{k-1}, a_k(= a_0)]\}$ Clearly, Φ_1 is balanced. Repeating as above, we get another closed path, etc. Thus, Φ is a cycle.

Conclusion 2.126 If Φ is a balanced collection of oriented intervals, then there exists an ordering of members of Φ for which Φ is a cycle.

Note 2.127 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\Omega \neq \mathbb{C}$. Let K be a nonempty compact subset of Ω.
For every $a \in K$, put $\varphi(a) \equiv \inf\{|z-a| : z \in \Omega^c\}$.

Problem 2.128 $\varphi : K \to [0, \infty)$ is continuous.

(Solution Let us fix any $a_0 \in K$. We have to show that φ is continuous at a_0. For this purpose, let us take any $\varepsilon > 0$. Let us take any $z \in D(a_0; \varepsilon)$. It suffices to show that $|\varphi(z) - \varphi(a_0)| < \varepsilon$.

Here,

$$\begin{aligned}\varphi(z) &= \inf\{|w-z| : w \in \Omega^c\} = \inf\{|w-a_0| + |a_0 - z| : w \in \Omega^c\} \\ &= \inf\{|w-a_0| : w \in \Omega^c\} + |a_0 - z| = \varphi(a_0) + |a_0 - z|,\end{aligned}$$

so $\varphi(z) - \varphi(a_0) \leq |z - a_0|$. Similarly, $\varphi(a_0) - \varphi(z) \leq |a_0 - z| (= |z - a_0|)$. It follows that $|\varphi(z) - \varphi(a_0)| \leq |z - a_0| (< \varepsilon)$. ∎)

Now, since K is compact, there exists $a \in K$ such that for every $z \in K$, $\varphi(a) \leq \varphi(z) (= \inf\{|w-z| : w \in \Omega^c\})$. Thus, for every $z \in K$, and for every $w \in \Omega^c$, $\varphi(a) \leq |w - z|$. Since $a \in K (\subset \Omega)$, and Ω is open, there exists $r > 0$ such that $D(a; r) \subset \Omega$, and hence $\Omega^c \subset (D(a; r))^c (= \{z : r \leq |z - a|\})$. It follows that

$$(0 <) r \leq \inf\{|z-a| : z \in \Omega^c\} (= \varphi(a)),$$

and hence $0 < \eta$ where $\eta \equiv \varphi(a)$.

Thus, for every $z \in K$, and, for every $w \in \Omega^c$, $\eta \leq |w - z|$, where $\eta > 0$.

Let us construct a grid of horizontal and vertical lines in the complex plane such that distance between two adjacent horizontal lines is $\frac{\eta}{2}$, and distance between two adjacent vertical lines is $\frac{\eta}{2}$.

Let $Q_1, \ldots, Q_m$ be all those squares (that is, closed 2-cells) of edge length $\frac{\eta}{2}$ that are formed by this grid, and have nonempty intersection with K.

Problem 2.129 Q_1 is contained in Ω.

(Solution If not, suppose otherwise that there exists $w \in \Omega^c$ such that $w \in Q_1$. We have to arrive at a contradiction. Since Q_1 has nonempty intersection with K, there exists $z \in K$ such that $z \in Q_1$. Since $z \in K$, and $w \in \Omega^c$, we have $\eta \leq |w - z|$. Since $z, w \in Q_1$, and Q_1 is a square of edge length $\frac{\eta}{2}$, we have

$$\underbrace{|w - z| \leq \sqrt{2} \cdot \frac{\eta}{2}} = \frac{\eta}{\sqrt{2}} < \eta \leq |w - z|.$$

This is a contradiction. ∎)

Similarly, each Q_r is contained in Ω.

Suppose that for each $r = 1, \ldots, m$, a_r is the center of square Q_r. Clearly, $K \subset Q_1 \cup \cdots \cup Q_m$.

There exists a nonzero complex number b such that for each $r = 1, \ldots, m$, the vertices of square Q_r are $a_r + b, a_r + ib, a_r - b$, and $a_r - ib$. It follows that for every $r = 1, \ldots, m$,

$$\partial Q_r = [a_r + ib, a_r - b] \dot{+} [a_r - b, a_r - ib] \dot{+} [a_r - ib, a_r + b] \dot{+} [a_r + b, a_r + ib].$$

By Conclusion 1.91, for every $r = 1, \ldots, m$, and, for every $z \notin Q_r$, $(\mathrm{Ind})_{\partial Q_r}(z) = 0$. By Lemma 1.92, and by Conclusion 1.212, for every $r = 1, \ldots, m$, and, for every $z \in (Q_r)^0$, $(\mathrm{Ind})_{\partial Q_r}(z) = 1$. Thus, for every $r = 1, \ldots, m$,

$$(\mathrm{Ind})_{\partial Q_r}(z) = \begin{cases} 1 & \mathrm{if\ z \in (Q_r)^0} \\ 0 & \mathrm{if\ z \notin Q_r.} \end{cases}$$

Here, $\cup_{r=1}^m \{[a_r + ib, a_r - b], [a_r - b, a_r - ib], [a_r - ib, a_r + b], [a_r + b, a_r + ib]\}$ is a balanced, finite collection of oriented intervals in $\mathbb{C}$. Let us remove those members of

$$\bigcup_{r=1}^m \{[a_r + ib, a_r - b], [a_r - b, a_r - ib], [a_r - ib, a_r + b], [a_r + b, a_r + ib]\}$$

whose opposites also belongs to

$$\cup_{r=1}^m \{[a_r + ib, a_r - b], [a_r - b, a_r - ib], [a_r - ib, a_r + b], [a_r + b, a_r + ib]\}.$$

Let Φ be the collection of the remaining members of

$$\cup_{r=1}^m \{[a_r + ib, a_r - b], [a_r - b, a_r - ib], [a_r - ib, a_r + b], [a_r + b, a_r + ib]\}.$$

Clearly, Φ is balanced. Now, by Conclusion 2.126, there exists an ordering of members of Φ for which Φ is a cycle.

Let us denote such a cycle by Γ. Clearly, for every $\gamma \in \Phi$, ran(γ) is contained in the open set $(\Omega - K)$, and hence the cycle Γ is contained in $(\Omega - K)$.

For every

$$\underbrace{z \in (\Gamma^*)^c}_{} = (\cup \{\mathrm{ran}(\gamma) : \gamma \in \Phi\})^c = \cap \{(\mathrm{ran}(\gamma))^c : \gamma \in \Phi\},$$

we have

$$\begin{aligned}(\mathrm{Ind})_\Gamma(z) &= (\mathrm{Ind})_{\partial Q_1}(z) + \cdots + (\mathrm{Ind})_{\partial Q_m}(z) \\ &= \begin{cases} 1 & \text{if } z \in (Q_r)^0 \text{ for some } r \in \{1, \ldots, m\} \\ 0 & \textit{if } z \notin Q_1 \cup \cdots \cup Q_m (\subset \Omega). \end{cases}\end{aligned}$$

It follows that for every $z \in \Omega^c$, $(\mathrm{Ind})_\Gamma(z) = 0$.

Next, let $z \in K(\subset Q_1 \cup \cdots \cup Q_m)$.

Case I: when $z \in (Q_r)^0$for some$r \in \{1, \ldots, m\}$. We have seen that $(\mathrm{Ind})_\Gamma(z) = 1$.
Case II: when $z \notin (Q_r)^0$ for every $r \in \{1, \ldots, m\}$. Since $z \in K \subset (Q_1 \cup \cdots \cup Q_m)$, there exists $r \in \{1, \ldots, m\}$ such that $z \in Q_r$. Since $z \in Q_r$, Q_r is a square, and $z \notin (Q_r)^0$, there exists a sequence $\{z_n\}$ in $(Q_r)^0$ such that $\lim_{n\to\infty} z_n = z$. By Lemma 1.32, and Conclusion 1.89, $(\mathrm{Ind})_\Gamma$ is a continuous function, so $(1 = \lim_{n\to\infty} 1 =) \lim_{n\to\infty} (\mathrm{Ind})_\Gamma(z_n) = (\mathrm{Ind})_\Gamma(z)$.

Thus, for every $z \in K$, $(\mathrm{Ind})_\Gamma(z) = 1$.
Let $f \in H(\Omega)$, and $z \in K$.

Problem 2.130 $f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$.

(Solution Since $z \in K(\subset \Omega)$, and cycle Γ is contained in $(\Omega - K)$, we have $z \in (\Omega - \Gamma^*)$. Now, since for every $w \in \Omega^c$, $(\mathrm{Ind})_\Gamma(w) = 0$, by Conclusion 1.204,

$$\frac{1}{2\pi i} \int\limits_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = f(z) \cdot (\mathrm{Ind})_\Gamma(z)(= f(z) \cdot 1 = f(z)).$$

∎)

Conclusion 2.131 Let Ω be a nonempty open subset of $\mathbb{C}$. Let K be a nonempty compact subset of Ω. Then there exists a cycle Γ in $(\Omega - K)$ such that for every $f \in H(\Omega)$, and for every $z \in K$,

$$f(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta.$$

Note 2.132 Let Ω be a nonempty open subset of $\mathbb{C}$. Let K be a nonempty compact subset of Ω.

Problem 2.133 $(\{\infty\} \cup (\mathbb{C} - K) =) \mathbb{S}^2 - K$ is an open subset of $\mathbb{S}^2$.

(Solution Since K is a nonempty compact subset of $\mathbb{C}$, K is closed and bounded. Since K is bounded, there exists an open neighborhood V of ∞ in $\mathbb{S}^2$ such that $V \cap K = \emptyset$. It follows that $V \subset (\mathbb{S}^2 - K)$, and hence ∞ is an interior point of $(\mathbb{S}^2 - K)$. Since K is a closed subset of $\mathbb{C}$, $(\mathbb{C} - K)(\subset (\mathbb{S}^2 - K))$ is an open set in $\mathbb{C}$, and hence every point of $(\mathbb{C} - K)$ is an interior point of $(\mathbb{S}^2 - K)$. Thus, every point of $(\mathbb{S}^2 - K)$ is an interior point of $(\mathbb{S}^2 - K)$. This shows that $\mathbb{S}^2 - K$ is an open subset of $\mathbb{S}^2$. ∎)

Problem 2.134 $(\{\infty\} \cup (\mathbb{C} - K) =) (\mathbb{S}^2 - K)$ has at most countable-many components.

(**Solution** Since the open set $(\mathbb{C} - K)$ is partitioned into countable-many components (which are open connected sets), out of which exactly one of them is unbounded. Thus, $(\mathbb{S}^2 - K)$ has at most countable-many components. ∎)

Now, suppose that $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ is a set, which contains some point in each component of $(\mathbb{S}^2 - K)$.

Problem 2.135 $C(K)$ is a Banach space with the supremum norm.

(**Solution** By Conclusion 2.248, Vol. 1, $C_0(K)$ is a Banach space, whose norm is defined by

$$\|f\| \equiv \sup\{|f(z)| : z \in K\}.$$

By Conclusion 2.240, we have $C_0(K) = C(K)$, and hence $C(K)$ is a Banach space with the supremum norm. ∎)

Let R be a rational function, whose poles are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. So, there exist polynomials P and Q such that R is of the form $\frac{P}{Q}$, where P, Q are polynomials with no common factor, and the zeros of Q (that is the poles of R) are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}(\subset (\mathbb{S}^2 - K))$. By the method of partial fractions decomposition, there exist polynomials $P_0, P_1, P_2, \ldots, P_m$ such that

1. $P_1, P_2, \ldots, P_m$ have no constant term,
2. for every $z \in K$, $R(z) = P_0(z) + P_1\left(\frac{1}{z-\alpha}\right) + P_2\left(\frac{1}{z-\beta}\right) + \cdots + P_m\left(\frac{1}{z-\gamma}\right)$, where $\alpha, \beta, \ldots, \gamma$ are distinct zeros of Q.

It follows that $R|_K \in C(K)$.

Let

$$M \equiv \{R|_K : R \text{ is a rational function, whose poles are in } \{\alpha_1, \alpha_2, \alpha_3, \ldots\}\}.$$

It is clear that M is a linear subspace of $C(K)$.

Let us fix any $f \in H(\Omega)$.

Problem 2.136 $f|_K \in \bar{M}(\subset C(K))$.

(**Solution** Let us take any bounded linear functional $\Phi : C(K) \to \mathbb{C}$ such that, for every rational function R, whose poles are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, $\Phi(R|_K) = 0$. By Lemma 2.219, Vol. 1, it suffices to show that $\Phi(f|_K) = 0$.

By Theorem 2.124, there exists a complex Borel measure μ over K such that for every $g \in C(K)$, $\Phi(g) = \int_K g \, d\mu$.

Since $f \in H(\Omega)$, we have $f|_K \in C(K)$, and hence $\Phi(f|_K) = \int_K f|_K d\mu$. Hence, it suffices to show that $\int_K f|_K d\mu = 0$.

Since for every rational function R, whose poles are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, $\left(\int_K R|_K d\mu =\right)\Phi(R|_K) = 0$, we have for every rational function R, whose poles are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, $\int_K R|_K d\mu = 0$.

By Conclusion 2.131, there exists a cycle Γ in $(\Omega - K)(\subset (\mathbb{C} - K) \subset \left(\mathbb{S}^2 - K\right))$ such that for every $z \in K$,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta.$$

It follows that

$$\begin{aligned}
\int_K f|_K \mathrm{d}\mu = \int_K f(z)\mathrm{d}\mu(z) &= \int_K \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \right) \mathrm{d}\mu(z) \\
&= \frac{1}{2\pi i} \int_K \left(\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \right) \mathrm{d}\mu(z) \\
&= \frac{1}{2\pi i} \int_{\Gamma} \left(\int_K \frac{f(\zeta)}{\zeta - z} \mathrm{d}\mu(z) \right) \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_{\Gamma} \left(f(\zeta) \int_K \frac{1}{\zeta - z} \mathrm{d}\mu(z) \right) \mathrm{d}\zeta \\
&= \frac{-1}{2\pi i} \int_{\Gamma} f(\zeta) \left(\int_K \frac{1}{z - \zeta} \mathrm{d}\mu(z) \right) \mathrm{d}\zeta \\
&= \frac{-1}{2\pi i} \int_{\Gamma} f(\zeta) h(\zeta) \mathrm{d}\zeta,
\end{aligned}$$

where

$$h : \zeta \mapsto \int_K \frac{1}{z - \zeta} \mathrm{d}\mu(z)$$

is a function from open set $\left(\mathbb{S}^2 - K\right)$ to $\mathbb{C}$. It suffices to show that $h = 0$ on $\left(\mathbb{S}^2 - K\right)$.

By Conclusion 1.66(b) (take K for X, identity map for φ, and $\left(\mathbb{S}^2 - K\right)$ for Ω), the function

$$h : \zeta \mapsto \int_K \frac{1}{z - \zeta} \mathrm{d}\mu(z)$$

from $\left(\mathbb{S}^2 - K\right)$ to $\mathbb{C}$ is representable by power series in $\left(\mathbb{S}^2 - K\right)$, and hence, by Conclusion 1.56, $h \in H\left(\mathbb{S}^2 - K\right)$.

Problem 2.137 $h = 0$ on $\left(\mathbb{S}^2 - K\right)$.

(Solution Let us take any $\zeta \in \left(\mathbb{S}^2 - K\right)$. We have to show that $h(\zeta) = 0$.

Since $\zeta \in \left(\mathbb{S}^2 - K\right)$, and $\left(\mathbb{S}^2 - K\right)$ is partitioned into components, there exists a positive integer j such that $\zeta \in V_j$, and V_j is a component of $\left(\mathbb{S}^2 - K\right)$. It follows that $\alpha_j \in V_j$, and V_j is an open set. Hence, there exists $r > 0$ such that

$$D\left(\alpha_j; r\right) \subset V_j\left(\subset \left(\mathbb{S}^2 - K\right)\right).$$

Case I: when $\alpha_j \neq \infty$. Here, for every $w \in D\left(\alpha_j; r\right)$,

$$\begin{aligned}
h(w) &= \int_K \frac{1}{z - w} \mathrm{d}\mu(z) \\
&= \int_K \left(\frac{1}{z - \alpha_j} + \frac{1}{1!} \frac{(-1)(0-1)}{\left(z - \alpha_j\right)^2} \left(w - \alpha_j\right) + \frac{1}{2!} \frac{(-1)(-2)(0-1)^2}{\left(z - \alpha_j\right)^3} \left(w - \alpha_j\right)^2 + \cdots \right) \mathrm{d}\mu(z) \\
&= \int_K \left(\frac{1}{z - \alpha_j} + \frac{1}{\left(z - \alpha_j\right)^2} \left(w - \alpha_j\right) + \frac{1}{\left(z - \alpha_j\right)^3} \left(w - \alpha_j\right)^2 + \cdots \right) \mathrm{d}\mu(z) \\
&= \int_K \left(\lim_{N \to \infty} \left(\sum_{n=0}^{N} \frac{\left(w - \alpha_j\right)^n}{\left(z - \alpha_j\right)^{n+1}} \right) \right) \mathrm{d}\mu(z) = \lim_{N \to \infty} \int_K \left(\sum_{n=0}^{N} \frac{\left(w - \alpha_j\right)^n}{\left(z - \alpha_j\right)^{n+1}} \right) \mathrm{d}\mu(z) \\
&= \lim_{N \to \infty} \left(\int_K \left(\sum_{n=0}^{N} \left(w - \alpha_j\right)^n \left(\frac{1}{z - \alpha_j} \right)^{n+1} \right) \mathrm{d}\mu(z) \right) = \lim_{N \to \infty} 0 = 0.
\end{aligned}$$

Since for every $w \in D\left(\alpha_j; r\right)\left(\subset V_j \subset \left(\mathbb{S}^2 - K\right)\right)$, $h(w) = 0$, $h \in H\left(\mathbb{S}^2 - K\right)$, and $\zeta \in \left(\mathbb{S}^2 - K\right)$, we have, by Conclusion 1.134, $h(\zeta) = 0$.

Case II: when $\alpha_j = \infty$. Here, for every $w \in D\left(a_j; r\right)(= D(\infty; r) \not\ni 0)$, we have $w \neq 0$, and $r < |w|$. Here,

$$D(\infty; r) \subset V_j\left(\subset \left(\mathbb{S}^2 - K\right)\right),$$

so for every $z \in K$, we have $|z| \leq r(<|w|)$, and hence $\left|\frac{z}{w}\right| < 1$. Here, for every $w \in D(\infty; r)$, we have

$$
\begin{aligned}
h(w) &= \int\limits_K \frac{1}{z-w}\mathrm{d}\mu(z) = -\int\limits_K \frac{1}{w}\frac{1}{1-\frac{z}{w}}\mathrm{d}\mu(z) \\
&= -\int\limits_K \frac{1}{w}\left(1+\frac{z}{w}+\left(\frac{z}{w}\right)^2+\cdots\right)\mathrm{d}\mu(z) \\
&= -\int\limits_K \lim_{N\to\infty}\left(\sum_{n=0}^{N}\left(\frac{1}{w^{n+1}}\right)z^n\right)\mathrm{d}\mu(z) \\
&= -\lim_{N\to\infty}\int\limits_K \left(\sum_{n=0}^{N}\left(\frac{1}{w^{n+1}}\right)z^n\right)\mathrm{d}\mu(z) = -\lim_{N\to\infty} 0 = 0.
\end{aligned}
$$

Since for every $w \in D(\infty; r)\left(\subset V_j \subset \left(\mathbb{S}^2 - K\right)\right), h(w) = 0, h \in H\left(\mathbb{S}^2 - K\right)$, and $\zeta \in \left(\mathbb{S}^2 - K\right)$, we have, by Conclusion 1.134, $h(\zeta) = 0$.

Thus, in all cases, $h(\zeta) = 0$. ∎)

Let us fix any $\varepsilon > 0$.

Since $f|_K \in \bar{M}$, there exists a rational function R, whose poles are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, and $(\sup\{|f(z) - R(z)| : z \in K\} =)\left\|f|_K - R|_K\right\| < \varepsilon$.. Hence, for every $z \in K$, $|f(z) - R(z)| < \varepsilon$.

Conclusion 2.138 Let Ω be a nonempty open subset of $\mathbb{C}$. Let K be a nonempty compact subset of Ω. Suppose that $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ is a set, which contains some point in each component of $\left(\mathbb{S}^2 - K\right)$. Let $f \in H(\Omega)$. Let $\varepsilon > 0$. Then there exists a rational function R, whose poles are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, and for every $z \in K$, $|f(z) - R(z)| < \varepsilon$.

This result, known as the **Runge's theorem**, is due to C. D. T. Runge (30.08.1856 – 03.01.1927).

2.12 Mittag-Leffler Theorem

Theorem 2.139 Let Ω be a nonempty open subset of $\mathbb{C}$. Let K be a nonempty compact subset of Ω. Let $\left(\mathbb{S}^2 - K\right)$ be connected. Let $f \in H(\Omega)$. Then there exists a sequence $\{R_n\}$ of polynomials such that $\lim_{n\to\infty} R_n(z) = f(z)$ uniformly on K.

Proof Since $\left(\mathbb{S}^2 - K\right)$ is connected, $\left(\mathbb{S}^2 - K\right)$ has only one component, namely $\left(\mathbb{S}^2 - K\right)(\ni\infty)$. By Conclusion 2.138, for every positive integer n, there exists a rational function R_n, whose poles are in $\{\infty\}$, and, for every $z \in K$, $|f(z) - R_n(z)| < \frac{1}{n}$. Since R_n is a rational function whose poles are in $\{\infty\}$, R_n is a polynomial.

It suffices to show that $\lim_{n\to\infty} R_n(z) = f(z)$ uniformly on K.

For this purpose, let us take any $\varepsilon > 0$. There exists a positive integer N such that for every positive integer $n \geq N$, $\frac{1}{n} < \varepsilon$. Now, since for every positive integer n, and for every $z \in K$, $|f(z) - R_n(z)| < \frac{1}{n}$, we have, for every positive integer $n \geq N$, and for every $z \in K$, $|f(z) - R_n(z)| < \varepsilon$. Thus, $\lim_{n\to\infty} R_n(z) = f(z)$ uniformly on K. ∎

Note 2.140 Let Ω be a nonempty open subset of $\mathbb{C}$. Suppose that A is a set, which contains one point in each component of $\left(\mathbb{S}^2 - \Omega\right)$. Let $f \in H(\Omega)$.

By Conclusion 2.53, there exists a sequence $\{K_1, K_2, K_3, \ldots\}$ of compact subsets of Ω satisfying

1. $(K_1)^0 \subset K_1 \subset (K_2)^0 \subset K_2 \subset (K_3)^0 \subset K_3 \subset \ldots \subset \Omega$,
2. $\Omega = \cup_{n=1}^{\infty} K_n$,
3. for every compact set K contained in Ω, there exists a positive integer N such that $K \subset K_N$,
4. for every positive integer n, and for every $z \in K_n$, $D\left(z; \frac{1}{n} - \frac{1}{n+1}\right) \subset K_{n+1}$,
5. for every positive integer n, if C is a component of $\left(\mathbb{S}^2 - K_n\right)$, then there exists a component D of $\left(\mathbb{S}^2 - \Omega\right)$ such that $D \subset C$.

Problem 2.141 For every positive integer n, every component of $\left(\mathbb{S}^2 - K_n\right)$ contains an element of A.

(Solution Let us take any positive integer n. Let C be a component of $\left(\mathbb{S}^2 - K_n\right)$. We have to show that C contains an element of A.

By 5, there exists a component D of $\left(\mathbb{S}^2 - \Omega\right)$ such that $D \subset C$. Since D is a component of $\left(\mathbb{S}^2 - \Omega\right)$, $D(\subset C)$ contains an element of A, and hence C contains an element of A. ∎)

By Conclusion 2.138, for every positive integer n, there exists a rational function R_n, whose poles are in A, and, for every $z \in K_n$, $|f(z) - R_n(z)| < \frac{1}{n}$.

Problem 2.142 $\lim_{n\to\infty} R_n = f$ uniformly on compact subsets of Ω.

(Solution For this purpose, let us take any nonempty compact subset K of Ω. Next, let us take any $\varepsilon > 0$. Since K is a compact subset of Ω, by 3, there exists a positive integer N such that $K \subset K_N \subset K_{N+1} \subset K_{N+2} \subset \ldots$, and $\cdots < \frac{1}{N+2} < \frac{1}{N+1} < \frac{1}{N} < \varepsilon$.

Now, for every $z \in K_N (\supset K)$,

$$|f(z) - R_N(z)| < \frac{1}{N} < \varepsilon.$$

Thus, for every $z \in K$, $|f(z) - R_N(z)| < \varepsilon$. Similarly, for every $z \in K$, $|f(z) - R_{N+1}(z)| < \varepsilon$. Also, for every $z \in K$, $|f(z) - R_{N+2}(z)| < \varepsilon$, etc.

Thus, $\lim_{n\to\infty} R_n = f$ uniformly on compact subsets of Ω. ∎)

Conclusion 2.143 Let Ω be a nonempty open subset of $\mathbb{C}$. Suppose that A is a set, which contains one point in each component of $\left(\mathbb{S}^2 - \Omega\right)$. Let $f \in H(\Omega)$. Then there exists a sequence $\{R_n\}$ of rational functions, whose poles are in A, such that $\lim_{n\to\infty} R_n = f$ uniformly on compact subsets of Ω.

In the special case, in which $\left(\mathbb{S}^2 - \Omega\right)$ is connected, we may take $A = \{\infty\}$, and hence, there exists a sequence $\{R_n\}$ of rational functions , whose poles are in $\{\infty\}$, such that $\lim_{n\to\infty} R_n = f$ uniformly on compact subsets of Ω. Since each R_n is a rational function, whose poles are in $\{\infty\}$, each R_n is a polynomial.

Conclusion 2.144 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $\left(\mathbb{S}^2 - \Omega\right)$ be connected. Let $f \in H(\Omega)$. Then there exists a sequence $\{P_n\}$ of polynomials such that $\lim_{n\to\infty} P_n = f$ uniformly on compact subsets of Ω.

Note 2.145 Let Ω be a nonempty open subset of $\mathbb{C}$. Let A be a subset of Ω, and A has no limit point in Ω.

It follows that each point of $(\Omega - A)$ is an interior point of $(\Omega - A)$, and hence $(\Omega - A)$ is an open subset of Ω.

By Conclusion 2.53, there exists a sequence $\{K_1, K_2, K_3, \ldots\}$ of compact subsets of Ω satisfying

1. $(K_1)^0 \subset K_1 \subset (K_2)^0 \subset K_2 \subset (K_3)^0 \subset K_3 \subset \cdots \subset \Omega$,
2. $\Omega = \cup_{n=1}^{\infty} K_n$,
3. for every compact set K contained in Ω, there exists a positive integer N such that $K \subset K_N$,
4. for every positive integer n, and for every $z \in K_n$, $D\left(z; \frac{1}{n} - \frac{1}{n+1}\right) \subset K_{n+1}$,
5. for every positive integer n, if C is a component of $\left(\mathbb{S}^2 - K_n\right)$, then there exists a component D of $\left(\mathbb{S}^2 - \Omega\right)$ such that $D \subset C$.
   Here, let us take any $\alpha_{C,n} \in D$. Thus, for every positive integer n, $\left\{\alpha_{C,n} : C \text{ is a component of} \left(\mathbb{S}^2 - K_n\right)\right\} \left(\subset \left(\mathbb{S}^2 - \Omega\right)\right)$ is a set which contains some point in each component of $\left(\mathbb{S}^2 - K_n\right)$.

Problem 2.146 $A \cap K_1$ is a finite set.

(**Solution** If not, otherwise let $A \cap K_1$ be an infinite set. We have to arrive at a contradiction. Since $A \cap K_1$ is an infinite set, $A \cap K_1 \subset K_1$, and K_1 is a compact subset of $\mathbb{C}$, $A \cap K_1$ has a limit point in $K_1 (\subset \Omega)$, and hence $A \cap K_1 (\subset A)$ has a limit point in Ω. Thus, A has a limit point in Ω. This is a contradiction. ∎)

Problem 2.147 $A \cap (K_2 - K_1)$ is a finite set.

(**Solution** If not, otherwise let $A \cap (K_2 - K_1)$ be an infinite set. We have to arrive at a contradiction. Since $A \cap (K_2 - K_1)$ is an infinite set, $A \cap (K_2 - K_1) \subset K_2$, and K_2 is a compact subset of $\mathbb{C}$, $A \cap (K_2 - K_1)$ has a limit point in $K_2 (\subset \Omega)$, and hence $A \cap (K_2 - K_1) (\subset A)$ has a limit point in Ω. Thus, A has a limit point in Ω. This is a contradiction. ∎)

Similarly, $A \cap (K_3 - K_2)$ is a finite set, $A \cap (K_4 - K_3)$ is a finite set, etc.

For every positive integer $n \geq 2$, put $A_n \equiv A \cap (K_n - K_{n-1})$. Next, put $A_1 \equiv A \cap K_1$. We have seen that each A_n is a finite set. It is clear that $\{A_1, A_2, A_3, \ldots\}$ is a partition of A.

Suppose that to each $\alpha \in A$, there exists a positive integer $m(\alpha)$, and a rational function

$$P_\alpha : z \mapsto \left(c_{1,\alpha}\left(\frac{1}{z-\alpha}\right)^1 + c_{2,\alpha}\left(\frac{1}{z-\alpha}\right)^2 + \cdots + c_{m(\alpha),\alpha}\left(\frac{1}{z-\alpha}\right)^{m(\alpha)} \right)$$

from $\mathbb{C} - \{\alpha\}$ to $\mathbb{C}$.

Since $A \cap K_1$ is a finite set, and each P_α is a rational function, $\sum_{\alpha \in (A \cap K_1)} P_\alpha$ is a rational function. Similarly, $\sum_{\alpha \in (A \cap (K_2 - K_1))} P_\alpha$ is a rational function, $\sum_{\alpha \in (A \cap (K_3 - K_2))} P_\alpha$ is a rational function, etc.

For every positive integer $n \geq 2$, put $Q_n \equiv \sum_{\alpha \in (A \cap (K_n - K_{n-1}))} P_\alpha \left(= \sum_{\alpha \in A_n} P_\alpha \right)$. Next, put $Q_1 \equiv \sum_{\alpha \in (A \cap K_1)} P_\alpha \left(= \sum_{\alpha \in A_1} P_\alpha \right)$.

We have seen that each Q_n is a rational function from open set $(\mathbb{C} - A_n)$ to $\mathbb{C}$, and hence, for every positive integer n, $Q_n \in H(\mathbb{C} - A_n)$. Also, for every positive integer $n \geq 2$,

$$\mathbb{C} - A_n = \mathbb{C} - (A \cap (K_n - K_{n-1})) \supset \mathbb{C} - (K_n - K_{n-1}) \supset K_{n-1}.$$

Thus, $Q_1 \in H(\mathbb{C} - A_1)$, $Q_2 \in H(\mathbb{C} - A_2), Q_3 \in H(\mathbb{C} - A_3), \ldots$, and

$$K_1 \subset (\mathbb{C} - A_2), K_2 \subset (\mathbb{C} - A_3), K_3 \subset (\mathbb{C} - A_4), \ldots.$$

Also, for every positive integer n, Q_n has a pole at each point of A_n.

Since $\sum_{\alpha \in (A \cap (K_2 - K_1))} P_\alpha$ is holomorphic in the open set $(\mathbb{C} - A_2)$, which contains the compact set K_1, by Conclusion 2.138, and by the condition 5, there exists a rational function R_2, whose poles are in $\left(\mathbb{S}^2 - \Omega\right)$, and, for every $z \in K_1$,

$$\left| \sum_{\alpha \in (A \cap (K_2 - K_1))} P_\alpha(z) - R_2(z) \right| < \frac{1}{2^2}.$$

Similarly, there exists a rational function R_3, whose poles are in $\left(\mathbb{S}^2 - \Omega\right)$, and, for every $z \in K_2 (\supset K_1)$,

$$\left| \sum_{\alpha \in (A \cap (K_3 - K_2))} P_\alpha(z) - R_3(z) \right| < \frac{1}{2^3}.$$

Also, there exists a rational function R_4, whose poles are in $\left(\mathbb{S}^2 - \Omega\right)$, and, for every $z \in K_3 (\supset K_2 \supset K_1)$,

$$\left| \sum_{\alpha \in (A \cap (K_4 - K_3))} P_\alpha(z) - R_4(z) \right| < \frac{1}{2^4}, \text{etc.}$$

Thus, for every positive integer $n \geq 2$, there exists a rational function R_n, whose poles are in $\left(\mathbb{S}^2 - \Omega\right)$, and, for every $z \in K_{n-1} (\supset \cdots \supset K_2 \supset K_1)$,

$$|Q_n(z) - R_n(z)| < \frac{1}{2^n}.$$

Let us observe that for every

$$\begin{aligned}\underbrace{z \in (\Omega - A)} &= \left(\bigcup_{n=1}^{\infty} K_n \right) - A \\ &= (K_1 - A) \cup ((K_2 - K_1) - A) \cup ((K_3 - K_2) - A) \cup \cdots,\end{aligned}$$

we have $z \in (K_1 - A)$ or $z \in ((K_2 - K_1) - A)$ or $z \in ((K_3 - K_2) - A)$ or ….

If

$$\underbrace{z \in (K_1 - A)} \subset (\mathbb{C} - A) \subset (\mathbb{C} - (A \cap K_1)) = (\mathbb{C} - A_1),$$

then $Q_1(z)$ is a complex number, because $Q_1 \in H(\mathbb{C} - A_1)$. Next, if $z \in (K_1 - A) (\subset K_1)$, then

$$|Q_2(z) - R_2(z)| < \frac{1}{2^2}.$$

Further, if $z \in (K_1 - A) (\subset K_1 \subset K_2)$, then $|Q_3(z) - R_3(z)| < \frac{1}{2^3}$, etc.

It follows that $(Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on $(K_1 - A)$, and hence, $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on $(K_1 - A)$.

If

$$\underbrace{z \in ((K_2 - K_1) - A)} = K_2 - (K_1 \cup A) \subset K_2 - (A \cap K_1) = (K_2 - A_1) \subset (\mathbb{C} - A_1),$$

then $Q_1(z)$ is a complex number, because $Q_1 \in H(\mathbb{C} - A_1)$. Next, if $z \in ((K_2 - K_1) - A) (\subset K_2)$, then

$$|Q_3(z) - R_3(z)| < \frac{1}{2^3}.$$

Further, if $z \in ((K_2 - K_1) - A)(\subset K_2 \subset K_3)$, then $|Q_4(z) - R_4(z)| < \frac{1}{2^4}$, etc. If

$$\underbrace{z \in ((K_2 - K_1) - A)} = K_2 - (K_1 \cup A) \subset K_2 - (A \cap (K_2 - K_1)) \subset \mathbb{C} - (A \cap (K_2 - K_1))$$

$$= (\mathbb{C} - A_2),$$

then $Q_2(z)$ is a complex number, because $Q_2 \in H(\mathbb{C} - A_2)$. Since R_2 is a rational function whose poles are in $\left(\mathbb{S}^2 - \Omega\right)$, for every $z \in ((K_2 - K_1) - A)(\subset \Omega)$, $R_2(z)$ is a complex number.

It follows that $(Q_3 - R_3) + (Q_4 - R_4) + \cdots$ converges uniformly on $((K_2 - K_1) - A)$, and hence $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + (Q_4 - R_4) + \cdots$ converges uniformly on $((K_2 - K_1) - A)$.

Similarly, $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + (Q_4 - R_4) + (Q_5 - R_5) + \cdots$ converges uniformly on $((K_3 - K_2) - A)$, etc.

Thus,

$$f : z \mapsto (Q_1(z) + (Q_2(z) - R_2(z)) + (Q_3(z) - R_3(z)) + \cdots)$$

is a function from $(\Omega - A)$ to C.

Problem 2.148 $f \in H(\Omega - A)$.

(Solution By Conclusion 1.172, it suffices to show that the sequence $\{Q_1, Q_1 + (Q_2 - R_2), Q_1 + (Q_2 - R_2) + (Q_3 - R_3), \ldots\}$ converges to f uniformly on compact subsets of Ω.

For this purpose, let us take any nonempty compact subset K of $(\Omega - A)$. It suffices to show that, $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on K.

By 3, there exists a positive integer N such that $K \subset K_{N+1}$. Since $K \subset (\Omega - A)$, we have $(K - A) = K$.

Since $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on $(K_1 - A)$, $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on $((K_2 - K_1) - A)$, $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on $((K_3 - K_2) - A), \ldots,$ and $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on $((K_{N+1} - K_N) - A)$, we find that $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on $(K_1 - A) \cup ((K_2 - K_1) - A) \cup \cdots \cup ((K_{N+1} - K_N) - A)$ $(= (K_{N+1} - A) \supset (K - A) = K)$, and hence $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ converges uniformly on K. ∎)

Since for every positive integer n, Q_n has a pole at each point of A_n, and R_n has poles in $\left(\mathbb{S}^2 - \Omega\right)$, $Q_1 + (Q_2 - R_2) + (Q_3 - R_3) + \cdots$ has a pole at each point of $(A_1 \cup A_2 \cup A_3 \cup \ldots)$. Now, since $\{A_1, A_2, A_3, \ldots\}$ is a partition of $A(\subset \Omega)$, f has a pole at each point of A, and has no other pole.

Thus, f is a meromorphic function in Ω.

Conclusion 2.149 Let Ω be a nonempty open subset of $\mathbb{C}$. Let A be a subset of Ω, and A has no limit point in Ω. Suppose that to each $\alpha \in A$, there exists a positive integer $m(\alpha)$, and a rational function

$$P_\alpha : z \mapsto \left(c_{1,\alpha}\left(\frac{1}{z-\alpha}\right)^1 + c_{2,\alpha}\left(\frac{1}{z-\alpha}\right)^2 + \cdots + c_{m(\alpha),\alpha}\left(\frac{1}{z-\alpha}\right)^{m(\alpha)} \right)$$

from $\mathbb{C} - \{\alpha\}$ to $\mathbb{C}$. Then there exists a meromorphic function f in Ω such that for every $\alpha \in A$, f has a pole at α, and f has no other pole. Also, its principal part is P_α.

Proof of the remaining part For this purpose, let us fix any $\beta \in A$. We must show that $f - P_\beta$ has removable singularity at β, that is

$$\begin{aligned}\underbrace{z \mapsto (f(z) - P_\beta(z))} &= (Q_1(z) + (Q_2(z) - R_2(z)) + (Q_3(z) - R_3(z)) + \cdots) - P_\beta(z) \\ &= \left(\sum_{\alpha \in A_1} P_\alpha(z) + \left(\sum_{\alpha \in A_2} P_\alpha(z) - R_2(z) \right) \right. \\ &\quad \left. + \left(\sum_{\alpha \in A_3} P_\alpha(z) - R_3(z) \right) + \cdots \right) - P_\beta(z)\end{aligned}$$

has removable singularity at β. Since $\beta \in A$, and $\{A_1, A_2, A_3, \ldots\}$ is a partition of $A(\subset \Omega)$, there exists a unique positive integer N such that $\beta \in A_N$. For simplicity, let $N = 2$. Thus, $\beta \in A_2$.

Clearly,

$$\begin{aligned}\underbrace{z \mapsto (f(z) - P_\beta(z))} = &\left(\left(\sum_{\alpha \in A_1} P_\alpha(z) \right) + \left(\left(\sum_{\alpha \in (A_2 - \{\beta\})} P_\alpha(z) \right) - R_2(z) \right) \right. \\ &\left. + \left(\sum_{\alpha \in A_3} P_\alpha(z) - R_3(z) \right) + \cdots \right)\end{aligned}$$

has removable singularity at β. ∎

This result, known as the **Mittag-Leffler theorem** is due to M. G. Mittag-Leffler (16.03.1846–07.07.1927).

2.13 Simply Connected

Note 2.150 Definition Let X be a topological space. Let $\gamma_0 : [0,1] \to X$ be a closed curve in X. It there exists a closed curve $\gamma_1 : [0,1] \to X$ such that

1. γ_1 is a constant function,
2. γ_0 and γ_1 are X-homotopic,

then we say that γ_0 *is a* ***null-homotopic*** *in* X.

Definition Let X be a topological space. If

1. X is connected,
2. every closed curve in X is null-homotopic in X,

then we say that X *is* ***simply connected***.

Problem 2.151 Every convex region is simply connected.

(Solution Let $\Omega(\subset \mathbb{C})$ be a region, which is convex. We have to show that Ω is simply connected. Since Ω is a region, Ω is connected. Next, let $\gamma_0 : [0,1] \to \Omega$ be a closed curve in Ω. We have to show that γ_0 is null-homotopic. Since Ω is a region, Ω is nonempty, and hence there exists $z_1 \in \Omega$.

Let γ_1 be the constant function z_1 defined on $[0,1]$. Clearly, $\gamma_1 : [0,1] \to \Omega$ is a closed curve in Ω. It suffices to show that γ_0 and γ_1 are Ω-homotopic.

Let

$$H : (s,t) \mapsto (1-t)\gamma_0(s) + t\gamma_1(s)(= (1-t)\gamma_0(s) + tz_1)$$

be a function from $[0,1] \times [0,1]$ to $\mathbb{C}$. Since $\gamma_0 : [0,1] \to \Omega$, $z_1 \in \Omega$, and Ω is convex, for every $(s,t) \in [0,1] \times [0,1]$,

$$(H(s,t) =)((1-t)\gamma_0(s) + tz_1) \in \Omega,$$

and hence $H : [0,1] \times [0,1] \to \Omega$.

Since $\gamma_0 : [0,1] \to \Omega$ is a closed curve, γ_0 is continuous, and hence

$$H : (s,t) \mapsto (1-t)\gamma_0(s) + tz_1$$

is continuous.

Next, for every $s \in [0,1]$, $H(s,0) = (1-0)\gamma_0(s) + 0z_1 = \gamma_0(s)$, and $H(s,1) = (1-1)\gamma_0(s) + 1z_1 = z_1 = \gamma_1(s)$. Thus, for every $s \in [0,1]$, $H(s,0) = \gamma_0(s)$, and $H(s,1) = \gamma_1(s)$.

Since $\gamma_0 : [0,1] \to \Omega$ is a closed curve, $\gamma_0(0) = \gamma_0(1)$. Now, for every $t \in [0,1]$,

$$H(0,t) = (1-t)\gamma_0(0) + tz_1 = (1-t)\gamma_0(1) + tz_1 = H(1,t).$$

Thus, for every $t \in [0,1]$, $H(0,t) = H(1,t)$.

Thus, γ_0 and γ_1 are Ω-homotopic. ∎)

Conclusion 2.152 Every convex region is simply connected.

Note 2.153 Let Ω be a region. Suppose that Ω is homeomorphism to $D(0;1)$.

Problem 2.154 Ω is simply connected.

(**Solution** Since Ω is homeomorphic to $D(0;1)$, there exists a function $\psi : \Omega \to D(0;1)$ such that ψ is 1-1, onto, continuous and ψ^{-1} is continuous.

Since Ω is a region, Ω is connected. Next, let $\gamma_0 : [0,1] \to \Omega$ be a closed curve in Ω. We have to show that γ_0 is null-homotopic.

Since $\psi : \Omega \to D(0;1)$ is 1-1, onto, and $0 \in D(0;1)$, $\psi^{-1}(0) \in \Omega$.

Let γ_1 be the constant function $\psi^{-1}(0)$ defined on $[0,1]$. Clearly, $\gamma_1 : [0,1] \to \Omega$ is a closed curve in Ω. It suffices to show that γ_0 and γ_1 are Ω-homotopic.

Let

$$H : (s,t) \mapsto \psi^{-1}((1-t)\psi(\gamma_0(s)))$$

be a function from $[0,1] \times [0,1]$ to Ω.

Since ψ, ψ^{-1}, and γ_0 are continuous, $H : [0,1] \times [0,1] \to \Omega$ is continuous.

Next, for every $s \in [0,1]$,

$$H(s,0) = \psi^{-1}((1-0)\psi(\gamma_0(s))) = \psi^{-1}(\psi(\gamma_0(s))) = \gamma_0(s),$$

and

$$H(s,1) = \psi^{-1}((1-1)\psi(\gamma_0(s))) = \psi^{-1}(0) = \gamma_1(s).$$

Thus, for every $s \in [0,1]$, $H(s,0) = \gamma_0(s)$, and $H(s,1) = \gamma_1(s)$.

Since $\gamma_0 : [0,1] \to \Omega$ is a closed curve, $\gamma_0(0) = \gamma_0(1)$. Now, for every $t \in [0,1]$,

$$H(0,t) = \psi^{-1}((1-t)\psi(\gamma_0(0))) = \psi^{-1}((1-t)\psi(\gamma_0(1))) = H(1,t).$$

Thus, for every $t \in [0,1]$, $H(0,t) = H(1,t)$. Thus, γ_0 and γ_1 are Ω-homotopic. ∎)

Conclusion 2.155 Let Ω be a region. Consider the following statements:

a. Ω is homeomorphic to $D(0;1)$,
b. Ω is simply connected,
c. for every closed path γ in Ω, and for every $\alpha \in (\mathbb{S}^2 - \Omega)$, $(\mathrm{Ind})_\gamma(\alpha) = 0$.

Then $a \Rightarrow b \Rightarrow c$.

Proof of the remaining part $b \Rightarrow c$: Let us take any closed path γ_0 in Ω. Let us take any $\alpha \in (\mathbb{S}^2 - \Omega)$. We have to show that $(\mathrm{Ind})_{\gamma_0}(\alpha) = 0$.

Since Ω is simply connected, and γ_0 is a closed path in Ω, γ_0 is null-homotopic in Ω, and hence there exists a closed curve $\gamma_1 : [0,1] \to \Omega$ such that

1. γ_1 is a constant function, say z_1,
2. γ_0 and γ_1 are Ω-homotopic.

Now, since Ω is a region, and $\alpha \in (\mathbb{S}^2 - \Omega)$, by Conclusion 1.214,

$$\underbrace{(\mathrm{Ind})_{\gamma_0}(\alpha) = (\mathrm{Ind})_{\gamma_1}(\alpha)} = \frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{1}{\zeta - \alpha}\mathrm{d}\zeta = \frac{1}{2\pi i}\int\limits_{\gamma_1} \frac{1}{z_1 - \alpha}\mathrm{d}\zeta = \frac{1}{2\pi i}0 = 0,$$

and hence $(\mathrm{Ind})_{\gamma_0}(\alpha) = 0$. ∎

Note 2.156 Let Ω be a region in $\mathbb{C}$. Suppose that for every closed path γ in Ω, and for every $\alpha \in (\mathbb{S}^2 - \Omega)$, $(\mathrm{Ind})_\gamma(\alpha) = 0$.

Problem 2.157 $(\mathbb{S}^2 - \Omega)$
is connected.

(Solution If not, otherwise suppose that $(\mathbb{S}^2 - \Omega)$ is not connected. We have to arrive at a contradiction.

Since Ω is a region in $\mathbb{C}$, Ω is a nonempty open subset of $\mathbb{C}$, and hence Ω is open in $\mathbb{S}^2$. It follows that $(\mathbb{S}^2 - \Omega)$ is closed in $\mathbb{S}^2$. Since $(\mathbb{S}^2 - \Omega)$ is not connected, there exist open sets G_1 and G_2 in $\mathbb{S}^2$ such that $G_1 \cap (\mathbb{S}^2 - \Omega) \neq \emptyset$, $G_2 \cap (\mathbb{S}^2 - \Omega) \neq \emptyset$, $G_1 \cap G_2 \cap (\mathbb{S}^2 - \Omega) \neq \emptyset$, and $(\infty \in)(\mathbb{S}^2 - \Omega) \subset G_1 \cup G_2$.

It follows that $\infty \in G_1$ or $\infty \in G_2$. For definiteness, let $\infty \in G_2$.

It follows that $\infty \notin G_1$. Since $(\mathbb{S}^2 - \Omega)$ is closed in $\mathbb{S}^2$, and G_1 is open in $\mathbb{S}^2$, $(\infty \in (\mathbb{S}^2 - (\Omega \cup G_1)) =)((\mathbb{S}^2 - \Omega) - G_1)$ is closed in $\mathbb{S}^2$. Similarly, $(\infty \notin (\mathbb{S}^2 - (\Omega \cup G_2)) =)((\mathbb{S}^2 - \Omega) - G_2)$ is closed in $\mathbb{S}^2$, and hence $(\Omega \cup G_2)$ is an open neighborhood of ∞ is $\mathbb{S}^2$. This shows that $((\mathbb{S}^2 - \Omega) - G_2 =)$ $(\mathbb{S}^2 - (\Omega \cup G_2))$ is a compact subset of $\mathbb{C}$. Let us put $H \equiv ((\mathbb{S}^2 - \Omega) - G_1)$, and $K \equiv ((\mathbb{S}^2 - \Omega) - G_2)$.

We have seen that $(\infty \in)H$ is a closed subset of $\mathbb{S}^2$, K is a compact subset of $\mathbb{C}$, H is nonempty, K is nonempty, $H \cap K = \emptyset$, and $H \cup K = (\mathbb{S}^2 - \Omega)$. It follows that

$$\begin{aligned}\Omega \cup K &= (\mathbb{S}^2 - (H \cup K)) \cup K = ((\mathbb{S}^2 - H) \cap (\mathbb{S}^2 - K)) \cup K \\ &= ((\mathbb{S}^2 - H) \cup K) \cap ((\mathbb{S}^2 - K) \cup K) \\ &= ((\mathbb{S}^2 - H) \cup K) \cap \mathbb{S}^2 = (\mathbb{S}^2 - H) \cup K = (\mathbb{S}^2 - H),\end{aligned}$$

and hence $(\mathbb{S}^2 - H) = \Omega \cup K$.

Here, $(\Omega \cup K =)(\mathbb{S}^2 - H)$ is an open subset of $\mathbb{C}$, and K is a nonempty compact subset of $\Omega \cup K$. By Conclusion 2.138, there exists a cycle Γ in $((\Omega \cup K) - K)(= \Omega)$ such that, for every $z \in K$,

$$1 = \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{\zeta - z}\mathrm{d}\zeta\big(= (\mathrm{Ind})_{\Gamma}(z)\big).$$

Now, since K is nonempty, there exists $\alpha \in K\big(\subset \left(\mathbb{S}^2 - \Omega\right)\big)$ such that $(\mathrm{Ind})_{\Gamma}(\alpha) = 1$. Since $\alpha \in \left(\mathbb{S}^2 - \Omega\right)$, by assumption, $(\mathrm{Ind})_{\Gamma}(\alpha) = 0$. This is a contradiction. ∎)

Conclusion 2.158 Let Ω be a region. Consider the following statements:

a. for every closed path γ in Ω, and, for every $\alpha \in \left(\mathbb{S}^2 - \Omega\right)$, $(\mathrm{Ind})_{\gamma}(\alpha) = 0$,
b. $\left(\mathbb{S}^2 - \Omega\right)$ is connected,
c. for every $f \in H(\Omega)$, there exists a sequence $\{P_n\}$ of polynomials such that $\lim_{n\to\infty} P_n = f$ uniformly on compact subsets of Ω,
d. for every $f \in H(\Omega)$, and, for every closed path γ in Ω, $\int_{\gamma} f(z)\mathrm{d}z = 0$.

Then $a \Rightarrow b \Rightarrow c \Rightarrow d$.

Proof of the remaining part $b \Rightarrow c$: Let $\left(\mathbb{S}^2 - \Omega\right)$ be connected. Let $f \in H(\Omega)$. Now, by Conclusion 2.144, there exists a sequence $\{P_n\}$ of polynomials such that $\lim_{n\to\infty} P_n = f$ uniformly on compact subsets of Ω.

$c \Rightarrow d$: Let $f \in H(\Omega)$, and let $\gamma : [0,1] \to \Omega$ be a closed path in Ω. We have to show that $\int_{\gamma} f(z)\mathrm{d}z = 0$. Here $\mathrm{ran}(\gamma)$ is a compact subset of Ω. By assumption, there exists a sequence $\{P_n\}$ of polynomials such that $\lim_{n\to\infty} P_n = f$ uniformly on $\mathrm{ran}(\gamma)$. Hence, by Conclusion 1.94,

$$\begin{aligned}\mathrm{LHS} &= \int_{\gamma} f(z)\mathrm{d}z = \int_{\gamma}\Big(\lim_{n\to\infty} P_n(z)\Big)\mathrm{d}z = \lim_{n\to\infty}\left(\int_{\gamma} P_n(z)\mathrm{d}z\right) = \lim_{n\to\infty} 0 = 0\\ &= \mathrm{RHS}.\end{aligned}$$

∎

Note 2.159 Let Ω be a region in $\mathbb{C}$. Suppose that for every $f \in H(\Omega)$, and, for every closed path γ in Ω, $\int_{\gamma} f(z)\mathrm{d}z = 0$.

Problem 2.160 For every $f \in H(\Omega)$, there exists $F \in H(\Omega)$ such that $F' = f$.

(Solution Let us take any $f \in H(\Omega)$. We have to construct a function $F : \Omega \to \mathbb{C}$ such that $F \in H(\Omega)$, and $F' = f$. For this purpose, let us fix any $a \in \Omega$.

Let us take any $z \in \Omega$. Let Γ be any path in Ω from a to z. Let Γ_1 be another path in Ω from a to z. Now, by assumption, the integral of f over the closed path Γ_2 is 0, where Γ_2 stands for Γ followed by the opposite of Γ_1, and hence $\int_{\Gamma} f(z)\mathrm{d}z = \int_{\Gamma_1} f(z)\mathrm{d}z$.

Let us define a function $F : \Omega \to \mathbb{C}$ as follows: For every $z \in \Omega$,

$$F(z) \equiv \begin{cases} \int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta & \text{if } z \neq a \\ 0 & \text{if } z = a. \end{cases}$$

Let us take any $z_0 \in (\Omega - \{a\})$.

Problem 2.161 $\lim_{z \to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0)$.

(**Solution** For this purpose, let us take any $\varepsilon > 0$.

Since $f : \Omega \to \mathbb{C}$ is continuous at $z_0 (\in (\Omega - \{a\}))$, there exists $\delta > 0$ such that $D(z_0; \delta) \subset (\Omega - \{a\})$, and for every $z \in D(z_0; \delta)$, $|f(z) - f(z_0)| < \varepsilon$. It follows that for every $z \in D'(z_0; \delta)$, and for every $\zeta \in \operatorname{ran}([z_0, z]) (\subset D(z_0; \delta))$, $|f(\zeta) - f(z_0)| < \varepsilon$.

It suffices to show that for every $z \in D'(z_0; \delta)$,

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \varepsilon.$$

For this purpose, let us fix any $z \in D'(z_0; \delta) (\subset D(z_0; \delta) \subset (\Omega - \{a\}))$. It follows that z, z_0, a are distinct. Since

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &= \left| \frac{1}{z - z_0} \left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[a,z_0]} f(\zeta)\mathrm{d}\zeta \right) - f(z_0) \right| \\ &= \left| \frac{1}{z - z_0} \left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z_0,a]} f(\zeta)\mathrm{d}\zeta \right) - f(z_0) \right| \\ &= \left| \frac{1}{z - z_0} \left(\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z,z_0]} f(\zeta)\mathrm{d}\zeta + \int\limits_{[z_0,a]} f(\zeta)\mathrm{d}\zeta \right) + \left(\frac{-1}{z - z_0} \int\limits_{[z,z_0]} f(\zeta)\mathrm{d}\zeta - f(z_0) \right) \right| \\ &= \left| \frac{1}{z - z_0}(0) + \left(\frac{1}{z - z_0} \int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0) \right) \right| = \left| \frac{1}{z - z_0} \left(\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - f(z_0)(z - z_0) \right) \right| \\ &= \left| \frac{1}{z - z_0} \left(\int\limits_{[z_0,z]} f(\zeta)\mathrm{d}\zeta - \int\limits_{[z_0,z]} f(z_0)\mathrm{d}\zeta \right) \right| = \left| \frac{1}{z - z_0} \int\limits_{[z_0,z]} (f(\zeta) - f(z_0))\mathrm{d}\zeta \right| \\ &= \frac{1}{|z - z_0|} \left| \int\limits_{[z_0,z]} (f(\zeta) - f(z_0))\mathrm{d}\zeta \right| \leq \frac{1}{|z - z_0|} (\varepsilon(\text{length of}[z_0, z])) = \frac{1}{|z - z_0|} (\varepsilon|z - z_0|) = \varepsilon, \end{aligned}$$

we have $\left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right| \leq \varepsilon$. ∎)

Thus, F' and f coincide on $(\Omega - \{a\})$.

Problem 2.162 It suffices to show that $\lim_{z\to a} \frac{F(z)-F(a)}{z-a} = f(a)$.

(**Solution** For this purpose, let us take any $\varepsilon > 0$.

Since $f : \Omega \to \mathbb{C}$ is continuous at a, there exists $\delta > 0$ such that $D(a;\delta) \subset \Omega$, and, for every $z \in D(a;\delta)$, $|f(z) - f(a)| < \varepsilon$. It follows that for every $z \in D'(a;\delta)$, and for every $\zeta \in \mathrm{ran}([a,z])(\subset D(a;\delta))$, $|f(\zeta) - f(a)| < \varepsilon$.

It suffices to show that for every $z \in D'(a;\delta)$, $\left|\frac{F(z)-F(a)}{z-a} - f(a)\right| \le \varepsilon$.

For this purpose, let us fix any $z \in D'(a;\delta)(\subset D(a;\delta) \subset \Omega)$. Since

$$\begin{aligned}\left|\frac{F(z)-F(a)}{z-a} - f(a)\right| &= \left|\frac{F(z)-0}{z-a} - f(a)\right| = \left|\frac{1}{z-a}\int\limits_{[a,z]} f(\zeta)\mathrm{d}\zeta - f(a)\right| \\ &= \left|\frac{1}{z-a}\int\limits_{[a,z]} (f(\zeta) - f(a))\mathrm{d}\zeta\right| = \frac{1}{|z-a|}\left|\int\limits_{[a,z]} (f(\zeta) - f(a))\mathrm{d}\zeta\right| \\ &\le \frac{1}{|z-a|}(\varepsilon(\text{length of}[a,z])) = \frac{1}{|z-a|}(\varepsilon|z-a|) = \varepsilon,\end{aligned}$$

we have $\left|\frac{F(z)-F(a)}{z-a} - f(a)\right| \le \varepsilon$. ∎)

Thus, $F \in H(\Omega)$ such that $F' = f$. ∎)

Conclusion 2.163 Let Ω be a region. Consider the following statements:

a. for every $f \in H(\Omega)$, and for every closed path γ in Ω, $\int_\gamma f(z)\mathrm{d}z = 0$,
b. for every $f \in H(\Omega)$, there exists $F \in H(\Omega)$ such that $F' = f$,
c. if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $g \in H(\Omega)$ such that $f = e^g$,
d. if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$.

Then $a \Rightarrow b \Rightarrow c \Rightarrow d$.

Proof of the remaining part $b \Rightarrow c$: Let $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$. Since $f \in H(\Omega)$, by Lemma 1.117, we have $f' \in H(\Omega)$. Now, since $\frac{1}{f} \in H(\Omega)$, we have $\left(\frac{f'}{f} =\right)\left(f' \cdot \frac{1}{f}\right) \in H(\Omega)$, and, for every $z \in \Omega$, $f(z) \neq 0$. Let us fix any $z_0 \in \mathbb{C}$. It follows that $f(z_0) \neq 0$. Also, $e^{-F(z_0)} \neq 0$. It follows that $f(z_0)e^{-F(z_0)}$ is a nonzero complex number. Now, by Conclusion 1.59, Vol. 1, there exists $\alpha \in \mathbb{C}$ such that $e^\alpha = f(z_0)e^{-F(z_0)}$, and hence

$$\frac{f(z_0)e^{-F(z_0)}}{e^\alpha} = 1.$$

Since $\frac{f'}{f} \in H(\Omega)$, by b, there exists $F \in H(\Omega)$ such that $F' = \frac{f'}{f}$.

Put $g \equiv F + \alpha$.

Now, since $F \in H(\Omega)$, we have $g \in H(\Omega)$, and hence $g' = F'\left(=\frac{f'}{f}\right)$. Thus,

$$(fe^{-g})' = f'e^{-g} + f \cdot (e^{-g}(-g')) = e^{-g}(f' - fg') = e^{-g} \cdot 0 = 0,$$

and hence $(fe^{-g})' = 0$ on the connected open set Ω. It follows that $fe^{-g} \in H(\Omega)$, and hence, by Conclusion 1.116, fe^{-g} is representable by power series in Ω. Now, by Lemma 1.60, fe^{-g} is a constant on an open disk in Ω. Since fe^{-g} is a constant on an open disk in Ω, and Ω is a region, by Theorem 1.135, fe^{-g} is a constant on Ω. It follows that for every $z \in \Omega$,

$$f(z)e^{-g(z)} = f(z_0)e^{-g(z_0)} = f(z_0)e^{-(F(z_0)+\alpha)} = \frac{f(z_0)e^{-F(z_0)}}{e^{\alpha}} = 1,$$

and hence $f = e^g$ on Ω.

$c \Rightarrow d$: Let $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$. By c, there exists $g \in H(\Omega)$ such that $f = e^g$. Hence, $f = \varphi^2$, where $\varphi \equiv e^{\frac{1}{2}g}$. It suffices to show that $\left((\exp) \circ \left(\frac{1}{2}g\right) =\right) e^{\frac{1}{2}g} \in H(\Omega)$. Since $g \in H(\Omega)$, and $(\exp) \in H(\Omega)$, we have

$$e^{\frac{1}{2}g} = \underbrace{\left((\exp) \circ \left(\frac{1}{2}g\right)\right) \in H(\Omega)},$$

and hence $e^{\frac{1}{2}g} \in H(\Omega)$. ∎

If we combine Conclusions 2.155, 2.158, and 2.163, we get the following remarkable

Conclusion 2.164 Let Ω be a region. Consider the following statements:

a. Ω is homeomorphic to $D(0;1)$,
b. Ω is simply connected,
c. for every closed path γ in Ω, and for every $\alpha \in \left(\mathbb{S}^2 - \Omega\right)$, $(\mathrm{Ind})_\gamma(\alpha) = 0$,
d. $\left(\mathbb{S}^2 - \Omega\right)$ is connected,
e. for every $f \in H(\Omega)$, there exists a sequence $\{P_n\}$ of polynomials such that $\lim_{n\to\infty} P_n = f$ uniformly on compact subsets of Ω,
f. for every $f \in H(\Omega)$, and for every closed path γ in Ω, $\int_\gamma f(z)\mathrm{d}z = 0$,
g. for every $f \in H(\Omega)$, there exists $F \in H(\Omega)$ such that $F' = f$,
h. if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $g \in H(\Omega)$ such that $f = e^g$,
i. if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$.

Then $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow f \Rightarrow g \Rightarrow h \Rightarrow i$.

2.14 Riemann Mapping Theorem

Theorem 2.165 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f \in H(\Omega)$. Suppose that f has no zero in Ω. Then $\ln|f|$ is harmonic in Ω.

Proof It suffices to show that $\ln|f|$ is harmonic in each open disk of Ω. For this purpose, let us take any open disk D such that $D \subset \Omega$. We have to show that $\ln|f|$ is harmonic in D.

Since D is a disk, D is a region. Clearly, D is homomorphic to the unit disk $D(0;1)$. Here, $f|_D \in H(D)$. Now, since f has no zero in $\Omega(\supset D)$, $\frac{1}{f|_D} \in H(D)$.

It follows, by Conclusion $(*)(a \Rightarrow h)$ of Note 2.159, that there exists $g \in H(D)$ such that for every $z \in D, f(z) = e^{g(z)}$. It follows that $\left|\left(f|_D\right)\right| = \left(e^{\left(\mathrm{Re}\left(g|_D\right)\right)}\right)$. Since $g|_D \in H(D)$, $\left(\ln\left|\left(f|_D\right)\right| =\right)\mathrm{Re}\left(g|_D\right)$ is harmonic in D. Thus, $\ln|f|$ is harmonic in Ω. ∎

Note 2.166 Let Ω be a region. Let $\Omega \neq \mathbb{C}$. Suppose that if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$.

Since $\Omega \neq \mathbb{C}$, there exists a complex number w_0 such that $w_0 \notin \Omega$. Let $f : z \mapsto (z - w_0)$ be a function from Ω to $\mathbb{C} - \{0\}$. Clearly, $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$. By assumption, there exists $\varphi \in H(\Omega)$ such that $\varphi^2 = f$, and hence for every $z \in \Omega$, we have $(\varphi(z))^2 = z - w_0$.

Problem 2.167 $\varphi : \Omega \to \mathbb{C}$
is 1-1.

(Solution For this purpose, let $\varphi(z_1) = \varphi(z_2)$. We have to show that $z_1 = z_2$. Since $\varphi(z_1) = \varphi(z_2)$, we have

$$z_1 - w_0 = \underbrace{(\varphi(z_1))^2 = (\varphi(z_2))^2} = z_2 - w_0,$$

and hence $z_1 = z_2$. ∎)

Problem 2.168 $\varphi(z_1) = -\varphi(z_2) \Rightarrow z_1 = z_2$.

(Solution Let $\varphi(z_1) = -\varphi(z_2)$. We have to show that $z_1 = z_2$. Since $\varphi(z_1) = -\varphi(z_2)$, we have

$$z_1 - w_0 = \underbrace{(\varphi(z_1))^2 = (\varphi(z_2))^2} = z_2 - w_0,$$

and hence $z_1 = z_2$. ∎)

Since Ω is a region, and $\varphi : \Omega \to \mathbb{C}$ is 1-1, $\varphi(\Omega)$ is not a singleton. Now, since $\varphi \in H(\Omega)$, by Conclusion 1.190, $\varphi(\Omega)$ is a region. It follows that $\varphi(\Omega)$ is a nonempty open subset of $\mathbb{C}$, and hence there exist a nonzero $a \in \varphi(\Omega)$, and a positive real number r such that $D(a;r) \subset \varphi(\Omega)$, and $0 \notin D[a;r]$.

Since $\varphi(\Omega)$ is an open set, and $D(-a;r)$ is an open set, $D(-a;r) \cap \varphi(\Omega)$ is an open set.

Problem 2.169 $D(-a;r) \cap \varphi(\Omega) = \emptyset$.

(Solution If not, otherwise, suppose that there exists $b \in D(-a;r) \cap \varphi(\Omega)$, and $b \neq 0$. We have to arrive at a contradiction. Since $b \in D(-a;r) \cap \varphi(\Omega)$, we have $b \in D(-a;r)$, and $b \in \varphi(\Omega)$. Since $b \in D(-a;r)$, we have $(|(-b) - a| =)|b - (-a)| < r$, and hence $(-b) \in D(a;r)(\subset \varphi(\Omega))$. Hence, there exists $a_1 \in \Omega$ such that $\varphi(a_1) = -b$. Since $b \in \varphi(\Omega)$, there exists $a_2 \in \Omega$ such that $\varphi(a_2) = b(= -\varphi(a_1))$. Since $\varphi(a_2) = -\varphi(a_1)$, we have $a_2 = a_1$, and hence $\varphi(a_2) = \varphi(a_1)$. Since $\varphi(a_2) = -\varphi(a_1)$, and $\varphi(a_2) = \varphi(a_1)$, we have $(b =)\varphi(a_2) = 0$, and hence $b = 0$. This is a contradiction. ∎)

Since $(-a) \in D(-a;r)$, and $D(-a;r) \cap \varphi(\Omega) = \emptyset$, we have $(-a) \notin \varphi(\Omega)$, and hence for every $z \in \Omega$, $-a \neq \varphi(z)$. Thus, for every $z \in \Omega$, $\varphi(z) + a \neq 0$. Now, let $\psi : z \mapsto \frac{r}{2(\varphi(z)+a)}$ be a function from Ω to $\mathbb{C}$. Since $\varphi \in H(\Omega)$, we have $\psi \in H(\Omega)$.

Problem 2.170 $\psi : \Omega \to \mathbb{C}$ is 1-1.

(Solution For this purpose, let $\psi(z_1) = \psi(z_2)$. We have to show that $z_1 = z_2$. Since

$$\frac{r}{2(\varphi(z_1)+a)} = \underbrace{\psi(z_1) = \psi(z_2)} = \frac{r}{2(\varphi(z_2)+a)},$$

we have $\frac{r}{2(\varphi(z_1)+a)} = \frac{r}{2(\varphi(z_2)+a)}$, and hence $\varphi(z_1) = \varphi(z_2)$. Now, since $\varphi : \Omega \to \mathbb{C}$ is 1-1, we have $z_1 = z_2$. ∎)

Problem 2.171 $\psi : \Omega \to D(0;1)$.

(Solution Let us take any $z \in \Omega$. We have to show that

$$\frac{\frac{r}{2}}{|\varphi(z)+a|} = \left|\frac{r}{2(\varphi(z)+a)}\right| = \underbrace{|\psi(z)| < 1},$$

that is $\frac{r}{2} < |\varphi(z) + a|$.

If not, otherwise let $|\varphi(z) + a| \leq \frac{r}{2}$. We have to arrive at a contradiction. Since $|\varphi(z) + a| \leq \frac{r}{2}$, we have $\varphi(z) \in D\left[-a; \frac{r}{2}\right](\subset D(-a;r))$, and hence $\varphi(z) \in D(-a;r) \cap \varphi(\Omega)(= \emptyset)$. This is a contradiction. ∎)

Thus, $\psi : \Omega \to D(0;1)$ is a 1-1 function, and $\psi \in H(\Omega)$.

Conclusion 2.172 Let Ω be a region. Let $\Omega \neq \mathbb{C}$. Suppose that if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$. Then there exists a 1-1 function $\psi : \Omega \to D(0;1)$ such that $\psi \in H(\Omega)$.

Note 2.173 Let Ω be a region. Let $\Omega \neq \mathbb{C}$. Suppose that if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$. Let Σ be the collection of all 1-1 functions $\psi : \Omega \to D(0;1)$ such that $\psi \in H(\Omega)$. (By Conclusion 2.172, Σ is nonempty.) Let $\psi \in \Sigma$. Let $\psi : \Omega \to D(0;1)$ be not onto. Let $z_0 \in \Omega$.

Since $\psi : \Omega \to D(0;1)$ is not onto, there exists $\alpha \in D(0;1)$ such that $\alpha \notin \psi(\Omega)$. Since $\alpha \in D(0;1)$, by Conclusions 2.11(2) and 2.11(3), $\varphi_\alpha|_{D(0;1)}: D(0;1) \to D(0;1)$ is 1-1, onto, and hence, by Conclusion 2.11(4),

$$\left(\varphi_\alpha|_{D(0;1)}\right)^{-1} = \varphi_{(-\alpha)}\Big|_{D(0;1)}.$$

For the sake of simplicity, we shall denote $\varphi_\alpha|_{D(0;1)}$ simply by φ_α.

Thus, $\varphi_\alpha : D(0;1) \to D(0;1)$ is 1-1, onto, and $(\varphi_\alpha)^{-1} = \varphi_{(-\alpha)}$. By Conclusion 2.11, $\varphi_\alpha \in H(D(0;1))$.

Problem 2.174 $(\varphi_\alpha \circ \psi) \in \Sigma$.

(Solution Since $\psi : \Omega \to D(0;1)$, and $\varphi_\alpha : D(0;1) \to D(0;1)$, we have $(\varphi_\alpha \circ \psi) : \Omega \to D(0;1)$. Since $\psi \in H(\Omega)$, and $\varphi_\alpha \in H(D(0;1))$, we have $(\varphi_\alpha \circ \psi) \in H(\Omega)$. It remains to show that $(\varphi_\alpha \circ \psi)$ is 1-1. Since $\psi \in \Sigma$, $\psi : \Omega \to D(0;1)$ is 1-1. Now, since $\varphi_\alpha : D(0;1) \to D(0;1)$ is 1-1, $(\varphi_\alpha \circ \psi)$ is 1-1. ∎)

Problem 2.175 $(\varphi_\alpha \circ \psi) : \Omega \to D(0;1)$
has no zero in Ω.

(Solution If not, suppose otherwise that there exists $z \in \Omega$ such that $(\varphi_\alpha \circ \psi)(z) = 0$. We have to arrive at a contradiction. Since

$$\varphi_\alpha(\psi(z)) = \underbrace{(\varphi_\alpha \circ \psi)(z) = 0} = \varphi_\alpha(\alpha),$$

we have $\varphi_\alpha(\psi(z)) = \varphi_\alpha(\alpha)$. Now, since $\varphi_\alpha : D(0;1) \to D(0;1)$ is 1-1, we have $(\psi(\Omega) \ni)\psi(z) = \alpha$, and hence $\alpha \in \psi(\Omega)$. This is a contradiction. ∎)

Since $(\varphi_\alpha \circ \psi) : \Omega \to D(0;1)$ has no zero in Ω, and $(\varphi_\alpha \circ \psi) \in \Sigma$, we have $(\varphi_\alpha \circ \psi), \frac{1}{(\varphi_\alpha \circ \psi)} \in H(\Omega)$. Now, since Ω is a simply connected region, by Conclusion 2.164 $(b \Rightarrow i)$, there exists $g \in H(\Omega)$ such that $g^2 = (\varphi_\alpha \circ \psi)$.

Problem 2.176 $g : \Omega \to \mathbb{C}$ is 1-1.

(Solution For this purpose, let $g(z_1) = g(z_2)$. We have to show that $z_1 = z_2$. Since $g(z_1) = g(z_2)$, we have

$$\varphi_\alpha(\psi(z_1)) = \underbrace{(g(z_1))^2 = (g(z_2))^2} = \varphi_\alpha(\psi(z_2)),$$

and hence

$$\varphi_\alpha(\psi(z_1)) = \varphi_\alpha(\psi(z_2)).$$

Now, since $\varphi_\alpha : D(0;1) \to D(0;1)$ is 1-1, we have $\psi(z_1) = \psi(z_2)$. Since $\psi \in \Sigma$, $\psi : \Omega \to D(0;1)$ is 1-1. Now, since $\psi(z_1) = \psi(z_2)$, we have $z_1 = z_2$. ∎)

Problem 2.177 $g : \Omega \to D(0;1)$.

(Solution Let us fix any $z \in \Omega$. We have to show that $|g(z)| < 1$. If not, otherwise let $1 \le |g(z)|$. We have to arrive at a contradiction. Since $1 \le |g(z)|$, we have $1 \le \left|(g^2)(z)\right| (= |(\varphi_\alpha \circ \psi)(z)|)$, and hence $1 \le |(\varphi_\alpha \circ \psi)(z)|$. Since $(\varphi_\alpha \circ \psi) : \Omega \to D(0;1)$, and $z \in \Omega$, we have $|(\varphi_\alpha \circ \psi)(z)| < 1$. This is a contradiction. ∎)

Since $g : \Omega \to D(0;1)$ is 1-1, and $g \in H(\Omega)$, we have $g \in \Sigma$.

Problem 2.178 $\left(\varphi_{g(z_0)} \circ g\right) \in \Sigma$.

(Solution Since $g : \Omega \to D(0;1)$, and $z_0 \in \Omega$, we have $g(z_0) \in D(0;1)$, and hence $\varphi_{g(z_0)} : D(0;1) \to D(0;1)$ is 1-1, onto, and $\varphi_{g(z_0)} \in H(D(0;1))$. Since $\varphi_{g(z_0)} : D(0;1) \to D(0;1)$, and $g : \Omega \to D(0;1)$, we have $\left(\varphi_{g(z_0)} \circ g\right) : \Omega \to D(0;1)$. Since $g, \varphi_{g(z_0)}$ are 1-1, $\left(\varphi_{g(z_0)} \circ g\right) : \Omega \to D(0;1)$ is 1-1. Since $g \in H(\Omega)$, and $\varphi_{g(z_0)} \in H(D(0;1))$, we have $\left(\varphi_{g(z_0)} \circ g\right) \in H(\Omega)$. Thus, $\left(\varphi_{g(z_0)} \circ g\right) \in \Sigma$. ∎)

Let $s : z \to z^2$ be a function from $D(0;1)$ to $D(0;1)$. Since $(\varphi_\alpha \circ \psi) = g^2 (= s \circ g)$, and $(\varphi_\alpha)^{-1} = \varphi_{(-\alpha)}$, we have

$$\underbrace{\psi = \varphi_{(-\alpha)} \circ (s \circ g)} = \left(\varphi_{(-\alpha)} \circ s\right) \circ g = \left(\varphi_{(-\alpha)} \circ s\right) \circ \left(\varphi_{-g(z_0)} \circ \left(\varphi_{g(z_0)} \circ g\right)\right)$$

$$= \left(\varphi_{(-\alpha)} \circ s \circ \varphi_{-g(z_0)}\right) \circ \left(\varphi_{g(z_0)} \circ g\right),$$

and hence $\psi = F \circ \psi_1$, where $F \equiv \left(\varphi_{(-\alpha)} \circ s \circ \varphi_{-g(z_0)}\right)$, and $\psi_1 \equiv \left(\varphi_{g(z_0)} \circ g\right) (\in \Sigma)$. It follows that

$$\begin{aligned}
\underbrace{\psi'(z_0) = (F'(\psi_1(z_0))) \cdot ((\psi_1)'(z_0))} &= \left(F'\left(\left(\varphi_{g(z_0)} \circ g\right)(z_0)\right)\right) \cdot ((\psi_1)'(z_0)) \\
&= \left(F'\left(\varphi_{g(z_0)}(g(z_0))\right)\right) \cdot ((\psi_1)'(z_0)) \\
&= (F'(0)) \cdot ((\psi_1)'(z_0)),
\end{aligned}$$

and hence

$$\psi'(z_0) = (F'(0)) \cdot \left((\psi_1)'(z_0)\right).$$

Since $\varphi_{-g(z_0)} : D(0;1) \to D(0;1)$, $s : D(0;1) \to D(0;1)$, $\varphi_{-\alpha} : D(0;1) \to D(0;1)$, and $F = \left(\varphi_{(-\alpha)} \circ s \circ \varphi_{-g(z_0)}\right)$, we have $F : D(0;1) \to D(0;1)$.

Problem 2.179 $F : D(0;1) \to D(0;1)$ is not 1-1.

(Solution Since $\varphi_{-g(z_0)} : D(0;1) \to D(0;1)$ is 1-1, onto, and $\frac{1}{2} \in D(0;1)$, there exists $a \in D(0;1)$ such that $\varphi_{-g(z_0)}(a) = \frac{1}{2}$. Again, there exists $b \in D(0;1)$ such that $\varphi_{-g(z_0)}(b) = \frac{-1}{2}$. Now, since $\varphi_{-g(z_0)}(a) \neq \varphi_{-g(z_0)}(b)$, we have $a \neq b$. Now, it suffices to show that $F(a) = F(b)$.

$$\begin{aligned}
\text{LHS} = F(a) &= \left(\varphi_{(-\alpha)} \circ s \circ \varphi_{-g(z_0)}\right)(a) = \varphi_{(-\alpha)}\left(s\left(\varphi_{-g(z_0)}(a)\right)\right) \\
&= \varphi_{(-\alpha)}\left(s\left(\frac{1}{2}\right)\right) = \varphi_{(-\alpha)}\left(\left(\frac{1}{2}\right)^2\right) \\
&= \varphi_{(-\alpha)}\left(\left(\frac{-1}{2}\right)^2\right) = \varphi_{(-\alpha)}\left(s\left(\frac{-1}{2}\right)\right) = \varphi_{(-\alpha)}\left(s\left(\varphi_{-g(z_0)}(b)\right)\right) \\
&= \left(\varphi_{(-\alpha)} \circ s \circ \varphi_{-g(z_0)}\right)(b) = F(b) = \text{RHS}.
\end{aligned}$$

∎)

By Conclusion 2.11, $\varphi_{(-\alpha)}, \varphi_{-g(z_0)} \in H(D(0;1))$, and the function $s : z \to z^2$ from $D(0;1)$ to $D(0;1)$ is a member of $H(D(0;1))$, so $(F =)$ $\left(\varphi_{(-\alpha)} \circ s \circ \varphi_{-g(z_0)}\right) \in H(D(0;1))$, and hence $F \in H(D(0;1))$. Now, since $F : D(0;1) \to D(0;1)$, we have $F \in H^\infty$, and $\|f\|_\infty \leq 1$.

Now, by Conclusion 2.16,

1. $|F'(0)| \leq \frac{1-|F(0)|^2}{1-|0|^2} \left(= 1 - |F(0)|^2 \leq 1\right)$,
2. if $|F'(0)| = \frac{1-|F(0)|^2}{1-|0|^2}$, then there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and, for every $z \in D(0;1)$, $F(z) = \varphi_{-F(0)}(\lambda\varphi_0(z))\left(= \varphi_{-F(0)}(\lambda z) = \frac{\lambda z + F(0)}{1 + \overline{F(0)}\lambda z}\right)$.

From 1, $|F'(0)| \leq 1$.

Problem 2.180 $|F'(0)| < 1$.

(Solution If not, otherwise, let $|F'(0)| = 1$. We have to arrive at a contradiction. Since $\left(1 - |F(0)|^2 \geq\right)|F'(0)| = 1$, we have $(0 \leq)|F(0)|^2 \leq 0$, and hence

$F(0) = 0$. It follows that $|F'(0)| = \frac{1-|F(0)|^2}{1-|0|^2}$, and hence, by 2, there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$, and, for every $z \in D(0;1)$,

$$\underbrace{F(z) = \frac{\lambda z + F(0)}{1 + \overline{F(0)}\lambda z}} = \frac{\lambda z + 0}{1 + \bar{0}\lambda z} = \lambda z.$$

Thus, $F : z \mapsto \lambda z$ is a function from $D(0;1)$ to $D(0;1)$, where $|\lambda| = 1$. It follows that $F : D(0;1) \to D(0;1)$ is 1-1. This is a contradiction. ∎)

Since $\psi'(z_0) = (F'(0)) \cdot ((\psi_1)'(z_0))$, we have $|\psi'(z_0)| = |F'(0)|\left|(\psi_1)'(z_0)\right|$. Since $\psi \in \Sigma$, $\psi : \Omega \to D(0;1)$ is 1-1, and $\psi \in H(\Omega)$, by Theorem 1.191, $\psi'(z_0) \neq 0$, and hence $0 < |\psi'(z_0)| \left(= |F'(0)|\left|(\psi_1)'(z_0)\right|\right)$. This shows that $(1 >)|F'(0)| = \frac{|\psi'(z_0)|}{\left|(\psi_1)'(z_0)\right|}$, and hence $|\psi'(z_0)| < \left|(\psi_1)'(z_0)\right|$.

Conclusion 2.181 Let Ω be a region. Let $\Omega \neq \mathbb{C}$. Suppose that if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$. Let Σ be the collection of all 1-1 functions $\psi : \Omega \to D(0;1)$ such that $\psi \in H(\Omega)$. Let $\psi \in \Sigma$. Let $\psi : \Omega \to D(0;1)$ be not onto. Let $z_0 \in \Omega$. Then there exists $\psi_1 \in \Sigma$ such that $|\psi'(z_0)| < \left|(\psi_1)'(z_0)\right|$.

Note 2.182 Let Ω be a region. Let $\Omega \neq \mathbb{C}$. Suppose that, if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$.

Let Σ be the collection of all 1-1 functions $\psi : \Omega \to D(0;1)$ such that $\psi \in H(\Omega)$. Let $z_0 \in \Omega$.

Problem 2.183 $\sup\{|\psi'(z_0)| : \psi \in \Sigma\}$ is a positive real number.

(Solution By Conclusion 2.172, Σ is nonempty, and hence $\{|\psi'(z_0)| : \psi \in \Sigma\}$ is a nonempty set of nonnegative real numbers. For every $\psi \in \Sigma$, we have $\psi : \Omega \to D(0;1)$ is 1-1, and $\psi \in H(\Omega)$, and hence by Theorem 1.191, $\psi'(z_0) \neq 0$. Thus, $\{|\psi'(z_0)| : \psi \in \Sigma\}$ is a nonempty set of positive real numbers. It suffices to show that $\{|\psi'(z_0)| : \psi \in \Sigma\}$ is bounded above.

Since $z_0 \in \Omega$, and Ω is an open set, there exists $r > 0$ such that $D[z_0;r] \subset \Omega$. Next, let us take any $\psi \in \Sigma$. Since $\psi \in \Sigma$, we have $\psi \in H(\Omega)$, and, for every $z \in \Omega$, $|\psi(z)| < 1$, and hence, by Conclusion 1.166, $|\psi'(z_0)| \leq \frac{1(1!)}{r^2}\left(= \frac{1}{r^2}\right)$. This shows that $\frac{1}{r^2}$ is an upper bound of $\{|\psi'(z_0)| : \psi \in \Sigma\}$, and hence $\{|\psi'(z_0)| : \psi \in \Sigma\}$ is bounded above. ∎)

By the definition of Σ, we have $\Sigma \subset H(\Omega)$. Also, by Conclusion 2.172, Σ is nonempty. Since for every $\psi \in \Sigma$, we have $\psi : \Omega \to D(0;1)$, Σ is uniformly bounded on Ω, and hence Σ is uniformly bounded on each compact subset of Ω. Now, by Conclusion 2.96, Σ is a normal family. Since $\sup\{|\psi'(z_0)| : \psi \in \Sigma\}$ is a positive real number, there exists a sequence $\{\psi_n\}$ in Σ such that $\lim_{n\to\infty}|(\psi_n)'(z_0)| = \eta$, where

$$\eta \equiv \sup\{|\psi'(z_0)| : \psi \in \Sigma\}(\in (0,\infty)).$$

Now, since Σ is a normal family, there exists a subsequence $\{\psi_{n_k}\}$ of $\{\psi_n\}$ such that $n_1 < n_2 < n_3 < \cdots$, and $\{\psi_{n_k}\}$ converges uniformly on every compact subset of Ω. It follows that there exists a function $h : \Omega \to \mathbb{C}$ such that $\{\psi_{n_k}\}$ converges to h uniformly on every compact subset of Ω. Now, by Conclusion 1.172,

1. $h \in H(\Omega)$,
2. $\left\{(\psi_{n_k})'\right\}$ converges to h' uniformly on compact subsets of Ω.

Since $\lim_{n\to\infty}|(\psi_n)'(z_0)| = \eta$, and $\left\{\left|(\psi_{n_k})'(z_0)\right|\right\}$ is a subsequence of $\{|(\psi_n)'(z_0)|\}$, $\left\{\left|(\psi_{n_k})'(z_0)\right|\right\}$ converges to η. By 2, $\{(\psi_n)'(z_0)\}$ converges to $h'(z_0)$, and hence $\{|(\psi_n)'(z_0)|\}$ converges to $|h'(z_0)|$. Now, since $\left\{\left|(\psi_{n_k})'(z_0)\right|\right\}$ converges to η, we have $|h'(z_0)| = \eta(>0)$, and hence $h : \Omega \to \mathbb{C}$ is not a constant function. Now, since $h \in H(\Omega)$, by Conclusion 1.190, $h(\Omega)$ is a region, and hence $h(\Omega)$ is an open set.

Since $\{\psi_{n_k}\}$ converges to h uniformly on every compact subset of Ω, for every $z \in \Omega$, $\lim_{k\to\infty}\psi_{n_k}(z) = h(z)$. Since each $\psi_{n_k} : \Omega \to D(0;1)$, we have, for every $z \in \Omega$, $(h(z) =)\left(\lim_{k\to\infty}\psi_{n_k}(z)\right) \in D[0;1]$, and hence, for every $z \in \Omega$, $h(z) \in D[0;1]$. Thus, $h : \Omega \to D[0;1]$. Now, since $h(\Omega)$ is an open set, $h(\Omega)$ is an open set contained in $D[0;1]$, we have $h(\Omega) \subset (D[0;1])^0(= D(0;1))$. Thus, $h : \Omega \to D(0;1)$.

Problem 2.184 $h : \Omega \to D(0;1)$ is 1-1.

(**Solution** If not, otherwise suppose that there exist $z_1, z_2 \in \Omega$ such that $z_1 \neq z_2$, and $h(z_1) = h(z_2)$. We have to arrive at a contradiction.

Since $z_1, z_2 \in \Omega$, $z_1 \neq z_2$, and Ω is open, there exists $r > 0$ such that $D[z_2;r] \subset \Omega$, and $z_1 \notin D[z_2;r]$.

Since $\{\psi_{n_k}\}$ converges to h uniformly on the compact subset $D[z_2;r]$ of Ω, and $\lim_{k\to\infty}\psi_{n_k}(z_1) = h(z_1)$, the sequence $\{\psi_{n_k} - \psi_{n_k}(z_1)\}$ of functions converges to $(h - h(z_1))$ uniformly on $D[z_2;r]$. Since each ψ_{n_k} is 1-1, and $z_1 \notin D[z_2;r](\supset D(z_2;r))$, each function $(\psi_{n_k} - \psi_{n_k}(z_1))$ has no zero in $D(z_2;r)$. Now, by Corollary 1.231, $(h - h(z_1)) = 0$, or $(h - h(z_1))$ has no zero on Ω. Since $h(z_1) = h(z_2)$, $z_2(\in \Omega)$ is a zero of $(h - h(z_1))$, and hence $(h - h(z_1)) = 0$. This shows that $h(\Omega) = \{h(z_1)\}$. Now, since $(\{h(z_1)\} =)h(\Omega)$ is an open set, $\{h(z_1)\}$ is an open set. This is a contradiction. ∎)

Problem 2.185 $h : \Omega \to D(0;1)$ is onto.

(Solution If not, otherwise let $h : \Omega \to D(0;1)$ be not onto. We have to arrive at a contradiction.

Since $h \in H(\Omega)$, and $h : \Omega \to D(0;1)$ is 1-1, we have $h \in \Sigma$ Now, since $h : \Omega \to D(0;1)$ is not onto, by Conclusion 2.181, there exists $\psi_1 \in \Sigma$ such that

$$\left|(\psi_1)'(z_0)\right| \leq \sup\{|\psi'(z_0)| : \psi \in \Sigma\} = \eta = |h'(z_0)| < \left|(\psi_1)'(z_0)\right|.$$

This is a contradiction. ∎)

Conclusion 2.186 Let Ω be a region. Let $\Omega \neq \mathbb{C}$. Suppose that if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$. Then there exists a function $h : \Omega \to D(0;1)$ such that h is 1-1, onto, and $h \in H(\Omega)$.

Definition Let Ω_1 and Ω_2 be any regions. If there exists a function $h : \Omega_1 \to \Omega_2$ such that h is 1-1, onto, and $h \in H(\Omega)$, then we say that Ω_1 *and* Ω_2 *are* ***conformally equivalent***.

Observe that if Ω_1 and Ω_2 are conformally equivalent, and $h : \Omega_1 \to \Omega_2$ such that h is 1-1, onto, and $h \in H(\Omega_1)$, then, by Theorem 1.191, $h^{-1} \in H(\Omega_2)$, and, for every $z \in \Omega_1$, $h'(z) \neq 0$.

Further, it is easy to observe that the relation of 'conformally equivalent' in the collection of all regions is an equivalence relation.

Now, the above Conclusion (2.187) can be restated as follows:

Conclusion 2.187 Let Ω be a simply connected region. Let $\Omega \neq \mathbb{C}$. Then Ω and $D(0;1)$ are conformally equivalent.

Proof Since Ω is a simply connected region, by Conclusion 2.164 $(b \Rightarrow i)$, if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$. It follows, by the above Conclusion 2.186, that there exists a function $h : \Omega \to D(0;1)$ such that h is 1-1, onto, and $h \in H(\Omega)$, and hence Ω and $D(0;1)$ are conformally equivalent. ∎

This result is known as the **Riemann mapping theorem**.

Theorem 2.188 Let Ω be a region. Then, the following statements are equivalent:

a. Ω is homeomorphic to $D(0;1)$,
b. Ω is simply connected,
c. for every closed path γ in Ω, and for every $\alpha \in \left(\mathbb{S}^2 - \Omega\right)$, $(\mathrm{Ind})_\gamma(\alpha) = 0$,
d. $\left(\mathbb{S}^2 - \Omega\right)$ is connected,
e. for every $f \in H(\Omega)$, there exists a sequence $\{P_n\}$ of polynomials such that $\lim_{n\to\infty} P_n = f$ uniformly on compact subsets of Ω,
f. for every $f \in H(\Omega)$, and for every closed path γ in Ω, $\int_\gamma f(z)\mathrm{d}z = 0$,
g. for every $f \in H(\Omega)$, there exists $F \in H(\Omega)$ such that $F' = f$,

h. if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $g \in H(\Omega)$ such that $f = e^g$,
i. if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$.

Proof In view of Conclusion 2.164, it suffices to show that $i \Rightarrow a$.

$i \Rightarrow a$: Case I: when $\Omega = \mathbb{C}$. Let $f : z \mapsto \frac{1}{1+|z|} z$ be a function from $(\Omega =)\mathbb{C}$ to $\mathbb{C}$. Since for every $z \in \mathbb{C}$,

$$|f(z)| = \left| \frac{1}{1+|z|} z \right| = \frac{1}{1+|z|} |z| < 1,$$

we have $f : \mathbb{C} \to D(0;1)$. It suffices to show that $f : \mathbb{C} \to D(0;1)$ is 1-1, onto, continuous, and f^{-1} is continuous.

1-1ness: For this purpose, let $f(z) = f(w)$. We have to show that $z = w$. Since

$$\frac{1}{1+|z|} z = \underbrace{f(z) = f(w)} = \frac{1}{1+|w|} w,$$

we have

$$\frac{1}{1+|z|} |z| = \underbrace{\left| \frac{1}{1+|z|} z \right| = \left| \frac{1}{1+|w|} w \right|} = \frac{1}{1+|w|} |w|,$$

and hence

$$|z|(1+|w|) = |w|(1+|z|).$$

This shows that $|z| = |w|$. Since $\frac{1}{1+|z|} z = \frac{1}{1+|w|} w \left(= \frac{1}{1+|z|} w \right)$, we have $\frac{1}{1+|z|} (z-w) = 0$. Now, since $\frac{1}{1+|z|} \neq 0$, we have $(z-w) = 0$, and hence $z = w$.

Onto-ness: For this purpose, let us take any $w \in D(0;1)$. Since

$$f\left(\frac{w}{1-|w|} \right) = \frac{1}{1 + \left| \frac{w}{1-|w|} \right|} \frac{w}{1-|w|} = \frac{1}{1 + \frac{|w|}{1-|w|}} \frac{w}{1-|w|} = \frac{1}{\frac{1}{1-|w|}} \frac{w}{1-|w|} = w,$$

we have $f\left(\frac{w}{1-|w|} \right) = w$, and $\frac{w}{1-|w|} \in \mathbb{C}$. Thus, $f : \mathbb{C} \to D(0;1)$ is onto.

Clearly, $f : z \mapsto \frac{1}{1+|z|} z$ is continuous. Since $f^{-1} : w \mapsto \frac{w}{1-|w|}$, f^{-1} is continuous.

Case II: when $\Omega \neq \mathbb{C}$. It is given that if $f \in H(\Omega)$, and $\frac{1}{f} \in H(\Omega)$, then there exists $\varphi \in H(\Omega)$ such that $f = \varphi^2$. Now, by Conclusion 2.186, there exists a function $h : \Omega \to D(0;1)$ such that h is 1-1, onto, and $h \in H(\Omega)$. Since $h \in H(\Omega)$, $h : \Omega \to D(0;1)$ is continuous. By Theorem 1.191, $h^{-1} \in H(D(0;1))$, and hence h^{-1} is continuous.

So, in all cases Ω is homeomorphic to $D(0;1)$. ∎

Note 2.189 Definition Let r and R be positive real numbers such that $r < R$. The set $\{z : r < |z| < R\}$ is denoted by $A(r, R)$, and is called the ***annulus*** *with* ***inner radius*** r *and* ***outer radius*** R.

Problem 2.190 Every annulus is a region.

(Solution Let us take an annulus $A(r, R)$. We have to show that $A(r, R)$ is a region, that is $A(r, R)$ is open and connected. Since $A(r, R) = \{z : r < |z| < R\} = D(0; R) - D[0; r]$, and $D(0; R) - D[0; r]$ is open, $A(r, R)$ is open. It is clear that $\{z : r < |z| < R\}$ is a path-connected subset of $\mathbb{C}$, and hence by Conclusion 1.25, $\{z : r < |z| < R\}$ is connected. Thus, $A(r, R)$ is a region. ∎)

Let $A(r, R)$, and $A(r_1, R_1)$ be two annuli. Let $\frac{R}{r} = \frac{R_1}{r_1}$.

Problem 2.191 $A(r, R)$ and $A(r_1, R_1)$ are conformally equivalent.

(Solution Consider the map

$$f : z \mapsto \left(\frac{r_1}{r} z\right) \left(= \left(\frac{R_1}{R} z\right) \right)$$

from $(A(r, R) =)\{z : r < |z| < R\}$ to $\{z : r_1 < |z| < R_1\}(= A(r_1, R_1))$. It suffices to show that

$$f : A(r, R) \to A(r_1, R_1)$$

is 1-1, onto, and $f \in H(A(r, R))$.

1-1ness: Let $f(z) = f(w)$. We have to show that $z = w$. Since $\frac{r_1}{r} z = \underbrace{f(z) = f(w)} = \frac{r_1}{r} w$, we have $\frac{r_1}{r}(z - w) = 0$. Now, since $\frac{r_1}{r} \neq 0$, we have $(z - w) = 0$, and hence $z = w$.

Onto-ness: For this purpose, let us take any $w \in A(r_1, R_1)$. It follows that $r_1 < |w| < R_1$, and hence

$$r = \underbrace{\frac{r}{r_1} r_1 < \frac{r}{r_1} |w| < \frac{r}{r_1} R_1} = r \frac{R_1}{r_1} = r \frac{R}{r} = R.$$

This shows that $r < \left| \frac{r}{r_1} w \right| < R$, and hence $\frac{r}{r_1} w \in A(r, R)$. It suffices to show that $f\left(\frac{r}{r_1} w\right) = w$.

$$\text{LHS} = f\left(\frac{r}{r_1} w\right) = \frac{r_1}{r} \left(\frac{r}{r_1} w\right) = w = \text{RHS}.$$

It is clear that $f : z \mapsto \left(\frac{r_1}{r}z\right)$ from $A(r,R)$ to $A(r_1,R_1)$ is a member of $H(A(r,R))$. ■)

Conclusion 2.192 Let $A(r,R)$, and $A(r_1,R_1)$ be two annuli. Let $\frac{R}{r} = \frac{R_1}{r_1}$. Then $A(r,R)$ and $A(r_1,R_1)$ are conformally equivalent.

2.15 Exercises

2.1 Let $f \in H^\infty$, $f(0) = 0$, and $\|f\|_\infty < 2$. Show that

a. for every $z \in D(0;1)$, $|f(z)| \leq 2|z|$,
b. $|f'(0)| \leq 2$.

2.2 Let $f : D(0;1) \to D(0;2)$ be 1-1, onto. Let $f \in H(D(0;1))$. Let $\alpha \in D(0;1)$, and $f(\alpha) = 0$. Show that there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 2$, and for every $z \in D(0;1)$,

$$f(z) = \lambda \frac{z-\alpha}{1-\bar{\alpha}z}.$$

2.3 Let $\{u_n\}$ be any sequence of complex numbers. Show that

$$|(1+u_1)(1+u_2)(1+u_3) - 1| \leq (1+|u_1|)(1+|u_2|)(1+|u_3|) - 1.$$

2.4 Show that for every positive integer p,

$$\left|(1-i)e^{\sum_{n=1}^{p}\frac{i^n}{n}} - 1\right| \leq 1.$$

2.5 Let $\{a_n\}$ be any sequence of nonzero complex numbers. Let $\lim_{n\to\infty}|a_n| = \infty$. Show that there exists an infinite sequence $\{p_n\}$ of nonnegative integers such that

a. for every $r > 0$, $\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+p_n} < \infty$,
b. if a occurs in the sequence $\{a_1, a_2, a_3, \ldots\}$ exactly m times, then $P : z \mapsto \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$ has a zero at $z = a$ of multiplicity m.

2.6 Let $f : D(0;1) \to \mathbb{C}$ be a mapping. Suppose that for every $z \in D'\left(\frac{1}{2};\frac{1}{2}\right)$, $f(z) \neq f\left(\frac{1}{2}\right)$. Suppose that $f'\left(\frac{1}{2}\right)$ exists, and $f'\left(\frac{1}{2}\right) \neq 0$. Show that for every real θ,

$$\lim_{r\to 0^+} e^{-i\theta} A\left[f\left(\frac{1}{2}+re^{i\theta}\right)-f\left(\frac{1}{2}\right)\right]$$

exists and is independent of θ.

2.7 Show that there exists a unique linear fractional transformation φ, which maps the ordered triplet $(1, i, -1)$ into $(2, i+1, \infty)$.

2.8 Let $\mathcal{F}$ be a nonempty subset of $H(D(0;1))$. Suppose that $\mathcal{F}$ is uniformly bounded on each compact subset of $D(0;1)$. Show that $\mathcal{F}$ is a normal family.

2.9 Suppose that $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ is a set, which contains some point in each component of $\left(\mathbb{S}^2 - D\left[0;\frac{1}{2}\right]\right)$. Let $f \in H(D(0;1))$. Let $\varepsilon > 0$. Show that there exists a rational function R, whose poles are in $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, and

$$|z| \le \frac{1}{2} \Rightarrow |f(z) - R(z)| < \varepsilon.$$

2.10 Suppose that whenever $f \in H(D(0;1))$, and $\frac{1}{f} \in H(D(0;1))$, there exists $\varphi \in H(D(0;1))$ such that $f = \varphi^2$. Show that there exists a function $\psi : D(0;1) \to D(0;1)$ such that

a. $\psi \in H(\Omega)$,
b. ψ is 1-1.

Chapter 3
Analytic Continuation

In this chapter, we introduce analytic continuation, and prove the monodromy theorem. A branch of logarithm function is also discussed here. We also prove Ostrowski's theorem, Hadamard's theorem, and Lindelöf's theorem. A discussion on modular functions and Riemann surfaces is included.

3.1 Analytic Continuation

Note 3.1

Definition By a ***function element*** we mean an ordered pair (f, D), where D is an open disk, and $f \in H(D)$.

Definition Let (f_0, D_0) and (f_1, D_1) be any function elements. If $D_0 \cap D_1 \neq \emptyset$ and, for every $z \in D_0 \cap D_1$, $f_0(z) = f_1(z)$, then we say that (f_0, D_0) *and* (f_1, D_1) *are the* ***direct continuation*** *of each other*, and we write: $(f_0, D_0) \sim (f_1, D_1)$.

Let $(f_0, D_0), (f_1, D_1), (f_2, D_2)$ be function elements. Let $(f_0, D_0) \sim (f_1, D_1)$, and $(f_1, D_1) \sim (f_2, D_2)$. Suppose that $D_0 \cap D_1 \cap D_2 \neq \emptyset$.

Problem 3.2 $(f_0, D_0) \sim (f_2, D_2)$.

(Solution It suffices to show:

1. $D_0 \cap D_2 \neq \emptyset$,
2. for every $z \in D_0 \cap D_2$, $f_0(z) = f_2(z)$, that is $f_0 = f_2$ on $D_0 \cap D_2$.

For 1: Since $(D_0 \cap D_2 \supset) D_0 \cap D_1 \cap D_2 \neq \emptyset$, we have $D_0 \cap D_2 \neq \emptyset$.
For 2: Since D_0, D_1, D_2 are open disks, and $D_0 \cap D_1 \cap D_2 \neq \emptyset$, $D_0 \cap D_1 \cap D_2$ is a region. By Theorem 4.135, it suffices to show that

a. $f_0 = f_2$ on $D_0 \cap D_1 \cap D_2$,
b. for every $z \in D_0 \cap D_2$, $(f_0)'(z), (f_2)'(z)$ exist.

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For (a): Let us take any $z \in D_0 \cap D_1 \cap D_2$. We have to show that $f_0(z) = f_2(z)$. It follows that $z \in D_0 \cap D_1$, and $z \in D_1 \cap D_2$. Since $z \in D_0 \cap D_1$, and $(f_0, D_0) \sim (f_1, D_1)$, we have $f_0(z) = f_1(z)$. Since $z \in D_1 \cap D_2$, and $(f_1, D_1) \sim (f_2, D_2)$, we have $f_1(z) = f_2(z)$. It follows that $f_0(z) = f_2(z)$.

For (b): Let us take any $z \in D_0 \cap D_2$ $(\subset D_0)$. Now, since (f_0, D_0) is a function element, $(f_0)'(z)$ exists. Similarly, $(f_2)'(z)$ exists. ∎)

Conclusion 3.3 Let $(f_0, D_0), (f_1, D_1), (f_2, D_2)$ be function elements. Let $(f_0, D_0) \sim (f_1, D_1)$, and $(f_1, D_1) \sim (f_2, D_2)$. Suppose that $D_0 \cap D_1 \cap D_2 \neq \emptyset$. Then $(f_0, D_0) \sim (f_2, D_2)$.

Note 3.4

Definition Let $D_0, D_1, \ldots, D_n$ be any open disks. If $D_0 \cap D_1 \neq \emptyset, D_1 \cap D_2 \neq \emptyset, \ldots, D_{n-1} \cap D_n \neq \emptyset$, then the finite sequence $\{D_0, D_1, \ldots, D_n\}$ is called a ***chain***.

Let $\{D_0, D_1, \ldots, D_n\}$ be a chain. Let (f_0, D_0) be a function element. Suppose that $(f_0, D_0), (f_1, D_1), \ldots, (f_n, D_n)$ are function elements such that $(f_0, D_0) \sim (f_1, D_1)$, $(f_1, D_1) \sim (f_2, D_2), \ldots, (f_{n-1}, D_{n-1}) \sim (f_n, D_n)$. Suppose that $(f_0, D_0), (g_1, D_1), \ldots, (g_n, D_n)$ are function elements such that $(f_0, D_0) \sim (g_1, D_1)$, $(g_1, D_1) \sim (g_2, D_2), \ldots, (g_{n-1}, D_{n-1}) \sim (g_n, D_n)$.

Problem 3.5 $f_1 = g_1, \ldots, f_n = g_n$.

(Solution Since $(f_0, D_0) \sim (f_1, D_1)$, we have $(f_1, D_1) \sim (f_0, D_0)$, and $D_0 \cap D_1 \neq \emptyset$. Now, since $(f_0, D_0) \sim (g_1, D_1)$, and $D_1 \cap D_0 \cap D_1 = D_0 \cap D_1 \neq \emptyset$, by Conclusion 3.3, $(f_1, D_1) \sim (g_1, D_1)$, and hence $f_1 = g_1$. Thus, $(g_1, D_1) \sim (g_2, D_2)$ becomes $(f_1, D_1) \sim (g_2, D_2)$, and hence $(g_2, D_2) \sim (f_1, D_1)$. Now, since $(f_1, D_1) \sim (f_2, D_2)$, and $D_2 \cap D_1 \cap D_2 = D_1 \cap D_2 \neq \emptyset$, by Conclusion 3.3, $(g_2, D_2) \sim (f_2, D_2)$, and hence $f_2 = g_2$, etc. ∎)

Conclusion 3.6 Let $\{D_0, D_1, \ldots, D_n\}$ be a chain. Let (f_0, D_0) be a function element. If there exist $f_1 \in H(D_1), \ldots, f_n \in H(D_n)$ such that $(f_0, D_0) \sim (f_1, D_1)$, $(f_1, D_1) \sim (f_2, D_2), \ldots, (f_{n-1}, D_{n-1}) \sim (f_n, D_n)$, then $f_1, \ldots, f_n$ are unique.
This conclusion justifies the following definition:

Definition Let $\{D_0, D_1, \ldots, D_n\}$ be a chain. Let (f_0, D_0) be a function element. If there exist function elements $(f_1, D_1), \ldots, (f_n, D_n)$ such that $(f_0, D_0) \sim (f_1, D_1)$, $(f_1, D_1) \sim (f_2, D_2), \ldots, (f_{n-1}, D_{n-1}) \sim (f_n, D_n)$, then we say that (f_n, D_n) *is the* ***analytic continuation*** *of* (f_0, D_0) *along the chain* $\{D_0, D_1, \ldots, D_n\}$.

Definition Let $\{D_0, D_1, \ldots, D_n\}$ be a chain. Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve (that is, γ is continuous). If there exists a partition $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$ of [0,1] such that

1. $\gamma(0)$ is the center of the disk D_0,
2. $\gamma(1)$ is the center of the disk D_n,
3. $\gamma([s_0, s_1]) \subset D_0$, $\gamma([s_1, s_2]) \subset D_1, \ldots, \gamma([s_{n-1}, s_n]) \subset D_{n-1}$, then we say that the *chain* $\{D_0, D_1, \ldots, D_n\}$ ***covers*** *the curve* γ.

Note 3.7 Let (f, D) be a function element. Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve. Let $\gamma(0)$ be the center of the disk D. Suppose that $\{D, A_1, \ldots, A_m\}$ is a chain that covers the curve γ, that is

1. $D \cap A_1 \neq \emptyset, A_1 \cap A_2 \neq \emptyset, \ldots, A_{m-1} \cap A_m \neq \emptyset$,
2. there exists a partition $\{0 = s_0 < s_1 < \cdots < s_m = 1\}$ of [0,1] such that
   a. $\gamma(1)$ is the center of the disk A_m,
   b. $\gamma([s_0, s_1]) \subset D$, $\gamma([s_1, s_2]) \subset A_1, \ldots, \gamma([s_{m-1}, s_m]) \subset A_{m-1}$.

Suppose that (g_m, A_m) is the analytic continuation of (f, D) along the chain $\{D, A_1, \ldots, A_m\}$, that is there exist function elements $(g_1, A_1), \ldots, (g_m, A_m)$ such that $(f, D) \sim (g_1, A_1)$, $(g_1, A_1) \sim (g_2, A_2), \ldots, (g_{m-1}, A_{m-1}) \sim (g_m, A_m)$.

Suppose that $\{D, B_1, \ldots, B_n\}$ is a chain that covers the curve γ, that is

1. $D \cap B_1 \neq \emptyset, B_1 \cap B_2 \neq \emptyset, \ldots, B_{n-1} \cap B_n \neq \emptyset$,
2. there exists a partition $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of [0,1] such that
   a. $\gamma(1)$ is the center of the disk B_n,
   b. $\gamma([t_0, t_1]) \subset D$, $\gamma([t_1, t_2]) \subset B_1, \ldots, \gamma([t_{n-1}, t_n]) \subset B_{n-1}$.

Suppose that (h_n, B_n) is the analytic continuation of (f, D) along the chain $\{D, B_1, \ldots, B_n\}$, that is there exist function elements $(h_1, B_1), \ldots, (h_n, B_n)$ such that $(f, D) \sim (h_1, B_1)$, $(h_1, B_1) \sim (h_2, B_2), \ldots, (h_{n-1}, B_{n-1}) \sim (h_n, B_n)$.

Put $g_0 \equiv f$, $h_0 \equiv f, A_0 \equiv D$, $B_0 \equiv D, s_{m+1} \equiv 1$, and $t_{n+1} \equiv 1$.

Problem 3.8 For every $i \in \{0, 1, \ldots, m\}$, and for every $j \in \{0, 1, \ldots, n\}$, if $[s_i, s_{i+1}] \cap [t_j, t_{j+1}] \neq \emptyset$ then $(g_i, A_i) \sim (h_j, B_j)$.

(Solution Let

$$N \equiv \{i + j : i \in \{0, 1, \ldots, m\}, j \in \{0, 1, \ldots, n\}, [s_i, s_{i+1}] \cap [t_j, t_{j+1}] \neq \emptyset, \text{ and } (g_i, A_i) \nsim (h_j, B_j)\}.$$

We have to show that N is an empty set. If not, otherwise, let N be a nonempty set of nonnegative integers. We have to arrive at a contradiction.

Consider the case when $[s_0, s_{0+1}] \cap [t_0, t_{0+1}] \neq \emptyset$. This case actually holds. Since $(f, D) \sim (f, D)$, we have $(g_0, A_0) \sim (h_0, B_0)$. Hence $0 \notin N$.

Thus, N is a nonempty subset of $\{1, 2, \ldots, m + n\}$, and hence N has a smallest element, say $i_0 + j_0$, where $[s_{i_0}, s_{i_0+1}] \cap [t_{j_0}, t_{j_0+1}] \neq \emptyset$, and $(g_{i_0}, A_{i_0}) \nsim (h_{j_0}, B_{j_0})$. Also, $1 \leq i_0 + j_0$.

Situation I: when $t_{j_0} \leq s_{i_0}$. Since $[s_{i_0}, s_{i_0+1}] \cap [t_{j_0}, t_{j_0+1}] \neq \emptyset$, we have $s_{i_0} \leq t_{j_0+1}$, and hence $s_{i_0} \in [t_{j_0}, t_{j_0+1}]$.

Problem 3.9 $1 \leq i_0$.

(Solution If not, otherwise, let $i_0 = 0$. We have to arrive at a contradiction. Since $i_0 = 0$, and $t_{j_0} \leq s_{i_0}$, we have $(0 \leq)\ t_{j_0} \leq s_0\ (= 0)$, and hence $t_{j_0} = 0$. This contradicts $1 \leq i_0 + j_0$. ∎)

It follows that $[s_{i_0-1}, s_{i_0}] \cap [t_{j_0}, t_{j_0+1}] \neq \emptyset$.

Since $i_0 + j_0$ is the smallest element of N, we have $((i_0-1)+j_0) \notin N$ (that is, $[s_{i_0-1}, s_{i_0}] \cap [t_{j_0}, t_{j_0+1}] = \emptyset$ or $(g_{i_0-1}, A_{i_0-1}) \sim (h_{j_0}, B_{j_0})$). Now, since $[s_{i_0-1}, s_{i_0}] \cap [t_{j_0}, t_{j_0+1}] \neq \emptyset$, we have $(g_{i_0-1}, A_{i_0-1}) \sim (h_{j_0}, B_{j_0})$.

Since $[s_{i_0}, s_{i_0+1}] \cap [t_{j_0}, t_{j_0+1}] \neq \emptyset$, we have $(A_{i_0} \cap B_{j_0} \supset \gamma([s_{i_0}, s_{i_0+1}]) \cap \gamma([t_{j_0}, t_{j_0+1}]) \supset) \gamma([s_{i_0}, s_{i_0+1}] \cap [t_{j_0}, t_{j_0+1}]) \ni \gamma(s_{i_0})$, and hence $\gamma(s_{i_0}) \in A_{i_0} \cap B_{j_0}$. Since $(\gamma(s_{i_0}) \in) \gamma([s_{i_0-1}, s_{i_0}]) \subset A_{i_0-1}$, and $\gamma(s_{i_0}) \in A_{i_0} \cap B_{j_0}$, we have $\gamma(s_{i_0}) \in A_{i_0} \cap B_{j_0} \cap A_{i_0-1}$.

Since $(g_{i_0-1}, A_{i_0-1}) \sim (h_{j_0}, B_{j_0})$, $(g_{i_0-1}, A_{i_0-1}) \sim (g_{i_0}, A_{i_0})$, and $A_{i_0-1} \cap B_{j_0} \cap A_{i_0} \ni \gamma(s_{i_0})$, by Conclusion 3.3, $(g_{i_0}, A_{i_0}) \sim (h_{j_0}, B_{j_0})$. This is a contradiction.

Situation II: when $s_{i_0} < t_{j_0}$. This is similar to Situation I.

Thus, in all situations, we get a contradiction. ∎)

It follows that for every $i \in \{0, 1, \ldots, m\}$, and for every $j \in \{0, 1, \ldots, n\}$, we have $[s_m, s_{m+1}] \cap [t_n, t_{n+1}] = \emptyset$ or $(g_m, A_m) \sim (h_n, B_n)$. Now, since $1 \in [s_m, s_{m+1}] \cap [t_n, t_{n+1}]$, we have $(g_m, A_m) \sim (h_n, B_n)$, and hence $g_m = h_n$ on $A_m \cap B_n$ $(\ni \gamma(1))$.

Conclusion 3.10 Let (f, D) be a function element. Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve. Let $\gamma(0)$ be the center of the disk D. Suppose that $\{D, A_1, \ldots, A_m\}$ is a chain that covers the curve γ. Suppose that (g_m, A_m) is the analytic continuation of (f, D) along the chain $\{D, A_1, \ldots, A_m\}$. That is, there exist function elements $(g_1, A_1), \ldots, (g_m, A_m)$ such that $(f, D) \sim (g_1, A_1)$, $(g_1, A_1) \sim (g_2, A_2), \ldots,$ $(g_{m-1}, A_{m-1}) \sim (g_m, A_m)$. Suppose that $\{D, B_1, \ldots, B_n\}$ is a chain that covers the curve γ. Suppose that (h_n, B_n) is the analytic continuation of (f, D) along the chain $\{D, B_1, \ldots, B_n\}$. That is, there exist function elements $(h_1, B_1), \ldots, (h_n, B_n)$ such that $(f, D) \sim (h_1, B_1)$, $(h_1, B_1) \sim (h_2, B_2), \ldots, (h_{n-1}, B_{n-1}) \sim (h_n, B_n)$. Then

1. for every $i \in \{0, 1, \ldots, m\}$, and, for every $j \in \{0, 1, \ldots, n\}$, if $[s_i, s_{i+1}] \cap [t_j, t_{j+1}] \neq \emptyset$, then $(g_i, A_i) \sim (h_j, B_j)$,
2. $g_m = h_n$ on $A_m \cap B_n$.

This conclusion justifies the following definition:

Definition Let (f_0, D_0) be a function element. Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve. Let $\gamma(0)$ be the center of the disk D_0. Suppose that $\{D_0, D_1, \ldots, D_n\}$ is a chain that covers the curve γ. If (f_n, D_n) is the analytic continuation of (f_0, D_0) along the chain $\{D_0, D_1, \ldots, D_n\}$, then we say that (f_n, D_n) *is an* ***analytic continuation*** *of* (f_0, D_0) *along* γ. We also say that (f_0, D_0) ***admits*** *an analytic continuation along* γ *to* (f_n, D_n).

Conclusion 3.10 can be restated as follows:

Conclusion 3.11 Let (f, D) be a function element. Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve. Let $\gamma(0)$ be the center of the disk D. Suppose that $\{D, A_1, \ldots, A_m\}$ is a chain that covers the curve γ. Suppose that (f, D) admits an analytic continuation along γ to

(g_m, A_m). Suppose that $\{D, B_1, \ldots, B_n\}$ is a chain that covers the curve γ. Suppose that (h_n, B_n) is an analytic continuation of (f, D) along γ to (h_n, B_n). Then $g_m = h_n$ on $A_m \cap B_n$.

Note 3.12

Definition Let X be a topological space. Let $\alpha, \beta \in X$. Let $\varphi : [0,1] \times [0,1] \to X$ be a continuous mapping such that for every $t \in [0,1]$, $\varphi(0,t) = \alpha$, and $\varphi(1,t) = \beta$. Now, for every $t \in [0,1]$, let $\gamma_t : s \mapsto \varphi(s,t)$ be a function from [0,1] to X. Since φ is continuous, each γ_t is continuous, and hence each $\gamma_t : [0,1] \to X$ is a curve in X from α to β. Here, we say that the *curves* γ_t *form a* ***one-parameter family*** $\{\gamma_t\}_{t\in[0,1]}$ ***of curves from*** α ***to*** β ***in*** X.

Let $\alpha, \beta \in \mathbb{C}$. Let $\varphi : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous mapping such that for every $t \in [0,1]$, $\varphi(0,t) = \alpha$, and $\varphi(1,t) = \beta$. For every $t \in [0,1]$, let $\gamma_t : s \mapsto \varphi(s,t)$ be a function from [0,1] to $\mathbb{C}$. In short, $\{\gamma_t\}_{t\in[0,1]}$ is a one-parameter family of curves from α to β in $\mathbb{C}$.

Let D be a disk with center $\alpha(= \varphi(0,t) = \gamma_t(0)$ for every $t \in [0,1])$. Let (f, D) be a function element. Suppose that for every $t \in [0,1]$, (f, D) admits an analytic continuation along γ_t to a function element (g_t, D_t).

Here, (f, D) admits an analytic continuation along γ_0 to a function element (g_0, D_0), and (f, D) admits an analytic continuation along γ_1 to a function element (g_1, D_1). Also, $\gamma_0(0) = \varphi(0,0) = \alpha$, $\gamma_0(1) = \varphi(1,0) = \beta$, and $\gamma_1(0) = \varphi(0,1) = \alpha$, $\gamma_1(1) = \varphi(1,1) = \beta$.

Problem 3.13 $g_0 = g_1$ on $D_0 \cap D_1$.

(Solution Let us fix any $t \in [0,1]$. Since (f, D) admits an analytic continuation along γ_t to a function element (g_t, D_t), there exists a chain $\{A_0 = D, A_1, \ldots, A_{n-1}, A_n = D_t\}$ that covers the curve γ_t, that is

1. $D \cap A_1 \neq \emptyset, A_1 \cap A_2 \neq \emptyset, \ldots, A_{n-1} \cap A_n \neq \emptyset$,
2. there exists a partition $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$ of [0,1] such that
   a. $\gamma_t(1)$ is the center of the disk A_n,
   b. $\gamma_t([s_0, s_1]) \subset D, \gamma_t([s_1, s_2]) \subset A_1, \ldots, \gamma_t([s_{n-1}, s_n]) \subset A_{n-1}$.

Also, there exist function elements $(h_1, A_1), \ldots, (h_n, A_n)$ such that $(f, D) \sim (h_1, A_1)$, $(h_1, A_1) \sim (h_2, A_2), \ldots, (h_{n-1}, A_{n-1}) \sim (h_n, A_n)$, and $h_n = g_t$.

a. **Problem 3.14** There exists $\varepsilon_0 > 0$ such that for every $z \in \gamma_t([s_0, s_1])$, and, for every $w \in D^c$, $\varepsilon_0 < |z - w|$.

(Solution Since γ_t is continuous, and $[s_0, s_1]$ is a compact set, $\gamma_t([s_0, s_1])$ is a compact subset of the open disk D. Here α is the center of disk D. Let $r\,(>0)$ be the radius of D. Thus, $(\gamma_t([s_0, s_1]) \subset) D = D(\alpha; r)$.

Since $\gamma_t([s_0, s_1])$ is compact, and $z \mapsto |z - \alpha|$ is continuous, there exists $s^* \in [s_0, s_1]$ such that for every $s \in [s_0, s_1]$,

$$\underbrace{|\gamma_t(s) - \alpha| \leq |\gamma_t(s^*) - \alpha|} < r.$$

It follows that

$$\gamma_t([s_0, s_1]) \subset D\left(\alpha; \frac{1}{2}(|\gamma_t(s^*) - \alpha| + r)\right) \ (\subset D(\alpha; r) = D).$$

Put $\varepsilon_0 \equiv \frac{1}{2}(r - |\gamma_t(s^*) - \alpha|)$ (>0).

Now, let us take any $z \in \gamma_t([s_0, s_1])$, and $w \in D^c$. We have to show that $\varepsilon_0 < |z - w|$. Since $z \in \gamma_t([s_0, s_1])$, we have $|z - \alpha| \leq |\gamma_t(s^*) - \alpha|$. Since $w \in D^c$, and $D = D(\alpha; r)$, we have $r \leq |w - \alpha|$. It follow that

$$\begin{aligned} |z - w| &\geq |w - \alpha| - |z - \alpha| \geq r - |z - \alpha| \\ &\geq r - |\gamma_t(s^*) - \alpha| > \frac{1}{2}(r - |\gamma_t(s^*) - \alpha|) = \varepsilon_0. \end{aligned}$$

Thus, $\varepsilon_0 < |z - w|$. ∎)

b. Similarly, there exists $\varepsilon_1 > 0$ such that, for every $z \in \gamma_t([s_1, s_2])$, and, for every $w \in (A_1)^c$, $\varepsilon_1 < |z - w|$, etc.

Put $\varepsilon \equiv \min\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n\}$ (>0).

Since $\varphi : [0,1] \times [0,1] \to \mathbb{C}$ is continuous, and $[0,1] \times [0,1]$ is compact, $\varphi : [0,1] \times [0,1] \to \mathbb{C}$ is uniformly continuous. Now, since $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s, u, v, w \in [0,1]$ satisfying $|s - v| < \delta$ and $|u - w| < \delta$, we have $(|\gamma_u(s) - \gamma_w(v)| =)\ |\varphi(s, u) - \varphi(v, w)| < \varepsilon$. It follows that for every $u, v, w \in [0,1]$ satisfying $|u - w| < \delta$, we have $|\gamma_u(v) - \gamma_w(v)| < \varepsilon$, and hence for every $v, w \in [0,1]$ satisfying $|t - w| < \delta$, we have $|\gamma_t(v) - \gamma_w(v)| < \varepsilon$.

Let us take any $w \in (t - \delta, t + \delta) \cap [0,1]$. Now, for every $v \in [0,1]$, we have

$$(|\varphi(v, t) - \varphi(v, w)| =)\ |\gamma_t(v) - \gamma_w(v)| < \varepsilon. \ (*)$$

Problem 3.15 The chain $\{A_0 = D, A_1, \ldots, A_n = D_t\}$ covers the curve γ_w.

(Solution It suffices to show that

a. $\gamma_w(1)$ is the center of the disk A_n,
b. $\gamma_w([s_0, s_1]) \subset D, \gamma_w([s_1, s_2]) \subset A_1, \ldots, \gamma_w([s_{n-1}, s_n]) \subset A_{n-1}$.

For a: Since $(\gamma_w(1) = \varphi(1, w) = \beta = \varphi(1, t) =)\gamma_t(1)$ is the center of the disk A_n, $\gamma_w(1)$ is the center of the disk A_n.

For b: We, first try to show that $\gamma_w([s_0, s_1]) \subset D$. If not, otherwise, suppose that there exists $s \in [s_0, s_1]$ such that $(\varphi(s, w) =)\gamma_w(s) \in D^c$. We have to arrive at a contradiction.

Since $s \in [s_0, s_1]$, we have $(\varphi(s,t) =) \gamma_t(s) \in \gamma_t([s_0, s_1])$, Now, since $\varphi(s,w) \in D^c$, we have, by a, $\varepsilon_0 < |\varphi(s,t) - \varphi(s,w)|$. However, from $(*)$, $|\varphi(s,t) - \varphi(s,w)| < \varepsilon\,(\leq \varepsilon_0)$. This is a contradiction.

Now, we shall try to show that $\gamma_w([s_1, s_2]) \subset A_1$. If not, suppose otherwise that there exists $s \in [s_1, s_2]$ such that $(\varphi(s,w) =)\gamma_w(s) \in (A_1)^c$. We have to arrive at a contradiction.

Since $s \in [s_1, s_2]$, we have $(\varphi(s,t) =) \gamma_t(s) \in \gamma_t([s_1, s_2])$. Now, since $\varphi(s,w) \in (A_1)^c$, we have, by b, $\varepsilon_1 < |\varphi(s,t) - \varphi(s,w)|$. However, from $(*)$, $|\varphi(s,t) - \varphi(s,w)| < \varepsilon\,(\leq \varepsilon_1)$. This is a contradiction, etc. ∎)

Since the chain $\{A_0 = D, A_1, \ldots, A_n = D_t\}$ covers the curve γ_w, by assumption, (g_w, D_w) is the analytic continuation of (f, D) along the chain $\{A_0 = D, A_1, \ldots, A_n = D_t\}$. That is, there exist function elements $(k_1, A_1), \ldots, (h_n, A_n)$ such that $(f,D) \sim (k_1, A_1)$, $(k_1, A_1) \sim (k_2, A_2), \ldots, (k_{n-1}, A_{n-1}) \sim (k_n, A_n)$, $g_w = k_n$, and $D_w = D_t = A_n$.

Also, $(f,D) \sim (h_1, A_1)$, $(h_1, A_1) \sim (h_2, A_2), \ldots, (h_{n-1}, A_{n-1}) \sim (h_n, A_n)$, $g_t = h_n$.

Now, by Conclusion 3.6, $(g_t =)\ h_n = k_n\ (= g_w)$.

c. Thus, for every $t \in [0,1]$, there exists an open interval J_t such that, $t \in J_t$, and, for every $w \in J_t \cap [0,1]$, $g_w = g_t$.

Since $\{J_t : t \in [0,1]\}$ is an open cover of the compact set [0,1], there exist finite-many $t_1, \ldots, t_k$ in [0,1] such that $\{J_{t_1}, \ldots, J_{t_k}\}$ covers [0,1]. Now, by c, $g_0 = g_1$ on $D_0 \cap D_1$. ∎)

Conclusion 3.16 Let $\alpha, \beta \in \mathbb{C}$. Let $\varphi : [0,1] \times [0,1] \to \mathbb{C}$ be a continuous mapping such that, for every $t \in [0,1]$, $\varphi(0,t) = \alpha$ and $\varphi(1,t) = \beta$. For every $t \in [0,1]$, let $\gamma_t : s \mapsto \varphi(s,t)$ be a function from [0,1] to $\mathbb{C}$. In short, $\{\gamma_t\}_{t \in [0,1]}$ is a one-parameter family of curves from α to β in $\mathbb{C}$. Let D be a disk with center $\alpha\ (= \varphi(0,t) = \gamma_t(0)$ for every $t \in [0,1])$. Let (f, D) be a function element. Suppose that for every $t \in [0,1]$, (f, D) admits an analytic continuation along γ_t to a functional element (g_t, D_t). Then $g_0 = g_1$ on $D_0 \cap D_1$.

Note 3.17 Let X be a simply connected space. Let Γ_0 and Γ_1 be curves in X, with common initial point α, and common end point β.

Let us take $[0,\pi]$ as the $\mathrm{dom}(\Gamma_0)$, and $\mathrm{dom}(\Gamma_1)$. Let $\Gamma : s \mapsto \begin{cases} \Gamma_0(s) \text{ if } s \in [0,\pi] \\ \Gamma_1(2\pi - s) \text{ if } s \in [\pi, 2\pi] \end{cases}$ be a well-defined mapping from $[0, 2\pi]$ to X. Clearly, Γ is a closed curve in X.

Since X is simply connected, and Γ is a closed curve in X, Γ is null-homotopic in X, and hence there exists a closed curve $\gamma_1 : [0, 2\pi] \to X$ such that

1. γ_1 is a constant function,
2. Γ and γ_1 are X-homotopic.

Suppose that for every $s \in [0, 2\pi]$, $\gamma_1(s) = c \ (\in X)$. From 2, there exists a mapping $H : [0, 2\pi] \times [0, 1] \to X$ such that

1. H is continuous,
2. for every $s \in [0, 2\pi]$, $H(s, 0) = \Gamma(s)$, and $H(s, 1) = \gamma_1(s) \ (= c)$,
3. for every $t \in [0, 1]$, $H(0, t) = H(2\pi, t)$.

Let $\Phi : re^{i\theta} \mapsto H(\theta, 1 - r)$ be a function from $D[0; 1] \ \left(= \left\{re^{i\theta} : r \in [0, 1], \theta \in [0, 2\pi]\right\}\right)$ to X. Since H is continuous, $\Phi : D[0; 1] \to X$ is continuous.

Observe that for every $t \in [0, 1], \theta \in [0, 2\pi]$, we have $\left((1 - t)e^{i\theta} + te^{-i\theta}\right) \in D[0; 1]$, and hence $\Phi\left((1 - t)e^{i\theta} + te^{-i\theta}\right) \in X$. Let $K : (\theta, t) \mapsto \Phi\left((1 - t)e^{i\theta} + te^{-i\theta}\right)$ be a mapping from $[0, \pi] \times [0, 1]$ to X. Clearly, $K : [0, \pi] \times [0, 1] \to X$ is a continuous map.

Here, for every $t \in [0, 1]$,

$$\begin{aligned} K(0, t) &= \Phi\left((1 - t)e^{i0} + te^{-i0}\right) = \Phi(1) = \Phi\left(1e^{i0}\right) = H(0, 1 - 1) = H(0, 0) \\ &= \Gamma(0) = \Gamma_0(0) = \alpha, \end{aligned}$$

so for every $t \in [0, 1]$, $K(0, t) = \alpha$. Here, for every $t \in [0, 1]$,

$$\begin{aligned} K(\pi, t) &= \Phi\left((1 - t)e^{i\pi} + te^{-i\pi}\right) = \Phi(-1) = \Phi\left(1e^{i\pi}\right) \\ &= H(\pi, 1 - 1) = H(\pi, 0) = \Gamma(\pi) = \Gamma_0(\pi) = \beta, \end{aligned}$$

so for every $t \in [0, 1]$, $K(\pi, t) = \beta$.

Now, for every $t \in [0, 1]$, let $\gamma_t : \theta \mapsto K(\theta, t)$ be a function from $[0, \pi]$ to X. Since K is continuous, each γ_t is continuous, and hence each $\gamma_t : [0, \pi] \to X$ is a curve in X from α to β. Thus, $\{\gamma_t\}_{t \in [0,1]}$ is a one-parameter family of curves from α to β in X.

Problem 3.18 $\gamma_0 = \Gamma_0$, and $\gamma_1 = \Gamma_1$.

(Solution For this purpose, let us take any $\theta \in [0, \pi]$. We have to show that $\gamma_0(\theta) = \Gamma_0(\theta)$, and $\gamma_1(\theta) = \Gamma_1(\theta)$.

$$\begin{aligned} \text{LHS} &= \gamma_0(\theta) = K(\theta, 0) = \Phi\left((1 - 0)e^{i\theta} + 0e^{-i\theta}\right) = \Phi\left(1e^{i\theta}\right) \\ &= H(\theta, 1 - 1) = H(\theta, 0) = \Gamma(\theta) = \Gamma_0(\theta) = \text{RHS}, \end{aligned}$$

and

$$\begin{aligned} \text{LHS} &= \gamma_1(\theta) = K(\theta, 1) = \Phi\left((1 - 1)e^{i\theta} + 1e^{-i\theta}\right) \\ &= \Phi\left(1e^{-i\theta}\right) = \Phi\left(1e^{-i\theta}\right) = \Phi\left(1e^{i(2\pi - \theta)}\right) \\ &= H((2\pi - \theta), 1 - 1) = H((2\pi - \theta), 0) = \Gamma(2\pi - \theta) \\ &= \Gamma_1(2\pi - (2\pi - \theta)) = \Gamma_1(\theta) = \text{RHS}.) \end{aligned}$$

Conclusion 3.19 Let X be a simply connected space. Let Γ_0 and Γ_1 be curves in X, with common initial point α, and common end point β. Then, there exists a one-parameter family $\{\gamma_t\}_{t\in[0,1]}$ of curves from α to β in X such that $\gamma_0 = \Gamma_0$, and $\gamma_1 = \Gamma_1$.

Note 3.20

Definition Let D be an open disk. Let $f \in H(D)$. Let β be a point of the boundary $\partial D\left(\equiv \overline{D} - D^0 = \overline{D} - D\right)$ of D. If there exist a disk D_1 with center at β, and a function $g \in H(D_1)$ such that for every $z \in D \cap D_1, f(z) = g(z)$, then we say that β *is a* ***regular point*** *of f*. If β *is not a* ***regular point*** *of f*, then we say that β *is a* ***singular point*** *of f*.

Let D be an open disk. Let $f \in H(D)$.

Problem 3.21 The collection R of all regular points of f is open in the boundary of D.

(Solution Let $\beta \in R$. By the definition of a regular point, there exists an open disk D_1 with center at β, and a function $g \in H(D_1)$ such that for every $z \in D \cap D_1, f(z) = g(z)$. It suffices to show that every point of $D_1 \cap (\partial D)$ is a regular point of f.

For this purpose, let us take any $\alpha \in D_1 \cap (\partial D)$ $(\subset \partial D)$. We have to show that α is a regular point of f.

Since $\alpha \in D_1 \cap (\partial D)$, we have $\alpha \in D_1$. Now, since D_1 is an open disk, there exists an open disk D_2 with center α such that D_2 is contained in D_1. Since $g \in H(D_1)$, and $D_2 \subset D_1$, we have $g|_{D_2} \in H(D_2)$. It suffices to show that for every $z \in D \cap D_2, f(z) = \left(g|_{D_2}\right)(z)$. For this purpose, let us take any $z \in D \cap D_2$ $(\subset D_2)$ We have to show that $f(z) = \left(g|_{D_2}\right)(z)$ $(= g(z))$. Since $z \in D \cap D_2$ $(\subset D \cap D_1)$, we have $f(z) = g(z)$. ∎)

Conclusion 3.22 Let D be an open disk. Let $f \in H(D)$. Then the collection R of all regular points of f is open in the boundary of D.

Note 3.23 Suppose that the radius of convergence of the power series $\left(a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\right)$ is $r\ (>0)$. Let $f : z \mapsto (a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots)$ be a function from $D(\alpha; r)$ to $\mathbb{C}$. (It follows that $f \in H(D(\alpha; r))$.)

Problem 3.24 There exists a singular point of f.

(Solution If not, suppose otherwise that every point of $\partial(D(\alpha; r))$ $(= \{z : |z-\alpha| < r\})$ is a regular point of f. We have to arrive at a contradiction.

It follows that for every $\beta \in \partial(D(\alpha; r))$, there exists a disk D_β with center at β, and a function $g_\beta \in H(D_\beta)$ such that for every $z \in D \cap D_\beta$, $f(z) = g_\beta(z)$. Here, $\{D_\beta : \beta \in \partial(D(\alpha; r))\}$ is an open cover of the compact set $\partial(D(\alpha; r))$ $(= \{z : |z-\alpha| = r\})$, so there exist finite-many $\beta_1, \ldots, \beta_n$ in $\partial(D(\alpha; r))$ such that $D_{\beta_1} \cup \cdots \cup D_{\beta_n} \supset \{z : |z-\alpha| = r\}$, and $f = g_{\beta_1}$ on $D \cap D_{\beta_1}, \ldots, f = g_{\beta_n}$ on $D \cap D_{\beta_n}$.

Problem 3.25 If $z \in D_{\beta_1} \cap D_{\beta_2}$, then $g_{\beta_1}(z) = g_{\beta_2}(z)$.

(**Solution** Let us take any $z \in D_{\beta_1} \cap D_{\beta_2}$. We have to show that $g_{\beta_1}(z) = g_{\beta_2}(z)$. Since $z \in D_{\beta_1} \cap D_{\beta_2}$, we have $D_{\beta_1} \cap D_{\beta_2} \neq \emptyset$. Now, since $\beta_1, \beta_2 \in \{z : |z - \alpha| < r\}$, D_{β_1} is an open disk with center β_1, and D_{β_2} is an open disk with center β_2, we have $D_{\beta_1} \cap D_{\beta_2} \cap D \neq \emptyset$.

Here, $(g_{\beta_1}, D_{\beta_1}) \sim (f, D)$, $(g_{\beta_2}, D_{\beta_2}) \sim (f, D)$, and $D_{\beta_1} \cap D_{\beta_2} \cap D \neq \emptyset$. By Conclusion 3.6, $(g_{\beta_1}, D_{\beta_1}) \sim (g_{\beta_2}, D_{\beta_2})$, and hence $g_{\beta_1} = g_{\beta_2}$ on $D_{\beta_1} \cap D_{\beta_2}$. Now, since $z \in D_{\beta_1} \cap D_{\beta_2}$, $g_{\beta_1}(z) = g_{\beta_2}(z)$. ∎)

It follows that

$$h : z \mapsto \begin{cases} f(z) \text{ if } z \in D(\alpha; r) \\ g_{\beta_1}(z) \text{ if } z \in D_{\beta_1} \\ \vdots \\ g_{\beta_n}(z) \text{ if } z \in D_{\beta_n}. \end{cases}$$

is a well-defined function from $D(\alpha; r) \cup D_{\beta_1} \cup \cdots \cup D_{\beta_n}$ $(\supset D(\alpha; r) \cup \{z : |z - \alpha| = r\} = D[\alpha; r])$ to $\mathbb{C}$. Since $D_{\beta_1} \cup \cdots \cup D_{\beta_n}$ contains $\{z : |z - \alpha| = r\}$, D_{β_1} is a an open disk with center at $\beta_1, \ldots, D_{\beta_n}$ is a an open disk with center at β_n, there exists $\varepsilon > 0$ such that $(D(\alpha; r + \varepsilon) - D(\alpha; r)) \subset D_{\beta_1} \cup \cdots \cup D_{\beta_n}$, and hence

$$D(\alpha; r + \varepsilon) \subset \left(D(\alpha; r) \cup D_{\beta_1} \cup \cdots \cup D_{\beta_n}\right)(= \mathrm{dom}(h)).$$

Since $f \in H(D(\alpha; r))$, $g_{\beta_1} \in H(D_{\beta_1}), \ldots, g_{\beta_n} \in H(D_{\beta_n})$, by the definition of h, $h \in H\left(D(\alpha; r) \cup D_{\beta_1} \cup \cdots \cup D_{\beta_n}\right)$. It follows, by Conclusion 4.116, that h is representable by power series in $D(\alpha; r) \cup D_{\beta_1} \cup \cdots \cup D_{\beta_n} (\supset D(\alpha; r + \varepsilon))$, and hence there exist complex numbers $d_0, d_1, d_2, d_3, \ldots$ such that for every $w \in D(\alpha; r + \varepsilon)$,

$$h(w) = d_0 + d_1(w - \alpha) + d_2(w - \alpha)^2 + d_3(w - \alpha)^3 + \cdots,$$

It follows that the power series $d_0 + d_1(z - \alpha) + d_2(z - \alpha)^2 + d_3(z - \alpha)^3 + \cdots$ has the radius of convergence $\geq r + \varepsilon$.

Here, for every $w \in D(\alpha; r)$,

$$\begin{aligned} &a_0 + a_1(w - \alpha) + a_2(w - \alpha)^2 + a_3(w - \alpha)^3 + \cdots = f(w) \\ &= \underbrace{h(w) = d_0 + d_1(w - \alpha) + d_2(w - \alpha)^2 + d_3(w - \alpha)^3 + \cdots}. \end{aligned}$$

Now, by Lemma 1.60, $a_0 = d_0$, $a_1 = d_1$, $a_2 = d_2, \ldots$, and hence the power series $a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + a_3(z-\alpha)^3 + \cdots$ has the radius of convergence $\geq r + \varepsilon\ (> r)$. This contradicts the assumption. ∎)

Conclusion 3.26 Suppose that the radius of convergence of the power series $\left(a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\right)$ is $r\ (>0)$. Let $f : z \mapsto (a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots)$ be a function from $D(\alpha; r)$ to $\mathbb{C}$. (Here, $f \in H(D(\alpha; r))$.) Then there exists a singular point of f.

3.2 Ostrowski's Theorem

Note 3.27 Let $\lambda, p_1, p_2, p_3, \ldots, q_1, q_2, q_3, \ldots$ be positive integers. Suppose that $(\lambda+1)p_1 < \lambda q_1$, $(\lambda+1)p_2 < \lambda q_2$, $(\lambda+1)p_3 < \lambda q_3, \ldots$. (It follows that $p_1 < q_1$, $p_2 < q_2$, $p_3 < q_3, \ldots$.) Let $p_1 < q_1 \leq p_2 < q_2 \leq p_3 < \cdots$. Suppose that the radius of convergence of the power series $(a_0 + a_1 z + a_2 z^2 + \cdots)$ is 1. Let

$$f : z \mapsto \left(a_0 + a_1 z + a_2 z^2 + \cdots\right)$$

be a function from $D(0;1)\ (\equiv D)$ to $\mathbb{C}$. Suppose that for every positive integer k, $(p_k < n < q_n \Rightarrow a_n = 0)$. Let $s_n(z)$ be the nth partial sum of the series $a_0 + a_1 z + a_2 z^2 + \cdots$. Let $1\ (\in \partial D)$ be a regular point of f.

Since $1\ (\in \partial D)$ is a regular point of f, there exist a disk D_1 with center at 1, and a function $h \in H(D_1)$ such that for every $z \in D \cap D_1$, $f(z) = h(z)$. Thus, f has a holomorphic extension $f^*\ (= f \cup h)$ to a region $\Omega\ (= D \cup D_1 \supset D \cup \{1\})$. Thus, Ω is a nonempty open subset of $\mathbb{C}$, and $f^* \in H(\Omega)$.

Let

$$\varphi : w \mapsto \frac{1}{2}\left(w^{\lambda} + w^{\lambda+1}\right)$$

be a function from $\mathbb{C}$ to $\mathbb{C}$. Clearly, $\varphi \in H(\mathbb{C})$.

Since Ω is a nonempty open subset of $\mathbb{C}$, and $\varphi \in H(\mathbb{C})$, $\varphi^{-1}(\Omega)$ is an open subset of $\mathbb{C}$. For every $w \in \varphi^{-1}(\Omega)$, we have $\varphi(w) \in \Omega$ and hence $f^*(\varphi(w)) \in \mathbb{C}$. For every $w \in \varphi^{-1}(\Omega)$, put $F(w) \equiv f^*(\varphi(w))$. Thus, $F : \varphi^{-1}(\Omega) \to \mathbb{C}$ is a function. Since $\varphi \in H(\mathbb{C})$, and $f^* \in H(\Omega)$, by the definition of F, $F \in H(\varphi^{-1}(\Omega))$.

Let us first try to show that $(D[0;1] - \{1\}) \subset \varphi^{-1}(D)$. For this purpose, let us take any $w \in (D[0;1] - \{1\})$. We have to show that $\varphi(w) \in D\ (= D(0;1))$, that is $|\varphi(w)| < 1$.

Since $w \in (D[0;1] - \{1\})$, we have $\mathrm{Re}(w) < 1$. Now,

$$|1+w|^2 = 1 + 2\mathrm{Re}(w) + |w|^2 < 1 + 2 \cdot 1 + |w|^2 = 3 + |w|^2 \leq 3 + 1^2 = 4,$$

so $|1+w| < 2$, and hence

$$|\varphi(w)| = \left|\frac{1}{2}\left(w^\lambda + w^{\lambda+1}\right)\right| = \frac{1}{2}|w|^\lambda |1+w| \leq \frac{1}{2}1^\lambda |1+w| = \frac{1}{2}|1+w| < \frac{1}{2} \cdot 2 = 1.$$

Thus, $|\varphi(w)| < 1$. Next, $\varphi(1) = \frac{1}{2}\left(1^\lambda + 1^{\lambda+1}\right) = 1$, so $\{1\} \subset \varphi^{-1}(\{1\})$. Thus,

$$\begin{aligned} D[0;1] &= (D[0;1] - \{1\}) \cup \{1\} \subset \varphi^{-1}(D) \cup \{1\} \subset \varphi^{-1}(D) \cup \varphi^{-1}(\{1\}) \\ &= \varphi^{-1}(D \cup \{1\}) \subset \varphi^{-1}(\Omega), \end{aligned}$$

and hence $D[0;1] \subset \varphi^{-1}(\Omega)$.

Problem 3.28 There exists $\varepsilon > 0$ such that $D(0; 1+\varepsilon) \subset \varphi^{-1}(\Omega)$.

(**Solution** If not, suppose otherwise that for every $\varepsilon > 0$, $D(0; 1+\varepsilon) \not\subset \varphi^{-1}(\Omega)$. We have to arrive at a contradiction.

There exists $z_1 \in D\left(0, 1+\frac{1}{1}\right)$ such that $z_1 \notin \varphi^{-1}(\Omega)$ Now, since $D[0;1] \subset \varphi^{-1}(\Omega)$, we have $z_1 \notin D[0;1]$. It follows that $1 < |z_1| < 1 + \frac{1}{1}$. There exists a positive integer n_1 such that $1 + \frac{1}{n_1} < |z_1|$. There exists $z_2 \in D\left(0, 1+\frac{1}{n_1}\right)$ such that $z_2 \notin \varphi^{-1}(\Omega)$. Now, since $D[0;1] \subset \varphi^{-1}(\Omega)$, we have $z_2 \notin D[0;1]$. It follows that $1 < |z_2| < 1 + \frac{1}{n_1} (< |z_1|)$. There exists a positive integer n_2 such that $1 + \frac{1}{n_2} < |z_2|$. There exists $z_3 \in D\left(0, 1+\frac{1}{n_2}\right)$ such that $z_3 \notin \varphi^{-1}(\Omega)$. Now, since $D[0;1] \subset \varphi^{-1}(\Omega)$, we have $z_3 \notin D[0;1]$. It follows that $1 < |z_3| < 1 + \frac{1}{n_2} (< |z_2|)$, etc.

Thus we get an infinite bounded sequence $\{z_n\}$ in $\left(\varphi^{-1}(\Omega)\right)^c$ such that $\lim_{n\to\infty}|z_n| = 1$. Since $\{z_n\}$ is an infinite bounded sequence, there exists a convergent subsequence $\{z_{n_k}\}$ of $\{z_n\}$, and hence there exists $a \in \mathbb{C}$ such that $\lim_{k\to\infty} z_{n_k} = a$. It follows that

$$1 = \lim_{n\to\infty}|z_n| = \underbrace{\lim_{k\to\infty}|z_{n_k}| = |a|},$$

and hence $a \in D[0;1] (\subset \varphi^{-1}(\Omega))$ This shows that $a \in \varphi^{-1}(\Omega)$.

Since $\{z_{n_k}\}$ is a sequence in the closed set $\left(\varphi^{-1}(\Omega)\right)^c$, and $\lim_{k\to\infty} z_{n_k} = a$, we have $a \in \left(\varphi^{-1}(\Omega)\right)^c$. This is a contradiction. ∎)

Since $F \in H(\varphi^{-1}(\Omega))$, by Conclusion 1.116, F is representable by power series in $\varphi^{-1}(\Omega)$ $(\supset D(0; 1+\varepsilon))$, and hence there exist complex numbers $d_0, d_1, d_2, d_3, \ldots$ such that for every $w \in D(0; 1+\varepsilon)$,

$$F(w) = d_0 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots.$$

It follows that the power series $d_0 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$ has the radius of convergence $\geq 1 + \varepsilon$.

Here, for every $w \in D(0;1)$ $(\subset (D[0;1] - \{1\}) \subset \varphi^{-1}(D))$, we have $\varphi(w) \in D$, and hence for every $w \in D(0;1)$,

$$\begin{aligned}
&\big(d_0 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots = F(w) =\big) f^*(\varphi(w)) \\
&= f(\varphi(w)) = a_0 + a_1 \varphi(w) + a_2 (\varphi(w))^2 + \cdots \\
&= a_0 + a_1 \varphi(w) + a_2 (\varphi(w))^2 + \cdots + a_{p_1} (\varphi(w))^{p_1} \\
&\quad + a_{q_1} (\varphi(w))^{q_1} + \cdots + a_{p_2} (\varphi(w))^{p_2} + a_{q_2} (\varphi(w))^{q_2} + \cdots.
\end{aligned}$$

Now, by Lemma 1.60, and $(\lambda+1)p_1 < \lambda q_1$, we have

$$a_0 + a_1 \varphi(w) + a_2 (\varphi(w))^2 + \cdots + a_{p_1} (\varphi(w))^{p_1} = d_0 + d_1 w + d_2 w^2 + \cdots + d_{(\lambda+1)p_1} w^{(\lambda+1)p_1},$$

$$a_0 + a_1 \varphi(w) + a_2 (\varphi(w))^2 + \cdots + a_{p_1} (\varphi(w))^{p_1} = \sum_{n=1}^{p_1} a_n (\varphi(w))^n = s_{p_1}(\varphi(w)),$$

and

$$d_0 + d_1 w + d_2 w^2 + \cdots + d_{(\lambda+1)p_1} w^{(\lambda+1)p_1} = \sum_{m=0}^{(\lambda+1)p_1} d_m w^m,$$

and hence for every $w \in D(0;1)$,

$$s_{p_1}(\varphi(w)) = \sum_{m=0}^{(\lambda+1)p_1} d_m w^m.$$

Similarly, for every $w \in D(0;1)$,

$$s_{p_2}(\varphi(w)) = \sum_{m=0}^{(\lambda+1)p_2} d_m w^m, \text{ etc.}$$

Since the power series

$$d_0 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$$

has the radius of convergence $\geq 1 + \varepsilon$, for every $w \in D(0; 1 + \varepsilon)$,

$$\left\{s_{p_k}(\varphi(w))\right\} = \underbrace{\left\{\sum\nolimits_{m=0}^{(\lambda+1)p_k} d_m w^m\right\}}$$

is convergent, and hence for every $w \in D(0; 1+\varepsilon)$, $\left\{s_{p_k}(\varphi(w))\right\}$ is convergent. It follows that for every $z \in \varphi(D(0; 1+\varepsilon))$, $\left\{s_{p_k}(z)\right\}$ is a convergent sequence.

By Conclusion 1.189, $\varphi(D(0; 1+\varepsilon))$ is an open neighborhood of $\varphi(1)$ $(= 1)$.

Conclusion 3.29 Let $\lambda, p_1, p_2, p_3, \ldots, q_1, q_2, q_3, \ldots$ be positive integers. Suppose that

$$(\lambda+1)p_1 < \lambda q_1,\ (\lambda+1)p_2 < \lambda q_2,\ (\lambda+1)p_3 < \lambda q_3, \ldots.$$

(It follows that $p_1 < q_1$, $p_2 < q_2$, $p_3 < q_3, \ldots$.)

Let $p_1 < q_1 \le p_2 < q_2 \le p_3 < \cdots$. Suppose that the radius of convergence of the power series $(a_0 + a_1 z + a_2 z^2 + \cdots)$ is 1. Let

$$f : z \mapsto \left(a_0 + a_1 z + a_2 z^2 + \cdots\right)$$

be a function from $D(0; 1)$ $(\equiv D)$ to $\mathbb{C}$. Suppose that

$$(p_1 < n < q_1) \Rightarrow a_n = 0,\ (p_2 < n < q_2) \Rightarrow a_n = 0, \text{ etc.}$$

Let $s_n(z)$ be the n-th partial sum of the series $a_0 + a_1 z + a_2 z^2 + \cdots$. Let 1 $(\in \partial D)$ be a regular point of f. Then there exists an open neighborhood V of 1 such that, for every $z \in V$, $\left\{s_{p_k}(z)\right\}$ is a convergent sequence.

This result, known as the **Ostrowski's theorem**, is due to A. M. Ostrowski (25.09.1893–20.11.1986).

3.3 Hadamard's Theorem

Note 3.30 Let $\lambda, p_1, p_2, p_3, \ldots$ be positive integers. Suppose that

$$\left(1+\frac{1}{\lambda}\right)p_1 < p_2,\ \left(1+\frac{1}{\lambda}\right)p_2 < p_3,\ \left(1+\frac{1}{\lambda}\right)p_3 < p_4, \ldots.$$

(It follows that $p_1 < p_2 < p_3 < \cdots$.) Suppose that the radius of convergence of the power series

$$\left(a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots\right)$$

is 1. Let

$$f : z \mapsto \left(a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots\right)$$

be a function from $D(0;1)$ $(\equiv D)$ to $\mathbb{C}$. (Here, $f \in H(D(0;1))$.)

Problem 3.31 Every point of ∂D is a singular point of f.

(**Solution** If not, suppose otherwise that there exists $\beta\,(\in \partial D)$ such that β is a regular point of f. For simplicity, suppose that $\beta = 1$. We have to arrive at a contradiction.

Put $q_1 \equiv p_2,\ q_2 \equiv p_3,\ q_3 \equiv p_4, \ldots$. Now, all the assumptions of Conclusion 3.29 are satisfied. Hence there exists an open neighborhood V of 1 such that for every $z \in V$, $\{s_{p_k}(z)\}$ is a convergent sequence where $s_{p_k}(z)$ denotes the p_k-th partial sum of the power series $a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots$. Hence, for every $z \in V$, the sequence $\{s_{p_1}(z), s_{p_2}(z), s_{p_3}(z), \ldots\}$ becomes

$$\{a_1 z^{p_1}, a_1 z^{p_1} + a_2 z^{p_2}, a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3}, \ldots\}.$$

Thus, for every $z \in V$, the series $a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots$ is convergent. Now, since V is an open neighborhood of 1, the radius of convergence of the power series $\left(a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots\right)$ is strictly greater than 1. This is a contradiction. ∎)

Conclusion 3.32 Let $\lambda, p_1, p_2, p_3, \ldots$ be positive integers. Suppose that

$$\left(1 + \frac{1}{\lambda}\right)p_1 < p_2,\ \left(1 + \frac{1}{\lambda}\right)p_2 < p_3,\ \left(1 + \frac{1}{\lambda}\right)p_3 < p_4, \ldots.$$

Suppose that the radius of convergence of the power series $\left(a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots\right)$ is 1. Let

$$f : z \mapsto \left(a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots\right)$$

be a function from $D(0;1)$ $(\equiv D)$ to $\mathbb{C}$. (Here, $f \in H(D(0;1))$.) Then every point of ∂D is a singular point of f.

Definition Let $f \in H(D(0;1))$. If every point of $\partial(D(0;1))$ is a singular point of f, then we say that $\partial(D(0;1))$ *is the* ***natural boundary*** *of* f.
Now, the above conclusion can be restated as the following:

Conclusion 3.33 Let $\lambda, p_1, p_2, p_3, \ldots$ be positive integers. Suppose that

$$\left(1 + \frac{1}{\lambda}\right)p_1 < p_2,\ \left(1 + \frac{1}{\lambda}\right)p_2 < p_3,\ \left(1 + \frac{1}{\lambda}\right)p_3 < p_4, \ldots.$$

Suppose that the radius of convergence of the power series

$$\left(a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots\right)$$

is 1. Let

$$f : z \mapsto \left(a_1 z^{p_1} + a_2 z^{p_2} + a_3 z^{p_3} + \cdots\right)$$

be a function from $D(0;1)$ ($\equiv D$) to $\mathbb{C}$. (Here, $f \in H(D(0;1))$.) Then, ∂D is the natural boundary of f.

This result, known as the **Hadamard's theorem**, is due to J. S. Hadamard (08.12.1865–17.10.1963).

3.4 Modular Function

Note 3.34 Let G be the set of all linear fractional transformations $\varphi_{(a,b,c,d)} : \mathbb{S}^2 \to \mathbb{S}^2$, where a, b, c, d are integers satisfying $ad - bc = 1$.

Thus, for every $\varphi_{(a,b,c,d)} \in G$, when $c \neq 0$,

$$\varphi_{(a,b,c,d)}(z) \equiv \begin{cases} \frac{az+b}{cz+d}\ (\neq \infty) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{a}{c}\ (\neq \infty) & \text{if } z = \infty, \end{cases}$$

and when $c = 0$,

$$\varphi_{(a,b,c,d)}(z) \equiv \begin{cases} \frac{az+b}{d}\ (\neq \infty) & \text{if } z \in C \\ \infty & \text{if } z = \infty. \end{cases}$$

Since a, b, c, d are real numbers, $\varphi_{(a,b,c,d)}$ sends real axis onto itself.

Problem 3.35 $\mathrm{Im}\left(\varphi_{(a,b,c,d)}(i)\right) > 0$.

(Solution Case I: when $c \neq 0$. Here,

$$\mathrm{Im}\left(\varphi_{(a,b,c,d)}(i)\right) = \mathrm{Im}\left(\frac{ai+b}{ci+d}\right) = \mathrm{Im}\left(\frac{(ai+b)(-ci+d)}{c^2+d^2}\right) = \frac{ad-bc}{c^2+d^2}$$
$$= \frac{1}{c^2+d^2} > 0.$$

Case II: when $c = 0$. Here, $\mathrm{Im}\left(\varphi_{(a,b,c,d)}(i)\right) = \mathrm{Im}\left(\frac{ai+b}{d}\right) = \frac{a}{d} = 1 > 0.$ ∎)

Problem 3.36 For every $\varphi_{(a,b,c,d)} \in G$, $\varphi_{(a,b,c,d)}(\Pi^+) = \Pi^+$, where $\Pi^+ \equiv \{x + iy : x, y \in \mathbb{R}, \text{ and } y > 0\}$. (Here Π^+ is called the *open* ***upper half plane***.)

(**Solution** Let us take any $\varphi_{(a,b,c,d)} \in G$ where a, b, c, d are integers satisfying $ad - bc = 1$. Let us take any $(x + iy) \in \Pi^+$ where $x, y \in \mathbb{R}$, and $y > 0$. We shall show that $\operatorname{Im}\left(\varphi_{(a,b,c,d)}(x+iy)\right) > 0$.

Case I: when $c \neq 0$. We have

$$\operatorname{Im}\left(\varphi_{(a,b,c,d)}(x+iy)\right) = \operatorname{Im}\left(\frac{a(x+iy)+b}{c(x+iy)+d}\right) = \frac{(ax+b)(-cy)+ay(cx+d)}{(cx+d)^2+(cy)^2}$$
$$= \frac{(ad-bc)y}{(cx+d)^2+(cy)^2} = \frac{y}{(cx+d)^2+(cy)^2} > 0.$$

Case II: when $c = 0$. We have

$$\operatorname{Im}\left(\varphi_{(a,b,c,d)}(x+iy)\right) = \operatorname{Im}\left(\frac{a(x+iy)+b}{d}\right) = \frac{a}{d}y = y > 0.$$

∎)

Thus, $\varphi_{(a,b,c,d)}(\Pi^+) \subset \Pi^+$. Further we get the formula:

$$\operatorname{Im}\left(\varphi_{(a,b,c,d)}(z)\right) = \frac{1}{|cz+d|^2}\operatorname{Im}(z).$$

Similarly, $\varphi_{(a,b,c,d)}(\Pi^-) \subset \Pi^-$, where $\Pi^- \equiv \{x + iy : x, y \in \mathbb{R}, \text{ and } y < 0\}$.

(**Definition** Here Π^- is called the *open* ***lower half plane***.)

Now, since $\varphi_{(a,b,c,d)}$ sends real axis onto itself, and $\varphi_{(a,b,c,d)} : \mathbb{S}^2 \to \mathbb{S}^2$ is 1-1, onto, we have

$$\varphi_{(a,b,c,d)}(\Pi^+) = \Pi^+,\ \varphi_{(a,b,c,d)}(\Pi^-) = \Pi^-, \text{ and } \varphi_{(a,b,c,d)}(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}.$$

Let $\varphi_{(a,b,c,d)} \in G$, where a, b, c, d are integers satisfying $ad - bc = 1$. It follows that $\varphi_{(d,-b-,c,a)} \in G$.

Problem 3.37 $\varphi_{(d,-b-,c,a)} \circ \varphi_{(a,b,c,d)} = \varphi_{(1,0,0,1)}$.

(**Solution** Case I: when $c = 0$. Here, $\frac{a}{d} = 1$. Next,

$$\varphi_{(a,b,c,d)}(z) \equiv \begin{cases} \frac{az+b}{d}\left(= \frac{a}{d}\left(z + \frac{b}{a}\right) = z + \frac{b}{a}\right) & \text{if } z \in C \\ \infty & \text{if } z = \infty. \end{cases}$$

This shows that $\varphi_{(a,b,c,d)} = \varphi_{\left(1,\frac{b}{a},0,1\right)}$. Similarly, $\varphi_{(d,-b-,c,a)} = \varphi_{\left(1,\frac{-b}{d},0,1\right)} = \varphi_{\left(1,\frac{-b}{a},0,1\right)}$. Hence,

$$\varphi_{(d,-b-,c,a)} \circ \varphi_{(a,b,c,d)} = \varphi_{\left(1,\frac{-b}{a},0,1\right)} \circ \varphi_{\left(1,\frac{b}{a},0,1\right)} = \varphi_{\left(1,\frac{-b}{a}+\frac{b}{a},0,1\right)} = \varphi_{(1,0,0,1)}.$$

Case II: when $c \neq 0$. Next,

$$\varphi_{(a,b,c,d)}(z) \equiv \begin{cases} \frac{az+b}{cz+d} \ (\neq \infty) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{a}{c} \ (\neq \infty) & \text{if } z = \infty, \end{cases}$$

and

$$\varphi_{(d,-b-,c,a)}(z) \equiv \begin{cases} \frac{dz-b}{-cz+a} \ (\neq \infty) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{a}{c}\right\}\right) \\ \infty & \text{if } z = \frac{a}{c} \\ \frac{-d}{c} \ (\neq \infty) & \text{if } z = \infty, \end{cases}$$

so

$$\begin{aligned} \left(\varphi_{(d,-b-,c,a)} \circ \varphi_{(a,b,c,d)}\right)(z) &= \varphi_{(d,-b-,c,a)}\left(\varphi_{(a,b,c,d)}(z)\right) \\ &= \begin{cases} \varphi_{(d,-b-,c,a)}\left(\frac{az+b}{cz+d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \varphi_{(d,-b-,c,a)}(\infty) & \text{if } z = \frac{-d}{c} \\ \varphi_{(d,-b-,c,a)}\left(\frac{a}{c}\right) & \text{if } z = \infty \end{cases} \\ &= \begin{cases} \frac{d\left(\frac{az+b}{cz+d}\right)-b}{-c\left(\frac{az+b}{cz+d}\right)+a} & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \frac{-d}{c} & \text{if } z = \frac{-d}{c} \\ \infty & \text{if } z = \infty \end{cases} \\ &= \begin{cases} \frac{d(az+b)-b(cz+d)}{-c(az+b)+a(cz+d)} & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \frac{-d}{c} & \text{if } z = \frac{-d}{c} \\ \infty & \text{if } z = \infty \end{cases} \\ &= \begin{cases} \frac{z}{1} & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \frac{-d}{c} & \text{if } z = \frac{-d}{c} \\ \infty & \text{if } z = \infty \end{cases} = \varphi_{(1,0,0,1)}(z). \end{aligned}$$

Thus, in all cases, $\varphi_{(d,-b-,c,a)} \circ \varphi_{(a,b,c,d)} = \varphi_{(1,0,0,1)}$. ∎)

This shows that every element of G has an inverse element in G. Further, we get the formula

$$\left(\varphi_{(a,b,c,d)}\right)^{-1} = \varphi_{(d,-b-,c,a)}.$$

Let $\varphi_{(a,b,c,d)} \in G$, where a,b,c,d are integers satisfying $ad - bc = 1$. Let $\varphi_{(e,f,g,h)} \in G$, where e,f,g,h are integers satisfying $eh - fg = 1$.

We shall try to show that $\left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right) \in G$.

Case I: when $c = 0$, and $g = 0$. Here, $ad = 1$, and hence $\frac{a}{d} = 1$. Similarly, $\frac{e}{h} = 1$. Also, $a, d, e, h \in \{-1, 1\}$. Now,

$$\varphi_{(a,b,c,d)}(z) \equiv \begin{cases} \frac{az+b}{d}\left(= z + \frac{b}{d} = \varphi_{\left(1,\frac{b}{d},0,1\right)}(z)\right) & \text{if } z \in \mathbb{C} \\ \infty\left(= \varphi_{\left(1,\frac{b}{d},0,1\right)}(z)\right) & \text{if } z = \infty. \end{cases}$$

This shows that $\varphi_{(a,b,c,d)} = \varphi_{\left(1,\frac{b}{d},0,1\right)}$. Similarly, $\varphi_{(e,f,g,h)} = \varphi_{\left(1,\frac{f}{h},0,1\right)}$. Hence,

$$\begin{aligned} \varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)} &= \varphi_{\left(1,\frac{b}{d}+\frac{f}{h},0,1\right)} = \varphi_{\left(1,\frac{bh+df}{dh},0,1\right)} \\ &= \varphi_{(dh,bh+df,0,dh)}\left(= \varphi_{(ea,eb+df,0,dh)}\right). \end{aligned}$$

Now, since $(dh)(dh) - (bh + df)0 = d^2h^2 = 1 \cdot 1 = 1$, $\varphi_{(ea,eb+df,0,dh)} \in G$, and hence

$$\left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right) \in G.$$

Case II: when $c \neq 0$, and $g = 0$. Here, $\frac{e}{h} = 1$. Also $e, h \in \{-1, 1\}$. It follows that $hc \neq 0$. Now,

$$\begin{aligned} \left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right)(z) &= \varphi_{(e,f,g,h)}\left(\varphi_{(a,b,c,d)}(z)\right) \\ &= \begin{cases} \varphi_{(e,f,0,h)}\left(\frac{az+b}{cz+d}\right) & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \varphi_{(e,f,0,h)}(\infty) & \text{if } z = \frac{-d}{c} \\ \varphi_{(e,f,0,h)}\left(\frac{a}{c}\right) & \text{if } z = \infty \end{cases} \\ &= \begin{cases} \frac{e\left(\frac{az+b}{cz+d}\right)+f}{h} & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{e\left(\frac{a}{c}\right)+f}{h} & \text{if } z = \infty. \end{cases} \\ &= \begin{cases} \frac{(ea+cf)z+(eb+df)}{hcz+hd} & \text{if } z \in \left(\mathbb{C} - \left\{\frac{-d}{c}\right\}\right) \\ \infty & \text{if } z = \frac{-d}{c} \\ \frac{ea+cf}{ch} & \text{if } z = \infty. \end{cases} \\ &= \varphi_{(ea+cf,eb+df,hc,hd)}(z). \end{aligned}$$

So,

$$\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)} = \varphi_{(ea+cf,eb+df,hc,hd)}.$$

Now, since

$$\begin{aligned}(ea+cf)(hd)-(eb+df)(hc) &= h((ea+cf)d-(eb+df)c) = he(ad-bc)\\ &= 1\cdot 1 = 1,\end{aligned}$$

$\varphi_{(ea+cf,eb+df,hc,hd)} \in G$, and hence $\left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right) \in G$.

Case III: when $c=0$, and $g \neq 0$. Here, $\frac{a}{d}=1$. Also $a,d \in \{-1,1\}$. It follows that $ga \neq 0$. Now,

$$\begin{aligned}
\left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right)(z) &= \varphi_{(e,f,g,h)}\left(\varphi_{(a,b,c,d)}(z)\right)\\
&= \begin{cases} \varphi_{(e,f,g,h)}\left(\frac{az+b}{d}\right) & \text{if } z \in C \\ \qquad\qquad = \varphi_{(e,f,g,h)}(\infty) & \text{if } z=\infty \end{cases}\\
&= \begin{cases} \begin{cases} \frac{e\left(\frac{az+b}{d}\right)+f}{g\left(\frac{az+b}{d}\right)+h} & \text{if } \frac{az+b}{d} \in \left(\mathbb{C}-\left\{\frac{-h}{g}\right\}\right) \\ \infty & \text{if } \frac{az+b}{d} = \frac{-h}{g} \end{cases} \\ \varphi_{(e,f,g,h)}(\infty) \text{ if } z=\infty \end{cases}\\
&= \begin{cases} \frac{e\left(\frac{az+b}{d}\right)+f}{g\left(\frac{az+b}{d}\right)+h} & \text{if } \frac{az+b}{d} \in \left(\mathbb{C}-\left\{\frac{-h}{g}\right\}\right) \\ \infty & \text{if } \frac{az+b}{d} = \frac{-h}{g} \\ \frac{e}{g} & \text{if } z=\infty \end{cases}\\
&= \begin{cases} \frac{eaz+(eb+df)}{gaz+(gb+dh)} & \text{if } z \in \left(\mathbb{C}-\left\{\left(\frac{-hd}{g}-b\right)\frac{1}{a}\right\}\right) \\ \infty & \text{if } z = \left(\frac{-hd}{g}-b\right)\frac{1}{a} \\ \frac{e}{g} & \text{if } z=\infty \end{cases}\\
&= \begin{cases} \frac{eaz+(eb+df)}{gaz+(gb+dh)} & \text{if } z \in \left(\mathbb{C}-\left\{-\frac{gb+dh}{ga}\right\}\right) \\ \infty & \text{if } z = -\frac{gb+dh}{ga} \\ \frac{ea}{ga} & \text{if } z=\infty \end{cases}\\
&= \varphi_{(ea,eb+df,ga,gb+dh)}(z).
\end{aligned}$$

So,

$$\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)} = \varphi_{(ea,eb+df,ga,gb+dh)}.$$

Now, since

$$(ea)(gb+dh)-(eb+df)(ga)=(eadh-dfga)=ad(eh-fg)=1\cdot 1=1,$$

$\varphi_{(ea,eb+df,ga,gb+dh)}\in G$, and hence $\left(\varphi_{(e,f,g,h)}\circ\varphi_{(a,b,c,d)}\right)\in G$.

Case IV: when $c\neq 0$, and $g\neq 0$. Now,

$$\begin{aligned}\left(\varphi_{(e,f,g,h)}\circ\varphi_{(a,b,c,d)}\right)(z)&=\varphi_{(e,f,g,h)}\left(\varphi_{(a,b,c,d)}(z)\right)\\&=\begin{cases}\varphi_{(e,f,g,h)}\left(\frac{az+b}{cz+d}\right) & \text{if } z\in\left(\mathbb{C}-\left\{\frac{-d}{c}\right\}\right)\\ \varphi_{(e,f,g,h)}(\infty) & \text{if } z=\frac{-d}{c}\\ \varphi_{(e,f,g,h)}\left(\frac{a}{c}\right) & \text{if } z=\infty.\end{cases}\end{aligned}$$

Subcase I: when $\frac{a}{c}=\frac{-h}{g}$, that is $ag+ch=0$. Here,

$$\begin{aligned}\left(\varphi_{(e,f,g,h)}\circ\varphi_{(a,b,c,d)}\right)(z)&=\begin{cases}\varphi_{(e,f,g,h)}\left(\frac{az+b}{cz+d}\right) & \text{if } z\in\left(\mathbb{C}-\left\{\frac{-d}{c}\right\}\right)\\ \frac{e}{g} & \text{if } z=\frac{-d}{c}\\ \infty & \text{if } z=\infty\end{cases}\\&=\begin{cases}\frac{e\left(\frac{az+b}{cz+d}\right)+f}{g\left(\frac{az+b}{cz+d}\right)+h} & \text{if } z\in\left(\mathbb{C}-\left\{\frac{-d}{c}\right\}\right)\\ \frac{e}{g} & \text{if } z=\frac{-d}{c}\\ \infty & \text{if } z=\infty\end{cases}\\&=\begin{cases}\frac{(ea+cf)z+(eb+df)}{0z+(gb+dh)} & \text{if } z\in\left(\mathbb{C}-\left\{\frac{-d}{c}\right\}\right)\\ \frac{e}{g} & \text{if } z=\frac{-d}{c}\\ \infty & \text{if } z=\infty\end{cases}\\&=\begin{cases}\frac{(ea+cf)z+(eb+df)}{gb+dh} & \text{if } z\in\mathbb{C}\\ \infty & \text{if } z=\infty\end{cases}\\&=\varphi_{(ea+cf,eb+df,0,gb+dh)}(z).\end{aligned}$$

So,

$$\left(\varphi_{(e,f,g,h)}\circ\varphi_{(a,b,c,d)}\right)=\varphi_{(ea+cf,eb+df,0,gb+dh)}.$$

Now, since

$$\begin{aligned}(ea+cf)(gb+dh)-(eb+df)(0) &= abeg+bcfg+adeh+cdfh\\ &= be(-ch)+bcfg+adeh+df(-ag)\\ &= -bc(eh-fg)+ad(eh-fg)\\ &= (ad-bc)(eh-fg)=1\cdot 1=1,\end{aligned}$$

we have

$$\varphi_{(ea+cf,eb+df,0,gb+dh)}\in G,$$

and hence $\left(\varphi_{(e,f,g,h)}\circ\varphi_{(a,b,c,d)}\right)\in G$.

Subcase II: when $\frac{a}{c}\neq\frac{-h}{g}$, that is $ag+ch\neq 0$.

Problem 3.38 $-\frac{bg+dh}{ag+ch}\neq-\frac{d}{c}$.

(**Solution** If not, otherwise, let $-\frac{bg+dh}{ag+ch}=-\frac{d}{c}$. We have to arrive at a contradiction. Here, $\frac{bg+dh}{ag+ch}=\frac{d}{c}$, so $bcg+cdh=adg+cdh$ or, $(g=g1=)g(ad-bc)=0$. Thus, $g=0$. This is a contradiction. ∎)

Now,

$$\begin{aligned}\left(\varphi_{(e,f,g,h)}\circ\varphi_{(a,b,c,d)}\right)(z) &= \begin{cases}\varphi_{(e,f,g,h)}\left(\frac{az+b}{cz+d}\right) & \text{if } z\in\left(\mathbb{C}-\left\{\frac{-d}{c}\right\}\right)\\ \varphi_{(e,f,g,h)}(\infty) & \text{if } z=\frac{-d}{c}\\ \varphi_{(e,f,g,h)}\left(\frac{a}{c}\right) & \text{if } z=\infty.\end{cases}\\
&= \begin{cases}\varphi_{(e,f,g,h)}\left(\frac{az+b}{cz+d}\right) & \text{if } z\in\left(=\mathbb{C}-\left\{\frac{-d}{c}\right\}\right)\\ \frac{e}{g} & \text{if } z=\frac{-d}{c}\\ \frac{e\left(\frac{a}{c}\right)+f}{g\left(\frac{a}{c}\right)+h} & \text{if } z=\infty\end{cases}\\
&= \begin{cases}\begin{cases}\frac{e\left(\frac{az+b}{cz+d}\right)+f}{g\left(\frac{az+b}{cz+d}\right)+h} & \text{if } \frac{az+b}{cz+d}\in\left(\mathbb{C}-\left\{\frac{-h}{g}\right\}\right)\\ \infty & \text{if } \frac{az+b}{cz+d}=\frac{-h}{g}\end{cases} & \text{if } z\in\left(\mathbb{C}-\left\{\frac{-d}{c}\right\}\right)\\ \frac{e}{g} \text{ if } z=\frac{-d}{c} & \\ \frac{ea+cf}{ag+ch} \text{ if } z=\infty & \end{cases}\end{aligned}$$

$$= \begin{cases} \begin{cases} \frac{e\left(\frac{az+b}{cz+d}\right)+f}{g\left(\frac{az+b}{cz+d}\right)+h} \text{ if } z \in \left(\mathbb{C} - \left\{-\frac{bg+dh}{ag+ch}, \frac{-d}{c}\right\}\right) \\ \infty \text{ if } z = -\frac{bg+dh}{ag+ch} \end{cases} \\ \frac{e}{g} \text{ if } z = \frac{-d}{c} \\ \frac{ea+cf}{ag+ch} \text{ if } z = \infty \end{cases}$$

$$= \begin{cases} \frac{(ea+cf)z+(eb+df)}{(ag+ch)z+(bg+dh)} \text{ if } z \in \left(\mathbb{C} - \left\{-\frac{bg+dh}{ag+ch}, -\frac{d}{c}\right\}\right) \\ \infty \text{ if } z = -\frac{bg+dh}{ag+ch} \\ \frac{e}{g} \text{ if } z = \frac{-d}{c} \\ \frac{ea+cf}{ag+ch} \text{ if } z = \infty \end{cases}$$

$$= \begin{cases} \frac{(ea+cf)z+(eb+df)}{(ag+ch)z+(bg+dh)} \text{ if } z \in \left(\mathbb{C} - \left\{-\frac{bg+dh}{ag+ch}\right\}\right) \\ \infty \text{ if } z = -\frac{bg+dh}{ag+ch} \\ \frac{ea+cf}{ag+ch} \text{ if } z = \infty \end{cases}$$

$$= \varphi_{(ea+cf,eb+df,ag+ch,bg+dh)}(z).$$

So,

$$\left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right) = \varphi_{(ea+cf,eb+df,ag+ch,bg+dh)}.$$

Now, since

$$\begin{aligned}(ea+cf)(bg+dh) - (eb+df)(ag+ch) &= abeg + bcfg + adeh + cdfh \\ &\quad - (abeg + adfg + bceh + cdfh) \\ &= -fg(ad-bc) + eh(ad-bc) \\ &= (ad-bc)(eh-fg) = 1 \cdot 1 = 1,\end{aligned}$$

we have $\varphi_{(ea+cf,eb+df,ag+ch,bg+dh)} \in G$, and hence $\left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right) \in G$.

Thus, in all cases, $\left(\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)}\right) \in G$. Further, we get the formula

$$\varphi_{(e,f,g,h)} \circ \varphi_{(a,b,c,d)} = \varphi_{ea+cf,eb+df,ag+ch,bg+dh}.$$

Conclusion 3.39 G is a group under composition of mapping as the binary operation.

Since for every $\varphi_{(a,b,c,d)} \in G$, $\varphi_{(a,b,c,d)}(\Pi^+) = \Pi^+$, it is customary to regard G as a group of transformations on Π^+.

Definition Let $f:\Pi^+\to\mathbb{C}$. Let $f\in H(\Pi^+)$. If, for every $\varphi\in G$ (and hence $\varphi:\Pi^+\to\Pi^+$), $f\circ\varphi=f$, then we say that *f is a* ***modular function***.

Note 3.40

Definition We shall denote $\varphi_{(1,0,2,1)}$ $(\in G)$ by σ, and $\varphi_{(1,2,0,1)}$ $(\in G)$ by τ. The group generated by $\{\sigma,\tau\}$ is denoted by Γ or $[\sigma,\tau]$. The set

$$\left\{\varphi_{(a,b,c,d)}:\varphi_{(a,b,c,d)}\in G,\ a\text{ is odd},\ d\text{ is odd},\ b\text{ is even},\ c\text{ is even}\right\}$$

is denoted by Γ_1. Thus, $\sigma,\tau\in\Gamma_1$.

Problem 3.41 Γ_1 is a subgroup of G.

(**Solution** Let $\varphi_{(2a+1,2b,2c,2d+1)},\varphi_{(2e+1,2f,2g,2h+1)}\in\Gamma_1$, where a,b,c,d,e,f,g,h are integers satisfying $(2a+1)(2d+1)-(2b)(2c)=1$, and $(2e+1)(2h+1)-(2f)(2g)=1$.

It suffices to show that $\varphi_{(2e+1,2f,2g,2h+1)}\circ\left(\left(\varphi_{(2a+1,2b,2c,2d+1)}\right)^{-1}\right)\in\Gamma_1$.

Here,

$$\varphi_{(2e+1,2f,2g,2h+1)}\circ\left(\left(\varphi_{(2a+1,2b,2c,2d+1)}\right)^{-1}\right)=\varphi_{(2e+1,2f,2g,2h+1)}\circ\varphi_{(2d+1,-2b,-2c,2a+1)}$$
$$=\varphi_{((2e+1)(2d+1)+(-2c)(2f),(2e+1)(-2b)+(2a+1)(2f),(2d+1)(2g)+(-2c)(2h+1),(-2b)(2g)+(2a+1)(2h+1))}\in\Gamma_1,$$

so $\varphi_{(2e+1,2f,2g,2h+1)}\circ\left(\left(\varphi_{(2a+1,2b,2c,2d+1)}\right)^{-1}\right)\in\Gamma_1$. ∎)

I. It follows that $\Gamma\subset\Gamma_1$.

Definition The set

$$\left\{z:0<\mathrm{Im}(z),\mathrm{Re}(z)\in[-1,1),z\in\left(D\left(\frac{-1}{2};\frac{1}{2}\right)\right)^c,z\in\left(D\left[\frac{1}{2};\frac{1}{2}\right]\right)^c\right\}(\subset\Pi^+)$$

will be denoted by Q. By drawing the graph of Q, it is easy to see that

$$Q\cap(Q+2)=\emptyset,Q\cap(Q-2)=\emptyset,Q\cap(Q+4)=\emptyset,\text{etc.}$$

Suppose that $\varphi_1,\varphi_2\in\Gamma_1$. Let $\varphi_1\neq\varphi_2$.

Problem 3.42 $\varphi_1(Q)\cap\varphi_2(Q)=\emptyset$, that is

$$\begin{aligned}Q\cap\left((\varphi_1)^{-1}\circ\varphi_2\right)(Q)&=(\varphi_1)^{-1}(\varphi_1(Q))\cap(\varphi_1)^{-1}(\varphi_2(Q))\\&=\underbrace{(\varphi_1)^{-1}(\varphi_1(Q)\cap\varphi_2(Q))=\emptyset}_{},\end{aligned}$$

that is

$$Q \cap \left((\varphi_1)^{-1} \circ \varphi_2 \right)(Q) = \emptyset.$$

(**Solution** Since $\varphi_1 \neq \varphi_2$, $(\varphi_1)^{-1} \circ \varphi_2$ is not the identity transformation. Put $\varphi_{(2a+1,2b,2c,2d+1)} \equiv (\varphi_1)^{-1} \circ \varphi_2$ $(\in \Gamma_1)$, where a,b,c,d are integers satisfying $(2a+1)(2d+1) - (2b)(2c) = 1$. We have to show that $Q \cap \varphi_{(2a+1,2b,2c,2d+1)}(Q) = \emptyset$, that is

$$\left\{ z : 0 < \mathrm{Im}(z), \mathrm{Re}(z) \in [-1,1), z \in \left(D\left(\frac{-1}{2}; \frac{1}{2} \right) \right)^c, z \in \left(D\left[\frac{1}{2}; \frac{1}{2} \right] \right)^c \right\}$$
$$\cap \left\{ \varphi_{(2a+1,2b,2c,2d+1)}(z) : 0 < \mathrm{Im}(z), \mathrm{Re}(z) \in [-1,1), \right.$$
$$\left. z \in \left(D\left(\frac{-1}{2}; \frac{1}{2} \right) \right)^c, z \in \left(D\left[\frac{1}{2}; \frac{1}{2} \right] \right)^c \right\} = \emptyset.$$

If not, suppose otherwise that there exists a complex number $z \equiv x + iy$ such that $z \in Q$, and $\varphi_{(2a+1,2b,2c,2d+1)}(z) \in Q$, Thus, $0 < y, -1 \leq x < 1, 1 \leq |2z + 1|$, $1 < |2z - 1|$. We have to arrive at a contradiction.

Case I: when $c = 0$. Since $(2a+1)(2d+1) - (2b)(2c) = 1$, we have $(2a+1)(2d+1) = 1$, and hence $(2a+1), (2d+1) \in \{-1, 1\}$, and $(2a+1) = (2d+1)$. Hence, for every $w \in \Pi^+$,

$$\left(\left((\varphi_1)^{-1} \circ \varphi_2 \right)(w) = \right) \varphi_{(2a+1,2b,2c,2d+1)}(w) = \varphi_{(2a+1,2b,0,2a+1)}(w) = w \pm 2b.$$

Now, since $(\varphi_1)^{-1} \circ \varphi_2$ is not the identity transformation, $b \neq 0$. Also, $\varphi_{(2a+1,2b,2c,2d+1)}(Q) = Q \pm 2b$. Since b is a nonzero integer, $Q \cap (Q \pm 2b) = \emptyset$, and hence $Q \cap \varphi_{(2a+1,2b,2c,2d+1)}(Q) = \emptyset$. This is a contradiction.

Case II: when $c = 2d + 1$. Since

$$c(2a+1-4b) = (2a+1)c - (2b)(2c) = \underbrace{(2a+1)(2d+1) - (2b)(2c) = 1}_{},$$

we have

$$\left. \begin{array}{c} 2a+1-4b = c = 1 \\ d = 0 \end{array} \right\} \text{ or } \left. \begin{array}{c} 2a+1-4b = c = -1 \\ d = -1 \end{array} \right\}.$$

If $\left. \begin{array}{c} 2a+1-4b = c = 1 \\ d = 0 \end{array} \right\}$, then for every $w \in \Pi^+$,

$$\begin{aligned}\varphi_{(2a+1,2b,2c,2d+1)}(w)-\sigma(w)&=\varphi_{(2a+1,2b,2,1)}(w)-\sigma(w)\\&=\frac{(2a+1)w+2b}{2w+1}-\frac{w}{2w+1}\\&=\frac{2aw+2b}{2w+1}=\frac{2aw+a}{2w+1}=\frac{a(2w+1)}{2w+1}=a=2b.\end{aligned}$$

If $\left.\begin{array}{c}2a+1-4b=c=-1\\ d=-1\end{array}\right\}$, then for every $w\in\Pi^+$,

$$\begin{aligned}\varphi_{(2a+1,2b,2c,2d+1)}(w)-\sigma(w)&=\varphi_{(2a+1,2b,-2,-1)}(w)-\sigma(w)\\&=\frac{(2a+1)w+2b}{-2w-1}-\frac{w}{2w+1}\\&=-\frac{2(a+1)w+2b}{2w+1}=-2\frac{(a+1)w+b}{2w+1}\\&=-2\frac{2bw+b}{2w+1}=-2b.\end{aligned}$$

So, in all situations, for every $w\in\Pi^+$, $\varphi_{(2a+1,2b,2c,2d+1)}(w)=\sigma(w)+2m$, where m is an integer. Since for every $w\in Q$,

$$\left|\sigma(w)-\frac{1}{2}\right|=\left|\frac{w}{2w+1}-\frac{1}{2}\right|=\left|\frac{-1}{2(2w+1)}\right|=\frac{1}{2}\frac{1}{|2w+1|}\le\frac{1}{2}\cdot 1=\frac{1}{2},$$

we have

$$\varphi_{(2a+1,2b,2c,2d+1)}(Q)-2m=\underbrace{\sigma(Q)\subset D\left[\frac{1}{2};\frac{1}{2}\right]},$$

and hence

$$\varphi_{(2a+1,2b,2c,2d+1)}(Q)\subset D\left[\frac{1}{2};\frac{1}{2}\right]+2m=D\left[\frac{1}{2}+2m;\frac{1}{2}\right].$$

Now, since $Q\cap\varphi_{(2a+1,2b,2c,2d+1)}(Q)\ne\varnothing$, we have $Q\cap D\left[\frac{1}{2}+2m;\frac{1}{2}\right]\ne\varnothing$, and hence

$$\begin{aligned}(\varnothing=)\Big\{w:0<\mathrm{Im}(w),\mathrm{Re}(w)\in[-1,1),w\in\left(D\left(\frac{-1}{2};\frac{1}{2}\right)\right)^c,&\\ w\in\left(D\left[\frac{1}{2};\frac{1}{2}\right]\right)^c\Big\}\cap D\left[\frac{1}{2}+2m;\frac{1}{2}\right]\ne\varnothing.&\end{aligned}$$

This is a contradiction.

Case III: when $c \neq 2d+1$, and $c \neq 0$. Here, $\frac{-(2d+1)}{2c} \neq \frac{-1}{2}$.

Problem 3.43 For every $w \in Q$, $1 < |(2c)w + (2d+1)|$.

(Solution If not, otherwise suppose that there exists $w \in Q$ such that $|(2c)w + (2d+1)| \leq 1$. We have to arrive at a contradiction.

Since

$$2|c|\left|w - \frac{-(2d+1)}{2c}\right| = \underbrace{|(2c)w + (2d+1)| \leq 1},$$

we have $w \in D\left[\frac{-(2d+1)}{2c}; \frac{1}{2|c|}\right]$, and hence $w \in Q \cap D\left[\frac{-(2d+1)}{2c}; \frac{1}{2|c|}\right]$. Thus, the closed disk $D\left[\frac{-(2d+1)}{2c}; \frac{1}{2|c|}\right]$, with center at the real number $\frac{-(2d+1)}{2c}$ $\left(\neq \frac{-1}{2}\right)$, intersects Q, and hence by the graph of region Q, $D\left(\frac{-(2d+1)}{2c}; \frac{1}{2|c|}\right)$ contains at least one of the points –1,0,1. It follows that

$$\left|-1 - \frac{-(2d+1)}{2c}\right| < \frac{1}{2|c|} \text{ or } \left|0 - \frac{-(2d+1)}{2c}\right| < \frac{1}{2|c|} \text{ or } \left|1 - \frac{-(2d+1)}{2c}\right| < \frac{1}{2|c|},$$

that is

$$|-2c + (2d+1)| < 1 \text{ or } |2d+1| < 1 \text{ or } |2c + (2d+1)| < 1.$$

These are impossible, because c, d are integers. ∎)

It follows that

$$\frac{\mathrm{Im}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right)}{\mathrm{Im}(z)} = \underbrace{\frac{1}{(|(2c)z + (2d+1)|)^2} < 1},$$

and hence

$$\mathrm{Im}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right) < \mathrm{Im}(z).$$

Problem 3.44 For every $w \in Q$, $1 \leq |(-2c)w + (2a+1)|$.

(Solution If not, otherwise suppose that there exists $w \in Q$ such that $|(-2c)w + (2a+1)| < 1$. We have to arrive at a contradiction Since

$$2|c|\left|w - \frac{(2a+1)}{2c}\right| = \underbrace{|(-2c)w + (2a+1)| < 1},$$

we have $w \in D\left(\frac{(2a+1)}{2c}; \frac{1}{2|c|}\right)$, and hence $w \in Q \cap D\left(\frac{(2a+1)}{2c}; \frac{1}{2|c|}\right)$.

Thus, the open disk $D\left(\frac{(2a+1)}{2c};\frac{1}{2|c|}\right)$, with center at the real number $\frac{(2a+1)}{2c}$, intersects Q, and hence by the graph of region Q, $D\left(\frac{(2a+1)}{2c};\frac{1}{2|c|}\right)$ contains at least one of the points –1,0,1. It follows that $\left|-1-\frac{(2a+1)}{2c}\right|<\frac{1}{2|c|}$ or $\left|0-\frac{(2a+1)}{2c}\right|<\frac{1}{2|c|}$ or $\left|1-\frac{(2a+1)}{2c}\right|<\frac{1}{2|c|}$, that is $|2c+(2a+1)|<1$ or $|2a+1|<1$ or $|2c-(2a+1)|<1$. These are impossible, because c,a are integers. ∎)

It follows that

$$\frac{\mathrm{Im}(z)}{\mathrm{Im}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right)}=\frac{\mathrm{Im}\left(\left(\varphi_{(2a+1,2b,2c,2d+1)}\right)^{-1}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right)\right)}{\mathrm{Im}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right)}$$
$$=\frac{\mathrm{Im}\left(\varphi_{(2d+1,-2b,-2c,2a+1)}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right)\right)}{\mathrm{Im}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right)}=\frac{1}{\underbrace{\left(\left|(-2c)\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right)+(2a+1)\right|\right)^2}}\leq 1,$$

and hence

$$\mathrm{Im}(z)\leq \mathrm{Im}\left(\varphi_{(2a+1,2b,2c,2d+1)}(z)\right).$$

This is a contradiction.

So, in all cases we get a contradiction. ∎)

Conclusion 3.45 Suppose that $\varphi_1,\varphi_2\in\Gamma_1$. Let $\varphi_1\neq\varphi_2$. Then, $\varphi_1(Q)\cap\varphi_2(Q)=\emptyset$.

On using the result I, and Conclusion 3.45, we get the following

Conclusion 3.46 Suppose that $\varphi_1,\varphi_2\in\Gamma$. Let $\varphi_1\neq\varphi_2$. Then $\varphi_1(Q)\cap\varphi_2(Q)=\emptyset$.

Note 3.47 Let Γ and Q be as above.

I. Problem 3.48 $\bigcup_{\varphi\in\Gamma}\varphi(Q)\subset\Pi^+$.

(Solution Since for every $\varphi\in\Gamma$ ($\subset G$), $\varphi:\Pi^+\to\Pi^+$ and $Q\subset\Pi^+$, for every $\varphi\in\Gamma$, $\varphi(Q)\subset\Pi^+$, and hence $\bigcup_{\varphi\in\Gamma}\varphi(Q)\subset\Pi^+$. ∎)

II. Problem 3.49 $\underbrace{\Pi^+\subset\bigcup_{\varphi\in\Gamma}\varphi(Q)}=\bigcup_{\varphi\in[\sigma,\tau]}\varphi(Q)$

$$=\bigcup\left\{\left(\underbrace{\alpha\circ\beta\circ\cdots\circ\gamma}_{\text{finite}-\text{many}}\right)(Q):\alpha,\beta,\ldots,\gamma\in\left\{\sigma,\tau,\sigma^{-1},\tau^{-1}\right\}\right\}$$

(Solution Since $\left(\varphi_{(1,2,0,1)} =\right)\tau : w \mapsto w+2$ is a 1-1 function from Π^+ onto Π^+, $\left(\varphi_{(1,-2,0,1)} =\right)\tau^{-1} : w \mapsto w-2$ is a 1-1 function from Π^+ onto Π^+, and hence for every integer n, $(\Gamma = [\sigma,\tau] \ni)\, \tau^n : w \mapsto (w+2n)$ is a 1-1 function from Π^+ onto Π^+. It follows that for every integer n, $\tau^n(Q) = Q+2n$, and hence

$$\bigcup_{\varphi\in\Gamma} \varphi(Q) \supset \bigcup_{n\in\mathbb{Z}} \tau^n(Q) = \bigcup_{n\in\mathbb{Z}} (Q+2n).$$

Thus, $\bigcup_{n\in\mathbb{Z}} (Q+2n) \subset \bigcup_{\varphi\in\Gamma} \varphi(Q)$.

a. Problem 3.50 σ sends the circle $\left\{w : w \in \Pi^+, \left|w - \frac{-1}{2}\right| = \frac{1}{2}\right\}$ $(\subset Q)$ onto the circle $\left\{w : w \in \Pi^+, \left|w - \frac{1}{2}\right| = \frac{1}{2}\right\}$.

(Solution Let us take any w satisfying $w \in \Pi^+, \left|w - \frac{-1}{2}\right| = \frac{1}{2}$. We have to show that

$$\frac{1}{2|2w+1|} = \left|\frac{w}{2w+1} - \frac{1}{2}\right| = \left|\sigma(w) - \frac{1}{2}\right| = \frac{1}{2},$$

that is $|2w+1| = 1$. Since $\left|w - \frac{-1}{2}\right| = \frac{1}{2}$, we have $|2w+1| = 1$. Thus, σ sends the circle $\left\{w : w \in \Pi^+, \left|w - \frac{-1}{2}\right| = \frac{1}{2}\right\}$ to the circle $\left\{w : w \in \Pi^+, \left|w - \frac{1}{2}\right| = \frac{1}{2}\right\}$.

Onto-ness: Let us take any w satisfying $w \in \Pi^+, \left|w - \frac{1}{2}\right| = \frac{1}{2}$, that is $|2w-1| = 1$. Since $w \in \Pi^+$, we have

$$\frac{w}{-2w+1} = \varphi_{(1,0,-2,1)}(w) \in \Pi^+.$$

Since

$$\left|\frac{w}{-2w+1} - \frac{-1}{2}\right| = \frac{1}{2|2w-1|} = \frac{1}{2\cdot 1} = \frac{1}{2},$$

and

$$\begin{aligned}\sigma\left(\frac{w}{-2w+1}\right) &= \varphi_{(1,0,2,1)}\left(\frac{w}{-2w+1}\right) = \varphi_{(1,0,2,1)}\left(\varphi_{(1,0,-2,1)}(w)\right) \\ &= \varphi_{(1,0,2,1)}\left(\left(\varphi_{(1,0,2,1)}\right)^{-1}(w)\right) = w,\end{aligned}$$

σ sends the circle $\left\{w : w \in \Pi^+, \left|w - \frac{-1}{2}\right| = \frac{1}{2}\right\}$ onto the circle $\left\{w : w \in \Pi^+, \left|w - \frac{1}{2}\right| = \frac{1}{2}\right\}$. ∎)

It follows that for every integer n, $(\tau^n \circ \sigma)$ sends

$$\left\{w : w \in \Pi^+, \left|w - \frac{-1}{2}\right| = \frac{1}{2}\right\}$$

onto

$$\left\{w : w \in \Pi^+, \left|w - \left(\frac{1}{2} + 2n\right)\right| = \frac{1}{2}\right\},$$

and hence for every integer n,

$$(\tau^n \circ \sigma)(Q) \supset \left\{w : w \in \Pi^+, \left|w - \left(\frac{1}{2} + 2n\right)\right| = \frac{1}{2}\right\}.$$

Here, for every integer n, $(\tau^n \circ \sigma) \in [\sigma, \tau]$ $(= \Gamma)$, so

$$\bigcup_{\varphi\in\Gamma} \varphi(Q) \supset \bigcup_{n\in\mathbb{Z}} (\tau^n \circ \sigma)(Q) \supset \bigcup_{n\in\mathbb{Z}} \left\{w : w \in \Pi^+, \left|w - \left(\frac{1}{2} + 2n\right)\right| = \frac{1}{2}\right\},$$

and hence

$$\bigcup\nolimits_{n\in\mathbb{Z}} \left\{w : w \in \Pi^+, \left|w - \left(\frac{1}{2} + 2n\right)\right| = \frac{1}{2}\right\} \subset \bigcup\nolimits_{\varphi\in\Gamma} \varphi(Q).$$

Now, since $\bigcup_{n\in\mathbb{Z}} (Q + 2n) \subset \bigcup_{\varphi\in\Gamma} \varphi(Q)$, we have

$$\begin{aligned}&\left\{w : w \in \Pi^+, \frac{1}{2} \le \left|w - \frac{2n+1}{2}\right| \text{ for every } n \in \mathbb{Z}\right\}\\ &= \underbrace{\left(\bigcup_{n\in\mathbb{Z}} (Q + 2n)\right) \cup \left(\bigcup_{n\in\mathbb{Z}} \left\{w : w \in \Pi^+, \left|w - \left(\frac{1}{2} + 2n\right)\right| = \frac{1}{2}\right\}\right) \subset \bigcup_{\varphi\in\Gamma} \varphi(Q),}\end{aligned}$$

and hence

$$\left\{w : w \in \Pi^+, \frac{1}{2} \le \left|w - \frac{2n+1}{2}\right| \text{ for every } n \in \mathbb{Z}\right\} \subset \bigcup_{\varphi\in\Gamma} \varphi(Q).$$

Problem 3.51 σ^{-1} sends $\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{1}{2}\right|\right\}$ $(\subset Q)$ onto $\left\{w : w \in \Pi^+, \left|w - \frac{-1}{2}\right| < \frac{1}{2}\right\}$.

(Solution Let us take any w satisfying $w \in \Pi^+, \frac{1}{2} < \left|w - \frac{1}{2}\right|$. We have to show that

$$\frac{1}{2|2w-1|} = \left|\frac{w}{-2w+1} - \frac{-1}{2}\right| = \left|\sigma^{-1}(w) - \frac{-1}{2}\right| < \frac{1}{2},$$

that is $1 < |2w-1|$. Since $\frac{1}{2} < \left|w - \frac{1}{2}\right|$, we have $1 < |2w-1|$. Thus, σ sends

$$\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{1}{2}\right|\right\}$$

to

$$\left\{w : w \in \Pi^+, \left|w - \frac{-1}{2}\right| < \frac{1}{2}\right\}.$$

Onto-ness: Let us take any w satisfying $w \in \Pi^+, \left|w - \frac{-1}{2}\right| < \frac{1}{2}$, that is $|2w+1| < 1$. Since $w \in \Pi^+$, we have

$$\frac{w}{2w+1} = \underbrace{\varphi_{(1,0,2,1)}(w) \in \Pi^+}.$$

Since

$$\left|\frac{w}{2w+1} - \frac{1}{2}\right| = \frac{1}{2|2w+1|} > \frac{1}{2 \cdot 1} = \frac{1}{2},$$

and

$$\begin{aligned}\sigma^{-1}\left(\frac{w}{2w+1}\right) &= \varphi_{(1,0,-2,1)}\left(\frac{w}{2w+1}\right) = \varphi_{(1,0,-2,1)}\left(\varphi_{(1,0,2,1)}(w)\right) \\ &= \left(\varphi_{(1,0,2,1)}\right)^{-1}\left(\varphi_{(1,0,2,1)}(w)\right) = w,\end{aligned}$$

σ^{-1} sends $\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{1}{2}\right|\right\}$ $(\subset Q)$ onto $\left\{w : w \in \Pi^+, \left|w - \frac{-1}{2}\right| < \frac{1}{2}\right\}$. ∎)

Now, for every integer n, $(\tau^n \circ \sigma^{-1})$ sends $\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{1}{2}\right|\right\}$ $(\subset Q)$ onto $\left\{w : w \in \Pi^+, \left|w - \left(\frac{-1}{2} + 2n\right)\right| < \frac{1}{2}\right\}$, and hence for every integer n,

$$\left(\tau^n \circ \sigma^{-1}\right)(Q) \supset \left\{w : w \in \Pi^+, \left|w - \left(\frac{-1}{2} + 2n\right)\right| < \frac{1}{2}\right\}.$$

Here, for every integer n, $(\tau^n \circ \sigma^{-1}) \in [\sigma, \tau]$ $(= \Gamma)$, so

$$\begin{aligned}\bigcup\nolimits_{\varphi \in \Gamma} \varphi(Q) &\supset \bigcup\nolimits_{n \in \mathbb{Z}} \left(\tau^n \circ \sigma^{-1}\right)(Q) \\ &\supset \bigcup\nolimits_{n \in \mathbb{Z}} \left\{w : w \in \Pi^+, \left|w - \left(\frac{-1}{2} + 2n\right)\right| < \frac{1}{2}\right\},\end{aligned}$$

and hence

b. $\bigcup_{n\in\mathbb{Z}} \left\{w : w \in \Pi^+, \left|w - \left(\frac{-1}{2} + 2n\right)\right| < \frac{1}{2}\right\} \subset \bigcup_{\varphi\in\Gamma} \varphi(Q)$.

Problem 3.52 σ sends $\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{-1}{2}\right|\right\}$ $(\subset Q)$ onto $\left\{w : w \in \Pi^+, \left|w - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

(Solution Let us take any w satisfying $w \in \Pi^+, \frac{1}{2} < \left|w - \frac{-1}{2}\right|$. We have to show that

$$\frac{1}{2|2w+1|} = \left|\frac{w}{2w+1} - \frac{1}{2}\right| = \underbrace{\left|\sigma(w) - \frac{1}{2}\right| < \frac{1}{2}},$$

that is $1 < |2w+1|$. Since $\frac{1}{2} < \left|w - \frac{-1}{2}\right|$, we have $1 < |2w+1|$. Thus, σ sends $\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{-1}{2}\right|\right\}$ $(\subset Q)$ to $\left\{w : w \in \Pi^+, \left|w - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

Onto-ness: Let us take any w satisfying $w \in \Pi^+, \left|w - \frac{1}{2}\right| < \frac{1}{2}$, that is $|2w-1| < 1$. Since $w \in \Pi^+$, we have

$$\frac{w}{-2w+1} = \underbrace{\varphi_{(1,0,-2,1)}(w) \in \Pi^+}.$$

Since

$$\left|\frac{w}{-2w+1} - \frac{-1}{2}\right| = \frac{1}{2|2w-1|} > \frac{1}{2\cdot 1} = \frac{1}{2},$$

and

$$\begin{aligned}\sigma\left(\frac{w}{-2w+1}\right) &= \varphi_{(1,0,2,1)}\left(\frac{w}{-2w+1}\right) = \varphi_{(1,0,2,1)}\left(\varphi_{(1,0,-2,1)}(w)\right)\\ &= \varphi_{(1,0,2,1)}\left(\left(\varphi_{(1,0,2,1)}\right)^{-1}(w)\right) = w,\end{aligned}$$

σ sends $\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{-1}{2}\right|\right\}$ $(\subset Q)$ onto $\left\{w : w \in \Pi^+, \left|w - \frac{1}{2}\right| < \frac{1}{2}\right\}$. ∎)

Now, for every integer n, $(\tau^n \circ \sigma)$ sends $\left\{w : w \in \Pi^+, \frac{1}{2} < \left|w - \frac{-1}{2}\right|\right\}$ $(\subset Q)$ onto $\left\{w : w \in \Pi^+, \left|w - \left(\frac{1}{2} + 2n\right)\right| < \frac{1}{2}\right\}$, and hence for every integer n,

$$(\tau^n \circ \sigma)(Q) \supset \left\{w : w \in \Pi^+, \left|w - \left(\frac{1}{2} + 2n\right)\right| < \frac{1}{2}\right\}.$$

Here, for every integer n, $(\tau^n \circ \sigma) \in [\sigma, \tau]\,(= \Gamma)$, so

$$\bigcup_{\varphi\in\Gamma} \varphi(Q) \supset \bigcup_{n\in\mathbb{Z}} (\tau^n \circ \sigma)(Q) \supset \bigcup_{n\in\mathbb{Z}} \left\{ w : w \in \Pi^+, \left| w - \left(\frac{1}{2} + 2n\right) \right| < \frac{1}{2} \right\},$$

and hence

c. $\bigcup_{n\in\mathbb{Z}} \left\{ w : w \in \Pi^+, \left| w - \left(\frac{1}{2} + 2n\right) \right| < \frac{1}{2} \right\} \subset \bigcup_{\varphi\in\Gamma} \varphi(Q)$.

It follows that

$$\begin{aligned} (\Pi^+ =) & \left\{ w : w \in \Pi^+, \frac{1}{2} \le \left| w - \frac{2n+1}{2} \right| \text{ for every } n \in \mathbb{Z} \right\} \\ & \cup \left(\bigcup_{n\in\mathbb{Z}} \left\{ w : w \in \Pi^+, \left| w - \left(\frac{-1}{2} + 2n\right) \right| < \frac{1}{2} \right\} \right) \\ & \cup \left(\bigcup_{n\in\mathbb{Z}} \left\{ w : w \in \Pi^+, \left| w - \left(\frac{1}{2} + 2n\right) \right| < \frac{1}{2} \right\} \right) \subset \bigcup_{\varphi\in\Gamma} \varphi(Q), \end{aligned}$$

and hence $\Pi^+ \subset \bigcup_{\varphi\in\Gamma} \varphi(Q)$. ∎)

Conclusion 3.53 Let Γ and Q be as above. Then $\bigcup_{\varphi\in\Gamma} \varphi(Q) = \Pi^+$.

Theorem 3.54 Let Γ and Γ_1 be as above. Then $\Gamma = \Gamma_1$.

Proof By Note 3.40, $\Gamma \subset \Gamma_1$. It remains to show that $\Gamma_1 \subset \Gamma$. If not, suppose otherwise that there exists $\varphi \in \Gamma_1$ such that $\varphi \notin \Gamma$. We have to arrive at a contradiction.

Here, for every $\psi \in \Gamma\ (\subset \Gamma_1)$, we have $\varphi, \psi \in \Gamma_1$, and $\varphi \neq \psi$. Hence, for every $\psi \in \Gamma$, by Conclusion 3.45, $\varphi(Q) \cap \psi(Q) = \emptyset$, and hence

$$\begin{aligned} \varphi\left(\frac{-1}{2} + i\frac{1}{2}\right) \in \varphi(Q) = \varphi(Q) \cap \Pi^+ = \varphi(Q) \cap \left(\bigcup_{\psi\in\Gamma} \psi(Q) \right) \\ = \underbrace{\bigcup_{\psi\in\Gamma} (\varphi(Q) \cap \psi(Q)) = \emptyset}. \end{aligned}$$

This gives a contradiction. ∎

3.5 Harnack's Theorem

Note 3.55 Let Ω be a region. For every positive integer n, suppose that $u_n : \Omega \to \mathbb{R}$ is a harmonic function. Let $u : \Omega \to \mathbb{R}$ be a function. Suppose that $\{u_n\}$ converges to u uniformly on compact subsets of Ω.

Problem 3.56 u is harmonic in Ω.

(**Solution** For this purpose, let us take any $a \in \Omega$. Now, since Ω is open, there exists a positive real number R such that $D[a;R] \subset \Omega$. It suffices to show that u is harmonic in $D(a;R)$.

For every positive integer n, $u_n : \Omega \to \mathbb{R}$ is a harmonic function, so, by Conclusion 1.289, for every positive integer n and for every $\left(a + re^{i\theta}\right) \in D(a;R)$ satisfying $r \in [0, R)$,

$$u_n\left(a + re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u_n\left(a + Re^{it}\right) \mathrm{d}t.$$

Now, since $\{u_n\}$ converges to u uniformly on compact subsets of Ω, for every $\left(a + re^{i\theta}\right) \in D(a;R)$ satisfying $r \in [0, R)$, we have

$$\begin{aligned} u\left(a + re^{i\theta}\right) &= \underbrace{\lim_{n\to\infty} u_n\left(a + re^{i\theta}\right) = \lim_{n\to\infty}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u_n\left(a + Re^{it}\right) \mathrm{d}t\right)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \left(\lim_{n\to\infty} u_n\left(a + Re^{it}\right)\right) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u\left(a + Re^{it}\right) \mathrm{d}t, \end{aligned}$$

and hence for every $\left(a + re^{i\theta}\right) \in D(a;R)$ satisfying $r \in [0, R)$, we have

$$u\left(a + re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u\left(a + Re^{it}\right) \mathrm{d}t.$$

Now, by Conclusion 1.286, it suffices to show that u is continuous on $\{z : |z - a| = R\}$.

Here, each u_n is harmonic, so each u_n is continuous. Since $\{z : |z - a| = R\}$ is a compact subset of Ω, and $\{u_n\}$ converges to u uniformly on compact subsets of Ω, $\{u_n\}$ converges to u uniformly on $\{z : |z - a| = R\}$. Now, since each u_n is continuous, $(u =)\ \lim_{n\to\infty} u_n$ is continuous on $\{z : |z - a| = R\}$, and hence u is continuous on $\{z : |z - a| = R\}$. ∎)

Conclusion 3.57 Let Ω be a region. For every positive integer n, suppose that $u_n : \Omega \to \mathbb{R}$ is a harmonic function. Let $u : \Omega \to \mathbb{R}$ be a function. Suppose that $\{u_n\}$ converges to u uniformly on compact subsets of Ω. Then u is harmonic in Ω.

Note 3.58 Let Ω be a region. For every positive integer n, suppose that $u_n : \Omega \to \mathbb{R}$ is a harmonic function. Let $0 \le u_1 \le u_2 \le u_3 \le \cdots$.

For every $z \in \Omega$, put $u(z) = \lim_{n\to\infty} u_n(z)$. Thus, $u : \Omega \to [0, \infty]$.

Let us take any $a \in \Omega$.

Since Ω is open, there exists a positive real number R such that $D[a; R] \subset \Omega$. Since for every real φ, $-1 \le \cos\varphi \le 1$, for every $r \in [0, R)$, we have $-2Rr \le -2Rr\cos\varphi \le 2Rr$, and hence for every $r \in [0, R)$, we have

$$\left(0 < (R-r)^2 = \right)\left(R^2 + r^2\right) - 2Rr \le \left(R^2 + r^2\right) - 2Rr\cos\varphi \le \left(R^2 + r^2\right) + 2Rr\left(= (R+r)^2\right).$$

It follows that for every $r \in [0, R)$, and for every real φ,

$$\frac{1}{(R+r)^2} \le \frac{1}{(R^2 + r^2) - 2Rr\cos\varphi} \le \frac{1}{(R-r)^2},$$

and hence for every $r \in [0, R)$, and for every real φ,

$$\frac{R-r}{R+r} = \underbrace{\frac{(R^2 - r^2)}{(R+r)^2} \le \frac{(R^2 - r^2)}{(R^2 + r^2) - 2Rr\cos\varphi} \le \frac{(R^2 - r^2)}{(R-r)^2}} = \frac{R+r}{R-r}.$$

Thus, for every $r \in [0, R)$, for every real θ, for every real t, and for every positive integer n, we have

$$\frac{R-r}{R+r} u_n\left(a + Re^{it}\right) \le \frac{(R^2 - r^2)}{R^2 - 2Rr\cos(\theta - t) + r^2} u_n\left(a + Re^{it}\right) \le \frac{R+r}{R-r} u_n\left(a + Re^{it}\right),$$

and hence for every $r \in [0, R)$, for every real θ, and for every positive integer n,

$$\begin{aligned}
\frac{R-r}{R+r}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(R^2 - 0^2)}{R^2 - 2R0\cos(\theta - t) + 0^2} u_n\left(a + Re^{it}\right)\mathrm{d}t &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R-r}{R+r} u_n\left(a + Re^{it}\right)\mathrm{d}t \\
\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(R^2 - r^2)}{R^2 - 2Rr\cos(\theta - t) + r^2} u_n\left(a + Re^{it}\right)\mathrm{d}t &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R+r}{R-r} u_n\left(a + Re^{it}\right)\mathrm{d}t \\
= \frac{R+r}{R-r}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(R^2 - 0^2)}{R^2 - 2R0\cos(\theta - t) + 0^2} u_n\left(a + Re^{it}\right)\mathrm{d}t.
\end{aligned}$$

Hence, for every $r \in [0, R)$, for every real θ, and for every positive integer n,

$$\frac{R-r}{R+r} u_n\left(a+0e^{i\theta}\right) \leq u_n\left(a+re^{i\theta}\right) \leq \frac{R+r}{R-r} u_n\left(a+0e^{i\theta}\right),$$

that is, for every $r \in [0, R)$, for every real θ, and, for every positive integer n,

$$\frac{R-r}{R+r} u_n(a) \leq u_n\left(a+re^{i\theta}\right) \leq \frac{R+r}{R-r} u_n(a).$$

It follows that for every $r \in [0, R)$, for every real θ,

$$\lim_{n\to\infty} \frac{R-r}{R+r} u_n(a) \leq \lim_{n\to\infty} u_n\left(a+re^{i\theta}\right) \leq \lim_{n\to\infty} \frac{R+r}{R-r} u_n(a),$$

that is, for every $r \in [0, R)$, for every real θ,

$$\frac{R-r}{R+r} u(a) \leq u\left(a+re^{i\theta}\right) \leq \frac{R+r}{R-r} u(a).$$

Thus, for every $a \in \Omega$, there exists an open neighborhood $D(a; R)$ of a such that for every $z \in D(a; R)$,

$$\frac{R-|z-a|}{R+|z-a|} u(a) \leq u(z) \leq \frac{R+|z-a|}{R-|z-a|} u(a) \quad (*).$$

It follows that every point of $\{z : z \in \Omega \text{ and } u(z) < \infty\}$ is an interior point, and that every point of $\{z : z \in \Omega \text{ and } u(z) = \infty\}$ is an interior point of $\{z : z \in \Omega \text{ and } u(z) = \infty\}$. Thus, $\{z : z \in \Omega \text{ and } u(z) < \infty\}$, and $\{z : z \in \Omega \text{ and } u(z) = \infty\}$ are both open sets. Also,

$$\{z : z \in \Omega \text{ and } u(z) < \infty\} \cup \{z : z \in \Omega \text{ and } u(z) = \infty\} = \Omega,$$

and

$$\{z : z \in \Omega \text{ and } u(z) < \infty\} \cap \{z : z \in \Omega \text{ and } u(z) = \infty\} = \emptyset.$$

Now, since Ω is connected,

$$\{z : z \in \Omega \text{ and } u(z) < \infty\} = \Omega \text{ or } \{z : z \in \Omega \text{ and } u(z) = \infty\} = \Omega.$$

Conclusion 3.59 Let Ω be a region. For every positive integer n, suppose that $u_n : \Omega \to \mathbb{R}$ is a harmonic function. Let $0 \le u_1 \le u_2 \le u_3 \le \cdots$. For every $z \in \Omega$, put $u(z) = \lim_{n\to\infty} u_n(z)$ $(\in [0,\infty])$. Then,

$$\{z : z \in \Omega \text{ and } u(z) < \infty\} = \Omega \text{ or } \{z : z \in \Omega \text{ and } u(z) = \infty\} = \Omega.$$

Let Ω be a region. For every positive integer n, suppose that $u_n : \Omega \to \mathbb{R}$ is a harmonic function. Let $0 \le u_1 \le u_2 \le u_3 \le \cdots$. For every $z \in \Omega$, put $u(z) = \lim_{n\to\infty} u_n(z)$ $(\in [0,\infty])$.

Then, by Conclusion 3.59, $\{z : z \in \Omega \text{ and } u(z) < \infty\} = \Omega$ or $\{z : z \in \Omega \text{ and } u(z) = \infty\} = \Omega$.

Case I: when $\{z : z \in \Omega \text{ and } u(z) = \infty\} \ne \Omega$. It follows that $\{z : z \in \Omega \text{ and } u(z) < \infty\} = \Omega$, and hence for every $z \in \Omega$, $\lim_{n\to\infty} u_n(z) < \infty$.

Problem 3.60 u is harmonic in Ω.

(Solution For this purpose, let us take any $a \in \Omega$. Since Ω is open, there exists a positive real number R such that $D[a;R] \subset \Omega$. It suffices to show that u is harmonic in $D(a;R)$.

For every positive integer n, $u_n : \Omega \to \mathbb{R}$ is a harmonic function, so, by Conclusion 1.289, for every positive integer n and for every $\left(a + re^{i\theta}\right) \in D(a;R)$ satisfying $r \in [0,R)$,

$$u_n\left(a + re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u_n\left(a + Re^{it}\right) \mathrm{d}t.$$

By Conclusions 1.266, 1.267, and Theorem 1.125, Vol. 1, for every $\left(a + re^{i\theta}\right) \in D(a;R)$ satisfying $r \in [0,R)$, we have

$$\begin{aligned}\left(a + re^{i\theta}\right) = \lim_{n\to\infty} u_n\left(a + re^{i\theta}\right) &= \lim_{n\to\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u_n\left(a + Re^{it}\right)\mathrm{d}t\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\lim_{n\to\infty} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u_n\left(a + Re^{it}\right)\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u\left(a + Re^{it}\right)\mathrm{d}t,\end{aligned}$$

and hence for every $\left(a + re^{i\theta}\right) \in D(a;R)$ satisfying $r \in [0,R)$, we have

$$u\left(a + re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u\left(a + Re^{it}\right) \mathrm{d}t.$$

From the inequality Note 3.58(∗), it is clear that u is continuous in Ω. It follows that u is continuous on $\{z : |z-a| = R\}$. Now, by Conclusion 1.286, u is harmonic in $D(a;R)$. ■)

Since u is continuous in Ω, by a theorem (cf. WR[5], Theorem 7.13), $\{u_n\}$ converges to u uniformly on the compact subsets of Ω.

Case II: when $\{z : z \in \Omega \text{ and } u(z) = \infty\} = \Omega$. It follows that for every $z \in \Omega$, $\lim_{n\to\infty} u_n(z) = \infty$.

Conclusion 3.61 Let Ω be a region. For every positive integer n, suppose that $u_n : \Omega \to \mathbb{R}$ is a harmonic function. Let $0 \le u_1 \le u_2 \le u_3 \le \cdots$. Then, either $\lim_{n\to\infty} u_n(z) = \infty$ for every $z \in \Omega$, or $\{u_n\}$ converges to a harmonic function uniformly on compact subsets Ω.

Theorem 3.62 Let Ω be a region. For every positive integer n, suppose that $u_n : \Omega \to \mathbb{R}$ is a harmonic function. Let $u_1 \le u_2 \le u_3 \le \cdots$. Then, either $\lim_{n\to\infty} u_n(z) = \infty$ for every $z \in \Omega$, or $\{u_n\}$ converges to a harmonic function uniformly on compact subsets Ω.

Proof Since $u_1 \le u_2 \le u_3 \le \cdots$, we have

$$0 \le (u_2 - u_1) \le (u_3 - u_1) \le (u_4 - u_1) \le \cdots.$$

Since each $u_n : \Omega \to \mathbb{R}$ is a harmonic function, each $(u_n - u_1) : \Omega \to \mathbb{R}$ is a harmonic function. Now, by Conclusions 3.61 and 3.59, either $\lim_{n\to\infty}(u_n - u_1)(z) = \infty$ for every $z \in \Omega$, or $\{u_n - u_1\}$ converges to a harmonic function uniformly on compact subsets Ω. It follows that either $\lim_{n\to\infty} u_n(z) = \infty$ for every $z \in \Omega$, or $\{u_n\}$ converges to a harmonic function uniformly on compact subsets Ω. ■

Conclusion 3.57 together with Theorem 3.62, known as the **Harnack's theorem**, is due to C. G. A. Harnack (07.05.1851–03.04.1888).

3.6 Nontangential and Radial Maximal Functions

Note 3.63 Definition Let $u : D(0;1) \to \mathbb{C}$ be a function. For every $r \in [0,1)$, the function

$$e^{i\theta} \mapsto u\left(re^{i\theta}\right)$$

from $\{z : |z| = 1\}$ to $\mathbb{C}$ is denoted by u_r. Thus, for every $r \in [0,1)$, $u_r : \{z : |z| = 1\} \to \mathbb{C}$, and for every real θ,

$$u_r\left(e^{i\theta}\right) = u\left(re^{i\theta}\right).$$

Let $p \in [1, \infty)$. Let $f : \{z : |z| = 1\} \to \mathbb{C}$ be a member of $L^p(\{z : |z| = 1\})$. Hence,

$$\left(\|f\|_p =\right) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(e^{it}\right)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} < \infty.$$

Here,

$$\underbrace{P[f] : re^{i\theta} \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f\left(e^{it}\right) \mathrm{d}t}= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} f\left(e^{it}\right) \mathrm{d}t$$

is a function from $D(0;1)$ to $\mathbb{C}$. Put $u \equiv P[f]$. Hence, for every $r \in [0,1)$, $u_r : \{z : |z| = 1\} \to \mathbb{C}$.

Problem 3.64 For every $r \in [0,1)$, $\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_r\left(e^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} \leq \|f\|_p$.

(Solution Here, for every $r \in [0,1)$,

$$u_r\left(e^{i\theta}\right) = u\left(re^{i\theta}\right) = (P[f])\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right) P_r(\theta - t)\,\mathrm{d}t = \int_{\{z:|z|=1\}} f \mathrm{d}\mu_s,$$

where $\mathrm{d}\mu_s \equiv \frac{1}{2\pi} P_r(\theta - t)\mathrm{d}t$. Now,

$$\left|u_r\left(e^{i\theta}\right)\right| = \left|\int_{\{z:|z|=1\}} f \mathrm{d}\mu_s\right| \leq \int_{\{z:|z|=1\}} |f| \mathrm{d}\mu_s$$

so, by Conclusion 2.10, Vol. 1,

$$\left|u_r\left(e^{i\theta}\right)\right|^p \leq \underbrace{\left(\int_{\{z:|z|=1\}} |f| \mathrm{d}\mu_s\right)^p \leq \int_{\{z:|z|=1\}} |f|^p \mathrm{d}\mu_s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(e^{it}\right)\right|^p P_r(\theta - t)\,\mathrm{d}t.$$

Now, for every $r \in [0,1)$,

$$\begin{aligned}
\int\limits_{-\pi}^{\pi} \left|u_r\left(e^{i\theta}\right)\right|^p \mathrm{d}\theta &\le \int\limits_{-\pi}^{\pi} \left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \left|f\left(e^{it}\right)\right|^p P_r(\theta-t)\mathrm{d}t\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(\int\limits_{-\pi}^{\pi} \left|f\left(e^{it}\right)\right|^p P_r(\theta-t)\mathrm{d}\theta\right)\mathrm{d}t \\
&= \int\limits_{-\pi}^{\pi}\left(\left|f\left(e^{it}\right)\right|^p \cdot \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_r(\theta-t)\mathrm{d}\theta\right)\mathrm{d}t \\
&= \int\limits_{-\pi}^{\pi}\left(\left|f\left(e^{it}\right)\right|^p \cdot \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} P_r(t-\theta)\mathrm{d}\theta\right)\mathrm{d}t \\
&= \int\limits_{-\pi}^{\pi}\left(\left|f\left(e^{it}\right)\right|^p \cdot 1\right)\mathrm{d}t = \int\limits_{-\pi}^{\pi}\left|f\left(e^{it}\right)\right|^p \mathrm{d}t,
\end{aligned}$$

so for every $r \in [0,1)$,

$$\left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \left|u_r\left(e^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} \le \left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \left|f\left(e^{it}\right)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} = \|f\|_p. \quad \blacksquare)$$

Conclusion 3.65 Let $p \in [1,\infty)$. Let $f : \{z : |z| = 1\} \to \mathbb{C}$ be a member of $L^p(\{z : |z| = 1\})$. Then, for every $r \in [0,1)$,

1. $(P[f])_r \in L^p(\{z : |z| = 1\})$,
2. $\left\|(P[f])_r\right\|_p \le \|f\|_p$,
3. $\lim_{r\to 1^-}\left\|(P[f])_r - f\right\|_p = 0$.

Proof of the remaining part (3): For this purpose, let us take any $\varepsilon > 0$. By Conclusion 2.50, Vol. 1, $C(\{z : |z| = 1\})$ is dense in $L^p(\{z : |z| = 1\})$ $(\ni f)$, and hence there exists

$$g \in C(\{z : |z| = 1\}) \ (\subset L^p(\{z : |z| = 1\}))$$

such that $\left(\|g - f\|_p =\right) \|f - g\|_p < \varepsilon$. Now, for every $r \in [0,1)$,

$$
\begin{aligned}
\left\|(P[f])_r - f\right\|_p &= \left\|\left((P[f])_r - g\right) + (g-f)\right\|_p \le \left\|(P[f])_r - g\right\|_p \\
&\quad + \|g-f\|_p < \left\|(P[f])_r - g\right\|_p + \varepsilon \\
&= \left\|\left((P[f])_r - (P[g])_r\right) + \left((P[g])_r - g\right)\right\|_p + \varepsilon \le \left\|(P[f])_r - (P[g])_r\right\|_p \\
&\quad + \left\|(P[g])_r - g\right\|_p + \varepsilon \\
&= \left\|(P[f] - P[g])_r\right\|_p + \left\|(P[g])_r - g\right\|_p + \varepsilon = \left\|(P[f-g])_r\right\|_p \\
&\quad + \left\|(P[g])_r - g\right\|_p + \varepsilon \\
&\le \|f-g\|_p + \left\|(P[g])_r - g\right\|_p + \varepsilon < \varepsilon + \left\|(P[g])_r - g\right\|_p + \varepsilon \\
&= \left\|(P[g])_r - g\right\|_p + 2\varepsilon = \left\|(Hg)_r - g\right\|_p + 2\varepsilon \le \left\|(Hg)_r - g\right\|_\infty + 2\varepsilon,
\end{aligned}
$$

where the symbol Hg is as defined in Note 1.271. It suffices to show that $\lim_{r\to 1^-}\left\|(Hg)_r - g\right\|_\infty = 0$.

Since $g \in C(\{z : |z| = 1\})$, by Conclusion 1.275, for every $w \in \{z : |z| = 1\}$, $\lim_{r\to 1^-}\left((Hg)_r - g\right)(w) = 0$, and hence $\lim_{r\to 1^-}\left\|(Hg)_r - g\right\|_\infty = 0$. ∎

Theorem 3.66 Let $f : \{z : |z| = 1\} \to \mathbb{C}$ be a member of $L^\infty(\{z : |z| = 1\})$. Then, for every $r \in [0, 1)$,

1. $(P[f])_r \in L^\infty(\{z : |z| = 1\})$,
2. $\left\|(P[f])_r\right\|_\infty \le f_\infty$.

Proof Here $f : \{z : |z| = 1\} \to \mathbb{C}$ is a member of $L^\infty(\{z : |z| = 1\})$, so $\left(\|f\|_\infty =\right) \sup\{|f(e^{it})| : t \in [-\pi, \pi]\} < \infty$. Next,

$$
\underbrace{P[f] : re^{i\theta} \mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f\left(e^{it}\right) \mathrm{d}t}= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} f\left(e^{it}\right) \mathrm{d}t
$$

is a function from $D(0; 1)$ to $\mathbb{C}$. Put $u \equiv P[f]$. Hence, for every $r \in [0, 1)$, $u_r : \{z : |z| = 1\} \to \mathbb{C}$.

Problem 3.67 For every $r \in [0, 1)$, $\sup\left\{\left|u_r\left(e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} \le \sup\{|f(e^{it})| : t \in [-\pi, \pi]\}$.

(Solution Here, for every $r \in [0, 1)$, and for every $\theta \in [-\pi, \pi]$,

$$\left|u_r\left(e^{i\theta}\right)\right| = \left|u\left(re^{i\theta}\right)\right| = \left|(P[f])\left(re^{i\theta}\right)\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(e^{it}\right)P_r(\theta - t)\mathrm{d}t\right|$$
$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(e^{it}\right)P_r(\theta - t)\right|\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(e^{it}\right)\right||P_r(\theta - t)|\mathrm{d}t$$
$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sup\left\{\left|f\left(e^{it}\right)\right| : t \in [-\pi, \pi]\right\}\right)|P_r(\theta - t)|\mathrm{d}t$$
$$= \left(\sup\left\{\left|f\left(e^{it}\right)\right| : t \in [-\pi, \pi]\right\}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\theta - t)|\mathrm{d}t$$
$$= \left(\sup\left\{\left|f\left(e^{it}\right)\right| : t \in [-\pi, \pi]\right\}\right)\frac{1}{2\pi}\int_{\pi}^{\pi} P_r(\theta - t)\mathrm{d}t$$
$$= \left(\sup\left\{\left|f\left(e^{it}\right)\right| : t \in [-\pi, \pi]\right\}\right)\cdot 1$$
$$= \sup\left\{\left|f\left(e^{it}\right)\right| : t \in [-\pi, \pi]\right\},$$

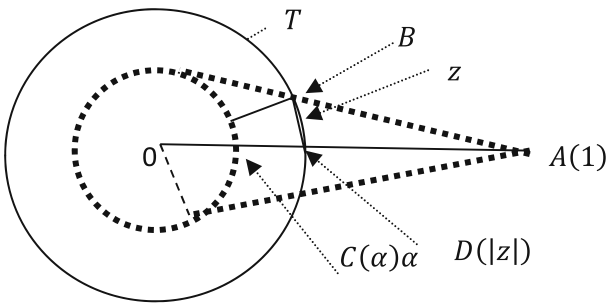
so $\sup\left\{\left|u_r\left(e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} \leq \sup\{|f(e^{it})| : t \in [-\pi, \pi]\}$. ∎)

This proves 1 and 2. ∎

Note 3.68 Definition Let $\alpha \in (0, 1)$. The set $D(0; \alpha) \cup \{(1 - t)1 + tz : t \in (0, 1) \text{ and } |z| < \alpha\}$ is denoted by Ω_α.

Clearly, Ω_α is the smallest set among convex open sets that contain $D(0; \alpha)$ and 1 in its boundary. Observe that, if $\alpha, \beta \in (0, 1)$, and $\alpha \leq \beta$, then $\Omega_\alpha \subset \Omega_\beta$. Further, $\bigcup_{\alpha\in(0,1)} \Omega_\alpha = D(0; 1)$, and $\bigcap_{\alpha\in(0,1)} \Omega_\alpha = [0, 1)$. Also, each Ω_α is a region contained in $D(0; 1)$. For every real t, and, for every $\alpha \in (0, 1)$, $e^{it}\Omega_\alpha (= \{e^{it}z : z \in \Omega_\alpha\})$ is also a region contained in $D(0; 1)$. Actually, each $e^{it}\Omega_\alpha$ is a rotated copy of Ω_α, with vertex e^{it}.

Here, Ω_α is called a ***nontangential approach region with vertex*** 1.



From the above figure, we find that for every $z \in (\Omega_\alpha - D(0;\alpha))$,

$$\frac{|z-|z||}{1-|z|} = \frac{|z-|z||}{DA} \le \frac{DB}{DA} = \frac{\sin\angle DAB}{\sin\angle DBA} = \frac{\sin\angle OAT}{\sin\angle DBA} = \frac{\frac{OT}{OA}}{\sin\angle DBA} = \frac{\frac{\alpha}{1}}{\sin\angle DBA}$$
$$= \frac{\alpha}{\sin\angle DBA} \le \frac{\alpha}{\sin\angle CTA},$$

and for every $z \in D(0;\alpha)$,

$$\frac{|z-|z||}{1-|z|} \le \frac{|z-|z||}{1-\alpha} \le \frac{|z|+|z|}{1-\alpha} = \frac{2|z|}{1-\alpha} \le \frac{2\alpha}{1-\alpha},$$

and hence for every $z \in \Omega_\alpha$,

$$\frac{|z-|z||}{1-|z|} \le \max\left\{\frac{\alpha}{\sin\angle CTA}, \frac{2\alpha}{1-\alpha}\right\} (\equiv \gamma_\alpha, \text{say}).$$

Conclusion 3.69 Let $\alpha \in (0,1)$. Then there exists $\gamma_\alpha > 0$ such that for every $z \in \Omega_\alpha$, $\frac{|z-|z||}{1-|z|} \le \gamma_\alpha$.

Definition Let $\alpha \in (0,1)$. Let $u : D(0;1) \to \mathbb{C}$ be a function. The function

$$e^{it} \mapsto \sup\{|u(w)| : w \in e^{it}\Omega_\alpha\}$$

from $\{z : |z| = 1\}$ to $[0,\infty]$ is denoted by $N_\alpha u$. Thus, $N_\alpha u : \{z : |z| = 1\} \to [0,\infty]$, and for every real t,

$$(N_\alpha u)(e^{it}) = \sup\{|u(e^{it}w)| : w \in \Omega_\alpha\}.$$

Here, $N_\alpha u$ is called the ***nontangential maximal function*** *of* u.

Definition Let $\alpha \in (0,1)$. Let $u : D(0;1) \to \mathbb{C}$ be a function. The function

$$e^{it} \mapsto \sup\{|u(re^{it})| : r \in [0,1)\}$$

from $\{z : |z| = 1\}$ to $[0,\infty]$ is denoted by $M_{\text{rad}}u$. Thus,

$$M_{\text{rad}}u : \{z : |z| = 1\} \to [0,\infty],$$

and for every real t,

$$(M_{\text{rad}}u)(e^{it}) = \sup\{|u(re^{it})| : r \in [0,1)\}.$$

Here, $M_{\text{rad}}u$ is called the ***radial maximal function*** *of* u.

Clearly, for every $z \in D(0;1)$, $(M_{\text{rad}}u)(z) \le (N_\alpha u)(z)$. Also, if $\alpha, \beta \in (0,1)$, and $\alpha \le \beta$, then $N_\alpha u \le N_\beta u(e^{it})$.

Note 3.70 From Note 1.263, let us recall that for every real θ, t, and $r \in [0, 1)$,

$$\underbrace{P_r(\theta - t) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}}$$

$$= \frac{1 - r^2}{1 - 2r(\cos\theta\cos t + \sin\theta\sin t) + r^2}$$
$$= \frac{1 - r^2}{1 - 2((r\cos\theta)\cos t + (rin\theta)\sin t) + r^2}$$
$$= \frac{1 - r^2}{1 - 2(x\cos t + y\sin t) + r^2} = \frac{1 - |z|^2}{1 - 2(x\cos t + y\sin t) + |z|^2}$$
$$= \frac{1 - |z|^2}{1 - 2\mathrm{Re}((x - iy)(\cos t + i\sin t)) + |z|^2} = \frac{1 - |z|^2}{1 - 2\mathrm{Re}(\bar{z}e^{it}) + |z|^2}$$
$$= \frac{1 - |z|^2}{1 - (\bar{z}e^{it} + ze^{-it}) + |z|^2} = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

Notation For every real θ, t, and $r \in [0, 1)$, $P_r(\theta - t)\left(= \frac{1-|z|^2}{|e^{it}-z|^2}\right)$ is also denoted by $P(z, e^{it})$, where $\left(re^{i\theta} =\right) z \in D(0; 1)$, and $e^{it} \in \{w : |w| = 1\}$.

Thus, for every $f : \{w : |w| = 1\} \to \mathbb{C}$ in $L^1(\{w : |w| = 1\})$, and for every $z \in D(0; 1)$,

$$(P[f])(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P\left(z, e^{it}\right)f\left(e^{it}\right)\,\mathrm{d}t.$$

Definition Let μ be a complex measure on $\{w : |w| = 1\}$. The function

$$P[\mathrm{d}\mu] : z \mapsto \int_{\{w:|w|=1\}} P\left(z, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right)$$

from $D(0; 1)$ to $\mathbb{C}$ is called the ***Poisson's integral*** *of* μ. Thus,

$$P[\mathrm{d}\mu] : D(0; 1) \to \mathbb{C},$$

and for every $z \in D(0; 1)$,

$$(P[\mathrm{d}\mu])(z) = \int_{\{w:|w|=1\}} P\left(z, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right),$$

where $P(z, e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$.

Let μ be a complex measure on $\{w : |w| = 1\}$.

I. Problem 3.71 $P[\mathrm{d}\mu] : D(0;1) \to \mathbb{C}$ is harmonic.

(Solution Case I: when μ is a positive measure. Since for every $re^{i\theta} \in D(0;1)$,

$$\begin{aligned}
(P[\mathrm{d}\mu])\left(re^{i\theta}\right) &= \int\limits_{\{w:|w|=1\}} P\left(re^{i\theta}, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right) = \int\limits_{\{w:|w|=1\}} P_r(\theta - t)\mathrm{d}\mu\left(e^{it}\right) \\
&= \int\limits_{\{w:|w|=1\}} \left(1 + \left(r^{|1|}e^{i(\theta-t)1} + r^{|-1|}e^{i(\theta-t)(-1)}\right) + \left(r^{|2|}e^{i(\theta-t)2} + r^{|-2|}e^{i(\theta-t)(-2)}\right) + \cdots\right)\mathrm{d}\mu\left(e^{it}\right) \\
&= \int\limits_{\{w:|w|=1\}} \left(1 + \left(r^1(2\cos(\theta-t))\right) + \left(r^2(2\cos 2(\theta-t))\right) + \cdots\right)\mathrm{d}\mu\left(e^{it}\right) \\
&= \int\limits_{\{w:|w|=1\}} \left(\mathrm{Re}\left(1 + 2\left(\left(re^{i\theta}e^{-it}\right)^1 + \left(re^{i\theta}e^{-it}\right)^2 + \cdots\right)\right)\right)\mathrm{d}\mu\left(e^{it}\right) \\
&= \int\limits_{\{w:|w|=1\}} \mathrm{Re}\left(1 + 2\left(\frac{re^{i\theta}e^{-it}}{1 - re^{i\theta}e^{-it}}\right)\right)\mathrm{d}\mu\left(e^{it}\right) \\
&= \int\limits_{\{w:|w|=1\}} \mathrm{Re}\left(\frac{1 + re^{i\theta}e^{-it}}{1 - re^{i\theta}e^{-it}}\right)\mathrm{d}\mu\left(e^{it}\right) = \int\limits_{\{w:|w|=1\}} \mathrm{Re}\left(\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\right)\mathrm{d}\mu\left(e^{it}\right) \\
&= \mathrm{Re}\left(\int\limits_{\{w:|w|=1\}} \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\mathrm{d}\mu\left(e^{it}\right)\right) = \mathrm{Re}\left(\int\limits_{\{w:|w|=1\}} \left(\frac{2e^{it}}{e^{it} - re^{i\theta}} - 1\right)\mathrm{d}\mu\left(e^{it}\right)\right) \\
&= \mathrm{Re}\left(2\int\limits_{\{w:|w|=1\}} \left(\frac{1}{e^{it} - re^{i\theta}}e^{it}\right)\mathrm{d}\mu\left(e^{it}\right) - \int\limits_{\{w:|w|=1\}} 1\mathrm{d}\mu\left(e^{it}\right)\right)
\end{aligned}$$

we have

$$(P[\mathrm{d}\mu]) : z \mapsto \mathrm{Re}\left(2\int\limits_{\{w:|w|=1\}} \left(\frac{1}{e^{it} - z}e^{it}\right)\mathrm{d}\mu\left(e^{it}\right) - \int\limits_{\{w:|w|=1\}} 1\mathrm{d}\mu\left(e^{it}\right)\right)$$

from $D(0;1)$ to $\mathbb{R}$. By Conclusion 1.66(b), the function

$$z \mapsto \int\limits_{\{w:|w|=1\}} \left(\frac{1}{e^{it} - z}e^{it}\right)\mathrm{d}\mu\left(e^{it}\right)$$

from $D(0;1)$ to $\mathbb{R}$ is representable by power series of z in $D(0;1)$, and hence

$$z \mapsto \left(2\int\limits_{\{w:|w|=1\}} \left(\frac{1}{e^{it} - z}e^{it}\right)\mathrm{d}\mu\left(e^{it}\right) - \int\limits_{\{w:|w|=1\}} 1\mathrm{d}\mu\left(e^{it}\right)\right)$$

from $D(0;1)$ to $\mathbb{R}$ is holomorphic. Now, by Conclusion 1.239,

$$P[\mathrm{d}\mu] : D(0;1) \to \mathbb{R}$$

is a harmonic function in $D(0;1)$.

Case II: when μ is a signed measure. Since

$$P[\mathrm{d}\mu] = P[d(\mu^+ - \mu^-)] = P[d(\mu^+)] - P[d(\mu^-)],$$

and, by Case I, $P[d(\mu^+)], P[d(\mu^-)]$ are harmonic in $D(0;1)$,

$$(P[\mathrm{d}\mu] =)\, P[d(\mu^+)] - P[d(\mu^-)]$$

is harmonic, and hence $P[\mathrm{d}\mu]$ is harmonic.

Case III: when μ is a complex measure. Since

$$P[\mathrm{d}\mu] = P[d(\mathrm{Re}(\mu) + i\,\mathrm{Im}(\mu))] = P[d(\mathrm{Re}(\mu))] + iP[d(\mathrm{Im}(\mu))],$$

and, by Case II, $P[d(\mathrm{Re}(\mu))], P[d(\mathrm{Im}(\mu))]$ are harmonic in $D(0;1)$,

$$(P[\mathrm{d}\mu] =) P[d(\mathrm{Re}(\mu))] + iP[d(\mathrm{Im}(\mu))]$$

is harmonic, and hence $P[\mathrm{d}\mu]$ is harmonic. ∎)

Let $\alpha \in (0,1)$.

II. Problem 3.72 There exists $c_\alpha > 0$ such that for every positive finite Borel measure μ on $\{w : |w| = 1\}$,

$$c_\alpha((N_\alpha(P[\mathrm{d}\mu]))(1)) \leq ((M_{\mathrm{rad}}(P[\mathrm{d}\mu]))(1)).$$

(**Solution** By Conclusion 3.69, there exists $\gamma_\alpha > 0$ such that for every $z \in \Omega_\alpha$, $\frac{|z-|z||}{1-|z|} \leq \gamma_\alpha$. We have to show that there exists $c_\alpha > 0$ such that

$$\begin{aligned} \sup\left\{\left|\int\limits_{\{w:|w|=1\}} c_\alpha \frac{1-|z|^2}{|e^{it}-z|^2}\mathrm{d}\mu\left(e^{it}\right)\right| : z \in \Omega_\alpha\right\} &= \sup\left\{\left|\int\limits_{\{w:|w|=1\}} c_\alpha P\left(z, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right)\right| : z \in \Omega_\alpha\right\} \\ &= \sup\left\{\left|c_\alpha \int\limits_{\{w:|w|=1\}} P\left(z, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right)\right| : z \in \Omega_\alpha\right\} \end{aligned}$$

$$\begin{aligned}
&= \sup\{|c_\alpha(P[\mathrm{d}\mu])(z)| : z \in \Omega_\alpha\} \\
&= \sup\{c_\alpha|(P[\mathrm{d}\mu])(z)| : z \in \Omega_\alpha\} \\
&= c_\alpha \cdot \sup\{|(P[\mathrm{d}\mu])(z)| : z \in \Omega_\alpha\} \\
&= \underbrace{c_\alpha((N_\alpha(P[\mathrm{d}\mu]))(1)) \leq ((M_{\mathrm{rad}}(P[\mathrm{d}\mu]))(1))} \\
&= \sup\{|(P[\mathrm{d}\mu])(r)| : r \in [0,1)\} \\
&= \sup\left\{\left|\int_{\{w:|w|=1\}} P\left(r, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right)\right| : r \in [0,1)\right\} \\
&= \sup\left\{\left|\int_{\{w:|w|=1\}} \frac{1-r^2}{|e^{it}-r|^2}\mathrm{d}\mu\left(e^{it}\right)\right| : r \in [0,1)\right\},
\end{aligned}$$

that is

$$\sup\left\{\left|\int_{\{w:|w|=1\}} c_\alpha \frac{1-|z|^2}{|e^{it}-z|^2}\mathrm{d}\mu\left(e^{it}\right)\right| : z \in \Omega_\alpha\right\} \leq \sup\left\{\left|\int_{\{w:|w|=1\}} \frac{1-r^2}{|e^{it}-r|^2}\mathrm{d}\mu\left(e^{it}\right)\right| : r \in [0,1)\right\}.$$

It suffices to show that there exists $c_\alpha > 0$ such that for every $z \in \Omega_\alpha$ and for every real t,

$$c_\alpha \frac{1-|z|^2}{|e^{it}-z|^2} \leq \frac{1-|z|^2}{|e^{it}-|z||^2},$$

that is for every $z \in \Omega_\alpha$ and for every real t,

$$\left|e^{it}-|z|\right| \leq \frac{1}{\sqrt{c_\alpha}}\left|e^{it}-z\right|.$$

Here, for every $z \in \Omega_\alpha$ and for every real t,

$$\begin{aligned}
\left|e^{it}-|z|\right| &\leq \left|e^{it}-z\right| + |z-|z|| \leq \left|e^{it}-z\right| + \gamma_\alpha(1-|z|) \\
&\leq \left|e^{it}-z\right| + \gamma_\alpha\left|e^{it}-z\right| = (1+\gamma_\alpha)\left|e^{it}-z\right| = \frac{1}{\sqrt{c_\alpha}}\left|e^{it}-z\right|,
\end{aligned}$$

where $c_\alpha \equiv \frac{1}{(1+\gamma_\alpha)^2}$. Thus, for every $z \in \Omega_\alpha$ and for every real t,

$$\left|e^{it}-|z|\right| \leq \frac{1}{\sqrt{c_\alpha}}\left|e^{it}-z\right|. \quad \blacksquare)$$

III. Problem 3.73 There exists $c_\alpha > 0$ such that for every positive finite Borel measure μ on $\{w : |w| = 1\}$, and for every real θ,

$$c_\alpha\big((N_\alpha(P[\mathrm{d}\mu]))\big(e^{i\theta}\big)\big) \le \big((M_{\mathrm{rad}}(P[\mathrm{d}\mu]))\big(e^{i\theta}\big)\big).$$

(Solution Let us fix any real θ. Let $\mu_\theta : E \mapsto \mu\big(e^{i\theta}E\big)$ be the positive finite Borel measure on $\{w : |w| = 1\}$. By II, there exists $c_\alpha > 0$ such that

$$c_\alpha\big((N_\alpha(P[\mathrm{d}\mu]))\big(e^{i\theta}\big)\big) = \underbrace{c_\alpha((N_\alpha(P[\mathrm{d}\mu_\theta]))(1)) \le ((M_{\mathrm{rad}}(P[\mathrm{d}\mu_\theta]))(1))}$$
$$= (M_{\mathrm{rad}}(P[\mathrm{d}\mu]))\big(e^{i\theta}\big),$$

and hence

$$c_\alpha\big((N_\alpha(P[\mathrm{d}\mu]))\big(e^{i\theta}\big)\big) \le (M_{\mathrm{rad}}(P[\mathrm{d}\mu]))\big(e^{i\theta}\big).$$

∎)

Conclusion 3.74 Let $\alpha \in (0,1)$. Then there exists $c_\alpha > 0$ such that for every positive finite Borel measure μ on $\{w : |w| = 1\}$, and for every real θ,

1. $c_\alpha\big((N_\alpha(P[\mathrm{d}\mu]))\big(e^{i\theta}\big)\big) \le (M_{\mathrm{rad}}(P[\mathrm{d}\mu]))\big(e^{i\theta}\big)$,
2. $(M_{\mathrm{rad}}(P[\mathrm{d}\mu]))\big(e^{i\theta}\big) \le \sup\Big\{\frac{\mu(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } e^{i\theta}, \text{ and } I \subset \{w : |w| = 1\}\Big\}$.

<u>Proof of the remaining part</u> <u>(2)</u>: It suffices to show that

$$\sup\left\{\left|\int\limits_{\{w:|w|=1\}} \frac{1-r^2}{|e^{it}-r|^2}\,\mathrm{d}\mu(e^{it})\right| : r \in [0,1)\right\}$$
$$= \sup\left\{\left|\int\limits_{\{w:|w|=1\}} P(r,e^{it})\mathrm{d}\mu(e^{it})\right| : r \in [0,1)\right\}$$
$$= \sup\{|(P[\mathrm{d}\mu])(r)| : r \in [0,1)\}$$
$$= \underbrace{(M_{\mathrm{rad}}(P[\mathrm{d}\mu]))(1) \le \sup\left\{\frac{\mu(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } 1, \text{ and } I \subset \{w : |w| = 1\}\right\}},$$

that is

$$\sup\left\{\left|\int\limits_{\{w:|w|=1\}} \frac{1-r^2}{|e^{it}-r|^2}\,\mathrm{d}\mu(e^{it})\right| : r \in [0,1)\right\}$$
$$\le \sup\left\{\frac{\mu(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } 1, \text{ and } I \subset \{w : |w| = 1\}\right\}.$$

For this purpose, let us fix any $r \in [0,1)$. It suffices to show that

$$\underbrace{\int\limits_{\{w:|w|=1\}} \tfrac{1-r^2}{|e^{it}-r|^2}\,\mathrm{d}\mu(e^{it}) = (1-r^2)\int\limits_{\{w:|w|=1\}} \tfrac{1}{|e^{it}-r|^2}\,\mathrm{d}\mu(e^{it}) = \left|\int\limits_{\{w:|w|=1\}} \frac{1-r^2}{|e^{it}-r|^2}\,\mathrm{d}\mu(e^{it})\right| \le \sup\left\{\frac{\mu(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } 1, \text{ and } I \subset \{w:|w|=1\}\right\}}$$

$$= 2\pi \cdot \sup\left\{\tfrac{\mu(I)}{m(I)} : I \text{ is an open arc with center } 1, \text{ and } I \subset \{w:|w|=1\}\right\},$$

that is

$$\int\limits_{\{w:|w|=1\}} \frac{1}{|e^{it}-r|^2}\,\mathrm{d}\mu(e^{it}) \le \frac{2\pi}{1-r^2}M$$

where $M \equiv \sup\left\{\frac{\mu(I)}{m(I)} : I \text{ is an open arc with center } 1, \text{ and } I \subset \{w:|w|=1\}\right\}$.

For every $j = 1,2,\ldots,n-1,n$, let I_j be an open arc with center 1, and $I_j \subset \{w:|w|=1\}$. Let $I_1 \subset I_2 \subset \cdots \subset I_{n-1} \subset I_n = \{w:|w|=1\}$. It follows that for every $j=1,2,\ldots,n$, $\mu(I_j) \le M \cdot m(I_j)$.

Clearly,

$$\inf\left\{\frac{1}{|e^{it}-r|^2} : t \in [-\pi,\pi]\right\} = \frac{1}{(1+r)^2} < 1, \text{ and, } \sup\left\{\frac{1}{|e^{it}-r|^2} : t \in [-\pi,\pi]\right\} = \frac{1}{(1-r)^2} > 1.$$

For every $j = 1,2,\ldots,n-1,n$, put

$$\underbrace{h_j \equiv \inf\left\{\frac{1}{|e^{it}-r|^2} : e^{it} \in I_j\right\}} \ge \inf\left\{\frac{1}{|e^{it}-r|^2} : t \in [-\pi,\pi]\right\} = \frac{1}{(1+r)^2} > 0.$$

Thus, each $h_j > 0$. Further, the function $t \mapsto \frac{1}{|e^{it}-r|^2}$ from $[-\pi,\pi]$ to $\left[\frac{1}{(1+r)^2}, \frac{1}{(1-r)^2}\right]$ is even. Also, the function $t \mapsto \frac{1}{|e^{it}-r|^2}$ from $[0,\pi]$ to $\left[\frac{1}{(1+r)^2}, \frac{1}{(1-r)^2}\right]$ is decreasing. Since $I_1 \subset I_2 \subset \cdots \subset I_{n-1} \subset I_n = \{w:|w|=1\}$, we have

$$\inf\left\{\frac{1}{|e^{it}-r|^2} : e^{it} \in I_1\right\} \ge \inf\left\{\frac{1}{|e^{it}-r|^2} : e^{it} \in I_2\right\} \ge \inf\left\{\frac{1}{|e^{it}-r|^2} : e^{it} \in I_3\right\} \ge \cdots,$$

and hence $\left(0 < \frac{1}{(1+r)^2} \le\right) h_n \le \cdots \le h_2 \le h_1$. Let us put $h_{n+1} = 0$. It follows that each $(h_j - h_{j+1}) \ge 0$. Since for every $j = 1,2,\ldots,n$,

$$h_j = \inf\left\{\frac{1}{|e^{it} - r|^2} : e^{it} \in I_j\right\},$$

for every $t \in I_j$ and for every $j = 1, 2, \ldots, n$, we have $h_j \leq \frac{1}{|e^{it}-r|^2}$. Since

$$(h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n} = h_1 \text{ on } I_1,$$

for every $e^{it} \in I_1$,

$$\big((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\big)\big(e^{it}\big) \leq \frac{1}{|e^{it} - r|^2}.$$

Since

$$(h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n} = h_2 \text{ on } (I_2 - I_1),$$

for every $e^{it} \in (I_2 - I_1)$,

$$\big((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\big)\big(e^{it}\big) \leq \frac{1}{|e^{it} - r|^2}.$$

Since

$$(h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n} = h_3 \text{ on } (I_3 - I_2),$$

for every $e^{it} \in (I_3 - I_2)$,

$$\big((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\big)\big(e^{it}\big) \leq \frac{1}{|e^{it} - r|^2}, \text{ etc.}$$

Hence, for every $e^{it} \in \{w : |w| = 1\}$,

$$\big((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\big)\big(e^{it}\big) \leq \frac{1}{|e^{it} - r|^2}.$$

Thus,

$$\begin{aligned}&\int\limits_{\{w:|w|=1\}} \big((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\big)\mathrm{d}\mu\big(e^{it}\big)\\ &= (h_1 - h_2)\mu(I_1) + (h_2 - h_3)\mu(I_2) + \cdots + (h_n - h_{n+1})\mu(I_n)\end{aligned}$$

$$\begin{aligned}
&\leq (h_1 - h_2)M \cdot m(I_1) + (h_2 - h_3)M \cdot m(I_2) + \cdots + (h_n - h_{n+1})M \cdot m(I_n) \\
&= M((h_1 - h_2)m(I_1) + (h_2 - h_3)m(I_2) + \cdots + (h_n - h_{n+1})m(I_n)) \\
&= M \int\limits_{w:|w|=1} \left((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\right)\left(e^{it}\right)\mathrm{d}m\left(e^{it}\right) \\
&\leq M \int\limits_{-\pi}^{\pi} \frac{1}{|e^{it} - r|^2}\mathrm{d}t = \frac{M}{1 - r^2} \int\limits_{-\pi}^{\pi} \frac{1 - \left|re^{i0}\right|^2}{|e^{it} - re^{i0}|^2}\mathrm{d}t = \frac{M}{1 - r^2} \int\limits_{-\pi}^{\pi} P_r(0 - t)\mathrm{d}t \\
&= \frac{M}{1 - r^2} \int\limits_{-\pi}^{\pi} P_r(-t)\mathrm{d}t = \frac{M}{1 - r^2} \int\limits_{-\pi}^{\pi} P_r(t)\mathrm{d}t = \frac{M}{1 - r^2} \cdot 2\pi.
\end{aligned}$$

Hence,

$$\int\limits_{\{w:|w|=1\}} \left((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\right) \mathrm{d}\mu\left(e^{it}\right) \leq \frac{2\pi}{1 - r^2}M.$$

We can choose the arcs I_j such that their end points form a sufficiently fine partition of $\{w : |w| = 1\}$, and

$$\int\limits_{\{w:|w|=1\}} \left((h_1 - h_2)\chi_{I_1} + (h_2 - h_3)\chi_{I_2} + \cdots + (h_n - h_{n+1})\chi_{I_n}\right) \mathrm{d}\mu$$

is as close to

$$\int\limits_{\{w:|w|=1\}} \frac{1}{|e^{it} - r|^2}\mathrm{d}\mu\left(e^{it}\right)$$

as we please. Hence,

$$\int\limits_{\{w:|w|=1\}} \frac{1}{|e^{it} - r|^2}\mathrm{d}\mu\left(e^{it}\right) \leq \frac{2\pi}{1 - r^2}M. \quad \blacksquare$$

Definition Let μ be a complex measure on $\{w : |w| = 1\}$. The function

$$e^{i\theta} \mapsto \sup\left\{\frac{|\mu|(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } e^{i\theta}, \text{ and } I \subset \{w : |w| = 1\}\right\}$$

from $\{w : |w| = 1\}$ to $[0, \infty)$ is denoted by $M\mu$. Thus, $M\mu : \{w : |w| = 1\} \to [0, \infty)$ and, for every real θ,

$$(M\mu)\left(e^{i\theta}\right) = \sup\left\{\frac{|\mu|(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } e^{i\theta}, \text{ and } I \subset \{w : |w| = 1\}\right\}.$$

Now, the above conclusion can be restated as follows:

Conclusion 3.75 Let $\alpha \in (0,1)$. Then there exists $c_\alpha > 0$ such that for every positive finite Borel measure μ on $\{w : |w| = 1\}$, and for every real θ,

$$c_\alpha\left((N_\alpha(P[\mathrm{d}\mu]))\left(e^{i\theta}\right)\right) \le (M_{\mathrm{rad}}(P[\mathrm{d}\mu]))\left(e^{i\theta}\right) \le (M\mu)\left(e^{i\theta}\right).$$

Note 3.76 Definition Let μ be a complex measure on $\{w : |w| = 1\}$. By $(\mathrm{d}\mu)\left(e^{i\theta}\right)$, we mean

$$\lim \frac{\mu(I)}{\frac{1}{2\pi}m(I)}$$

as the open arcs I, with center $e^{i\theta}$ and $I \subset \{w : |w| = 1\}$, shrink to $e^{i\theta}$. Thus, $\mathrm{d}\mu : \{w : |w| = 1\} \to \mathbb{C}$. Here, $\mathrm{d}\mu$ is called the ***derivative*** *of* μ *on* $\{w : |w| = 1\}$.

Let μ be a positive Borel measure on $\{w : |w| = 1\}$. It follows that $P[\mathrm{d}\mu] : D(0;1) \to \mathbb{C}$ is a function. Suppose that $(\mathrm{d}\mu)(1) = 0$. Let $\alpha \in (0,1)$. Let $\{z_n\}$ be a sequence in $\Omega_\alpha(\subset D(0;1))$ satisfying $\lim_{n\to\infty} z_n = 1$.

Problem 3.77 $\lim_{n\to\infty}(P[\mathrm{d}\mu])(z_n) = 0$.

(Solution For this purpose, let us take any $\varepsilon > 0$.

For every Borel set E in $\{w : |w| = 1\}$, put $\mu_0(E) \equiv \mu(E \cap I_0)$. Clearly, μ_0 is a positive Borel measure on $\{w : |w| = 1\}$. Also, $(\mu - \mu_0)$ is a positive Borel measure on $\{w : |w| = 1\}$. By Conclusion 3.74, there exists $c_\alpha > 0$ such that

$$\begin{aligned}&\underbrace{c_\alpha((N_\alpha(P[\mathrm{d}\mu_0]))(1)) \le \sup\left\{\frac{\mu_0(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } 1, \text{ and } I \subset \{w : |w| = 1\}\right\}}\\ &= \sup\left\{\frac{\mu(I\cap I_0)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } 1, \text{ and } I \subset \{w : |w| = 1\}\right\}\\ &= \sup\left\{\frac{\mu(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } 1,\ I \subset I_0, \text{ and } I \subset \{w : |w| = 1\}\right\}.\end{aligned}$$

Since $(\mathrm{d}\mu)(1) = 0$, there exists an open arc I_0, with center 1 and $I_0 \subset \{w : |w| = 1\}$ such that, for every arc open arc I, with center 1 and $I \subset I_0$ we have

$$\left|\frac{\mu(I)}{\frac{1}{2\pi}m(I)} - 0\right| < c_\alpha \varepsilon,$$

and hence

$$c_\alpha((N_\alpha(P[\mathrm{d}\mu_0]))(1)) \le \underbrace{\sup\left\{\frac{\mu(I)}{\frac{1}{2\pi}m(I)} : I \text{ is an open arc with center } 1,\ I \subset I_0,\ \text{and } I \subset \{w : |w| = 1\}\right\} \le c_\alpha\varepsilon}.$$

It follows that

$$\begin{aligned}\sup\{(P[\mathrm{d}\mu_0])(w) : w \in \Omega_\alpha\} &= \sup\{|(P[\mathrm{d}\mu_0])(w)| : w \in \Omega_\alpha\} \\ &= \underbrace{(N_\alpha(P[\mathrm{d}\mu_0]))(1) \le \varepsilon}.\end{aligned}$$

Hence, for every $w \in \Omega_\alpha$, $(P[\mathrm{d}\mu_0])(w) \le \varepsilon$. It follows that for every positive integer n, $(P[\mathrm{d}\mu_0])(z_n) \le \varepsilon$. It suffices to show that

$$\begin{aligned}&\lim_{n\to\infty} \int\limits_{\{w:|w|=1\}-I_0} \frac{1-|z_n|^2}{|e^{it}-z_n|^2} d(\mu-\mu_0)\left(e^{it}\right) = \lim_{n\to\infty} \int\limits_{\{w:|w|=1\}-I_0} P\left(z_n, e^{it}\right) d(\mu-\mu_0)\left(e^{it}\right) \\ &= \lim_{n\to\infty} \int\limits_{\{w:|w|=1\}} P\left(z_n, e^{it}\right) d(\mu-\mu_0)\left(e^{it}\right) = \lim_{n\to\infty} (P[d(\mu-\mu_0)])(z_n) \\ &= \underbrace{\lim_{n\to\infty} (P[d(\mu-\mu_0)])(z_n) = 0},\end{aligned}$$

that is

$$\int\limits_{\{w:|w|=1\}-I_0} \frac{1-|z_n|^2}{|e^{it}-z_n|^2} d(\mu-\mu_0)\left(e^{it}\right) = 0.$$

For every positive integer n, $z_n \in \Omega_\alpha$, so by Conclusion 3.69, for every positive integer n,

$$\frac{|z_n - |z_n||}{1-|z_n|} \le \gamma_\alpha,$$

and hence for every positive integer n,

$$\frac{1-|z_n|^2}{|e^{it}-z_n|^2} = \frac{(1-|z_n|)(1+|z_n|)}{|e^{it}-z_n|^2} \le \frac{\frac{|z_n-|z_n||}{\gamma_\alpha}(1+|z_n|)}{|e^{it}-z_n|^2} = \frac{(1+|z_n|)}{\gamma_\alpha|e^{it}-z_n|^2}|z_n - |z_n||.$$

Since $\lim_{n\to\infty} z_n = 1$,

$$\lim_{n\to\infty} \frac{(1+|z_n|)}{\gamma_\alpha|e^{it}-z_n|^2}|z_n - |z_n|| = \lim_{n\to\infty} \frac{(1+|z_n|)}{\gamma_\alpha|e^{it}-z_n|^2}|1-|1|| = 0,$$

and hence

$$\lim_{n\to\infty} \frac{1-|z_n|^2}{|e^{it}-z_n|^2} = 0.$$

This shows that

$$\int\limits_{\{w:|w|=1\}-I_0} \frac{1-|z_n|^2}{|e^{it}-z_n|^2} d(\mu-\mu_0)\left(e^{it}\right) = 0. \quad \blacksquare)$$

Conclusion 3.78 Let μ be a positive Borel measure on $\{w : |w| = 1\}$. It follows that $P[\mathrm{d}\mu] : D(0;1) \to \mathbb{C}$ is a function. Suppose that $(\mathrm{d}\mu)(1) = 0$. Let $\alpha \in (0,1)$. Let $\{z_n\}$ be a sequence in Ω_α $(\subset D(0;1))$ satisfying $\lim_{n\to\infty} z_n = 1$ Then $\lim_{n\to\infty}(P[\mathrm{d}\mu])(z_n) = 0$.

Similarly, we get the following

Conclusion 3.79 Let μ be a positive Borel measure on $\{w : |w| = 1\}$. It follows that $P[\mathrm{d}\mu] : D(0;1) \to \mathbb{C}$ is a function. Let θ be any real number. Suppose that $(\mathrm{d}\mu)\left(e^{i\theta}\right) = 0$. Let $\alpha \in (0,1)$. Let $\{z_n\}$ be a sequence in $e^{i\theta}\Omega_\alpha$ $(\subset D(0;1))$ satisfying $\lim_{n\to\infty} z_n = e^{i\theta}$. Then $\lim_{n\to\infty}(P[\mathrm{d}\mu])(z_n) = 0$.

Definition Let $F : D(0;1) \to \mathbb{C}$ be a function. Let θ be a real number. Let λ be a complex number. If for every $\alpha \in (0,1)$ and for every sequence $\{z_n\}$ in $e^{i\theta}\Omega_\alpha$ $(\subset D(0;1))$ satisfying $\lim_{n\to\infty} z_n = e^{i\theta}$, $\lim_{n\to\infty} F(z_n) = \lambda$, then we say that *$F$ has **non-tangential limit** λ at $e^{i\theta}$*.

Now, Conclusion 3.79 can be restated as Conclusion 3.80 below.

Conclusion 3.80 Let μ be a positive Borel measure on $\{w : |w| = 1\}$. It follows that $P[\mathrm{d}\mu] : D(0;1) \to \mathbb{C}$ is a function. Let θ be any real number. Suppose that $(\mathrm{d}\mu)\left(e^{i\theta}\right) = 0$. Then $P[\mathrm{d}\mu]$ has non-tangential limit 0 at $e^{i\theta}$.

3.7 Lebesgue Point

Note 3.81 Definition Let $f \in L^1(\{w : |w| = 1\})$. Let $e^{i\theta} \in \{w : |w| = 1\}$. If

$$\lim \frac{\frac{1}{2\pi}\int\limits_I \left|f - f\left(e^{i\theta}\right)\right| \mathrm{d}m}{m(I)}$$

as the open arcs I, with center $e^{i\theta}$ and $I \subset \{w : |w| = 1\}$ shrink to $e^{i\theta}$ vanishes, then we say that *$e^{i\theta}$ is a **Lebesgue point** of f*.

Let

$$f : \{w : |w| = 1\} \to \mathbb{C}$$

be a member of $L^1(\{w : |w| = 1\})$. It follows that

$$P[f] : z \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} P\left(z, e^{it}\right) f\left(e^{it}\right) \mathrm{d}t$$

is a function from $D(0;1)$ to $\mathbb{C}$. Let $e^{i\theta} \in \{w : |w| = 1\}$. Let $e^{i\theta}$ be a Lebesgue point of f.

Problem 3.82 $P[f]$ has non-tangential limit $f\left(e^{i\theta}\right)$ at $e^{i\theta}$.

(Solution For this purpose, let us take any $\alpha \in (0,1)$, and let us take any sequence $\{z_n\}$ in $e^{i\theta}\Omega_\alpha$ $(\subset D(0;1))$ satisfying $\lim_{n\to\infty} z_n = e^{i\theta}$. We have to show that $\lim_{n\to\infty}(P[f])(z_n) = f\left(e^{i\theta}\right)$.

Case I: when $f\left(e^{i\theta}\right) = 0$. Here, we have to show that $\lim_{n\to\infty}(P[f])(z_n) = 0$. Since $f \in L^1(\{w : |w| = 1\})$,

$$\mu : E \mapsto \frac{1}{2\pi} \int_E |f| \mathrm{d}m$$

is a positive Borel measure on $\{w : |w| = 1\}$. Next,

$$(\mathrm{d}\mu)\left(e^{i\theta}\right) = \lim \frac{\mu(I)}{\frac{1}{2\pi} m(I)}$$

as the open arcs I, with center $e^{i\theta}$ and $I \subset \{w : |w| = 1\}$, shrink to $e^{i\theta}$. Now, since

$$\frac{\mu(I)}{\frac{1}{2\pi} m(I)} = \frac{\frac{1}{2\pi} \int_I |f| \mathrm{d}m}{\frac{1}{2\pi} m(I)} = \frac{\int_I |f| \mathrm{d}m}{m(I)} = \frac{\int_I |f - 0| \mathrm{d}m}{m(I)} = \frac{\int_I \left|f - f\left(e^{i\theta}\right)\right| \mathrm{d}m}{m(I)},$$

we have

$$(D\mu)\left(e^{i\theta}\right) = \lim \frac{\int_I \left|f - f\left(e^{i\theta}\right)\right| \mathrm{d}m}{m(I)}$$

as the open arcs I, with center $e^{i\theta}$ and $I \subset \{w : |w| = 1\}$, shrink to $e^{i\theta}$. Since $e^{i\theta}$ is a Lebesgue point of f,

$$\lim \frac{\int_I \left|f - f\left(e^{i\theta}\right)\right| \mathrm{d}m}{m(I)}$$

as the open arcs I, with center $e^{i\theta}$ and $I \subset \{w : |w| = 1\}$ shrink to $e^{i\theta}$ vanishes, and hence $(\mathrm{d}\mu)\left(e^{i\theta}\right) = 0$. Now, by Conclusion 3.80,

$$P[\mathrm{d}\mu] : z \mapsto \int\limits_{\{w:|w|=1\}} \frac{1-|z|^2}{|e^{it}-z|^2} \mathrm{d}\mu\left(e^{it}\right)$$

has non-tangential limit 0 at $e^{i\theta}$. Now, since $\alpha \in (0,1)$, and $\{z_n\}$ is a sequence in $e^{i\theta}\Omega_\alpha$ $(\subset D(0;1))$ satisfying $\lim_{n\to\infty} z_n = e^{i\theta}$, we have $\lim_{n\to\infty}(P[\mathrm{d}\mu])(z_n) = 0$. It suffices to show that for every positive integer n,

$$\begin{aligned} 0 \le \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P\left(z_n, e^{it}\right) f\left(e^{it}\right) \mathrm{d}t &= \underbrace{(P[f])(z_n) \le (P[\mathrm{d}\mu])(z_n)} \\ &= \int\limits_{\{w:|w|=1\}} P\left(z_n, e^{it}\right) \mathrm{d}\mu\left(e^{it}\right) \\ &= \int\limits_{\{w:|w|=1\}} P\left(z_n, e^{it}\right) \frac{1}{2\pi} \left|f\left(e^{it}\right)\right| \mathrm{d}m \\ &= \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P\left(z_n, e^{it}\right) \left|f\left(e^{it}\right)\right| \mathrm{d}m, \end{aligned}$$

that is for every positive integer n,

$$\int\limits_{-\pi}^{\pi} P\left(z_n, e^{it}\right) f\left(e^{it}\right) \mathrm{d}t \le \int\limits_{\{w:|w|=1\}} P\left(z_n, e^{it}\right) \left|f\left(e^{it}\right)\right| \mathrm{d}m.$$

For every positive integer n,

$$\begin{aligned} \int\limits_{-\pi}^{\pi} P\left(z_n, e^{it}\right) f\left(e^{it}\right) \mathrm{d}t &= \left| \int\limits_{-\pi}^{\pi} P\left(z_n, e^{it}\right) f\left(e^{it}\right) \mathrm{d}t \right| \le \int\limits_{-\pi}^{\pi} \left| P\left(z_n, e^{it}\right) f\left(e^{it}\right) \right| \mathrm{d}t \\ &= \int\limits_{-\pi}^{\pi} \left| P\left(z_n, e^{it}\right) \right| \left| f\left(e^{it}\right) \right| \mathrm{d}t \\ &= \int\limits_{-\pi}^{\pi} P\left(z_n, e^{it}\right) \left| f\left(e^{it}\right) \right| \mathrm{d}t \\ &= \int\limits_{\{w:|w|=1\}} P\left(z_n, e^{it}\right) \left| f\left(e^{it}\right) \right| \mathrm{d}m \end{aligned}$$

so for every positive integer n,

$$\int_{-\pi}^{\pi} P(z_n, e^{it}) f(e^{it}) \mathrm{d}t \leq \int_{\{w:|w|=1\}} P(z_n, e^{it}) |f(e^{it})| \mathrm{d}m.$$

Case II: when $f(e^{i\theta}) \neq 0$. Let

$$g : w \mapsto (f(w) - f(e^{i\theta}))$$

be the function from $\{w : |w| = 1\}$ to $\mathbb{C}$. Since $f \in L^1(\{w : |w| = 1\})$, we have $g \in L^1(\{w : |w| = 1\})$. Since $e^{i\theta}$ is a Lebesgue point of f,

$$\begin{aligned}
\lim \frac{\frac{1}{2\pi}\int_I |g - g(e^{i\theta})| \mathrm{d}m}{m(I)} &= \lim \frac{\frac{1}{2\pi}\int_I |g - 0| \mathrm{d}m}{m(I)} = \lim \frac{\frac{1}{2\pi}\int_I |g| \mathrm{d}m}{m(I)} \\
&= \underbrace{\lim \frac{\frac{1}{2\pi}\int_I |f - f(e^{i\theta})| \mathrm{d}m}{m(I)} = 0}
\end{aligned}$$

and hence $e^{i\theta}$ is a Lebesgue point of g. Also, $g(e^{i\theta}) = 0$. Now, by Case I,

$$\lim_{n\to\infty} (P[g])(z_n) = 0.$$

Since for every positive integer n,

$$\begin{aligned}
(P[g])(z_n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} g(e^{it}) \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} (f(e^{it}) - f(e^{i\theta})) \mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} f(e^{it}) \mathrm{d}t - f(e^{i\theta}) \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} \mathrm{d}t \\
&= (P[f])(z_n) - f(e^{i\theta}) \cdot 1,
\end{aligned}$$

we have

$$(0 =) \lim_{n\to\infty} (P[g])(z_n) = \lim_{n\to\infty} (P[f])(z_n) - f(e^{i\theta}),$$

and hence

$$\lim_{n\to\infty}(P[f])(z_n)=f\left(e^{i\theta}\right).$$

∎)

Conclusion 3.83 Let $f:\{w:|w|=1\}\to\mathbb{C}$ be a member of $L^1(\{w:|w|=1\})$. Let $e^{i\theta}\in\{w:|w|=1\}$. Let $e^{i\theta}$ be a Lebesgue point of f. Then $P[f]$ has non-tangential limit $f\left(e^{i\theta}\right)$ at $e^{i\theta}$.

3.8 Representation Theorems

Theorem 3.84 Let X be a separable Banach space. Let $\{\Lambda_n\}$ be a sequence of linear functionals on X. Suppose that $\sup_n\|\Lambda_n\|=M<\infty$. Then there exists a subsequence $\{\Lambda_{n_k}\}$ of $\{\Lambda_n\}$ such that

1. for every $x\in X$, $\lim_{k\to\infty}\Lambda_{n_k}(x)$ exists,
2. the function $\Lambda:x\mapsto\lim_{k\to\infty}\Lambda_{n_k}(x)$ from X to $\mathbb{C}$ is a linear functional on X,
3. $\|\Lambda\|\le M$.

Proof

Problem 3.85 The collection $\{\Lambda_1,\Lambda_2,\Lambda_3,\ldots\}$ is pointwise bounded.

(Solution For this purpose, let us take any $a\in X$. Since $\sup_n\|\Lambda_n\|=M<\infty$, for every positive integer n, $\|\Lambda_n\|\le M$, and hence for every positive integer n, $(|\Lambda_n(a)|\le)\ \|\Lambda_n\|\|a\|\le M\|a\|$. Thus, for every positive integer n, $|\Lambda_n(a)|\le(M\|a\|)$. This shows that the collection $\{\Lambda_1,\Lambda_2,\Lambda_3,\ldots\}$ is pointwise bounded. ∎)

Problem 3.86 The collection $\{\Lambda_1,\Lambda_2,\Lambda_3,\ldots\}$ is equicontinuous.

(Solution Case I: when $M\neq 0$. Let us take any $\varepsilon>0$. Let us take any $x,y\in X$ satisfying $\|x-y\|<\frac{\varepsilon}{M}$. It suffices to show that for every positive integer n, $|\Lambda_n(x)-\Lambda_n(y)|<\varepsilon$. Since for every positive integer n,

$$|\Lambda_n(x)-\Lambda_n(y)|=|\Lambda_n(x-y)|\le\|\Lambda_n\|\|x-y\|\le M\|x-y\|<\varepsilon,$$

for every positive integer n, $|\Lambda_n(x)-\Lambda_n(y)|<\varepsilon$.

Case II: when $M=0$. Now, since $\sup_n\|\Lambda_n\|=M$, for every positive integer n, $\|\Lambda_n\|=0$, and hence for every positive integer n, $\Lambda_n=0$. Thus, $\{\Lambda_1,\Lambda_2,\Lambda_3,\ldots\}=\{0\}$. Now, since $\{0\}$ is trivially equicontinuous, $\{\Lambda_1,\Lambda_2,\Lambda_3,\ldots\}$ is equicontinuous.

Thus, in all cases, the collection $\{\Lambda_1,\Lambda_2,\Lambda_3,\ldots\}$ is equicontinuous. ∎)

For 1: By Conclusion 2.89, there exists a subsequence $\{\Lambda_{n_k}\}$ of $\{\Lambda_n\}$ such that $\{\Lambda_{n_k}\}$ converges uniformly on every compact subset of X. It follows that for every $x \in X$, $\lim_{k\to\infty} \Lambda_{n_k}(x)$ exists, because each singleton set is a compact set.
For 2: Let us take any $x, y \in X$, and let us take any $\alpha, \beta \in \mathbb{C}$. We have to show that

$$\lim_{k\to\infty} \Lambda_{n_k}(\alpha x + \beta y) = \alpha\left(\lim_{k\to\infty} \Lambda_{n_k}(x)\right) + \beta\left(\lim_{k\to\infty} \Lambda_{n_k}(y)\right).$$

$$\begin{aligned}\text{LHS} &= \lim_{k\to\infty} \Lambda_{n_k}(\alpha x + \beta y) = \lim_{k\to\infty} (\alpha(\Lambda_{n_k}(x)) + \beta(\Lambda_{n_k}(y))) \\ &= \alpha\left(\lim_{k\to\infty} \Lambda_{n_k}(x)\right) + \beta\left(\lim_{k\to\infty} \Lambda_{n_k}(y)\right) = \text{RHS}.\end{aligned}$$

For 3: Let us take any $x \in X$. It suffices to show that $|\Lambda(x)| \le M\|x\|$, that is

$$\lim_{k\to\infty} |\Lambda_{n_k}(x)| = \underbrace{\left|\lim_{k\to\infty} \Lambda_{n_k}(x)\right| \le M\|x\|}_{},$$

that is

$$\lim_{k\to\infty} |\Lambda_{n_k}(x)| \le M\|x\|.$$

Since $\sup_n \|\Lambda_n\| = M < \infty$, for every positive integer k, $\|\Lambda_{n_k}\| \le M$, and hence for every positive integer k,

$$|\Lambda_{n_k}(x)| \le \underbrace{\|\Lambda_{n_k}\|\|x\| \le M\|x\|}_{}.$$

Thus, for every positive integer k, $|\Lambda_{n_k}(x)| \le (M\|x\|)$. It follows that $\lim_{k\to\infty} |\Lambda_{n_k}(x)| \le M\|x\|$. ∎

Theorem 3.87 $C(\{w : |w| = 1\})$ is a separable Banach space.

Proof By Conclusions 2.240, Vol. 1 and 2.248, Vol. 1, $C(\{w : |w| = 1\})$ is a Banach space, whose norm is defined by

$$\|f\|_\infty \equiv \sup\{|f(w)| : |w| = 1\}.$$

It remains to show that $C(\{w : |w| = 1\})$ is separable.

For this purpose, let us take any $f \in C(\{w : |w| = 1\})$, and $\varepsilon > 0$.

Here, we can think of f as a 2π-periodic continuous function defined on $\mathbb{R}$. By Conclusion 2.156, Vol. 1, there exists a trigonometric polynomial $P : \mathbb{R} \to \mathbb{C}$ such that for every real t, $|P(t) - f(t)| < \frac{\varepsilon}{3}$. It follows that $\|P - f\|_\infty \le \frac{\varepsilon}{3}$. Since P is a trigonometric polynomial, there exist real numbers $a_{-N}, \ldots, a_{-1}, a_0, a_1, \ldots, a_N$, and $b_{-N}, \ldots, b_{-1}, b_0, b_1, \ldots, b_N$ such that for every real t,

$$\begin{aligned}P(t) = (a_{-N} + ib_{-N})e^{i(-N)t} + \cdots + (a_{-1} + ib_{-1})e^{i(-1)t} \\ + (a_0 + ib_0) + (a_1 + ib_1)e^{i1t} + \cdots + (a_N + ib_N)e^{iNt}.\end{aligned}$$

There exist rational numbers $r_{-N}, \ldots, r_{-1}, r_0, r_1, \ldots, r_N$, and $s_{-N}, \ldots, s_{-1}, s_0, s_1, \ldots, s_N$ such that for every $n \in \{-N, \ldots, -1, 0, 1, \ldots, N\}$, $|r_n - a_n| < \frac{\varepsilon}{4(2N+1)}$, and $|s_n - b_n| < \frac{\varepsilon}{4(2N+1)}$. For every real t, put

$$\begin{aligned}Q(t) \equiv (r_{-N} + is_{-N})e^{i(-N)t} + \cdots + (r_{-1} + is_{-1})e^{i(-1)t} \\ + (r_0 + is_0) + (r_1 + is_1)e^{i1t} + \cdots + (r_N + is_N)e^{iNt}.\end{aligned}$$

Here, $Q : \mathbb{R} \to \mathbb{C}$ is a trigonometric polynomial, so $Q \in C(\{w : |w| = 1\})$. Also, for every real t,

$$\begin{aligned}|Q(t) - P(t)| &= \left| \sum_{n=-N}^{N} ((r_n - a_n) + i(s_n - b_n))e^{int} \right| \le \sum_{n=-N}^{N} \left|((r_n - a_n) + i(s_n - b_n))e^{int}\right| \\ &= \sum_{n=-N}^{N} |(r_n - a_n) + i(s_n - b_n)| \le \sum_{n=-N}^{N} (|r_n - a_n| + |s_n - b_n|) \\ &< \sum_{n=-N}^{N} \left(\frac{\varepsilon}{4(2N+1)} + \frac{\varepsilon}{4(2N+1)} \right) = \sum_{n=-N}^{N} \frac{\varepsilon}{2(2N+1)} = \frac{\varepsilon}{2},\end{aligned}$$

so for every real t, $|Q(t) - P(t)| < \frac{\varepsilon}{2}$, and hence $\|Q - P\|_\infty \le \frac{\varepsilon}{2}$. It follows that

$$\|Q - f\|_\infty \le \|Q - P\|_\infty + \|P - f\|_\infty \le \frac{\varepsilon}{2} + \frac{\varepsilon}{3} < \varepsilon,$$

and hence $\|Q - f\|_\infty < \varepsilon$. Thus, the collection, say $\mathcal{C}$, of all trigonometric polynomials with coefficients in $\mathbb{Q}$, is a dense subset of $C(\{w : |w| = 1\})$. Further, since $\mathbb{Q}$ is countable, $\mathcal{C}$ is countable. Hence, $C(\{w : |w| = 1\})$ is separable. ∎

Theorem 3.88 Let $p \in [1, \infty)$. Then, $L^p(\{w : |w| = 1\})$ is a separable Banach space.

Proof By Theorem 3.87, $C(\{w : |w| = 1\})$ is a separable Banach space, whose norm is defined by

$$\|f\|_\infty \equiv \sup\{|f(w)| : |w| = 1\},$$

so there exists a countable dense subset, say E, of $C(\{w : |w| = 1\})$. By Conclusions 2.240, Vol.1, and 2.50, Vol.1, $C(\{w : |w| = 1\})$ is a dense subset of $L^p(\{w : |w| = 1\})$. It suffices to show that E is dense in $L^p(\{w : |w| = 1\})$.

For this purpose, let us take any $f \in L^p(\{w : |w| = 1\})$, and $\varepsilon > 0$.

Since $C(\{w : |w| = 1\})$ is a dense subset of $L^p(\{w : |w| = 1\})$, there exists $g \in C(\{w : |w| = 1\})$ such that $\|g - f\|_p < \frac{\varepsilon}{2}$. Since E is dense in the Banach space $C(\{w : |w| = 1\})$ with the norm $\| \|_\infty$, and $g \in C(\{w : |w| = 1\})$, there exists $h \in E$ such that $\|h - g\|_\infty < \frac{\varepsilon}{2}$. It suffices to show that

$$\|h - f\|_p < \varepsilon.$$

Here,

$$\|h - f\|_p \le \|h - g\|_p + \|g - f\|_p < \|h - g\|_p + \frac{\varepsilon}{2}, \text{ and } \|h - g\|_\infty < \frac{\varepsilon}{2}.$$

It suffices to show that

$$\|h - g\|_p \le \|h - g\|_\infty.$$

Since

$$\begin{aligned}
\|h - g\|_p &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|(h-g)\left(e^{i\theta}\right)\right|^p \mathrm{d}\theta\right)^{\frac{1}{p}} \le \left(\frac{1}{2\pi}\left(\sup\left\{\left|(h-g)\left(e^{i\theta}\right)\right|^p : \theta \in [-\pi, \pi]\right\}\right) \cdot 2\pi\right)^{\frac{1}{p}} \\
&= \left(\sup\left\{\left|(h-g)\left(e^{i\theta}\right)\right|^p : \theta \in [-\pi, \pi]\right\}\right)^{\frac{1}{p}} \\
&= \left(\left(\sup\left\{\left|(h-g)\left(e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\}\right)^p\right)^{\frac{1}{p}} \\
&= \sup\left\{\left|(h-g)\left(e^{i\theta}\right)\right| : \theta \in [-\pi, \pi]\right\} \\
&= \sup\{|(h-g)(w)| : |w| = 1\} = \|h - g\|_\infty,
\end{aligned}$$

we have

$$\|h - g\|_p \le \|h - g\|_\infty. \qquad \blacksquare$$

Note 3.89 Let $u : D[0; 1] \to \mathbb{R}$ be a continuous function. Let u be harmonic in $D(0; 1)$.

Since u is continuous, for every $r \in (0, 1)$,

$$u_r : e^{i\theta} \mapsto u\left(re^{i\theta}\right)$$

from $\{w : |w| = 1\}$ to $\mathbb{C}$ is continuous. It follows that for every $g \in C(\{w : |w| = 1\})$, and for every $r \in (0, 1)$, $gu_r : \{w : |w| = 1\} \to \mathbb{C}$ is continuous, and hence for every $g \in C(\{w : |w| = 1\})$, and for every $r \in (0, 1)$,

$$\left(\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} gu_r \mathrm{d}m\right) \in \mathbb{C}.$$

Thus, for every $r \in (0,1)$,

$$\Lambda_r : g \mapsto \left(\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} gu_r \mathrm{d}m\right)$$

is a function from $C(\{w : |w| = 1\})$ to $\mathbb{C}$. By Theorem 3.87, $C(\{w : |w| = 1\})$ is a separable Banach space, whose norm is defined by

$$\|f\|_\infty \equiv \sup\{|f(w)| : |w| = 1\}.$$

Clearly, each Λ_r is a linear functional on the separable Banach space $C(\{w : |w| = 1\})$.

Since $u : D[0;1] \to \mathbb{R}$ is continuous, there exists a positive real number M such that

$$\max\{|u(w)| : |w| \leq 1\} = M\,(<\infty).$$

Now, for every $r \in (0,1)$,

$$\begin{aligned}
\|u_r\|_1 &= \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} |u_r| \mathrm{d}m = \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} \left|u_r\left(e^{it}\right)\right| \mathrm{d}m\left(e^{it}\right) \\
&= \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} \left|u\left(re^{it}\right)\right| \mathrm{d}m\left(e^{it}\right) \leq M,
\end{aligned}$$

so for every $r \in (0,1)$, $\|u_r\|_1 \leq M$. Also, for every $r \in (0,1)$,

$$\begin{aligned}
\|\Lambda_r\| &= \sup\left\{|\Lambda_r(g)| : \|g\|_\infty = 1\right\} \\
&= \sup\left\{\left|\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} gu_r \mathrm{d}m\right| : \|g\|_\infty = 1\right\} \\
&\leq \sup\left\{\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} |gu_r| \mathrm{d}m : \|g\|_\infty = 1\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup\left\{\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} |g||u_r|\mathrm{d}m : \|g\|_\infty = 1\right\}\\
&\le \sup\left\{\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} (\sup\{|g(w)| : |w| = 1\})|u_r|\mathrm{d}m : \|g\|_\infty = 1\right\}\\
&= \sup\left\{\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} \|g\|_\infty|u_r|\mathrm{d}m : \|g\|_\infty = 1\right\}\\
&= \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} 1\cdot|u_r|\mathrm{d}m\\
&= \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} |u_r|\mathrm{d}m = \|u_r\|_1 \le M,
\end{aligned}
$$

so for every $r \in (0,1)$, $\|\Lambda_r\| \le M$. Since $C(\{w : |w| = 1\})$ is a separable Banach space, and $\left\{\Lambda_{\left(1-\frac{1}{n}\right)}\right\}$ is a sequence of linear functionals on $C(\{w : |w| = 1\})$ satisfying $\sup_n\left\|\Lambda_{\left(1-\frac{1}{n}\right)}\right\| \le M < \infty$, by Theorem 3.84, there exists a subsequence $\left\{\Lambda_{\left(1-\frac{1}{n_k}\right)}\right\}$ of $\left\{\Lambda_{\left(1-\frac{1}{n}\right)}\right\}$ such that

1. for every $g \in C(\{w : |w| = 1\})$, $\lim_{k\to\infty}\Lambda_{\left(1-\frac{1}{n_k}\right)}(g)$ exists,
2. the function $\Lambda : g \mapsto \lim_{k\to\infty}\Lambda_{\left(1-\frac{1}{n_k}\right)}(g)$ from $C(\{w : |w| = 1\})$ to $\mathbb{C}$ is a linear functional on $C(\{w : |w| = 1\})$,
3. $\|\Lambda\| \le M$.

Thus, $\Lambda : C(\{w : |w| = 1\}) \to \mathbb{C}$ is a bounded linear functional. Now, by Theorem 2.124, there exists a complex Borel measure μ on $\{w : |w| = 1\}$ such that

1. for every $g \in C(\{w : |w| = 1\})$, $\lim_{k\to\infty}\frac{1}{2\pi}\int_{\{w:|w|=1\}} gu_{\left(1-\frac{1}{n_k}\right)}\mathrm{d}m = \lim_{k\to\infty}\Lambda_{\left(1-\frac{1}{n_k}\right)}(g) = \underbrace{\Lambda(g) = \int_{\{w:|w|=1\}} g\mathrm{d}\mu}$,
2. $|\mu|(\{w : |w| = 1\}) = \|\Lambda\|$.

From Note 3.70(I), the function

$$P[\mathrm{d}\mu] : z \mapsto \int\limits_{\{w:|w|=1\}} P\left(z, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right)$$

from $D(0;1)$ to $\mathbb{C}$ is harmonic.

Problem 3.90 $u = P[\mathrm{d}\mu]$ on $D(0;1)$.

(Solution For this purpose, let us take any $z \in D(0;1)$. We have to show that

$$\begin{aligned}\underbrace{u(z) = \int\limits_{\{w:|w|=1\}} P\left(z, e^{it}\right)\mathrm{d}\mu\left(e^{it}\right)} &= \int\limits_{\{w:|w|=1\}} \frac{1-|z|^2}{|e^{it}-z|^2}\mathrm{d}\mu\left(e^{it}\right)\\ &= \lim_{k\to\infty}\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} \frac{1-|z|^2}{|e^{it}-z|^2}u_{\left(1-\frac{1}{n_k}\right)}\left(e^{it}\right)\mathrm{d}m\left(e^{it}\right)\\ &= \lim_{k\to\infty}\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} P\left(z, e^{it}\right)u\left(\left(1-\frac{1}{n_k}\right)e^{it}\right)\mathrm{d}m\left(e^{it}\right),\end{aligned}$$

that is

$$u(z) = \lim_{k\to\infty}\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} P\left(z, e^{it}\right)u\left(\left(1-\frac{1}{n_k}\right)e^{it}\right)\mathrm{d}m\left(e^{it}\right).$$

Since $u : D[0;1] \to \mathbb{R}$ is continuous, for every positive integer k, the function

$$h_k : w \mapsto u\left(\left(1-\frac{1}{n_k}\right)w\right)$$

from $D[0;1]$ to $\mathbb{R}$ is continuous. Further, since u is harmonic in $D(0;1)$, each h_k is harmonic in $D(0;1)$. Next, by Conclusions 1.282 and 1.284, for every positive integer k, and for every $z \in D(0;1)$,

$$\begin{aligned}u\left(\left(1-\frac{1}{n_k}\right)z\right) &= \underbrace{h_k(z) = \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} P\left(z, e^{it}\right)h_k\left(e^{it}\right)\mathrm{d}m\left(e^{it}\right)}\\ &= \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} P\left(z, e^{it}\right)u\left(\left(1-\frac{1}{n_k}\right)e^{it}\right)\mathrm{d}m\left(e^{it}\right),\end{aligned}$$

and hence for every positive integer k, and for every $z \in D(0;1)$,

$$u\left(\left(1-\frac{1}{n_k}\right)z\right) = \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P(z,e^{it})u\left(\left(1-\frac{1}{n_k}\right)e^{it}\right)\mathrm{d}m(e^{it}).$$

It follows that

$$\begin{aligned}\text{RHS} &= \lim_{k\to\infty} \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P(z,e^{it})u\left(\left(1-\frac{1}{n_k}\right)e^{it}\right)\mathrm{d}m(e^{it}) \\ &= \lim_{k\to\infty} u\left(\left(1-\frac{1}{n_k}\right)z\right) = u((1-0)z) \\ &= u(z) = \text{LHS}.\end{aligned}$$

∎)

Conclusion 3.91 Let $u : D[0;1] \to \mathbb{R}$ be a continuous function. Let u be harmonic in $D(0;1)$. Then there exists a complex Borel measure μ on $\{w : |w| = 1\}$ such that $u = P[\mathrm{d}\mu]$ on $D(0;1)$.

Note 3.92 Let $p \in (1,\infty]$. Let $u : D[0;1] \to \mathbb{R}$ be a continuous function. Let u be harmonic in $D(0;1)$. Let $\sup\left\{\|u_r\|_p : r \in (0,1)\right\} = M < \infty$.

Let $q\,(\in [1,\infty))$ be the exponent conjugate to p. Since $q \in [1,\infty)$, by Theorem 3.88, $L^q(\{w : |w| = 1\})$ is a separable Banach space.

Since u is continuous, for every $r \in (0,1)$, $u_r : e^{i\theta} \mapsto u(re^{i\theta})$ from $\{w : |w| = 1\}$ to $\mathbb{C}$ is continuous, and hence each $u_r \in L^p(\{w : |w| = 1\})$. Now, by Lemma 2.21, Vol. 1, and Lemma 2.23, Vol. 1, for every $g \in L^q(\{w : |w| = 1\})$, and for every $r \in (0,1)$, $gu_r : \{w : |w| = 1\} \to \mathbb{C}$ is a member of $L^1(\{w : |w| = 1\})$, and hence for every $g \in L^q(\{w : |w| = 1\})$, and for every $r \in (0,1)$,

$$\left(\frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} gu_r \mathrm{d}m\right) \in \mathbb{C}.$$

Thus, for every $r \in (0,1)$,

$$\Lambda_r : g \mapsto \left(\frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} gu_r \mathrm{d}m\right)$$

is a function from $L^q(\{w : |w| = 1\})$ to $\mathbb{C}$. Clearly, each Λ_r is a linear functional on the separable Banach space $L^q(\{w : |w| = 1\})$. Since $\sup\left\{\|u_r\|_p : r \in (0,1)\right\} = M < \infty$, for every $r \in (0,1)$, we have $\|u_r\|_p \le M$. Also, for every $r \in (0,1)$,

$$\begin{aligned}
\|\Lambda_r\| &= \sup\left\{|\Lambda_r(g)| : \|g\|_q = 1\right\} \\
&= \sup\left\{\left|\frac{1}{2\pi}\int_{\{w:|w|=1\}} gu_r \mathrm{d}m\right| : \|g\|_g = 1\right\} \\
&\le \sup\left\{\frac{1}{2\pi}\int_{\{w:|w|=1\}} |gu_r| \mathrm{d}m : \|g\|_q = 1\right\} \\
&= \sup\left\{u_r g_1 : \|g\|_q = 1\right\} \le \sup\left\{\|u_r\|_p \|g\|_q : \|g\|_q = 1\right\} \\
&= \|u_r\|_p \cdot 1 = \|u_r\|_p \le M,
\end{aligned}$$

so for every $r \in (0,1)$, $\|\Lambda_r\| \le M$. Since $L^q(\{w : |w| = 1\})$ is a separable Banach space, and, $\left\{\Lambda_{\left(1-\frac{1}{n}\right)}\right\}$ is a sequence of linear functionals on $L^q(\{w : |w| = 1\})$ satisfying $\sup_n \left\|\Lambda_{\left(1-\frac{1}{n}\right)}\right\| \le M < \infty$, by Theorem 3.84, there exists a subsequence $\left\{\Lambda_{\left(1-\frac{1}{n_k}\right)}\right\}$ of $\left\{\Lambda_{\left(1-\frac{1}{n}\right)}\right\}$ such that

1. for every $g \in L^q(\{w : |w| = 1\})$, $\lim_{k\to\infty} \Lambda_{\left(1-\frac{1}{n_k}\right)}(g)$ exists,
2. the function $\Lambda : g \mapsto \lim_{k\to\infty} \Lambda_{\left(1-\frac{1}{n_k}\right)}(g)$ from $L^q(\{w : |w| = 1\})$ to $\mathbb{C}$ is a linear functional on $L^q(\{w : |w| = 1\})$,
3. $\|\Lambda\| \le M$.

Thus, $\Lambda : L^q(\{w : |w| = 1\}) \to \mathbb{C}$ is a bounded linear functional. Now, since $q \in [1, \infty)$, by Conclusion 3.80, Vol. 1, there exists a function $f : \{w : |w| = 1\} \to \mathbb{C}$ such that

1. $f \in L^p(\{w : |w| = 1\})$,
2. for every $g \in L^q(\{w : |w| = 1\})$, $\left(\lim\limits_{k\to\infty}\left(\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} gu_{\left(1-\frac{1}{n_k}\right)} \mathrm{d}m\right) = \lim\limits_{k\to\infty} \Lambda_{\left(1-\frac{1}{n_k}\right)}(g) =\right) \Lambda(g) = \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} (g \cdot f)\mathrm{d}m$,
3. $\|f\|_p = \|\Lambda\| \ (\le M)$.

From Note 4.265, the function $P[f] : z \mapsto \frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} P(z, e^{it}) f(e^{it}) \mathrm{d}m(e^{it})$ from $D(0;1)$ to $\mathbb{C}$ is harmonic.

Problem 3.93 $u = P[f]$ on $D(0;1)$.

(Solution For this purpose, let us take any $z \in D(0;1)$. We have to show that

$$\underbrace{u(z) = \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P\big(z, e^{it}\big) f\big(e^{it}\big) \mathrm{d}m\big(e^{it}\big)} = \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} \frac{1-|z|^2}{|e^{it}-z|^2} f\big(e^{it}\big) \mathrm{d}m\big(e^{it}\big)$$

$$= \lim_{k\to\infty} \left(\frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} \frac{1-|z|^2}{|e^{it}-z|^2} u_{\left(1-\frac{1}{n_k}\right)}\big(e^{it}\big) \mathrm{d}m\big(e^{it}\big) \right)$$

$$= \lim_{k\to\infty} \left(\frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} \frac{1-|z|^2}{|e^{it}-z|^2} u\left(\left(1-\frac{1}{n_k}\right) e^{it} \right) \mathrm{d}m\big(e^{it}\big) \right)$$

$$= \lim_{k\to\infty} \left(\frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P\big(z, e^{it}\big) u\left(\left(1-\frac{1}{n_k}\right) e^{it} \right) \mathrm{d}m\big(e^{it}\big) \right),$$

that is

$$u(z) = \lim_{k\to\infty} \left(\frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P\big(z, e^{it}\big) u\left(\left(1-\frac{1}{n_k}\right) e^{it} \right) \mathrm{d}m\big(e^{it}\big) \right).$$

Since $u : D[0;1] \to \mathbb{R}$ is continuous, for every positive integer k, the function

$$h_k : w \mapsto u\left(\left(1-\frac{1}{n_k}\right) w \right)$$

from $D[0;1]$ to $\mathbb{R}$ is continuous. Further, since u is harmonic in $D(0;1)$, each h_k is harmonic in $D(0;1)$.

Next, by Conclusions 1.282 and 1.284, for every positive integer k, and for every $z \in D(0;1)$,

$$u\left(\left(1-\frac{1}{n_k}\right) z \right) = \underbrace{h_k(z) = \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P\big(z, e^{it}\big) h_k\big(e^{it}\big) \mathrm{d}m\big(e^{it}\big)}$$

$$= \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P\big(z, e^{it}\big) u\left(\left(1-\frac{1}{n_k}\right) e^{it} \right) \mathrm{d}m\big(e^{it}\big),$$

and hence for every positive integer k, and for every $z \in D(0;1)$,

$$u\left(\left(1-\frac{1}{n_k}\right) z \right) = \frac{1}{2\pi} \int\limits_{\{w:|w|=1\}} P\big(z, e^{it}\big) u\left(\left(1-\frac{1}{n_k}\right) e^{it} \right) \mathrm{d}m\big(e^{it}\big).$$

It follows that

$$\text{RHS} = \lim_{k\to\infty}\left(\frac{1}{2\pi}\int\limits_{\{w:|w|=1\}} P\left(z,e^{it}\right)u\left(\left(1-\frac{1}{n_k}\right)e^{it}\right)\mathrm{d}m\left(e^{it}\right)\right)$$
$$= \lim_{k\to\infty} u\left(\left(1-\frac{1}{n_k}\right)z\right) = u((1-0)z) = u(z) = \text{LHS}. \qquad \blacksquare)$$

Conclusion 3.94 Let $p \in (1,\infty]$. Let $u : D[0;1] \to \mathbb{R}$ be a continuous function. Let u be harmonic in $D(0;1)$. Let $\sup\left\{\|u_r\|_p : r \in (0,1)\right\} = M < \infty$. Then there exists a function $f : \{w : |w| = 1\} \to \mathbb{C}$ such that $f \in L^p(\{w : |w| = 1\})$, and $u = P[f]$ on $D(0;1)$.

Note 3.95 Let $f : D[0;1] \to \mathbb{C}$ be a continuous function. Let $f \in H^\infty$, that is $f|_{D(0;1)}$ is a member of $H(D(0;1))$ ($\subset C(D(0;1))$).

(Clearly, H^∞ is a linear space. Also, H^∞ is a normed linear space, under the norm given by $\|f\|_\infty \equiv \sup\{|f(z)| : z \in D(0;1)\}$.)

Since $f|_{D(0;1)} \in H(D(0;1))$, by Conclusion 1.239, f is harmonic in $D(0;1)$, and hence $\mathrm{Re}(f), \mathrm{Im}(f)$ are harmonic in $D(0;1)$. Since $f : D[0;1] \to \mathbb{C}$ is continuous, $\mathrm{Re}(f) : D[0;1] \to \mathbb{R}$, and $\mathrm{Im}(f) : D[0;1] \to \mathbb{R}$ are continuous.

For every $r \in (0,1)$,

$$\|(\mathrm{Re}(f))_r\|_\infty = \sup\left\{\left|\left((\mathrm{Re}(f))_r\right)\left(e^{it}\right)\right| : t \in \mathbb{R}\right\} = \sup\left\{\left|(\mathrm{Re}(f))\left(re^{it}\right)\right| : t \in \mathbb{R}\right\}$$
$$\leq \sup\{|(\mathrm{Re}(f))(z)| : z \in D(0;1)\} \leq \sup\{|f(z)| : z \in D(0;1)\} = \|f\|_\infty,$$

so

$$\sup\left\{\|(\mathrm{Re}(f))_r\|_\infty : r \in (0,1)\right\} \leq \|f\|_\infty < \infty.$$

Now, by Conclusion 3.94 there exists a function $g_1 : \{w : |w| = 1\} \to \mathbb{C}$ such that $g_1 \in L^\infty(\{w : |w| = 1\})$, and $\mathrm{Re}(f) = P[g_1]$ on $D(0;1)$. Similarly, there exists a function $g_2 : \{w : |w| = 1\} \to \mathbb{C}$ such that $g_2 \in L^\infty(\{w : |w| = 1\})$, and $\mathrm{Im}(f) = P[g_2]$ on $D(0;1)$.

Let us put $g \equiv g_1 + ig_2$. Clearly,

$$g \in L^\infty(\{w : |w| = 1\}) \left(\subset L^1(\{w : |w| = 1\})\right)$$

such that $(f =)\,\mathrm{Re}(f) + i\mathrm{Im}(f) = P[g]$ on $D(0;1)$, and hence $g \in L^1(\{w : |w| = 1\})$, and $f = P[g]$ on $D(0;1)$. Now, by Conclusion 3.83, if $e^{i\theta}$ is a Lebesgue point of g, then $P[g]$ has non-tangential limit $g\left(e^{i\theta}\right)$ at $e^{i\theta}$, and hence if $e^{i\theta}$ is a Lebesgue point of g, then

$$\lim_{r\to 1^-} f\left(re^{i\theta}\right) = \underbrace{\lim_{r\to 1^-} (P[g])\left(re^{i\theta}\right) = g\left(e^{i\theta}\right)}.$$

Thus, whenever $e^{i\theta}$ is a Lebesgue point of g,

$$\lim_{r\to 1^-} f\left(re^{i\theta}\right) = g\left(e^{i\theta}\right).$$

Next, by Conclusion 3.120, Vol. 1, almost every point of $\{w : |w| = 1\}$ is a Lebesgue point of g, and hence $\lim_{r\to 1^-} f\left(re^{i\theta}\right) = g\left(e^{i\theta}\right)$ a.e. on $\{w : |w| = 1\}$.

By Theorem 3.66, for every $r \in [0, 1)$,

1. $(P[g])_r \in L^\infty(\{z : |z| = 1\})$,
2. $\|(P[g])_r\|_\infty \le \|g\|_\infty$,

so $(\|f\|_\infty =)\ \|P[g]\|_\infty \le \|g\|_\infty$, and hence $\|f\|_\infty \le \|g\|_\infty$.

Problem 3.96 $\|g\|_\infty \le \|f\|_\infty$, that is $\|g\|_\infty \le \sup\{|f(z)| : z \in D(0; 1)\}$, where $g \in L^\infty(\{w : |w| = 1\})$.

(**Solution** By Conclusion 2.20, Vol. 1, it suffices to show that $|g(w)| \le \sup\{|f(z)| : z \in D(0; 1)\}$ holds a.e. on $\{w : |w| = 1\}$.

Here, for almost all points $e^{i\theta}$ of $\{w : |w| = 1\}$,

$$\left|g\left(e^{i\theta}\right)\right| = \left|\lim_{r\to 1^-} f\left(re^{i\theta}\right)\right| = \lim_{r\to 1^-} \left|f\left(re^{i\theta}\right)\right| \le \sup\{|f(z)| : z \in D(0; 1)\},$$

so for almost all points $e^{i\theta}$ of $\{w : |w| = 1\}$,

$$\left|g\left(e^{i\theta}\right)\right| \le \sup\{|f(z)| : z \in D(0; 1)\}.$$

∎)

Conclusion 3.97 Let $f \in H^\infty$. Then there exists $f^* \in L^\infty(\{w : |w| = 1\})$ such that for almost all points $e^{i\theta}$ of $\{w : |w| = 1\}$,

$$f^*\left(e^{i\theta}\right) = \lim_{r\to 1^-} f\left(re^{i\theta}\right),$$

and $\|f\|_\infty = \|f^*\|_\infty$. Further, if I $(\subset \{w : |w| = 1\})$ is an arc, and

$$\lim_{r\to 1^-} f\left(re^{i\theta}\right) = \underbrace{f^*\left(e^{i\theta}\right) = 0}$$

for almost all $e^{i\theta}$ on I, then $f = 0$ on $D(0; 1)$.

Proof of the remaining part: Let us take any arc $I(\subset \{w : |w| = 1\})$. Suppose that $f^*\left(e^{i\theta}\right) = 0$ for almost all points $e^{i\theta}$ on I. We have to show that $f = 0$ on $D(0; 1)$.

There exists a positive integer n_0 such that $\frac{2\pi}{n_0}$ is strictly less than the length $\ell(I)$ of I. Let

$$F : z \mapsto \left(f\left(e^{i\frac{2\pi}{n_0}\times 1} z \right) \cdot f\left(e^{i\frac{2\pi}{n_0}\times 2} z \right) \cdot \cdots \cdot f\left(e^{i\frac{2\pi}{n_0}\times n_0} z \right) \right)$$

be a function from $D[0;1]$ to $\mathbb{C}$.

Since $f \in H^\infty$, we have $F \in H^\infty$. It follows from the preceding result, that there exists $F^* \in L^\infty(\{w : |w| = 1\})$ such that for almost all points $e^{i\theta}$ of $\{w : |w| = 1\}$, $F^*\left(e^{i\theta}\right) = \lim_{r\to 1^-} F\left(re^{i\theta}\right)$, and $\|F\|_\infty = \|F^*\|_\infty$. Since for almost all $e^{i\theta}$ on I, $f^*\left(e^{i\theta}\right) = 0$, for almost all points $e^{i\theta}$ of $\{w : |w| = 1\}$, we have

$$\begin{aligned}
F^*\left(e^{i\theta}\right) &= \lim_{r\to 1^-} F\left(re^{i\theta}\right) \\
&= \lim_{r\to 1^-} \left(f\left(e^{i\frac{2\pi}{n_0}\times 1} re^{i\theta} \right) \cdot f\left(e^{i\frac{2\pi}{n_0}\times 2} re^{i\theta} \right) \cdot \cdots \cdot f\left(e^{i\frac{2\pi}{n_0}\times n_0} re^{i\theta} \right) \right) \\
&= \lim_{r\to 1^-} \left(f\left(re^{i\frac{2\pi}{n_0}\times 1 + \theta} \right) \cdot f\left(re^{i\frac{2\pi}{n_0}\times 2 + \theta} \right) \cdot \cdots \cdot f\left(re^{i\frac{2\pi}{n_0}\times n_0 + \theta} \right) \right) \\
&= \lim_{r\to 1^-} \left(f\left(re^{i\left(\frac{2\pi}{n_0}\times 1 + \theta\right)} \right) \right) \cdot \lim_{r\to 1^-} \left(f\left(re^{i\left(\frac{2\pi}{n_0}\times 2 + \theta\right)} \right) \right) \cdot \cdots \cdot \lim_{r\to 1^-} \left(f\left(re^{i\left(\frac{2\pi}{n_0}\times n_0 + \theta\right)} \right) \right) \\
&= f^*\left(e^{i\left(\frac{2\pi}{n_0}\times 1 + \theta\right)} \right) \cdot f^*\left(e^{i\left(\frac{2\pi}{n_0}\times 2 + \theta\right)} \right) \cdot \cdots \cdot f^*\left(e^{i\left(\frac{2\pi}{n_0}\times n_0 + \theta\right)} \right),
\end{aligned}$$

and for almost all points $e^{i\theta}$ of $\{w : |w| = 1\}$,

$$F^*\left(e^{i\theta}\right) = f^*\left(e^{i\left(\frac{2\pi}{n_0}\times 1 + \theta\right)} \right) \cdot f^*\left(e^{i\left(\frac{2\pi}{n_0}\times 2 + \theta\right)} \right) \cdot \cdots \cdot f^*\left(e^{i\left(\frac{2\pi}{n_0}\times n_0 + \theta\right)} \right).$$

Since $\frac{2\pi}{n_0} < l(I)$, for every real θ, at least one member of

$$\left\{ \frac{2\pi}{n_0} \times 1 + \theta, \frac{2\pi}{n_0} \times 2 + \theta, \ldots, \frac{2\pi}{n_0} \times n_0 + \theta \right\}$$

lies in I, and hence for almost all points $e^{i\theta}$ of

$$\begin{aligned}
&\{w : |w| = 1\},\ \left(F^*\left(e^{i\theta}\right) =\right) f^*\left(e^{i\left(\frac{2\pi}{n_0}\times 1 + \theta\right)} \right) \cdot f^*\left(e^{i\left(\frac{2\pi}{n_0}\times 2 + \theta\right)} \right) \cdot \cdots \\
&\quad \cdot f^*\left(e^{i\left(\frac{2\pi}{n_0}\times n_0 + \theta\right)} \right) \\
&\quad = 0.
\end{aligned}$$

Thus, for almost all points $e^{i\theta}$ of $\{w : |w| = 1\}$, $F^*\left(e^{i\theta}\right) = 0$. This shows that $\left(\sup\{|F(z)| : z \in D(0;1)\} = \|F\|_\infty =\right) \|F^*\|_\infty = 0$, and hence for all $z \in D(0;1)$,

$$f\left(e^{i\frac{2\pi}{n_0}\times 1}z\right) \cdot f\left(e^{i\frac{2\pi}{n_0}\times 2}z\right) \cdot \cdots \cdot f\left(e^{i\frac{2\pi}{n_0}\times n_0}z\right) = \underbrace{F(z) = 0}.$$

Since for all $z \in D\left[0;\frac{1}{2}\right]$,

$$f\left(e^{i\frac{2\pi}{n_0}\times 1}z\right) \cdot f\left(e^{i\frac{2\pi}{n_0}\times 2}z\right) \cdot \cdots \cdot f\left(e^{i\frac{2\pi}{n_0}\times n_0}z\right) = 0,$$

and $D\left[0;\frac{1}{2}\right]$ is uncountable,

$$\left\{z : z \in D\left[0;\frac{1}{2}\right] \text{ and } f\left(e^{i\frac{2\pi}{n_0}\times n_0}z\right) = 0\right\}$$

is uncountable, and hence

$$\left\{z : z \in D\left[0;\frac{1}{2}\right] \text{ and } f(z) = 0\right\}$$

is uncountable. It follows, by Conclusion 1.134, that $f = 0$ on $D(0;1)$. ∎

3.9 Lindelöf's Theorem

Note 3.98 Let $\Gamma : [0,1] \to (D(0;1) \cup \{1\})$ $(\subset \mathbb{C})$ be a curve, that is $\Gamma : [0,1] \to (D(0;1) \cup \{1\})$ is continuous. Suppose that $\Gamma([0,1)) \subset D(0;1)$, and $\Gamma(1) = 1$. Let $g : D(0;1) \to D(0;1)$ be a holomorphic function. Hence, $|g| < 1$.

It follows that $\mathrm{dom}(g \circ \Gamma) = [0,1)$, and hence 1 is a limit point of $\mathrm{dom}(g \circ \Gamma)$. Also, $(g \circ \Gamma) : [0,1) \to D(0;1)$.

Next, suppose that $\lim_{t\to 1^-}(g \circ \Gamma)(t) = 0$. We shall try to show that $\lim_{r\to 1^-} g(r) = 0$.

For this purpose, let us take any $\varepsilon > 0$.

Problem 3.99 There exists $t_0 \in (0,1)$ such that

$$t \in (t_0, 1) \Rightarrow \left(|g(\Gamma(t))| < \varepsilon^4 \text{ and } \frac{1}{2} < (\mathrm{Re}(\Gamma))(t_0) \le (\mathrm{Re}(\Gamma))(t) < 1\right).$$

(Solution Since $\lim_{t\to 1^-}(g \circ \Gamma)(t) = 0$, there exists $t_1 \in (0,1)$ such that $t \in (t_1, 1) \Rightarrow |(g \circ \Gamma)(t) - 0| < \varepsilon^4$. Thus, $t \in (t_1, 1) \Rightarrow |g(\Gamma(t))| < \varepsilon^4$.

Since $\Gamma : [0,1] \to (D(0;1) \cup \{1\})$ is continuous, $\mathrm{Re}(\Gamma) : [0,1] \to ((-1,1) \cup \{1\})\ (= (-1,1])$ is continuous. Also,

$$(\mathrm{Re}(\Gamma))([0,1)) \subset (-1,1), \text{ and } (\mathrm{Re}(\Gamma))(1) = 1.$$

Since $\mathrm{Re}(\Gamma) : [0,1] \to (-1,1]$ is continuous,

$$\lim_{t \to 1^-} (\mathrm{Re}(\Gamma))(t) = (\mathrm{Re}(\Gamma))(1)\ (= 1).$$

Since $\lim_{t \to 1^-} (\mathrm{Re}(\Gamma))(t) = 1$, there exists $s_0 \in (t_1, 1)$ such that

$$s \in (s_0, 1] \Rightarrow \left(1 - \frac{1}{2}\right) < (\mathrm{Re}(\Gamma))(s).$$

Since $\mathrm{Re}(\Gamma) : [0,1] \to (-1,1]$ is continuous, there exists $t_0 \in \left[\frac{s_0+1}{2}, 1\right]$ $(\subset (s_0, 1])$ such that

$$s \in \left[\frac{s_0+1}{2}, 1\right] \Rightarrow (\mathrm{Re}(\Gamma))(t_0) \le (\mathrm{Re}(\Gamma))(s)\ (\le 1).$$

Also, $\left(1 - \frac{1}{2}\right) < (\mathrm{Re}(\Gamma))(t_0)$.

Problem 3.100 $t_0 \neq 1$.

(Solution If not, otherwise, let $t_0 = 1$. We have to arrive at a contradiction. Since

$$s \in \left[\frac{s_0+1}{2}, 1\right] \Rightarrow (\mathrm{Re}(\Gamma))(t_0) \le (\mathrm{Re}(\Gamma))(s)\ (\le 1),$$

and $(\mathrm{Re}(\Gamma))([0,1)) \subset (-1,1)$,
we have

$$1 = (\mathrm{Re}(\Gamma))(1) = \underbrace{(\mathrm{Re}(\Gamma))(t_0) \le (\mathrm{Re}(\Gamma))\left(\frac{\frac{s_0+1}{2} + 1}{2}\right)} < 1.$$

This is a contradiction. ∎)
Thus, $t_0 < 1$. Here,

$$t \in (t_0, 1) \Rightarrow \frac{1}{2} < (\mathrm{Re}(\Gamma))(t_0) \le (\mathrm{Re}(\Gamma))(t) < 1.$$

Since

$$(t_0, 1) \subset \left[\frac{s_0 + 1}{2}, 1\right) \subset (s_0, 1) \subset (t_1, 1),$$

we have

$$t \in (t_0, 1) \Rightarrow |g(\Gamma(t))| < \varepsilon^4.$$

Thus, $t \in (t_0, 1) \Rightarrow \left(|g(\Gamma(t))| < \varepsilon^4 \text{ and } \frac{1}{2} < (\mathrm{Re}(\Gamma))(t_0) \le (\mathrm{Re}(\Gamma))(t) < 1\right)$. ∎)
Let us take any $r \in ((\mathrm{Re}(\Gamma))(t_0), 1)$. It suffices to show that $|g(r)| \le \varepsilon$.

Since $\frac{1}{2} < (\mathrm{Re}(\Gamma))(t_0) < r < 1$, we have $1 < 2r < 2$, and hence $D(0;1) \cap D(2r;1) \neq \emptyset$. Thus, $D(0;1) \cap D(2r;1)$ is a nonempty open set such that

$$z \in D(0;1) \cap D(2r;1) \Rightarrow \bar{z} \in D(0;1) \cap D(2r;1).$$

Also, $D(0;1) \cap D(2r;1)$ is connected.

Observe that if $z \in D(0;1) \cap D(2r;1)$, then $|2r - z| < 1$, and $(|2r - (2r - z)| =)$ $|z| < 1$. Thus,

$$z \in D(0;1) \cap D(2r;1) \Rightarrow (2r - z) \in D(0;1) \cap D(2r;1).$$

Let us put $\Omega \equiv D(0;1) \cap D(2r;1)$ $(\subset D(0;1))$.

Thus, $\Omega(\subset D(0;1))$ is a region symmetric about the x-axis, symmetric about the line $x = r$, and $z \in \Omega \Rightarrow (\bar{z} \in \Omega,\ (2r - z) \in \Omega, \text{ and } (2r - \bar{z}) \in \Omega)$.

Now, since $g : D(0;1) \to D(0;1)$, for every $z \in \Omega$, we have $g(z) \in D(0;1)$, $g(\bar{z}) \in D(0;1)$, $\overline{g(\bar{z})} \in D(0;1)$, $g(2r - z) \in D(0;1)$, $g\left(\overline{2r - z}\right) \in D(0;1)$, and $\overline{g\left(\overline{2r - z}\right)} \in D(0;1)$. Thus, for every $z \in \Omega$, the product $g(z) \cdot \overline{g(\bar{z})} \cdot g(2r - z) \cdot \overline{g(2r - \bar{z})} \in D(0;1)$.

Let

$$h : z \cdot \left(g(z) \cdot \overline{g(\bar{z})} \cdot g(2r - z) \cdot \overline{g(2r - \bar{z})}\right)$$

be a function from Ω to $D(0;1)$. Here h is a nonnegative real number on $\Omega \cap \mathbb{R}$ $(= (2r - 1, 1) \subset (0, 1))$.

Observe that for every $z \in \Omega$,

$$\begin{aligned} h(2r - z) &= \left(g(2r - z) \cdot \overline{g(2r - \bar{z})} \cdot g(2r - (2r - z)) \cdot \overline{g(2r - (2r - \bar{z}))}\right) \\ &= \left(g(2r - z) \cdot \overline{g(2r - \bar{z})} \cdot g(z) \cdot \overline{g(\bar{z})}\right) = h(z), \end{aligned}$$

so for every $z \in \Omega$, $h(2r - z) = h(z)$.

Next, for every $z \in \Omega$,

$$\begin{aligned}h(\bar{z}) &= \left(g(\bar{z}) \cdot \overline{g(z)} \cdot g(2r-\bar{z}) \cdot \overline{g(2r-z)}\right) \\ &= \left(g(z) \cdot \overline{g(\bar{z})} \cdot g(2r-z) \cdot \overline{g(2r-\bar{z})}\right)^{-} = \overline{h(z)},\end{aligned}$$

so for every $z \in \Omega$, $h(\bar{z}) = \overline{h(z)}$.

Problem 3.101 $h \in H(\Omega)$.

(Solution Let us fix any $z \in \Omega$. Since

$$\begin{aligned}&\lim_{k\to 0} \frac{\overline{g\left(\overline{z+k}\right)} - \overline{g(\bar{z})}}{k} = \lim_{k\to 0} \frac{\overline{g(\bar{z}+\bar{k})-g(\bar{z})}}{k} = \lim_{k\to 0}\left(\left(\frac{g(\bar{z}+\bar{k})-g(\bar{z})}{\bar{k}}\right)^{-}\right) \\ &= \left(\lim_{k\to 0} \frac{g(\bar{z}+\bar{k})-g(\bar{z})}{\bar{k}}\right)^{-} = \left(\lim_{k\to 0} \frac{g(\bar{z}+\bar{k})-g(\bar{z})}{\bar{k}}\right)^{-} = \overline{g'(\bar{z})},\end{aligned}$$

the function $z \mapsto \overline{g(\bar{z})}$ from Ω to $D(0;1)$ is a member of $H(\Omega)$. Similarly, the function $z \mapsto g(2r-z)$ from Ω to $D(0;1)$ is a member of $H(\Omega)$, and the function $z \mapsto \overline{g(2r-\bar{z})}$ from Ω to $D(0;1)$ is a member of $H(\Omega)$. It follows that the function

$$h : z \mapsto \left(g(z) \cdot \overline{g(\bar{z})} \cdot g(2r-z) \cdot \overline{g(2r-\bar{z})}\right)$$

from Ω to $D(0;1)$ is a member of $H(\Omega)$. Thus, $h \in H(\Omega)$. ∎)

Since $r \in ((\mathrm{Re}(\Gamma))(t_0), 1)$, we have $\left(\frac{1}{2} <\right) (\mathrm{Re}(\Gamma))(t_0) < r < 1$, and hence $r \in D(0;1)$. Since $(0 <)\ 2r - r < 1$, we have $|2r - r| < 1$, and hence $r \in D(2r;1)$. Thus, $r \in D(0;1) \cap D(2r;1)\ (= \Omega)$. Since $r \in \Omega$, we have

$$h(r) = g(r) \cdot \overline{g(\bar{r})} \cdot g(2r-r) \cdot \overline{g(2r-\bar{r})} = g(r) \cdot \overline{g(r)} \cdot g(r) \cdot \overline{g(r)} = |g(r)|^4,$$

and hence $h(r) = |g(r)|^4$. It suffices to show that $|h(r)| \le \varepsilon^4$.

Since

$$(\mathrm{Re}(\Gamma))(t_0) < r < 1\ (= (\mathrm{Re}(\Gamma))(1)),$$

and

$$\mathrm{Re}(\Gamma) : [0,1] \to (-1,1]$$

is continuous, there exists $s_0 \in (t_0, 1)$ such that $(\mathrm{Re}(\Gamma))(s_0) = r$. It follows that $(\mathrm{Re}(\Gamma))^{-1}(r)$ is a nonempty closed subset of the compact set $[0,1]$, and hence

$$\left(\sup\left((\mathrm{Re}(\Gamma))^{-1}(r)\right)\right) \in \left((\mathrm{Re}(\Gamma))^{-1}(r)\right)^{-} = \left((\mathrm{Re}(\Gamma))^{-1}(r)\right).$$

It follows that $\max\left((\mathrm{Re}(\Gamma))^{-1}(r)\right)$ exists. Put $t_1 \equiv \max\left((\mathrm{Re}(\Gamma))^{-1}(r)\right)$.

It follows that $(\text{Re}(\Gamma))(t_1) = r$. Also, if $(\text{Re}(\Gamma))(s) = r$ then $s \leq t_1$. Since $(\text{Re}(\Gamma))(s_0) = r$, we have $s_0 \leq t_1$. Since $s_0 \in (t_0, 1)$, we have $t_0 < s_0 \ (\leq t_1)$, and hence $t_0 < t_1$. Since $(\text{Re}(\Gamma))(t_1) = r < 1$, we have $t_1 \in [0, 1)$. Thus $0 < t_0 < t_1 < 1$. Since $t_1 = \max\left((\text{Re}(\Gamma))^{-1}(r)\right)$,

$$r \notin \{(\text{Re}(\Gamma))(s) : s \in (t_1, 1]\},$$

and hence $r \notin \{\Gamma(s) : s \in (t_1, 1]\}$. Here, $1 \in \{\Gamma(s) : s \in (t_1, 1]\}$.

Put $E_1 \equiv \{\Gamma(s) : s \in [t_1, 1]\}$, and $E_2 \equiv \left\{\overline{\Gamma(s)} : s \in [t_1, 1]\right\}$. Let E be the union of $(E_1 \cup E_2)$ and the reflection of $(E_1 \cup E_2)$ in the line $x = r$.

There exists a closed curve γ such that $\text{ran}(\gamma) = E$. Since γ is continuous on a closed bounded interval, $\text{ran}(\gamma)$ $(= E)$ is compact, and hence E is compact. It follows that the union, say K, of E and all bounded components of $(\text{ran}(\gamma))^c$ $(= e^c)$ is a compact set, and the boundary $\partial K = E$ $(= \text{ran}(\gamma))$.

Problem 3.102 For every $z \in \Omega \cap E$, $|h(z)| < \varepsilon^4$.

(Solution Let us fix any $z \in \Omega \cap E$. We have to show that

$$\begin{aligned} |g(z)||g(\bar{z})||g(2r - z)||g(2r - \bar{z})| &= \left|g(z) \cdot \overline{g(\bar{z})} \cdot g(2r - z) \cdot \overline{g(2r - \bar{z})}\right| \\ &= \underbrace{|h(z)| < \varepsilon^4}, \end{aligned}$$

that is

$$|g(z)||g(\bar{z})||g(2r - z)||g(2r - \bar{z})| < \varepsilon^4.$$

Now, since $|g| < 1$ on $D(0; 1)$, and $\Omega(\subset D(0; 1))$, it suffices to show that $(|g(z)| < \varepsilon^4$ or $|g(\bar{z})| < \varepsilon^4$ or $|g(2r - z)| < \varepsilon^4$ or $|g(2r - \bar{z})| < \varepsilon^4)$.

Since $z \in \Omega \cap E$ $(\subset \Omega = D(0; 1) \cap D(2r; 1))$, we have $z \neq 1$ and $z \neq 2r - 1$.

Since $z \in \Omega \cap E$, and E is the union of $(E_1 \cup E_2)$ and the reflection of $(E_1 \cup E_2)$ in the line $x = r$, $z \in (E_1 \cup E_2)$ or z is a member of the reflection of $(E_1 \cup E_2)$ in the line $x = r$.

Case I: when $z \in (E_1 \cup E_2)$. It follows that $z \in E_1$ or $z \in E_2$.

Subcase Ia: when $z \in E_1$. It follows that there exists $s \in [t_1, 1]$ such that $\Gamma(s) = z$. Since $z \neq 1$, we have $s \neq 1$. Now, since $s \in [t_1, 1]$, we have $(t_0 <)\ t_1 \leq s < 1$ and hence $(|g(z)| =)\ |g(\Gamma(s))| < \varepsilon^4$. Thus, $(|g(z)| < \varepsilon^4$ or $|g(\bar{z})| < \varepsilon^4$ or $|g(2r - z)| < \varepsilon^4$ or $|g(2r - \bar{z})| < \varepsilon^4)$.

Subcase Ib: when $z \in E_2$. It follows that there exists $s \in [t_1, 1]$ such that $\overline{\Gamma(s)} = z$. Since $z \neq 1$, we have $s \neq 1$. Now, since $s \in [t_1, 1]$, we have $(t_0 <)\ t_1 \leq s < 1$, and hence $(|g(\bar{z})| =)\ |g(\Gamma(s))| < \varepsilon^4$. Thus, $(|g(z)| < \varepsilon^4$ or $|g(\bar{z})| < \varepsilon^4$ or $|g(2r - z)| < \varepsilon^4$ or $|g(2r - \bar{z})| < \varepsilon^4)$.

Case II: when z is a member of the reflection of $(E_1 \cup E_2)$ in the line $x = r$.

Subcase IIa: when z is a member of the reflection of E_1 in the line $x = r$. It follows that there exists $s \in [t_1, 1]$ such that

$$2r - \overline{\Gamma(s)} = \underbrace{(2r - \mathrm{Re}(\Gamma(s))) + i\,\mathrm{Im}(\Gamma(s)) = z}.$$

Now, since $z \neq 2r - 1$, we have $s \neq 1$. Since $s \in [t_1, 1]$, we have $(t_0 <)\ t_1 \leq s < 1$, and hence $(|g(2r - \bar{z})| =)\ |g(\Gamma(s))| < \varepsilon^4$. Thus, $(|g(z)| < \varepsilon^4$ or $|g(\bar{z})| < \varepsilon^4$ or $|g(2r - z)| < \varepsilon^4$ or $|g(2r - \bar{z})| < \varepsilon^4)$.

Subcase IIb: when z is a member of the reflection of E_2 in the line $x = r$. It follows that there exists $s \in [t_1, 1]$ such that $(2r - \Gamma(s) =)\ (2r - \mathrm{Re}(\Gamma(s))) - i\,\mathrm{Im}(\Gamma(s)) = z$. Now, since $z \neq 2r - 1$, we have $s \neq 1$. Since $s \in [t_1, 1]$, we have $(t_0 <)\ t_1 \leq s < 1$, and hence $(|g(2r - z)| =)\ |g(\Gamma(s))| < \varepsilon^4$. Thus, $(|g(z)| < \varepsilon^4$ or $|g(\bar{z})| < \varepsilon^4$ or $|g(2r - z)| < \varepsilon^4$ or $|g(2r - \bar{z})| < \varepsilon^4)$.

Thus, in all cases, $(|g(z)| < \varepsilon^4$ or $|g(\bar{z})| < \varepsilon^4$ or $|g(2r - z)| < \varepsilon^4$ or $|g(2r - \bar{z})| < \varepsilon^4)$ holds. ∎)

For every real $c \in (0, 1)$, let

$$h_c : z \mapsto \begin{cases} h(z) \cdot ((1 - z)(z - (2r - 1)))^c & \text{if } z \in \Omega \\ 0 & \text{if } z = 1 \text{ or } (2r - 1) \end{cases}$$

be a function from $\Omega \cup \{(2r - 1), 1\}$ to $\mathbb{C}$.

Since $h : \Omega \to D(0; 1)$ is continuous, by the definition of h_c, for every real $c \in (0, 1)$, $h_c : \Omega \cup \{(2r - 1), 1\} \to \mathbb{C}$ is continuous, and hence h_c is continuous at all points of K. Since $(r \in)\ K^0 \subset \Omega$, and $h \in H(\Omega)$, by the definition of h_c, for every real $c \in (0, 1)$, h_c is holomorphic in K^0. Now, by Conclusion 1.156, there exists $z_0 \in \partial K$ such that $|h_c(r)| \leq |h_c(z_0)|$.

Since $\frac{1}{2} < r < 1$, for every $z \in \partial K\ (= E)$, we have $|z - (2r - 1)| < 1$, and $|1 - z| < 1$. Now, since for every $z \in \Omega \cap E$, $|h(z)| < \varepsilon^4$, by the definition of h_c, for every real $c \in (0, 1)$, and for every $z \in (\partial K - \{(2r - 1), 1\})$, $|h_c(z)| \leq |h(z)| < \varepsilon^4$. It follows that for every real $c \in (0, 1)$, and for every $z \in \partial K$, $|h_c(z)| < \varepsilon^4$. Since $z_0 \in \partial K$, for every real $c \in (0, 1)$, $(|h_c(r)| \leq)\ |h_c(z_0)| < \varepsilon^4$, and hence for every real $c \in (0, 1)$, $|h_c(r)| < \varepsilon^4$. This shows that $\lim_{c \to 0} |h_c(r)| \leq \varepsilon^4$.

Since the function $c \mapsto h_c(r)\ (= h(r) \cdot ((1 - r)(r - (2r - 1)))^c = h(r) \cdot (1 - r)^{2c})$ from [0,1] to $\mathbb{R}$ is continuous, we have $(\varepsilon^4 \geq)\ \lim_{c \to 0} |h_c(r)| = |h_0(r)|$ $\left(= \left|h(r) \cdot (1 - r)^{2 \cdot 0}\right| = |h(r)|\right)$, and hence $|h(r)| \leq \varepsilon^4$.

Conclusion 3.103 Let $\Gamma : [0, 1] \to (D(0; 1) \cup \{1\})\ (\subset \mathbb{C})$ be a curve. Suppose that $\Gamma([0, 1)) \subset D(0; 1)$, and $\Gamma(1) = 1$. Let $g : D(0; 1) \to D(0; 1)$ be a holomorphic function. Suppose that $\lim_{t \to 1^-} (g \circ \Gamma)(t) = 0$. Then $\lim_{r \to 1^-} g(r) = 0$.

Theorem 3.104 Let $\Gamma : [0,1] \to (D(0;1) \cup \{1\})\ (\subset \mathbb{C})$ be a curve. Suppose that $\Gamma([0,1)) \subset D(0;1)$, and $\Gamma(1) = 1$. Let $g : D(0;1) \to \mathbb{C}$ be a bounded holomorphic function. Let $L \in \mathbb{C}$. Suppose that $\lim_{t \to 1^-}(g \circ \Gamma)(t) = L$. Then

$$\lim_{r \to 1^-} g(r) = L.$$

Proof Since $g : D(0;1) \to \mathbb{C}$ is bounded, $(g - L) : D(0;1) \to \mathbb{C}$ is bounded, and hence there exists a positive real number M such that $|M(g-L)| < 1$. Thus, $M(g-L) : D(0;1) \to D(0;1)$. Since $g : D(0;1) \to \mathbb{C}$ is a holomorphic function, $M(g-L) : D(0;1) \to \mathbb{C}$ is a holomorphic function. Here,

$$\begin{aligned}\lim_{t \to 1^-}((M(g-L)) \circ \Gamma)(t) &= M \lim_{t \to 1^-}(((g-L)) \circ \Gamma)(t) \\ &= M \lim_{t \to 1^-}((g \circ \Gamma)(t) - L) \\ &= M\left(\left(\lim_{t \to 1^-}(g \circ \Gamma)(t)\right) - L\right) = M(L-L) = 0,\end{aligned}$$

so $\lim_{t \to 1^-}((M(g-L)) \circ \Gamma)(t) = 0$. Now, by Conclusion 3.103,

$$M\left(\lim_{r \to 1^-} g(r) - L\right) = M \lim_{r \to 1^-}(g-L)(r) = \underbrace{\lim_{r \to 1^-}(M(g-L))(r)} = 0,$$

and hence $\lim_{r \to 1^-} g(r) = L$. ∎

The above theorem, known as the **Lindelöf's theorem**, is due to E. L. Lindelöf (07.03.1870–04.03.1946).

3.10 Monodromy Theorem

Note 3.105 Definition Let Ω be a simply connected region. Let β be a boundary point of Ω, that is $\beta \in \partial\Omega = \overline{\Omega} - \Omega^0 = \overline{\Omega} - \Omega$, that is β is a limit point of Ω, and β is not in Ω. If, for every convergent sequence $\{\alpha_n\}$ in Ω satisfying $\lim_{n \to \infty} \alpha_n = \beta$, there exist a curve $\gamma : [0,1] \to \mathbb{C}$ and a convergent sequence $\{t_n\}$ in $(0,1)$ such that

1. $\gamma([0,1)) \subset \Omega$,
2. $0 < t_1 < t_2 < t_3 < \cdots$,
3. $\lim_{n \to \infty} t_n = 1$,
4. for every positive integer n, $\gamma(t_n) = \alpha_n$,

then we say that β *is a* ***simple boundary point*** *of* Ω.

Example: Observe that $(D(0;1) - [0,1))$ is a simply connected region. Here, $\partial(D(0;1) - [0,1)) = \{z : |z| = 1\} \cup [0,1)$, and the set of all simple boundary points of $(D(0;1) - [0,1))$ is $(0,1]$.

Let Ω be a bounded simply connected region. Let β be a simple boundary point of Ω (and hence $\beta \notin \Omega$). Let $f : \Omega \to D(0;1)$ be a 1-1 function from Ω onto $D(0;1)$. Suppose that f is a conformal mapping; that is, $f \in H(\Omega)$, and for every $z \in \Omega, f'(z) \neq 0$.

Problem 3.106 There exists $a \in \{w : |w| = 1\}$ such that the 1-1 function $F_a :$ $z \mapsto \begin{cases} f(z) \text{ if } z \in \Omega \\ a \text{ if } z = \beta \end{cases}$ from $\Omega \cup \{\beta\}$ onto $D(0;1) \cup \{a\}$ is continuous.

(**Solution** If not, suppose otherwise that for every $a \in \{w : |w| = 1\}$, the 1-1 function

$$F_a : z \mapsto \begin{cases} f(z) \text{ if } z \in \Omega \\ a \text{ if } z = \beta \end{cases}$$

from $\Omega \cup \{\beta\}$ onto $D(0;1) \cup \{a\}$ is not continuous at a. We have to arrive at a contradiction.

Since $f : \Omega \to D(0;1)$ is a 1-1 and onto function, $f^{-1} : D(0;1) \to \Omega$ is 1-1 and onto. Next, since for every $z \in \Omega, f'(z) \neq 0$, by Conclusion 1.181, $f^{-1} \in H(D(0;1))$. Since $f^{-1} : D(0;1) \to \Omega$, and Ω is bounded, f^{-1} is bounded, and hence $f^{-1} \in H^\infty$. Since $f \in H(\Omega)$, $f : \Omega \to D(0;1)$ is continuous. Since $f^{-1} \in H(D(0;1)), f^{-1} : D(0;1) \to \Omega$ is continuous. It follows that $f : \Omega \to D(0;1)$ is a homeomorphism.

Problem 3.107 There exists a convergent sequence $\{\alpha_n\}$ in Ω such that

1. $\lim_{n\to\infty} \alpha_n = \beta$,
2. $(\lim_{n\to\infty} f(\alpha_{2n+1})), (\lim_{n\to\infty} f(\alpha_{2n})) \in \{w : |w| = 1\}$, and $(\lim_{n\to\infty} f(\alpha_{2n})) \neq (\lim_{n\to\infty} f(\alpha_{2n+1}))$.

(**Solution** Let us fix any $a \in \{w : |w| = 1\}$. Since the 1-1 function

$$F_a : z \mapsto \begin{cases} f(z) \text{ if } z \in \Omega \\ a \text{ if } z = \beta \end{cases}$$

from $\Omega \cup \{\beta\}$ onto $D(0;1) \cup \{a\}$ is not continuous at a, and $\beta \in (\overline{\Omega} - \Omega)$, there exists $\varepsilon > 0$ such that for every positive integer n, there exists $\alpha_n \in (\Omega \cup \{\beta\})$ such that $\alpha_n \in D(\beta, \frac{1}{n})$ and $F_a(\alpha_n) \notin D(F_a(\beta); \varepsilon)$ $(= D(a; \varepsilon))$. It follows that for every positive integer n, $F_a(\alpha_n) \neq a$, and hence, by the definition of F_a, each $\alpha_n \in \Omega$. Thus, for every positive integer n, $\alpha_n \in \Omega$, $\alpha_n \in D(\beta, \frac{1}{n})$ and $f(\alpha_n) \notin D(a; \varepsilon)$.

Since for every positive integer n, $\alpha_n \in D(\beta, \frac{1}{n})$, we have $\lim_{n\to\infty} \alpha_n = \beta$. Now, since $\beta \in (\overline{\Omega} - \Omega)$, and each $\alpha_n \in \Omega$, $\{a_1, a_2, a_3, \ldots\}$ is an infinite subset of Ω.

Now, since $f : \Omega \to D(0;1)$ is 1-1, $\{f(a_1), f(a_2), f(a_3), \ldots\}$ is an infinite subset of $D(0;1)$ $(\subset D[0;1])$. Since $\{f(a_1), f(a_2), f(a_3), \ldots\}$ is an infinite subset of the compact set $D[0;1]$, there exists a subsequence $\{f(a_{k_n})\}$ of $\{f(a_n)\}$ such that $\lim_{n\to\infty} f(a_{k_n}) \in D[0;1]$. Since for every positive integer $n, f(\alpha_n) \notin D(a;\varepsilon)$, $\lim_{n\to\infty} f(a_{k_n}) \neq a$.

Problem 3.108 $|\lim_{n\to\infty} f(a_{k_n})| = 1$.

(**Solution** If not, otherwise, let $|\lim_{n\to\infty} f(a_{k_n})| < 1$. We have to arrive at a contradiction.

Since $|\lim_{n\to\infty} f(a_{k_n})| < 1$, we have $(\lim_{n\to\infty} f(a_{k_n})) \in D(0;1)$. Now, since $f^{-1} : D(0;1) \to \Omega$ is continuous, and $\{f(a_{k_n})\}$ is a convergent sequence in $D(0;1)$, we have

$$\beta = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} a_{k_n} = \underbrace{\lim_{n\to\infty} f^{-1}(f(a_{k_n})) = f^{-1}\left(\left(\lim_{n\to\infty} f(a_{k_n})\right)\right)} \in \Omega,$$

and hence $\beta \in \Omega$. This is a contradiction. ∎)

Since $\{a_{k_n}\}$ is a subsequence of $\{a_n\}$, and $\lim_{n\to\infty} \alpha_n = \beta$, we have $\lim_{n\to\infty} a_{k_n} = \beta$. For every positive integer n, let us put $b_n \equiv a_{k_n}$.

Thus, for every $a \in \{w : |w| = 1\}$, there exists a convergent sequence $\{b_n\}$ in Ω such that $\lim_{n\to\infty} b_n = \beta$, $\lim_{n\to\infty} f(b_n) \neq a$, and $(\lim_{n\to\infty} f(b_n)) \in \{w : |w| = 1\}$.

Again, there exists a convergent sequence $\{c_n\}$ in Ω such that $\lim_{n\to\infty} c_n = \beta$, $(\lim_{n\to\infty} f(c_n)) \neq (\lim_{n\to\infty} f(b_n))$, and $(\lim_{n\to\infty} f(c_n)) \in \{w : |w| = 1\}$.

For every positive integer n, put $\alpha_n \equiv \begin{cases} b_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ c_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$

Thus, the sequence $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ becomes $\{b_1, c_1, b_2, c_2, \ldots\}$.

Clearly, $\{\alpha_n\}$ is a convergent sequence in Ω such that $\lim_{n\to\infty} \alpha_n = \beta$, $(\lim_{n\to\infty} f(\alpha_{2n+1})), (\lim_{n\to\infty} f(\alpha_{2n})) \in \{w : |w| = 1\}$, and $(\lim_{n\to\infty} f(\alpha_{2n})) \neq (\lim_{n\to\infty} f(\alpha_{2n+1}))$. ∎)

Since β is a simple boundary point of Ω, and $\{\alpha_n\}$ is a convergent sequence in Ω satisfying $\lim_{n\to\infty} \alpha_n = \beta$, there exist a curve $\gamma : [0,1] \to \mathbb{C}$ and a convergent sequence $\{t_n\}$ in $(0,1)$ such that

1. $\gamma([0,1)) \subset \Omega$,
2. $0 < t_1 < t_2 < t_3 < \cdots$,
3. $\lim_{n\to\infty} t_n = 1$,
4. for every positive integer n, $\gamma(t_n) = \alpha_n$.

Let J_1 and J_2 be the two open arcs such that

$$J_1 \cup J_2 = \{w : |w| = 1\} - \left\{\left(\lim_{n\to\infty} f(\alpha_{2n+1})\right), \left(\lim_{n\to\infty} f(\alpha_{2n})\right)\right\}.$$

Since $\gamma([0,1)) \subset \Omega$ and $f : \Omega \to D(0;1)$, we have

$$\{f(\gamma(t)) : t \in [0,1)\} \subset D(0;1).$$

Since $(\lim_{n\to\infty} f(\gamma(t_{2n+1}))), (\lim_{n\to\infty} f(\gamma(t_{2n}))) \in \{w : |w| = 1\}$, and $(\lim_{n\to\infty} f(\gamma(t_{2n+1}))) \neq (\lim_{n\to\infty} f(\gamma(t_{2n})))$, one of the two open arcs, say J, has the following property:

For every radius of $D(0;1)$ that ends at a point, say $e^{i\theta}$, of J, the intersection of this radius with $\{f(\gamma(t)) : t \in [0,1)\}$ $(\subset D(0;1))$ has a limit point $e^{i\theta}$ on $\{w : |w| = 1\}$.

Also,

$$\gamma(1) = \gamma\left(\lim_{n\to\infty} t_n\right) = \lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} \alpha_n = \beta,$$

so

$$\lim_{t\to 1^-} f^{-1}(f(\gamma(t))) = \lim_{t\to 1^-} \gamma(t) = \underbrace{\gamma(1) = \beta},$$

and hence

$$\lim_{t\to 1^-} f^{-1}(f(\gamma(t))) = \beta.$$

This shows that for every $e^{i\theta} \in J$, $\lim_{r\to 1^-} f^{-1}\left(re^{i\theta}\right) = \beta$, and hence for every $e^{i\theta} \in J$, $\lim_{r\to 1^-} (f^{-1} - \beta)\left(re^{i\theta}\right) = 0$. Now, by Conclusion 3.97, $(f^{-1} - \beta) = 0$ on $D(0;1)$, and hence f^{-1} is not 1-1. This is a contradiction. ∎)

Conclusion 3.109 Let Ω be a bounded simply connected region. Let $f : \Omega \to D(0;1)$ be a 1-1 function from Ω onto $D(0;1)$. Suppose that f is a conformal mapping. Then

a. if β is a simple boundary point of Ω, then f has a continuous extension to $\Omega \cup \{\beta\}$ such that $|f(\beta)| = 1$,
b. if β_1 and β_2 are distinct simple boundary points of Ω, and if f has a continuous extension to $\Omega \cup \{\beta_1, \beta_2\}$ as in Conclusion 3.109(a), then $f(\beta_1) \neq f(\beta_2)$.

Proof of the remaining part

b. Let β_1 and β_2 be distinct simple boundary points of Ω. By Conclusion 3.109 (a), there exist $a, b \in \{w : |w| = 1\}$ such that the function

$$F : z \mapsto \begin{cases} f(z) & \text{if } z \in \Omega \\ a & \text{if } z = \beta_1 \\ b & \text{if } z = \beta_2 \end{cases}$$

from $\Omega \cup \{\beta_1, \beta_2\}$ onto $D(0;1) \cup \{a, b\}$ is continuous, $(|a| =)\ |F(\beta_1)| = 1$, and $(|b| =)\ |F(\beta_2)| = 1$. We have to show that $a \neq b$.

If not, otherwise, let $a = b$. We have to arrive at a contradiction.

Here, the function

$$F : z \mapsto \begin{cases} f(z) & \text{if } z \in \Omega \\ a & \text{if } z = \beta_1 \\ a & \text{if } z = \beta_2 \end{cases}$$

from $\Omega \cup \{\beta_1, \beta_2\}$ onto $D(0;1) \cup \{a\}$ is continuous. There exists a convergent sequences $\{\alpha_{1n}\}$ and $\{\alpha_{2n}\}$ in Ω such that $\lim_{n\to\infty} \alpha_{1n} = \beta_1$, and $\lim_{n\to\infty} \alpha_{2n} = \beta_2$. Since β_1 is a simple boundary point of Ω, and $\{\alpha_{1n}\}$ is a convergent sequence in Ω satisfying $\lim_{n\to\infty} \alpha_{1n} = \beta_1$, there exist a curve $\gamma_1 : [0,1] \to \mathbb{C}$, and a convergent sequence $\{t_{1n}\}$ in $(0,1)$ such that

1. $\gamma_1([0,1)) \subset \Omega$,
2. $0 < t_{11} < t_{12} < t_{13} < \cdots$,
3. $\lim_{n\to\infty} t_{1n} = 1$,
4. for every positive integer n, $\gamma_1(t_{1n}) = \alpha_{1n}$.

Similarly, there exist a curve $\gamma_2 : [0,1] \to \mathbb{C}$ and a convergent sequence $\{t_{2n}\}$ in $(0,1)$ such that

1. $\gamma_2([0,1)) \subset \Omega$,
2. $0 < t_{21} < t_{22} < t_{23} < \cdots$,
3. $\lim_{n\to\infty} t_{2n} = 1$,
4. for every positive integer n, $\gamma_2(t_{2n}) = \alpha_{2n}$.

Here,

$$\gamma_1(1) = \gamma_1\left(\lim_{n\to\infty} t_{1n}\right) = \lim_{n\to\infty} \gamma_1(t_{1n}) = \lim_{n\to\infty} \alpha_{1n} = \beta_1,$$

so $\gamma_1(1) = \beta_1$. Similarly, $\gamma_2(1) = \beta_2$. Since $|a| = 1$,

$$\gamma_1([0,1]) \subset \Omega \cup \{\beta_1\} \quad (\subset \Omega \cup \{\beta_1, \beta_2\}),$$

and

$$F : \Omega \cup \{\beta_1, \beta_2\} \to D(0;1) \cup \{a\}$$

is continuous,

$$\left(\left(\frac{1}{a}F\right) \circ \gamma_1\right) : [0,1] \to D(0;1) \cup \{1\}$$

is continuous, and hence

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right):[0,1]\to D(0;1)\cup\{1\}$$

is a curve. Similarly, $\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right):[0,1]\to D(0;1)\cup\{1\}$ is a curve. Since

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right)(1)=\frac{1}{a}F(\gamma_1(1))=\frac{1}{a}F(\beta_1)=\frac{1}{a}\cdot a=1,$$

and

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right)(1)=\frac{1}{a}F(\gamma_2(1))=\frac{1}{a}F(\beta_2)=\frac{1}{a}\cdot a=1,$$

we have

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right)(1)=\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right)(1)=1.$$

Also,

$$\begin{aligned}\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right)([0,1))&=\left(\frac{1}{a}F\right)(\gamma_1([0,1)))\subset\frac{1}{a}F(\Omega)=\frac{1}{a}f(\Omega)\subset\frac{1}{a}D(0;1)\\&=D(0;1),\end{aligned}$$

so,

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right)([0,1))\subset D(0;1).$$

Similarly, $\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right)([0,1))\subset D(0;1)$. Since $f^{-1}:D(0;1)\to\Omega$ is a bounded holomorphic function,

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right):[0,1]\to D(0;1)\cup\{1\}$$

is a curve,

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right)([0,1))\subset D(0;1),$$

and $\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right)(1)=1$, by Theorem 3.104,

$$\underbrace{\lim_{r\to 1^-} f^{-1}(r) = \lim_{t\to 1^-}\left(f^{-1}\circ\left(\left(\frac{1}{a}F\right)\circ\gamma_1\right)\right)(t)} = \frac{1}{a}\lim_{t\to 1^-}\left(f^{-1}(F(\gamma_1(t)))\right)$$

$$= \frac{1}{a}\lim_{t\to 1^-}\left(f^{-1}(f(\gamma_1(t)))\right)$$

$$= \frac{1}{a}\lim_{t\to 1^-}\gamma_1(t) = \frac{1}{a}\gamma_1(1) = \frac{1}{a}\cdot\beta_1,$$

and hence

$$\lim_{r\to 1^-} f^{-1}(r) = \frac{1}{a}\cdot\beta_1.$$

Since $f^{-1} : D(0;1) \to \Omega$ is a bounded holomorphic function,

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right) : [0,1] \to D(0;1)\cup\{1\}$$

is a curve,

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right)([0,1)) \subset D(0;1),$$

and

$$\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right)(1) = 1,$$

by Theorem 3.104,

$$\frac{1}{a}\cdot\beta_1 = \underbrace{\lim_{r\to 1^-} f^{-1}(r) = \lim_{t\to 1^-}\left(f^{-1}\circ\left(\left(\frac{1}{a}F\right)\circ\gamma_2\right)\right)(t)}$$

$$= \frac{1}{a}\lim_{t\to 1^-}\left(f^{-1}(F(\gamma_2(t)))\right)$$

$$= \frac{1}{a}\lim_{t\to 1^-}\left(f^{-1}(f(\gamma_2(t)))\right)$$

$$= \frac{1}{a}\lim_{t\to 1^-}\gamma_2(t) = \frac{1}{a}\gamma_2(1) = \frac{1}{a}\cdot\beta_2,$$

and hence $\beta_1 = \beta_2$. This is a contradiction. ∎

Note 3.110 Let Ω be a bounded simply connected region. Suppose that every boundary point of Ω is simple. Let $f : \Omega \to D(0;1)$ be a 1-1 function from Ω onto $D(0;1)$. Suppose that f is a conformal mapping.

By Conclusion 3.109, there exists a 1-1 function $F : \Omega \cup (\partial\Omega) \left(= \overline{\Omega}\right) \to D[0;1]$ such that

1. for every $\alpha \in \Omega$, $F(\alpha) = f(\alpha)$,
2. for every convergent sequence $\{\alpha_n\}$ in Ω satisfying $(\lim_{n\to\infty} \alpha_n) \in (\Omega \cup (\partial\Omega)) \left(= \overline{\Omega}\right)$, $(\lim_{n\to\infty} f(\alpha_n) =)\ \lim_{n\to\infty} F(\alpha_n) = F(\lim_{n\to\infty} \alpha_n)$,
3. for every $\beta \in \partial\Omega$, $|F(\beta)| = 1$.

Problem 3.111 $F : \overline{\Omega} \to D[0;1]$ is continuous.

(**Solution** For this purpose, let us take any convergent sequence $\{z_n\}$ in $\overline{\Omega}$ such that $(\lim_{n\to\infty}, z_n) \in \overline{\Omega}\ (= \Omega \cup (\partial\Omega))$. We have to show that $\lim_{n\to\infty} F(z_n) = F(\lim_{n\to\infty} z_n)$.

Since $z_1 \in \overline{\Omega}\ (= \Omega \cup (\partial\Omega))$, either $z_1 \in \Omega$ or $z_1 \in \partial\Omega$.

Case I: when $z_1 \in \Omega$. We shall take $\alpha_1 \equiv z_1$. Thus, $\alpha_1 \in \Omega$, $|\alpha_1 - z_1| < \frac{1}{1}$, and $|f(\alpha_1) - F(z_1)| < \frac{1}{1}$.

Case II: when $z_1 \in \partial\Omega$. There exists a sequence $\{\beta_n\}$ in Ω such that $\lim_{n\to\infty} \beta_n = z_1$. By 2, $\lim_{n\to\infty} f(\beta_n) = F(z_1)$, and hence there exists a positive integer N_1 such that

$$n \geq N_1 \Rightarrow |f(\beta_n) - F(z_1)| < \frac{1}{1}.$$

Since $\lim_{n\to\infty} \beta_n = z_1$, there exists a positive integer $N_2 \geq N_1$ such that

$$n \geq N_2 \Rightarrow |\beta_n - z_1| < \frac{1}{1}.$$

Thus, $\left|\beta_{N_2} - z_1\right| < \frac{1}{1}$, and $\left|f\left(\beta_{N_2}\right) - F(z_1)\right| < \frac{1}{1}$. Let us put $\alpha_1 \equiv \beta_{N_2}\ (\in \Omega)$. Thus, $\alpha_1 \in \Omega$, $|\alpha_1 - z_1| < \frac{1}{1}$, and $|f(\alpha_1) - F(z_1)| < \frac{1}{1}$.

So, in all cases, there exists $\alpha_1 \in \Omega$ such that $|\alpha_1 - z_1| < \frac{1}{1}$, and $|f(\alpha_1) - F(z_1)| < \frac{1}{1}$.

Similarly, there exists $\alpha_2 \in \Omega$ such that $|\alpha_2 - z_2| < \frac{1}{2}$, and $|f(\alpha_2) - F(z_2)| < \frac{1}{2}$; there exists $\alpha_3 \in \Omega$ such that $|\alpha_3 - z_3| < \frac{1}{3}$, and $|f(\alpha_3) - F(z_3)| < \frac{1}{3}$ etc.

It follows that $\lim_{n\to\infty}(\alpha_n - z_n) = 0$, and $\lim_{n\to\infty}(f(\alpha_n) - F(z_n)) = 0$. Since $\lim_{n\to\infty}(\alpha_n - z_n) = 0$, and $\{z_n\}$ is convergent, we have $(\lim_{n\to\infty} \alpha_n) = (\lim_{n\to\infty} z_n)$ $\left(\in \overline{\Omega}\right)$, and hence, by 2, $\lim_{n\to\infty} f(\alpha_n) = F(\lim_{n\to\infty} \alpha_n)$. Now, since $\lim_{n\to\infty}(f(\alpha_n) - F(z_n)) = 0$, we have

$$\lim_{n\to\infty} F(z_n) = \lim_{n\to\infty} (f(\alpha_n) - (f(\alpha_n) - F(z_n))) = F\left(\lim_{n\to\infty} \alpha_n\right) - 0 = F\left(\lim_{n\to\infty} \alpha_n\right)$$
$$= F\left(\lim_{n\to\infty} z_n\right),$$

and hence $\lim_{n\to\infty} F(z_n) = F(\lim_{n\to\infty} z_n)$. ∎)

Conclusion 3.112 Let Ω be a bounded simply connected region. Suppose that every boundary point of Ω is simple. Let $f : \Omega \to D(0;1)$ be a 1-1 function from Ω onto $D(0;1)$. Suppose that f is a conformal mapping. Then there exists a continuous 1-1 function $F : \Omega \cup (\partial\Omega) \left(= \overline{\Omega}\right) \to D[0;1]$ such that

1. for every $\alpha \in \Omega$, $F(\alpha) = f(\alpha)$,
2. for every $\beta \in \partial\Omega$, $|F(\beta)| = 1$,
3. F maps $\overline{\Omega}$ onto $D[0;1]$,
4. $F^{-1} : D[0;1] \to \overline{\Omega}$ is continuous.

(Thus, the extension F of f to $\overline{\Omega}$ is a homeomorphism.)

Proof of the remaining part

3: We have to show that $F\left(\overline{\Omega}\right) = D[0;1]$.

Since Ω is bounded, the closed set $\overline{\Omega}$ is bounded, and hence $\overline{\Omega}$ is compact. Now, since $F : \overline{\Omega} \to D[0;1]$ is continuous, $(D(0;1) = f(\Omega) = F(\Omega) \subset)\ F\left(\overline{\Omega}\right)$ is a compact subset of $D[0;1]$. It follows that $(D[0;1] =)\overline{D(0;1)} \subset \overline{F\left(\overline{\Omega}\right)}$ $\left(= F\left(\overline{\Omega}\right) \subset D[0;1]\right)$, and hence $F\left(\overline{\Omega}\right) = D[0;1]$.

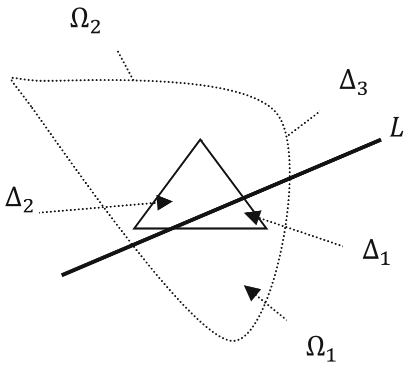
4: Since $F : \overline{\Omega} \to D[0;1]$ is a 1-1, onto, continuous map from compact set $\overline{\Omega}$ to $D[0;1]$, by a theorem, (e.g., [5], Theorem 4.17), $F^{-1} : D[0;1] \to \overline{\Omega}$ is continuous. ∎

Note 3.113 Let Ω be a region. Let L be a straight line. Let $\Omega - L$ be the union of two disjoint regions Ω_1 and Ω_2. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let f be holomorphic in Ω_1, and f be holomorphic in Ω_2.

Problem 3.114 $f \in H(\Omega)$.

(Solution By Conclusion 1.123, it suffices to show that for every triangle Δ satisfying $\Delta \subset \Omega$, $\int_{\partial\Delta} f(z)dz = 0$.

Case I: when $\Delta \subset \Omega_1$ $(\subset \Omega)$. By Conclusion 1.104, $\int_{\partial\Delta} f(z)dz = 0$.
Case II: when $\Delta \subset \Omega_2$ $(\subset \Omega)$. By Conclusion (1.104), $\int_{\partial\Delta} f(z)dz = 0$.
Case III: when $\Delta \cap \Omega_1 \neq \emptyset$, and $\Delta \cap \Omega_2 \neq \emptyset$.



It is clear from the above Figure that

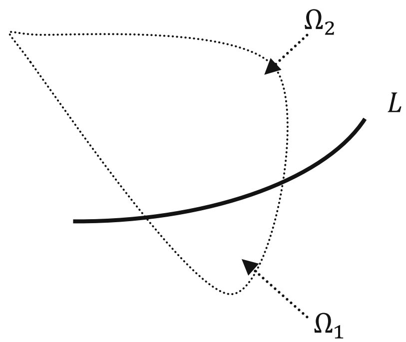
$$\int_{\partial\Delta} f(z)dz = \int_{\partial\Delta_1} f(z)dz + \int_{\partial\Delta_2} f(z)dz + \int_{\partial\Delta_3} f(z)dz.$$

Since $f : \Omega \to \mathbb{C}$ is continuous, by Conclusion 1.104, $\int_{\partial\Delta_1} f(z)dz = 0$, $\int_{\partial\Delta_2} f(z)dz = 0$, and $\int_{\partial\Delta_3} f(z)dz = 0$, and hence $\int_{\partial\Delta} f(z)dz = 0$.

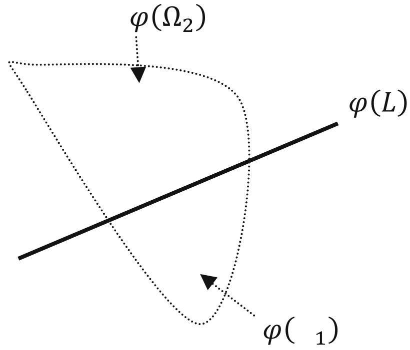
Thus, in all cases, $\int_{\partial\Delta} f(z)dz = 0$. ∎)

Conclusion 3.115 Let Ω be a region. Let L be a straight line or a circular arc. Let $\Omega - L$ be the union of two disjoint regions Ω_1 and Ω_2. Let $f : \Omega \to \mathbb{C}$ be a continuous function. Let f be holomorphic in Ω_1, and f be holomorphic in Ω_2. Then $f \in H(\Omega)$.

Proof of the remaining part: Let L be a circular arc.



There exists a linear fractional transformation φ that sends circular arc L into straight line $\varphi(L)$.



It follows that $(f \circ (\varphi^{-1})) : \varphi(\Omega) \to \mathbb{C}$ is a continuous function. Further, $(f \circ (\varphi^{-1}))$ is holomorphic in $\varphi(\Omega_1)$, and $(f \circ (\varphi^{-1}))$ is holomorphic in $\varphi(\Omega_2)$. It follows, from previous discussion that $(f \circ (\varphi^{-1})) \in H(\varphi(\Omega))$, and hence $(f =)\ (f \circ (\varphi^{-1})) \circ \varphi \in H(\Omega)$. Thus, $f \in H(\Omega)$. ∎

Note 3.116 Let Ω be a simply connected region. Let (f, D) be a function element. Let $D \subset \Omega$. Suppose that (f, D) admits an analytic continuation along every curve in Ω that starts at the center of D.

Let α be the center of D. Let β be any point of Ω.

Let Γ_0 and Γ_1 be any curves in Ω, with common initial point α, and common end point β. Now, by Conclusion 3.19, there exists a continuous function $\varphi : [0,1] \times [0,1] \to \Omega$ such that

1. for every $t \in [0,1]$, $\varphi(0,t) = \alpha$, and $\varphi(1,t) = \beta$,
2. if, for every $t \in [0,1]$, $\gamma_t : s \mapsto \varphi(s,t)$ is a curve from [0,1] to Ω, then $\gamma_0 = \Gamma_0$, and $\gamma_1 = \Gamma_1$.

By assumption, for every $t \in [0,1]$, (f, D) admits an analytic continuation along γ_t to a functional element (g_t, D_t), where $(\beta = \varphi(1,t) =)\ \gamma_t(1)$ is the center of the disk D_t. It follows, by Conclusion 3.16, that $g_0 = g_1$ on $D_0 \cap D_1\ (\ni \beta)$, and hence $g_0(\beta) = g_1(\beta)$.

Thus, the analytic continuations of (f, D) along $\gamma_0\ (= \Gamma_0)$ and $\gamma_1\ (= \Gamma_1)$ lead to the same function element, say, (g_β, D_β), where β is the center of D_β. Thus, for every $\beta \in \Omega$, $g_\beta(\beta)$ is independent of the curve in Ω with initial point α, and end point β. Hence, $g : \beta \mapsto g_\beta(\beta)$ is a well-defined function from Ω to $\mathbb{C}$.

Problem 3.117 $g : \Omega \to \mathbb{C}$ is an extension of $f : D \to \mathbb{C}$.

(Solution For this purpose, let us take any $z \in D$. We have to show that $(g_z(z) =)\ g(z) = f(z)$, that is $g_z(z) = f(z)$.

Since $z \in D$, there exists an open disk D_1 such that z is the center of D_1, and $D_1 \subset D$. Let $\gamma : t \mapsto (1-t)\alpha + tz$ from [0,1] to D be the straight line with initial point α, and end point z. Clearly, γ is a curve in D. Since z is the center of D_1, and $z \in D$, we have $D_1 \cap D \neq \emptyset$, and hence $\{D, D_1\}$ is a chain. Since $\gamma(0)\ (= \alpha)$ is the center of the disk D, $\gamma(1)\ (= z)$ is the center of the disk D_1, and $\gamma([0,1]) \subset D$,

$\{D, D_1\}$ is a chain that covers γ. Clearly, $(f, D) \sim \left(f|_{D_1}, D_1\right)$, so $\left(f|_{D_1}, D_1\right)$ is an analytic continuation of (f, D) along the chain $\{D, D_1\}$. It follows that (f, D) admits an analytic continuation along γ to $\left(f|_{D_1}, D_1\right)$. Now, since (f, D) admits an analytic continuation along some curve to (g_z, D_z), where z is the center of D_z, by Conclusion 3.11, $g_z(z) = \left(f|_{D_1}\right)(z)\ (= f(z))$. Thus, $g_z(z) = f(z)$. ∎)

Problem 3.118 $g : \Omega \to \mathbb{C}$ is holomorphic in Ω.

(Solution For this purpose, let us fix any $\beta \in \Omega$. We have to show that $g'(\beta)$ exists.

Let Γ be any curve in Ω, with initial point α, and end point β. The analytic continuations of (f, D) along Γ lead to the function element (g_β, D_β), where β is the center of D_β $(\subset \Omega)$.

It suffices to show that $g_\beta = g$ on D_β.

For this purpose, let us take any $\beta_1 \in D_\beta$. We have to show that $g_\beta(\beta_1) = g(\beta_1)\ \left(= g_{\beta_1}(\beta_1)\right)$, that is $g_\beta(\beta_1) = g_{\beta_1}(\beta_1)$.

Since $\beta_1 \in D_\beta$, and β_1 is the center of D_{β_1}, there exists an open disk $D^*_{\beta_1}$ with center β_1 such that $D^*_{\beta_1} \subset D_\beta \cap D_{\beta_1}$. It follows that $D_\beta \cap D^*_{\beta_1} \neq \emptyset$, and hence $\left\{D_\beta, D^*_{\beta_1}\right\}$ is a chain.

Let γ be the straight line with initial point β, and end point β_1. Clearly, γ is a curve in D_β.

Since β is the center of the disk D_β, β_1 is the center of the disk $D^*_{\beta_1}$, and $\operatorname{ran}(\gamma) \subset D_\beta$, $\left\{D_\beta, D^*_{\beta_1}\right\}$ is a chain that covers γ. Clearly, $(g_\beta, D_\beta) \sim \left(g_\beta|_{D^*_{\beta_1}}, D^*_{\beta_1}\right)$, so $\left(g_\beta|_{D^*_{\beta_1}}, D^*_{\beta_1}\right)$ is an analytic continuation of (g_β, D_β) along the chain $\left\{D_\beta, D_{\beta_1}\right\}$. It follows that (g_β, D_β) admits an analytic continuation along γ to $\left(g_\beta|_{D^*_{\beta_1}}, D^*_{\beta_1}\right)$. Since (f, D) admits an analytic continuation along Γ to (g_β, D_β), and (g_β, D_β) admits an analytic continuation along γ to $\left(g_\beta|_{D^*_{\beta_1}}, D^*_{\beta_1}\right)$, (f, D) admits an analytic continuation along $\Gamma \cup \gamma$ to $\left(g_\beta|_{D^*_{\beta_1}}, D^*_{\beta_1}\right)$. Now, since (f, D) admits an analytic continuation along Γ to $\left(g_{\beta_1}, D_{\beta_1}\right)$, where β_1 is the center of D_{β_1}, by Conclusion 3.11, $g_{\beta_1}(\beta_1) = \left(g_\beta|_{D^*_{\beta_1}}\right)(\beta_1)\ \left(= g_\beta(\beta_1)\right)$. Thus, $g_{\beta_1}(\beta_1) = g_\beta(\beta_1)$. ∎)

Conclusion 3.119 Let Ω be a simply connected region. Let (f, D) be a function element. Let $D \subset \Omega$. Suppose that (f, D) admits an analytic continuation along every curve in Ω that starts at the center of D. Then there exists an extension g of f to Ω such that $g \in H(\Omega)$. ∎

This result is known as the **monodromy theorem**.

3.11 Applications

Note 3.120

1. Problem 3.121 For every $s, t \in (0, \infty)$, and, for every $z \in \mathbb{C}$, $(st)^z = s^z t^z$.

(Solution LHS $= (st)^z = \left(e^{\ln(st)}\right)^z = e^{(\ln(st))z} = e^{(\ln s + \ln t)z} = e^{(\ln s)z + (\ln t)z} = e^{(\ln s)z} e^{(\ln t)z} = s^z t^z =$ RHS. ■)

2. Problem 3.122 For every $t \in (0, \infty)$, and for every $z \in \mathbb{C}$, $|t^z| = t^{\mathrm{Re}(z)}$.

(Solution LHS $= \left|e^{(\ln t)z}\right| = \left|e^{(\ln t)\mathrm{Re}(z) + i(\ln t)\mathrm{Im}(z)}\right| = \left|e^{(\ln t)\mathrm{Re}(z)} e^{i(\ln t)\mathrm{Im}(z)}\right|$

$$= \left|e^{(\ln t)\mathrm{Re}(z)}\right|\left|e^{i(\ln t)\mathrm{Im}(z)}\right| = \left|e^{(\ln t)\mathrm{Re}(z)}\right| 1 = e^{(\ln t)\mathrm{Re}(z)} = t^{\mathrm{Re}(z)} = \text{RHS}. \quad ■)$$

3. Let $t \in (0, \infty)$, and let $f : z \mapsto t^z$ be a function from $\mathbb{C}$ to $\mathbb{C}$.

Problem 3.123 $f' : z \mapsto t^z(\ln t)$ is a function from $\mathbb{C}$ to $\mathbb{C}$.

(Solution We have to show that $\frac{d(t^z)}{dz} = t^z(\ln t)$.

$$\text{LHS} = \frac{d(t^z)}{dz} = \frac{d\left(e^{(\ln t)z}\right)}{dz} = \left(e^{(\ln t)z}\right)((\ln t)1) = t^z(\ln t) = \text{RHS}.$$

■)

4. Let $z \in \mathbb{C}$, and let $f : t \mapsto t^z$ be a function from $(0, \infty)$ to $\mathbb{C}$.

Problem 3.124 $f' : t \mapsto z t^{z-1}$ is a function from $(0, \infty)$ to $\mathbb{C}$.

(Solution We have to show that $\frac{d(t^z)}{\mathrm{d}t} = z t^{z-1}$.

$$\text{LHS} = \frac{d(t^z)}{\mathrm{d}t} = \frac{d\left(e^{(\ln t)z}\right)}{\mathrm{d}t} = \left(e^{(\ln t)z}\right)\left(\left(\frac{1}{t}\right)z\right) = t^z\left(\left(\frac{1}{t}\right)z\right) = z t^{z-1} = \text{RHS}.$$

■)

5. Problem 3.125 For every $x \in (-1, \infty)$, $\int_0^1 e^{-t} t^x \, \mathrm{d}t \left(= \int_0^1 e^{-t} e^{(\ln t)x} \, \mathrm{d}t\mathrm{d}t = \int_0^1 e^{(-t + x(\ln t))} \, \mathrm{d}t \geq 0\right)$ is a nonnegative real number; that is, for every $x \in (-1, \infty)$, $0 \leq \int_0^1 e^{-t} t^x \mathrm{d}t < \infty$.

(Solution For this purpose, let us fix any $x \in (-1, \infty)$. Now,

$$0 \leq \int_0^1 e^{-t} t^x \,\mathrm{d}t = \lim_{\varepsilon \to 0^+} \left(\int_\varepsilon^1 e^{-t} t^x \,\mathrm{d}t \right) \leq \lim_{\varepsilon \to 0^+} \left(\int_\varepsilon^1 e^0 t^x \,\mathrm{d}t \right) = \lim_{\varepsilon \to 0^+} \left(\int_\varepsilon^1 t^x \,\mathrm{d}t \right)$$
$$= \lim_{\varepsilon \to 0^+} \left(\frac{t^{x+1}}{x+1} \bigg|_{t=\varepsilon}^{t=1} \right) = \lim_{\varepsilon \to 0^+} \left(\frac{1}{x+1} \left(1^{(x+1)} - \varepsilon^{(x+1)} \right) \right) = \lim_{\varepsilon \to 0^+} \left(\frac{1}{x+1} \left(1 - \varepsilon^{(x+1)} \right) \right) = \frac{1}{x+1}(1-0) = \frac{1}{x+1} < \infty,$$

so $\int_0^1 e^{-t} t^x \,\mathrm{d}t$ is a nonnegative real number. ∎)

6. Problem 3.126 For every $x \in (-1, \infty)$, $\int_1^\infty e^{-t} t^x \,\mathrm{d}t \left(= \int_1^\infty e^{-t} e^{(\ln t)x} \,\mathrm{d}t = \int_1^\infty e^{(-t + x(\ln t))} \,\mathrm{d}t \geq 0 \right)$ is a nonnegative real number; that is, for every $x \in (-1, \infty)$, $0 \leq \int_1^\infty e^{-t} t^x \mathrm{d}t < \infty$.

(Solution For this purpose, let us fix any $x \in (-1, \infty)$. Let us take any positive integer N such that $x + 1 < N$. Now,

$$0 \leq \int_1^\infty e^{-t} t^x \,\mathrm{d}t = \lim_{A \to \infty} \left(\int_1^A e^{-t} t^x \,\mathrm{d}t \right) = \lim_{A \to \infty} \left(\int_1^A \frac{1}{1 + t + \frac{t^2}{2!} + \cdots} t^x \mathrm{d}t \right) \leq \lim_{A \to \infty} \left(\int_1^A \frac{1}{\frac{t^N}{N!}} t^x \,\mathrm{d}t \right)$$
$$= N! \lim_{A \to \infty} \left(\int_1^A t^{x-N} \,\mathrm{d}t \right) = N! \lim_{A \to \infty} \left(\frac{1}{x-N+1} t^{x-N+1} \bigg|_{t=1}^{t=A} \right) = \frac{N!}{x-N+1} \lim_{A \to \infty} \left(A^{x-N+1} - 1^{x-N+1} \right)$$
$$= \frac{N!}{x-N+1} \lim_{A \to \infty} \left(\left(\frac{1}{A} \right)^{N-(x+1)} - 1 \right) = \frac{N!}{x-N+1}(0-1) = \frac{N!}{N-(x+1)} < \infty,$$

so, $\int_1^\infty e^{-t} t^x \,\mathrm{d}t$ is a nonnegative real number. ∎)

7. From 5 and 6, for every $x \in (-1, \infty)$, $\int_0^\infty e^{-t} t^x \,\mathrm{d}t < \infty$. It follows that for every $x \in (0, \infty)$, $\int_0^\infty e^{-t} t^{x-1} \mathrm{d}t < \infty$.

8. Problem 3.127 For every $z \in \mathbb{C}$ satisfying $\mathrm{Re}(z) > 0$, $\int_0^\infty e^{-t} t^{z-1} \,\mathrm{d}t$ is a complex number.

(Solution For this purpose, let us fix any $z \in \mathbb{C}$ satisfying $\mathrm{Re}(z) > 0$. Since for every $t \in (0, \infty)$, $\left| e^{-t} t^{z-1} \right| = \left| e^{-t} \right| \left| t^{z-1} \right| = e^{-t} \left| t^{z-1} \right| = e^{-t} t^{\mathrm{Re}(z)-1}$, for every $t \in (0, \infty)$, $\left| e^{-t} t^{z-1} \right| = e^{-t} t^{x-1}$, where $x \equiv \mathrm{Re}(z)$ (> 0). Now, it suffices to show that for every $x \in (0, \infty)$, $\int_0^\infty e^{-t} t^{x-1} \mathrm{d}t < \infty$. This is true, by 7. ∎)

Conclusion 3.128 $f : z \mapsto \int_0^\infty e^{-t} t^{z-1} \,\mathrm{d}t$ is a function from $\{z : \mathrm{Re}(z) > 0\}$ to $\mathbb{C}$.

Note 3.129 Definition Let (X, d) be a metric space. Let A be a nonempty subset of X. For every $x \in X$, 0 is a lower bound of the nonempty set $\{d(x, y) : y \in A\}$ of nonnegative real numbers, and hence $\inf\{d(x, y) : y \in A\}$ exists. We denote $\inf\{d(x, y) : y \in A\}$ by $d(x, A)$, and $d(x, A)$ is called the ***distance*** *of x to A*.

Let (X, d) be a metric space. Let A be a nonempty subset of X.

1. Problem 3.130 For every $x \in X$, $d(x, A) = d(x, \overline{A})$.

(Solution Let us fix any $x \in X$. We have to show that $d(x, A) = d(x, \overline{A})$. Since $A \subset \overline{A}$, $\{d(x,y) : y \in A\} \subset \{d(x,y) : y \in \overline{A}\}$, and hence

$$d(x, \overline{A}) = \underbrace{\inf\{d(x,y) : y \in \overline{A}\} \leq \inf\{d(x,y) : y \in A\}} = d(x, A).$$

Thus, $d(x, \overline{A}) \leq d(x, A)$. We claim that $d(x, A) = d(x, \overline{A})$.

If not, otherwise let $\left(\inf\{d(x,y) : y \in \overline{A}\} =\right) d(x, \overline{A}) < d(x, A)$. We have to arrive at a contradiction. $\inf\{d(x,y) : y \in \overline{A}\} < d(x, A)$, there exists $y \in \overline{A}$ such that $d(x,y) < d(x,A)$, and hence $0 < (d(x,A) - d(x,y))$. Now, since $y \in \overline{A}$, there exists $z \in A$ such that $d(y,z) < (d(x,A) - d(x,y))$, and hence

$$d(x,z) \leq \underbrace{d(y,z) + d(x,y) < d(x,A)} = \inf\{d(x,w) : w \in A\} \leq d(x,z).$$

Thus, $d(x,z) < d(x,z)$. This is a contradiction. Hence, $d(x, A) = d(x, \overline{A})$. ∎)

2. Problem 3.131 For every $x \in X$, $d(x, A) = 0 \Leftrightarrow x \in \overline{A}$.

(Solution Let us fix any $x \in X$. Let $d(x, A) = 0$. We have to show that $x \in \overline{A}$. Since

$$\inf\{d(x,y) : y \in A\} = \underbrace{d(x, A) = 0},$$

we have $\inf\{d(x,y) : y \in A\} = 0$, and hence for every positive integer n, there exists $a_n \in A$ such that $d(x, a_n) < \frac{1}{n}$. Thus, $\{a_n\}$ is a sequence in A such that $\lim_{n\to\infty} a_n = x$. It follows that $(x =)$ $(\lim_{n\to\infty} a_n) \in \overline{A}$, and hence $x \in \overline{A}$.

Conversely, let $x \in \overline{A}$. We have to show that $(\inf\{d(x,y) : y \in A\} =)$ $d(x, A) = 0$, that is $\inf\{d(x,y) : y \in A\} = 0$. If not, otherwise let $0 < \inf\{d(x,y) : y \in A\}$. We have to arrive at a contradiction.

There exists a real number r such that $0 < r < \inf\{d(x,y) : y \in A\}$. Now, since $x \in \overline{A}$, there exists $z \in A$ such that $(\inf\{d(x,y) : y \in A\} \leq)$ $d(x,z)$ $< r$ $(< \inf\{d(x,y) : y \in A\})$. This gives a contradiction. ∎)

3. Problem 3.132 For every $x, y \in X$, $|d(x, A) - d(y, A)| \leq d(x, y)$.

(Solution Let us fix any $x, y \in X$. We have to show that $|d(x,A) - d(y,A)| \leq d(x,y)$.

For every, $a \in A$, $d(x,a) \leq d(x,y) + d(y,a)$, we have

$$d(x,A) = \underbrace{\inf\{d(x,a) : a \in A\} \leq \inf\{d(x,y) + d(y,a) : a \in A\}}$$

$$= d(x,y) + \inf\{d(y,a) : a \in A\} = d(x,y) + d(y,A),$$

so $d(x,A) - d(y,A) \leq d(x,y)$.

Similarly,

$$-(d(x,A)-d(y,A)) = \underbrace{d(y,A)-d(x,A) \leq d(y,x)} = d(x,y),$$

so

$$-(d(x,A)-d(y,A)) \leq d(x,y).$$

It follows that

$$(|d(x,A)-d(y,A)| =) \max\{d(x,A)-d(y,A), -(d(x,A)-d(y,A))\} \leq d(x,y),$$

and hence $|d(x,A)-d(y,A)| \leq d(x,y)$. ∎)

4. From 3, it is clear that the function $x \mapsto d(x,A)$ from X to $[0,\infty)$ is uniformly continuous, and hence, the function $x \mapsto d(x,A)$ from X to $[0,\infty)$ is continuous.

Definition Let (Ω, d) be a complete metric space. Let G be a nonempty open subset of $\mathbb{C}$. The collection of all continuous functions $f : G \to \Omega$ will be denoted by $C(G,\Omega)$.

Let (Ω, d) be a complete metric space. Let G be a nonempty open subset of $\mathbb{C}$.

By Conclusion 2.53, there exists a sequence $\{K_1, K_2, K_3, \ldots\}$ of nonempty compact subsets of G satisfying

1. $(K_1)^0 \subset K_1 \subset (K_2)^0 \subset K_2 \subset (K_3)^0 \subset K_3 \subset \cdots \subset G$,
2. $G = \bigcup_{n=1}^{\infty} K_n$,
3. for every compact set K contained in G, there exists a positive integer N such that $K \subset K_N$,
4. for every positive integer n, and for every $z \in K_n$, $D\left(z; \frac{1}{n} - \frac{1}{n+1}\right) \subset K_{n+1}$,
5. for every positive integer n, if C is a component of $\left(\mathbb{S}^2 - K_n\right)$, then there exists a component D of $\left(\mathbb{S}^2 - \Omega\right)$ such that $D \subset C$.

Let us take any $f, g \in C(G,\Omega)$.

It follows that $f : G \to \Omega$ and $g : G \to \Omega$ are continuous functions. Also, the metric $d : \Omega \times \Omega \mapsto [0,\infty)$ is continuous. It follows that the function $z \mapsto d(f(z), g(z))$ from G to $[0,\infty)$ is continuous. Now, since K_1 is a nonempty compact subset of G, $\{d(f(z), g(z)) : z \in K_1\}$ is a nonempty compact set of nonnegative real numbers, and hence $\{d(f(z), g(z)) : z \in K_1\}$ is a nonempty bounded above set of nonnegative real numbers. It follows that $\sup\{d(f(z), g(z)) : z \in K_1\}$ exists and is a nonnegative real number. For every $f, g \in C(G,\Omega)$, put

$$\rho_1(f,g) \equiv \sup\{d(f(z), g(z)) : z \in K_1\} \ (\geq 0).$$

Clearly, $\rho_1 : C(G,\Omega) \times C(G,\Omega) \to [0,\infty)$ has the following properties:

1. for every $f \in C(G,\Omega)$, $\rho_1(f,f) = 0$,
2. if $\rho_1(f,g) = 0$ then $f = g$ on K_1,
3. for every $f,g \in C(G,\Omega)$, $\rho_1(f,g) = \rho_1(g,f)$,
4. for every $f,g,h \in C(G,\Omega)$, $\rho_1(f,g) \le \rho_1(f,h) + \rho_1(h,g)$.

For 1: Let us take any $f \in C(G,\Omega)$. We have to show that $(\sup\{0\} = \sup\{d(f(z),f(z)) : z \in K_1\} =)\ \rho_1(f,f) = 0$, that is $\sup\{0\} = 0$. This is trivially true.

For 2: Let $f,g \in C(G,\Omega)$, and $(\sup\{d(f(z),g(z)) : z \in K_1\} =)\ \rho_1(f,g) = 0$. We have to show that $f = g$ on K_1. Since $\sup\{d(f(z),g(z)) : z \in K_1\} = 0$, for every $z \in K_1$, $d(f(z),g(z)) = 0$. It follows that $f = g$ on K_1.

For 3. This is clear.

For 4: Let us take any $f,g,h \in C(G,\Omega)$. We have to show that

$$\rho_1(f,g) \le \rho_1(f,h) + \rho_1(h,g),$$

that is

$$\sup\{d(f(z),g(z)) : z \in K_1\} \le \rho_1(f,h) + \rho_1(h,g),$$

that is

$$\rho_1(f,h) + \rho_1(h,g) \text{ is an upper bound of } \{d(f(z),g(z)) : z \in K_1\}.$$

For this purpose, let us take any $z \in K_1$. We have to show that $d(f(z),g(z)) \le \rho_1(f,h) + \rho_1(h,g)$. Since

$$\underbrace{d(f(z),g(z)) \le d(f(z),h(z)) + d(h(z),g(z))}$$
$$\le \sup\{d(f(z),h(z)) : z \in K_1\} + \sup\{d(h(z),g(z)) : z \in K_1\} = \rho_1(f,h) + \rho_1(h,g),$$

we have $d(f(z),g(z)) \le \rho_1(f,h) + \rho_1(h,g)$.

For every $f,g \in C(G,\Omega)$, put

$$\rho_2(f,g) \equiv \sup\{d(f(z),g(z)) : z \in K_2\}\ (\ge 0).$$

Clearly, $\rho_2 : C(G,\Omega) \times C(G,\Omega) \to [0,\infty)$ has the following properties:

1. for every $f \in C(G,\Omega)$, $\rho_2(f,f) = 0$,
2. if $\rho_2(f,g) = 0$ then $f = g$ on K_2,
3. for every $f,g \in C(G,\Omega)$, $\rho_2(f,g) = \rho_2(g,f)$,
4. for every $f,g,h \in C(G,\Omega)$, $\rho_2(f,g) \le \rho_2(f,h) + \rho_2(h,g)$, etc.

For every $f, g \in C(G, \Omega)$, put

$$\rho(f,g) \equiv \frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)} + \frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)} + \frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)} + \cdots \quad \left(\leq \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1\right)$$

Problem 3.133 $\rho : C(G,\Omega) \times C(G,\Omega) \to [0,\infty)$ is a metric over $C(G,\Omega)$, that is

1. for every $f \in C(G,\Omega)$, $\rho(f,f) = 0$,
2. if $\rho(f,g) = 0$ then $f = g$,
3. for every $f, g \in C(G,\Omega)$, $\rho(f,g) = \rho(g,f)$,
4. for every $f, g, h \in C(G,\Omega)$, $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$, etc.

(Solution For 1: Let us take any $f \in C(G,\Omega)$. We have to show that

$$\begin{aligned} 0+0+0+\cdots &= \frac{1}{2}\frac{0}{1+0} + \frac{1}{4}\frac{0}{1+0} + \frac{1}{8}\frac{0}{1+0} + \cdots \\ &= \frac{1}{2}\frac{\rho_1(f,f)}{1+\rho_1(f,f)} + \frac{1}{4}\frac{\rho_2(f,f)}{1+\rho_2(f,f)} + \frac{1}{8}\frac{\rho_3(f,f)}{1+\rho_3(f,f)} + \cdots \\ &= \underbrace{\rho(f,f) = 0}. \end{aligned}$$

This is clear.

For 2: Let $f, g \in C(G,\Omega)$, and

$$\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)} + \frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)} + \frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)} + \cdots = \underbrace{\rho(f,g) = 0},$$

that is for every positive integer n,

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)} = 0,$$

that is for every positive integer n, $\rho_n(f,g) = 0$, that is for every positive integer n, $f = g$ on K_n. We have to show that $f = g$ on G. Since for every positive integer n, $f = g$ on K_n, we have $f = g$ on $\bigcup_{n=1}^{\infty} K_n$ $(= G)$, and hence $f = g$.

For 3: This is clear.

For 4: Let us take any $f, g, h \in C(G,\Omega)$. We have to show that $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$, that is

$$\begin{aligned} &\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)} + \frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)} + \frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)} + \cdots \\ &\quad \leq \left(\frac{1}{2}\frac{\rho_1(f,h)}{1+\rho_1(f,h)} + \frac{1}{4}\frac{\rho_2(f,h)}{1+\rho_2(f,h)} + \frac{1}{8}\frac{\rho_3(f,h)}{1+\rho_3(f,h)} + \cdots\right) \\ &\quad + \left(\frac{1}{2}\frac{\rho_1(h,g)}{1+\rho_1(h,g)} + \frac{1}{4}\frac{\rho_2(h,g)}{1+\rho_2(h,g)} + \frac{1}{8}\frac{\rho_3(h,g)}{1+\rho_3(h,g)} + \cdots\right), \end{aligned}$$

that is

$$
\begin{aligned}
0 \leq \frac{1}{2}&\left(\frac{\rho_1(f,h)}{1+\rho_1(f,h)}+\frac{\rho_1(h,g)}{1+\rho_1(h,g)}-\frac{\rho_1(f,g)}{1+\rho_1(f,g)}\right)\\
&+\frac{1}{4}\left(\frac{\rho_2(f,h)}{1+\rho_2(f,h)}+\frac{\rho_2(h,g)}{1+\rho_2(h,g)}-\frac{\rho_2(f,g)}{1+\rho_2(f,g)}\right)\\
&+\frac{1}{8}\left(\frac{\rho_3(f,h)}{1+\rho_3(f,h)}+\frac{\rho_3(h,g)}{1+\rho_3(h,g)}-\frac{\rho_3(f,g)}{1+\rho_3(f,g)}\right)+\cdots.
\end{aligned}
$$

It suffices to show that for every positive integer n,

$$0 \leq \frac{\rho_n(f,h)}{1+\rho_n(f,h)}+\frac{\rho_n(h,g)}{1+\rho_n(h,g)}-\frac{\rho_n(f,g)}{1+\rho_n(f,g)},$$

that is for every positive integer n,

$$
\begin{aligned}
0 \leq\ & \rho_n(f,h)(1+\rho_n(h,g))(1+\rho_n(f,g))+\rho_n(h,g)(1+\rho_n(f,g))(1+\rho_n(f,h))\\
&-\rho_n(f,g)(1+\rho_n(f,h))(1+\rho_n(h,g)).
\end{aligned}
$$

Here, for every positive integer n,

$$
\begin{aligned}
&\rho_n(f,h)(1+\rho_n(h,g))(1+\rho_n(f,g))+\rho_n(h,g)(1+\rho_n(f,g))(1+\rho_n(f,h))\\
&\quad-\rho_n(f,g)(1+\rho_n(f,h))(1+\rho_n(h,g))\\
&\qquad=\rho_n(f,h)(1+\rho_n(h,g)+\rho_n(f,g)+\rho_n(h,g)\rho_n(f,g))\\
&\qquad\quad+\rho_n(h,g)(1+\rho_n(f,g)+\rho_n(f,h)+\rho_n(f,g)\rho_n(f,h))\\
&\qquad\quad-\rho_n(f,g)(1+\rho_n(f,h)+\rho_n(h,g)+\rho_n(f,h)\rho_n(h,g))\\
&=\rho_n(f,h)(1+\rho_n(h,g)+\rho_n(f,g))+\rho_n(h,g)(1+\rho_n(f,g)+\rho_n(f,h)+\rho_n(f,g)\rho_n(f,h))\\
&\quad-\rho_n(f,g)(1+\rho_n(f,h)+\rho_n(h,g))\\
&=(\rho_n(f,h)+\rho_n(h,g)-\rho_n(f,g))+2\rho_n(f,h)\rho_n(h,g)+\rho_n(h,g)\rho_n(f,g)\rho_n(f,h)\\
&\quad\geq \rho_n(f,h)+\rho_n(h,g)-\rho_n(f,g)\geq 0,
\end{aligned}
$$

so for every positive integer n,

$$
\begin{aligned}
0 \leq\ & \rho_n(f,h)(1+\rho_n(h,g))(1+\rho_n(f,g))+\rho_n(h,g)(1+\rho_n(f,g))(1+\rho_n(f,h))\\
&-\rho_n(f,g)(1+\rho_n(f,h))(1+\rho_n(h,g)).
\end{aligned}
$$

∎)

I. Thus $(C(G,\Omega),\rho)$ is a metric space.

Let us take any $\varepsilon > 0$.

Since $\lim_{n\to\infty}\frac{1}{2^n} = 0$, there exists a positive integer N such that $n \geq N \Rightarrow \frac{1}{2^n} < \frac{\varepsilon}{2}$. Since $\lim_{t\to 0^+}\frac{t}{1+t} = 0$, there exists $\delta > 0$ such that $t \in [0,\delta) \Rightarrow \frac{t}{1+t} < \frac{\varepsilon}{2}$.

Now, let us take any $f,g \in C(G,\Omega)$ satisfying $(\rho_N(f,g) =) \sup\{d(f(z), g(z)) : z \in K_N\} < \delta$.

Problem 3.134 $\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)} + \frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)} + \frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)} + \cdots = \underbrace{\rho(f,g) < \varepsilon}$.

(Solution Since $K_1 \subset K_2 \subset K_3 \subset \cdots \subset G$, we have

$$\{d(f(z),g(z)) : z \in K_1\} \subset \{d(f(z),g(z)) : z \in K_2\}$$
$$\subset \{d(f(z),g(z)) : z \in K_3\} \subset \cdots,$$

and hence

$$\sup\{d(f(z),g(z)) : z \in K_1\}$$
$$\leq \sup\{d(f(z),g(z)) : z \in K_2\}$$
$$\leq \sup\{d(f(z),g(z)) : z \in K_3\} \leq \cdots.$$

It follows that

$$\rho_1(f,g) \leq \rho_2(f,g) \leq \rho_3(f,g) \leq \cdots.$$

Now, since $\rho_N(f,g) < \delta$, we have $n \leq N \Rightarrow \rho_n(f,g) < \delta$, and hence

$$n \leq N \Rightarrow \frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \frac{\varepsilon}{2}.$$

Now,

$$\begin{aligned}
&\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)} + \frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)} + \frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)} + \cdots \\
&= \left(\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)} + \frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)} + \cdots + \frac{1}{2^N}\frac{\rho_N(f,g)}{1+\rho_N(f,g)}\right) \\
&+ \left(\frac{1}{2^{N+1}}\frac{\rho_{N+1}(f,g)}{1+\rho_{N+1}(f,g)} + \frac{1}{2^{N+2}}\frac{\rho_{N+2}(f,g)}{1+\rho_{N+2}(f,g)} + \cdots\right) \\
< &\left(\frac{1}{2}\frac{\varepsilon}{2} + \frac{1}{4}\frac{\varepsilon}{2} + \cdots + \frac{1}{2^N}\frac{\varepsilon}{2}\right) + \left(\frac{1}{2^{N+1}}\frac{\rho_{N+1}(f,g)}{1+\rho_{N+1}(f,g)} + \frac{1}{2^{N+2}}\frac{\rho_{N+2}(f,g)}{1+\rho_{N+2}(f,g)} + \cdots\right) \\
< &\frac{\varepsilon}{2} + \left(\frac{1}{2^{N+1}}\frac{\rho_{N+1}(f,g)}{1+\rho_{N+1}(f,g)} + \frac{1}{2^{N+2}}\frac{\rho_{N+2}(f,g)}{1+\rho_{N+2}(f,g)} + \cdots\right) \\
&\leq \frac{\varepsilon}{2} + \left(\frac{1}{2^{N+1}}\cdot 1 + \frac{1}{2^{N+2}}\cdot 1 + \cdots\right) \\
= &\frac{\varepsilon}{2} + \frac{1}{2^N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

so $\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)}+\frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)}+\frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)}+\cdots<\varepsilon$. ∎)

Put $K\equiv K_N$.

II. Thus, for every $\varepsilon>0$, there exists $\delta>0$, and a compact subset K of G such that for every $f,g\in C(G,\Omega)$ satisfying $\sup\{d(f(z),g(z)):z\in K\}<\delta$, $\rho(f,g)<\varepsilon$.

Let K be any compact subset of G Let δ be any positive real number.

Since K is a compact subset of G, there exists a positive integer N such that $K\subset K_N$.

Since $\lim_{t\to 0^+}\frac{t}{1-t}=0$, there exists $\varepsilon>0$ such that $t\in[0,2^N\varepsilon)\Rightarrow\frac{t}{1-t}<\frac{\delta}{2}$.

Now, let us take any $f,g\in C(G,\Omega)$ satisfying

$$\frac{1}{2^N}\frac{\rho_N(f,g)}{1+\rho_N(f,g)}\le\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)}+\frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)}+\frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)}+\cdots=\underbrace{\rho(f,g)<\varepsilon}.$$

Problem 3.135 $\sup\{d(f(z),g(z)):z\in K\}<\delta$.

(Solution For this purpose, let us take any $w\in K$. It suffices to show that $d(f(w),g(w))\le\frac{\delta}{2}$. Since $\frac{1}{2^N}\frac{\rho_N(f,g)}{1+\rho_N(f,g)}<\varepsilon$, we have $\frac{\rho_N(f,g)}{1+\rho_N(f,g)}\in[0,2^N\varepsilon)$, and hence

$$\begin{aligned}d(f(w),g(w))&\le\sup\{d(f(z),g(z)):z\in K\}\\&\le\sup\{d(f(z),g(z)):z\in K_N\}=\rho_N(f,g)=\underbrace{\frac{\frac{\rho_N(f,g)}{1+\rho_N(f,g)}}{1-\frac{\rho_N(f,g)}{1+\rho_N(f,g)}}<\frac{\delta}{2}}.\end{aligned}$$

Thus, $d(f(w),g(w))<\frac{\delta}{2}$. ∎)

III. Thus, for every compact subset K of G, and for every $\delta>0$, there exists $\varepsilon>0$ such that for every $f,g\in C(G,\Omega)$ satisfying $\rho(f,g)<\varepsilon$,

$$\sup\{d(f(z),g(z)):z\in K\}<\delta.$$

Let $\mathcal{O}$ be an open set in the metric space $(C(G,\Omega),\rho)$. Let $f\in\mathcal{O}$.

It follows that there exists $\varepsilon>0$ such that for every $g\in C(G,\Omega)$ satisfying $\rho(f,g)<\varepsilon$, $g\in\mathcal{O}$. By II, there exists $\delta>0$, and a compact subset K of G such that for every $g\in C(G,\Omega)$ satisfying $\sup\{d(f(z),g(z)):z\in K\}<\delta$, $\rho(f,g)<\varepsilon$, and hence $g\in\mathcal{O}$. Thus,

$$\{g:\sup\{d(f(z),g(z)):z\in K\}<\delta\}\subset\mathcal{O}.$$

IV. In short, if $\mathcal{O}$ is an open set in the metric space $(C(G,\Omega),\rho)$, then for every $f\in\mathcal{O}$ there exist $\delta>0$, and a compact subset K of G such that $\{g:d(f(z),g(z))<\delta \text{ for every } z\in K\}\subset\mathcal{O}$.

Let $\mathcal{O}$ be a subset of the $C(G,\Omega)$. Suppose that for every $f \in \mathcal{O}$ there exist $\delta > 0$, and a compact subset K of G such that $\{g : d(f(z), g(z)) < \delta$ for every $z \in K\} \subset \mathcal{O}$.

Problem 3.136 $\mathcal{O}$ is an open set in the metric space $(C(G,\Omega), \rho)$.

(Solution For this purpose, let us take any $f \in \mathcal{O}$. We have to show that f is an interior point of $\mathcal{O}$. Since $f \in \mathcal{O}$, by assumption, there exist $\delta > 0$, and a compact subset K of G such that

$$\{g : d(f(z), g(z)) < \delta \text{ for every } z \in K\} \subset \mathcal{O} \quad \cdots (*).$$

Let $g \in C(G,\Omega)$ such that

$$\frac{1}{2}\frac{\rho_1(f,g)}{1+\rho_1(f,g)} + \frac{1}{4}\frac{\rho_2(f,g)}{1+\rho_2(f,g)} + \frac{1}{8}\frac{\rho_3(f,g)}{1+\rho_3(f,g)} + \cdots = \underbrace{\rho(f,g) < \delta}.$$

It suffices to show that $g \in \mathcal{O}$. By III, there exists $\varepsilon > 0$ such that for every $h \in C(G,\Omega)$ satisfying $\rho(f,h) < \varepsilon$, $\sup\{d(f(z), h(z)) : z \in K\} < \delta$. Now, since $g \in C(G,\Omega)$, and $\rho(f,g) < \delta$, we have $\sup\{d(f(z), g(z)) : z \in K\} < \delta$. It follows that for every $z \in K$, $d(f(z), g(z)) < \delta$, and hence, by $(*)$, $g \in \mathcal{O}$. ∎)

V. By III, and IV, it is clear that the topology inherited by metric ρ is independent of the choice of compact sets $K_1, K_2, K_3, \ldots$.

Let $\{f_n\}$ be a sequence in the metric space $C(G,\Omega)$ with metric ρ. Let $f \in C(G,\Omega)$. Suppose that $\lim_{n\to\infty} f_n = f$.

Problem 3.137 $\{f_n\}$ converges to f uniformly on compact subsets of G, in the sense that for every nonempty compact subset K of G, and for every $\varepsilon > 0$, there exists a positive integer $N\ (\equiv N(K,\varepsilon))$ such that $((n \geq N$, and $z \in K) \Rightarrow d(f_n(z), f(z)) < \varepsilon)$.

(Solution For this purpose, let us take any compact subset K of G, and $\varepsilon > 0$. By III, there exists $\delta > 0$ such that for every $f, g \in C(G,\Omega)$ satisfying $\rho(f,g) < \delta$, $\sup\{d(f(z), g(z)) : z \in K\} < \varepsilon$. Since $\lim_{n\to\infty} f_n = f$, there exists a positive integer N such that

$$(n \geq N \Rightarrow \rho(f_n, f) < \delta).$$

It follows that

$$(n \geq N \Rightarrow \sup\{d(f_n(z), f(z)) : z \in K\} < \varepsilon),$$

and hence

$$(n \geq N, \text{ and } z \in K) \Rightarrow d(f_n(z), f(z)) < \varepsilon.$$

∎)

VI. In short, if a sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to $f \in C(G,\Omega)$, then $\{f_n\}$ converges to f uniformly on compact subsets of G.

Let $\{f_n\}$ be a sequence in the metric space $C(G,\Omega)$ with metric ρ. Let $f \in C(G,\Omega)$. Suppose that $\{f_n\}$ converges to f uniformly on compact subsets of G.

Problem 3.138 $\lim_{n\to\infty} f_n = f$ in the metric space $(C(G,\Omega),\rho)$.

(Solution For this purpose, let us take any $\varepsilon > 0$. By II, there exists $\delta > 0$, and a compact subset K of G such that for every $f, g \in C(G,\Omega)$ satisfying $\sup\{d(f(z), g(z)) : z \in K\} < \delta$, $\rho(f,g) < \varepsilon$. Now, since $\{f_n\}$ converges to f uniformly on compact subsets of G, there exists a positive integer N $(\equiv N(K,\delta))$ such that $((n \geq N, \text{ and } z \in K) \Rightarrow d(f_n(z), f(z)) < \delta)$.

Thus, for every $\varepsilon > 0$, there exists $\delta > 0$, and a compact subset K of G such that for every $n \geq N$, $\sup\{d(f(z), f_n(z)) : z \in K\} \leq \delta$. It follows, by II, that $n \geq N \Rightarrow \rho(f, f_n) < \varepsilon$. Hence, $\lim_{n\to\infty} f_n = f$. ∎)

VII. In short, if a sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to $f \in C(G,\Omega)$ uniformly on compact subsets of G, then $\lim_{n\to\infty} f_n = f$ in the metric space $(C(G,\Omega),\rho)$.

Let (X,d) and (Ω,d_1) be any metric spaces. For every positive integer n, let $f_n : X \to \Omega$ be a continuous function. Let $f : X \to \Omega$ be a function. Suppose that the sequence $\{f_n\}$ converges to f uniformly on X.

Problem 3.139 $f : X \to \Omega$ is continuous.

(Solution For this purpose, let us take any $a \in X$. We have to show that $f : X \to \Omega$ is continuous at a. For this purpose, let us take any $\varepsilon > 0$. Since $\{f_n\}$ converges to f uniformly on X, there exists a positive integer N such that

$$n \geq N \Rightarrow (d_1(f_n(x), f(x)) < \varepsilon \text{ for every } x \in X).$$

It follows that for every $x \in X$, $d_1(f_N(x), f(x)) < \varepsilon$. Since $f_N : X \to \Omega$ is continuous at a, there exists $\delta > 0$ such that

$$(d(x,a) < \delta \Rightarrow d_1(f_N(x), f_N(a)) < \varepsilon).$$

It suffices to show that $d(x,a) < \delta \Rightarrow d_1(f(x), f(a)) < 3\varepsilon$.

Let us fix any x in X such that $d(x,a) < \delta$. We have to show that $d_1(f(x), f(a)) < 3\varepsilon$.

Since $d(x,a) < \delta$, we have $d_1(f_N(x), f_N(a)) < \varepsilon$. Also, $d_1(f_N(x), f(x)) < \varepsilon$, and $d_1(f_N(a), f(a)) < \varepsilon$. Hence,

$$\begin{aligned} d_1(f(x),f(a)) &\le d_1(f(x),f_N(x)) + d_1(f_N(x),f(a)) < \varepsilon + d_1(f_N(x),f(a)) \\ &\le \varepsilon + (d_1(f_N(x),f_N(a)) + d_1(f_N(a),f(a))) < \varepsilon + (\varepsilon + \varepsilon) = 3\varepsilon. \end{aligned}$$

Thus, $d_1(f(x), f(a)) < 3\varepsilon$. ∎)

VIII. In short: Let (X, d) and (Ω, d_1) be any metric spaces. For every positive integer n, let $f_n : X \to \Omega$ be a continuous function. Let $f : X \to \Omega$ be a function. Suppose that the sequence $\{f_n\}$ converges to f uniformly on X. Then $f : X \to \Omega$ is continuous.

Let $\{f_n\}$ be a Cauchy sequence in the metric space $(C(G,\Omega), \rho)$. Let us take any nonempty compact subset K of G. Let us take any $\varepsilon > 0$.

By III, there exists $\delta > 0$ such that for every $f, g \in C(G,\Omega)$ satisfying $\rho(f,g) < \delta$, $\sup\{d(f(z), g(z)) : z \in K\} < \varepsilon$. Since $\{f_n\}$ is a Cauchy sequence in the metric space $(C(G,\Omega), \rho)$, there exists a positive integer N such that $m, n \ge N \Rightarrow \rho(f_m, f_n) < \delta$. It follows that $m, n \ge N \Rightarrow \sup\{d(f_m(z), f_n(z)) : z \in K\} < \varepsilon$. Hence for every $z \in K$, $\{f_n(z)\}$ is a Cauchy sequence in the complete metric space (Ω, d).

It follows that for every $z \in G$, (and hence $\{z\}$ is a compact subset of G), $\{f_n(z)\}$ is a Cauchy sequence in the complete metric space (Ω, d). Now, for every $z \in G$, there exists $f(z) \in \Omega$ such that $\lim_{n\to\infty} f_n(z) = f(z)$. Thus, $f : G \to \Omega$.

Problem 3.140 $f : G \to \Omega$ is continuous.

(Solution For this purpose, let us fix any z_0 in G. We have to show that $f : G \to \Omega$ is continuous at z_0.

Since $z_0 \in G$, and G is an open subset of $\mathbb{C}$, there exists $\delta > 0$ such that $D[z_0;\delta] \subset G$. Also, $D[z_0;\delta]$ is a compact subset of G.

Problem 3.141 $\{f_n\}$ converges to f uniformly on $D[z_0;\delta]$.

(Solution For this purpose, let us take any $\varepsilon > 0$. For every $z \in G\ (\supset D[z_0;\delta])$, $m, n \ge N \Rightarrow d(f_m(z), f_n(z)) < \varepsilon$, metric d is continuous, and, for every $z \in G$, $\lim_{n\to\infty} f_n(z) = f(z)$, we have, for every $z \in D[z_0;\delta]$, $m \ge N \Rightarrow d(f_m(z), f(z)) \le \varepsilon$. Thus, $\{f_n\}$ converges to f uniformly on $D[z_0;\delta]$. ∎)

Now, by VIII, f is continuous on $D[z_0;\delta]\ (\supset D(z_0;\delta) \ni z_0)$, and hence $f : G \to \Omega$ is continuous at z_0. ∎)

Thus, $f \in C(G,\Omega)$.

Problem 3.142 $\lim_{n\to\infty} \rho(f_n, f) = 0$.

(Solution By VII, it suffices to show that $\{f_n\}$ converges to f uniformly on compact subsets of G. For this purpose, let us take any nonempty compact subset K of G. Next, let us take any $\varepsilon > 0$. For every $z \in G\ (\supset K)$, $m, n \ge N \Rightarrow d(f_m(z), f_n(z)) < \varepsilon$, metric d is continuous, and, for every $z \in G$,

$\lim_{n\to\infty} f_n(z) = f(z)$, we have, for every $z \in K$, $m \geq N \Rightarrow d(f_m(z), f(z)) \leq \varepsilon$. Thus, $\{f_n\}$ converges to f uniformly on K. ∎)

Thus, $\{f_n\}$ converges to $f \in C(G, \Omega)$.

IX. In short, $(C(G,\Omega), \rho)$ is a complete metric space.

Problem 3.143 $H(G) \subset C(G, \mathbb{C})$. Also, $H(G)$ is a complete metric space.

(Solution In view of IX, it suffices to show that $H(G)$ is a closed subset of $C(G, \mathbb{C})$. For this purpose, let us take any convergent sequence $\{f_n\}$ in $H(G)$. Let $f \in C(G, \mathbb{C})$, and $\lim_{n\to\infty} f_n = f$. It suffices to show that $f \in H(G)$.

By VI, $\{f_n\}$ converges to f uniformly on compact subsets of G. Now, by Conclusion 1.174, $f \in H(\Omega)$. ∎)

X. Conclusion 3.144 $H(G)$ is a complete metric space.

Note 3.145 Let X be a nonempty set. For every positive integer n, let $f_n : X \to \mathbb{C}$ be a function. Let $f : X \to \mathbb{C}$ be a function. Let a be a real number. Suppose that the sequence $\{f_n\}$ converges to f uniformly on X. Suppose that for every $x \in X$, $\mathrm{Re}(f(x)) \leq a$.

Problem 3.146 The sequence $\{e^{f_n}\}$ converges to e^f uniformly on X.

(Solution For this purpose, let us take any $\varepsilon > 0$. Now, since $\lim_{z\to 0} e^z = 1$, there exists $\delta > 0$ such that

$$|z| < \delta \Rightarrow |e^z - 1| < \frac{\varepsilon}{e^a} \quad \cdots (*).$$

Since the sequence $\{f_n\}$ converges to f uniformly on X, there exists a positive integer N such that

$$(n \geq N, \text{ and } x \in X) \Rightarrow |f_n(x) - f(x)| < \delta \quad \cdots (**).$$

Let us fix any positive integer $n \geq N$, and any $x \in X$. It suffices to show that $\left|e^{f_n(x)} - e^{f(x)}\right| < \varepsilon$.

By $(**)$, $|f_n(x) - f(x)| < \delta$, and hence by $(*)$,

$$\frac{\left|e^{f_n(x)} - e^{f(x)}\right|}{e^a} \leq \frac{\left|e^{f_n(x)} - e^{f(x)}\right|}{e^{\mathrm{Re}(f(x))}} = \frac{\left|e^{f_n(x)} - e^{f(x)}\right|}{\left|e^{f(x)}\right|} = \left|\frac{e^{f_n(x)} - e^{f(x)}}{e^{f(x)}}\right|$$
$$= \underbrace{\left|e^{f_n(x) - f(x)} - 1\right| < \frac{\varepsilon}{e^a}}.$$

Thus, $\left|e^{f_n(x)} - e^{f(x)}\right| < \varepsilon$. ∎)

Conclusion 3.147 Let X be a nonempty set. For every positive integer n, let $f_n : X \to \mathbb{C}$ be a function. Let $f : X \to \mathbb{C}$ be a function. Let a be a real number. Suppose that the sequence $\{f_n\}$ converges to f uniformly on X. Suppose that for every $x \in X$, $\mathrm{Re}(f(x)) \le a$. Then the sequence $\{e^{f_n}\}$ converges to e^f uniformly on X.

3.12 Branch of the Logarithm Function

Note 3.148 Definition Let z be a nonzero complex number. By Conclusion 1.57, Vol. 1, there exists a real number θ_0 such that $\frac{z}{|z|} = e^{i\theta_0}$, and hence $\{\theta : \theta \in \mathbb{R} \text{ and } z = |z|e^{i\theta}\}$ is a nonempty set of real numbers.

Problem 3.149 $\{\theta : \theta \in \mathbb{R} \text{ and } z = |z|e^{i\theta}\} = \{\theta_0 + 2n\pi : n \in \mathbb{Z}\}$.

(**Solution** Let $\theta \in \text{LHS}$, that is $\theta \in \mathbb{R}$ and $z = |z|e^{i\theta}$. We have to show that $\theta \in \text{RHS}$, that is $\left(\frac{(\theta-\theta_0)i}{2\pi i} =\right) \frac{(\theta-\theta_0)}{2\pi}$ is an integer. By Conclusion 1.53(b), Vol. 1, it suffices to show that $e^{(\theta-\theta_0)i} = 1$. Since $z = |z|e^{i\theta}$, and $\frac{z}{|z|} = e^{i\theta_0}$, we have $e^{i\theta} = e^{i\theta_0}$, and hence $e^{(\theta-\theta_0)i} = 1$. ∎)

The nonempty set $\{\theta : \theta \in \mathbb{R} \text{ and } z = |z|e^{i\theta}\}$ is denoted by $\arg(z)$.

Let z,w be nonzero complex numbers.

I. Problem 3.150 If $\theta \in \arg(z)$, and $\varphi \in \arg(w)$, then $(\theta + \varphi) \in \arg(zw)$.

(**Solution** Let $\theta \in \arg(z)$, and $\varphi \in \arg(w)$, that is $\theta, \varphi \in \mathbb{R}$, $z = |z|e^{i\theta}$, and $w = |w|e^{i\varphi}$. It follows that $(\theta + \varphi) \in \mathbb{R}$. It remains to show that $zw = |zw|e^{i(\theta+\varphi)}$.

$$\begin{aligned}\text{LHS} &= zw = |z|e^{i\theta} \cdot |w|e^{i\varphi} = |z||w|e^{i\theta}e^{i\varphi} = |zw|e^{i\theta}e^{i\varphi} = |zw|e^{i\theta+i\varphi} = |zw|e^{i(\theta+\varphi)} \\ &= \text{RHS}.\end{aligned}$$

∎)

II. Problem 3.151 If $\theta \in \arg(z)$, then $(-\theta) \in \arg\left(\frac{1}{z}\right)$.

(**Solution** Let $\theta \in \arg(z)$, that is $\theta \in \mathbb{R}$, and $z = |z|e^{i\theta}$. It follows that $(-\theta) \in \mathbb{R}$. It remains to show that $\frac{1}{z} = \left|\frac{1}{z}\right|e^{i(-\theta)}$.

$$\text{LHS} = \frac{1}{z} = \frac{1}{|z|e^{i\theta}} = \left|\frac{1}{z}\right|\frac{1}{e^{i\theta}} = \left|\frac{1}{z}\right|e^{i(-\theta)} = \text{RHS}.$$

∎)

III. Problem 3.152 If $\theta \in \arg(z)$, and $\varphi \in \arg(w)$, then $(\theta - \varphi) \in \arg\left(\frac{z}{w}\right)$.

(Solution By II, $(-\varphi) \in \arg\left(\frac{1}{w}\right)$, and hence by I,

$$((\theta - \varphi) =)\ (\theta + (-\varphi)) \in \arg\left(z \cdot \frac{1}{w}\right)\ \left(= \arg\left(\frac{z}{w}\right)\right). \text{ Thus, } (\theta - \varphi) \in \arg\left(\frac{z}{w}\right).$$

Definition Let z_0 be a complex number, and α be a nonzero complex number. The set

$$\{z_0 + t\alpha : 0 \le t\}$$

is denoted by $L(z_0; \alpha)$. Here, we say that $L(z_0; \alpha)$ *is a* ***half-line*** *with* ***initial point*** z_0 *and the* ***direction*** $\arg(\alpha)$.

Let $z_1, z_2, \alpha, \beta \in \mathbb{C}$, and let α, β be nonzero.

Problem 3.153 $L(z_1; \alpha) = L(z_2; \beta) \Rightarrow (z_1 = z_2$ and $\arg(\alpha) = \arg(\beta))$.

(Solution Let $L(z_1; \alpha) = L(z_2; \beta)$. We claim that $z_1 = z_2$. If not, otherwise let $z_1 \neq z_2$. We have to arrive at a contradiction.

Since $z_1 \in L(z_1; \alpha)\ (= L(z_2; \beta))$, $\frac{z_1 - z_2}{\beta}$ is a positive real. Similarly, $\frac{z_2 - z_1}{\alpha}$ is a positive real. It follows that $\left(\frac{-\alpha}{\beta} =\right) \frac{\frac{z_1 - z_2}{\beta}}{\frac{z_2 - z_1}{\alpha}}$ is a positive real, and hence $(L(z_1; \alpha) =)\ L(z_2; \beta) = L(z_2; -\alpha)\ (\ni z_2)$. Thus, there exists a real number $t > 0$ such that $z_2 = z_1 + t\alpha$. Here, $L(z_1; \alpha) = L(z_2; \beta) = L(z_2; -\alpha) \ni (z_2 + 2t(-\alpha)) = (z_1 + t\alpha) + 2t(-\alpha) = (z_1 + (-t)\alpha) \notin L(z_1; \alpha)$, so we get a contradiction. Thus, our claim $z_1 = z_2$ is true.

Thus, $L(z_1; \alpha) = L(z_1; \beta)$.

Case I: when $\alpha = \beta$. In this case, $\arg(\alpha) = \arg(\beta)$.

Case II: when $\alpha \neq \beta$. There exists a real number θ_1 such that $\frac{\alpha}{|\alpha|} = e^{i\theta_1}$. There exists a real number θ_2 such that $\frac{\beta}{|\beta|} = e^{i\theta_2}$. We claim that

$$\{\theta_1 + 2n\pi : n \in \mathbb{Z}\} = \underbrace{\arg(\alpha) \subset \arg(\beta)} = \{\theta_2 + 2n\pi : n \in \mathbb{Z}\}.$$

If not, suppose otherwise that there exists an integer n_1 such that $\frac{(\theta_1 + 2n_1\pi) - \theta_2}{2\pi}$ is not an integer. We have to arrive at a contradiction. Since $\frac{((\theta_1 + 2n_1\pi) - \theta_2)i}{2\pi i}$ is not an integer, by Conclusion 1.53,

$$\frac{\alpha}{|\alpha|}\frac{|\beta|}{\beta} = e^{i\theta_1}e^{-i\theta_2} \cdot 1 = e^{i\theta_1}e^{-i\theta_2}e^{2n_1\pi i} = \underbrace{e^{((\theta_1 + 2n_1\pi) - \theta_2)i} \neq 1},$$

and hence $\frac{\alpha}{\beta} \neq \left|\frac{\alpha}{\beta}\right|$. Thus, $\frac{\alpha}{\beta}$ is not a positive real. Since $((z_1 + \alpha) \in)\ L(z_1; \alpha) = L(z_1; \beta)$, we have $(z_1 + \alpha) \in L(z_1; \beta)$, and hence $\left(\frac{\alpha}{\beta} =\right) \frac{(z_1 + \alpha) - z_1}{\beta}$ is a positive real. Thus, $\frac{\alpha}{\beta}$ is a positive real. This is a contradiction.

Thus, our claim $\arg(\alpha) \subset \arg(\beta)$ is true. Similarly, $\arg(\beta) \subset \arg(\alpha)$ is true. Thus, $\arg(\alpha) = \arg(\beta)$. Hence, in all cases, $\arg(\alpha) = \arg(\beta)$. ∎)

Conclusion 3.154 Let $z_1, z_2, \alpha, \beta \in \mathbb{C}$, and let α, β be nonzero. Then $L(z_1; \alpha) = L(z_2; \beta) \Rightarrow (z_1 = z_2 \text{ and } \arg(\alpha) = \arg(\beta))$.

Note 3.155 **Definition** Let w be a nonzero complex number. By Conclusion 1.59, Vol. 1, $\{z : e^z = w\}$ is a nonempty set, and is denoted by $\log(w)$.

Let w be a nonzero complex number.

Problem 3.156 $(\{z : e^z = w\} =) \log(w) = \ln|w| + i\arg(w)$.

(Solution There exists a real number θ_0 such that $\frac{w}{|w|} = e^{i\theta_0}$. We have to show that

$$\{z : e^z = w\} = \{\ln|w| + i(\theta_0 + 2n\pi) : n \in \mathbb{Z}\}.$$

For this purpose, let us take any complex number z satisfying $e^z = w$. We have to show that $\frac{z - (\ln|w| + i\theta_0)}{2\pi i}$ is an integer. By Conclusion 1.53(a), Vol. 1, we have to show that $e^{z - (\ln|w| + i\theta_0)} = 1$.

$$\begin{aligned} \text{LHS} &= e^{z - (\ln|w| + i\theta_0)} = e^z \frac{1}{e^{(\ln|w| + i\theta_0)}} = w \frac{1}{e^{(\ln|w| + i\theta_0)}} = \frac{w}{e^{\ln|w|} e^{i\theta_0}} = \frac{w}{|w| e^{i\theta_0}} = 1 \\ &= \text{RHS}. \end{aligned}$$

Thus, $\{z : e^z = w\} \subset \{\ln|w| + i(\theta_0 + 2n\pi) : n \in \mathbb{Z}\}$. It remains to show that $\{\ln|w| + i(\theta_0 + 2n\pi) : n \in \mathbb{Z}\} \subset \{z : e^z = w\}$.

For this purpose, let us take any integer n. We have to show that $e^{\ln|w| + i(\theta_0 + 2n\pi)} = w$.

$$\text{LHS} = e^{\ln|w| + i(\theta_0 + 2n\pi)} = e^{\ln|w|} e^{i\theta_0} e^{2n\pi i} = |w| e^{i\theta_0} e^{2n\pi i} = w e^{2n\pi i} = w1 = w = \text{RHS}.$$

∎)

Conclusion 3.157 Let w be a nonzero complex number. Then $\log(w) = \ln|w| + i\arg(w)$.

Note 3.158 Let w_1, w_2 be nonzero complex numbers.

Problem 3.159 If $z_1 \in \log(w_1)$, and $z_2 \in \log(w_2)$, then

$$\begin{aligned} \underbrace{(z_1 + z_2) \in \log(w_1 w_2)} &= \ln|w_1 w_2| + i\arg(w_1 w_2) = \ln(|w_1||w_2|) + i\arg(w_1 w_2) \\ &= (\ln|w_1| + \ln|w_2|) + i\arg(w_1 w_2) \end{aligned}$$

(Solution Since $z_1 \in \log(w_1)$, by Conclusion 3.157, there exists $\theta_1 \in \arg(w_1)$ such that $z_1 = \ln|w_1| + i\theta_1$. Similarly, there exists $\theta_2 \in \arg(w_2)$ such that $z_2 = \ln|w_2| + i\theta_2$. Since $\theta_1 \in \arg(w_1)$, and $\theta_2 \in \arg(w_2)$, by Conclusion 3.154(I), $(\theta_1 + \theta_2) \in \arg(w_1 w_2)$. It suffices to show that $(z_1 + z_2) = (\ln|w_1| + \ln|w_2|) + i(\theta_1 + \theta_2)$.

$$\begin{aligned}\text{LHS} &= z_1 + z_2 = (\ln|w_1| + i\theta_1) + (\ln|w_2| + i\theta_2) = (\ln|w_1| + \ln|w_2|) + i(\theta_1 + \theta_2) \\ &= \text{RHS}.\end{aligned}$$

∎)

Problem 3.160 If $z_1 \in \log(w_1)$, then

$$\begin{aligned}\underbrace{(-z_1) \in \log\left(\frac{1}{w_1}\right)} &= \ln\left|\frac{1}{w_1}\right| + i\arg\left(\frac{1}{w_1}\right) = \ln\frac{1}{|w_1|} + i\arg\left(\frac{1}{w_1}\right) \\ &= (-\ln|w_1|) + i\arg\left(\frac{1}{w_1}\right).\end{aligned}$$

(**Solution** Since $z_1 \in \log(w_1)$, by the conclusion of Note 3.39, there exists $\theta_1 \in \arg(w_1)$ such that $z_1 = \ln|w_1| + i\theta_1$. Since $\theta_1 \in \arg(w_1)$, by Conclusion 3.154(II), $(-\theta_1) \in \arg\left(\frac{1}{w_1}\right)$. It suffices to show that $-z_1 = (-\ln|w_1|) + i(-\theta_1)$.

$$\text{LHS} = -z_1 = -(\ln|w_1| + i\theta_1) = (-\ln|w_1|) + i(-\theta_1) = \text{RHS}.$$

∎)

Conclusion 3.161 Let w_1, w_2 be nonzero complex numbers. Then

1. if $z_1 \in \log(w_1)$, and $z_2 \in \log(w_2)$, then $(z_1 + z_2) \in \log(w_1 w_2)$,
2. if $z_1 \in \log(w_1)$, then $(-z_1) \in \log\left(\frac{1}{w_1}\right)$.

Note 3.162 **Definition** Let z be a nonzero complex number. Let c be any complex number. The set

$$\{e^{ac} : a \in \log(z)\}$$

is denoted by $z^{[c]}$.

I. Let w be a nonzero complex number. There exists a real number θ_0 such that $\frac{w}{|w|} = e^{i\theta_0}$.

Clearly,

$$\begin{aligned} w^{[2]} &= \left\{e^{2a} : a \in \log(w)\right\} = \left\{e^{2a} : a \in (\ln|w| + i\arg(w))\right\} \\ &= \left\{e^{2a} : a \in (\ln|w| + \{i(\theta_0 + 2n\pi) : n \in \mathbb{Z}\})\right\} \\ &= \left\{e^{2a} : a \in \{\ln|w| + i(\theta_0 + 2n\pi) : n \in \mathbb{Z}\}\right\} \\ &= \left\{e^{2(\ln|w| + i(\theta_0 + 2n\pi))} : n \in \mathbb{Z}\right\} \\ &= \left\{e^{2(\ln|w| + i\theta_0)}\right\} = \left\{\left(e^{(\ln|w| + i\theta_0)}\right)^2\right\} = \left\{w^2\right\}. \end{aligned}$$

Similarly, for every integer n, $w^{[n]} = \{w^n\}$.
Clearly,

$$\begin{aligned} w^{\left[\frac{1}{3}\right]} = \left\{e^{\frac{1}{3}a} : a \in \log(w)\right\} &= \left\{e^{\frac{1}{3}a} : a \in (\ln|w| + i\arg(w))\right\} \\ &= \left\{e^{\frac{1}{3}a} : a \in (\ln|w| + \{i(\theta_0 + 2n\pi) : n \in \mathbb{Z}\})\right\} \\ &= \left\{e^{\frac{1}{3}a} : a \in \{\ln|w| + i(\theta_0 + 2n\pi) : n \in \mathbb{Z}\}\right\} \\ &= \left\{e^{\frac{1}{3}(\ln|w| + i(\theta_0 + 2n\pi))} : n \in \mathbb{Z}\right\} \\ &= \left\{e^{\frac{1}{3}(\ln|w| + i\theta_0)} e^{\frac{2n\pi i}{3}} : n \in \mathbb{Z}\right\} \\ &= \left\{e^{\frac{1}{3}(\ln|w| + i\theta_0)}, e^{\frac{1}{3}(\ln|w| + i\theta_0)} e^{\frac{2\pi i}{3}}, e^{\frac{1}{3}(\ln|w| + i\theta_0)} e^{\frac{4\pi i}{3}}\right\}. \end{aligned}$$

Thus,

$$w^{\left[\frac{1}{3}\right]} = \left\{e^{\frac{1}{3}(\ln|w| + i\theta_0)}, e^{\frac{1}{3}(\ln|w| + i\theta_0)} e^{\frac{2\pi i}{3}}, e^{\frac{1}{3}(\ln|w| + i\theta_0)} e^{\frac{4\pi i}{3}}\right\}, \text{ etc.}$$

II. Let w be a nonzero complex number. Let $\alpha \in \mathbb{R}$.

Problem 3.163 $\alpha \in \arg(w) \Leftrightarrow w \in L(0; e^{i\alpha})$.

(Solution Let $\alpha \in \arg(w)$. We have to show that $w \in L(0; e^{i\alpha})$ $(= \{te^{i\alpha} : 0 \le t\})$, that is $\frac{w}{e^{i\alpha}}$ is a nonnegative real number. Since $\alpha \in \arg(w)$, we have $w = |w|e^{i\alpha}$. It follows that $\frac{w}{e^{i\alpha}}$ $(= |w|)$ is a nonnegative real number.

Conversely, let $w \in L(0; e^{i\alpha})$ $(= \{te^{i\alpha} : 0 \le t\})$, that is $\frac{w}{e^{i\alpha}}$ is a nonnegative real number. We have to show that $\alpha \in \arg(w)$, that is $w = |w|e^{i\alpha}$. Let $\frac{w}{e^{i\alpha}} = t_0$, where $t_0 \ge 0$. It follows that $(|w| =)$ $\left|\frac{w}{e^{i\alpha}}\right| = t_0$ $\left(= \frac{w}{e^{i\alpha}}\right)$, and hence $w = |w|e^{i\alpha}$. ∎)

There exists a real number θ_0 such that $\frac{w}{|w|} = e^{i\theta_0}$. Hence,

$$\arg(w) = \{\theta_0 + 2n\pi : n \in \mathbb{Z}\} = \{\cdots, \theta_0 - 4\pi, \theta_0 - 2\pi, \theta_0, \theta_0 + 2\pi, \theta_0 + 4\pi, \ldots\}.$$

Thus,

$$w \notin L\left(0; e^{i\alpha}\right) \Leftrightarrow \alpha \notin \{\cdots, \theta_0 - 4\pi, \theta_0 - 2\pi, \theta_0, \theta_0 + 2\pi, \theta_0 + 4\pi, \ldots\}.$$

It follows that for every complex number $w \notin L(0; e^{i\alpha})$,

$$(\alpha, \alpha + 2\pi) \cap \{\cdots, \theta_0 - 4\pi, \theta_0 - 2\pi, \theta_0, \theta_0 + 2\pi, \theta_0 + 4\pi, \ldots\}$$

is a singleton set, and hence $(\alpha, \alpha + 2\pi) \cap \arg(w)$ is a singleton set.

Also, for every nonzero complex number w,

$$(\alpha, \alpha + 2\pi] \cap \{\cdots, \theta_0 - 4\pi, \theta_0 - 2\pi, \theta_0, \theta_0 + 2\pi, \theta_0 + 4\pi, \ldots\}$$

is a singleton set, and hence $(\alpha, \alpha + 2\pi] \cap \arg(w)$ is a singleton set.

Definition This singleton set is denoted by $\{\mathrm{Arg}_\alpha(w)\}$.

For example, $\mathrm{Arg}_{3.7\pi}(i) = 4.5\pi$, $\mathrm{Arg}_{4.5\pi}(i) = 6.5\pi$, $\mathrm{Arg}_0(1) = 2\pi$.

IIa. Thus, for every real α, $\mathrm{Arg}_\alpha : (\mathbb{C} - \{0\}) \to (\alpha, \alpha + 2\pi]$ is a function. For every real α, and for every $w \in (\mathbb{C} - \{0\})$, we have $\mathrm{Arg}_\alpha(w) \in \arg(w)$, and $\mathrm{Arg}_\alpha(w) \in (\alpha, \alpha + 2\pi]$, and hence $w = |w|e^{i(\mathrm{Arg}_\alpha(w))}$.

Here, $\mathrm{Arg}_0 : (\mathbb{C} - \{0\}) \to (0, 2\pi]$ is a function. Also,

$$(\mathbb{C} - L(0;1)) = \{w : 0 < \mathrm{Im}(w)\} \cup \{w : \mathrm{Im}(w) < 0\} \cup \{w : \mathrm{Re}(w) < 0\}.$$

Let us fix any $w_0 \in (\mathbb{C} - \{0\})$.

It follows that

$$w_0 = |w_0|e^{i(\mathrm{Arg}_0(w_0))} = |w_0|\cos(\mathrm{Arg}_0(w_0)) + i|w_0|\sin(\mathrm{Arg}_0(w_0)),$$

and hence

$$\mathrm{Re}(w_0) = |w_0|\cos(\mathrm{Arg}_0(w_0)) \text{ and } \mathrm{Im}(w_0) = |w_0|\sin(\mathrm{Arg}_0(w_0)).$$

Problem 3.164 Arg_0 is continuous at w_0.

(**Solution** Case I: when $w_0 \in \{w : 0 < \mathrm{Im}(w)\}$. Since $0 < \mathrm{Im}(w_0)$ $(= |w_0| \sin(\mathrm{Arg}_0(w_0)))$, we have $0 < \sin(\mathrm{Arg}_0(w_0))$. Now, since $\mathrm{Arg}_0(w_0) \in (0, 2\pi)$, we have $\mathrm{Arg}_0(w_0) \in (0, \pi)$. Since

$$\cos : (0, \pi) \to (-1, 1)$$

is a strictly decreasing continuous function,

$$\cos^{-1} : (-1, 1) \to (0, \pi)$$

is a strictly decreasing continuous function. Now, since

$$\underbrace{w \mapsto \mathrm{Arg}_0(w)} = \cos^{-1}(\cos(\mathrm{Arg}_0(w))) = \cos^{-1}\left(\frac{\mathrm{Re}(w)}{|w|}\right)$$

is a continuous function from $\{w : 0 < \mathrm{Im}(w)\}$ to $(0, \pi)$, and $w_0 \in \{w : 0 < \mathrm{Im}(w)\}$, Arg_0 is continuous at w_0.

Case II: when $w_0 \in \{w : \mathrm{Im}(w) < 0\}$. Since $(|w_0|\sin(\mathrm{Arg}_0(w_0)) =)$ $\mathrm{Im}(w_0) < 0$, we have $\sin(\mathrm{Arg}_0(w_0)) < 0$. Now, since $\mathrm{Arg}_0(w_0) \in (0, 2\pi)$, we have $\mathrm{Arg}_0(w_0) \in (\pi, 2\pi)$. Since $\cos : (0, \pi) \to (-1, 1)$ is a strictly decreasing continuous function, $\cos^{-1} : (-1,1) \to (0,\pi)$ is a strictly decreasing continuous function. Now, since

$$\underbrace{w \mapsto \mathrm{Arg}_0(w)} = \cos^{-1}(\cos(\mathrm{Arg}_0(w) - \pi)) + \pi = \cos^{-1}\left(-\frac{\mathrm{Re}(w)}{|w|}\right) + \pi$$

is a continuous function from $\{w : \mathrm{Im}(w) < 0\}$ to $(\pi, 2\pi)$, and $w_0 \in \{w : \mathrm{Im}(w) < 0\}$, Arg_0 is continuous at w_0.

Case III: when $w_0 \in \{w : \mathrm{Re}(w) < 0\}$. Since $(|w_0|\cos(\mathrm{Arg}_0(w_0)) =)$ $\mathrm{Re}(w_0) < 0$, we have $\cos(\mathrm{Arg}_0(w_0)) < 0$. Now, since $\mathrm{Arg}_0(w_0) \in (0, 2\pi)$, we have $\mathrm{Arg}_0(w_0) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Since $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$ is a strictly increasing continuous function, $\tan^{-1} : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a strictly increasing continuous function. Now, since

$$\underbrace{w \mapsto \mathrm{Arg}_0(w)} = \tan^{-1}(\tan(\mathrm{Arg}_0(w) - \pi)) + \pi = \tan^{-1}\left(\frac{\mathrm{Im}(w)}{\mathrm{Re}(w)}\right) + \pi$$

is a continuous function from $\{w : \mathrm{Re}(w) < 0\}$ to $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $w_0 \in \{w : \mathrm{Re}(w) < 0\}$, Arg_0 is continuous at w_0.

So in all cases, Arg_0 is continuous at w_0. ∎)

III. Thus, $\mathrm{Arg}_0 : (\mathbb{C} - \{0\}) \to (0, 2\pi]$ is continuous at all points of $(\mathbb{C} - L(0;1))$.

Let α be any real number.

Problem 3.165 $\mathrm{Arg}_\alpha : (\mathbb{C} - \{0\}) \to (\alpha, \alpha + 2\pi]$ is continuous on $(\mathbb{C} - L(0; e^{i\alpha}))$.

(Solution By II, for every $w \in (\mathbb{C} - L(0; e^{i\alpha}))$, we have $\alpha \notin \arg(w)$, and hence $w \neq |w|e^{i\alpha}$. Thus, for every $w \in (\mathbb{C} - L(0; e^{i\alpha}))$, $we^{-i\alpha} \neq |w|$ $(= |we^{-i\alpha}|)$, and hence $we^{-i\alpha} \in (\mathbb{C} - L(0;1))$. Thus, $g : w \mapsto we^{-i\alpha}$ is a continuous function from $(\mathbb{C} - L(0; e^{i\alpha}))$ to $(\mathbb{C} - L(0;1))$.

Problem 3.166 For every $w \in (\mathbb{C} - L(0; e^{i\alpha}))$, $\mathrm{Arg}_\alpha(w) = \mathrm{Arg}_0(g(w)) + \alpha$.

(Solution For this purpose, let us take any $w \in (\mathbb{C} - L(0; e^{i\alpha}))$. We have to show that

$$\mathrm{Arg}_\alpha(w) = \mathrm{Arg}_0\left(we^{-i\alpha}\right) + \alpha.$$

Here,

$$w = |w|e^{i(\mathrm{Arg}_\alpha(w))},$$

and

$$|w|e^{i(\mathrm{Arg}_\alpha(w))}e^{-i\alpha} = \underbrace{we^{-i\alpha} = \left|we^{-i\alpha}\right|e^{i(\mathrm{Arg}_0(we^{-i\alpha}))}} = |w|e^{i(\mathrm{Arg}_0(we^{-i\alpha}))},$$

so

$$e^{i(\mathrm{Arg}_\alpha(w))} = e^{i(\mathrm{Arg}_0(we^{-i\alpha}) + \alpha)}.$$

Since

$$\mathrm{Arg}_0\left(we^{-i\alpha}\right) \in (0, 2\pi),$$

we have

$$\left(\mathrm{Arg}_0\left(we^{-i\alpha}\right) + \alpha\right) \in (\alpha, \alpha + 2\pi).$$

Next, $\mathrm{Arg}_\alpha(w) \in (\alpha, \alpha + 2\pi)$. Now, since $e^{i(\mathrm{Arg}_\alpha(w))} = e^{i(\mathrm{Arg}_0(we^{-i\alpha}) + \alpha)}$, we have $\mathrm{Arg}_\alpha(w) = \mathrm{Arg}_0(g(w)) + \alpha$. ∎)

Now, since Arg_0, g are continuous, $\mathrm{Arg}_\alpha : (\mathbb{C} - \{0\}) \to (\alpha, \alpha + 2\pi]$ is continuous on $(\mathbb{C} - L(0; e^{i\alpha}))$. ∎)

IV. Thus, $\mathrm{Arg}_\alpha : (\mathbb{C} - \{0\}) \to (\alpha, \alpha + 2\pi]$ is continuous on $(\mathbb{C} - L(0; e^{i\alpha}))$.

Definition Let $\alpha \in \mathbb{R}$. The function $w \mapsto (\ln|w| + i\,\mathrm{Arg}_\alpha(w))$ from $(\mathbb{C} - \{0\})$ to $\mathbb{C}$ is denoted by Log_α.

Continuous 3.167 For every $w \in (\mathbb{C} - \{0\})$, $\mathrm{Log}_\alpha(w) = \ln|w| + i\,\mathrm{Arg}_\alpha(w)$. Log_α and Arg_α are continuous on the region $(\mathbb{C} - L(0; e^{i\alpha}))$. Further, if $e^z = w$, then there exists an integer n such that $z = \mathrm{Log}_\alpha(w) + 2n\pi i$.

Note 3.168 **Definition** Let G be a region, that is, G is a nonempty open connected subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a continuous function. If for every $w \in G$, $e^{f(w)} = w$ then we say that *f is a **branch of the logarithm***.

The restriction of $\mathrm{Arg}_{-\pi}$ to $(\mathbb{C} - L(0; -1))$ is a continuous function, and is denoted by Arg. Arg is called the ***principal argument***.

Here, for every $w \in (\mathbb{C} - L(0; -1))\ (= \mathbb{C} - (-\infty, 0])$, we have $\mathrm{Arg}(w) \in (-\pi, -\pi + 2\pi)\ (= (-\pi, \pi))$, and $w = |w|e^{i\mathrm{Arg}(w)}$.

The restriction on $\mathrm{Log}_{-\pi}$ to $(\mathbb{C} - L(0; -1))$ is a continuous function, and is denoted by Log. Log is called the ***principal log.***

Now, since for every $w \in (\mathbb{C} - (-\infty, 0])$, $e^{\mathrm{Log}(w)} = e^{(\ln|w| + i\,\mathrm{Arg}(w))}$ $\left(= e^{\ln|w|}e^{i\,\mathrm{Arg}(w)} = |w|e^{i\mathrm{Arg}(w)} = w\right)$. Log is a branch of the logarithm, and is called the ***principal branch*** *of the logarithm.*

By Note 3.162(IIa), for every $w \in (\mathbb{C} - (-\infty, 0])$,

$$\begin{aligned}\log(w) &= \ln|w| + i\arg(w) = \ln|w| + i\{\mathrm{Arg}_{-\pi}(w) + 2n\pi : n \in \mathbb{Z}\} \\ &= \ln|w| + i\{\mathrm{Arg}(w) + 2n\pi : n \in \mathbb{Z}\},\end{aligned}$$

so for every $w \in (\mathbb{C} - (-\infty, 0])$,

$$\log(w) = (\ln|w| + i\,\mathrm{Arg}(w)) + \{2n\pi i : n \in \mathbb{Z}\} = \mathrm{Log}(w) + \{2n\pi i : n \in \mathbb{Z}\}.$$

I. Since for every $z \in \mathbb{C}$, $e^z \neq 0$, if $f : G \to \mathbb{C}$ is a branch of the logarithm, then $0 \notin G$.

II. Let $f : G \to \mathbb{C}$ be a branch of the logarithm. Let $k \in \mathbb{Z}$. Let $g : w \mapsto (f(w) + 2k\pi i)$ be a function from G to $\mathbb{C}$.

Problem 3.169 $g : G \to \mathbb{C}$ is a branch of the logarithm.

(**Solution** Since $f : G \to \mathbb{C}$ be a branch of the logarithm, $f : G \to \mathbb{C}$ is continuous, and hence $g : w \mapsto (f(w) + 2k\pi i)$ is continuous. Next, let us take any $w \in G$. It remains to show that $e^{g(w)} = w$. Here,

$$\text{LHS} = e^{g(w)} = e^{f(w)+2k\pi i} = e^{f(w)}e^{2k\pi i} = e^{f(w)} \cdot 1 = e^{f(w)} = w = \text{RHS}.$$

∎)

III. Let $f : G \to \mathbb{C}$, and $g : G \to \mathbb{C}$ be branches of the logarithm.

Problem 3.170 There exists $k_0 \in \mathbb{Z}$ such that for every $w \in G$, $g(w) = f(w) + 2k_0\pi i$.

(**Solution** Here, for every $w \in G$, $e^{f(w)} = w$ and $e^{g(w)} = w$, so, for every $w \in G$, $e^{g(w)-f(w)} = 1$. It follows that for every $w \in G$, there exists an integer $k(w)$ such that $g(w) - f(w) = 2k(w)\pi i$. Thus,

$$k : w \mapsto \frac{1}{2\pi i}(g - f)(w)$$

is a function from G to $\mathbb{Z}$. Since $f : G \to \mathbb{C}$, and $g : G \to \mathbb{C}$ are branches of the logarithm, $f : G \to \mathbb{C}$, and $g : G \to \mathbb{C}$ are continuous, and hence

$$k : w \mapsto \frac{1}{2\pi i}(g - f)(w)$$

is a continuous function from G to $\mathbb{Z}$. Now, since G is a nonempty connected subset of $\mathbb{C}$, $k(G)$ is a connected subset of $\mathbb{Z}$, and hence $k(G)$ is a singleton set, say $\{k_0\}$. Thus, for every $w \in G$, $g(w) = f(w) + 2k_0\pi i$. ∎)

Conclusion 3.171 Let $f : G \to \mathbb{C}$, and $g : G \to \mathbb{C}$ be branches of the logarithm. Then, there exists $k_0 \in \mathbb{Z}$ such that for every $w \in G$, $g(w) = f(w) + 2k_0\pi i$.

Note 3.172 Let G and Ω be nonempty open subsets of $\mathbb{C}$. Let $f : G \to \Omega$, and $g : \Omega \to \mathbb{C}$ be any functions. Suppose that for every $w \in G$, $g(f(w)) = w$. Let $g \in H(\Omega)$ and, for every $z \in \Omega$, $g'(z) \neq 0$. Let $f : G \to \Omega$ be continuous.

Problem 3.173 $f \in H(G)$ and, for every $w \in G$, $\lim_{h\to 0}\frac{f(w+h)-f(w)}{h} = \frac{1}{g'(f(w))}$.

(Solution Let us first observe that f is 1-1.

If not, suppose otherwise that there exist $w_1, w_2 \in G$ such that $w_2 \neq w_2$, and $f(w_1) = f(w_2)$. We have to arrive at a contradiction. Since $f(w_1) = f(w_2)$, we have $(w_1 =)\ g(f(w_1)) = g(f(w_2))\ (= w_2)$, and hence $w_1 = w_2$. This is a contradiction.

Hence, f is 1-1. Let us take any $w \in G$. We have to show that

$$\lim_{h\to 0}\frac{f(w+h)-f(w)}{h} = \frac{1}{g'(f(w))}.$$

$$\text{LHS} = \lim_{h\to 0}\frac{f(w+h)-f(w)}{h} = \lim_{h\to 0}\frac{f(w+h)-f(w)}{g(f(w+h))-g(f(w))}$$
$$= \lim_{k\to 0}\frac{k}{g(f(w)+k)-g(f(w))} = \frac{1}{g'(f(w))} = \text{RHS}. \qquad \blacksquare)$$

Conclusion 3.174 Let G and Ω be nonempty open subsets of $\mathbb{C}$. Let $f : G \to \Omega$, and $g : \Omega \to \mathbb{C}$ be any functions. Suppose that for every $w \in G$, $g(f(w)) = w$. Let $g \in H(\Omega)$ and, for every $z \in \Omega$, $g'(z) \neq 0$. Let $f : G \to \Omega$ be continuous. Then $f \in H(G)$ and, for every $w \in G, f'(w) = \frac{1}{g'(f(w))}$.

Let G be a region. Let $f : G \to \mathbb{C}$ be a branch of the logarithm.

It follows that $f : G \to \mathbb{C}$ is continuous. Also, for every $w \in G$, $e^{f(w)} = w$. Further, the function $g : z \mapsto e^z$ from $\mathbb{C}$ to $\mathbb{C}$ is a member of $H(\mathbb{C})$, and, for every $z \in \mathbb{C}$, $g'(z) = e^z \neq 0$. Hence, by Conclusion 3.174, $f \in H(G)$ and, for every $w \in G, f'(w) = \frac{1}{e^{f(w)}}\ \left(= \frac{1}{w}\right)$.

Conclusion 3.175 Let G be a region. Let $f : G \to \mathbb{C}$ be a branch of the logarithm. Then $f \in H(G)$ and, for every $w \in G, f'(w) = \frac{1}{w}$. In particular, for every $z \in (\mathbb{C} - (-\infty, 0])$, $(\text{Log})'(z) = \frac{1}{z}$.

3.13 Riemann Surface of the Logarithm Function

[Let us have some **intuitive discussion**.

We want to draw a graph of equation $w = e^z$ from z-plane to w-plane. Imagine that z-plane has a Cartesian coordinate system and w-plane has a polar coordinate system.

For fixed real y_0,

$$(x + iy_0) \mapsto e^{(x+iy_0)}\ \left(= (e^x)e^{iy_0} \neq 0\right),$$

so every horizontal line in the z-plane is mapped in a 1-1 manner onto a ray emanating from origin in the w-plane. Hence, for every real α, the mapping $z \mapsto e^z$ sends the 'horizontal strip' $\mathbb{R} \times (\alpha, \alpha + 2\pi)$ of the z-plane in a 1-1 manner onto the 'slit w-plane' $(\mathbb{C} - L(0; e^{i\alpha}))$. Evidently, this correspondence is 'continuous'.

Further, for every real α, the mapping $w \mapsto \mathrm{Log}_\alpha(w)$ sends the 'slit w-plane' $(\mathbb{C} - L(0; e^{i\alpha}))$ in a 1-1 manner onto the 'horizontal strip' $\mathbb{R} \times (\alpha, \alpha + 2\pi)$ in the z-plane. Evidently, this correspondence is continuous.

Also, for every real α, the mapping $w \mapsto \mathrm{Log}_\alpha(w)$ sends the 'punctured w-plane' $(\mathbb{C} - \{0\})$ in a 1-1 manner onto 'horizontal strip' $\mathbb{R} \times (\alpha, \alpha + 2\pi]$ in the z-plane. Evidently, this correspondence is 'continuous from below' at all nonzero points of the slit.

Observe that

1. the mapping $f_1 : w \mapsto \mathrm{Log}(w)$ sends the '1st slit w-plane' $(\mathbb{C} - L(0; -1))$ in a 1-1 manner onto the 'horizontal strip' $\mathbb{R} \times (-\pi, \pi)$ in the z-plane. Evidently, this correspondence is continuous,
2. the mapping $f_2 : w \mapsto \mathrm{Log}(w) + 2\pi$ sends the '2nd slit w-plane' $(\mathbb{C} - L(0; -1))$ in a 1-1 manner onto the 'horizontal strip' $\mathbb{R} \times (\pi, 3\pi)$ in the z-plane. Evidently, this correspondence is continuous.

Let us imagine that the slit of the '1st slit w-plane' is made up of two edges, namely 'top edge' and 'bottom edge'. Here, let us fix any $r \in L(0; -1)$. Observe that $\lim_{\varepsilon \to 0} f_1(r + i\varepsilon) = \pi$, and $\lim_{\varepsilon \to 0} f_1(r - i\varepsilon) = -\pi$, $\lim_{\varepsilon \to 0} f_2(r - i\varepsilon) = (-\pi) + 2\pi\ (= \pi)$.

Thus, $\lim_{\varepsilon \to 0} f_1(r + i\varepsilon) = \lim_{\varepsilon \to 0} f_2(r - i\varepsilon)$.

Here, $\lim_{\varepsilon \to 0} f_1(r + i\varepsilon)$ corresponds with top edge of the 1st slit w-plane, and $\lim_{\varepsilon \to 0} f_2(r - i\varepsilon)$ corresponds with bottom edge of the 2nd slit w-plane.

On observing the fact $\lim_{\varepsilon \to 0} f_1(r + i\varepsilon) = \lim_{\varepsilon \to 0} f_2(r - i\varepsilon)$, Riemann had an interesting geometrical idea:

If we place a 2nd slit w-plane over a 1st slit w-plane, and glue together the top edge of the 1st slit w-plane with the bottom edge of the 2nd slit w-plane, then we get a new 'spiral stairway shaped surface', say S_1, which has some interesting properties. There exists a 1-1 function F_1 from S_1 onto $\mathbb{R} \times (-\pi, 3\pi)$ such that

1. F_1 is an extension of f_1,
2. F_1 is an extension of f_2,
3. F_1 is continuous at all points of S_1.

On repeating the above procedure, we get an 'unending spiral stairway shaped surface', say S, and a 1-1 function F from S onto $\mathbb{R} \times \mathbb{R}\ (\approx \mathbb{C})$ such that

1. F is an extension of each f_n,
2. F is continuous at all points of S.

This 'imaginary' surface S is referred to as the **Riemann surface of the logarithm function**.

Similarly, we can construct the Riemann surface of other 'multi-valued functions', like $z^{\frac{1}{2}}, (z-1)^{\frac{1}{3}}$, etc.]

3.14 Infinite Product Converges Absolutely

Note 3.176 For every positive integer n, let $\mathrm{Re}(z_n) > 0$. Suppose that the infinite product $\prod_{n=1}^{\infty} z_n \left(= \lim_{n\to\infty}\left(\prod_{k=1}^{n} z_k\right)\right)$ converges to a nonzero complex number.

Problem 3.177 $\lim_{n\to\infty} z_n = 1$.

(Solution It suffices to show that $\lim_{n\to\infty} z_{n+1} = 1$. Suppose that $\lim_{n\to\infty}\left(\prod_{k=1}^{n} z_k\right) = z$, where z is a nonzero complex number. It follows that $\lim_{n\to\infty}\left(\prod_{k=1}^{n+1} z_k\right) = z$, and hence

$$\left(\lim_{n\to\infty} z_{n+1} =\right) \log_{n\to\infty} \frac{\prod_{k=1}^{n+1} z_k}{\prod_{k=1}^{n} z_k} = \frac{\lim_{n\to\infty}\left(\prod_{k=1}^{n+1} z_k\right)}{\lim_{n\to\infty}\left(\prod_{k=1}^{n} z_k\right)} \left(= \frac{z}{z} = 1\right).$$

Thus, $\lim_{n\to\infty} z_{n+1} = 1$. ∎)

Conclusion 3.178 For every positive integer n, let $\mathrm{Re}(z_n) > 0$. Suppose that the infinite product $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number. Then, $\lim_{n\to\infty} z_n = 1$.

For every positive integer n, let $\mathrm{Re}(z_n) > 0$.

(It follows that each $z_n \in (\mathbb{C} - (-\infty, 0])$, and hence for every positive integer, $\mathrm{Log}(z_n) \in \mathbb{C}$.)

Suppose that

$$\left(\lim_{n\to\infty}\left(\sum_{k=1}^{n} \mathrm{Log}(z_k)\right) =\right) \mathrm{Log}(z_1) + \mathrm{Log}(z_2) + \mathrm{Log}(z_3) + \cdots$$

is convergent.

Problem 3.179 The infinite product $\prod_{n=1}^{\infty} z_n \left(= \lim_{n\to\infty}\left(\prod_{k=1}^{n} z_n\right)\right)$ converges to a nonzero complex number.

(Solution Suppose that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \mathrm{Log}(z_k)\right) = s,$$

where $s \in \mathbb{C}$. Now, since exponential is a continuous function from $\mathbb{C}$ to $\mathbb{C} - \{0\}$,

$$\prod_{n=1}^{\infty} z_n = \lim_{n\to\infty}\left(\prod_{k=1}^{n} z_k\right) = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(e^{\mathrm{Log}(z_k)}\right)\right) = \underbrace{\lim_{n\to\infty} e^{\left(\sum_{k=1}^{n} \mathrm{Log}(z_k)\right)} = e^s}_{} \neq 0.$$

Thus, $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number. ∎)

Conclusion 3.180 For every positive integer n, let $\mathrm{Re}(z_n) > 0$. Suppose that $\sum_{n=1}^{\infty} \mathrm{Log}(z_n)$ is convergent. Then, $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number.

For every positive integer n, let $\mathrm{Re}(z_n) > 0$.

(It follows that each $z_n \in (\mathbb{C} - (-\infty, 0])$, and hence for every positive integer, $\mathrm{Log}(z_n) \in \mathbb{C}$.)

Suppose that $\prod_{n=1}^{\infty} z_n \left(= \lim_{n\to\infty}\left(\prod_{k=1}^{n} z_k\right)\right)$ converges to a nonzero complex number.

Problem 3.181 $\lim_{n\to\infty} \mathrm{Log}(z_n) = 0$.

(**Solution** By Conclusion 3.178, $\lim_{n\to\infty} z_n = 1 \ (\in (\mathbb{C} - (-\infty, 0]))$. Now, since $\{z_n\}$ is a sequence in $(\mathbb{C} - (-\infty, 0])$, and $\mathrm{Log} : (\mathbb{C} - (-\infty, 0]) \to \mathbb{C}$ is a continuous function, we have $\lim_{n\to\infty} \mathrm{Log}(z_n) = \mathrm{Log}1 \ (= 0)$, and hence $\lim_{n\to\infty} \mathrm{Log}(z_n) = 0$. ∎)

Conclusion 3.182 For every positive integer n, let $\mathrm{Re}(z_n) > 0$. Suppose that $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number. Then, $\lim_{n\to\infty} \mathrm{Log}(z_n) = 0$.

For every positive integer n, let $\mathrm{Re}(z_n) > 0$. Suppose that $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number.

Problem 3.183 $\mathrm{Log}(z_1) + \mathrm{Log}(z_2) + \mathrm{Log}(z_3) + \cdots$ is convergent.

(**Solution** For every positive integer n, put $S_n \equiv \sum_{k=1}^{n} \mathrm{Log}(z_k)$, and $p_n \equiv \prod_{k=1}^{n} z_k$. We have to show that $\{S_n\}$ is convergent.

Since $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number, each z_n is nonzero. It follows that $\{p_n\}$ is a sequence in $(\mathbb{C} - \{0\})$ such that $(\lim_{n\to\infty} p_n) \in (\mathbb{C} - \{0\})$. Also, for every positive integer n,

$$e^{S_n} = e^{\sum_{k=1}^{n} \mathrm{Log}(z_k)} = \prod_{k=1}^{n} e^{\mathrm{Log}(z_k)} = \prod_{k=1}^{n} z_k = p_n,$$

so for every positive integer n, $e^{S_n} = p_n$. Since $(\lim_{n\to\infty} p_n) \neq 0$, there exists a real number α such that $(\lim_{n\to\infty} p_n) \in (\mathbb{C} - L(0; e^{i\alpha}))$. By Note 3.162(IV), the function $\mathrm{Log}_\alpha : (\mathbb{C} - \{0\}) \to \{x + iy : x \in \mathbb{R} \text{ and } y \in [\alpha, \alpha + 2\pi)\}$ is continuous. Now, since $\{p_n\}$ is a convergent sequence in $(\mathbb{C} - \{0\})$ such that $(\lim_{n\to\infty} p_n) \in (\mathbb{C} - \{0\})$, $\{\mathrm{Log}_\alpha(p_n)\}$ is convergent.

Since for every positive integer n, $e^{S_n} = p_n$, for every positive integer n, there exists an integer k_n such that $S_n = \mathrm{Log}_\alpha(p_n) + 2k_n\pi i$. It follows that $\{S_n - 2k_n\pi i\}$ is convergent.

It suffices to show that all but finite-many k_n's are equal, that is

$$\begin{aligned}\frac{1}{2\pi i}\lim_{n\to\infty}\mathrm{Log}(z_n) &= \frac{1}{2\pi i}\left(\lim_{n\to\infty}\mathrm{Log}(z_n)-\lim_{n\to\infty}\mathrm{Log}_\alpha(p_n)+\lim_{n\to\infty}\mathrm{Log}_\alpha(p_n)\right)\\ &= \frac{1}{2\pi i}\left(\lim_{n\to\infty}\mathrm{Log}(z_n)-\lim_{n\to\infty}\mathrm{Log}_\alpha(p_n)+\lim_{n\to\infty}\mathrm{Log}_\alpha(p_{n-1})\right)\\ &= \frac{1}{2\pi i}\lim_{n\to\infty}(\mathrm{Log}(z_n)-\mathrm{Log}_\alpha(p_n)+\mathrm{Log}_\alpha(p_{n-1}))\\ &= \lim_{n\to\infty}\left(\frac{1}{2\pi i}(S_n-\mathrm{Log}_\alpha(p_n))-\frac{1}{2\pi i}(S_{n-1}-\mathrm{Log}_\alpha(p_{n-1}))\right)\\ &= \underbrace{\lim_{n\to\infty}(k_n-k_{n-1})=0}, \end{aligned}$$

that is $\lim_{n\to\infty}\mathrm{Log}(z_n)=0$. This is known to be true, by Conclusion 3.182. ∎)

Conclusion 3.184 For every positive integer n, let $\mathrm{Re}(z_n)>0$. Suppose that $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number. Then, $\mathrm{Log}(z_1)+\mathrm{Log}(z_2)+\mathrm{Log}(z_3)+\cdots$ is convergent.

Note 3.185 By Conclusion 3.175, in some neighborhood of 0, the function $f : z\mapsto \mathrm{Log}(1+z)$ is holomorphic, so, by Conclusion 1.116, and Lemma 1.60, for every z in that neighborhood of 0,

$$\begin{aligned}\mathrm{Log}(1+z) = \underbrace{f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(z-0)^n} &= \mathrm{Log}(1+0)+\tfrac{1}{1!}\tfrac{1}{(1+0)}z+\tfrac{1}{2!}\tfrac{1(-1)}{(1+0)^2}z^2+\cdots\\ &= 0+z-\tfrac{z^2}{2}+\tfrac{z^3}{3}-\cdots = z-\tfrac{z^2}{2}+\tfrac{z^3}{3}-\cdots.\end{aligned}$$

Now, since the radius of convergence of series $z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots$ is $\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^{n+1}}{n}\right|}$, and

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{(-1)^{n+1}}{n}\right|}=\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{n}}=\frac{1}{1}=1,$$

for every $z\in D(0;1)$,

$$\underbrace{\mathrm{Log}(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots}=z\left(1-\frac{z}{2}+\frac{z^2}{3}-\cdots\right),$$

and hence for every $z\in\left(D\left[0;\frac{1}{2}\right]-\{0\}\right)$,

$$\left|\frac{|\text{Log}(1+z)|}{|z|}-1\right|=\left|\left|\frac{\text{Log}(1+z)}{z}\right|-|1|\right|\le\left|\frac{\text{Log}(1+z)}{z}-1\right|=\left|-\frac{z}{2}+\frac{z^2}{3}-\cdots\right|\le\frac{|z|}{2}+\frac{|z|^2}{3}+\cdots$$
$$\le\frac{|z|}{2}+\frac{|z|^2}{2}+\frac{|z|^3}{2}\cdots\le\frac{1}{2}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

Thus, for every $z\in\left(D\left[0;\frac{1}{2}\right]-\{0\}\right)$, $1-\frac{1}{2}\le\frac{|\text{Log}(1+z)|}{|z|}\le 1+\frac{1}{2}$, and hence

$$|z|\le\frac{1}{2}\Rightarrow\frac{1}{2}|z|\le|\text{Log}(1+z)|\le\frac{3}{2}|z|.$$

Conclusion 3.186 For every complex number z satisfying $|z|\le\frac{1}{2}$, $\frac{1}{2}|z|\le|\text{Log}(1+z)|\le\frac{3}{2}|z|$.

Note 3.187 For every positive integer n, let $-1<\text{Re}(z_n)$. It follows that $\text{Re}(1+z_n)>0$. Suppose that $|\text{Log}(1+z_1)|+|\text{Log}(1+z_2)|+|\text{Log}(1+z_3)|+\cdots$ is convergent.

It follows that $\text{Log}(1+z_1)+\text{Log}(1+z_2)+\text{Log}(1+z_3)+\cdots$ is convergent. Now, by Conclusion 3.180, $\prod_{n=1}^{\infty}(1+z_n)$ converges to a nonzero complex number, and hence, by Conclusion 3.178, $\lim_{n\to\infty}(1+z_n)=1$. It follows that $\lim_{n\to\infty}z_n=0$, and hence there exists a positive integer N such that $n\ge N\Rightarrow|z_n|\le\frac{1}{2}$. Now, by Conclusion 3.186,

$$n\ge N\Rightarrow\frac{1}{2}|z_n|\le|\text{Log}(1+z_n)|.$$

Thus, since $|\text{Log}(1+z_1)|+|\text{Log}(1+z_2)|+|\text{Log}(1+z_3)|+\cdots$ is convergent, $\sum_{n=1}^{\infty}|z_n|$ is convergent.

Conclusion 3.188 For every positive integer n, let $-1<\text{Re}(z_n)$. $\sum_{n=1}^{\infty}\text{Log}(1+z_n)$ converges absolutely if and only if $\sum_{n=1}^{\infty}z_n$ converges absolutely.

Proof of the remaining part: Let $\sum_{n=1}^{\infty}z_n$ converges absolutely, that is $\sum_{n=1}^{\infty}|z_n|<\infty$. We have to show that $\sum_{n=1}^{\infty}\text{Log}(1+z_n)$ converges absolutely, that is $\sum_{n=1}^{\infty}|\text{Log}(1+z_n)|<\infty$.

Since $\sum_{n=1}^{\infty}|z_n|<\infty$, we have $\lim_{n\to\infty}|z_n|=0$, and hence there exists a positive integer N such that $n\ge N\Rightarrow|z_n|\le\frac{1}{2}$. Now, by Conclusion 3.186,

$$n\ge N\Rightarrow|\text{Log}(1+z_n)|\le\frac{3}{2}|z_n|.$$

Thus, since $\sum_{n=1}^{\infty}|z_n|<\infty$, we have $\sum_{n=1}^{\infty}|\text{Log}(1+z_n)|<\infty$. ∎

Definition For every positive integer n, let $\text{Re}(z_n)>0$. If $\sum_{n=1}^{\infty}|\text{Log}(z_n)|<\infty$, then we say that the *infinite product* $\prod_{n=1}^{\infty}z_n$ ***converges absolutely***.

For every positive integer n, let $\text{Re}(z_n)>0$. Suppose that $\prod_{n=1}^{\infty}z_n$ converges absolutely.

Problem 3.189 $\prod_{n=1}^{\infty} z_n$ is convergent.

(Solution Since $\prod_{n=1}^{\infty} z_n$ converges absolutely, by definition, $\sum_{n=1}^{\infty} |\mathrm{Log}(z_n)| < \infty$, and hence $\sum_{n=1}^{\infty} \mathrm{Log}(z_n)$ is convergent. Now, by Conclusion 3.180, $\prod_{n=1}^{\infty} z_n$ is convergent. ∎)

Conclusion 3.190 For every positive integer n, let $\mathrm{Re}(z_n) > 0$. If $\prod_{n=1}^{\infty} z_n$ converges absolutely, then $\prod_{n=1}^{\infty} z_n$ is convergent.

For every positive integer n, let $\mathrm{Re}(z_n) > 0$. Suppose that $\prod_{n=1}^{\infty} z_n$ converges absolutely.

Problem 3.191 $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely.

(Solution Since $\prod_{n=1}^{\infty} z_n$ converges absolutely, by definition,

$$\sum_{n=1}^{\infty} |\mathrm{Log}(1 + (z_n - 1))| = \underbrace{\sum_{n=1}^{\infty} |\mathrm{Log}(z_n)| < \infty}.$$

Since $\mathrm{Re}(z_n) > 0$, we have $-1 < \mathrm{Re}(z_n - 1)$. Now, since $\sum_{n=1}^{\infty} \mathrm{Log}(1 + (z_n - 1))$ converges absolutely, by Conclusion 3.188, $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely. ∎)

Conclusion 3.192 For every positive integer n, let $\mathrm{Re}(z_n) > 0$. $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely.

<u>Proof of the remaining part</u>: Let $\sum_{n=1}^{\infty} (z_n - 1)$ converge absolutely, that is, $\sum_{n=1}^{\infty} |z_n - 1| < \infty$. We have to show that $\prod_{n=1}^{\infty} z_n$ converges absolutely, that is $\sum_{n=1}^{\infty} |\mathrm{Log}(z_n)| < \infty$.

Since for every positive integer n, $\mathrm{Re}(z_n) > 0$, for every positive integer n, $-1 < \mathrm{Re}(z_n - 1)$. Now, since $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely, by Conclusion 3.188, $\left(\sum_{n=1}^{\infty} \mathrm{Log}(z_n) =\right) \sum_{n=1}^{\infty} \mathrm{Log}(1 + (z_n - 1))$ converges absolutely, and hence $\sum_{n=1}^{\infty} |\mathrm{Log}(z_n)| < \infty$. ∎

Exercises

3.1 Suppose that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is a positive real number r. Let $f : z \mapsto \sum_{n=0}^{\infty} a_n z^n$ be the function from $D(0; r)$ to $\mathbb{C}$. Show that there exists a singular point of f.

3.2 Let $p_1, p_2, p_3, \ldots, q_1, q_2, q_3, \ldots$ be positive integers. Suppose that

$$2p_1 < q_1,\ 2p_2 < q_2,\ 2p_3 < q_3, \ldots.$$

Let $p_1 < q_1 \le p_2 < q_2 \le p_3 < \cdots$. Suppose that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is 1. Let

$$f : z \mapsto \sum_{n=0}^{\infty} a_n z^n$$

be a function from $D(0;1)$ to $\mathbb{C}$. Suppose that for every positive integer k, $(p_k < n < q_k) \Rightarrow a_n = 0$. Let $s_n(z)$ be the n-th partial sum of the series $\sum_{n=0}^{\infty} a_n z^n$. Let 1 be a regular point of f. Show that there exists an open neighborhood V of 1 such that for every $z \in V$, $\{s_{p_k}(z)\}$ is a convergent sequence.

3.3 Let $p_1, p_2, p_3, \ldots$ be positive integers such that $2p_1 < p_2 < 2p_2 < p_3 < 2p_3 < p_4 < \cdots$. Suppose that the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^{p_n}$ is 1. Let

$$f : z \mapsto \sum_{n=1}^{\infty} a_n z^{p_n}$$

be a function from $D(0;1)$ to $\mathbb{C}$. Show that every point z satisfying $|z| = 1$ is a singular point of f.

3.4 Let G be the set of all linear fractional transformations $\varphi_{(a,b,c,d)} : \mathbb{S}^2 \to \mathbb{S}^2$, where a,b,c,d are integers satisfying $ad - bc = 1$. Let $\varphi_{(a,b,c,d)} \in G$, where a, b,c,d are integers satisfying $ad - bc = 1$. Show that

$$\varphi_{(a,b,c,d)} \circ \varphi_{(d,-b-,c,a)} = \varphi_{(1,0,0,1)}.$$

3.5 Let $\Gamma : [0,1] \to (D(0;2) \cup \{2\})$ be a curve. Suppose that $\Gamma([0,1)) \subset D(0;2)$, and $\Gamma(1) = 2$. Let $g : D(0;2) \to D(0;2)$ be a holomorphic function. Suppose that $\lim_{t \to 2^-} (g \circ \Gamma)(t) = 0$. Show that $\lim_{r \to 2^-} g(r) = 0$.

3.6 Let Ω be a bounded simply connected region. Let $f : \Omega \to D(0;1)$ be a 1-1 function from Ω onto $D(0;1)$. Suppose that f is a conformal mapping. Let β be a simple boundary point of Ω. Show that f has a continuous extension to $\Omega \cup \{\beta\}$ such that $|f(\beta)| = 1$.

3.7 Show that for every positive integer n,

$$z \mapsto \int_0^{\infty} e^{-t} t^{z-1}\, dt$$

is a function from $\{z : \mathrm{Re}(z) > \frac{1}{n}\}$ to $\mathbb{C}$.

3.8 Let (X, ρ) be a metric space. Let A be a nonempty subset of X. Show that for every $x \in X$,

$$\rho(x, A) = \rho\left(x, \overline{A}\right).$$

3.9 Let $f : G \to \mathbb{C}$, and $g : G \to \mathbb{C}$ be branches of the logarithm. Show that there exists an integer k such that for every $w \in G$,

$$g(w) = f(w) + 2k\pi i.$$

3.10 Suppose tha, for every positive integer n, $\mathrm{Re}(z_n) > 0$. Suppose that $\sum_{n=1}^{\infty} \mathrm{Log}(z_n)$ is convergent. Show that $\prod_{n=1}^{\infty} z_n$ converges to a nonzero complex number.

Chapter 4
Special Functions

Our main interest in this chapter is to acquaint the reader with three things: the Riemann hypothesis, the challenging analytic proof of the celebrated prime number theorem, and a proof of Picard's little theorem. Through these theorems, we have endeavored to demonstrate the power of complex analysis. Although this is our last chapter, we shall proceed here in an enough slow pace.

4.1 Laurent Series

Note 4.1

Example 4.2 Let

$$f : z \mapsto \frac{1}{1-z}$$

be a function from $(\mathbb{C} - \{1\})$ to $\mathbb{C}$. Clearly, f is representable by the power series

$$1 + z + z^2 + z^3 + \cdots$$

in the open disk $D(0; 1)$. Let

$$g : z \mapsto \frac{1}{1-z} \left(= -\frac{1}{z} \frac{1}{1 - \frac{1}{z}} \right)$$

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be a function from $(\mathbb{C}-\{0,1\})$ to $\mathbb{C}$. Clearly, g is representable by the power series

$$-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\left(=-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\right)$$

in the open region $\left\{z : \left|\frac{1}{z}\right|<1\right\}(=\{z : 1<|z|\})$. In short, in the disk $D(0;1), \frac{1}{1-z}$ is representable by a series in nonnegative powers of z, and, in the region $\{z : 1<|z|\}, \frac{1}{1-z}$ is representable by a series in negative powers of z.

Example 4.3 Let

$$f : z \mapsto \frac{1}{(z-1)(z-3)}\;\left(=\frac{1}{2}\left(\frac{1}{z-3}-\frac{1}{z-1}\right)\right)$$

be a function from $(\mathbb{C}-\{1,3\})$ to $\mathbb{C}$. Let

$$f_0 : z \mapsto \frac{1}{2}\frac{1}{z-3}\left(=\frac{-1}{6}\frac{1}{1-\frac{z}{3}}\right)$$

be a function from $(\mathbb{C}-\{3\})$ to $\mathbb{C}$. Clearly, f_0 is representable by the power series

$$\frac{-1}{6}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right)\left(=\frac{-1}{6}+\frac{-1}{6}\frac{z}{3}+\frac{-1}{6}\frac{z^2}{3^2}+\frac{-1}{6}\frac{z^3}{3^3}+\cdots\right)$$

in $\left\{z : \left|\frac{z}{3}\right|<1\right\}(=\{z : |z|<3\}=D(0;3))$. Let

$$f_1 : z \mapsto \frac{1}{2}\frac{1}{1-z}$$

be a function from $(\mathbb{C}-\{0,1\})$ to $\mathbb{C}$. By Example 4.2, in the region $\{z : 1<|z|\}, f_1$ is representable by a series

$$-\frac{1}{2}\frac{1}{z}-\frac{1}{2}\frac{1}{z^2}-\frac{1}{2}\frac{1}{z^3}-\cdots$$

in negative powers of z. Thus, in the 'annulus' $\{z : 1<|z|<3\}$,

$$\begin{aligned} f(z) &= \left(\frac{-1}{6}+\frac{-1}{6}\frac{z}{3}+\frac{-1}{6}\frac{z^2}{3^2}+\frac{-1}{6}\frac{z^3}{3^3}+\cdots\right)+\left(-\frac{1}{2}\frac{1}{z}-\frac{1}{2}\frac{1}{z^2}-\frac{1}{2}\frac{1}{z^3}-\cdots\right) \\ &= \sum_{n=0}^{\infty} c_n z^n + \sum_{n=-1}^{-\infty} c_n z^n, \end{aligned}$$

where

$$c_n \equiv \begin{cases} \frac{-1}{6\cdot 3^n} \text{ if } n \text{ is a nonnegative integer} \\ -\frac{1}{2} \text{ if } n \text{ is a negative integer.} \end{cases}$$

Hence, in the annulus $\{z : 1 < |z| < 3\}$,

$$\begin{aligned} f(z) &= c_0 + \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty} c_{-n} z^{-n} = c_0 + \sum_{n=1}^{\infty} (c_n z^n + c_{-n} z^{-n}) \\ &= c_0 + \left(c_1 z^1 + c_{-1} z^{-1}\right) + \left(c_2 z^2 + c_{-2} z^{-2}\right) + \cdots \\ &= \lim_{n\to\infty} \left(c_0 + \left(c_1 z^1 + c_{-1} z^{-1}\right) + \cdots + \left(c_n z^n + c_{-n} z^{-n}\right)\right) \\ &= \lim_{n\to\infty} \left(\sum_{k=-n}^{n} c_k z^k\right). \end{aligned}$$

Thus, in the annulus $\{z : 1 < |z| < 3\}$,

$$f(z) = \lim_{n\to\infty} \left(\sum_{k=-n}^{n} c_k z^k\right).$$

Roughly, we say that in the annulus $\{z : 1 < |z| < 3\}$,

$$\frac{1}{(z-1)(z-3)}$$

is representable by a series in both positive and negative powers of z.

Definition Let G be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : G \to \mathbb{C}$ be a function. If for every $z_0 \in G$, there exists $r > 0$ such that $\{f_n\}$ converges uniformly on $D(z_0; r)$, then we say that $\{f_n\}$ ***converges locally uniformly*** *in G*.

Let G be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : G \to \mathbb{C}$ be a function. Suppose that $\{f_n\}$ converges locally uniformly in G.

Problem 4.4 $\{f_n\}$ converges uniformly on compact subsets of G.

(Solution For this purpose, let us take any nonempty compact subset K of G, and $\varepsilon > 0$. By the given assumption, for every $a \in K$, there exists $r_a > 0$ such that $\{f_n\}$ converges uniformly on $D(a; r_a)$. Since $\{D(a; r_a) : a \in K\}$ is an open cover of the compact set K, there exists finite-many $a_1, \ldots, a_n \in K$ such that

$$K \subset D(a_1; r_{a_1}) \cup \cdots \cup D(a_n; r_{a_n}).$$

Since $\{f_n\}$ converges uniformly on $D(a_1; r_{a_1})$, there exists a positive integer N_1 such that

$$(m, n \geq N_1, \text{and } z \in D(a_1; r_{a_1})) \Rightarrow |f_n(z) - f_m(z)| < \varepsilon.$$

Since $\{f_n\}$ converges uniformly on $D(a_2; r_{a_2})$, there exists a positive integer N_2 such that

$$(m, n \geq N_2, \text{and } z \in D(a_2; r_{a_2})) \Rightarrow |f_n(z) - f_m(z)| < \varepsilon, \text{etc.}$$

Put $N \equiv \max\{N_1, \ldots, N_n\}$. Now, let us take any $z \in K$. Let $m, n \geq N$. It suffices to show that $|f_n(z) - f_m(z)| < \varepsilon$. Since

$$z \in K \subset D(a_1; r_{a_1}) \cup \cdots \cup D(a_n; r_{a_n}),$$

there exists $j \in \{1, \ldots, n\}$ such that $z \in D(a_j; r_{a_j})$. For simplicity, suppose that $j = 1$. Now, since $z \in D(a_1; r_{a_1})$, and $m, n \geq N(\geq N_1)$, we have $|f_n(z) - f_m(z)| < \varepsilon$. ∎)

Conclusion 4.5 Let G be a nonempty open subset of $\mathbb{C}$. For every positive integer n, let $f_n : G \to \mathbb{C}$ be a function. $\{f_n\}$ converges locally uniformly in G if and only if $\{f_n\}$ converges uniformly on compact subsets of G.

Proof of the remaining part Suppose that $\{f_n\}$ converges uniformly on compact subsets of G We have to show that $\{f_n\}$ converges locally uniformly in G For this purpose, let us take any $z_0 \in G$. Since G is open, there exists $r > 0$ such that

$$D\left[z_0; \frac{r}{2}\right] \subset \underbrace{D(z_0; r) \subset G.}$$

Since $D\left[z_0; \frac{r}{2}\right]$ is a compact subset of G, by assumption, $\{f_n\}$ converges uniformly on $D\left[z_0; \frac{r}{2}\right] \left(\supset D\left(z_0; \frac{r}{2}\right)\right)$, and hence $\{f_n\}$ converges uniformly on $D\left(z_0; \frac{r}{2}\right)$. ∎

Conclusion 4.6 Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series, where z_0 is the 'center' of the series. Let $w \neq z_0$, and $\sum_{n=0}^{\infty} a_n (w - z_0)^n$ be convergent. Then,

1. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely on $D(z_0; |w - z_0|)$,
2. if $0 < r < |w - z_0|$ then $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on $D[z_0; r]$,
3. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on compact subsets of $D(z_0; |w - z_0|)$,
4. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges locally uniformly in $D(z_0; |w - z_0|)$.

Proof Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

1. Since $\sum_{n=0}^{\infty} a_n (w - z_0)^n$ is convergent, and R is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, we have $|w - z_0| \leq R$. Since R is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely on $D(z_0; R)$. Now, since $|w - z_0| \leq R$, $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely on $D(z_0; |w - z_0|)$.

2. Let us fix any real number r satisfying $0<r<|w-z_0|$. We have to show that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on $D[z_0;r]$. Since $0<r<|w-z_0|$, we have $(r+z_0) \in D(z_0;|w-z_0|)$, and hence, by 1,

$$\left(\sum_{n=0}^{\infty} |a_n|r^n = \right) \sum_{n=0}^{\infty} |a_n((r+z_0)-z_0)^n|$$

is convergent. Thus, $\sum_{n=0}^{\infty} |a_n|r^n$ is convergent. For every nonnegative integer n, and for every $z \in D[z_0;r]$,

$$|a_n(z-z_0)^n| = |a_n||z-z_0|^n \le |a_n|r^n,$$

so for every nonnegative integer n, and for every $z \in D[z_0;r]$, we have

$$|a_n(z-z_0)^n| \le |a_n|r^n.$$

Now, since $\sum_{n=0}^{\infty} |a_n|r^n$ is convergent, by Weierstrass M-test, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on $D[z_0;r]$.

3. Let us fix any nonempty compact subset K of $D(z_0;|w-z_0|)$. We have to show that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on K. Since $z \mapsto |z-z_0|$ is a continuous real-valued function, and K is a compact set, there exists $z^* \in K(\subset D(z_0;|w-z_0|))$ such that

$$z \in K \Rightarrow |z-z_0| \le |z^*-z_0|(<|w-z_0|).$$

Put $r \equiv \frac{1}{2}(|z^*-z_0|+|w-z_0|)$. Thus, $|z^*-z_0|<r<|w-z_0|$. It follows that $K \subset D[z_0;r] \subset D(z_0;|w-z_0|)$. Now, by 2, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on $D[z_0;r]$ $(\supset K)$, and hence $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on K.

4. For every nonnegative integer n, let

$$f_n : z \mapsto \sum_{k=0}^{n} a_k(z-z_0)^k$$

be a function from open set $D(z_0;|w-z_0|)$ to $\mathbb{C}$. By (3), $\{f_n\}$ converges uniformly on compact subsets of $D(z_0;|w-z_0|)$, and hence, by Conclusion 4.5, $\{f_n\}$ converges locally uniformly in $D(z_0;|w-z_0|)$. That is, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges locally uniformly in $D(z_0;|w-z_0|)$. ∎

Conclusion 4.7 Let $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ be a series. Let $w \neq z_0$, and $\sum_{n=1}^{\infty} a_{-n}(w-z_0)^{-n}$ be convergent. Then,

1. $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges absolutely on $\{z : |w-z_0|<|z-z_0|\}$,
2. if r is a real number satisfying $|w-z_0|<r$, then $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on $\{z : r \le |z-z_0|\}$,

3. $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on compact subsets of $\{z : |w-z_0| < |z-z_0|\}$,
4. $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges locally uniformly in $\{z : |w-z_0| < |z-z_0|\}$.

Proof Let

$$([0,\infty] \ni) R \equiv \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}.$$

(Here, by $\frac{1}{0}$ we shall mean ∞, and, by $\frac{1}{\infty}$ we shall mean 0.)

1. Since $\sum_{n=1}^{\infty} a_{-n}(w-z_0)^{-n}$ is convergent, we have

$$\begin{aligned}\frac{1}{|w-z_0|}\left(\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}\right) &= \limsup_{n\to\infty} \frac{1}{|w-z_0|} \sqrt[n]{|a_{-n}|} \\ &= \limsup_{n\to\infty} \sqrt[n]{|a_{-n}(w-z_0)^{-n}|} \le 1,\end{aligned}$$

and hence $R \le |w-z_0|$. Let us fix any complex number z satisfying $|w-z_0| < |z-z_0|$. We have to show that

$$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

converges absolutely. It suffices to show that

$$\frac{R}{|z-z_0|} = \frac{1}{|z-z_0|}\left(\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}\right) = \underbrace{\limsup_{n\to\infty} \sqrt[n]{|a_{-n}(z-z_0)^{-n}|} < 1},$$

that is $\frac{R}{|z-z_0|} < 1$. Since $R \le |w-z_0|$, and $|w-z_0| < |z-z_0|$, we have $\frac{R}{|z-z_0|} < 1$.

2. Let us fix any real r satisfying $|w-z_0| < r$. We have to show that $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on $\{z : r \le |z-z_0|\}$. Since $0 < |w-z_0| < r$, we have

$$(r+z_0) \in \{z : |w-z_0| < |z-z_0|\},$$

and hence, by 1,

$$\left(\sum_{n=1}^{\infty} |a_{-n}| r^{-n} =\right) \sum_{n=1}^{\infty} |a_{-n}((r+z_0)-z_0)^{-n}|$$

is convergent. Thus, $\sum_{n=1}^{\infty} |a_{-n}| r^{-n}$ is convergent. For every nonnegative integer n, and for every $z \in \{z : r \le |z-z_0|\}$,

$$|a_{-n}(z - z_0)^{-n}| = |a_{-n}||z - z_0|^{-n} \le |a_{-n}|r^{-n},$$

so for every nonnegative integer n, and for every $z \in \{z : r \le |z - z_0|\}$,

$$|a_{-n}(z - z_0)^{-n}| \le |a_{-n}|r^{-n}.$$

Now, since $\sum_{n=1}^{\infty} |a_{-n}|r^{-n}$ is convergent, by Weierstrass M-test, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on $\{z : r \le |z - z_0|\}$.

3. Let us fix any nonempty compact subset K of $\{z : |w - z_0| < |z - z_0|\}$. We have to show that $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on K. Since $z \mapsto |z - z_0|$ is a continuous real-valued function, and K is a compact set, there exists

$$z^* \in K(\subset \{z : |w - z_0| < |z - z_0|\})$$

such that

$$z \in K \Rightarrow (|w - z_0| <)\ |z^* - z_0| \le |z - z_0|.$$

Put $r \equiv \frac{1}{2}(|z^* - z_0| + |w - z_0|)$. Thus, $|w - z_0| < r < |z^* - z_0|$. It follows that

$$K \subset \{z : |z^* - z_0| \le |z - z_0|\} \subset \{z : r \le |z - z_0|\} \subset \{z : |w - z_0| \le |z - z_0|\}.$$

Since $|w - z_0| < r$, by 2, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on $\{z : r \le |z - z_0|\}(\supset K)$, and hence $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on K.

4. For every nonnegative integer n, let

$$f_n : z \mapsto \sum_{k=1}^{n} a_{-k}(z - z_0)^{-k}$$

be a function from open set $\{z : |w - z_0| < |z - z_0|\}$ to $\mathbb{C}$. By 3, $\{f_n\}$ converges uniformly on compact subsets of $\{z : |w - z_0| < |z - z_0|\}$, and hence, by Conclusion 4.5, $\{f_n\}$ converges locally uniformly in

$$\{z : |w - z_0| < |z - z_0|\},$$

that is $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges locally uniformly in $\{z : |w - z_0| < |z - z_0|\}$. ∎

Note 4.8

Definition Let $z_0 \in \mathbb{C}$. For every integer n, let $a_n \in \mathbb{C}$. Let $w \in \mathbb{C}$. By the ***Laurent series*** $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ ***is convergent*** *at* w, we mean that

1. $\sum_{n=0}^{\infty} a_n(w - z_0)^n$ is convergent,
2. $\sum_{n=1}^{\infty} a_{-n}(w - z_0)^{-n}$ is convergent.

Definition Suppose that the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is convergent at w. (It is clear that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n(w-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(w-z_0)^{-n} \\
&= \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k(w-z_0)^k\right) + \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_{-k}(w-z_0)^{-k}\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=-n}^{n} a_k(w-z_0)^k\right)
\end{aligned}$$

exists.)

By the ***sum of*** $\sum_{n=-\infty}^{\infty} a_n(w-z_0)^n$, we mean $\lim_{n\to\infty}\left(\sum_{k=-n}^{n} a_k(w-z_0)^k\right)$.

Let $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be a Laurent series with 'center' z_0. We know that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is

$$\begin{cases} \text{absolutely convergent if} |z-z_0| < \dfrac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} \\ \text{divergent if} \dfrac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} < |z-z_0|. \end{cases}$$

Similarly, $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ is

$$\begin{cases} \text{absolutely convergent if} \left|\frac{1}{z-z_0}\right| < \dfrac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_{-n}|}} \\ \text{divergent if} \dfrac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_{-n}|}} < \left|\frac{1}{z-z_0}\right|, \end{cases}$$

that is $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ is

$$\begin{cases} \text{absolutely convergent if } \limsup\limits_{n\to\infty} \sqrt[n]{|a_{-n}|} < |z-z_0| \\ \text{divergent if } |z-z_0| < \limsup\limits_{n\to\infty} \sqrt[n]{|a_{-n}|}, \end{cases}$$

It follows that $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is absolutely convergent if

1. $$\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|} < \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

2. $$\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|} < |z-z_0| < \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Conclusion 4.9 Let $R_1 \equiv \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$, and $R_2 \equiv \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$. Let $R_1 < R_2$. Then $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ converges absolutely on the 'annulus' $\{z : R_1 < |z - z_0| < R_2\}$.

Conclusion 4.10 Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series, where z_0 is the 'center' of the series. Let $R_2 \equiv \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$. Then,

1. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely on $D(z_0; R_2)$,
2. if $0 < s < R_2$ then $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on $D[z_0; s]$,
3. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on compact subsets of $D(z_0; R_2)$,
4. $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges locally uniformly in $D(z_0; R_2)$.

Proof

1. Here R_2 is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, so $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely on $D(z_0; R_2)$.
2. Let s be a real number satisfying $0 < s < R_2$. We have to show that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on $D[z_0; s]$. Put $w \equiv \frac{1}{2}(s + R_2) + z_0$. Since R_2 is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, and $(z_0 \neq) w \in D(z_0; R_2)$, by 1, $\sum_{n=0}^{\infty} a_n (w - z_0)^n$ is convergent. Now, since

$$|w - z_0| = \frac{1}{2}(s + R_2) > s > 0,$$

by Conclusion 4.6, $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on $D[z_0; s]$.
3. Let us fix any nonempty compact subset K of $D(z_0; R_2)$. We have to show that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on K. Since $z \mapsto |z - z_0|$ is a continuous real-valued function, and K is a compact set, there exists $z^* \in K (\subset D(z_0; R_2))$ such that

$$z \in K \Rightarrow |z - z_0| \leq |z^* - z_0| (< R_2).$$

Put $r \equiv \frac{1}{2}(|z^* - z_0| + R_2)$. Thus, $|z^* - z_0| < r < R_2$. It follows that $K \subset D[z_0; r] \subset D(z_0; R_2)$. Now, by 2, $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on $D[z_0; r]$ $(\supset K)$, and hence $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on K.
4. For every nonnegative integer n, let

$$f_n : z \mapsto \sum_{k=0}^{n} a_k (z - z_0)^k$$

be a function from open set $D(z_0; R_2)$ to $\mathbb{C}$. By 3, $\{f_n\}$ converges uniformly on compact subsets of $D(z_0; R_2)$, and hence, by Conclusion 4.5, $\{f_n\}$ converges locally uniformly in $D(z_0; R_2)$, that is $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges locally uniformly in $D(z_0; R_2)$. ∎

Conclusion 4.11 Let $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ be a series. Let $R_1 \equiv \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$. Then,

1. $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges absolutely on $\{z : R_1 < |z-z_0|\}$.
2. if $R_1 < r$, then $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on $\{z : r \le |z-z_0|\}$,
3. $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on compact subsets of $\{z : R_1 < |z-z_0|\}$,
4. $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges locally uniformly in $\{z : R_1 < |z-z_0|\}$.

Proof

1. $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges absolutely, whenever

$$\begin{aligned}\frac{1}{|z-z_0|}R_1 &= \frac{1}{|z-z_0|}\left(\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}\right) = \limsup_{n\to\infty} \frac{1}{|z-z_0|}\sqrt[n]{|a_{-n}|} \\ &= \underbrace{\limsup_{n\to\infty} \sqrt[n]{|a_{-n}(z-z_0)^{-n}|} < 1,}\end{aligned}$$

and hence $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges absolutely on $\{z : R_1 < |z-z_0|\}$.

2. Let us fix any real r satisfying $R_1 < r$. We have to show that $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on $\{z : r \le |z-z_0|\}$. Since $0 \le R_1 < r$, we have $(r+z_0) \in \{z : R_1 < |z-z_0|\}$, and hence, by 1,

$$\left(\sum_{n=1}^{\infty} |a_{-n}|r^{-n} = \right)\sum_{n=1}^{\infty} |a_{-n}((r+z_0)-z_0)^{-n}|$$

is convergent. Thus, $\sum_{n=1}^{\infty} |a_{-n}|r^{-n}$ is convergent. For every nonnegative integer n, and for every $z \in \{w : r \le |w-z_0|\}$,

$$|a_{-n}(z-z_0)^{-n}| = |a_{-n}||z-z_0|^{-n} \le |a_{-n}|r^{-n},$$

so for every nonnegative integer n, and for every $z \in \{z : r \le |z-z_0|\}$,

$$|a_{-n}(z-z_0)^{-n}| \le |a_{-n}|r^{-n}.$$

Now, since $\sum_{n=1}^{\infty} |a_{-n}|r^{-n}$ is convergent, by Weierstrass M-test, $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on $\{w : r \le |w-z_0|\}$.

3. Let us fix any nonempty compact subset K of $\{z : R_1 < |z-z_0|\}$. We have to show that $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on K. Since $z \mapsto |z-z_0|$ is a continuous real-valued function, and K is a compact set, there exists $z^* \in K(\subset \{z : R_1 < |z-z_0|\})$ such that

$$z \in K \Rightarrow (R_1 <)|z^* - z_0| \leq |z - z_0|.$$

Put $r \equiv \frac{1}{2}(|z^* - z_0| + R_1)$. Thus, $R_1 < r < |z^* - z_0|$. It follows that

$$K \subset \{z : |z^* - z_0| \leq |z - z_0|\} \subset \{z : r \leq |z - z_0|\}.$$

Since $R_1 < r$, by 2, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on $\{z : r \leq |z - z_0|\} (\supset K)$, and hence $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on K.

4. For every nonnegative integer n, let

$$f_n : z \mapsto \sum_{k=1}^{n} a_{-k}(z - z_0)^{-k}$$

be a function from open set $\{z : R_1 < |z - z_0|\}$ to $\mathbb{C}$. By 3, $\{f_n\}$ converges uniformly on compact subsets of $\{z : R_1 < |z - z_0|\}$, and hence, by Conclusion 4.5, $\{f_n\}$ converges locally uniformly in $\{z : R_1 < |z - z_0|\}$, that is $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges locally uniformly in $\{z : R_1 < |z - z_0|\}$. ∎

Conclusion 4.12 Let $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ be a Laurent series. Let $R_1 \equiv \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$, and $R_2 \equiv \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$. Let $R_1 < R_2$. Then

1. $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges absolutely on $\{z : R_1 < |z - z_0| < R_2\}$,
2. if r, s are real numbers satisfying $R_1 < r < s < R_2$, then $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges uniformly on $\{z : r \leq |z - z_0| \leq s\}$,
3. $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on compact subsets of $\{z : R_1 < |z - z_0| < R_2\}$,
4. $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges locally uniformly in $\{z : R_1 < |z - z_0| < R_2\}$.

Proof

1. By Conclusion 4.9, $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges absolutely on $\{z : R_1 < |z - z_0| < R_2\}$.
2. Let us take any real numbers r,s satisfying $R_1 < r < s < R_2$. We have to show that $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges uniformly on $\{z : r \leq |z - z_0| \leq s\}$.
   Since $R_1 < r$, by Conclusion 4.11, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on $\{z : r \leq |z - z_0|\}$. Since $0 < s < R_2$, by Conclusion 4.10, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on $D[z_0; s]$. Since $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on $D[z_0; s]$, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on $\{z : r \leq |z - z_0|\}$ and $r < s$, $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges uniformly on $\{z : r \leq |z - z_0| \leq s\}$.

3. Let us fix any nonempty compact subset K of $\{z : R_1 < |z - z_0| < R_2\}$. We have to show that $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges uniformly on K. Since $K \subset \{z : R_1 < |z - z_0| < R_2\}$, we have $K \subset \{z : R_1 < |z - z_0|\}$, and $K \subset \{z : |z - z_0| < R_2\}$. Since $K \subset \{z : R_1 < |z - z_0|\}$, and K is compact, by Conclusion 4.11, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on K. Since $K \subset \{z : |z - z_0| < R_2\}$, and K is compact, by Conclusion 4.10, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on K. Since $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformly on K, and $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges uniformly on K, $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges uniformly on K.
4. For every nonnegative integer n, let $f_n : z \mapsto \sum_{k=-n}^{n} a_k(z - z_0)^k$ be a function from open set $\{z : R_1 < |z - z_0| < R_2\}$ to $\mathbb{C}$. By 3, $\{f_n\}$ converges uniformly on compact subsets of $\{z : R_1 < |z - z_0| < R_2\}$, and hence, by Conclusion 4.5, $\{f_n\}$ converges locally uniformly in $\{z : R_1 < |z - z_0| < R_2\}$, that is $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges locally uniformly in $\{z : R_1 < |z - z_0| < R_2\}$. ∎

Let $R_1 < R_2$, where $R_1 \equiv \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$, and $R_2 \equiv \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$. For every nonnegative integer n, let

$$f_n : z \mapsto \sum_{k=-n}^{n} a_k(z - z_0)^k$$

be a function from open set $\{z : R_1 < |z - z_0|\}$ to $\mathbb{C}$. By Conclusion 4.12, $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges absolutely on $\{z : R_1 < |z - z_0| < R_2\}$. Thus,

$$f : z \mapsto \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

is a function from $\{z : R_1 < |z - z_0| < R_2\}$ to $\mathbb{C}$. Here, each $f_n \in H(\{z : R_1 < |z - z_0| < R_2\})$, and $\{f_n\}$ converges to f uniformly on compact subsets of $\{z : R_1 < |z - z_0| < R_2\}$. So, by Conclusion 1.174,

$$f \in H(\{z : R_1 < |z - z_0| < R_2\}).$$

Let us take any positive real number r satisfying $R_1 < r < R_2$. Let C_r be the circle $\{z : |z - z_0| = r\}$ oriented anticlockwise. Since $\{f_n\}$ converges to f uniformly on compact subset C_r of $\{z : R_1 < |z - z_0| < R_2\}$, we have

$$\begin{aligned}
\int_{C_r} f(z)\mathrm{d}z &= \lim_{n\to\infty}\left(\int_{C_r} f_n(z)\mathrm{d}z\right) = \lim_{n\to\infty}\left(\int_{C_r}\left(\sum_{k=-n}^{n} a_k(z-z_0)^k\right)\mathrm{d}z\right)\\
&= \lim_{n\to\infty}\left(\sum_{k=-n}^{n} a_k\left(\int_{C_r}(z-z_0)^k\mathrm{d}z\right)\right) = \sum_{k=-\infty}^{\infty} a_k\left(\int_{C_r}(z-z_0)^k\mathrm{d}z\right)\\
&= \left(a_0\int_{C_r} 1\mathrm{d}z + a_1\int_{C_r}(z-z_0)\mathrm{d}z + a_2\int_{C_r}(z-z_0)^2\mathrm{d}z + \cdots\right)\\
&\quad + \left(a_{-1}\int_{C_r}\frac{1}{z-z_0}\mathrm{d}z + a_{-2}\int_{C_r}\frac{1}{(z-z_0)^2}\mathrm{d}z + \cdots\right)\\
&= (a_0\cdot 0 + a_1\cdot 0 + a_2\cdot 0 + \cdots) + (a_{-1}\cdot 2\pi i + a_{-2}\cdot 0 + \cdots) = a_{-1}\cdot 2\pi i,
\end{aligned}$$

and hence $a_{-1} = \frac{1}{2\pi i}\int_{C_r} f(z)\mathrm{d}z$. Since $\{f_n\}$ converges to f uniformly on compact subset C_r of $\{z : R_1 < |z-z_0| < R_2\}$, we have

$$\begin{aligned}
\int_{C_r}\frac{1}{z-z_0}f(z)\mathrm{d}z &= \lim_{n\to\infty}\left(\int_{C_r}\frac{1}{z-z_0}f_n(z)\mathrm{d}z\right)\\
&= \lim_{n\to\infty}\left(\int_{C_r}\left(\sum_{k=-n}^{n} a_k(z-z_0)^{k-1}\right)\mathrm{d}z\right)\\
&= \lim_{n\to\infty}\left(\sum_{k=-n}^{n} a_k\left(\int_{C_r}(z-z_0)^{k-1}\mathrm{d}z\right)\right)\\
&= \sum_{k=-\infty}^{\infty} a_k\left(\int_{C_r}(z-z_0)^{k-1}\mathrm{d}z\right)\\
&= \left(a_0\int_{C_r}\frac{1}{z-z_0}\mathrm{d}z + a_1\int_{C_r} 1\mathrm{d}z + a_2\int_{C_r}(z-z_0)\mathrm{d}z + \cdots\right)\\
&\quad + \left(a_{-1}\int_{C_r}\frac{1}{(z-z_0)^2}\mathrm{d}z + \cdots\right)\\
&= (a_0\cdot 2\pi i + a_1\cdot 0 + a_2\cdot 0 + \cdots)\\
&\quad + (a_{-1}\cdot 0 + a_{-2}\cdot 0 + \cdots) = a_0\cdot 2\pi i,
\end{aligned}$$

and hence $a_0 = \frac{1}{2\pi i}\int_{C_r} \frac{1}{z-z_0} f(z)\mathrm{d}z$. Similarly, $a_1 = \frac{1}{2\pi i}\int_{C_r} \frac{1}{(z-z_0)^2} f(z)\mathrm{d}z$, etc. Also,

$$a_{-2} = \tfrac{1}{2\pi i}\int_{C_r} (z-z_0) f(z)\mathrm{d}z, a_{-3} = \tfrac{1}{2\pi i}\int_{C_r} (z-z_0)^2 f(z)\mathrm{d}z, \text{ etc.}$$

Conclusion 4.13 Let $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be a Laurent series. Let $R_1 < R_2$, where $R_1 \equiv \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}$, and $R_2 \equiv \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$. Let

$$f : z \mapsto \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

be a function from annulus $\{z : R_1 < |z-z_0| < R_2\}$ to $\mathbb{C}$. Then

1. $f \in H(\{z : R_1 < |z-z_0| < R_2\})$,
2. if $R_1 < r < R_2$, and C_r is the circle $\{z : |z-z_0| = r\}$ oriented anticlockwise, then

$$\begin{aligned} a_{-1} &= \frac{1}{2\pi i}\int\limits_{C_r} f(z)\mathrm{d}z, a_0 = \frac{1}{2\pi i}\int\limits_{C_r} \frac{1}{z-z_0} f(z)\mathrm{d}z, a_1 \\ &= \frac{1}{2\pi i}\int\limits_{C_r} \frac{1}{(z-z_0)^2} f(z)\mathrm{d}z, \ldots. \end{aligned}$$

and

$$a_{-2} = \frac{1}{2\pi i}\int\limits_{C_r} (z-z_0) f(z)\mathrm{d}z, a_{-3} = \frac{1}{2\pi i}\int\limits_{C_r} (z-z_0)^2 f(z)\mathrm{d}z, \ldots.$$

4.2 Laurent's Theorem

Note 4.14

Example 4.15 Let us try to find the annulus of convergence of the Laurent series $\sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{n^2} z^n$. Here,

$$\begin{aligned} R_1 &= \limsup_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{2}\right)^{(-n)^2}\right|} = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{2^{n^2}}\right|} = \limsup_{n\to\infty} \frac{1}{\left(2^{n^2}\right)^{\frac{1}{n}}} \\ &= \limsup_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \frac{1}{2^n} = 0, \end{aligned}$$

and

$$R_2 = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{2}\right)^{n^2}\right|}} = \infty,$$

so, the punctured plane centered at 0 is the required annulus of convergence.

Definition Let $a<b$. Let $\gamma : [a,b] \to \mathbb{C}$ be any path. Let $\varphi : \text{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Clearly, $(\mathbb{C} - \text{ran}(\gamma))$ is an open subset of $\mathbb{C}$. Let

$$f : z \mapsto \int_\gamma \frac{\varphi(\zeta)}{\zeta - z} d\zeta$$

be a function from $(\mathbb{C} - \text{ran}(\gamma))$ to $\mathbb{C}$. We shall say that *f is the* ***Cauchy integral*** *of* φ *over* γ.

Let $a<b$. Let $\gamma : [a,b] \to \mathbb{C}$ be any path. Let $\varphi : \text{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Let $f : (\mathbb{C} - \text{ran}(\gamma)) \to \mathbb{C}$ be the Cauchy integral of φ over γ.

Let us fix any $z_0 \in \mathbb{C}$.

Since $\gamma : [a,b] \to \mathbb{C}$ is a path, $\gamma : [a,b] \to \mathbb{C}$ is continuous. Now, since $[a,b]$ is a compact set, $\text{ran}(\gamma)$ is a compact set. Next, since $z \mapsto |z - z_0|$ is a continuous real-valued function, and $\text{ran}(\gamma)$ is compact, there exists $z^* \in \text{ran}(\gamma)$ such that

$$\zeta \in \text{ran}(\gamma) \Rightarrow |\zeta - z_0| \le |z^* - z_0|.$$

Put $R \equiv |z^* - z_0|$. Thus, for every $\zeta \in \text{ran}(\gamma)$, we have $|\zeta - z_0| \le R$. Let us fix any complex number z_1 satisfying $R < |z_1 - z_0|$. It follows that $z_1 \ne z_0$. Now, for every $\zeta \in \text{ran}(\gamma)$, we have $|\zeta - z_0| < |z_1 - z_0|$, and hence for every $\zeta \in \text{ran}(\gamma)$,

$$\left|\frac{\zeta - z_0}{z_1 - z_0}\right| < 1.$$

It follows that for every $\zeta \in \text{ran}(\gamma)$,

$$\begin{aligned}
\frac{\varphi(\zeta)}{\zeta - z_1} &= \varphi(\zeta)\frac{1}{(\zeta - z_0) - (z_1 - z_0)} = \frac{-\varphi(\zeta)}{(z_1 - z_0)}\frac{1}{1 - \frac{\zeta - z_0}{z_1 - z_0}} \\
&= \frac{-\varphi(\zeta)}{(z_1 - z_0)}\left(1 + \frac{\zeta - z_0}{z_1 - z_0} + \left(\frac{\zeta - z_0}{z_1 - z_0}\right)^2 + \cdots\right) \\
&= \frac{-\varphi(\zeta)}{(z_1 - z_0)}\left(1 + \frac{1}{z_1 - z_0}(\zeta - z_0) + \frac{1}{(z_1 - z_0)^2}(\zeta - z_0)^2 + \cdots\right)
\end{aligned}$$

By Conclusion 4.10, the series

$$1+\frac{1}{z_1-z_0}(\zeta-z_0)+\frac{1}{(z_1-z_0)^2}(\zeta-z_0)^2+\cdots$$

converges uniformly on compact subsets of

$$\underbrace{D\left(z_0;\frac{1}{\limsup_{n\to\infty}\sqrt[n]{\left|\frac{-1}{(z_1-z_0)^n}\right|}}\right)}=D\left(z_0;\frac{1}{\limsup_{n\to\infty}\left(\frac{1}{|z_1-z_0|^n}\right)^{\frac{1}{n}}}\right)$$

$$=D\left(z_0;\frac{1}{\frac{1}{|z_1-z_0|}}\right)=D(z_0;|z_1-z_0|),$$

and hence

$$\left(\frac{\varphi(\zeta)}{\zeta-z_1}=\right)\frac{-\varphi(\zeta)}{(z_1-z_0)}\left(1+\frac{1}{z_1-z_0}(\zeta-z_0)+\frac{1}{(z_1-z_0)^2}(\zeta-z_0)^2+\cdots\right)$$

converges uniformly on compact subsets of $D(z_0;|z_1-z_0|)$. It follows that

$$(f(z_1)=)\int_\gamma\frac{\varphi(\zeta)}{\zeta-z_1}\mathrm{d}\zeta=\frac{-1}{z_1-z_0}\int_\gamma\varphi(\zeta)\mathrm{d}\zeta+\frac{-1}{(z_1-z_0)^2}$$
$$\int_\gamma(\zeta-z_0)\varphi(\zeta)\mathrm{d}\zeta+\frac{-1}{(z_1-z_0)^3}\int_\gamma(\zeta-z_0)^2\varphi(\zeta)\mathrm{d}\zeta+\cdots$$

Thus, for every $z\in\{w:R<|w-z_0|\}$, we get the expansion

$$f(z)=a_{-1}\frac{1}{z-z_0}+a_{-2}\frac{1}{(z-z_0)^2}+a_{-3}\frac{1}{(z-z_0)^3}+\cdots,$$

where, for every positive integer n,

$$a_{-n}\equiv-\int_\gamma(\zeta-z_0)^{n-1}\varphi(\zeta)\mathrm{d}\zeta.$$

Conclusion 4.16 Let $a<b$. Let $\gamma:[a,b]\to\mathbb{C}$ be any path. Let $\varphi:\operatorname{ran}(\gamma)\to\mathbb{C}$ be a continuous function. Let $f:(\mathbb{C}-\operatorname{ran}(\gamma))\to\mathbb{C}$ be the Cauchy integral of φ over γ. Let $z_0\in\mathbb{C}$. Let $R\equiv\max\{|\zeta-z_0|:\zeta\in\operatorname{ran}(\gamma)\}$. Then, for every $z\in\{w:R<|w-z_0|\}$,

$$f(z) = a_{-1}\frac{1}{z-z_0} + a_{-2}\frac{1}{(z-z_0)^2} + a_{-3}\frac{1}{(z-z_0)^3} + \cdots,$$

where, for every positive integer n,

$$a_{-n} \equiv -\int_{\gamma} (\zeta - z_0)^{n-1}\varphi(\zeta)\mathrm{d}\zeta.$$

Note 4.17 Let R_1, R_2 be nonnegative real numbers satisfying $R_1 < R_2$. Let $f \in H(\{z : R_1 < |z - z_0| < R_2\})$. Let $R_1 < r < R_2$, and let C_r be the circle $\{z : |z - z_0| = r\}$ oriented anticlockwise.

Problem 4.18 $\int_{C_r} f(z)\mathrm{d}z$ is independent of r.

(**Solution** Let us introduce the parametrization $\theta \mapsto \left(z_0 + re^{i\theta}\right)$ from $[0, 2\pi]$ to C_r. It suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)\left(0 + r\left(e^{i\theta}i\right)\right)\mathrm{d}\theta\right) = 0,$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_0^{2\pi} re^{i\theta}f\left(z_0 + re^{i\theta}\right)\mathrm{d}\theta\right) = 0.$$

Observe that

$$\begin{aligned}\frac{\partial}{\partial r}\left(re^{i\theta}f\left(z_0 + re^{i\theta}\right)\right) &= e^{i\theta}\frac{\partial}{\partial r}\left(rf\left(z_0 + re^{i\theta}\right)\right)\\ &= e^{i\theta}\left(1 \cdot f\left(z_0 + re^{i\theta}\right) + r\left(f'\left(z_0 + re^{i\theta}\right)\left(0 + 1 \cdot e^{i\theta}\right)\right)\right)\\ &= e^{i\theta}\left(f\left(z_0 + re^{i\theta}\right) + r\left(f'\left(z_0 + re^{i\theta}\right)\left(e^{i\theta}\right)\right)\right),\end{aligned}$$

and

$$(r, \theta) \mapsto e^{i\theta}\left(f\left(z_0 + re^{i\theta}\right) + r\left(f'\left(z_0 + re^{i\theta}\right)\left(e^{i\theta}\right)\right)\right)$$

is continuous, so

$$(r, \theta) \mapsto re^{i\theta}f\left(z_0 + re^{i\theta}\right)$$

has continuous partial derivative with respect to r. Hence, differentiation under the integral sign is valid here. Thus,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_0^{2\pi} re^{i\theta}f\left(z_0+re^{i\theta}\right)\mathrm{d}\theta\right) &= \int_0^{2\pi}\frac{\partial}{\partial r}\left(re^{i\theta}f\left(z_0+re^{i\theta}\right)\right)\mathrm{d}\theta \\
&= \int_0^{2\pi} e^{i\theta}\left(f\left(z_0+re^{i\theta}\right)+r\left(f'\left(z_0+re^{i\theta}\right)\left(e^{i\theta}\right)\right)\right)\mathrm{d}\theta \\
&= \int_0^{2\pi} f\left(z_0+re^{i\theta}\right)\cdot e^{i\theta}\mathrm{d}\theta+\int_0^{2\pi} e^{i\theta}r\left(f'\left(z_0+re^{i\theta}\right)\left(e^{i\theta}\right)\right)\mathrm{d}\theta \\
&= \left(\left(f\left(z_0+re^{i\theta}\right)\frac{e^{i\theta}}{i}\right)\Bigg|_{\theta=0}^{\theta=2\pi}-\int_0^{2\pi} f'\left(z_0+re^{i\theta}\right)\left(0+r\left(e^{i\theta}i\right)\right)\cdot\frac{e^{i\theta}}{i}\mathrm{d}\theta\right) \\
&\quad +\int_0^{2\pi} e^{i\theta}r\left(f'\left(z_0+re^{i\theta}\right)\left(e^{i\theta}\right)\right)\mathrm{d}\theta \\
&= \left(f\left(z_0+re^{i\theta}\right)\frac{e^{i\theta}}{i}\right)\Bigg|_{\theta=0}^{\theta=2\pi} = \left(f\left(z_0+re^{i2\pi}\right)\frac{e^{i2\pi}}{i}\right) \\
&\quad -\left(f\left(z_0+re^{i0}\right)\frac{e^{i0}}{i}\right) \\
&= \left(f(z_0+r1)\frac{1}{i}\right)-\left(f(z_0+r1)\frac{1}{i}\right)=0,
\end{aligned}
$$

and hence

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_0^{2\pi} re^{i\theta}f\left(z_0+re^{i\theta}\right)\mathrm{d}\theta\right)=0.
$$

∎)

Conclusion 4.19 Let R_1, R_2 be nonnegative real numbers satisfying $R_1 < R_2$. Let $f \in H(\{z : R_1 < |z-z_0| < R_2\})$. Let $R_1 < r < R_2$, and let C_r be the circle $\{z : |z-z_0| = r\}$ oriented anticlockwise. Then $\int_{C_r} f(z)\mathrm{d}z$ is independent of r.

Note 4.20 Let R_1, R_2 be nonnegative real numbers satisfying $R_1 < R_2$. Let $f \in H(\{z : R_1 < |z-z_0| < R_2\})$. For every $r \in (R_1, R_2)$, suppose that C_r denotes the circle $\{z : |z-z_0| = r\}$ oriented anticlockwise. Let $R_1 < r_1 < r_2 < R_2$.

Problem 4.21 For every $w \in \{z : r_1 < |z-z_0| < r_2\}$,

$$
f(w)=\frac{1}{2\pi i}\int_{C_{r_2}}\frac{f(\zeta)}{\zeta-w}\mathrm{d}\zeta-\frac{1}{2\pi i}\int_{C_{r_1}}\frac{f(\zeta)}{\zeta-w}\mathrm{d}\zeta.
$$

(Solution Let us fix any w satisfying $(R_1 <)\ r_1 < |w - z_0| < r_2\ (< R_2)$. We have to show that

$$f(w) = \frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(\zeta)}{\zeta - w} d\zeta - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{f(\zeta)}{\zeta - w} d\zeta.$$

Since $f \in H(\{z : R_1 < |z - z_0| < R_2\})$, and $w \in \{z : R_1 < |z - z_0| < R_2\}$, we have $f'(w) \in \mathbb{C}$. Let us define a function $g : \{z : R_1 < |z - z_0| < R_2\} \to \mathbb{C}$ as follows: For every $\zeta \in \{z : R_1 < |z - z_0| < R_2\}$,

$$g(\zeta) \equiv \begin{cases} \frac{1}{\zeta - w}(f(\zeta) - f(w)) & \text{if } \zeta \neq w \\ f'(w) & \text{if } \zeta = w. \end{cases}$$

Since f is continuous at all points of $\{z : R_1 < |z - z_0| < R_2\}$,

$$\zeta \mapsto \frac{1}{\zeta - w}(f(\zeta) - f(w))$$

is continuous on $(\{z : R_1 < |z - z_0| < R_2\} - \{w\})$, and hence $g : \{z : R_1 < |z - z_0| < R_2\} \to \mathbb{C}$ is continuous at all points of $(\{z : R_1 < |z - z_0| < R_2\} - \{w\})$. Since

$$g(w) = \underbrace{f'(w) = \lim_{\zeta \to w} \frac{1}{\zeta - w}(f(\zeta) - f(w))} = \lim_{\zeta \to w} g(\zeta),$$

we have $\lim_{\zeta \to w} g(\zeta) = g(w)$, and hence g is continuous at w. Thus, $g : \{z : R_1 < |z - z_0| < R_2\} \to \mathbb{C}$ is continuous. Clearly, for every $\zeta \in (\{z : R_1 < |z - z_0| < R_2\} - \{w\})$,

$$g'(\zeta) = \frac{(f'(\zeta) - 0)(\zeta - w) - (f(\zeta) - f(w))(1 - 0)}{(\zeta - w)^2},$$

so $g \in H(\{z : R_1 < |z - z_0| < R_2\} - \{w\})$. It follows, by Theorem 1.110, that

$$\int_{C_{r_2}} g(z) dz - \int_{C_{r_1}} g(z) dz = 0,$$

that is

$$\begin{aligned}\int\limits_{C_{r_2}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}z - f(w)\cdot 2\pi i &= \int\limits_{C_{r_2}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}z - f(w)\int\limits_{C_{r_2}} \frac{1}{\zeta - w}\mathrm{d}z \\ &= \int\limits_{C_{r_2}} \frac{1}{\zeta - w}(f(\zeta) - f(w))\mathrm{d}z = \int\limits_{C_{r_2}} g(z)\mathrm{d}z \\ &= \int\limits_{C_{r_1}} g(z)\mathrm{d}z = \int\limits_{C_{r_1}} \frac{1}{\zeta - w}(f(\zeta) - f(w))\mathrm{d}z \\ &= \int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}z - f(w)\int\limits_{C_{r_1}} \frac{1}{\zeta - w}\mathrm{d}z \\ &= \int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}z - f(w)\cdot 0 = \int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}z,\end{aligned}$$

and hence

$$\int\limits_{C_{r_2}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}z - \int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}z = f(w)\cdot 2\pi i.$$

Thus,

$$f(w) = \frac{1}{2\pi i}\int\limits_{C_{r_2}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta - \frac{1}{2\pi i}\int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta.$$

∎)

Conclusion 4.22 Let R_1, R_2 be nonnegative real numbers satisfying $R_1 < R_2$. Let $f \in H(\{z : R_1 < |z - z_0| < R_2\})$. For every $r \in (R_1, R_2)$, suppose that C_r denotes the circle $\{z : |z - z_0| = r\}$ oriented anticlockwise. Let $R_1 < r_1 < r_2 < R_2$. Then, for every $w \in \{z : r_1 < |z - z_0| < r_2\}$,

$$f(w) = \frac{1}{2\pi i}\int\limits_{C_{r_2}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta - \frac{1}{2\pi i}\int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta.$$

Note 4.23 Let $a < b$. Let $\gamma : [a, b] \to \mathbb{C}$ be any path. Let $\varphi : \mathrm{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Let $f : (\mathbb{C} - \mathrm{ran}(\gamma)) \to \mathbb{C}$ be the Cauchy integral of φ over γ.

Let us fix any $z_0 \in (\mathbb{C} - \mathrm{ran}(\gamma))$.

Since $\gamma : [a,b] \to \mathbb{C}$ is a path, $\gamma : [a,b] \to \mathbb{C}$ is continuous. Now, since $[a,b]$ is a compact set, $\text{ran}(\gamma)$ is a compact set. Next, since $z \mapsto |z - z_0|$ is a continuous real-valued function, and $\text{ran}(\gamma)$ is compact, there exists $z^* \in \text{ran}(\gamma)$ such that

$$\zeta \in \text{ran}(\gamma) \Rightarrow (0<)\ |z^* - z_0| \leq |\zeta - z_0|.$$

Put $R \equiv |z^* - z_0|$. Thus, for every $\zeta \in \text{ran}(\gamma),\ 0 < R \leq |\zeta - z_0|$.

Let us fix any complex number z_1 satisfying $0 < |z_1 - z_0| < R$. It follows that $z_1 \neq z_0$. Now, for every $\zeta \in \text{ran}(\gamma),\ 0 < |z_1 - z_0| < |\zeta - z_0|$, and hence for every $\zeta \in \text{ran}(\gamma), \left|\frac{z_1 - z_0}{\zeta - z_0}\right| < 1$. It follows that for every $\zeta \in \text{ran}(\gamma)$,

$$\begin{aligned}
\frac{\varphi(\zeta)}{\zeta - z_1} &= \varphi(\zeta)\frac{1}{(\zeta - z_0) - (z_1 - z_0)} = \frac{\varphi(\zeta)}{(\zeta - z_0)}\frac{1}{1 - \frac{z_1 - z_0}{\zeta - z_0}} \\
&= \frac{\varphi(\zeta)}{(\zeta - z_0)}\left(1 + \frac{z_1 - z_0}{\zeta - z_0} + \left(\frac{z_1 - z_0}{\zeta - z_0}\right)^2 + \cdots\right) \\
&= \frac{\varphi(\zeta)}{(\zeta - z_0)}\left(1 + (z_1 - z_0)(\zeta - z_0)^{-1} + (z_1 - z_0)^2(\zeta - z_0)^{-2} + \cdots\right) \\
&= \frac{\varphi(\zeta)}{(z_1 - z_0)}\left((z_1 - z_0)(\zeta - z_0)^{-1} + (z_1 - z_0)^2(\zeta - z_0)^{-2} + \cdots\right)
\end{aligned}$$

By Conclusion 4.11, the series

$$(z_1 - z_0)(\zeta - z_0)^{-1} + (z_1 - z_0)^2(\zeta - z_0)^{-2} + \cdots$$

converges uniformly on compact subsets of

$$\left\{z : \limsup_{n\to\infty} \sqrt[n]{|(z_1 - z_0)^n|} < |z - z_0|\right\}\ (= \{z : |z_1 - z_0| < |z - z_0|\}),$$

and hence

$$\left(\frac{\varphi(\zeta)}{\zeta - z_1} =\right)\ \frac{\varphi(\zeta)}{(z_1 - z_0)}\left((z_1 - z_0)(\zeta - z_0)^{-1} + (z_1 - z_0)^2(\zeta - z_0)^{-2} + \cdots\right)$$

converges uniformly on compact subsets of $\{z : |z_1 - z_0| < |z - z_0|\}$. It follows that

$$\begin{gathered}
(f(z_1) =) \int_\gamma \frac{\varphi(\zeta)}{\zeta - z_1}\,\mathrm{d}\zeta = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)}\,\mathrm{d}\zeta + (z_1 - z_0) \\
\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^2}\,\mathrm{d}\zeta + (z_1 - z_0)^2\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z_0)^3}\,\mathrm{d}\zeta + \cdots
\end{gathered}$$

Thus, for every $z \in \{w : |z - z_0| < R\}$, we get the expansion

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

where, for every nonnegative integer n,

$$a_n \equiv \int_\gamma (\zeta - z_0)^{-n-1} \varphi(\zeta) \mathrm{d}\zeta.$$

Conclusion 4.24 Let $a < b$. Let $\gamma : [a, b] \to \mathbb{C}$ be any path. Let $\varphi : \operatorname{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Let $f : (\mathbb{C} - \operatorname{ran}(\gamma)) \to \mathbb{C}$ be the Cauchy integral of φ over γ. Let $z_0 \in \mathbb{C}$. Let $R \equiv \min\{|\zeta - z_0| : \zeta \in \operatorname{ran}(\gamma)\}$. Then, for every $z \in \{w : |w - z_0| < R\}$,

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

where, for every nonnegative integer n,

$$a_n \equiv \int_\gamma (\zeta - z_0)^{-n-1} \varphi(\zeta) \mathrm{d}\zeta.$$

Note 4.25 Let R_1, R_2 be nonnegative real numbers satisfying $R_1 < R_2$. Let $f \in H(\{z : R_1 < |z - z_0| < R_2\})$. For every $\rho \in (R_1, R_2)$, suppose that C_r denotes the circle $\{z : |z - z_0| = \rho\}$ oriented anticlockwise. Let $R_1 < r_1 < r < r_2 < R_2$.

Let us fix any $w \in \{z : r_1 < |z - z_0| < r_2\}$. Here, $|w - z_0| < r_2$, and $w \neq z_0$. Now, by Conclusion 4.24,

$$\frac{1}{2\pi i} \int_{C_{r_2}} \frac{f(\zeta)}{\zeta - w} \mathrm{d}\zeta = a_0 + a_1(w - z_0) + a_2(w - z_0)^2 + \cdots,$$

where, for every nonnegative integer n,

$$a_n \equiv \frac{1}{2\pi i} \int_{C_{r_2}} (\zeta - z_0)^{-n-1} f(\zeta) \mathrm{d}\zeta.$$

By Problem 4.18, for every nonnegative integer n,

$$\int_{C_{r_2}} (\zeta - z_0)^{-n-1} f(\zeta) \mathrm{d}\zeta$$

is independent of r_2, and, r_2 can be chosen arbitrarily close to r, so, for every nonnegative integer n,

$$a_n = \frac{1}{2\pi i} \int\limits_{C_r} (\zeta - z_0)^{-n-1} f(\zeta) \mathrm{d}\zeta.$$

Thus, $f_2 : z \mapsto \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a function from $\{z : |z - z_0| < r_2\}$ to $\mathbb{C}$. Since, $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is a power series in $(z - z_0)$, we have $f_2 \in H(\{z : |z - z_0| < r_2\})$. Here, $r_1 < |w - z_0|$, so, by Conclusion 4.16,

$$\frac{-1}{2\pi i} \int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w} \mathrm{d}\zeta = a_{-1} \frac{1}{w - z_0} + a_{-2} \frac{1}{(w - z_0)^2} + a_{-3} \frac{1}{(w - z_0)^3} + \cdots,$$

where, for every negative integer n,

$$a_n \equiv \frac{1}{2\pi i} \int\limits_{C_{r_1}} (\zeta - z_0)^{-n-1} f(\zeta) \mathrm{d}\zeta.$$

By Problem 4.18, for every negative integer n,

$$\int\limits_{C_{r_1}} (\zeta - z_0)^{-n-1} f(\zeta) \mathrm{d}\zeta$$

is independent of r_1, and, r_1 can be chosen arbitrarily close to r, so, for every negative integer n,

$$a_n = \frac{1}{2\pi i} \int\limits_{C_r} (\zeta - z_0)^{-n-1} f(\zeta) \mathrm{d}\zeta.$$

Thus,

$$f_1 : z \mapsto \sum_{n=0}^{\infty} a_{-n} (z - z_0)^{-n}$$

is a function defined on $\{z : r_1 < |z - z_0|\}$. Since $\sum_{n=0}^{\infty} a_{-n} (z - z_0)^{-n}$ is a power series in $\frac{1}{(z - z_0)}$, and $z \mapsto \frac{1}{(z - z_0)}$ is holomorphic in $\{z : r_1 < |z - z_0|\}$, we have $f_1 \in H(\{z : r_1 < |z - z_0|\})$. By Conclusion 4.22,

$$
\begin{aligned}
f(w) &= \frac{1}{2\pi i}\int\limits_{C_{r_2}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta - \frac{1}{2\pi i}\int\limits_{C_{r_1}} \frac{f(\zeta)}{\zeta - w}\mathrm{d}\zeta \\
&= \left(a_0 + a_1(w - z_0) + a_2(w - z_0)^2 + \cdots\right) \\
&\quad + \left(a_{-1}\frac{1}{z - z_0} + a_{-2}\frac{1}{(z - z_0)^2} + a_{-3}\frac{1}{(z - z_0)^3} + \cdots\right).
\end{aligned}
$$

Hence, for every complex number z satisfying $r_1 < |z - z_0| < r_2$,

$$f(z) = f_2(z) + f_1(z).$$

Conclusion 4.26 Let R_1, R_2 be nonnegative real numbers satisfying $R_1 < R_2$. Let

$$f \in H(\{z : R_1 < |z - z_0| < R_2\}).$$

For every $r \in (R_1, R_2)$, suppose that C_r denotes the circle $\{z : |z - z_0| = r\}$ oriented anticlockwise. Let $R_1 < r_1 < r < r_2 < R_2$. Then,

1. there exist $f_2 \in H(\{z : |z - z_0| < r_2\})$, and $f_1 \in H(\{z : r_1 < |z - z_0|\})$ such that, for every complex number z satisfying $r_1 < |z - z_0| < r_2, f(z) = f_1(z) + f_2(z)$.
2. for the annulus $\{z : r_1 < |z - z_0| < r_2\}$, there exists Laurent representation $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, where, for every integer $n, a_n \equiv \frac{1}{2\pi i}\int_{C_r} (\zeta - z_0)^{-n-1} f(\zeta)\mathrm{d}\zeta$,
3. the coefficients are unique.

Proof of the uniqueness part (3) Suppose that for every complex number z satisfying $r_1 < |z - z_0| < r_2, f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$. Next, suppose that for every complex number z satisfying $r_1 < |z - z_0| < r_2$, $f(z) = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$. We have to show that, for every integer n, $a_n = b_n$.

Since for every z satisfying $r_1 < |z - z_0| < r_2, \sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ is convergent, we have

$$r_2 \leq \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Similarly,

$$\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|} \leq r_1.$$

Let us fix any complex number z_1 satisfying $r_1 < |z_1 - z_0| < r_2$. Let us put

$$r \equiv |z_1 - z_0|, \rho_1 \equiv \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}, \text{ and}$$
$$\rho_2 \equiv \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Here, we have $\rho_1 \leq r_1 < r < r_2 \leq \rho_2$, so $\rho_1 < r < \rho_2$, and hence, by Conclusion 4.13, for every integer n,

$$a_n = \frac{1}{2\pi i} \int_{C_r} (z - z_0)^{-n-1} f(z) \mathrm{d}z.$$

Similarly, for every integer n,

$$b_n = \frac{1}{2\pi i} \int_{C_r} (z - z_0)^{-n-1} f(z) \mathrm{d}z.$$

It follows that for every integer n, $a_n = b_n$. ∎

This result, known as **Laurent's theorem**, is due to P. A. Laurent (18.04.1813–02.09.1854, French).

4.3 Evaluation of Improper Riemann Integrals

Note 4.27

Example 4.28 Let

$$f : z \mapsto \frac{z}{e^z - 1}.$$

Here,

$$\begin{aligned}\mathrm{dom}(f) &= \{z : e^z - 1 \neq 0\} = \{z : e^z \neq 1\} = \mathbb{C} - \{z : e^z = 1\} \\ &= \mathbb{C} - \{\cdots, -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \ldots\} = \mathbb{C} - 2\pi i\mathbb{Z}.\end{aligned}$$

It is clear that

$$\mathbb{C} - \{\cdots, -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \ldots\}$$

is a nonempty open subset of $\mathbb{C}$, and

$$f \in H(\mathbb{C} - \{\cdots, -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \ldots\}).$$

Observe that

$$D'(2\pi i; 1) \subset (\mathbb{C} - \{\cdots, -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \ldots\}),$$

so $2\pi i$ is an isolated singularity of f. Similarly, each $2k\pi i$ is an isolated singularity of f.

Definition Let G be a nonempty open subset of $\mathbb{C}$. Let $f \in H(G)$. If there exists $R > 0$ such that $\{z : R < |z|\} \subset G$, then we say that ∞ *is an* ***isolated singularity*** *of* f.

In Example 4.28, ∞ is not an isolated singularity of f.

Since 0 is a zero of $z \mapsto (e^z - 1)$, and 0 is not a zero of $z \mapsto \frac{\mathrm{d}}{\mathrm{d}z}(e^z - 1)(= e^z)$, the entire function $z \mapsto (e^z - 1)$ has a 'simple' zero at the origin. Hence, by Theorem 1.136, there exists $g \in H(\mathbb{C})$ such that, for every $z \in \mathbb{C}, e^z - 1 = z(g(z))$, and $g(0) \neq 0$. It follows that $\frac{1}{g}$ is holomorphic in some neighborhood of 0. In Example 4.28, $\frac{1}{g}$ coincides with f in some punctured disk of 0. This shows that 0 is a removable singularity of

$$f : z \mapsto \frac{z}{e^z - 1}.$$

Conclusion 4.29 0 is a removable singularity of $f : z \mapsto \frac{z}{e^z - 1}$.

Note 4.30 Let G be a region. Let $a \in G$. Suppose that $f : (G - \{a\}) \to \mathbb{C}$ has an isolated singularity at a, and f has a pole of order m at a. It follows, from Conclusion 1.143, that there exist complex numbers $c_0, c_1, \ldots, c_{m-1}$ such that $c_0 \neq 0$, and the function

$$F : z \mapsto \left(f(z) - \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)} \right) \right)$$

from $(G - \{a\})$ to $\mathbb{C}$ has a removable singularity at a, and hence

$$\underbrace{g : z \mapsto (z-a)^m f(z)} = (z-a)^m F(z) + \left(c_0 + c_1(z-a) + \cdots + c_{m-1}(z-a)^{m-1} \right)$$

from $(G - \{a\})$ to $\mathbb{C}$ has a removable singularity at a. We can extend g holomorphically to G by defining $g(a) \equiv c_0 (\neq 0)$. It follows that $\frac{1}{g} \in H(G)$, and hence $z \mapsto \frac{(z-a)^m}{g(z)}$ is a member of $H(G)$. This shows that $\frac{1}{f}$ has a removable singularity at a. Also, the holomorphic extension of $\frac{1}{f}$ to a neighborhood of a has a zero of order m.

Conclusion 4.31 Let G be a region. Let $a \in G$. Suppose that $f \in H(G - \{a\})$ has a pole of order m at a. Then, $\frac{1}{f}$ has a removable singularity at a. Also, the holomorphic extension of $\frac{1}{f}$ to a neighborhood of a has a zero of order m, and hence, $\lim_{z \to a}|f(z)| = \infty$.

Example 4.32 Let

$$f : z \mapsto \frac{z}{1 + z + z^2 + z^3 + z^4}.$$

Here,

$$\begin{aligned}\mathrm{dom}(f) &= \left\{z : 1 + z + z^2 + z^3 + z^4 \neq 0\right\} = \mathbb{C} - \left\{z : 1 + z + z^2 + z^3 + z^4 = 0\right\} \\ &= \mathbb{C} - \left\{z : (1 - z)\left(1 + z + z^2 + z^3 + z^4\right) = 0, \text{and } (1 - z) \neq 0\right\} \\ &= \mathbb{C} - \left\{z : 1 - z^5 = 0, \text{and } z \neq 1\right\} = \mathbb{C} - \left\{e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}\right\} \\ &= \mathbb{C} - \left\{\alpha, \alpha^2, \alpha^3, \alpha^4\right\},\end{aligned}$$

where $\alpha \equiv e^{\frac{2\pi i}{5}}$. It follows that the set of all isolated singularities of f is $\left\{\alpha, \alpha^2, \alpha^3, \alpha^4, \infty\right\}$. Now, we want to classify isolated singularity α of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at α. Here,

$$f(z) = \frac{z}{1 + z + z^2 + z^3 + z^4} = \frac{z}{(z - \alpha)(z - \alpha^2)(z - \alpha^3)(z - \alpha^4)} = \frac{1}{(z - \alpha)}\phi(z),$$

where

$$\phi : z \mapsto \frac{z}{(z - \alpha^2)(z - \alpha^3)(z - \alpha^4)}.$$

Here

$$\phi(\alpha) = \frac{\alpha}{(\alpha - \alpha^2)(\alpha - \alpha^3)(\alpha - \alpha^4)} \neq 0.$$

Thus,

$$\begin{aligned}f(z) &= \frac{z}{1 + z + z^2 + z^3 + z^4} = \frac{1}{(z - \alpha)}\left(\phi(\alpha) + \frac{\phi'(\alpha)}{1!}(z - \alpha) + \cdots\right) \\ &= \phi(\alpha)\frac{1}{(z - \alpha)} + \frac{\phi'(\alpha)}{1!} + \cdots,\end{aligned}$$

where $\phi(\alpha) \neq 0$, and hence α is a simple pole of f. Similarly, $\alpha^2, \alpha^3, \alpha^4$ are simple poles of f. Further,

$$\mathrm{Res}(f;\alpha)=\frac{\alpha}{(\alpha-\alpha^2)(\alpha-\alpha^3)(\alpha-\alpha^4)},$$
$$\mathrm{Res}\left(f;\alpha^2\right)=\frac{\alpha^2}{(\alpha^2-\alpha)(\alpha^2-\alpha^3)(\alpha^2-\alpha^4)},$$
$$\mathrm{Res}\left(f;\alpha^3\right)=\frac{\alpha^3}{(\alpha^3-\alpha)(\alpha^3-\alpha^2)(\alpha^3-\alpha^4)},$$
$$\mathrm{Res}\left(f;\alpha^4\right)=\frac{\alpha^4}{(\alpha^4-\alpha)(\alpha^4-\alpha^2)(\alpha^4-\alpha^3)}.$$

Next, we want to classify isolated singularity ∞ of *f*. Since $g : z \mapsto f\left(\frac{1}{z}\right)$ and

$$\begin{aligned} f\left(\frac{1}{z}\right) &= \frac{\left(\frac{1}{z}\right)}{1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\left(\frac{1}{z}\right)^4} = \frac{z^3}{1+z+z^2+z^3+z^4} \\ &= z^3\left(1+\left(z+z^2+z^3+z^4\right)\right)^{-1} = z^3\left(1-\left(z+z^2+z^3+z^4\right)\right. \\ &\quad \left. +\left(z+z^2+z^3+z^4\right)^2+\cdots\right) = z^3-z^4+\cdots, \end{aligned}$$

g has removable singularity at the origin, and hence *f* has removable singularity at ∞.

Example 4.33 Let $f : z \mapsto e^{\frac{1}{z}}$ be a function from $\mathbb{C}-\{0\}$ to $\mathbb{C}$. Clearly, the set of all isolated singularities of *f* is $\{0,\infty\}$. Now, we want to classify isolated singularity 0 of *f*.

Let us obtain the Laurent representation of *f* corresponding to a punctured disk in $\mathbb{C}$ centered at 0. Here,

$$f(z)=e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\cdots.$$

Here, since the coefficient of $(z-0)^{-1}$ is nonzero, 0 in not a removable singularity of *f*. Since, the coefficient of each $(z-0)^{-n}$ is nonzero, 0 is an essential singularity of *f*. Also, $\mathrm{Res}(f;0)=1$. Next, we want to classify the isolated singularity ∞ of *f*. Since $g : z \mapsto f\left(\frac{1}{z}\right)\left(=e^z=1+z+\frac{1}{2!}z^2+\cdots\right)$, *g* has removable singularity at the origin, and hence *f* has removable singularity at ∞.

Example 4.34 Let

$$f : z \mapsto \frac{z^3}{z-2}$$

be a function from $\mathbb{C} - \{2\}$ to $\mathbb{C}$. Clearly, the set of all isolated singularities of f is $\{2, \infty\}$. Now, we want to classify isolated singularity 2 of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at 2. Here,

$$\begin{aligned} f(z) &= \frac{z^3}{z-2} = \frac{1}{z-2}((z-2)+2)^3 \\ &= \frac{1}{z-2}\left((z-2)^3 + 6(z-2)^2 + 12(z-2) + 8\right) \\ &= (z-2)^2 + 6(z-2) + 12 + \frac{8}{z-2}. \end{aligned}$$

This shows that 2 is a simple pole of f, and $\operatorname{Res}(f; 2) = 8$. Since

$$g : z \mapsto f\left(\frac{1}{z}\right)\left(= \frac{z-2}{z^3} = \frac{1}{z^2} - \frac{2}{z^3}\right),$$

origin is a pole of order 2 of g, ∞ is hence a pole of order 2 of f.

Note 4.35 Let G be a region. Let $a \in G$. Suppose that $f, g \in H(G)$. Let a be a simple zero of g.

It follows that there exists a unique function $g_1 : G \to \mathbb{C}$ such that

a. $g_1 \in H(G)$, b. for every $z \in G$, $g(z) = (z-a)(g_1(z))$, c. $g_1(a) \neq 0$.

Thus, there exists an open neighborhood $N(a)$ of a such that, for every $z \in (N(a) - \{a\})$,

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)(g_1(z))} = \frac{1}{(z-a)}\varphi(z),$$

where $\varphi : z \mapsto \frac{f(z)}{g_1(z)}$ is a holomorphic function defined in $N(a)$. Now, for every $z \in (N(a) - \{a\})$,

$$\begin{aligned} \frac{f(z)}{g(z)} &= \frac{1}{(z-a)}\left(\varphi(a) + \frac{\varphi'(a)}{1!}(z-a) + \cdots\right) \\ &= \frac{\varphi(a)}{(z-a)} + \frac{\varphi'(a)}{1!} + \cdots = \frac{\frac{f(a)}{g_1(a)}}{(z-a)} + \frac{\varphi'(a)}{1!} + \cdots, \end{aligned}$$

and hence $\operatorname{Res}\left(\frac{f}{g}; a\right) = \frac{f(a)}{g_1(a)}$. Since for every $z \in G$, $g(z) = (z-a)(g_1(z))$, we have $g'(a) = g_1(a)$, and hence $\operatorname{Res}\left(\frac{f}{g}; a\right) = \frac{f(a)}{g'(a)}$.

Conclusion 4.36 Let G be a region. Let $a \in G$. Suppose that $f, g \in H(G)$. Let a be a simple zero of g. Then

$$\operatorname{Res}\left(\frac{f}{g};a\right)=\frac{f(a)}{g'(a)}.$$

Note 4.37 Let G be a region. Let $a\in G$. Suppose that $f\in H(G)$. Let m be a positive integer. Suppose that f has a zero of order m at a.

It follows that there exists a unique function $f_1: G\to\mathbb{C}$ such that a. $f_1\in H(G)$, b. for every $z\in G$, $f(z)=(z-a)^m(f_1(z))$, c. $f_1(a)\neq 0$. Hence, for every $z\in G$,

$$f'(z)=m(z-a)^{m-1}(f_1(z))+(z-a)^m\big((f_1)'(z)\big).$$

Thus, there exists an open neighborhood $N(a)$ of a such that for every $z\in(N(a)-\{a\})$,

$$\begin{aligned}\frac{f'(z)}{f(z)}=\frac{m\frac{f_1(z)}{z-a}+(f_1)'(z)}{f_1(z)}&=\frac{m}{z-a}+\frac{(f_1)'(z)}{f_1(z)}\\&=\frac{m}{z-a}+\left(\frac{(f_1)'(a)}{f_1(a)}+(\cdots)(z-a)+\cdots\right),\end{aligned}$$

and hence $\operatorname{Res}\left(\frac{f'}{f};a\right)=m$.

Conclusion 4.38 Let G be a region. Let $a\in G$. Suppose that $f\in H(G)$. Let m be a positive integer. Suppose that f has a zero of order m at a. Then

$$\operatorname{Res}\left(\frac{f'}{f};a\right)=m.$$

Example 4.39 Let C be the 'unit circle' with anticlockwise orientation. Let a, b be points in the interior of C. Let k be a positive integer. We want to calculate

$$\int_C\frac{z^k}{(z-a)(z-b)}\,dz.$$

Case I: when $a\neq b$. Let

$$f:z\mapsto\frac{z^k}{(z-a)(z-b)}$$

be a function from $(\mathbb{C}-\{a,b\})$ to $\mathbb{C}$. Clearly, f has simple poles at a and b. Also, $f:(\mathbb{C}-\{a,b\})\to\mathbb{C}$ is a meromorphic function. By Conclusion 4.36, $\operatorname{Res}(f;a)=\frac{a^k}{(a-a)1+1(a-b)}=\frac{a^k}{a-b}$. Similarly, $\operatorname{Res}(f;b)=\frac{b^k}{b-a}=-\frac{b^k}{a-b}$. By Conclusion 1.225,

$$\frac{1}{2\pi i}\int_C f(z)\mathrm{d}z = (\mathrm{Res}(f;a))\big((\mathrm{Ind})_C(a)\big) + (\mathrm{Res}(f;b))$$
$$\big((\mathrm{Ind})_C(b)\big)\left(= \frac{a^k}{a-b}\cdot 1 + \left(-\frac{b^k}{a-b}\right)\cdot 1 = \frac{a^k-b^k}{a-b}\right).$$

Thus,

$$\int_C \frac{z^k}{(z-a)(z-b)}\mathrm{d}z = \frac{a^k-b^k}{a-b}.$$

Case II: when $a = b$ Let

$$f : z \mapsto \frac{z^k}{(z-a)(z-b)} \quad \left(= \frac{z^k}{(z-a)^2}\right)$$

be a function from $(\mathbb{C} - \{a\})$ to $\mathbb{C}$. Clearly, f has a pole of order 2 at a. Also, $f : (\mathbb{C} - \{a\}) \to \mathbb{C}$ is a meromorphic function, and $\mathrm{Res}(f;a) = 0$. By Conclusion 1.225,

$$\frac{1}{2\pi i}\int_C f(z)\mathrm{d}z = (\mathrm{Res}(f;a))\big((\mathrm{Ind})_C(a)\big)\ (= 0\cdot 1 = 0).$$

Thus,

$$\int_C \frac{z^k}{(z-a)(z-b)}\mathrm{d}z = \begin{cases} \frac{a^k-b^k}{a-b} & \text{if } a \neq b \\ 0 & \text{if } a = b. \end{cases}$$

Example 4.40 Let

$$f : z \mapsto \frac{z+1}{z^4(z^2+1)}$$

be a function from $\mathbb{C} - \{0, i, -i\}$ to $\mathbb{C}$. Clearly, the set of all isolated singularities of f is $\{0, i, -i, \infty\}$. Now, we want to classify isolated singularity 0 of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at 0. Here,

$$
\begin{aligned}
f(z) &= \frac{1}{z^4}\left(\frac{z+1}{z^2+1}\right) = \frac{1}{z^5}\left(1+\frac{1}{z}\right)\left(1+\frac{1}{z^2}\right)^{-1} \\
&= \frac{1}{z^5}\left(1+\frac{1}{z}\right)\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots\right) \\
&= \frac{1}{z^5}\left(\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots\right)+\left(\frac{1}{z}-\frac{1}{z^3}+\frac{1}{z^5}-\cdots\right)\right) \\
&= \frac{1}{z^5}\left(1+\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}-\cdots\right) = \frac{1}{z^5}+\frac{1}{z^6}-\cdots.
\end{aligned}
$$

This shows that 0 is a pole of order 5 of f, and $\mathrm{Res}(f;0) = 0$. Next,

$$
\begin{aligned}
\mathrm{Res}(f;i) &= \left.\frac{z+1}{4z^3(z^2+1)+z^4 2z}\right|_{z=i} = \frac{i+1}{2i} = \frac{1}{2}(1-i), \text{ and} \\
\mathrm{Res}(f;-i) &= \left.\frac{z+1}{4z^3(z^2+1)+z^4 2z}\right|_{z=-i} = \frac{-i+1}{-2i} = \frac{1}{2}(1+i).
\end{aligned}
$$

Since $g : z \mapsto f\left(\frac{1}{z}\right)$, and

$$
\begin{aligned}
f\left(\frac{1}{z}\right) &= \frac{z^5(1+z)}{1+z^2} = z^5(1+z)\left(1-z^2+z^4-\cdots\right) \\
&= z^5\left(\left(1-z^2+z^4-\cdots\right)+\left(z-z^3+z^5-\cdots\right)\right) \\
&= z^5\left(1+z-z^2-z^3+\cdots\right) = z^5+z^6-z^7-\cdots,
\end{aligned}
$$

g has removable singularity at the origin, and hence f has removable singularity at ∞.

Example 4.41 Let $f : z \mapsto \mathrm{Log}(z)\ (= \ln|z| + i\,\mathrm{Arg}(z))$ be a function from $(\mathbb{C} - (-\infty, 0])$ to $\mathbb{C}$. It is clear that 0 is not an isolated singularity of f. Let α be any real number. Let

$$g : z \mapsto \mathrm{Log}_\alpha(z)(= \ln|z| + i\mathrm{Arg}_\alpha(z))$$

be a function from $(\mathbb{C} - L(0; e^{i\alpha}))$ to $\mathbb{C}$. It is clear that 0 is not an isolated singularity of g. Actually, no point of $L(0; e^{i\alpha})$ is an isolated singularity of g.

Example 4.42 Let

$$f : z \mapsto \frac{1}{\sin\left(\frac{\pi}{z}\right)}$$

be a function from $\left(\mathbb{C} - \left\{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots\right\}\right)$ to $\mathbb{C}$. It is clear that the set of all isolated singularities is $\left\{\infty, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots\right\}$. Now, we want to classify the isolated singularity $\frac{1}{n}$ of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at $\frac{1}{n}$. Here,

$$\begin{aligned}
f(z) &= \frac{1}{\sin\left(\frac{\pi}{z}\right)} = \frac{1}{\sin\left(\frac{\pi}{\left(z-\frac{1}{n}\right)+\frac{1}{n}}\right)} = \frac{1}{\sin\left(\frac{n\pi}{n\left(z-\frac{1}{n}\right)+1}\right)} \\
&= \frac{1}{\sin\left(n\pi\left(1-n\left(z-\frac{1}{n}\right)+\cdots\right)\right)} = \frac{(-1)^{n+1}}{\sin\left(\pi n^2\left(z-\frac{1}{n}\right)-\cdots\right)} \\
&= \frac{(-1)^{n+1}}{\pi n^2\left(z-\frac{1}{n}\right)}\left(1-\frac{1}{3!}\left(\pi n^2\left(z-\frac{1}{n}\right)\right)^2+\cdots\right)^{-1} \\
&= \frac{(-1)^{n+1}}{\pi n^2\left(z-\frac{1}{n}\right)}\left(1+\frac{1}{3!}\left(\pi n^2\left(z-\frac{1}{n}\right)\right)^2-\cdots\right) \\
&= \frac{(-1)^{n+1}}{\pi n^2\left(z-\frac{1}{n}\right)}+\frac{(-1)^{n+1}}{3!}\pi n^2\left(z-\frac{1}{n}\right)-\cdots
\end{aligned}$$

It follows that for every integer n, $\frac{1}{n}$ is an isolated singularity of f, and $\mathrm{Res}\left(f;\frac{1}{n}\right) = \frac{(-1)^{n+1}}{\pi n^2}$.

Example 4.43 Let us evaluate

$$\int_C z^2 \sin\left(\frac{1}{z}\right)\mathrm{d}z,$$

where C denotes the unit circle $|z| = 1$ oriented anticlockwise.

Let

$$f : z \mapsto z^2 \sin\left(\frac{1}{z}\right)$$

be a function from $(\mathbb{C} - \{0\})$ to $\mathbb{C}$. It is clear that 0 is the only isolated singularity of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at 0. Here,

$$f(z) = z^2 \sin\left(\frac{1}{z}\right) = z^2\left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \cdots\right) = z - \frac{1}{3!}\frac{1}{z} + \cdots,$$

so $\mathrm{Res}(f;0) = -\frac{1}{3!}$, and hence, by residue theorem,

$$\int_C z^2 \sin\left(\frac{1}{z^2}\right)\mathrm{d}z = 2\pi i\left(-\frac{1}{3!}\right) = \frac{-\pi i}{3}.$$

Example 4.44 Let us evaluate $\int_C e^{\frac{1}{z^2}} \mathrm{d}z$, where C denotes the unit circle $|z| = 1$ oriented anticlockwise.

Let $f : z \mapsto e^{\frac{1}{z^2}}$ be a function from $(\mathbb{C} - \{0\})$ to $\mathbb{C}$. It is clear that 0 is the only isolated singularity of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at 0. Here,

$$f(z) = e^{\frac{1}{z^2}} = 1 + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \cdots,$$

so origin is an essential singularity of f, and $\mathrm{Res}(f; 0) = 0$. Now, by the residue theorem,

$$\int_C e^{\frac{1}{z^2}} \mathrm{d}z = 2\pi i(0) = 0.$$

Example 4.45 Let us evaluate $\int_C \frac{1}{z(z-3)^4} \mathrm{d}z$, where C denotes the circle $|z - 3| = 2$ oriented anticlockwise.

Let

$$f : z \mapsto \frac{1}{z(z-3)^4}$$

be a function from $(\mathbb{C} - \{0, 3\})$ to $\mathbb{C}$. It is clear that 0, 3 are the only isolated singularities of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at 3. Here,

$$\begin{aligned} f(z) &= \frac{1}{z(z-3)^4} = \frac{1}{(z-3)^4} \cdot \frac{1}{3} \cdot \frac{1}{1 + \frac{1}{3}(z-3)} \\ &= \frac{1}{(z-3)^4} \cdot \frac{1}{3}\left(1 - \frac{1}{3}(z-3) + \left(\frac{1}{3}(z-3)\right)^2 - \left(\frac{1}{3}(z-3)\right)^3 + \cdots\right), \end{aligned}$$

so $\mathrm{Res}(f; 3) = -\frac{1}{3^4}$, and hence, by the residue theorem,

$$\int_C \frac{1}{z(z-3)^4} \mathrm{d}z = 2\pi i\left(-\frac{1}{3^4}\right) = -\frac{2}{81}\pi i.$$

Example 4.46 Let us evaluate

$$\int_C \frac{2z-3}{z(z-1)} \mathrm{d}z,$$

where C denotes the circle $|z| = 2$ oriented anticlockwise.

Let

$$f : z \mapsto \frac{2z-3}{z(z-1)} \quad \left(= \frac{3}{z} + \frac{-1}{z-1} \right)$$

be a function from $(\mathbb{C} - \{0,1\})$ to $\mathbb{C}$. It is clear that 0, 1 are the only isolated singularities of f. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at 0. Here,

$$f(z) = \frac{3}{z} + \frac{-1}{z-1} = \frac{3}{z} + \left(1 + z + z^2 + \cdots\right)$$

so $\operatorname{Res}(f;0) = 3$. Let us obtain the Laurent representation of f corresponding to a punctured disk in $\mathbb{C}$ centered at 1. Here,

$$\begin{aligned} f(z) = \frac{3}{z} + \frac{-1}{z-1} = \frac{-1}{z-1} + 3\frac{1}{1+(z-1)} = \frac{-1}{z-1} \\ + 3\left(1 - (z-1) + (z-1)^2 - \cdots\right), \end{aligned}$$

so $\operatorname{Res}(f;1) = -1$. Now, by the residue theorem,

$$\int_C \frac{2z-3}{z(z-1)} \mathrm{d}z = 2\pi i(3 + (-1)) = 4\pi i.$$

Note 4.47

Definition Let $a_1, \ldots, a_n$ be n distinct complex numbers. Let $f \in H(\mathbb{C} - \{a_1, \ldots, a_n\})$. Suppose that each a_k is an isolated singularity of f. Let Γ be a cycle in $\mathbb{C} - \{a_1, \ldots, a_n\}$ such that each a_k is interior to Γ.

There exists a positive real number R_1 such that Γ lies inside the circle $\{z : |z| = R_1\}$. Now, since $f \in H(\mathbb{C} - \{a_1, \ldots, a_n\})$, f is holomorphic in the region $\{z : R_1 < |z|\}$. It follows that ∞ is an isolated singularity of f.

Let us take any real number $R_0 > R_1$. Let C_0 be the circle $\{z : |z| = R_0\}$ with clockwise orientation. We denote $\frac{1}{2\pi i}\int_{C_0} f(z)\mathrm{d}z$ by $\operatorname{Res}(f;\infty)$, and this is called the ***residue** of f at **infinity***.

Since f is holomorphic in the region bounded by Γ and C_0, we have

$$\int_\Gamma f(z)\mathrm{d}z + 2\pi i \cdot \operatorname{Res}(f;\infty) = \underbrace{\int_\Gamma f(z)\mathrm{d}z + \int_{C_0} f(z)\mathrm{d}z = 0.}$$

By Conclusion 4.26, for the annulus $\{z : R_1 < |z| < R_0\}$, there exists Laurent representation $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, where, for every integer n,

$$\underbrace{c_n \equiv \frac{1}{2\pi i} \int_{-C_0} \zeta^{-n-1} f(\zeta)\mathrm{d}\zeta}= -\frac{1}{2\pi i} \int_{C_0} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta.$$

It follows that for every complex number z satisfying $0 < |z| < \frac{1}{R_1}$, we have $f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{z^n}$, and hence for every z satisfying $0 < |z| < \frac{1}{R_1}$, we have

$$\underbrace{\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{z^{n+2}}} = \cdots + c_{-2} + \frac{c_{-1}}{z} + \frac{c_0}{z^2} + \cdots.$$

Thus,

$$\begin{aligned}\underbrace{\mathrm{Res}(g;0) = c_{-1}} &= -\frac{1}{2\pi i} \int_{C_0} \frac{f(\zeta)}{\zeta^{(-1)+1}} \mathrm{d}\zeta = -\frac{1}{2\pi i} \int_{C_0} f(\zeta)\mathrm{d}\zeta = -\mathrm{Res}(f;\infty) \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(z)\mathrm{d}z,\end{aligned}$$

where $g : z \mapsto \frac{1}{z^2} f\left(\frac{1}{z}\right)$. In short, we write

$$\int_{\Gamma} f(z)\mathrm{d}z = 2\pi i \,\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right).$$

Conclusion 4.48 Let $a_1, \ldots, a_n$ be n distinct complex numbers. Let $f \in H(\mathbb{C} - \{a_1, \ldots, a_n\})$. Suppose that each a_k is an isolated singularity of f. Let Γ be a cycle in $\mathbb{C} - \{a_1, \ldots, a_n\}$ such that each a_k is interior to Γ. Then,

$$\int_{\Gamma} f(z)\mathrm{d}z = 2\pi i \,\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right); 0\right).$$

Example 4.49 Using the preceding conclusion, let us evaluate

$$\int_{C} \frac{2z-3}{z(z-1)} \mathrm{d}z,$$

where C denotes the circle $|z| = 2$ oriented anticlockwise. Let

$$f : z \mapsto \frac{2z-3}{z(z-1)}$$

be a function from $(\mathbb{C} - \{0,1\})$ to $\mathbb{C}$. It is clear that 0, 1 are the only isolated singularity of f. Further, 0, 1 are interior to C. Next,

$$\begin{aligned}\frac{1}{z^2}f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\frac{2\frac{1}{z}-3}{\frac{1}{z}\left(\frac{1}{z}-1\right)} = \frac{2-3z}{z(1-z)} = \frac{1}{z}(2-3z)\left(1+z+z^2+\cdots\right)\\ &= \frac{1}{z}\left(2-z-z^2+\cdots\right) = \frac{2}{z} - 1 - z + \cdots,\end{aligned}$$

so $\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right);0\right) = 2$. Now, by Conclusion 4.48,

$$\int_C \frac{2z-3}{z(z-1)}\mathrm{d}z = 2\pi i \cdot 2 = 4\pi i.$$

Note 4.50 Let G be a region. Let $f \in H(G)$. Let a be an isolated singularity of f. It follows that $G \cup \{a\}$ is an open set. Suppose that f has a pole of order m at a.

Problem 4.51 There exists $\varphi \in H(G \cup \{a\})$ such that $\varphi(a) \neq 0$ and, for every $z \in G$, $f(z) = \frac{1}{(z-a)^m}\varphi(z)$.

(Solution Since f has a pole of order m at a, there exist complex numbers $c_0, c_1, \ldots, c_{m-1}$ such that $c_0 \neq 0$, and the function

$$F : z \mapsto \left(f(z) - \left(\frac{c_0}{(z-a)^m} + \frac{c_1}{(z-a)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-a)}\right)\right)$$

from G to $\mathbb{C}$ has a removable singularity at a. Hence, there exists $a^* \in \mathbb{C}$ such that

$$F_1 : z \mapsto \begin{cases} F(z) & \text{if } z \in G \\ a^* & \text{if } z = a \end{cases}$$

is a member of $H(G \cup \{a\})$. Let

$$\varphi : z \mapsto (z-a)^m F_1(z) + \left(c_0 + c_1(z-a) + \cdots + c_{m-1}(z-a)^{m-1}\right)$$

be a function from $G \cup \{a\}$ to $\mathbb{C}$. Clearly, $\varphi \in H(G \cup \{a\})$. Next, $\varphi(a) = c_0 \neq 0$. Also, for every $z \in G$,

$$\begin{aligned}\frac{1}{(z-a)^m}\varphi(z) &= \frac{1}{(z-a)^m}\left((z-a)^m F_1(z)\right.\\ &\quad \left.+\left(c_0+c_1(z-a)+\cdots+c_{m-1}(z-a)^{m-1}\right)\right)\\ &= F_1(z)+\left(\frac{c_0}{(z-a)^m}+\frac{c_1}{(z-a)^{m-1}}+\cdots+\frac{c_{m-1}}{(z-a)}\right)\\ &= F(z)+\left(\frac{c_0}{(z-a)^m}+\frac{c_1}{(z-a)^{m-1}}+\cdots+\frac{c_{m-1}}{(z-a)}\right)\\ &= \left(f(z)-\left(\frac{c_0}{(z-a)^m}+\frac{c_1}{(z-a)^{m-1}}+\cdots+\frac{c_{m-1}}{(z-a)}\right)\right)\\ &\quad +\left(\frac{c_0}{(z-a)^m}+\frac{c_1}{(z-a)^{m-1}}+\cdots+\frac{c_{m-1}}{(z-a)}\right)=f(z),\end{aligned}$$

so for every $z \in G$,

$$f(z) = \frac{1}{(z-a)^m}\varphi(z).$$

∎)

Conclusion 4.52 Let G be a region. Let $f \in H(G)$. Let a be an isolated singularity of f. It follows that $G \cup \{a\}$ is an open set. Let m be a positive integer. Then f has a pole of order m at a if and only if there exists $\varphi \in H(G \cup \{a\})$ such that $\varphi(a) \neq 0$ and, for every $z \in G$,

$$f(z) = \frac{1}{(z-a)^m}\varphi(z).$$

Further,

$$\operatorname{Res}(f;a) = \begin{cases} \frac{\varphi^{(m-1)}(a)}{(m-1)!} & \text{if } m \geq 2 \\ \varphi(a) & \text{if } m = 1. \end{cases}$$

Proof of the remaining part Suppose that there exists $\varphi \in H(G \cup \{a\})$ such that $\varphi(a) \neq 0$ and, for every $z \in G$, $f(z) = \frac{1}{(z-a)^m}\varphi(z)$. We have to show that f has a pole of order m at a.

Since $\varphi \in H(G \cup \{a\})$, we have, for every $z \in (G \cup \{a\})$,

$$\varphi(z) = \varphi(a) + \frac{\varphi'(a)}{1!}(z-a) + \frac{\varphi''(a)}{2!}(z-a)^2 + \cdots,$$

and hence for every $z \in G$,

$$\begin{aligned} f(z) &= \frac{1}{(z-a)^m}\left(\varphi(a) + \frac{\varphi'(a)}{1!}(z-a) + \frac{\varphi''(a)}{2!}(z-a)^2 + \cdots\right) \\ &= \frac{\varphi(a)}{(z-a)^m} + \frac{\varphi'(a)}{1!}\frac{1}{(z-a)^{m-1}} + \cdots + \frac{\varphi^{(m-1)}(a)}{(m-1)!}\frac{1}{(z-a)} \\ &\quad + \frac{\varphi^{(m)}(a)}{m!} + \frac{\varphi^{(m+1)}(a)}{(m+1)!}(z-a) + \cdots. \end{aligned}$$

Here, $\varphi(a) \neq 0$, so it remains to show that

$$F : z \mapsto \left(f(z) - \left(\frac{\varphi(a)}{(z-a)^m} + \frac{\varphi'(a)}{1!}\frac{1}{(z-a)^{m-1}} + \cdots + \frac{\varphi^{(m-1)}(a)}{(m-1)!}\frac{1}{(z-a)}\right)\right)$$
$$\left(= \frac{\varphi^{(m)}(a)}{m!} + \frac{\varphi^{(m+1)}(a)}{(m+1)!}(z-a) + \cdots\right)$$

from G to $\mathbb{C}$ has a removable singularity at a. Suppose that

$$F_1 : z \mapsto \begin{cases} F(z) & \text{if } z \in G \\ \frac{\varphi^{(m)}(a)}{m!} & \text{if } z = a \end{cases}$$

is a function from $G \cup \{a\}$ to $\mathbb{C}$. Thus,

$$F_1 : z \mapsto \frac{\varphi^{(m)}(a)}{m!} + \frac{\varphi^{(m+1)}(a)}{(m+1)!}(z-a) + \cdots$$

is a function from $G \cup \{a\}$ to $\mathbb{C}$. Clearly, $F_1 \in H(G \cup \{a\})$. Thus, $F : G \to \mathbb{C}$ has a removable singularity at a.

Since for every $z \in G$,

$$\begin{aligned} f(z) &= \frac{\varphi(a)}{(z-a)^m} + \frac{\varphi'(a)}{1!}\frac{1}{(z-a)^{m-1}} + \cdots + \frac{\varphi^{(m-1)}(a)}{(m-1)!}\frac{1}{(z-a)} \\ &\quad + \frac{\varphi^{(m)}(a)}{m!} + \frac{\varphi^{(m+1)}(a)}{(m+1)!}(z-a) + \cdots, \end{aligned}$$

we have

$$\operatorname{Res}(f; a) = \begin{cases} \frac{\varphi^{(m-1)}(a)}{(m-1)!} & \text{if } m \geq 2 \\ \varphi(a) & \text{if } m = 1. \end{cases}$$

∎

Theorem 4.53 Let $p : \mathbb{C} \to \mathbb{C}$, and $q : \mathbb{C} \to \mathbb{C}$ be polynomials. Let $2 \le \deg(q) - \deg(p)$. Then the sum of residues of the rational function $\frac{p}{q}$, taken over all of its poles in $\mathbb{C}$, is zero.

Proof Suppose that for every $z \in \mathbb{C}$,

$$p(z) \equiv a_0 + a_1 z + \cdots + a_m z^m,$$

where $a_0, a_1, \ldots, a_m$ are complex numbers, $a_m \neq 0$, and $\deg(p) = m$. Next, suppose that for every $z \in \mathbb{C}$,

$$q(z) \equiv b_0 + b_1 z + \cdots + b_m z^m + b_{m+1} z^{m+1} + b_{m+2} z^{m+2} + \cdots + b_n z^n,$$

where each b_k is a complex number, $b_n \neq 0$, $\deg(q) = n$, and $m + 2 \le n$. Let $\beta_1, \ldots, \beta_n$ be the n zeros of the polynomial q.

It follows that the set of all isolated singularities in $\mathbb{C}$ of the rational function $\frac{p}{q}$ is $\{\beta_1, \ldots, \beta_n\}$.

Let R be a positive real number such that $\{\beta_1, \ldots, \beta_n\} \subset D(0, R)$. Let C be the circle $\{z : |z| = R\}$ with anticlockwise orientation. Here, $\frac{p}{q} \in H(\mathbb{C} - \{\beta_1, \ldots, \beta_n\})$.

By Conclusion 4.48,

$$2\pi i \left(\text{sum of residues of the rational function } \frac{p}{q}, \text{taken over all of its poles in } \mathbb{C} \right)$$
$$= \int_C \frac{p(z)}{q(z)} \mathrm{d}z = 2\pi i \operatorname{Res}\left(\frac{1}{z^2} \frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)}; 0 \right).$$

It suffices to show that $\operatorname{Res}\left(\frac{1}{z^2} \frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)}; 0 \right) = 0$. Since

$$\begin{aligned}
\frac{1}{z^2} \frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)} &= \frac{1}{z^2} \frac{a_0 + a_1 \frac{1}{z} + \cdots + a_m \frac{1}{z^m}}{b_0 + b_1 \frac{1}{z} + \cdots + b_m \frac{1}{z^m} + b_{m+1} \frac{1}{z^{m+1}} + b_{m+2} \frac{1}{z^{m+2}} + \cdots + b_n \frac{1}{z^n}} \\
&= \frac{1}{z^{2+m-n}} \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m} + b_{m+1} z^{n-m-1} + b_{m+2} z^{n-m-2} + \cdots + b_n} \\
&= \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^{m+2} + b_1 z^{m+1} + \cdots + b_n z^{2+m-n}} \\
&= \frac{\frac{a_0}{b_0 z^2}\left(b_0 z^{m+2} + b_1 z^{m+1} + \cdots + b_n z^{2+m-n}\right) + \left(a_1 - \frac{a_0 b_1}{b_0}\right) z^{m-1} + (\cdots) z^{m-2} + \cdots}{b_0 z^{m+2} + b_1 z^{m+1} + \cdots + b_n z^{2+m-n}} \\
&= 0 \cdot \frac{1}{z} + \frac{a_0}{b_0} \frac{1}{z^2} + (\cdots) \frac{1}{z^3} + \cdots,
\end{aligned}$$

so $\mathrm{Res}\left(\frac{1}{z^2}\frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)};0\right) = 0.$ ∎

Example 4.54 Let

$$f : z \mapsto \frac{z^3 - 2z}{(z-i)^4}$$

be a function from region $(\mathbb{C} - \{i\})$ to $\mathbb{C}$. It is clear that i is the only isolated singularity of f. For every $z \in (\mathbb{C} - \{i\}), f(z) = \frac{1}{(z-i)^4}\varphi(z)$ where $\varphi : z \mapsto (z^3 - 2z)$ is a function from $\mathbb{C}$ to $\mathbb{C}$. Clearly, $\varphi \in H(\mathbb{C})$, and $\varphi(i) = (i^3 - 2i) = -3i \neq 0$. It follows, by Conclusion 4.52, that f has a pole of order 4 at i. Further, $\mathrm{Res}(f;i) = \frac{\varphi'''(i)}{3!} = \frac{1}{6}(6) = 1$.

Example 4.55 On taking the branch $\log z \equiv \ln|z| + i\theta \ (|z| > 0, 0 < \theta < 2\pi)$ of the logarithmic function, let $f : z \mapsto \frac{(\log z)^3}{(z^2+1)}$ be a function from region $(\mathbb{C} - \{-i, i\})$ to $\mathbb{C}$.

It is clear that $-i, i$ are the only isolated singularity of f. For every $z \in (\mathbb{C} - \{-i, i\}), f(z) = \frac{1}{z-i}\varphi(z)$, where

$$\varphi : z \mapsto \frac{(\ln|z| + i\theta)^3}{z+i}$$

is a function from $(\mathbb{C} - \{-i\})$ to $\mathbb{C}$. Clearly, $\varphi \in H(\mathbb{C} - \{-i\})$, and

$$\varphi(i) = \frac{\left(\ln|i| + i\frac{\pi}{2}\right)^3}{i+i} = \frac{-i\left(\frac{\pi}{2}\right)^3}{2i} = -\frac{\pi^3}{16} \neq 0.$$

It follows, by Conclusion 4.52, that f has a simple pole at i. Further, $\mathrm{Res}(f;i) = \varphi(i) = -\frac{\pi^3}{16}$. For every $z \in (\mathbb{C} - \{-i, i\}), f(z) = \frac{1}{z+i}\varphi(z)$, where $\varphi : z \mapsto \frac{(\ln|z| + i\theta)^3}{z-i}$ is a function from $(\mathbb{C} - \{i\})$ to $\mathbb{C}$. Clearly, $\varphi \in H(\mathbb{C} - \{i\})$, and

$$\varphi(-i) = \frac{\left(\ln|-i| + i\frac{3\pi}{2}\right)^3}{-i-i} = \frac{-i\left(\frac{3\pi}{2}\right)^3}{-2i} = \frac{27\pi^3}{16} \neq 0.$$

It follows, by Conclusion 4.52, that f has a simple pole at $-i$. Further, $\mathrm{Res}(f;-i) = \varphi(-i) = \frac{27\pi^3}{16}$.

Note 4.56 Let G be a region. Let $f \in H(G)$. Let a be an isolated singularity of f. It follows that $G \cup \{a\}$ is an open set. Suppose that a is a removable singularity of f.

It follows that there exists a function $f_1 : G \cup \{a\} \to \mathbb{C}$ such that $f_1|_G = f$, and $f_1 \in H(G \cup \{a\})$. Since a is an isolated singularity of f, there exists $\varepsilon > 0$ such that $D'(a;\varepsilon) \subset G$, and hence $D(a;\varepsilon) \subset (G \cup \{a\})$. Now, since $f_1 \in H(G \cup \{a\})$, f_1 is holomorphic in $D\left(a;\frac{\varepsilon}{2}\right)$, and hence f_1 is holomorphic in $D'\left(a;\frac{\varepsilon}{2}\right)$. Since f_1 is holomorphic in

$$D'\left(a;\frac{\varepsilon}{2}\right) \ \ (\subset D'(a;\varepsilon) \subset G),$$

and $f_1|_G = f$, f is holomorphic in $D'\left(a;\frac{\varepsilon}{2}\right)$. Since $f_1 \in H(G \cup \{a\})$, f_1 is continuous on $(G \cup \{a\})\left(\supset D\left[a;\frac{\varepsilon}{2}\right]\right)$, and hence

$$f\left(D'\left(a;\frac{\varepsilon}{2}\right)\right) = f_1\left(D'\left(a;\frac{\varepsilon}{2}\right)\right) \subset \underbrace{f_1\left(D\left[a;\frac{\varepsilon}{2}\right]\right)}$$

is bounded. It follows that $f\left(D'\left(a;\frac{\varepsilon}{2}\right)\right)$ is bounded.

Conclusion 4.57 Let G be a region. Let $f \in H(G)$. Let a be an isolated singularity of f. Suppose that a is a removable singularity of f. Then there exists $\varepsilon > 0$ such that f is holomorphic in $D'(a;\varepsilon)$, and f is bounded in $D'(a;\varepsilon)$.

Note 4.58

Definition Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function. By $\int_0^\infty f(x)\mathrm{d}x$, we mean

$$\lim_{R\to\infty}\left(\int_0^R f(x)\mathrm{d}x\right).$$

Definition Let $f : (-\infty, 0] \to \mathbb{R}$ be a continuous function. By $\int_{-\infty}^0 f(x)\mathrm{d}x$, we mean

$$\lim_{R\to\infty}\left(\int_{-R}^0 f(x)\mathrm{d}x\right).$$

Definition Let $f : (-\infty,\infty) \to \mathbb{R}$ be a continuous function. By $\int_{-\infty}^\infty f(x)\mathrm{d}x$, we mean

$$\int_{-\infty}^0 f(x)\mathrm{d}x + \int_0^\infty f(x)\mathrm{d}x.$$

Definition Let $f : (-\infty,\infty) \to \mathbb{R}$ be a continuous function. By p.v. $\int_{-\infty}^\infty f(x)\mathrm{d}x$, we mean

$$\lim_{R\to\infty}\left(\int_{-R}^{R} f(x)\mathrm{d}x\right),$$

and this is called the ***Cauchy principal value*** *of integral* $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$.

Let $f : (-\infty, \infty) \to \mathbb{R}$ be a continuous function. Let f be an even function. Suppose that p.v. $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ exists.

Problem 4.59 $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ exists, and $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x$.

(Solution Here,
$\left(2\lim_{R\to\infty}\left(\int_0^R f(x)\mathrm{d}x\right) = \lim_{R\to\infty}\left(2\int_0^R f(x)\mathrm{d}x\right) =\right) \lim_{R\to\infty}\left(\int_{-R}^R f(x)\mathrm{d}x\right)$
exists, so

$$\left(\lim_{R\to\infty}\left(\int_{-R}^{0} f(x)\mathrm{d}x\right) =\right) \lim_{R\to\infty}\left(\int_{0}^{R} f(x)\mathrm{d}x\right)$$

exists, and hence

$$\lim_{R\to\infty}\left(\int_{-R}^{0} f(x)\mathrm{d}x\right)$$

exists. It follows that
$\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ exists, and $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x$. ∎)

Conclusion 4.60 Let $f : (-\infty, \infty) \to \mathbb{R}$ be a continuous function. Let f be an even function. Suppose that p.v. $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ exists. Then

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x \,\text{exists, and} \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x.$$

Example 4.61 Let us evaluate

$$\int_{0}^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x.$$

Here, integrand is of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with real coefficients, and they have no common factor. Further, $q(z)$ has no real zero, but $q(z)$ has a zero above the real axis. In our example, zeros of $q(z)$ are

$$e^{i\frac{\pi}{6}}, e^{i\frac{3\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{7\pi}{6}}, e^{i\frac{9\pi}{6}}, e^{i\frac{11\pi}{6}}.$$

The zeros of $q(z)$ above the real axis are $e^{i\frac{\pi}{6}}, e^{i\frac{3\pi}{6}}, e^{i\frac{5\pi}{6}}$.

Problem 4.62 $\int_0^\infty \frac{x^2}{1+x^6}\mathrm{d}x$ is convergent.

(Solution Since for every positive $R > 1$,

$$\begin{aligned}\int_0^R \frac{x^2}{1+x^6}\mathrm{d}x &= \int_0^1 \frac{x^2}{1+x^6}\mathrm{d}x + \int_1^R \frac{x^2}{1+x^6}\mathrm{d}x \le \int_0^1 \frac{x^2}{1+x^6}\mathrm{d}x \\ &\quad + \int_1^R \frac{x^2}{1+x^6}\mathrm{d}x \le 1 + \int_1^R \frac{x^2}{x^6}\mathrm{d}x = 1 + \int_1^R x^{-4}\mathrm{d}x \\ &= 1 + \frac{1}{-3}\frac{1}{x^3}\bigg|_1^R = 1 + \frac{1}{-3}\left(\frac{1}{R^3} - 1\right) \le 1 + \frac{1}{3} = \frac{4}{3},\end{aligned}$$

$\int_0^\infty \frac{x^2}{1+x^6}\mathrm{d}x$ is convergent. ∎)

Prior knowledge of convergence will not be used below. Observe that the integrand $\frac{x^2}{1+x^6}$ in our example coincides with the function $f : z \mapsto \frac{z^2}{1+z^6}$ on the real axis. Also, $f : z \mapsto \frac{z^2}{1+z^6}$ is a holomorphic function in

$$\left(\mathbb{C} - \left\{e^{i\frac{\pi}{6}}, e^{i\frac{3\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{7\pi}{6}}, e^{i\frac{9\pi}{6}}, e^{i\frac{11\pi}{6}}\right\}\right).$$

For every $r > 0$, by S_r we shall mean the semicircle in the upper half-plane with center 0 and radius r, oriented anticlockwise. Let us fix any $R > 1$. Let Γ_R be the closed curve consisting of $[-R, R]$ followed by S_R. Here, isolated singularities of $f : z \mapsto \frac{z^2}{1+z^6}$ in the interior of Γ_R are $e^{i\frac{\pi}{6}}, e^{i\frac{3\pi}{6}}, e^{i\frac{5\pi}{6}}$. By the residue theorem,

$$\begin{aligned}&\left(\int_{-R}^R \frac{x^2}{1+x^6}\mathrm{d}x + \int_{S_R} \frac{z^2}{1+z^6}\mathrm{d}z = \int_{[-R,R]_R} \frac{z^2}{1+z^6}\mathrm{d}z + \int_{S_R} \frac{z^2}{1+z^6}\mathrm{d}z =\right) \int_{\Gamma_R} \frac{z^2}{1+z^6}\mathrm{d}z \\ &= 2\pi i\left(\mathrm{Res}\left(f; e^{i\frac{\pi}{6}}\right) + \mathrm{Res}\left(f; e^{i\frac{3\pi}{6}}\right) + \mathrm{Res}\left(f; e^{i\frac{5\pi}{6}}\right)\right) \\ &= 2\pi i\left(\frac{z^2}{6z^5}\bigg|_{z=e^{i\frac{\pi}{6}}} + \frac{z^2}{6z^5}\bigg|_{z=e^{i\frac{3\pi}{6}}} + \frac{z^2}{6z^5}\bigg|_{z=e^{i\frac{5\pi}{6}}}\right) \\ &= 2\pi i\frac{1}{6}\left(\frac{1}{z^3}\bigg|_{z=e^{i\frac{\pi}{6}}} + \frac{1}{z^3}\bigg|_{z=e^{i\frac{3\pi}{6}}} + \frac{1}{z^3}\bigg|_{z=e^{i\frac{5\pi}{6}}}\right) \\ &= \frac{\pi i}{3}\left(\frac{1}{e^{i\frac{\pi}{2}}} + \frac{1}{e^{i\frac{3\pi}{2}}} + \frac{1}{e^{i\frac{5\pi}{2}}}\right) = \frac{\pi i}{3}\left(\frac{1}{i} + \frac{1}{-i} + \frac{1}{i}\right) = \frac{\pi}{3}.\end{aligned}$$

Now, since

$$\left|\int\limits_{S_R} \frac{z^2}{1+z^6}\mathrm{d}z\right| \leq \int\limits_{S_R} \left|\frac{z^2}{1+z^6}\right|\mathrm{d}z = \int\limits_{S_R} |z^2| \frac{1}{|1-z^6)}\mathrm{d}z \leq \int\limits_{S_R} |z^2| \frac{1}{||1|-z^6|}\mathrm{d}z$$

$$= \int\limits_{S_R} |z|^2 \frac{1}{\left|1-|z|^6\right|}\mathrm{d}z \leq \left(R^2 \frac{1}{|1-R^6|}\cdot \pi R = \frac{\pi R^3}{R^6-1} = \pi \frac{\frac{1}{R^3}}{1-\frac{1}{R^6}} \to 0 \,\mathrm{as}\, R\to\infty\right),$$

we have

$$\text{p.v.} \int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x = \underbrace{\lim_{R\to\infty}\left(\int\limits_{-R}^{R} \frac{x^2}{1+x^6}\mathrm{d}x\right) = \frac{\pi}{3}}.$$

Since p.v. $\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x = \frac{\pi}{3}$, and $x \mapsto \frac{x^2}{1+x^6}$ is an even function, $\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x$ exists, and

$$2\int\limits_{0}^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x = \underbrace{\int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x = \text{p.v.} \int\limits_{-\infty}^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x} = \frac{\pi}{3}.$$

Thus, $\int_0^{\infty} \frac{x^2}{1+x^6}\mathrm{d}x = \frac{\pi}{6}$.

Example 4.63 Let us evaluate

$$\int\limits_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2}\mathrm{d}x.$$

Here, integrand is of the form $\frac{p(x)}{q(x)}\cos ax$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and they have no common factor. Further, $q(z)$ has no real zero, but $q(z)$ has a zero above the real axis.

In our example, zeros of $q(z)$ are $i, -i$. The zero of $q(z)$ above the real axis is i.

Problem 4.64 $\int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2}\mathrm{d}x$ is convergent.

(Solution Since for every $R > 1$,

$$\int_{-R}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx = 2\int_{0}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx = 2\int_{0}^{1} \frac{\cos 3x}{(1+x^2)^2}\,dx$$
$$+2\int_{1}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx \le 2\int_{0}^{1} \left|\frac{\cos 3x}{(1+x^2)^2}\right| dx + 2\int_{1}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx$$
$$\le 2\int_{0}^{1} 1\,dx + 2\int_{1}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx \le 2 + 2\int_{1}^{R} \frac{|\cos 3x|}{(1+x^2)^2}\,dx \le 2$$
$$+2\int_{1}^{R} \frac{1}{1+x^2}\,dx = 2 + 2\left(\tan^{-1} R - \tan^{-1} 1\right) \le 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2},$$

$\int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2}\,dx$ is convergent. ∎)

Prior knowledge of convergence will not be used below. Observe that the integrand

$$\frac{\cos 3x}{(1+x^2)^2}$$

in our example coincides with $\mathrm{Re}(f)$ on the real axis, where $f : z \mapsto \frac{e^{i3z}}{(1+z^2)^2}$. Also, $f : z \mapsto \frac{e^{i3z}}{(1+z^2)^2}$ is a holomorphic function in $(\mathbb{C} - \{i, -i\})$. For every $r > 0$, by S_r we shall mean the semicircle in the upper half-plane with center 0 and radius r, oriented anticlockwise.

Let us fix any $R > 1$. Let Γ_R be the closed curve consisting of $[-R, R]$ followed by S_R. Here, the isolated singularity of

$$f : z \mapsto \frac{e^{i3z}}{(1+z^2)^2} \quad \left(= \frac{1}{(z-i)^2} \frac{e^{i3z}}{(z+i)^2} \right)$$

in the interior of Γ_R is i. By the residue theorem,

$$\int_{-R}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx + \int_{S_R} \frac{e^{i3z}}{(1+z^2)^2}\,dz = \int_{-R}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx + i\cdot 0$$

$$+ \int_{S_R} \frac{e^{i3z}}{(1+z^2)^2}\,dz = \int_{-R}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx + i\int_{-R}^{R} \frac{\sin 3x}{(1+x^2)^2}\,dx$$

$$+ \int_{S_R} \frac{e^{i3z}}{(1+z^2)^2}\,dz = \int_{-R}^{R} \frac{e^{i3x}}{(1+x^2)^2}\,dx + \int_{S_R} \frac{e^{i3z}}{(1+z^2)^2}\,dz$$

$$= \int_{[-R,R]} \frac{e^{i3z}}{(1+z^2)^2}\,dz + \int_{S_R} \frac{e^{i3z}}{(1+z^2)^2}\,dz = \int_{\Gamma_R} \frac{e^{i3z}}{(1+z^2)^2}\,dz$$

$$= 2\pi i(\mathrm{Res}(f;i)) = 2\pi i\left(\left.\frac{\frac{d}{dz}\left(\frac{e^{i3z}}{(z+i)^2}\right)}{1!}\right|_{z=i}\right)$$

$$= 2\pi i\left(\left.\frac{3ie^{i3z}(z+i)^2 - e^{i3z}2(z+i)}{(z+i)^4}\right|_{z=i}\right)$$

$$= 2\pi i\left(\frac{3ie^{-3}(-4) - e^{-3}2(2i)}{}\right) = 2\pi i\left(\frac{16ie^{-3}}{16}\right) = 2\pi e^{-3}$$

Now, since

$$\left|\int_{S_R} \frac{e^{i3z}}{(1+z^2)^2}\,dz\right| \le \int_{S_R} \left|\frac{e^{i3z}}{(1+z^2)^2}\right| dz = \int_{S_R} \left|e^{i3z}\right| \frac{1}{|1-(-z^2)|^2}\,dz$$

$$\le \int_{S_R} e^{-3y}\frac{1}{||1|-|-z^2||^2}\,dz = \int_{S_R} e^{-3y}\frac{1}{\left|1-|z|^2\right|^2}dz$$

$$\le e^0\frac{1}{|1-R^2|^2}\cdot\pi R = \frac{1}{(R^2-1)^2}\cdot\pi R = \frac{1}{\left(1-\frac{1}{R^2}\right)^2}\cdot\pi\frac{1}{R^3} \to 0 \text{ as } R\to\infty,$$

we have

$$\left(\text{p.v.}\int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2}\,dx = \right) \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{\cos 3x}{(1+x^2)^2}\,dx\right) = 2\pi e^{-3}.$$

Since

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2} \mathrm{d}x = 2\pi e^{-3},$$

and $x \mapsto \frac{\cos 3x}{(1+x^2)^2}$ is an even function, $\int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2} \mathrm{d}x$ exists, and

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2} \mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2} \mathrm{d}x \left(= 2\pi e^{-3}\right).$$

Thus,

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(1+x^2)^2} \mathrm{d}x = \frac{2\pi}{e^3}.$$

Note 4.65 Let R be a real number satisfying $0 < R$. Let z_0 be a complex number. Let $f \in H(\{z : 0 < |z - z_0| < R\})$. Suppose that f has a simple pole at z_0. Let $R_2 \in (0, R)$. Suppose that for the punctured disk $\{z : 0 < |z - z_0| < R_2\}$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent representation of $f(z)$. For every $\rho > 0$, let C_ρ be an arc of the circle $\{z : |z - z_0| = \rho\}$ of angle α, oriented anticlockwise.

Problem 4.66 $\lim_{\rho \to 0} \left(\int_{C_\rho} f(z) \mathrm{d}z \right) = a_{-1} \alpha i.$

Solution Since f has a simple pole at z_0, and, in the punctured disk $\{z : 0 < |z - z_0| < R_2\}$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

we have $a_{-2} = a_{-3} = a_{-4} = \cdots = 0$. It follows that in $\{z : 0 < |z - z_0| < R_2\}$, we have

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z),$$

where $g : z \mapsto \sum_{n=0}^{\infty} a_n (z - z_0)^n$ from $D(z_0; R_2)$ to $\mathbb{C}$. Hence, for every $\rho \in (0, R_2)$, there exists a positive real number M such that for every $z \in D\left[z_0; \frac{R_2}{2}\right]$, $|g(z)| \leq M$. Since

$$0 \leq \lim_{\rho \to 0} \left| \int\limits_{C_\rho} g(z)\mathrm{d}z \right| \leq \lim_{\rho \to 0} \left(\int\limits_{C_\rho} |g(z)|\mathrm{d}z \right) \leq \lim_{\rho \to 0} (M \cdot \alpha\rho) = 0,$$

we have

$$\lim_{\rho \to 0} \left(\int\limits_{C_\rho} f(z)\mathrm{d}z \right) - a_{-1} \lim_{\rho \to 0} \left(\int\limits_{C_\rho} \frac{1}{z - z_0} \mathrm{d}z \right) = \underbrace{\lim_{\rho \to 0} \left(\int\limits_{C_\rho} g(z)\mathrm{d}z \right) = 0.}$$

It suffices to show that

$$\lim_{\rho \to 0} \left(\int\limits_{C_\rho} \frac{1}{z - z_0} \mathrm{d}z \right) = \alpha i.$$

$$\mathrm{LHS} = \lim_{\rho \to 0} \left(\int\limits_0^\alpha \frac{1}{(z_0 + \rho e^{i\theta}) - z_0} \left(0 + \rho e^{i\theta} i\right) \mathrm{d}\theta \right) = \lim_{\rho \to 0} \left(\int\limits_0^\alpha i\mathrm{d}\theta \right) = \alpha i = \mathrm{RHS}.$$

∎)

Conclusion 4.67 Let R be a real number satisfying $0 < R$. Let z_0 be a real number. Let $f \in H(\{z : 0 < |z - z_0| < R\})$. Suppose that f has a simple pole at z_0. Let $R_2 \in (0, R)$. Suppose that for the punctured disk $\{z : 0 < |z - z_0| < R_2\}$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent representation of $f(z)$. For every $\rho > 0$, let C_ρ be an arc of the circle $\{z : |z - z_0| = \rho\}$ of angle α, oriented anticlockwise. Then,

$$\lim_{\rho \to 0} \left(\int\limits_{C_\rho} f(z)\mathrm{d}z \right) = a_{-1}\alpha i.$$

Note 4.68 Let R_0 be a positive real number. Let G be an open set containing $\{z : R_0 < |z|, \text{and } 0 \le \mathrm{Im}(z)\}$. Let $f \in H(G)$. For every R satisfying $R_0 < R$, let S_R be the semicircle $\left\{z : z = Re^{i\theta}, \text{ and } 0 \le \theta \le \pi\right\}$ oriented anticlockwise. Suppose that for every R satisfying $R_0 < R$, there exists $M_R > 0$ such that, for every $z \in S_R$, $|f(z)| \le M_R$ and $\lim_{R\to\infty} M_R = 0$. Let $a > 0$.

Problem 4.69 $\lim_{R\to\infty}\left(\int_{S_R} f(z)e^{iaz}\mathrm{d}z\right) = 0.$

(Solution Since the graph of $\sin x$ is above the straight line graph of $\frac{2}{\pi}x$ over the interval $\left[0, \frac{\pi}{2}\right]$, for every $\theta \in \left[0, \frac{\pi}{2}\right]$, we have $\frac{2}{\pi}\theta \le \sin\theta$. It follows that for every $\theta \in \left[0, \frac{\pi}{2}\right]$, and for every positive ρ, we have $(-\rho\sin\theta) \le \left(-\rho\frac{2}{\pi}\theta\right)$, and hence, for every $\theta \in \left[0, \frac{\pi}{2}\right]$, and for every positive ρ, $e^{-\rho\sin\theta} \le e^{\frac{-2\rho}{\pi}\theta}$. Now, for every positive ρ,

$$\underbrace{\int_0^{\frac{\pi}{2}} e^{-\rho\sin\theta}\mathrm{d}\theta \le \int_0^{\frac{\pi}{2}} e^{\frac{-2\rho}{\pi}\theta}\mathrm{d}\theta} = \frac{1}{\frac{-2\rho}{\pi}} e^{\frac{-2\rho}{\pi}\theta}\Big|_0^{\frac{\pi}{2}} = \frac{-\pi}{2\rho}\left(e^{-\rho} - 1\right) = \frac{\pi}{2\rho}\left(1 - \frac{1}{e^{\rho}}\right).$$

Thus, for every positive ρ,

$$\begin{aligned}\int_0^{\pi} e^{-\rho\sin\theta}\mathrm{d}\theta &= \int_0^{\frac{\pi}{2}} e^{-\rho\sin\theta}\mathrm{d}\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-\rho\sin\theta}\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} e^{-\rho\sin\theta}\mathrm{d}\theta \\ &\quad + \int_{\frac{\pi}{2}}^{0} e^{-\rho\sin(\pi-\varphi)}(-1)\mathrm{d}\varphi \\ &= \int_0^{\frac{\pi}{2}} e^{-\rho\sin\theta}\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} e^{-\rho\sin\varphi}\mathrm{d}\varphi = 2\int_0^{\frac{\pi}{2}} e^{-\rho\sin\theta}\mathrm{d}\theta \\ &\le 2\cdot\frac{\pi}{2\rho}\left(1 - \frac{1}{e^{\rho}}\right) = \frac{\pi}{\rho}\left(1 - \frac{1}{e^{\rho}}\right) \le \frac{\pi}{\rho}.\end{aligned}$$

Hence, for every positive ρ,

$$\int_0^{\pi} e^{-\rho\sin\theta}\mathrm{d}\theta \le \frac{\pi}{\rho}.$$

For every R satisfying $R_0 < R$, and for every $\left(Re^{i\theta} =\right) z \in S_R$, we have

$$
\begin{aligned}
0 \leq \left| \int_{S_R} f(z) e^{iaz} \mathrm{d}z \right| &= \left| \int_0^\pi f\left(Re^{i\theta}\right) e^{iaRe^{i\theta}} \left(Rie^{i\theta}\right) \mathrm{d}\theta \right| \\
&\leq \int_0^\pi \left| f\left(Re^{i\theta}\right) e^{iaRe^{i\theta}} \left(Rie^{i\theta}\right) \right| \mathrm{d}\theta \\
&= \int_0^\pi Re^{-aR\sin\theta} \left| f\left(Re^{i\theta}\right) \right| \mathrm{d}\theta \leq \int_0^\pi Re^{-aR\sin\theta} M_R \mathrm{d}\theta \\
&= RM_R \int_0^\pi e^{-aR\sin\theta} \mathrm{d}\theta \leq RM_R \frac{\pi}{aR} = \frac{\pi}{a} M_R \to 0 \text{ as } R \to \infty.
\end{aligned}
$$

Thus,

$$\lim_{R\to\infty} \left(\int_{S_R} f(z) e^{iaz} \mathrm{d}z \right) = 0.$$

∎)

Conclusion 4.70 Let R_0 be a positive real number. Let G be an open set containing $\{z : R_0 < |z|, \text{and } 0 \leq \mathrm{Im}(z)\}$. Let $f \in H(G)$. For every R satisfying $R_0 < R$, let S_R be the semicircle

$$\left\{z : z = Re^{i\theta}, \text{ and } 0 \leq \theta \leq \pi\right\}$$

oriented anticlockwise. Suppose that for every R satisfying $R_0 < R$, there exists $M_R > 0$ such that, for every $z \in S_R$, $|f(z)| \leq M_R$ and $\lim_{R\to\infty} M_R = 0$. Let $a > 0$. Then,

$$\lim_{R\to\infty} \left(\int_{S_R} f(z) e^{iaz} \mathrm{d}z \right) = 0.$$

Example 4.71 Let us evaluate

$$\int_0^\infty \frac{\sin x}{x} \mathrm{d}x.$$

Observe that the integrand $\frac{\sin x}{x}$ in our example coincides with $\mathrm{Im}(f)$ on the real axis, where $f : z \mapsto \frac{e^{iz}}{z}$. Also, $f : z \mapsto \frac{e^{iz}}{z}$ is a holomorphic function in $(\mathbb{C} - \{0\})$.

For every $r > 0$, let S_r be the semicircle in the upper half-plane with center 0 and radius r, oriented anticlockwise.

Let ρ, R be any real numbers such that $0<\rho<R$. Let $\Gamma_{\rho,R}$ be the closed curve consisting of $[-R, -\rho]$ followed by $-S_\rho$ followed by $[\rho, R]$ followed by S_R. Here, the isolated singularity of

$$\underbrace{f : z \mapsto \frac{e^{iz}}{z}} = \frac{1}{z} e^{iz} = \frac{1}{z}\left(1 + iz - \frac{z^2}{2!} - \cdots\right) = \frac{1}{z} + i - \frac{z}{2!} - \cdots$$

is 0. Also, real number 0 is a simple pole of f, and $\operatorname{Res}(f; 0) = 1$. Now, by Conclusion 4.67,

$$\lim_{\rho\to 0}\left(\int_{S_\rho} f(z)\mathrm{d}z\right) = 1 \cdot \pi i.$$

Next, by the Cauchy theorem,

$$- \int_{S_\rho} f(z)\mathrm{d}z + \int_\rho^R \frac{2i\sin x}{x}\mathrm{d}x + \int_{S_R} \frac{e^{iz}}{z}\mathrm{d}z = -\int_{S_\rho} f(z)\mathrm{d}z$$

$$+ \int_\rho^R \frac{e^{ix} - e^{-ix}}{x}\mathrm{d}x + \int_{S_R} \frac{e^{iz}}{z}\mathrm{d}z = -\int_\rho^R \frac{e^{-ix}}{x}\mathrm{d}x + \int_{-S_\rho} f(z)\mathrm{d}z$$

$$+ \int_\rho^R \frac{e^{ix}}{x}\mathrm{d}x + \int_{S_R} \frac{e^{iz}}{z}\mathrm{d}z = \int_R^\rho \frac{e^{i(-x)}}{-x}(-1)\mathrm{d}x + \int_{-S_\rho} f(z)\mathrm{d}z$$

$$+ \int_\rho^R \frac{e^{ix}}{x}\mathrm{d}x + \int_{S_R} \frac{e^{iz}}{z}\mathrm{d}z = \int_{-R}^{-\rho} \frac{e^{ix}}{x}\mathrm{d}x + \int_{-S_\rho} f(z)\mathrm{d}z$$

$$+ \int_\rho^R \frac{e^{ix}}{x}\mathrm{d}x + \int_{S_R} \frac{e^{iz}}{z}\mathrm{d}z = \int_{[-R,-\rho]} \frac{e^{iz}}{z}\mathrm{d}z$$

$$+ \int_{-S_\rho} f(z)\mathrm{d}z + \int_{[\rho,R]} \frac{e^{iz}}{z}\mathrm{d}z + \int_{S_R} f(z)\mathrm{d}z = \int_{[-R,-\rho]} f(z)\mathrm{d}z$$

$$+ \int_{-S_\rho} f(z)\mathrm{d}z + \int_{[\rho,R]} f(z)\mathrm{d}z + \int_{S_R} f(z)\mathrm{d}z = \underbrace{\int_{\Gamma_{\rho,R}} f(z)\mathrm{d}z = 0,}$$

so

$$2i\int_0^R \frac{\sin x}{x}\mathrm{d}x + \int_{S_R}\frac{e^{iz}}{z}\mathrm{d}z = \underbrace{\lim_{\rho\to 0}\left(\int_\rho^R \frac{2i\sin x}{x}\mathrm{d}x + \int_{S_R}\frac{e^{iz}}{z}\mathrm{d}z\right) = 1.\pi i},$$

and hence

$$\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2} - \frac{1}{2i}\lim_{R\to\infty}\int_{S_R}\frac{1}{z}e^{iz}\mathrm{d}z.$$

Since for every $z \in S_R$, $\left|\frac{1}{z}\right| = \frac{1}{R}$, and $\lim_{R\to\infty}\frac{1}{R} = 0$, by Conclusion 4.70,

$$2i\left(\frac{\pi}{2} - \int_0^\infty \frac{\sin x}{x}\mathrm{d}x\right) = 2i\left(\frac{\pi}{2} - \lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\mathrm{d}x\right) = \underbrace{\lim_{R\to\infty}\int_{S_R}\frac{1}{z}e^{iz}\mathrm{d}z = 0},$$

and hence

$$\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}.$$

Example 4.72 Let us evaluate p.v. $\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+2x+2}\mathrm{d}x$.

Observe that the integrand

$$\frac{x\sin x}{x^2+2x+2}$$

in our example coincides with $\mathrm{Im}(f)$ on the real axis, where

$$f : z \mapsto \frac{ze^{iz}}{z^2+2z+2}$$

Also,

$$f : z \mapsto \frac{ze^{iz}}{z^2+2z+2}\quad\left(= \frac{z}{(z-(-1+i))(z-(-1-i))}e^{iz}\right)$$

is a holomorphic function in $(\mathbb{C} - \{-1+i, -1-i\})$. Clearly, $-1+i$, and $-1-i$ are isolated singularities of f.

For every $r > 0$, let S_r be the semicircle in the upper half-plane with center 0 and radius r, oriented anticlockwise.

Let us fix any $R > |-1+i| \left(= \sqrt{2}\right)$. Let Γ_R be the closed curve consisting of $[-R, R]$ followed by S_R.

Here, the isolated singularity of

$$f : z \mapsto \frac{z}{(z-(-1+i))(z-(-1-i))} e^{iz}$$

in the interior of Γ_R is $(-1+i)$. Also,

$$\begin{aligned}\operatorname{Res}(f; -1+i) &= \frac{(-1+i)}{((-1+i)-(-1-i))} e^{i(-1+i)} = \frac{(-1+i)}{2i} e^{-1}(\cos 1 - i \sin 1)\\ &= \frac{1}{2e}(1+i)(\cos 1 - i \sin 1).\end{aligned}$$

By the residue theorem,

$$\begin{aligned}&\left(\int_{-R}^{R} \frac{x\cos x}{x^2+2x+2} \mathrm{d}x + i \int_{-R}^{R} \frac{x \sin x}{x^2+2x+2} \mathrm{d}x\right) + \int_{S_R} \frac{ze^{iz}}{z^2+2z+2} \mathrm{d}z\\ &= \int_{-R}^{R} \frac{xe^{ix}}{x^2+2x+2} \mathrm{d}x + \int_{S_R} \frac{ze^{iz}}{z^2+2z+2} \mathrm{d}z = \int_{[-R,R]} \frac{ze^{iz}}{z^2+2z+2} \mathrm{d}z\\ &\quad + \int_{S_R} \frac{ze^{iz}}{z^2+2z+2} \mathrm{d}z = \underbrace{\int_{\Gamma_R} \frac{ze^{iz}}{z^2+2z+2} \mathrm{d}z = 2\pi i(\operatorname{Res}(f; -1+i))}\\ &= 2\pi i\left(\frac{1}{2e}(1+i)(\cos 1 - i \sin 1)\right) = \frac{\pi}{e}(i-1)(\cos 1 - i \sin 1)\\ &= \frac{\pi}{e}((-\cos 1 + \sin 1) + i(\cos 1 + \sin 1)),\end{aligned}$$

so

$$\int_{-R}^{R} \frac{x \sin x}{x^2+2x+2} \mathrm{d}x = \frac{\pi}{e}(\cos 1 + \sin 1) - \operatorname{Im}\left(\int_{S_R} g(z) e^{iz} \mathrm{d}z\right),$$

where

$$g : z \mapsto \frac{z}{(z-(-1+i))(z-(-1-i))}$$

is a holomorphic function in $(\mathbb{C} - \{-1+i, -1-i\})$.

For every R satisfying $\sqrt{2} < R$, and for every $z \in S_R$, we have

$$
\begin{aligned}
|g(z)| &= \left| \frac{z}{(z-(-1+i))(z-(-1-i))} \right| = \frac{|z|}{|z-(-1+i)||z-(-1-i)|} \\
&\leq \frac{|z|}{||z|-1+i|\,||z|-1-i|} = \frac{|z|}{\left||z|-\sqrt{2}\right|\left||z|-\sqrt{2}\right|} \\
&= \frac{|z|}{\left(|z|-\sqrt{2}\right)^2} = \frac{R}{\left(R-\sqrt{2}\right)^2} = \frac{1}{R}\frac{1}{\left(1-\sqrt{2}\frac{1}{R}\right)^2},
\end{aligned}
$$

and

$$\lim_{R\to\infty} \frac{1}{R}\frac{1}{\left(1-\sqrt{2}\frac{1}{R}\right)^2} = 0,$$

so, by Conclusion 4.70,

$$\lim_{R\to\infty}\left(\int_{S_R} g(z)e^{iz}\mathrm{d}z\right) = 0,$$

and hence

$$
\begin{aligned}
&\frac{\pi}{e}(\cos 1 + \sin 1) - \left(\text{p.v.} \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+2x+2}\mathrm{d}x\right) = \frac{\pi}{e}(\cos 1 + \sin 1) \\
&\quad - \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{x\sin x}{x^2+2x+2}\mathrm{d}x\right) = \lim_{R\to\infty}\left(\frac{\pi}{e}(\cos 1 + \sin 1) - \int_{-R}^{R} \frac{x\sin x}{x^2+2x+2}\mathrm{d}x\right) \\
&= \lim_{R\to\infty}\left(\mathrm{Im}\left(\int_{S_R} g(z)e^{iz}\mathrm{d}z\right)\right) = \underbrace{\mathrm{Im}\left(\lim_{R\to\infty}\left(\int_{S_R} g(z)e^{iz}\mathrm{d}z\right)\right)} = 0.
\end{aligned}
$$

Thus,

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+2x+2}\mathrm{d}x = \frac{\pi}{e}(\cos 1 + \sin 1).$$

Example 4.73 Let us evaluate

$$\int_0^\infty \frac{\ln x}{(4+x^2)^2}\,dx.$$

Observe that the integrand

$$\frac{\ln x}{(4+x^2)^2}$$

in our example coincides with f on the positive real axis, where

$$\underbrace{f : z \mapsto \frac{\log_{\frac{-\pi}{2}}(z)}{(4+z^2)^2}} = \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{(4+z^2)^2} = \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{(z+2i)^2(z-2i)^2}$$

is a function from

$$\underbrace{\left(\mathbb{C} - \left(L\left(0; e^{i\left(-\frac{\pi}{2}\right)}\right) \cup \{-2i, 2i\}\right)\right)} = (\mathbb{C} - (\{0 + t(-i) : 0 \le t\} \cup \{-2i, 2i\}))$$
$$= (\mathbb{C} - (\{-ti : 0 \le t\} \cup \{2i\}))$$

to $\mathbb{C}$. We know that

$$f : (\mathbb{C} - (\{-ti : 0 \le t\} \cup \{2i\})) \to \mathbb{C}$$

is a holomorphic function in

$$(\mathbb{C} - (\{-ti : 0 \le t\} \cup \{2i\})).$$

For every $r > 0$, let S_r be the semicircle in the upper half-plane with center 0 and radius r, oriented anticlockwise.

Let ρ, R be any real numbers such that $0 < \rho < |2i| < R$. Let $\Gamma_{\rho,R}$ be the closed curve consisting of $[-R, -\rho]$ followed by $-S_\rho$ followed by $[\rho, R]$ followed by S_R.

Here, the isolated singularity of

$$f : z \mapsto \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{(z+2i)^2(z-2i)^2}$$

in the interior of $\Gamma_{\rho,R}$ is $2i$. Further, f has a pole of order 2 at $2i$, and

$$\operatorname{Res}(f;2i) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\frac{\ln|z| + i\operatorname{Arg}_{\frac{-\pi}{2}}(z)}{(z+2i)^2}\right)\Bigg|_{z=2i} = \frac{\frac{1}{z}(z+2i)^2 - \left(\ln|z| + i\operatorname{Arg}_{\frac{-\pi}{2}}(z)\right)\cdot 2(z+2i)}{(z+2i)^4}\Bigg|_{z=2i}$$

$$= \frac{\frac{1}{2i}(-16) - \left(\ln 2 + i\frac{\pi}{2}\right)\cdot 2(4i)}{256} = \frac{8i - \left(i\ln 2 - \frac{\pi}{2}\right)8}{256} = \frac{\pi}{64} + i\left(\frac{1-\ln 2}{32}\right).$$

By the residue theorem,

$$-\int_{S_\rho} f(z)\mathrm{d}z + 2\int_\rho^R \frac{\ln x}{(4+x^2)^2}\mathrm{d}x + i\pi\int_\rho^R \frac{1}{(4+x^2)^2}\mathrm{d}x + \int_{S_R} f(z)\mathrm{d}z$$

$$= \int_{-S_\rho} f(z)\mathrm{d}z + \int_\rho^R \frac{2\ln x + i\pi}{(4+x^2)^2}\mathrm{d}x + \int_{S_R} f(z)\mathrm{d}z$$

$$= \int_R^\rho \frac{\ln x + i\pi}{\left(4+(-x)^2\right)^2}(-1)\mathrm{d}x + \int_{-S_\rho} f(z)\mathrm{d}z + \int_\rho^R \frac{\ln x + i0}{(4+x^2)^2}\mathrm{d}x + \int_{S_R} f(z)\mathrm{d}z$$

$$= \int_{-R}^{-\rho} \frac{\ln(-x) + i\pi}{(4+x^2)^2}\mathrm{d}x + \int_{-S_\rho} f(z)\mathrm{d}z + \int_\rho^R \frac{\ln x + i0}{(4+x^2)^2}\mathrm{d}x + \int_{S_R} f(z)\mathrm{d}z$$

$$= \int_{[-R,-\rho]} \frac{\ln|z| + i\operatorname{Arg}_{\frac{-\pi}{2}}(z)}{(4+z^2)^2}\mathrm{d}z + \int_{-S_\rho} f(z)\mathrm{d}z + \int_{[\rho,R]} \frac{\ln|z| + i\operatorname{Arg}_{\frac{-\pi}{2}}(z)}{(4+z^2)^2}\mathrm{d}z + \int_{S_R} f(z)\mathrm{d}z$$

$$= \underbrace{\int_{\Gamma_{\rho,R}} f(z)\mathrm{d}z = 2\pi i(\operatorname{Res}(f;2i))}= 2\pi i\left(\frac{\pi}{64} + i\left(\frac{1-\ln 2}{32}\right)\right) = \frac{\ln 2 - 1}{16}\pi + i\frac{\pi^2}{32},$$

so

$$2\int_\rho^R \frac{\ln x}{(4+x^2)^2}\mathrm{d}x = \frac{\ln 2 - 1}{16}\pi + \operatorname{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right) - \operatorname{Re}\left(\int_{S_R} f(z)\mathrm{d}z\right).$$

Now, we shall try to show that

1. $\lim_{\rho\to 0}\left(\operatorname{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right)\right) = 0,$
2. $\lim_{R\to\infty}\left(\operatorname{Re}\left(\int_{S_R} f(z)\mathrm{d}z\right)\right) = 0.$

For 1: Here it suffices to show that

$$\lim_{\rho\to 0}\left|\operatorname{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right)\right| = 0.$$

Since

$$
\begin{aligned}
0 &\le \left|\mathrm{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right)\right| \le \left|\int_{S_\rho} f(z)\mathrm{d}z\right| = \left|\int_0^\pi f\left(\rho e^{i\theta}\right)\left(\rho i e^{i\theta}\right)\mathrm{d}\theta\right| \\
&\le \int_0^\pi \left|f\left(\rho e^{i\theta}\right)\left(\rho i e^{i\theta}\right)\right|\mathrm{d}\theta = \int_0^\pi \left|f\left(\rho e^{i\theta}\right)\right|\rho\mathrm{d}\theta = \rho\int_0^\pi \left|\frac{\ln\left|\rho e^{i\theta}\right| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}\left(\rho e^{i\theta}\right)}{\left(4+\left(\rho e^{i\theta}\right)^2\right)^2}\right|\mathrm{d}\theta \\
&= \rho\int_0^\pi \left|\frac{\ln\rho + i\theta}{\left(4-\left(-\rho^2 e^{i2\theta}\right)\right)^2}\right|\mathrm{d}\theta = \rho\int_0^\pi \frac{|\ln\rho + i\theta|}{\left|4-\left(-\rho^2 e^{i2\theta}\right)\right|^2}\mathrm{d}\theta \le \rho\int_0^\pi \frac{|\ln\rho + i\theta|}{\left||4|-|-\rho^2 e^{i2\theta}|\right|^2}\mathrm{d}\theta \\
&= \rho\int_0^\pi \frac{|\ln\rho + i\theta|}{|4-\rho^2|^2}\mathrm{d}\theta \le \rho\int_0^\pi \frac{\sqrt{(\ln\rho)^2 + \left(\frac{3\pi}{2}\right)^2}}{(4-\rho^2)^2}\mathrm{d}\theta = \rho\frac{\sqrt{(\ln\rho)^2 + \left(\frac{3\pi}{2}\right)^2}}{(4-\rho^2)^2}\pi \\
&= \frac{\sqrt{(\rho\ln\rho)^2 + \left(\frac{3\pi}{2}\rho\right)^2}}{(4-\rho^2)^2}\pi,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{\rho\to 0}\left(\frac{\sqrt{(\rho\ln\rho)^2 + \left(\frac{3\pi}{2}\rho\right)^2}}{(4-\rho^2)^2}\pi\right) &= \frac{\sqrt{\left(\lim_{\rho\to 0}\left(\frac{\ln\rho}{\frac{1}{\rho}}\right)\right)^2 + (0)^2}}{(4-0)^2}\pi \\
&= \frac{\pi}{16}\lim_{\rho\to 0}\frac{\frac{1}{\rho}}{\frac{-1}{\rho^2}} = -\frac{\pi}{16}\lim_{\rho\to 0}\rho = 0,
\end{aligned}
$$

we have

$$
\lim_{\rho\to 0}\left|\mathrm{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right)\right| = 0.
$$

For 2: Here it suffices to show that

$$
\lim_{R\to\infty}\left|\mathrm{Re}\left(\int_{S_R} f(z)\mathrm{d}z\right)\right| = 0.
$$

Since

$$0 \leq \left|\mathrm{Re}\left(\int_{S_R} f(z)\mathrm{d}z\right)\right| \leq \left|\int_{S_R} f(z)\mathrm{d}z\right| = \left|\int_0^\pi f\left(Re^{i\theta}\right)\left(Rie^{i\theta}\right)\mathrm{d}\theta\right| \leq \int_0^\pi \left|f\left(Re^{i\theta}\right)\left(Rie^{i\theta}\right)\right|\mathrm{d}\theta$$

$$= \int_0^\pi \left|f\left(Re^{i\theta}\right)\right|R\mathrm{d}\theta = R\int_0^\pi \left|\frac{\ln\left|Re^{i\theta}\right| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}\left(Re^{i\theta}\right)}{\left(4+\left(Re^{i\theta}\right)^2\right)^2}\right|\mathrm{d}\theta$$

$$= R\int_0^\pi \left|\frac{\ln R + i\theta}{\left(4-\left(-R^2e^{i2\theta}\right)\right)^2}\right|\mathrm{d}\theta = R\int_0^\pi \frac{|\ln R + i\theta|}{\left|4-\left(-R^2e^{i2\theta}\right)\right|^2}\mathrm{d}\theta \leq R\int_0^\pi \frac{|\ln R + i\theta|}{\left||4|-\left|-R^2e^{i2\theta}\right|\right|^2}\mathrm{d}\theta$$

$$= R\int_0^\pi \frac{|\ln R + i\theta|}{\left|4-R^2\right|^2}\mathrm{d}\theta \leq R\int_0^\pi \frac{\sqrt{(\ln R)^2 + \left(\frac{3\pi}{2}\right)^2}}{\left(R^2-4\right)^2}\mathrm{d}\theta = R\frac{\sqrt{(\ln R)^2 + \left(\frac{3\pi}{2}\right)^2}}{\left(R^2-4\right)^2}\pi$$

$$= \frac{1}{R^2}\frac{\sqrt{\left(\frac{\ln R}{R}\right)^2 + \left(\frac{3\pi}{2}\frac{1}{R}\right)^2}}{\left(1-4\frac{1}{R^2}\right)^2}\pi,$$

and

$$\lim_{R\to\infty}\left(\frac{1}{R^2}\frac{\sqrt{\left(\frac{\ln R}{R}\right)^2 + \left(\frac{3\pi}{2}\frac{1}{R}\right)^2}}{\left(1-4\frac{1}{R^2}\right)^2}\pi\right) = 0 \cdot \frac{\sqrt{(0)^2 + \left(\frac{3\pi}{2}\cdot 0\right)^2}}{(1-4\cdot 0)^2}\pi = 0,$$

we have

$$\lim_{R\to\infty}\left|\mathrm{Re}\left(\int_{S_R} f(z)\mathrm{d}z\right)\right| = 0.$$

Thus,

$$2\int_0^\infty \frac{\ln x}{\left(4+x^2\right)^2}\mathrm{d}x = \frac{\ln 2 - 1}{16}\pi + 0 - 0,$$

and hence

$$\int_0^\infty \frac{\ln x}{\left(4+x^2\right)^2}\mathrm{d}x = \frac{\ln 2 - 1}{32}\pi.$$

Example 4.74 Let us evaluate

$$\int_0^\infty \frac{\ln x}{x^2-1}\,\mathrm{d}x.$$

Observe that the integrand

$$\frac{\ln x}{x^2-1}$$

in our example coincides with f on the positive real axis, where

$$\underbrace{f : z \mapsto \frac{\log_{\frac{-\pi}{2}}(z)}{z^2-1}} = \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{z^2-1} = \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{(z+1)(z-1)}$$

is a function from

$$\begin{aligned}\underbrace{\left(\mathbb{C} - \left(L\left(0; e^{i\left(-\frac{\pi}{2}\right)}\right) \cup \{-1,1\}\right)\right)} &= (\mathbb{C} - (\{0 + t(-i) : 0 \le t\} \cup \{-1,1\})) \\ &= (\mathbb{C} - (\{-ti : 0 \le t\} \cup \{-1,1\}))\end{aligned}$$

to $\mathbb{C}$. We know that

$$f : (\mathbb{C} - (\{-ti : 0 \le t\} \cup \{-1,1\})) \to \mathbb{C}$$

is a holomorphic function in

$$(\mathbb{C} - (\{-ti : 0 \le t\} \cup \{-1,1\})).$$

Since

$$f : (\mathbb{C} - (\{-ti : 0 \le t\} \cup \{-1,1\})) \to \mathbb{C}$$

is a holomorphic function, f has an isolated singularity at 1. Next,

$$\begin{aligned}f(z) &= \frac{\log_{\frac{-\pi}{2}}(z)}{z^2-1} = \frac{1}{z-1}\left(\frac{\log_{\frac{-\pi}{2}}(z)}{z+1}\right) = \frac{1}{z-1}\left(\frac{\log_{\frac{-\pi}{2}}(1+(z-1))}{2+(z-1)}\right) \\ &= \frac{1}{2(z-1)}\left(\frac{\log_{\frac{-\pi}{2}}(1+(z-1))}{1+\frac{(z-1)}{2}}\right) = \frac{1}{2(z-1)}\left((z-1) - \frac{1}{2}(z-1)^2 + \cdots\right) \\ &\quad \left(1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \cdots\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2}(z-1) + \frac{1}{3}(z-1)^2 - \cdots\right)\left(1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \cdots\right) \\ &= \frac{1}{2} - \frac{(z-1)}{2} + \cdots,\end{aligned}$$

so f has removable singularity at 1. Hence, without any loss of generality, we can assume that $f(1) = \frac{1}{2}$. Thus,

$$f : (\mathbb{C} - (\{-ti : 0 \le t\} \cup \{-1\})) \to \mathbb{C}$$

is a holomorphic function. Let ε, ρ, R be any positive real numbers such that

$$-R < -1 - \varepsilon < -1 + \varepsilon < -\rho < \rho < R.$$

Let C_ε be the semicircle in the upper half-plane with center -1 and radius ε, oriented anticlockwise. Let S_ρ be the semicircle in the upper half-plane with center 0 and radius ρ, oriented anticlockwise. Let S_R be the semicircle in the upper half-plane with center 0 and radius R, oriented anticlockwise.

Let Γ be the closed 'indented' curve consisting of $[-R, -1-\varepsilon]$ followed by $-C_\varepsilon$ followed by $[-1+\varepsilon, -\rho]$ followed by $-S_\rho$ followed by $[\rho, R]$ followed by S_R.

Here,

$$f : z \mapsto \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{(z+1)(z-1)}$$

has no isolated singularity in the interior of Γ. By the Cauchy theorem,

$$\begin{aligned}
&\int\limits_{-R}^{-1-\varepsilon} \frac{\ln(-x) + i\pi}{x^2 - 1}\mathrm{d}x - \int\limits_{C_\varepsilon} f(z)\mathrm{d}z + \int\limits_{-1+\varepsilon}^{-\rho} \frac{\ln(-x) + i\pi}{x^2 - 1}\mathrm{d}x - \int\limits_{S_\rho} f(z)\mathrm{d}z \\
&\quad + \int\limits_{\rho}^{R} \frac{\ln x + i0}{x^2 - 1}\mathrm{d}x + \int\limits_{S_R} f(z)\mathrm{d}z = \int\limits_{[-R,-1-\varepsilon]} \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{z^2 - 1}\mathrm{d}z \\
&\quad + \int\limits_{-C_\varepsilon} f(z)\mathrm{d}z + \int\limits_{[-1+\varepsilon,-\rho]} \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{z^2 - 1}\mathrm{d}z \\
&\quad + \int\limits_{-S_\rho} f(z)\mathrm{d}z + \int\limits_{[\rho,R]} \frac{\ln|z| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}(z)}{z^2 - 1}\mathrm{d}z + \int\limits_{S_R} f(z)\mathrm{d}z = \underbrace{\int\limits_{\Gamma} f(z)\mathrm{d}z = 0}_{},
\end{aligned}$$

so

$$\begin{aligned}
&\int\limits_{C_\varepsilon} f(z)\mathrm{d}z + \int\limits_{S_\rho} f(z)\mathrm{d}z - \int\limits_{S_R} f(z)\mathrm{d}z = \int\limits_{-R}^{-1-\varepsilon} \frac{\ln(-x) + i\pi}{x^2 - 1}\mathrm{d}x \\
&\quad + \int\limits_{-1+\varepsilon}^{-\rho} \frac{\ln(-x) + i\pi}{x^2 - 1}\mathrm{d}x + \int\limits_{\rho}^{R} \frac{\ln x}{x^2 - 1}\mathrm{d}x,
\end{aligned}$$

and hence

$$\mathrm{Re}\left(\int_{C_\varepsilon} f(z)\mathrm{d}z\right) + \mathrm{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right) - \mathrm{Re}\left(\int_{S_R} f(z)\mathrm{d}z\right)$$
$$= \int_{-R}^{-1-\varepsilon} \frac{\ln(-x)}{x^2-1}\mathrm{d}x + \int_{-1+\varepsilon}^{-\rho} \frac{\ln(-x)}{x^2-1}\mathrm{d}x + \int_{\rho}^{R} \frac{\ln x}{x^2-1}\mathrm{d}x$$

Now, we shall try to show that

1. $\lim_{\rho\to 0}\left(\mathrm{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right)\right) = 0,$
2. $\lim_{R\to\infty}\left(\mathrm{Re}\left(\int_{S_R} f(z)\mathrm{d}z\right)\right) = 0.$

For 1: Here it suffices to show that

$$\lim_{\rho\to 0}\left|\mathrm{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right)\right| = 0.$$

Since

$$0 \le \left|\mathrm{Re}\left(\int_{S_\rho} f(z)\mathrm{d}z\right)\right| \le \left|\int_{S_\rho} f(z)\mathrm{d}z\right| = \left|\int_0^\pi f\left(\rho e^{i\theta}\right)\left(\rho i e^{i\theta}\right)\mathrm{d}\theta\right|$$
$$\le \int_0^\pi \left|f\left(\rho e^{i\theta}\right)\left(\rho i e^{i\theta}\right)\right|\mathrm{d}\theta = \int_0^\pi \left|f\left(\rho e^{i\theta}\right)\right|\rho\mathrm{d}\theta = \rho\int_0^\pi \left|\frac{\ln\left|\rho e^{i\theta}\right| + i\,\mathrm{Arg}_{\frac{-\pi}{2}}\left(\rho e^{i\theta}\right)}{\left(\rho e^{i\theta}\right)^2 - 1}\right|\mathrm{d}\theta$$
$$= \rho\int_0^\pi \left|\frac{\ln\rho + i\theta}{\rho^2 e^{i2\theta} - 1}\right|\mathrm{d}\theta = \rho\int_0^\pi \frac{|\ln\rho + i\theta|}{|\rho^2 e^{i2\theta} - 1|}\mathrm{d}\theta \le \rho\int_0^\pi \frac{|\ln\rho + i\theta|}{||\rho^2 e^{i2\theta}| - |1||}\mathrm{d}\theta = \rho\int_0^\pi \frac{|\ln\rho + i\theta|}{|\rho^2 - 1|}\mathrm{d}\theta$$
$$\le \rho\int_0^\pi \frac{\sqrt{(\ln\rho)^2 + \left(\frac{3\pi}{2}\right)^2}}{1-\rho^2}\mathrm{d}\theta = \rho\frac{\sqrt{(\ln\rho)^2 + \left(\frac{3\pi}{2}\right)^2}}{1-\rho^2}\pi = \frac{\sqrt{(\rho\ln\rho)^2 + \left(\frac{3\pi}{2}\rho\right)^2}}{1-\rho^2}\pi,$$

and

$$\lim_{\rho\to 0}\left(\frac{\sqrt{(\rho\ln\rho)^2 + \left(\frac{3\pi}{2}\rho\right)^2}}{1-\rho^2}\pi\right) = \frac{\sqrt{\left(\lim_{\rho\to 0}\left(\frac{\ln\rho}{\frac{1}{\rho}}\right)\right)^2 + (0)^2}}{1-0}\pi = \pi\lim_{\rho\to 0}\frac{\frac{1}{\rho}}{\frac{-1}{\rho^2}}$$
$$= -\pi\lim_{\rho\to 0}\rho = 0,$$

we have

$$\lim_{\rho\to 0}\left|\operatorname{Re}\left(\int\limits_{S_\rho} f(z)\mathrm{d}z\right)\right| = 0.$$

For 2: Here it suffices to show that

$$\lim_{R\to\infty}\left|\operatorname{Re}\left(\int\limits_{S_R} f(z)\mathrm{d}z\right)\right| = 0.$$

Since

$$\begin{aligned}
0 \le \left|\operatorname{Re}\left(\int\limits_{S_R} f(z)\mathrm{d}z\right)\right| &= \left|\int\limits_{S_R} f(z)\mathrm{d}z\right| = \left|\int\limits_0^\pi f\left(Re^{i\theta}\right)\left(Rie^{i\theta}\right)\mathrm{d}\theta\right| \\
&\le \int\limits_0^\pi \left|f\left(Re^{i\theta}\right)\left(Rie^{i\theta}\right)\right|\mathrm{d}\theta = \int\limits_0^\pi \left|f\left(Re^{i\theta}\right)\right|R\mathrm{d}\theta = R\int\limits_0^\pi \left|\frac{\ln\left|Re^{i\theta}\right| + i\operatorname{Arg}_{\frac{-\pi}{2}}\left(Re^{i\theta}\right)}{\left(Re^{i\theta}\right)^2 - 1}\right|\mathrm{d}\theta \\
&= R\int\limits_0^\pi \left|\frac{\ln R + i\theta}{R^2e^{i2\theta} - 1}\right|\mathrm{d}\theta \le R\int\limits_0^\pi \frac{|\ln R + i\theta|}{\left|\left|R^2e^{i2\theta}\right| - |1|\right|}\mathrm{d}\theta = R\int\limits_0^\pi \frac{|\ln R + i\theta|}{R^2 - 1}\mathrm{d}\theta \\
&\le R\int\limits_0^\pi \frac{\sqrt{(\ln R)^2 + \left(\frac{3\pi}{2}\right)^2}}{R^2 - 1}\mathrm{d}\theta = R\frac{\sqrt{(\ln R)^2 + \left(\frac{3\pi}{2}\right)^2}}{R^2 - 1}\pi = \frac{\sqrt{\left(\frac{\ln R}{R}\right)^2 + \left(\frac{3\pi}{2}\frac{1}{R}\right)^2}}{1 - \frac{1}{R^2}}\pi,
\end{aligned}$$

and

$$\lim_{R\to\infty}\left(\frac{\sqrt{\left(\frac{\ln R}{R}\right)^2 + \left(\frac{3\pi}{2}\frac{1}{R}\right)^2}}{1 - \frac{1}{R^2}}\pi\right) = \frac{\sqrt{(0)^2 + \left(\frac{3\pi}{2}\cdot 0\right)^2}}{(1 - 4\cdot 0)}\pi = 0,$$

so

$$\lim_{R\to\infty}\left|\operatorname{Re}\left(\int\limits_{S_R} f(z)\mathrm{d}z\right)\right| = 0.$$

Hence,

$$\operatorname{Re}\left(\int_{C_\varepsilon} f(z)\mathrm{d}z\right) + 0 - 0 = \int_{-\infty}^{-1-\varepsilon} \frac{\ln(-x)}{x^2-1}\mathrm{d}x + \int_{-1+\varepsilon}^{0} \frac{\ln(-x)}{x^2-1}\mathrm{d}x + \int_{0}^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x.$$

Clearly,

$$f : z \mapsto \frac{\ln|z| + i\operatorname{Arg}_{\frac{-\pi}{2}}(z)}{(z+1)(z-1)}$$

has a simple pole at -1, and

$$\operatorname{Res}(f;-1) = \left.\frac{\ln|z| + i\operatorname{Arg}_{\frac{-\pi}{2}}(z)}{z-1}\right|_{z=-1} = \frac{0+i\pi}{-2} = -i\frac{\pi}{2}.$$

Now, by Conclusion 4.67,

$$\underbrace{\lim_{\varepsilon\to 0}\left(\int_{C_\varepsilon} f(z)\mathrm{d}z\right) = \left(-i\frac{\pi}{2}\right)\pi i}_{} = \frac{\pi^2}{2},$$

and hence

$$\lim_{\varepsilon\to 0}\left(\operatorname{Re}\left(\int_{C_\varepsilon} f(z)\mathrm{d}z\right)\right) = \frac{\pi^2}{2}.$$

Hence,

$$\frac{\pi^2}{2} = \int_{-\infty}^{-1} \frac{\ln(-x)}{x^2-1}\mathrm{d}x + \int_{-1}^{0} \frac{\ln(-x)}{x^2-1}\mathrm{d}x + \int_{0}^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x = \int_{-\infty}^{0} \frac{\ln(-x)}{x^2-1}\mathrm{d}x + \int_{0}^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x$$
$$= \int_{\infty}^{0} \frac{\ln x}{(-x)^2-1}(-1)\mathrm{d}x + \int_{0}^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x = 2\int_{0}^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x.$$

Thus,

$$\int_{0}^{\infty} \frac{\ln x}{x^2-1}\mathrm{d}x = \frac{\pi^2}{4}.$$

Example 4.75 Let $b \in (1, \infty)$. Recall that for every $x \in (0, \infty), x^b$ means $e^{b \ln x}(> 0)$. We shall try to evaluate

$$\int_0^\infty \frac{1}{1+x^b} \mathrm{d}x.$$

Problem 4.76 $\int_0^\infty \frac{1}{1+x^b} \mathrm{d}x$ is convergent.

(Solution Since for every positive $R > 1$, we have

$$\begin{aligned}
\int_0^R \frac{1}{1+x^b} \mathrm{d}x &= \int_0^1 \frac{1}{1+x^b} \mathrm{d}x + \int_1^R \frac{1}{1+x^b} \mathrm{d}x \leq \int_0^1 1 \mathrm{d}x + \int_1^R \frac{1}{1+x^b} \mathrm{d}x \\
&\leq 1 + \int_1^R \frac{1}{x^b} \mathrm{d}x = 1 + \frac{1}{-b+1} \frac{1}{x^{b-1}} \bigg|_1^R = 1 + \frac{1}{-b+1} \left(\frac{1}{R^{b-1}} - 1 \right) \\
&= 1 + \frac{1}{b-1} \left(1 - \frac{1}{R^{b-1}} \right) \leq 1 + \frac{1}{b-1} = \frac{b}{b-1},
\end{aligned}$$

$\int_0^\infty \frac{1}{1+x^b} \mathrm{d}x$ is convergent. ∎)

Prior knowledge of convergence will not be used below. Observe that the integrand $\frac{1}{1+x^b}$ in our example coincides with the function

$$\begin{aligned}
\underbrace{f : z \mapsto \frac{1}{1 + e^{b \log_{\frac{\pi}{b}-\pi}(z)}}} &= \frac{1}{1 + e^{b\left(\ln|z| + i \operatorname{Arg}_{\frac{\pi}{b}-\pi}(z)\right)}} = \frac{1}{1 + e^{b \ln|z| + ib \operatorname{Arg}_{\frac{\pi}{b}-\pi}(z)}} \\
&= \frac{1}{1 + e^{b \ln|z|} \cdot e^{ib \operatorname{Arg}_{\frac{\pi}{b}-\pi}(z)}}
\end{aligned}$$

on the positive real axis. Clearly, f is holomorphic in the region

$$\begin{aligned}
\underbrace{\left(\mathbb{C} - \left(L\left(0; e^{i\left(\frac{\pi}{b}-\pi\right)}\right) \cup \left\{ e^{i\frac{\pi}{b}} \right\} \right) \right)} &= \mathbb{C} - \left(\left\{ 0 + t\left(e^{i\left(\frac{\pi}{b}-\pi\right)} \right) : 0 \leq t \right\} \cup \left\{ e^{i\frac{\pi}{b}} \right\} \right) \\
&= \mathbb{C} - \left(\left\{ -te^{i\frac{\pi}{b}} : 0 \leq t \right\} \cup \left\{ e^{i\frac{\pi}{b}} \right\} \right)
\end{aligned}$$

to $\mathbb{C}$. Thus,

$$f : \left(\mathbb{C} - \left(\left\{ -te^{i\frac{\pi}{b}} : 0 \leq t \right\} \cup \left\{ e^{i\frac{\pi}{b}} \right\} \right) \right) \to \mathbb{C}$$

is a holomorphic function in

$$\left(\mathbb{C} - \left(\left\{-te^{i\frac{\pi}{b}} : 0 \le t\right\} \cup \left\{e^{i\frac{\pi}{b}}\right\}\right)\right).$$

Since $\frac{\pi}{b} \in \left(\frac{\pi}{b} - \pi, \frac{\pi}{b} + \pi\right), e^{i\frac{\pi}{b}}$ is the only isolated singularity of f. Since

$$\left.\frac{d\left(1+z^b\right)}{\mathrm{d}z}\right|_{z=e^{i\frac{\pi}{b}}} = b\left(e^{i\frac{\pi}{b}}\right)^{b-1} = be^{i\frac{\pi}{b}(b-1)} \neq 0,$$

$e^{i\frac{\pi}{b}}$ is a simple zero of $z \mapsto \left(1+z^b\right)$. Hence, by Conclusion 4.36,

$$\mathrm{Res}\left(f; e^{i\frac{\pi}{b}}\right) = \frac{1}{0 + b\left(e^{i\frac{\pi}{b}}\right)^{b-1}} = \frac{1}{be^{i\frac{\pi}{b}(b-1)}} = \frac{1}{b}e^{-i\frac{\pi}{b}(b-1)}.$$

For every $r > 0$, let A_r be the circular arc

$$\left\{re^{i\theta} : \theta \in \left(0, 2\frac{\pi}{b}\right)\right\},$$

oriented anticlockwise.

Let ρ, R be any real numbers such that $0 < \rho < \left|e^{i\frac{\pi}{b}}\right| < R$. Let $\Gamma_{\rho,R}$ be the closed curve consisting of $[\rho, R]$ followed by A_R followed by $\left[Re^{i\frac{2\pi}{b}}, \rho e^{i\frac{2\pi}{b}}\right]$ followed by $-A_\rho$.

Since $e^{i\frac{\pi}{b}}$ is the only isolated singularity of f in the interior of $\Gamma_{\rho,R}$, by the residue theorem,

$$\begin{aligned}
&\left(\left(1 - \cos\frac{2\pi}{b}\right) - i\sin\frac{2\pi}{b}\right)\int_\rho^R \frac{1}{1+x^b}\,\mathrm{d}x + \int_{A_R} f(z)\mathrm{d}z - \int_{A_\rho} f(z)\mathrm{d}z \\
&= \left(1 - e^{i\frac{2\pi}{b}}\right)\int_\rho^R \frac{1}{1+x^b}\,\mathrm{d}x + \int_{A_R} f(z)\mathrm{d}z - \int_{A_\rho} f(z)\mathrm{d}z \\
&= \int_\rho^R \frac{1}{1+x^b}\,\mathrm{d}x + \int_{A_R} f(z)\mathrm{d}z + e^{i\frac{2\pi}{b}}\int_R^\rho \frac{1}{1+x^b}\,\mathrm{d}x - \int_{A_\rho} f(z)\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\rho}^{R} \frac{1}{1+x^b}\,\mathrm{d}x + \int_{A_R} f(z)\mathrm{d}z + e^{i\frac{2\pi}{b}} \int_0^1 \frac{1}{1+(R+(\rho-R)t)^b}(\rho-R)\mathrm{d}t - \int_{A_\rho} f(z)\mathrm{d}z \\
&= \int_{\rho}^{R} \frac{1}{1+x^b}\,\mathrm{d}x + \int_{A_R} f(z)\mathrm{d}z + \int_0^1 \frac{1}{1+\left((1-t)Re^{i\frac{2\pi}{b}}+t\rho e^{i\frac{2\pi}{b}}\right)^b}(\rho-R)e^{i\frac{2\pi}{b}}\mathrm{d}t \\
&\quad - \int_{A_\rho} f(z)\mathrm{d}z = \int_{[\rho,R]} f(z)\mathrm{d}z + \int_{A_R} f(z)\mathrm{d}z + \int_{\left[Re^{i\frac{2\pi}{b}},\rho e^{i\frac{2\pi}{b}}\right]} f(z)\mathrm{d}z + \int_{-A_\rho} f(z)\mathrm{d}z \\
&= \underbrace{\int_{\Gamma_{\rho,R}} f(z)\mathrm{d}z = 2\pi i\left(\mathrm{Res}\left(f; e^{i\frac{\pi}{b}}\right)\right)} = 2\pi i \cdot \frac{1}{b} e^{-i\frac{\pi}{b}(b-1)} \\
&= 2\pi i \cdot \frac{1}{b} e^{-i\pi} e^{i\frac{\pi}{b}} = 2\pi i \cdot \frac{1}{b}(-1)\left(\cos\frac{\pi}{b} + i\sin\frac{\pi}{b}\right) \\
&= \frac{2\pi}{b}\left(-i\cos\frac{\pi}{b} + \sin\frac{\pi}{b}\right),
\end{aligned}
$$

so

$$
\left(1-\cos\frac{2\pi}{b}\right)\int_{\rho}^{R} \frac{1}{1+x^b}\,\mathrm{d}x + \mathrm{Re}\left(\int_{A_R} f(z)\mathrm{d}z\right) - \mathrm{Re}\left(\int_{A_\rho} f(z)\mathrm{d}z\right) = \frac{2\pi}{b}\sin\frac{\pi}{b}.
$$

Now, we shall try to show that

1. $\lim_{\rho\to 0}\left(\mathrm{Re}\left(\int_{A_\rho} f(z)\mathrm{d}z\right)\right) = 0,$
2. $\lim_{R\to\infty}\left(\mathrm{Re}\left(\int_{A_R} f(z)\mathrm{d}z\right)\right) = 0.$

For 1: Here it suffices to show that

$$
\lim_{\rho\to 0}\left|\mathrm{Re}\left(\int_{A_\rho} f(z)\mathrm{d}z\right)\right| = 0.
$$

Since

$$0 \le \left|\operatorname{Re}\left(\int_{A_\rho} f(z)\mathrm{d}z\right)\right| \le \left|\int_{A_\rho} f(z)\mathrm{d}z\right| = \left|\int_0^{\frac{2\pi}{b}} f\left(\rho e^{i\theta}\right)\left(\rho i e^{i\theta}\right)\mathrm{d}\theta\right|$$

$$\le \int_0^{\frac{2\pi}{b}} \left|f\left(\rho e^{i\theta}\right)\left(\rho i e^{i\theta}\right)\right|\mathrm{d}\theta = \int_0^{\frac{2\pi}{b}} \left|f\left(\rho e^{i\theta}\right)\right|\rho\mathrm{d}\theta = \rho\int_0^{\frac{2\pi}{b}} \left|\frac{1}{1+e^{b\ln|\rho e^{i\theta}|}\cdot e^{ib\operatorname{Arg}_{\frac{\pi}{b}-\pi}\left(\rho e^{i\theta}\right)}}\right|\mathrm{d}\theta$$

$$= \rho\int_0^{\frac{2\pi}{b}} \left|\frac{1}{1+\rho^b\cdot e^{ib\theta}}\right|\mathrm{d}\theta \le \rho\int_0^{\frac{2\pi}{b}} \frac{1}{||1|-|-\rho^b\cdot e^{ib\theta}||}\mathrm{d}\theta = \rho\int_0^{\frac{2\pi}{b}} \frac{1}{|1-\rho^b|}\mathrm{d}\theta$$

$$= \rho\frac{1}{1-\rho^b}\frac{2\pi}{b} = \frac{2\pi}{b}\frac{\rho}{1-\rho^b},$$

and

$$\lim_{\rho\to 0}\left(\frac{2\pi}{b}\frac{\rho}{1-\rho^b}\right) = \frac{2\pi}{b}\frac{0}{1-0} = 0,$$

we have

$$\lim_{\rho\to 0}\left|\operatorname{Re}\left(\int_{A_\rho} f(z)\mathrm{d}z\right)\right| = 0.$$

For 2: Here it suffices to show that

$$\lim_{R\to\infty}\left|\operatorname{Re}\left(\int_{A_R} f(z)\mathrm{d}z\right)\right| = 0.$$

Since

$$0 \le \left|\operatorname{Re}\left(\int_{A_R} f(z)\mathrm{d}z\right)\right| \le \left|\int_{A_R} f(z)\mathrm{d}z\right| = \left|\int_0^{\frac{2\pi}{b}} f\left(Re^{i\theta}\right)\left(Rie^{i\theta}\right)\mathrm{d}\theta\right|$$

$$\le \int_0^{\frac{2\pi}{b}} \left|f\left(Re^{i\theta}\right)\left(Rie^{i\theta}\right)\right|\mathrm{d}\theta = \int_0^{\frac{2\pi}{b}} \left|f\left(Re^{i\theta}\right)\right|R\mathrm{d}\theta = R\int_0^{\frac{2\pi}{b}} \left|\frac{1}{1+e^{b\ln|Re^{i\theta}|}\cdot e^{ib\operatorname{Arg}_{\frac{\pi}{b}-\pi}\left(Re^{i\theta}\right)}}\right|\mathrm{d}\theta$$

$$= R\int_0^{\frac{2\pi}{b}} \left|\frac{1}{1+R^b\cdot e^{ib\theta}}\right|\mathrm{d}\theta \le R\int_0^{\frac{2\pi}{b}} \frac{1}{||1|-|-R^b\cdot e^{ib\theta}||}\mathrm{d}\theta = R\int_0^{\frac{2\pi}{b}} \frac{1}{|R^b-1|}\mathrm{d}\theta$$

$$= R\frac{1}{R^b-1}\frac{2\pi}{b} = \frac{2\pi}{b}\frac{\frac{1}{R^{b-1}}}{1-\frac{1}{R^b}},$$

and

$$\lim_{R\to\infty}\left(\frac{2\pi}{b}\frac{\frac{1}{R^{b-1}}}{1-\frac{1}{R^b}}\right)=\frac{2\pi}{b}\frac{0}{1-0}=0,$$

we have

$$\lim_{R\to\infty}\left|\mathrm{Re}\left(\int\limits_{A_R} f(z)\mathrm{d}z\right)\right|=0.$$

It follows that

$$2\left(\sin^2\frac{\pi}{b}\right)\int\limits_0^\infty \frac{1}{1+x^b}\mathrm{d}x=\underbrace{\left(1-\cos\frac{2\pi}{b}\right)\int\limits_0^\infty \frac{1}{1+x^b}\mathrm{d}x+0-0=\frac{2\pi}{b}\sin\frac{\pi}{b},}$$

and hence

$$\int\limits_0^\infty \frac{1}{1+x^b}\mathrm{d}x=\frac{\pi}{b}\frac{1}{\sin\frac{\pi}{b}}.$$

Example 4.77 Let $a\in(0,1)$. Recall that for every $x\in(0,\infty), x^{-a}$ means $e^{-a\ln x}$ (>0). We shall try to evaluate

$$\int\limits_0^\infty \frac{x^{-a}}{1+x}\mathrm{d}x.$$

Let us substitute $y\equiv x^{1-a}$. Now,

$$\int\limits_0^\infty \frac{x^{-a}}{1+x}\mathrm{d}x=\int\limits_0^\infty \frac{\left(y^{\frac{1}{1-a}}\right)^{-a}}{1+y^{\frac{1}{1-a}}}\left(\frac{1}{1-a}y^{\frac{1}{1-a}-1}\right)\mathrm{d}y=\frac{1}{1-a}\int\limits_0^\infty \frac{y^{\frac{-a}{1-a}}}{1+y^{\frac{1}{1-a}}}y^{\frac{a}{1-a}}\mathrm{d}y$$
$$=\frac{1}{1-a}\int\limits_0^\infty \frac{1}{1+y^{\frac{1}{1-a}}}\mathrm{d}y=\frac{1}{1-a}\int\limits_0^\infty \frac{1}{1+x^b}\mathrm{d}x,$$

where $b\equiv\frac{1}{1-a}\in(1,\infty)$. Next, on using the result of Example 4.76, we have

$$\underbrace{\int_0^\infty \frac{x^{-a}}{1+x}\mathrm{d}x = \frac{1}{1-a}\frac{\pi}{b}\frac{1}{\sin\frac{\pi}{b}}} = \frac{1}{1-a}\frac{\pi}{\frac{1}{1-a}}\frac{1}{\sin\pi(1-a)} = \pi\frac{1}{\sin(\pi-\pi a)} = \frac{\pi}{\sin\pi a},$$

and hence

$$\int_0^\infty \frac{x^{-a}}{1+x}\mathrm{d}x = \frac{\pi}{\sin\pi a}.$$

Example 4.78 Taking into account multiplicities, let us find the number of zeros of polynomial

$$z^7 + 4z^4 + z^3 + 1$$

in the annulus $\{z : 1 < |z| < 2\}$.

Since for every complex number z satisfying $|z| = 1$,

$$\left|4z^4 - \left(z^7 + 4z^4 + z^3 + 1\right)\right| < \left|4z^4\right|,$$

and the function $z \mapsto 4z^4$ has 4 zeros inside $\{z : |z| < 1\}$, by Theorem 1.229, the number of zeros of polynomial $z^7 + 4z^4 + z^3 + 1$ inside $\{z : |z| < 1\}$ is 4.

Since for every complex number z satisfying $|z| = 2$,

$$\left|z^7 - \left(z^7 + 4z^4 + z^3 + 1\right)\right| < \left|z^7\right|,$$

and the function $z \mapsto z^7$ has 7 zeros inside $\{z : |z| < 2\}$, by Theorem 1.229, the number of zeros of polynomial $z^7 + 4z^4 + z^3 + 1$ inside $\{z : |z| < 2\}$ is 4. Thus, all the 7 zeros of $z^7 + 4z^4 + z^3 + 1$ are inside $\{z : |z| < 2\}$. Now, since the number of zeros of polynomial $z^7 + 4z^4 + z^3 + 1$ inside $\{z : |z| < 1\}$ is 4, the number of zeros of polynomial $z^7 + 4z^4 + z^3 + 1$ inside $\{z : 1 < |z| < 2\}$ is $(7 - 4)$ $(= 3)$. Thus, the number of zeros of polynomial $z^7 + 4z^4 + z^3 + 1$ inside $\{z : 1 < |z| < 2\}$ is 3.

4.4 Meromorphic Function as a Quotient of Holomorphic Functions

Note 4.79 Let Ω be a nonempty open subset of $\mathbb{S}^2 (= \mathbb{C} \cup \{\infty\})$. Let $\Omega \neq \mathbb{S}^2$. Let A be a nonempty finite subset of Ω. Corresponding to each $\alpha \in A$, suppose that $m(\alpha)$ is a positive integer. Let $\infty \in \Omega$, and $\infty \notin A$.
Since $\Omega \neq \mathbb{S}^2$, there exists $\beta \in \left(\mathbb{S}^2 - \Omega\right)$.

For simplicity, let $A \equiv \{\alpha_1, \alpha_2\}$, where α_1, α_2 are distinct complex numbers. Let

$$f : z \mapsto \begin{cases} \frac{(z-\alpha_1)^{m(\alpha_1)}(z-\alpha_2)^{m(\alpha_2)}}{(z-\beta)^{m(\alpha_1)+m(\alpha_2)}} & \text{if } z \in (\Omega - \{\infty\}) \\ 1 & \text{if } z = \infty, \end{cases}$$

be a function from Ω to $\mathbb{C}$.

Problem 4.80 For every $z \in (\Omega - \{\infty\})$, $f'(z)$ exists. Also, $f'(\infty)$ exists.

(Solution Let $\hat{f} : z \mapsto f\left(\frac{1}{z}\right)$ be a function. Since

$$\begin{aligned}\hat{f}(z) &= f\left(\frac{1}{z}\right) = \frac{\left(\frac{1}{z} - \alpha_1\right)^{m(\alpha_1)}\left(\frac{1}{z} - \alpha_2\right)^{m(\alpha_2)}}{\left(\frac{1}{z} - \beta\right)^{m(\alpha_1)+m(\alpha_2)}} = \frac{(1-\alpha_1 z)^{m(\alpha_1)}(1-\alpha_2 z)^{m(\alpha_2)}}{(1-\beta z)^{m(\alpha_1)+m(\alpha_2)}} \\ &= \frac{\left(1 - m(\alpha_1)(\alpha_1 z) + (\cdots)(\alpha_1 z)^2 - \cdots\right)\left(1 - m(\alpha_2)(\alpha_2 z) + (\cdots)(\alpha_2 z)^2 - \cdots\right)}{1 - (m(\alpha_1) + m(\alpha_2))(\beta z) + (\cdots)(\beta z)^2 - \cdots} \\ &= 1 + (\cdots)z + (\cdots)z^2 + \cdots,\end{aligned}$$

0 is a removable singularity of $\hat{f}$, and hence $f'(\infty)$ exists. ∎)

This shows that $f \in H(\Omega)$. It is clear that $f^{-1}(0) = \{\alpha_1, \alpha_2\}\ (= A)$ and, at each $\alpha \in A$, f has a zero of order $m(\alpha)$.

Conclusion 4.81 Let Ω be a nonempty open subset of $\mathbb{S}^2\ (= \mathbb{C} \cup \{\infty\})$. Let $\Omega \neq \mathbb{S}^2$. Let A be a nonempty finite subset of Ω. Corresponding to each $\alpha \in A$, suppose that $m(\alpha)$ is a positive integer. Let $\infty \in \Omega$, and $\infty \notin A$. Then there exists $h \in H(\Omega)$ such that

1. $h^{-1}(0) = A$,
2. at each $\alpha \in A$, h has a zero of order $m(\alpha)$.

Note 4.82 Let Ω be a nonempty open subset of $\mathbb{S}^2\ (= \mathbb{C} \cup \{\infty\})$. Let $\Omega \neq \mathbb{S}^2$. Let A be an infinite countable subset of Ω. Suppose that A has no limit point in Ω. Corresponding to each $\alpha \in A$, suppose that $m(\alpha)$ is a positive integer.

Let us consider the case when $\infty \in \Omega$, and $\infty \notin A$.

Let $\{\alpha_n\}$ be a sequence whose terms are in A, and in which each $\alpha \in A$ is listed precisely $m(\alpha)$ times. Since Ω is an open subset of $\mathbb{S}^2\ (= \mathbb{C} \cup \{\infty\})$, $\Omega \neq \mathbb{S}^2$, and $\mathbb{S}^2$ is a compact set, $\left(\mathbb{S}^2 - \Omega\right)$ is a closed subset of compact set $\mathbb{S}^2$, and hence $\left(\mathbb{S}^2 - \Omega\right)$ is a compact subset of $\mathbb{S}^2$.

Since $\left(\mathbb{S}^2 - \Omega\right)$ is a compact subset of $\mathbb{S}^2$, $\{\alpha_n\}$ is a sequence in A, $A \subset \Omega$, and, for every positive integer n, the function $\beta \mapsto |\beta - \alpha_n|$ from $\left(\mathbb{S}^2 - \Omega\right)$ to $(0, \infty)$ is continuous, for every positive integer n, there exists $\beta_n \in \left(\mathbb{S}^2 - \Omega\right)$ such that

$$\underbrace{\beta \in \left(\mathbb{S}^2 - \Omega\right)}\ (= (\mathbb{C} - \Omega)) \Rightarrow 0 < |\beta_n - \alpha_n| \leq |\beta - \alpha_n|.$$

Problem 4.83 $\lim_{n\to\infty}|\beta_n - \alpha_n| = 0$.

(**Solution** If not, otherwise, suppose that there exists $\varepsilon > 0$ such that for every positive integer m, there exists a positive integer $n \geq m$ satisfying $\varepsilon \leq |\beta_n - \alpha_n|$. We have to arrive at a contradiction.

It follows that $\{\alpha_n : \varepsilon \leq |\beta_n - \alpha_n|\}$ is an infinite set. Thus, there exists an 'infinite' subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that for every positive integer k, we have $\varepsilon \leq \left|\beta_{n_k} - \alpha_{n_k}\right|$.

Since A has no limit point in Ω, and $\infty \in \Omega$, ∞ is not a limit point of A. Since ∞ is not a limit point of A, and $\infty \notin A$, there exists R such that $\{z : R < |z|\}$ contains no point of A, and hence $A \subset D[0; R]$. Thus, A is an infinite bounded set of complex numbers. It follows that $\{\alpha_{n_k} : k = 1, 2, 3, \ldots\}$ is an infinite bounded set of complex numbers. Now, by the Weierstrass theorem, there exists $z_0 \in \mathbb{C}$ such that z_0 is a limit point of $\{\alpha_{n_k} : k = 1, 2, 3, \ldots\}$. Hence, there exists a subsequence $\left\{\alpha_{n_{k_l}}\right\}$ of $\{\alpha_{n_k}\}$ such that $\lim_{l\to\infty} \alpha_{n_{k_l}} = z_0$. Since z_0 is a limit point of $\{\alpha_{n_k} : k = 1, 2, 3, \ldots\}$ $(\subset A)$, z_0 is a limit point of A, and hence $z_0 \notin \Omega$. Thus, $z_0 \in \left(\mathbb{S}^2 - \Omega\right)$. It follows that for every positive integer l,

$$\varepsilon \leq \left|\beta_{n_{k_l}} - \alpha_{n_{k_l}}\right| \leq \left|z_0 - \alpha_{n_{k_l}}\right|.$$

Since for every positive integer l, $\varepsilon \leq \left|z_0 - \alpha_{n_{k_l}}\right|$, $\lim_{l\to\infty} \alpha_{n_{k_l}} = z_0$ is false. This is a contradiction. ∎)

Observe that for every $z \in (\Omega - \{\infty\})$, and for every positive integer n, $z - \beta_n \neq 0$, because each $\beta_n \in \left(\mathbb{S}^2 - \Omega\right)$ $(= (\mathbb{C} - \Omega))$. Thus, for every positive integer n, the function

$$f_n : z \mapsto E_n\left(\frac{\alpha_n - \beta_n}{z - \beta_n}\right)$$

from $(\Omega - \{\infty\})$ to $\mathbb{C}$ is a member of $H(\Omega - \{\infty\})$, where E_n denotes the elementary factor. Since for every positive integer n, 1 is the only zero of E_n, no f_n is identically 0 on any region of $(\Omega - \{\infty\})$. Also, for every positive integer n, we have $(f_n)^{-1}(0) = \{\alpha_n\}$.

Problem 4.84 $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on compact subsets of $(\Omega - \{\infty\})$.

(Solution For this purpose, let us take any nonempty compact subset K of $(\Omega - \{\infty\})$. Since $K, (\mathbb{C} - \Omega)$ are compact subsets of $\mathbb{C}$, and $K \cap (\mathbb{C} - \Omega) = \emptyset$, we have

$$\inf\{|w - z| : z \in K \text{ and } w \in (\mathbb{C} - \Omega)\} > 0.$$

Now, since $\lim_{n\to\infty} |\beta_n - \alpha_n| = 0$, there exists a positive integer N such that for every $n \geq N$, we have

$$\underbrace{|\beta_n - \alpha_n| < \frac{1}{2}\inf\{|w - z| : z \in K \text{ and } w \in (\mathbb{C} - \Omega)\}}$$
$$= \inf\left\{\frac{1}{2}|w - z| : z \in K \text{ and } w \in (\mathbb{C} - \Omega)\right\} \leq \frac{1}{2}|\beta_n - z|$$

for every $z \in K$. Thus, for every $n \geq N$, and for every $z \in K$, we have $\left|\frac{\beta_n - \alpha_n}{\beta_n - z}\right| \leq \frac{1}{2}$. Now, by Conclusion 2.42, for every $n \geq N$, and for every $z \in K$, we have

$$|f_n(z) - 1| = \underbrace{\left|E_n\left(\frac{\beta_n - \alpha_n}{\beta_n - z}\right) - 1\right| \leq \left|\frac{\beta_n - \alpha_n}{\beta_n - z}\right|^{n+1}} \leq \left(\frac{1}{2}\right)^{n+1}.$$

Thus, for every $n \geq N$, and for every $z \in K$, we have $|f_n(z) - 1| \leq \left(\frac{1}{2}\right)^{n+1}$. Next, since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+1}$ is convergent, by the Weierstrass M-test, $\sum_{n=1}^{\infty} |f_n - 1|$ converges uniformly on K. ∎)

Hence, by Conclusion 2.38,

1. $\left(\prod_{n=1}^{\infty} f_n\right) \in H(\Omega - \{\infty\})$,
2. $\left(\prod_{n=1}^{\infty} f_n\right)^{-1}(0) = A$,
3. at each $\alpha \in A$, $\left(\prod_{n=1}^{\infty} f_n\right)$ has a zero of order $m(\alpha)$.

Since $\left(\prod_{n=1}^{\infty} f_n\right)$ has a removable singularity at ∞, $\left(\prod_{n=1}^{\infty} f_n\right)$ can be extended to $h : \Omega \to \mathbb{C}$ such that $h \in H(\Omega)$. Also,

1. $h^{-1}(0) = A$,
2. at each $\alpha \in A$, h has a zero of order $m(\alpha)$.

Conclusion 4.85 Let Ω be a nonempty open subset of $\mathbb{S}^2$ $(= \mathbb{C} \cup \{\infty\})$. Let $\Omega \neq \mathbb{S}^2$. Let A be an infinite countable subset of Ω. Suppose that A has no limit point in Ω. Corresponding to each $\alpha \in A$, suppose that $m(\alpha)$ is a positive integer. Then there exists $h \in H(\Omega)$ such that

1. $h^{-1}(0) = A$,
2. at each $\alpha \in A$, h has a zero of order $m(\alpha)$.

Note 4.86 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a meromorphic function.

It follows that there exists a subset A of Ω such that

1. A has no limit point in Ω,
2. for every $z \in (\Omega - A), f'(z)$ exists,
3. for every $\alpha \in A$, f has a pole at α.

By Conclusion 1.219, A is countable. For every $\alpha \in A$, let $m(\alpha)$ be the order of the pole of f at α. By Conclusion 4.81, and Conclusion 4.85, there exists $h \in H(\Omega)$ such that

1. $h^{-1}(0) = A$,
2. at each $\alpha \in A$, h has a zero of order $m(\alpha)$.

Since for every $\alpha \in A$, f has a pole at α of order $m(\alpha)$, and h has a zero at α of order $m(\alpha), f \cdot h$ has a removable singularity at each $\alpha \in A$, and hence $f \cdot h$ can be extended to $g : \Omega \to \mathbb{C}$ such that $g \in H(\Omega)$. Thus, $g \in H(\Omega), h \in H(\Omega)$, and $f = \frac{g}{h}$ on $(\Omega - A)$.

Conclusion 4.87 Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ be a meromorphic function. Then there exist $g, h \in H(\Omega)$ such that $f = \frac{g}{h}$ on $(\Omega - A)$.

4.5 Euler's Gamma Function

Note 4.88

Problem 4.89 For every $x \in (0, \infty)$,

$$\int_0^\infty t^{x-1} e^{-t} \mathrm{d}t < \infty.$$

(Solution Let us fix any $x \in (0, \infty)$. Let us choose a positive integer N such that $x < N$. Since for every $t \in (0, \infty)$,

$$t^{x-1}e^{-t} = t^{x-1}\frac{1}{e^t} = t^{x-1}\frac{1}{1 + t + \frac{t^2}{2!} + \cdots} \le t^{x-1}\frac{1}{\frac{t^N}{N!}} = N!t^{(x-N)-1},$$

we have

$$\int_1^\infty t^{x-1}e^{-t}\mathrm{d}t \le \int_1^\infty N!t^{(x-N)-1}\mathrm{d}t = N!\frac{1}{(x-N)}\frac{1}{t^{(N-x)}}\Big|_{t=1}^{t=\infty}$$
$$= N!\frac{1}{(x-N)}\left(\lim_{t\to\infty}\frac{1}{t^{(N-x)}} - \frac{1}{1^{(N-x)}}\right)$$
$$= N!\frac{1}{(x-N)}(0-1) = \frac{N!}{N-x} < \infty.$$

It suffices to show that $\int_0^1 t^{x-1}e^{-t}\mathrm{d}t < \infty$. Since

$$\int_0^1 t^{x-1}e^{-t}\mathrm{d}t \le \int_0^1 t^{x-1}\cdot 1\mathrm{d}t = \frac{1}{x}t^x\big|_{t=0}^{t=1} = \frac{1}{x}(1^x - 0^x) = \frac{1}{x} < \infty,$$

we have $\int_0^1 t^{x-1}e^{-t}\mathrm{d}t < \infty$. ∎)

Since for every $x, t \in (0,\infty)$,

$$t^{x-1}e^{-t} = e^{(x-1)\ln t}e^{-t} = e^{(x-1)(\ln t)-t} > 0,$$

by Lemma 1.151, Vol. 1, for every $x \in (0,\infty)$, we have $\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t \ne 0$. Thus, for every $x \in (0,\infty)$, $\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ is a positive real number.

Definition For every $x \in (0,\infty)$, the positive real number $\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ is denoted by $\Gamma(x)$. Thus, $\Gamma : (0,\infty) \to (0,\infty)$.

Let us fix any $x \in (0,\infty)$. It follows that $(x+1) \in (0,\infty)$, and hence

$$\Gamma(x+1) = \int_0^\infty t^{(x+1)-1}e^{-t}\mathrm{d}t = \int_0^\infty t^x e^{-t}\mathrm{d}t = \left(t^x\frac{e^{-t}}{-1}\right)\Big|_{t=0}^{t=\infty} - \int_0^\infty \left(xt^{x-1}\right)\left(\frac{e^{-t}}{-1}\right)\mathrm{d}t$$
$$= -\left(\lim_{t\to\infty}\frac{t^x}{e^t} - \frac{0^x}{e^0}\right) + x\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t = -\left(\lim_{t\to\infty}\frac{t^x}{e^t} - 0\right) + x\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$$
$$= -(0-0) + x\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t = x\,\Gamma(x).$$

Conclusion 4.90 For every $x \in (0, \infty), \Gamma(x+1) = x\,\Gamma(x)$.

Since

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} \mathrm{d}t = \int_0^\infty e^{-t} \mathrm{d}t = \left(\frac{e^{-t}}{-1} \right) \Big|_{t=0}^{t=\infty}$$
$$= -\left(\lim_{t \to \infty} \frac{1}{e^t} - \frac{1}{e^0} \right) = -(0-1) = 1,$$

we have $\Gamma(1) = 1$. Now, by Conclusion 4.90, $\Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1!$. Next, $\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2!$. Similarly, $\Gamma(4) = 3!$, etc.

Conclusion 4.91

1. $\Gamma(1) = 1$,
2. For every positive integer n, $\Gamma(n+1) = n!$.

Since $\Gamma : (0, \infty) \to (0, \infty)$, and $\ln : (0, \infty) \to \mathbb{R}$, we have $(\ln \circ \Gamma) : (0, \infty) \to \mathbb{R}$.

Problem 4.92 $(\ln \circ \Gamma) : (0, \infty) \to \mathbb{R}$ is convex.

(Solution For this purpose, let us fix any $\lambda \in [0, 1]$, and $x, y \in (0, \infty)$. We have to show that

$$\underbrace{\ln(\Gamma((1-\lambda)x + \lambda y)) \leq (1-\lambda)(\ln(\Gamma(x))) + \lambda(\ln(\Gamma(y)))} = \ln\left((\Gamma(x))^{(1-\lambda)} \right) + \ln\left((\Gamma(y))^{\lambda} \right)$$
$$= \ln\left((\Gamma(x))^{(1-\lambda)} \cdot (\Gamma(y))^{\lambda} \right),$$

that is

$$\Gamma((1-\lambda)x + \lambda y) \leq (\Gamma(x))^{(1-\lambda)} \cdot (\Gamma(y))^{\lambda}.$$

Observe that

$$\int_0^\infty \left(t^{(x-1)} e^{-t} \right)^{(1-\lambda)} \cdot \left(t^{(y-1)} e^{-t} \right)^{\lambda} \mathrm{d}t = \int_0^\infty t^{(1-\lambda)(x-1) + \lambda(y-1)} e^{-t} \mathrm{d}t$$
$$= \int_0^\infty t^{((1-\lambda)x + \lambda y) - 1} e^{-t} = \Gamma((1-\lambda)x + \lambda y),$$

and

$$(\Gamma(x))^{(1-\lambda)}\cdot(\Gamma(y))^{\lambda}=\left(\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t\right)^{(1-\lambda)}\cdot\left(\int_0^\infty t^{y-1}e^{-t}\mathrm{d}t\right)^{\lambda}.$$

So, we have to show that

$$\int_0^\infty \left(t^{(x-1)}e^{-t}\right)^{(1-\lambda)}\cdot\left(t^{(y-1)}e^{-t}\right)^{\lambda}\mathrm{d}t \leq \left(\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t\right)^{(1-\lambda)}\cdot\left(\int_0^\infty t^{y-1}e^{-t}\mathrm{d}t\right)^{\lambda}.$$

By Problem 2.22, Vol. 1, we have

$$\int_0^\infty \left(t^{(x-1)}e^{-t}\right)^{(1-\lambda)}\cdot\left(t^{(y-1)}e^{-t}\right)^{\lambda}\mathrm{d}t \leq \left(\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t\right)^{(1-\lambda)}\cdot\left(\int_0^\infty t^{y-1}e^{-t}\mathrm{d}t\right)^{\lambda}.$$

∎)

Conclusion 4.93 $(\ln\circ\Gamma):(0,\infty)\to\mathbb{R}$ is convex.

Let $f:(0,\infty)\to(0,\infty)$ be a function satisfying the following conditions:

1. $f(1)=1$,
2. for every $x\in(0,\infty), f(x+1)=xf(x)$,
3. $(\ln\circ f):(0,\infty)\to\mathbb{R}$ is convex.

Problem 4.94 $f=\Gamma$.

(Solution For this purpose, let us fix any $a\in(0,\infty)$. We have to show that

$$f(a)=\Gamma(a)\left(=\int_0^\infty t^{a-1}e^{-t}\mathrm{d}t\right),$$

that is

$$\int_0^\infty t^{a-1}e^{-t}\mathrm{d}t=f(a).$$

By (1), (2), we have $f(1)=1, f(2)=1, f(3)=2!, f(4)=3!$, etc.

Case I: when $a\in(0,1)$. Since $(\ln\circ f):(0,\infty)\to\mathbb{R}$ is convex, by Problem 2.2, Vol. 1, for every positive integer n,

$$\begin{aligned}\ln n &= \ln(n!) - \ln((n-1)!) = \ln(f(n+1)) - \ln(f(n))\\ &= \underbrace{\frac{\ln(f(n+1)) - \ln(f(n))}{(n+1)-n} \leq \frac{\ln(f((n+1)+a)) - \ln(f(n+1))}{((n+1)+a)-(n+1)}}\\ &= \frac{\ln(f((n+1)+a)) - \ln(f(n+1))}{a},\end{aligned}$$

and hence for every positive integer n,

$$\ln n \leq \frac{\ln(f((n+1)+a)) - \ln(f(n+1))}{a}.$$

Since $0 < a < 1$, and $(\ln \circ \Gamma) : (0, \infty) \to \mathbb{R}$ is convex, for every positive integer n,

$$\begin{aligned}\ln(f((n+1)+a)) &= \underbrace{\ln(f((1-a)(n+1)+a(n+2))) \leq (1-a)(\ln(f(n+1))) + a(\ln(f(n+2)))}\\ &= \ln(f(n+1)) - a\ln(f(n+1)) + a(\ln(f(n+2)))\\ &= \ln(f(n+1)) - a\ln(n!) + a\ln((n+1)!) = \ln(f(n+1)) + a\ln(n+1),\end{aligned}$$

and hence for every positive integer n,

$$\frac{\ln(f((n+1)+a)) - \ln(f(n+1))}{a} \leq \ln(n+1).$$

Thus, for every positive integer n,

$$\ln n \leq \frac{\ln(f((n+1)+a)) - \ln(f(n+1))}{a} \leq \ln(n+1).$$

By (1), for every positive integer n,

$$\begin{aligned}\ln(f((n+1)+a)) &= \ln((n+a)f(n+a)) = \ln(n+a) + \ln(f(n+a))\\ &= \ln(n+a) + (\ln((n-1)+a) + \ln(f((n-1)+a)))\\ &= \ln((n+a)((n-1)+a)) + \ln(f((n-1)+a))\\ &= \cdots = \ln((n+a)((n-1)+a)\cdots a) + \ln(f(a)),\end{aligned}$$

so for every positive integer n,

$$\ln n \leq \frac{(\ln((n+a)((n-1)+a)\cdots a) + \ln(f(a))) - \ln(n!)}{a} \leq \ln(n+1),$$

and hence for every positive integer n,

$$a \ln n \leq \ln(f(a)) - (\ln(n!) - \ln((n+a)((n-1)+a)\cdots a)) \leq a \ln(n+1).$$

Thus, for every positive integer n,

$$\begin{aligned} 0 \leq \ln(f(a)) - (\ln(n!) - \ln((n+a)((n-1)+a)\cdots a)) \\ - a \ln n \leq a \ln(n+1) - a \ln n \left(= a \ln\left(1 + \frac{1}{n}\right)\right), \end{aligned}$$

that is for every positive integer n,

$$0 \leq \ln(f(a)) - \ln \frac{n!n^a}{(n+a)((n-1)+a)\cdots a} \leq a \ln\left(1 + \frac{1}{n}\right).$$

Now, since

$$\lim_{n\to\infty} a \ln\left(1 + \frac{1}{n}\right) = 0,$$

we have

$$\ln(f(a)) = \lim_{n\to\infty} \ln \frac{n!n^a}{(n+a)((n-1)+a)\cdots a} \left(= \ln\left(\lim_{n\to\infty} \frac{n!n^a}{(n+a)((n-1)+a)\cdots a}\right)\right),$$

and hence

$$f(a) = \lim_{n\to\infty} \frac{n!n^a}{(n+a)((n-1)+a)\cdots a}.$$

Case II: when $a \in (1, \infty)$. There exists a positive integer N such that $(a - N) \in (0, 1)$. Now, by (2),

$$\begin{aligned} f(a) &= (a-1)f(a-1) = \cdots = (a-1)\cdots(a-N)f((a-N)) \\ &= (a-1)\cdots(a-N) \lim_{n\to\infty} \frac{n!n^{(a-N)}}{(n+(a-N))((n-1)+(a-N))\cdots(a-N)}. \end{aligned}$$

Case III: when $a = 1$. By (1), $f(a) = 1$.

Thus, f is 'uniquely' determined in all cases. Now, since $\Gamma : (0, \infty) \to (0, \infty)$ is a function satisfying the following conditions:

1. $\Gamma(1) = 1$,
2. for every $x \in (0, \infty)$, $\Gamma(x+1) = x\,\Gamma(x)$,
3. $(\ln \circ \Gamma) : (0, \infty) \to \mathbb{R}$ is convex,

we have $f = \Gamma$. Further, for every $a \in (0, 1)$,

$$\Gamma(a) = \lim_{n\to\infty} \frac{n!n^a}{(n+a)((n-1)+a)\cdots a}.$$

∎)

Conclusion 4.95 Let $f : (0, \infty) \to (0, \infty)$ be a function satisfying the following conditions:

1. $f(1) = 1$,
2. for every $x \in (0, \infty)$, $f(x+1) = xf(x)$,
3. $(\ln \circ f) : (0, \infty) \to \mathbb{R}$ is convex.

Then, $f = \Gamma$. Further, for every $x \in (0, \infty)$, $\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}$.

Proof of the remaining part It has been shown above that, for every $x \in (0, 1)$,

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

Since

$$\lim_{n\to\infty} \frac{n!n^1}{1(1+1)\cdots(1+n)} = \lim_{n\to\infty} \frac{n^1}{\cdot(1+n)} = 1 = \Gamma(1),$$

we have, for every $x \in (0, 1]$,

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

Next, let $x \in (1, 2]$. We have to show that

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

$$\begin{aligned}
\text{LHS} &= \Gamma(x) = (x-1)\Gamma(x-1) = (x-1)\lim_{n\to\infty}\frac{n!n^{x-1}}{(x-1)\cdots((x-1)+n)} \\
&= (x-1)\lim_{n\to\infty}\frac{1}{n}\frac{(x+n)}{(x-1)}\frac{n!n^x}{x\cdots(x+n)} = \lim_{n\to\infty}\left(\frac{x}{n}+1\right)\frac{n!n^x}{x\cdots(x+n)} \\
&= \left(\lim_{n\to\infty}\left(\frac{x}{n}+1\right)\right)\left(\lim_{n\to\infty}\frac{n!n^x}{x\cdots(x+n)}\right) = 1\cdot\lim_{n\to\infty}\frac{n!n^x}{x\cdots(x+n)} \\
&= \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)} = \text{RHS}.
\end{aligned}$$

Thus, for every $x \in (0,2]$,

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

Similarly, for every $x \in (0,3]$,

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}, \text{etc.}$$

Hence, for every $x \in (0,\infty)$,

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

∎

4.6 Beta Function

Note 4.96

Definition For every $x, y \in (0,\infty)$, by $B(x,y)$ we mean

$$\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

Here, $B(x,y)$ is called the value of ***beta function*** at (x,y).

1. Problem 4.97 For every $y \in (0,\infty)$, $B(1,y) = \frac{1}{y}$.

(Solution Let us fix any $y \in (0,\infty)$. We have to show that $B(1,y) = \frac{1}{y}$.

$$\text{LHS} = B(1,y) = \int_0^1 t^{1-1}(1-t)^{y-1}\mathrm{d}t = \int_0^1 (1-t)^{y-1}\mathrm{d}t = \frac{1}{-1}\frac{(1-t)^y}{y}\bigg|_{t=0}^{t=1}$$
$$= \frac{1}{-y}(0-1) = \frac{1}{y} = \text{RHS}.$$

∎)

2. Problem 4.98 For every $y \in (0,\infty)$, the function $\zeta \mapsto \ln(B(\zeta,y))$ from $(0,\infty)$ to $\mathbb{R}$ is convex.

(Solution For this purpose, let us take any $\zeta_1,\zeta_2 \in (0,\infty)$. Next, let us take any $s \in (0,1)$. We have to show that

$$\underbrace{\ln(B((1-s)\zeta_1 + s\zeta_2, y)) \le (1-s)\cdot\ln(B(\zeta_1,y)) + s\cdot\ln(B(\zeta_2,y))}$$
$$= \ln\Big((B(\zeta_1,y))^{(1-s)}\cdot(B(\zeta_2,y))^s\Big),$$

that is

$$\int_0^1 t^{((1-s)\zeta_1+s\zeta_2)-1}(1-t)^{y-1}\mathrm{d}t = \underbrace{B((1-s)\zeta_1+s\zeta_2,y) \le (B(\zeta_1,y))^{(1-s)}\cdot(B(\zeta_2,y))^s}$$
$$= \left(\int_0^1 t^{\zeta_1-1}(1-t)^{y-1}\mathrm{d}t\right)^{(1-s)}\cdot\left(\int_0^1 t^{\zeta_2-1}(1-t)^{y-1}\mathrm{d}t\right)^s,$$

that is

$$\int_0^1 t^{((1-s)\zeta_1+s\zeta_2)-1}(1-t)^{y-1}\mathrm{d}t \le \left(\int_0^1 t^{\zeta_1-1}(1-t)^{y-1}\mathrm{d}t\right)^{(1-s)}\cdot\left(\int_0^1 t^{\zeta_2-1}(1-t)^{y-1}\mathrm{d}t\right)^s.$$

By Problem 2.12, Vol. 1, we have

$$\int_0^1 t^{((1-s)\zeta_1+s\zeta_2)-1}(1-t)^{y-1}\mathrm{d}t = \int_0^1 \left(\left(t^{\zeta_1-1}(1-t)^{y-1}\right)^{(1-s)}\cdot\left(\left(t^{\zeta_2-1}(1-t)^{y-1}\right)^s\right)\right)\mathrm{d}t$$
$$\le \left(\int_0^1 \left(\left(t^{\zeta_1-1}(1-t)^{y-1}\right)^{(1-s)}\right)^{\frac{1}{1-s}}\mathrm{d}t\right)^{\frac{1}{\frac{1}{1-s}}}\left(\int_0^1 \left(\left(t^{\zeta_2-1}(1-t)^{y-1}\right)^s\right)^{\frac{1}{s}}\mathrm{d}t\right)^{\frac{1}{\frac{1}{s}}}$$
$$= \left(\int_0^1 t^{\zeta_1-1}(1-t)^{y-1}\mathrm{d}t\right)^{(1-s)}\cdot\left(\int_0^1 t^{\zeta_2-1}(1-t)^{y-1}\mathrm{d}t\right)^s,$$

so

$$\int_0^1 t^{((1-s)\zeta_1+s\zeta_2)-1}(1-t)^{y-1}\mathrm{d}t \le \left(\int_0^1 t^{\zeta_1-1}(1-t)^{y-1}\mathrm{d}t\right)^{(1-s)} \cdot \left(\int_0^1 t^{\zeta_2-1}(1-t)^{y-1}\mathrm{d}t\right)^{s}.$$

■)

3. Problem 4.99 For every $x,y\in(0,\infty),\ B(x+1,y)=\frac{x}{x+y}B(x,y)$.

(Solution Let us fix any $x,y\in(0,\infty)$. We have to show that $B(x+1,y)=\frac{x}{x+y}B(x,y)$. Since

$$\begin{aligned}
B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \int_0^1 t^{x-1}(1-t)^{y-1}(t+(1-t))\mathrm{d}t \\
&= \int_0^1 t^{x}(1-t)^{y-1}\mathrm{d}t + \int_0^1 t^{x-1}(1-t)^{y}\mathrm{d}t \\
&= B(x+1,y) + \int_0^1 (1-t)^{y}t^{x-1}\mathrm{d}t = B(x+1,y) + \left((1-t)^{y}\frac{t^x}{x}\right)\Bigg|_{t=0}^{t=1} \\
&\quad - \int_0^1 \left(y(1-t)^{y-1}(-1)\right)\frac{t^x}{x}\mathrm{d}t \\
&= B(x+1,y) + (0-0) - \int_0^1 \left(y(1-t)^{y-1}(-1)\right)\frac{t^x}{x}\mathrm{d}t \\
&= B(x+1,y) + \frac{y}{x}\int_0^1 t^{x}(1-t)^{y-1}\mathrm{d}t \\
&= B(x+1,y) + \frac{y}{x}B(x+1,y) = \frac{x+y}{x}B(x+1,y),
\end{aligned}$$

we have $B(x+1,y)=\frac{x}{x+y}B(x,y)$. ■)

4. For fixed $y\in(0,\infty)$, let $f_y:\zeta\mapsto\frac{\Gamma(\zeta+y)}{\Gamma(y)}B(\zeta,y)$ be a function from $(0,\infty)$ to $(0,\infty)$.

Problem 4.100 a. $f_y(1)=1$, b. for every $\zeta\in(0,\infty), f_y(\zeta+1)=\zeta f_y(\zeta)$, c. $(\ln\circ f_y):(0,\infty)\to\mathbb{R}$ is convex.

(Solution For a: We have to show that $\frac{\Gamma(1+y)}{\Gamma(y)}B(1,y)=1$.

$$\text{LHS} = \frac{\Gamma(1+y)}{\Gamma(y)} B(1,y) = \frac{\Gamma(y+1)}{\Gamma(y)} \frac{1}{y} = \frac{y\Gamma(y)}{\Gamma(y)} \frac{1}{y} = 1 = \text{RHS}.$$

For b: Let us take any $\zeta \in (0, \infty)$. We have to show that $f_y(\zeta + 1) = \zeta f_y(\zeta)$.

$$\begin{aligned}\text{LHS} = f_y(\zeta+1) &= \frac{\Gamma((\zeta+1)+y)}{\Gamma(y)} B((\zeta+1),y) \\ &= \frac{\Gamma((\zeta+y)+1)}{\Gamma(y)} B(\zeta+1,y) = \frac{(\zeta+y)\Gamma(\zeta+y)}{\Gamma(y)} B(\zeta+1,y) \\ &= \frac{(\zeta+y)\Gamma(\zeta+y)}{\Gamma(y)} \frac{\zeta}{\zeta+y} B(\zeta,y) = \zeta \cdot \frac{\Gamma(\zeta+y)}{\Gamma(y)} B(\zeta,y) \\ &= \zeta \cdot f_y(\zeta) = \text{RHS}.\end{aligned}$$

For c: For this purpose, let us fix any $s \in (0, 1)$, and $\zeta_1, \zeta_2 \in (0, \infty)$. We have to show that

$$\ln\big(f_y((1-s)\zeta_1 + s\zeta_2)\big) \le (1-s)\big(\ln\big(f_y(\zeta_1)\big)\big) + s\big(\ln\big(f_y(\zeta_2)\big)\big).$$

Observe that

$$\begin{aligned}&(\ln\circ\Gamma)(((1-s)\zeta_1 + s\zeta_2) + y) + (\ln\circ B)((1-s)\zeta_1 + s\zeta_2, y) - \ln(\Gamma(y)) \\ &\quad = \ln(\Gamma(((1-s)\zeta_1 + s\zeta_2) + y)) + \ln(B((1-s)\zeta_1 + s\zeta_2, y)) - \ln(\Gamma(y)) \\ &\quad = \ln\left(\frac{\Gamma(((1-s)\zeta_1 + s\zeta_2) + y)}{\Gamma(y)} B((1-s)\zeta_1 + s\zeta_2, y)\right) = \ln\big(f_y((1-s)\zeta_1 + s\zeta_2)\big),\end{aligned}$$

and

$$\begin{aligned}&(1-s)\big(\ln\big(f_y(\zeta_1)\big)\big) + s\big(\ln\big(f_y(\zeta_2)\big)\big) = (1-s) \\ &\qquad \left(\ln\left(\frac{\Gamma(\zeta_1+y)}{\Gamma(y)} B(\zeta_1,y)\right)\right) + s\left(\ln\left(\frac{\Gamma(\zeta_2+y)}{\Gamma(y)} B(\zeta_2,y)\right)\right) \\ &\quad = (1-s)(\ln(\Gamma(\zeta_1+y)) + \ln(B(\zeta_1,y)) - \ln(\Gamma(y))) \\ &\qquad + s(\ln(\Gamma(\zeta_2+y)) + \ln(B(\zeta_2,y)) - \ln(\Gamma(y))) \\ &\quad = ((1-s)((\ln\circ\Gamma)(\zeta_1+y)) + s((\ln\circ\Gamma)(\zeta_2+y))) \\ &\qquad + ((1-s)((\ln\circ B)(\zeta_1,y)) + s((\ln\circ B)(\zeta_2,y))) - (\ln\circ\Gamma)(y).\end{aligned}$$

Hence, we have to show that

$$\begin{aligned}&(\ln\circ\Gamma)(((1-s)\zeta_1 + s\zeta_2) + y) + (\ln\circ B)((1-s)\zeta_1 + s\zeta_2, y) \\ &\quad \le ((1-s)((\ln\circ\Gamma)(\zeta_1+y)) + s((\ln\circ\Gamma)(\zeta_2+y))) \\ &\qquad + ((1-s)((\ln\circ B)(\zeta_1,y)) + s((\ln\circ B)(\zeta_2,y))).\end{aligned}$$

Since $(\ln \circ \Gamma)$ and $\zeta \mapsto (\ln \circ B)(\zeta, y)$ are convex, we have

$$\begin{aligned}
&(\ln \circ \Gamma)(((1-s)\zeta_1 + s\zeta_2) + y) + (\ln \circ B)((1-s)\zeta_1 + s\zeta_2, y) \\
&\quad = (\ln \circ \Gamma)((1-s)(\zeta_1 + y) + s(\zeta_2 + y)) + (\ln \circ B)((1-s)\zeta_1 + s\zeta_2, y) \\
&\quad \le ((1-s)((\ln \circ \Gamma)(\zeta_1 + y)) + s((\ln \circ \Gamma)(\zeta_2 + y))) \\
&\qquad + (\ln \circ B)((1-s)\zeta_1 + s\zeta_2, y) \\
&\quad \le ((1-s)((\ln \circ \Gamma)(\zeta_1 + y)) + s((\ln \circ \Gamma)(\zeta_2 + y))) \\
&\qquad + ((1-s)((\ln \circ B)(\zeta_1, y)) + s((\ln \circ B)(\zeta_2, y))).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\ln \circ \Gamma)(((1-s)\zeta_1 + s\zeta_2) + y) + (\ln \circ B)((1-s)\zeta_1 + s\zeta_2, y) \\
&\quad \le ((1-s)((\ln \circ \Gamma)(\zeta_1 + y)) + s((\ln \circ \Gamma)(\zeta_2 + y))) \\
&\qquad + ((1-s)((\ln \circ B)(\zeta_1, y)) + s((\ln \circ B)(\zeta_2, y))).
\end{aligned}$$

■)

Now, by Conclusion 4.95, for every $y \in (0, \infty)$, we have $f_y = \Gamma$, and hence, for every $x, y \in (0, \infty)$,

$$\frac{\Gamma(x+y)}{\Gamma(y)} B(x,y) = \underbrace{f_y(x) = \Gamma(x).}$$

Thus, for every $x, y \in (0, \infty)$,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Conclusion 4.101 For every $x, y \in (0, \infty), B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Problem 4.102 $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(Solution By Conclusion 4.101,

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)}.$$

Since

$$\pi = 2\int_0^{\frac{\pi}{2}} \mathrm{d}\theta = \int_{\frac{\pi}{2}}^{0} \frac{1}{\cos\theta\sin\theta}(2\cos\theta(-\sin\theta))\mathrm{d}\theta$$
$$= \int_{\frac{\pi}{2}}^{0} \frac{1}{\sqrt{\cos^2\theta(1-\cos^2\theta)}} d\left(\cos^2\theta\right)$$
$$= \int_0^1 \frac{1}{\sqrt{t(1-t)}}\mathrm{d}t = \int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{1}{2}-1}\mathrm{d}t = B\left(\frac{1}{2},\frac{1}{2}\right),$$

and

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(1)} = \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{1} = \left(\Gamma\left(\frac{1}{2}\right)\right)^2,$$

we have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. ∎)

Conclusion 4.103

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
2. $\int_{-\infty}^{\infty} e^{-s^2}\mathrm{d}s = \sqrt{\pi}$.
3. For every $x \in (0,\infty), \Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)$.

Proof of the remaining parts

(2) $\text{RHS} = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1}e^{-t}\mathrm{d}t$

$$= \int_0^\infty \frac{1}{\sqrt{t}}e^{-t}\mathrm{d}t = \int_0^\infty \frac{1}{\sqrt{s^2}}e^{-s^2}\mathrm{d}\left(s^2\right)$$
$$= \int_0^\infty \frac{1}{s}e^{-s^2}\cdot(2s)\mathrm{d}s = 2\int_0^\infty e^{-s^2}\mathrm{d}s = \int_{-\infty}^\infty e^{-s^2}\mathrm{d}s = \text{LHS}.$$

(3) Let us take any $x \in (0,\infty)$. Let

$$f : x \mapsto \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)$$

be a function from $(0,\infty)$ to $(0,\infty)$. By Conclusion 4.95, it suffices to show that

a. $f(1) = 1$,
b. for every $x \in (0, \infty), f(x+1) = xf(x)$,
c. $(\ln \circ f) : (0, \infty) \to \mathbb{R}$ is convex.

For a: Since

$$f(1) = \frac{2^{1-1}}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+1}{2}\right) = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)\Gamma(1) = \frac{1}{\sqrt{\pi}}\sqrt{\pi}\Gamma(1) = \Gamma(1) = 1,$$

we have $f(1) = 1$.

For b: Let us take any $x \in (0, \infty)$. We have to show that $f(x+1) = xf(x)$, that is

$$\frac{2^{(x+1)-1}}{\sqrt{\pi}}\Gamma\left(\frac{(x+1)}{2}\right)\Gamma\left(\frac{(x+1)+1}{2}\right) = x \cdot \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right),$$

that is

$$\left(\Gamma\left(\frac{x}{2}+1\right) = \right)\Gamma\left(\frac{(x+1)+1}{2}\right) = x \cdot \frac{1}{2}\Gamma\left(\frac{x}{2}\right),$$

that is

$$\Gamma\left(\frac{x}{2}+1\right) = \frac{x}{2}\Gamma\left(\frac{x}{2}\right).$$

This is known to be true.

For c: For this purpose, let us fix any $s \in (0,1)$, and $\zeta_1, \zeta_2 \in (0, \infty)$. We have to show that

$$\ln(f((1-s)\zeta_1 + s\zeta_2)) \leq (1-s)(\ln(f(\zeta_1))) + s(\ln(f(\zeta_2))).$$

Observe that

$$\begin{aligned}
&((1-s)\zeta_1 + s\zeta_2)(\ln 2) - \ln 2 - \ln\sqrt{\pi} + (\ln \circ \Gamma)\left(\frac{(1-s)\zeta_1 + s\zeta_2}{2}\right) \\
&\quad + (\ln \circ \Gamma)\left(\frac{((1-s)\zeta_1 + s\zeta_2) + 1}{2}\right) \\
&\quad \ln\left(\frac{2^{((1-s)\zeta_1 + s\zeta_2)-1}}{\sqrt{\pi}}\Gamma\left(\frac{(1-s)\zeta_1 + s\zeta_2}{2}\right)\Gamma\left(\frac{((1-s)\zeta_1 + s\zeta_2) + 1}{2}\right)\right) \\
&\quad = \ln(f((1-s)\zeta_1 + s\zeta_2)),
\end{aligned}$$

and

$$\begin{aligned}
&(1-s)(\ln(f(\zeta_1))) + s(\ln(f(\zeta_2))) = (1-s)\ln \\
&\quad \left(\frac{2^{\zeta_1-1}}{\sqrt{\pi}}\Gamma\left(\frac{\zeta_1}{2}\right)\Gamma\left(\frac{\zeta_1+1}{2}\right)\right) + s\,\ln\left(\frac{2^{\zeta_2-1}}{\sqrt{\pi}}\Gamma\left(\frac{\zeta_2}{2}\right)\Gamma\left(\frac{\zeta_2+1}{2}\right)\right) \\
&\quad = (1-s)\left((\zeta_1-1)(\ln 2) - \ln\sqrt{\pi} + (\ln\circ\Gamma)\left(\frac{\zeta_1}{2}\right) + (\ln\circ\Gamma)\left(\frac{\zeta_1+1}{2}\right)\right) \\
&\qquad + s\left((\zeta_2-1)(\ln 2) - \ln\sqrt{\pi} + (\ln\circ\Gamma)\left(\frac{\zeta_2}{2}\right) + (\ln\circ\Gamma)\left(\frac{\zeta_2+1}{2}\right)\right) \\
&\quad = ((1-s)\zeta_1 + s\zeta_2)(\ln 2) - \ln 2 - \ln\sqrt{\pi} \\
&\qquad + \left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1}{2}\right)\right) + s\left((\ln\circ\Gamma)\left(\frac{\zeta_2}{2}\right)\right)\right) \\
&\qquad + \left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1+1}{2}\right)\right) + s\left((\ln\circ\Gamma)\left(\frac{\zeta_2+1}{2}\right)\right)\right).
\end{aligned}$$

So, it suffices to show that

$$\begin{aligned}
&(\ln\circ\Gamma)\left(\frac{(1-s)\zeta_1+s\zeta_2}{2}\right) + (\ln\circ\Gamma)\left(\frac{((1-s)\zeta_1+s\zeta_2)+1}{2}\right) \\
&\quad \le \left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1}{2}\right)\right) + s\left((\ln\circ\Gamma)\left(\frac{\zeta_2}{2}\right)\right)\right) \\
&\qquad + \left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1+1}{2}\right)\right) + s\left((\ln\circ\Gamma)\left(\frac{\zeta_2+1}{2}\right)\right)\right).
\end{aligned}$$

By Conclusion 4.93,

$$\begin{aligned}
&(\ln\circ\Gamma)\left(\frac{(1-s)\zeta_1+s\zeta_2}{2}\right) + (\ln\circ\Gamma)\left(\frac{((1-s)\zeta_1+s\zeta_2)+1}{2}\right) \\
&\quad = (\ln\circ\Gamma)\left((1-s)\frac{\zeta_1}{2} + s\frac{\zeta_2}{2}\right) + (\ln\circ\Gamma)\left((1-s)\frac{\zeta_1+1}{2} + s\frac{\zeta_2+1}{2}\right) \\
&\quad \le \left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1}{2}\right)\right) + s\left((\ln\circ\Gamma)\left(\frac{\zeta_2}{2}\right)\right)\right) \\
&\qquad + (\ln\circ\Gamma)\left((1-s)\frac{\zeta_1+1}{2} + s\frac{\zeta_2+1}{2}\right) \\
&\quad \le \left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1}{2}\right)\right) + s\left((\ln\circ\Gamma)\left(\frac{\zeta_2}{2}\right)\right)\right) \\
&\qquad + \left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1+1}{2}\right)\right) + s\left((\ln\circ\Gamma)\left(\frac{\zeta_2+1}{2}\right)\right)\right),
\end{aligned}$$

so

$$(\ln\circ\Gamma)\left(\frac{(1-s)\zeta_1+s\zeta_2}{2}\right)+(\ln\circ\Gamma)\left(\frac{((1-s)\zeta_1+s\zeta_2)+1}{2}\right)$$
$$\leq\left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1}{2}\right)\right)+s\left((\ln\circ\Gamma)\left(\frac{\zeta_2}{2}\right)\right)\right)$$
$$+\left((1-s)\left((\ln\circ\Gamma)\left(\frac{\zeta_1+1}{2}\right)\right)+s\left((\ln\circ\Gamma)\left(\frac{\zeta_2+1}{2}\right)\right)\right).$$

∎

Note 4.104 Let $z(\equiv x+iy,\text{where}\, x,y\,\text{are real})$ be a complex number such that $\operatorname{Re}(z)>0$. It follows that, for every $t\in(0,\infty)$,

$$\begin{aligned}\left|e^{-t}t^{z-1}\right| &= |e^{-t}|\left|t^{z-1}\right| = e^{-t}\left|t^{z-1}\right| = e^{-t}\left|e^{(\ln t)((\operatorname{Re}(z)-1)+i\operatorname{Im}(z))}\right| \\ &= e^{-t}\left|e^{(\ln t)((\operatorname{Re}(z)-1))}e^{i(\ln t)\operatorname{Im}(z)}\right| = e^{-t}\left|e^{(\ln t)((\operatorname{Re}(z)-1))}\right|\left|e^{i(\ln t)\operatorname{Im}(z)}\right| \\ &= e^{-t}\left|e^{(\ln t)((\operatorname{Re}(z)-1))}\right|\cdot 1 = e^{-t}e^{(\ln t)((\operatorname{Re}(z)-1))} = e^{-t}t^{\operatorname{Re}(z)-1} = t^{x-1}e^{-t},\end{aligned}$$

and, by Problem 4.89, $\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t<\infty$, and hence $\int_0^\infty \left|e^{-t}t^{z-1}\right|\mathrm{d}t<\infty$. Thus, for every complex number z satisfying $\operatorname{Re}(z)>0$, $\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t$ is absolutely convergent.

Definition For every complex number z satisfying $\operatorname{Re}(z)>0$, the complex number $\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t$ is denoted by $\Gamma(z)$. Thus, $\Gamma:\{z:\operatorname{Re}(z)>0\}\to\mathbb{C}$.

Problem 4.105 $\Gamma\in H(\{z:\operatorname{Re}(z)>0\})$.

(Solution Suppose that for every positive integer n,

$$\Gamma_n : z\mapsto\left(\int_{\frac{1}{n}}^{n} e^{-t}t^{z-1}\mathrm{d}t\right)$$

is a function from $\{z:\operatorname{Re}(z)>0\}$ to $\mathbb{C}$. We want to show that for every positive integer $n, \Gamma_n\in H(\{z:\operatorname{Re}(z)>0\})$. For this purpose, let us fix any positive integer N. We have to show that $\Gamma_N\in H(\{z:\operatorname{Re}(z)>0\})$. By Conclusion 1.123, it suffices to show that

1. $\Gamma_N:\{z:\operatorname{Re}(z)>0\}\to\mathbb{C}$ is a continuous function,
2. for every triangle Δ satisfying $\Delta\subset\{z:\operatorname{Re}(z)>0\}$, $\int_{\partial\Delta}\Gamma_N(z)\mathrm{d}z=0$.

For 1: Let us fix any $z_0\in\{z:\operatorname{Re}(z)>0\}$. We have to show that Γ_N is continuous at z_0. For this purpose, let us take any $\varepsilon>0$. Observe that the function

$$\underbrace{\varphi:(t,z)\mapsto e^{-t}t^{z-1}} = e^{-t}e^{(z-1)(\ln t)} = e^{-t+(z-1)(\ln t)}$$

is continuous from $(0,\infty)\times\{z:\mathrm{Re}(z)>0\}$ to $\mathbb{C}$. There exists $r>0$ such that $D[z_0;r]\subset\{z:\mathrm{Re}(z)>0\}$. Since $\varphi:(0,\infty)\times\{z:\mathrm{Re}(z)>0\}\to\mathbb{C}$ is continuous, and $\left[\frac{1}{n},n\right]\times D[z_0;r]$ is a compact subset of $(0,\infty)\times\{z:\mathrm{Re}(z)>0\}$, there exists $\delta\in(0,r)$ such that, for every $t\in\left[\frac{1}{n},n\right]$, and for every $z\in D[z_0;\delta]$, we have

$$\left|e^{-t}t^{z-1}-e^{-t}t^{z_0-1}\right|=\underbrace{|\varphi(t,z)-\varphi(t,z_0)|<\frac{\varepsilon}{N}.}$$

Thus, for every $t\in\left[\frac{1}{n},n\right]$, and for every $z\in D[z_0;\delta]$, we have

$$\left|e^{-t}t^{z-1}-e^{-t}t^{z_0-1}\right|<\frac{\varepsilon}{N}.$$

Now, for every $z\in D[z_0;\delta]$, we have

$$\begin{aligned}|\Gamma_N(z)-\Gamma_N(z_0)|&=\left|\int_{\frac{1}{N}}^{N}e^{-t}t^{z-1}\mathrm{d}t-\int_{\frac{1}{N}}^{N}e^{-t}t^{z_0-1}\mathrm{d}t\right|\\&=\left|\int_{\frac{1}{N}}^{N}\left(e^{-t}t^{z-1}-e^{-t}t^{z_0-1}\right)\mathrm{d}t\right|\\&\le\int_{\frac{1}{N}}^{N}\left|e^{-t}t^{z-1}-e^{-t}t^{z_0-1}\right|\mathrm{d}t\le\frac{\varepsilon}{N}\cdot\left(N-\frac{1}{N}\right)<\varepsilon,\end{aligned}$$

so, for every $z\in D[z_0;\delta]$, $|\Gamma_N(z)-\Gamma_N(z_0)|<\varepsilon$. This shows that Γ_N is continuous at z_0.

For 2: Let us take any triangle Δ satisfying $\Delta\subset\{z:\mathrm{Re}(z)>0\}$. We have to show that $\int_{\partial\Delta}\Gamma_N(z)\mathrm{d}z=0$.

$$\begin{aligned}\mathrm{LHS}&=\int_{\partial\Delta}\Gamma_N(z)\mathrm{d}z=\int_{\partial\Delta}\left(\int_0^N e^{-t}t^{z-1}\mathrm{d}t\right)\mathrm{d}z\\&=\int_0^N\left(\int_{\partial\Delta}e^{-t}t^{z-1}\mathrm{d}z\right)\mathrm{d}t=\int_0^N\left(e^{-t}\int_{\partial\Delta}t^{z-1}\mathrm{d}z\right)\mathrm{d}t\\&=\int_0^N\left(e^{-t}\int_{\partial\Delta}e^{(\ln t)(z-1)}\mathrm{d}z\right)\mathrm{d}t=\int_0^N\left(e^{-t}\cdot 0\right)\mathrm{d}t=0=\mathrm{RHS}.\end{aligned}$$

Now, by Conclusion 1.172, it suffices to show that $\{\Gamma_n\}$ converges to Γ uniformly on compact subset of $\{w : \mathrm{Re}(w) > 0\}$. For this purpose, let us take any nonempty compact subset K of $\{w : \mathrm{Re}(w) > 0\}$. We have to show that $\{\Gamma_n\}$ converges to Γ uniformly on K. Here, for every $z \in K$,

$$\lim_{n\to\infty} \Gamma_n(z) = \lim_{n\to\infty}\left(\int_{\frac{1}{n}}^{n} e^{-t}t^{z-1}\mathrm{d}t\right) = \int_{\frac{1}{n}}^{\infty} e^{-t}t^{z-1}\mathrm{d}t = \Gamma(z),$$

so, for every $z \in K$, we have $\lim_{n\to\infty}\Gamma_n(z) = \Gamma(z)$. Since K is a compact subset of $\{w : \mathrm{Re}(w) > 0\}$, there exists a positive integer N such that, for every $z \in \{w : \mathrm{Re}(w) > 0\}$, we have $\mathrm{Re}(z) - 1 \le N$. It follows that $\lim_{t\to\infty}\frac{t^N}{e^t} = 0$. Next, for every $z \in K$,

$$\begin{aligned}
|\Gamma_n(z) - \Gamma(z)| &= \left|\int_{\frac{1}{n}}^{n} e^{-t}t^{z-1}\mathrm{d}t - \int_{\frac{1}{n}}^{\infty} e^{-t}t^{z-1}\mathrm{d}t\right| \\
&= \left|-\int_{n}^{\infty} e^{-t}t^{z-1}\mathrm{d}t\right| = \left|\int_{n}^{\infty} e^{-t}t^{z-1}\mathrm{d}t\right| \\
&\le \int_{n}^{\infty} \left|e^{-t}t^{z-1}\right|\mathrm{d}t = \int_{n}^{\infty} |e^{-t}|\left|t^{\mathrm{Re}(z)-1}\right|\mathrm{d}t \\
&= \int_{n}^{\infty} e^{-t}\left|t^{\mathrm{Re}(z)-1}\right|\mathrm{d}t \le \int_{n}^{\infty} \frac{t^N}{e^t}\mathrm{d}t,
\end{aligned}$$

so for every $z \in K$, we have

$$|\Gamma_n(z) - \Gamma(z)| \le \int_{n}^{\infty} \frac{t^N}{e^t}\mathrm{d}t.$$

Further, since $\lim_{t\to\infty}\frac{t^N}{e^t} = 0$, we have $\lim_{n\to\infty}\left(\int_n^\infty \frac{t^N}{e^t}\mathrm{d}t\right) = 0$, and hence $\{\Gamma_n\}$ converges to Γ uniformly on K. ∎)

Conclusion 4.106 $\Gamma \in H(\{z : \mathrm{Re}(z) > 0\})$.

Let $z \in \{w : \mathrm{Re}(w) > 0\}$.

It follows that $(z+1) \in \{w : \mathrm{Re}(w) > 0\}$, and hence

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^{(z+1)-1}\mathrm{d}t = \int_0^\infty t^z e^{-t}\mathrm{d}t = \left(t^z\frac{e^{-t}}{-1}\right)\Bigg|_{t=0}^{t=\infty} - \int_0^\infty \left(zt^{z-1}\frac{e^{-t}}{-1}\right)\mathrm{d}t$$
$$= -\left(\lim_{t\to\infty}\frac{t^z}{e^t} - 0\right) + z\int_0^\infty t^z e^{-t}\mathrm{d}t \int_0^\infty e^{-t}t^{z-1}\mathrm{d}t = -\lim_{t\to\infty}\frac{t^z}{e^t} + z\Gamma(z).$$

Since

$$\lim_{t\to\infty}\left|\frac{t^z}{e^t}\right| = \lim_{t\to\infty}\frac{|t^z|}{|e^t|} = \lim_{t\to\infty}\frac{|t^z|}{e^t} = \lim_{t\to\infty}\frac{e^{(\ln t)\mathrm{Re}(z)}}{e^t} = \lim_{t\to\infty}\frac{t^{\mathrm{Re}(z)}}{e^t} = 0,$$

we have

$$z\Gamma(z) - \Gamma(z+1) = \underbrace{\lim_{t\to\infty}\frac{t^z}{e^t}} = 0,$$

and hence $\Gamma(z+1) = z\Gamma(z)$.

Conclusion 4.107 For every $z \in \{w : \mathrm{Re}(w) > 0\}, \Gamma(z+1) = z\Gamma(z)$.

Since $\Gamma \in H(\{z : \mathrm{Re}(z) > 0\})$, the function

$$z \mapsto \frac{\Gamma(z+1)}{z}\ (= \Gamma(z))$$

is holomorphic in $\{z : \mathrm{Re}(z) > -1\} - \{0\}$, and hence the formula

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

extends Γ holomorphically from $\{w : \mathrm{Re}(w) > 0\}$ to $\{z : \mathrm{Re}(z) > -1\} - \{0\}$. Similarly, the formula $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ extends Γ holomorphically from $\{z : \mathrm{Re}(z) > -1\} - \{0\}$ to $\{z : \mathrm{Re}(z) > -2\} - \{0, -1\}$, etc., Thus, the formula $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ extends Γ holomorphically from $\{z : \mathrm{Re}(z) > 0\}$ to $\mathbb{C} - \{0, -1, -2, \ldots\}$.

It follows that $\Gamma \in H(\mathbb{C} - \{0, -1, -2, \ldots\})$. Further, $\{0, -1, -2, \ldots\}$ has no limit point in $\mathbb{C}$. Also, each member of $\{0, -1, -2, \ldots\}$ is a simple pole of the '***gamma function***' Γ.

Conclusion 4.108 The gamma function $\Gamma : (\mathbb{C} - \{0, -1, -2, \ldots\}) \to \mathbb{C}$ is meromorphic in $\mathbb{C}$. Also, each member of $\{0, -1, -2, \ldots\}$ is a simple pole of Γ. Further,

$$\mathrm{Res}(\Gamma; 0) = \lim_{z\to 0}(z-0)\Gamma(z) = \lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0}\Gamma(z+1) = \Gamma(0+1) = \Gamma(1) = 1,$$

so

$$\mathrm{Res}(\Gamma; 0) = 1.$$

Problem 4.109 Let $a, A \in (0, \infty)$, and $a < A$. Then, for every $\alpha, \beta \in (0, 1)$ satisfying $\alpha < \beta$, and for every complex number z satisfying $\mathrm{Re}(z) \in [a, A]$, $\left|\int_\alpha^\beta e^{-t} t^{z-1} \mathrm{d}t\right| \le \frac{1}{a}(\beta^a - \alpha^a)$.

(Solution Let us take any $\alpha, \beta \in (0, 1)$ satisfying $\alpha < \beta$. Let z be a complex number satisfying $\mathrm{Re}(z) \in [a, A]$. We have to show that

$$\left|\int_\alpha^\beta e^{-t} t^{z-1} \mathrm{d}t\right| \le \frac{1}{a}(\beta^a - \alpha^a).$$

Since

$$\begin{aligned}
\left|\int_\alpha^\beta e^{-t} t^{z-1} \mathrm{d}t\right| &\le \int_\alpha^\beta \left|e^{-t} t^{z-1}\right| \mathrm{d}t = \int_\alpha^\beta \left|e^{-t} e^{(\ln t)(z-1)}\right| \mathrm{d}t \\
&= \int_\alpha^\beta \left|e^{-t} e^{(\ln t)((\mathrm{Re}(z)-1) + i\,\mathrm{Im}(z))}\right| \mathrm{d}t = \int_\alpha^\beta \left|e^{-t} e^{(\ln t)(\mathrm{Re}(z)-1)} e^{i(\ln t)(\mathrm{Im}(z))}\right| \mathrm{d}t \\
&= \int_\alpha^\beta e^{-t} e^{(\ln t)(\mathrm{Re}(z)-1)} \mathrm{d}t \le \int_\alpha^\beta e^{-t} e^{(\ln t)(a-1)} \mathrm{d}t \le \int_\alpha^\beta e^0 e^{(\ln t)(a-1)} \mathrm{d}t \\
&= \int_\alpha^\beta t^{(a-1)} \mathrm{d}t = \frac{1}{a} t^a \Big|_{t=\alpha}^{t=\beta} = \frac{1}{a}(\beta^a - \alpha^a),
\end{aligned}$$

we have

$$\left|\int_\alpha^\beta e^{-t} t^{z-1} \mathrm{d}t\right| \le \frac{1}{a}(\beta^a - \alpha^a).$$

∎

Since $\lim_{t \to 0} t^a = 0$, for every $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that $0 < t < \delta \Rightarrow t^a < a\varepsilon$, and hence for every $\alpha, \beta \in (0, 1)$ satisfying $\alpha < \beta < \delta$, we have

$$a \left|\int_\alpha^\beta e^{-t} t^{z-1} \mathrm{d}t\right| \le (\beta^a - \alpha^a) \le \underbrace{|\beta^a - \alpha^a| < \beta^a < a\varepsilon.}$$

Conclusion 4.110 Let $a, A \in (0, \infty)$, and $a < A$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every real α, β satisfying $0 < \alpha < \beta < \delta$, and for every complex number z satisfying $\mathrm{Re}(z) \in [a, A]$,

$$\left| \int_{\alpha}^{\beta} e^{-t} t^{z-1} \mathrm{d}t \right| < \varepsilon.$$

Let $a, A \in (0, \infty)$, and $a < A$. Let $\varepsilon > 0$. Since

$$\lim_{t\to\infty} e^{-\frac{1}{2}t} t^{(A-1)} = \lim_{t\to\infty} \frac{t^{(A-1)}}{e^{\frac{1}{2}t}} = 0,$$

there exists a positive integer N such that for every real number $t \in (N, \infty)$, we have $e^{-\frac{1}{2}t} t^{(A-1)} < 1$. Since the function $t \mapsto e^{-\frac{1}{2}t} t^{(A-1)}$ from compact set $[1, N]$ to $[0, \infty)$ is continuous, there exists a real number c_1 such that for every real $t \in [1, N]$, we have $e^{-\frac{1}{2}t} t^{(A-1)} < c_1$. Thus, for every real number $t \in [1, \infty)$, we have $e^{-\frac{1}{2}t} t^{(A-1)} < c$, where $c \equiv \max\{c_1, 1\}$.

Next, since for every complex number z satisfying $\mathrm{Re}(z) \in [a, A]$, and for every $t \in [1, \infty)$,

$$\left| e^{-t} t^{z-1} \right| = e^{-t} e^{(\ln t)(\mathrm{Re}(z)-1)} \le e^{-t} e^{(\ln t)(A-1)} = e^{-\frac{1}{2}t} \left(e^{-\frac{1}{2}t} t^{(A-1)} \right) < e^{-\frac{1}{2}t} c,$$

so for every $\alpha, \beta \in (1, \infty)$ satisfying $\alpha < \beta$, and for every complex number z satisfying $\mathrm{Re}(z) \in [a, A]$, we have

$$\begin{aligned} \left| \int_{\alpha}^{\beta} e^{-t} t^{z-1} \mathrm{d}t \right| &\le \int_{\alpha}^{\beta} \left| e^{-t} t^{z-1} \right| \mathrm{d}t \le \int_{\alpha}^{\beta} e^{-\frac{1}{2}t} c \mathrm{d}t = c \frac{1}{\frac{-1}{2}} e^{-\frac{1}{2}t} \Big|_{t=\alpha}^{t=\beta} \\ &= -2c \left(e^{-\frac{1}{2}\beta} - e^{-\frac{1}{2}\alpha} \right) = 2c \left(e^{-\frac{1}{2}\alpha} - e^{-\frac{1}{2}\beta} \right). \end{aligned}$$

Thus, for every $\alpha, \beta \in (1, \infty)$ satisfying $\alpha < \beta$, and for every complex number z satisfying $\mathrm{Re}(z) \in [a, A]$, we have

$$\left| \int_{\alpha}^{\beta} e^{-t} t^{z-1} \mathrm{d}t \right| \le 2c \left(e^{-\frac{1}{2}\alpha} - e^{-\frac{1}{2}\beta} \right).$$

Since $\lim_{t\to\infty} e^{-\frac{1}{2}t} = 0$, there exists $\kappa > 1$ such that, for every real $t > \kappa$, we have $e^{-\frac{1}{2}t} < \frac{\varepsilon}{4c}$. It follows that for every $\alpha, \beta \in (\kappa, \infty)$ satisfying $\alpha < \beta$, we have

$$2c\left(e^{-\frac{1}{2}\alpha}-e^{-\frac{1}{2}\beta}\right)\leq 2c\left(e^{-\frac{1}{2}\alpha}+e^{-\frac{1}{2}\beta}\right)<2c\left(\frac{\varepsilon}{4c}+e^{-\frac{1}{2}\beta}\right)<2c\left(\frac{\varepsilon}{4c}+\frac{\varepsilon}{4c}\right)=\varepsilon.$$

Hence, for every $\alpha,\beta\in(\kappa,\infty)$ satisfying $\alpha<\beta$, and for every complex number z satisfying $\mathrm{Re}(z)\in[a,A]$, we have

$$\left|\int_{\alpha}^{\beta}e^{-t}t^{z-1}\mathrm{d}t\right|<\varepsilon.$$

Conclusion 4.111 Let $a,A\in(0,\infty)$, and $a<A$. Let $\varepsilon>0$. Then there exists a real number κ such that, for every $\alpha,\beta\in(\kappa,\infty)$ satisfying $\alpha<\beta$, and for every complex number z satisfying $\mathrm{Re}(z)\in[a,A]$,

$$\left|\int_{\alpha}^{\beta}e^{-t}t^{z-1}\mathrm{d}t\right|<\varepsilon.$$

On using Conclusions 4.110, and 4.111, we get the following

Conclusion 4.112 For every $\varepsilon>0$, there exists a positive integer N such that, for every positive integer $n\geq N$, and for every complex number z satisfying $\mathrm{Re}(z)\in\left[\frac{1}{n},n\right]$,

$$\left|\int_{0}^{\infty}e^{-t}t^{z-1}\mathrm{d}t-\int_{\frac{1}{n}}^{n}e^{-t}t^{z-1}\mathrm{d}t\right|<\varepsilon.$$

4.7 Euler's Constant

Note 4.113 Let $f:[1,\infty)\to(0,\infty)$ be a decreasing function. Let $\lim_{x\to\infty}f(x)=0$. For every positive integer n, put

$$s_n\equiv f(1)+\cdots+f(n),\quad t_n\equiv\int_1^n f(x)\mathrm{d}x,\ \text{and}\ d_n\equiv s_n-t_n.$$

a. Problem 4.114 For every positive integer n, $0<f(n+1)\leq d_{n+1}\leq d_n\leq f(1)$.

(Solution Let us fix any positive integer n. Since $f:[1,\infty)\to(0,\infty)$, we have $0<f(n+1)$. Since

$$\begin{aligned}
f(n+1) - d_{n+1} &= f(n+1) - (s_{n+1} - t_{n+1}) = f(n+1) \\
&\quad - \left((f(1) + \cdots + f(n+1)) - \int_1^{n+1} f(x)\mathrm{d}x \right) = f(n+1) \\
&\quad - \left((f(1) + \cdots + f(n+1)) - \left(\int_1^2 f(x)\mathrm{d}x + \cdots + \int_n^{n+1} f(x)\mathrm{d}x \right) \right) \\
&= \left(\int_1^2 f(x)\mathrm{d}x + \cdots + \int_n^{n+1} f(x)\mathrm{d}x \right) - (f(1) + \cdots + f(n)) \\
&= \left(\int_1^2 f(x)\mathrm{d}x - f(1) \right) + \cdots + \left(\int_n^{n+1} f(x)\mathrm{d}x - f(n) \right) \\
&\le \left(\int_1^2 f(1)\mathrm{d}x - f(1) \right) + \cdots + \left(\int_n^{n+1} f(n)\mathrm{d}x - f(n) \right) \\
&= 0 + \cdots + 0 = 0,
\end{aligned}$$

we have

$$f(n+1) \le d_{n+1}.$$

Next,

$$\begin{aligned}
d_{n+1} - d_n &= (s_{n+1} - t_{n+1}) - (s_n - t_n) = (s_{n+1} - s_n) - (t_{n+1} - t_n) \\
&= f(n+1) - (t_{n+1} - t_n) = f(n+1) - \left(\int_1^{n+1} f(x)\mathrm{d}x - \int_1^n f(x)\mathrm{d}x \right) \\
&= f(n+1) - \int_n^{n+1} f(x)\mathrm{d}x \le f(n+1) - \int_n^{n+1} f(n+1)\mathrm{d}x = 0,
\end{aligned}$$

so $d_{n+1} \le d_n$. Finally,

$$\begin{aligned}
d_n - f(1) = (s_n - t_n) - f(1) &= \left((f(1) + \cdots + f(n)) - \int_1^n f(x)\mathrm{d}x \right) - f(1) \\
&= (f(2) + \cdots + f(n)) - \int_1^n f(x)\mathrm{d}x = (f(2) + \cdots + f(n))
\end{aligned}$$

$$
\begin{aligned}
&-\left(\int_1^2 f(x)\mathrm{d}x + \cdots + \int_{n-1}^n f(x)\mathrm{d}x\right)\\
&= \left(f(2) - \int_1^2 f(x)\mathrm{d}x\right) + \cdots + \left(f(n) - \int_{n-1}^n f(x)\mathrm{d}x\right)\\
&\leq \left(f(2) - \int_1^2 f(2)\mathrm{d}x\right) + \cdots + \left(f(n) - \int_{n-1}^n f(n)\mathrm{d}x\right)\\
&= 0 + \cdots + 0 = 0,
\end{aligned}
$$

so $d_n \leq f(1)$. ∎)

b. Problem 4.115 The series $f(1) + f(2) + \cdots$ converges if and only if the sequence $\left\{\int_1^n f(x)\mathrm{d}x\right\}$ converges.

(Solution Let the series $f(1) + f(2) + \cdots$ be convergent; that is, the sequence $\{s_n\}$ is convergent. We have to show that $\{t_n\}$ is convergent. By Problem 3.114, $\{d_n\}$ is a monotonically decreasing, and bounded below sequence, so $\{d_n\}$ is convergent. Now, since $\{s_n\}$ is convergent, $(\{t_n\} =)\{s_n - d_n\}$ is convergent, and hence $\{t_n\}$ is convergent.

Conversely, let $\{t_n\}$ be convergent. We have to show that $\{s_n\}$ is convergent. By Problem (3.114), $\{d_n\}$ is a monotonically decreasing and bounded below sequence, so $\{d_n\}$ is convergent. Now, since $\{t_n\}$ is convergent, we have $(\{s_n\} =)\{t_n + d_n\}$ is convergent, and hence $\{s_n\}$ is convergent. ∎)

c. Problem 4.116 For every positive integer k, $0 \leq d_k - \lim_{n\to\infty} d_n \leq f(k)$.

(Solution Let us fix any positive integer k.

By Problem (a), $\{d_n\}$ is a monotonically decreasing and bounded below sequence, so $\lim_{n\to\infty} d_n$ exists, and $\lim_{n\to\infty} d_n \leq d_k$. It follows that

$$0 \leq d_k - \lim_{n\to\infty} d_n \left(= \sum_{n=k}^{\infty} (d_n - d_{n+1})\right).$$

It remains to show that $\sum_{n=k}^{\infty} (d_n - d_{n+1}) \leq f(k)$. We have seen above that for every positive integer n,

$$0 \leq \underbrace{d_n - d_{n+1} = \int_n^{n+1} f(x)\mathrm{d}x - f(n+1)}_{= f(n) - f(n+1)} \leq \int_n^{n+1} f(n)\mathrm{d}x - f(n+1)$$

and hence

$$\underbrace{\sum_{n=k}^{\infty}(d_n - d_{n+1}) \leq \sum_{n=k}^{\infty}(f(n) - f(n+1))} = f(k) - \lim_{n\to\infty} f(n) = f(k) - 0 = f(k).$$

It follows that $\sum_{n=k}^{\infty}(d_n - d_{n+1}) \leq f(k)$. ∎)

Conclusion 4.117 Let $f : [1,\infty) \to (0,\infty)$ be a decreasing function. Let $\lim_{x\to\infty} f(x) = 0$. For every positive integer n, put $s_n \equiv f(1) + \cdots + f(n), t_n \equiv \int_1^n f(x)\mathrm{d}x$, and $d_n \equiv s_n - t_n$. Then,

a. for every positive integer n, $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$,
b. $\lim_{n\to\infty} d_n$ exists,
c. the series $f(1) + f(2) + \cdots$ converges if and only if the sequence $\left\{\int_1^n f(x)\mathrm{d}x\right\}$ converges,
d. for every positive integer k, $0 \leq d_k - \lim_{n\to\infty} d_n \leq f(k)$.

Notation Let $\{a_n\}$ be any sequence of complex numbers. Let $\{b_n\}$ be any sequence of nonnegative real numbers.

1. If there exists a positive real number M such that, for every positive integer n, $|a_n| \leq Mb_n$, then we write: $a_n = O(b_n)$, and this is read: 'a_n ***is big oh of*** b_n.'
2. If each b_n is nonzero, and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then we write: $a_n = o(b_n)$, and this is read: '***is little oh of*** b_n.'

The above statement (d) can be expressed as: $d_k - \lim_{n\to\infty} d_n = O(f(k))$ or, $d_k = \lim_{n\to\infty} d_n + O(f(k))$.

Example In Conclusion 4.117, suppose that for every $x \in [1,\infty), f(x) \equiv \frac{1}{x}$. Clearly, $f : [1,\infty) \to (0,\infty)$ is a decreasing function, and $\lim_{x\to\infty} f(x) = 0$. Here, for every positive integer n,

$$\begin{aligned} d_n &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \int_1^n \frac{1}{x}\mathrm{d}x = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - (\ln n - \ln 1) \\ &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \ln n. \end{aligned}$$

Now, since $\lim_{n\to\infty} d_n$ exists, $\lim_{n\to\infty}\left(\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \ln n\right)$ exists.

Notation $\lim_{n\to\infty}\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\ln n\right)$ is denoted by γ, and is called the ***Euler's constant***.

Since for every positive integer k, $0\le d_k - \lim_{n\to\infty} d_n \le f(k)$, we have $0\le d_2 - \gamma \le f(2)$, and hence

$$0\le\left(\left(1+\frac{1}{2}\right)-\ln 2\right)-\gamma\le\frac{1}{2}.$$

It follows that

$$(1-\ln 2)\le\gamma\le\left(1+\frac{1}{2}\right)-\ln 2.$$

If we take $\ln 2\approx 0.69$, then γ lies between 0.31 and 0.81. 'Whether γ is rational or irrational' is still an open question. Here, we can write:

$$1+\frac{1}{2}+\cdots+\frac{1}{n}=\ln n+\gamma+O\left(\frac{1}{n}\right).$$

Example Let s be a positive real number different from 1. In Conclusion 4.117, suppose that for every $x\in[1,\infty), f(x)\equiv\frac{1}{x^s}$. Clearly, $f:[1,\infty)\to(0,1]$ is a decreasing function, and $\lim_{x\to\infty} f(x)=0$. We know that $\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots$ converges if and only if $s\in(1,\infty)$.

Notation For every $s\in(1,\infty)$, the sum of the convergent series $\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots$ is denoted by $\zeta(s)$. Thus, $\zeta:(1,\infty)\to(1,\infty)$ is a function, and ζ is called the ***Riemann zeta function***.

Here, for every positive integer n,

$$\begin{aligned}d_n&=\left(1+\frac{1}{2^s}+\cdots+\frac{1}{n^s}\right)-\int_1^n\frac{1}{x^s}\,\mathrm{d}x\\&=\left(1+\frac{1}{2^s}+\cdots+\frac{1}{n^s}\right)-\frac{1}{-s+1}\left(\frac{1}{n^{s-1}}-1\right),\end{aligned}$$

so for every $s\in(1,\infty)$, we have

$$\begin{aligned}\lim_{n\to\infty} d_n&=\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)-\lim_{n\to\infty}\frac{1}{-s+1}\left(\frac{1}{n^{s-1}}-1\right)\\&=\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)-\frac{1}{-s+1}(0-1)=\zeta(s)-\frac{1}{s-1}.\end{aligned}$$

Since, for every positive integer k,

$$0 \le d_k - \lim_{n\to\infty} d_n \le f(k),$$

we have, for every positive integer n,

$$0 \le \left(\left(1+\frac{1}{2^s}+\cdots+\frac{1}{n^s}\right)-\frac{1}{-s+1}\left(\frac{1}{n^{s-1}}-1\right)\right)-\left(\zeta(s)-\frac{1}{s-1}\right)\le\frac{1}{n^s}.$$

Hence, for every positive integer n,

$$0 \le \left(\left(1+\frac{1}{2^s}+\cdots+\frac{1}{n^s}\right)-\frac{1}{-s+1}\left(\frac{1}{n^{s-1}}\right)\right)-\zeta(s)\le\frac{1}{n^s}.$$

In other words,

$$\left(1+\frac{1}{2^s}+\cdots+\frac{1}{n^s}\right)=\zeta(s)-\frac{1}{s-1}\left(\frac{1}{n^{s-1}}\right)+O\left(\frac{1}{n^s}\right).$$

4.8 Riemann Zeta Function

Note 4.118

Problem 4.119 Suppose that f has a simple pole at z_0. Let g be holomorphic in a neighborhood of z_0. Then $\mathrm{Res}(fg; z_0) = \mathrm{Res}(f; z_0) \cdot g(z_0)$.

(Solution Since, f has a simple pole at z_0, there exist complex numbers $c_0, c_1, c_2, \ldots$ such that

$$f(z) = \mathrm{Res}(f; z_0) \cdot \frac{1}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

in a deleted neighborhood of z_0. Since, g is holomorphic in a neighborhood of z_0, there exist complex numbers $b_1, b_2, \ldots$ such that

$$g(z) = g(z_0) + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots$$

in a neighborhood of z_0. It follows that, in a deleted neighborhood of z_0,

$$
\begin{aligned}
((fg)(z) =)f(z) \cdot g(z) = (\mathrm{Res}(f; z_0) \cdot g(z_0)) &\frac{1}{z - z_0} \\
&+ (\mathrm{Res}(f; z_0) \cdot b_1 + c_0 \cdot g(z_0)) \\
&+ (\cdots)(z - z_0) + (\cdots)(z - z_0)^2 + \cdots,
\end{aligned}
$$

and hence

$$\mathrm{Res}(fg; z_0) = \mathrm{Res}(f; z_0) \cdot g(z_0).$$

∎)

Conclusion 4.120 Suppose that f has a simple pole at z_0. Let g be holomorphic in a neighborhood of z_0. Then, $\mathrm{Res}(fg; z_0) = \mathrm{Res}(f; z_0).g(z_0)$.

Let Ω be a region. Let Γ be a cycle in Ω. Suppose that for every $a \in \Omega^c$, $(\mathrm{Ind})_\Gamma(a) = 0$. Let A be a closed subset of Ω. Suppose that $A \cap \mathrm{ran}(\Gamma) = \emptyset$, and every point of A is isolated. Let $f \in H(\Omega - A)$. Suppose that f has a simple pole or removable singularity at each point of A. Let $z \in \Omega - (A \cup \mathrm{ran}(\Gamma))$.

Problem 4.121 $\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} \mathrm{d}w = (f(z))\big((\mathrm{Ind})_\Gamma(z)\big) + \sum_{p \in A} \left(\mathrm{Res}(f; p) \cdot \frac{1}{p-z}\right)\big((\mathrm{Ind})_\Gamma(p)\big)$.

(**Solution** Let $g : w \mapsto \frac{1}{w-z} f(w)$. It follows that g is holomorphic in Ω except for simple pole at z, and simple poles at points of A. Now, by Conclusion 1.120, $\mathrm{Res}(g; z) = 1 \cdot f(z)$ and, for every $p \in A$, $\mathrm{Res}(g; p) = \mathrm{Res}(f; p) \cdot \frac{1}{p-z}$. By Conclusion 4.225,

$$
\begin{aligned}
\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} \mathrm{d}w &= \frac{1}{2\pi i} \int_\Gamma g(w) \mathrm{d}w \\
&= (\mathrm{Res}(g; z))\big((\mathrm{Ind})_\Gamma(z)\big) + \sum_{p \in A} (\mathrm{Res}(g; p))\big((\mathrm{Ind})_\Gamma(p)\big) \\
&= (f(z))\big((\mathrm{Ind})_\Gamma(z)\big) + \sum_{p \in A} \left(\mathrm{Res}(f; p) \cdot \frac{1}{p - z}\right)\big((\mathrm{Ind})_\Gamma(p)\big).
\end{aligned}
$$

∎)

Conclusion 4.122 Let Ω be a region. Let Γ be a cycle in Ω. Suppose that for every $a \in \Omega^c$, $(\mathrm{Ind})_\Gamma(a) = 0$. Let A be a closed subset of Ω. Suppose that $A \cap \mathrm{ran}(\Gamma) = \emptyset$, and every point of A is isolated. Let $f \in H(\Omega - A)$. Suppose that f has a simple pole or removable singularity at each point of A. Let $z \in \Omega - (A \cup \mathrm{ran}(\Gamma))$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} \mathrm{d}w = (f(z))\big((\mathrm{Ind})_\Gamma(z)\big) + \sum_{p \in A} \left(\mathrm{Res}(f; p) \cdot \frac{1}{p - z}\right)\big((\mathrm{Ind})_\Gamma(p)\big).$$

Let us take any complex number $z \equiv x + iy$, where x, y are real numbers, and $z \notin \mathbb{Z}$. Since

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{\frac{1}{2}\left(e^{i\pi z} + e^{-i\pi z}\right)}{\frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right)} = i\frac{e^{2i\pi z} + 1}{e^{2i\pi z} - 1} = i\frac{e^{2i\pi(x+iy)} + 1}{e^{2i\pi(x+iy)} - 1}$$
$$= i\frac{e^{2i\pi x}\frac{1}{e^{2\pi y}} + 1}{e^{2i\pi x}\frac{1}{e^{2\pi y}} - 1},$$

and

$$\lim_{y\to\infty} i\frac{e^{2i\pi x}\frac{1}{e^{2\pi y}} + 1}{e^{2i\pi x}\frac{1}{e^{2\pi y}} - 1} = i\frac{e^{2i\pi x}\cdot 0 + 1}{e^{2i\pi x}\cdot 0 - 1} = -i,$$

we have

$$\lim_{y\to\infty} \cot(\pi z) = -i.$$

It follows that there exists $A_1 > 1$ such that

$$y \geq A_1 \Rightarrow (|\cot(\pi z)| - 1 = |\cot(\pi z)| - |-i| \leq)\ |\cot(\pi z) + i| < 1,$$

and hence

$$y \geq A_1 \Rightarrow |\cot(\pi z)| < 2.$$

Next, since

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{\frac{1}{2}\left(e^{i\pi z} + e^{-i\pi z}\right)}{\frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right)} = i\frac{1 + e^{-2i\pi z}}{1 - e^{-2i\pi z}} = i\frac{1 + e^{-2i\pi(x+iy)}}{1 - e^{-2i\pi(x+iy)}}$$
$$= i\frac{1 + e^{-2i\pi x}\frac{1}{e^{2\pi(-y)}}}{1 - e^{-2i\pi x}\frac{1}{e^{2\pi(-y)}}},$$

and

$$\lim_{y\to-\infty} i\frac{1 + e^{-2i\pi x}\frac{1}{e^{2\pi(-y)}}}{1 - e^{-2i\pi x}\frac{1}{e^{2\pi(-y)}}} = i\frac{1 + e^{-2i\pi x}\cdot 0}{1 - e^{-2i\pi x}\cdot 0} = i,$$

we have $\lim_{y\to-\infty}\cot(\pi z) = i$. It follows that there exists $A > A_1$ (> 1) such that

$$y \leq -A \Rightarrow (|\cot(\pi z)| - 1 = |\cot(\pi z)| - |i| \leq)\ |\cot(\pi z) - i| < 1,$$

and hence

$$y \leq -A \Rightarrow |\cot(\pi z)| < 2.$$

Since

$$y \geq A_1 \Rightarrow |\cot(\pi z)| < 2, y \leq -A \Rightarrow |\cot(\pi z)| < 2, \text{ and } A > A_1,$$

we have

$$(y \leq -A \text{ or } A \leq y) \Rightarrow |\cot(\pi z)| < 2.$$

Let us denote

$$\underbrace{\{x + iy : x \in [0,1], 1 \leq |y| \leq A\} \cup \left\{\frac{1}{2} + iy : y \in [-1,1]\right\}}_{= ([0,1] \times ([1,A] \cup [-A,-1])) \cup \left(\left\{\frac{1}{2}\right\} \times [-1,1]\right)}$$

by K. Clearly, K is a compact set, $\cot(\pi z)$ is defined on K, and is continuous on the compact set K. It follows that there exists real number $M > 2$ such that

$$z \in K \Rightarrow |\cot(\pi z)| < M.$$

Since

$$\cot(\pi(z+1)) = \cot(\pi z + \pi) = \cot(\pi z),$$

and

$$z \in K \Rightarrow |\cot(\pi z)| < M,$$

we have

$$z \in (K+1) \Rightarrow |\cot(\pi z)| < M.$$

Similarly,

$$z \in (K+2) \Rightarrow |\cot(\pi z)| < M,$$

etc. Thus,

$$z \in (\cup_{n \in \mathbb{Z}} (K+n)) \Rightarrow |\cot(\pi z)| < M.$$

Next,

$$(y \leq -A \text{ or } A \leq y) \Rightarrow |\cot(\pi z)| < 2,$$

and $M > 2$, and we have

$$z \in \left(\cup_{n\in\mathbb{Z}}(K+n)\right) \cup \{x+iy : A \leq |y|\} \Rightarrow |\cot(\pi z)| < M.$$

Since

$$K = ([0,1] \times ([1,A] \cup [-A,-1])) \cup \left(\left\{\frac{1}{2}\right\} \times [-1,1]\right),$$

we have

$$\begin{aligned}
&\left(\cup_{n\in\mathbb{Z}}(K+n)\right) \cup \{x+iy : A \leq |y|\} \\
&\quad = \left(\cup_{n\in\mathbb{Z}}\left(\left(([0,1] \times ([1,A] \cup [-A,-1])) \cup \left(\left\{\frac{1}{2}\right\} \times [-1,1]\right)\right) + n\right)\right) \cup \{x+iy : A \leq |y|\} \\
&\quad = (\mathbb{R} \times ([1,A] \cup [-A,-1])) \cup \left(\left\{\frac{1}{2}+n : n \in \mathbb{Z}\right\} \times [-1,1]\right) \cup \{x+iy : A \leq |y|\} \\
&\quad = (\{x+iy : 1 \leq |y| \leq A\} \cup \{x+iy : A \leq |y|\}) \cup \left(\left\{\frac{1}{2}+n : n \in \mathbb{Z}\right\} \times [-1,1]\right) \\
&\quad = \{x+iy : 1 \leq |y|\} \cup \left(\left\{\frac{1}{2}+n : n \in \mathbb{Z}\right\} \times [-1,1]\right).
\end{aligned}$$

Thus, for every $z \in \{x+iy : 1 \leq |y|\} \cup \left(\left\{\frac{1}{2}+n : n \in \mathbb{Z}\right\} \times [-1,1]\right)$, we have $|\cot(\pi z)| < M$.

Conclusion 4.123 There exists real number $M > 0$ such that for every

$$z \in \{x+iy : 1 \leq |y|\} \cup \left(\left\{\frac{1}{2}+n : n \in \mathbb{Z}\right\} \times [-1,1]\right), |\cot(\pi z)| < M.$$

Let us fix any $z \in (\mathbb{C} - \mathbb{Z})$.

Let Q_1 be the boundary of the square with lower-left corner $\left(-\left(1+\frac{1}{2}\right), -\left(1+\frac{1}{2}\right)\right)$ and side length $2 \times 1 + 1 (= 3)$, that is, Q_1 is a cycle

$$\begin{aligned}
&\left[-\left(1+\frac{1}{2}\right) - i\left(1+\frac{1}{2}\right), \left(1+\frac{1}{2}\right) - i\left(1+\frac{1}{2}\right)\right] \\
&\quad \dot{+} \left[\left(1+\frac{1}{2}\right) - i\left(1+\frac{1}{2}\right), \left(1+\frac{1}{2}\right) + i\left(1+\frac{1}{2}\right)\right] \\
&\quad \dot{+} \left[\left(1+\frac{1}{2}\right) + i\left(1+\frac{1}{2}\right), -\left(1+\frac{1}{2}\right) + i\left(1+\frac{1}{2}\right)\right] \\
&\quad \dot{+} \left[-\left(1+\frac{1}{2}\right) + i\left(1+\frac{1}{2}\right), -\left(1+\frac{1}{2}\right) - i\left(1+\frac{1}{2}\right)\right].
\end{aligned}$$

Suppose that z is inside Q_1. By Conclusion 4.122,

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{Q_1}\frac{\pi\cot\pi w}{w-z}\mathrm{d}w &= (\pi\cot\pi z)\Big((\mathrm{Ind})_{Q_1}(z)\Big)\\
&\quad+\sum_{n\in\mathbb{Z}}\left(\mathrm{Res}(\pi\cot\pi w;n)\cdot\frac{1}{n-z}\right)\Big((\mathrm{Ind})_{Q_1}(n)\Big) = (\pi\cot\pi z)\Big((\mathrm{Ind})_{Q_1}(z)\Big)\\
&\quad+\left(\mathrm{Res}(\pi\cot\pi w;-1)\cdot\frac{1}{-1-z}\right)\Big((\mathrm{Ind})_{Q_1}(-1)\Big)\\
&\quad+\left(\mathrm{Res}(\pi\cot\pi w;0)\cdot\frac{1}{0-z}\right)\Big((\mathrm{Ind})_{Q_1}(0)\Big)\\
&\quad+\left(\mathrm{Res}(\pi\cot\pi w;1)\cdot\frac{1}{1-z}\right)\Big((\mathrm{Ind})_{Q_1}(1)\Big)\\
&= (\pi\cot\pi z)1+\left(\mathrm{Res}(\pi\cot\pi w;-1)\cdot\frac{1}{-1-z}\right)1\\
&\quad+\left(\mathrm{Res}(\pi\cot\pi w;0)\cdot\frac{1}{0-z}\right)1+\left(\mathrm{Res}(\pi\cot\pi w;1)\cdot\frac{1}{1-z}\right)1,\end{aligned}$$

so

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{Q_1}\frac{\pi\cot\pi w}{w-z}dw &= \pi\cot\pi z-\mathrm{Res}(\pi\cot\pi w;-1)\cdot\frac{1}{z+1}\\
&\quad-\mathrm{Res}(\pi\cot\pi w;0)\cdot\frac{1}{z}-\mathrm{Res}(\pi\cot\pi w;1)\cdot\frac{1}{z-1}.\end{aligned}$$

Since

$$\begin{aligned}\pi\cot\pi w &= \pi\frac{\cos\pi w}{\sin\pi w} = \pi\frac{1-\frac{1}{2!}(\pi w)^2+\cdots}{\pi w-\frac{1}{3!}(\pi w)^3+\cdots} = \frac{1}{w}\frac{1-\frac{1}{2}(\pi w)^2+\cdots}{1-\frac{1}{6}(\pi w)^2+\cdots}\\
&= \frac{1}{w}\left(1-\frac{1}{3}(\pi w)^2+\cdots\right) = \frac{1}{w}-\frac{\pi^2}{3}w+\cdots,\end{aligned}$$

we have $\mathrm{Res}(\pi\cot\pi w;0)=1$. Since

$$\pi\cot\pi w = \pi\cot(\pi w+\pi) = \pi\cot\pi(w+1) = \frac{1}{w+1}-\frac{\pi^2}{3}(w+1)+\cdots,$$

we have

$$\mathrm{Res}(\pi\cot\pi w;-1)=1.$$

Since

$$\pi \cot \pi w = \pi \cot(\pi w - \pi) = \pi \cot \pi(w-1) = \frac{1}{w-1} - \frac{\pi^2}{3}(w-1) + \cdots,$$

we have $\mathrm{Res}(\pi \cot \pi w; 1) = 1$. It follows that

$$\frac{1}{2\pi i}\int_{Q_1} \frac{\pi \cot \pi w}{w-z}\mathrm{d}w = \pi \cot \pi z - 1 \cdot \frac{1}{z+1} - 1 \cdot \frac{1}{z} - 1 \cdot \frac{1}{z-1}$$

$$= \pi \cot \pi z - \left(\frac{1}{z+1} + \frac{1}{z} + \frac{1}{z-1}\right) = \pi \cot \pi z - \sum_{n=-1}^{1} \frac{1}{z-n},$$

that is

$$\pi \cot \pi z = \sum_{n=-1}^{1} \frac{1}{z-n} + \frac{1}{2\pi i}\int_{Q_1} \frac{\pi \cot \pi w}{w-z}\mathrm{d}w.$$

Similarly,

$$\pi \cot \pi z = \sum_{n=-2}^{2} \frac{1}{z-n} + \frac{1}{2\pi i}\int_{Q_2} \frac{\pi \cot \pi w}{w-z}\mathrm{d}w, \text{ etc.}$$

Problem 4.124 $\lim_{N\to\infty} \int_{Q_N} \frac{\cot \pi w}{w-z}\mathrm{d}w = 0.$

(Solution Observe that the integral of an even function over Q_N is equal to 0, and $w \mapsto \frac{\cot \pi w}{w}$ is an even function. It follows that $\int_{Q_N} \frac{\cot \pi w}{w}\mathrm{d}w = 0$. Here,

$$\int_{Q_N} \frac{\cot \pi w}{w-z}\mathrm{d}w = \int_{Q_N} \frac{\cot \pi w}{w-z}\mathrm{d}w - 0 = \int_{Q_N} \frac{\cot \pi w}{w-z}\mathrm{d}w - \int_{Q_N} \frac{\cot \pi w}{w}\,\mathrm{d}w$$

$$= \int_{Q_N} \left(\frac{\cot \pi w}{w-z} - \frac{\cot \pi w}{w}\right)\mathrm{d}w = \int_{Q_N} \left(\frac{1}{w-z} - \frac{1}{w}\right)\cot \pi w\,\mathrm{d}w$$

$$= \int_{Q_N} \frac{z}{(w-z)w}\cot \pi w\,\mathrm{d}w = z\int_{Q_N} \frac{\cot \pi w}{(w-z)w}\mathrm{d}w,$$

so it suffices to show that

$$\lim_{N\to\infty} \int_{Q_N} \frac{\cot \pi w}{(w-z)w}\mathrm{d}w = 0.$$

For every positive integer N, and for every $w \in Q_N$, by Conclusion 4.123, there exists a real number $M > 0$ such that $|\cot(\pi w)| < M$. For every positive integer N satisfying $2|z| \leq N$, and, for every $w \in Q_N$, we have

$$\left|\frac{\cot \pi w}{(w-z)w}\right| = |\cot \pi w| \frac{1}{|(w-z)w|} \leq M \frac{1}{|(w-z)w|} = M \frac{1}{|w|} \frac{1}{|w-z|} \leq M \frac{1}{N} \frac{1}{|w-z|},$$

and

$$|w-z| \geq |w| - |z| \geq N - |z| \geq N - \frac{N}{2} = \frac{N}{2},$$

so

$$\underbrace{\left|\frac{\cot \pi w}{(w-z)w}\right| \leq M \frac{1}{N} \frac{2}{N}} = 2M \frac{1}{N^2}.$$

Hence, for every positive integer N satisfying $2|z| \leq N$, we have

$$\underbrace{\left|\int\limits_{Q_N} \frac{\cot \pi w}{(w-z)w} \mathrm{d}w\right| \leq 2M \frac{1}{N^2} \cdot 4(2N+1)} = 8M \frac{1}{N}\left(2 + \frac{1}{N}\right) \to 0 \text{ as } N \to \infty.$$

Thus,

$$\lim_{N \to \infty} \int\limits_{Q_N} \frac{\cot \pi w}{(w-z)w} \mathrm{d}w = 0.$$

∎)

It follows that

$$\pi \cot \pi z = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Conclusion 4.125 For every $z \in (\mathbb{C} - \mathbb{Z})$,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Problem 4.126 Let $z_1, z_2, \ldots$ be any complex numbers. Then

$$|1 - z_1| < \frac{1}{2} \Rightarrow |1 - z_1 z_2| \leq 2(|1 - z_1| + |1 - z_2|).$$

(Solution Let $|z_1| - 1 = |z_1| - |1| \leq \underbrace{|1 - z_1| < \frac{1}{2}}$. We have to show that

$$|1 - z_1 z_2| \leq 2(|1 - z_1| + |1 - z_2|).$$

Here, $|z_1| \leq \frac{3}{2} \leq 2$. Now,

$$\begin{aligned}|1 - z_1 z_2| &= |(1 - z_1) + z_1(1 - z_2)| \leq |1 - z_1| + |z_1||1 - z_2| \leq |1 - z_1| \\ &\quad + 2|1 - z_2| \leq 2|1 - z_1| + 2|1 - z_2| = 2(|1 - z_1| + |1 - z_2|),\end{aligned}$$

so

$$|1 - z_1 z_2| \leq 2(|1 - z_1| + |1 - z_2|).$$

∎)

Problem 4.127 $|1 - z_1| + |1 - z_2| < \frac{1}{2} \Rightarrow |1 - z_1 z_2 z_3| \leq 2(|1 - z_1| + |1 - z_2| + |1 - z_3|)$.

(Solution Let

$$|1 - z_1| \leq \underbrace{|1 - z_1| + |1 - z_2| < \frac{1}{2}}.$$

We have to show that

$$|1 - z_1 z_2 z_3| \leq 2(|1 - z_1| + |1 - z_2| + |1 - z_3|).$$

Since $|1 - z_1| < \frac{1}{2}$, by Problem 4.126, we have

$$|z_1 z_2| - 1 = |z_1 z_2| - |1| \leq \underbrace{|1 - z_1 z_2| \leq 2(|1 - z_1| + |1 - z_2|)} \leq 2 \cdot \frac{1}{2} = 1,$$

and hence $|z_1 z_2| \leq 2$. Here

$$\begin{aligned}|1 - z_1 z_2 z_3| &= |(1 - z_1 z_2) + z_1 z_2(1 - z_3)| \leq |1 - z_1 z_2| + |z_1 z_2||1 - z_3| \\ &\leq 2(|1 - z_1| + |1 - z_2|) + |z_1 z_2||1 - z_3| \leq 2(|1 - z_1| + |1 - z_2|) \\ &\quad + 2|1 - z_3| = 2(|1 - z_1| + |1 - z_2| + |1 - z_3|),\end{aligned}$$

so

$$|1 - z_1 z_2 z_3| \le 2(|1 - z_1| + |1 - z_2| + |1 - z_3|).$$

∎)

Similarly,

$$|1 - z_1| + |1 - z_2| + |1 - z_3| < \frac{1}{2} \Rightarrow |1 - z_1 z_2 z_3 z_4|$$
$$\le 2(|1 - z_1| + |1 - z_2| + |1 - z_3| + |1 - z_4|), \text{etc.}$$

Conclusion 4.128 Let $z_1, z_2, \ldots$ be any complex numbers. Then, for every positive integer $N \ge 2$,

$$\sum_{n=1}^{N-1} |1 - z_n| < \frac{1}{2} \Rightarrow \left|1 - \prod_{n=1}^{N} z_n\right| \le 2 \sum_{n=1}^{N} |1 - z_n|.$$

Let $z_1, z_2, \ldots$ be any complex numbers. Suppose that $|1 - z_1| + |1 - z_2| + \cdots$ is convergent.

I. Problem 4.129 $\prod_{n=1}^{\infty} z_n \left(= \lim_{n\to\infty} \prod_{j=1}^{n} z_j\right)$ exists.

(Solution Since $|1 - z_1| + |1 - z_2| + \cdots$ is convergent, there exists a positive number N such that $|1 - z_{N+1}| + |1 - z_{N+2}| + \cdots < \frac{1}{2}$, and hence, by Conclusion 4.128, for every positive integer n,

$$|z_{N+1} \cdots z_{N+n}| - 1 = |z_{N+1} \cdots z_{N+n}| - |1|$$
$$= \underbrace{|1 - (z_{N+1} \cdots z_{N+n})| \le 2(|1 - z_{N+1}| + \cdots + |1 - z_{N+n}|)} \le 2 \cdot \frac{1}{2} = 1.$$

It follows that, for every positive integer n, $|z_{N+1} \cdots z_{N+n}| \le 2$.

We have to show that $\prod_{n=1}^{\infty} z_n$ exists, that is

$$\{z_1, z_1 z_2, \ldots, (z_1 \cdots z_N), (z_1 \cdots z_N) z_{N+1}, (z_1 \cdots z_N) z_{N+1} z_{N+2}, \ldots\}$$

is Cauchy, that is

$$\{(z_1 \cdots z_N) z_{N+1}, (z_1 \cdots z_N) z_{N+1} z_{N+2}, \ldots\}$$

is Cauchy, that is $\{z_{N+1}, z_{N+1} z_{N+2}, \ldots\}$ is Cauchy. Now, for every positive integers m, n satisfying $m < n$, we have

$$\left|(z_{N+1} \cdots z_{N+m})\left(z_{N+(m+1)} \cdots z_{N+n}\right) - (z_{N+1} \cdots z_{N+m})\right|$$
$$= |z_{N+1} \cdots z_{N+m}|\left|1 - z_{N+(m+1)} \cdots z_{N+n}\right| \le 2\left|1 - z_{N+(m+1)} \cdots z_{N+n}\right|$$
$$\le 2 \cdot 2\left(\left|1 - z_{N+(m+1)}\right| + \cdots + |1 - z_{N+n}|\right),$$

so for every positive integers m, n satisfying $m < n$, we have

$$|(z_{N+1} \cdots z_{N+n}) - (z_{N+1} \cdots z_{N+m})| \leq 4\left(\left|1 - z_{N+(m+1)}\right| + \cdots + |1 - z_{N+n}|\right).$$

Now, since

$|1 - z_1| + |1 - z_2| + \cdots$ is convergent, $\{z_{N+1}, z_{N+1}z_{N+2}, \ldots\}$ is Cauchy. ∎)

II. Problem 4.130 $\prod_{n=1}^{\infty} z_n \neq 0$, if each z_n is nonzero.

(**Solution** Let each z_n be nonzero. We have to show that $\prod_{n=1}^{\infty} z_n \neq 0$. Since $|1 - z_1| + |1 - z_2| + \cdots$ is convergent, there exists a positive number N such that $|1 - z_{N+1}| + |1 - z_{N+2}| + \cdots < \frac{1}{2}$, and hence, by Conclusion 4.128, for every positive integer n,

$$|1 - (z_{N+1} \cdots z_{N+n})| \leq 2(|1 - z_{N+1}| + \cdots + |1 - z_{N+n}|).$$

Now, by Problem 4.129,

$$\left|1 - \prod_{n=1}^{\infty} z_{N+n}\right| = \underbrace{\left|1 - \lim_{n\to\infty}(z_{N+1} \cdots z_{N+n})\right| \leq 2(|1 - z_{N+1}| + |1 - z_{N+2}| + \cdots)} < 2 \cdot \frac{1}{2} = 1.$$

It follows that $\prod_{n=1}^{\infty} z_{N+n} \neq 0$. Since $\prod_{n=1}^{\infty} z_{N+n} \neq 0$, and each z_n is nonzero, we have $\prod_{n=1}^{\infty} z_n \neq 0$. ∎)

Conclusion 4.131 Let $z_1, z_2, \ldots$ be any complex numbers. Suppose that $|1 - z_1| + |1 - z_2| + \cdots$ is convergent. Then

I. $\prod_{n=1}^{\infty} z_n$ exists,
II. if each z_n is nonzero, then $\prod_{n=1}^{\infty} z_n$ is nonzero.

Let S be a nonempty set. For every positive integer n, let $u_n : S \to \mathbb{C}$ be a function. Suppose that $\sum_{n=1}^{\infty} |1 - u_n|$ converges uniformly of S.

I. Problem 4.132 $\left\{\prod_{n=1}^{N} u_n\right\}$ converges uniformly to $\prod_{n=1}^{\infty} u_n$ on S.

(**Solution** Since $|1 - u_1| + |1 - u_2| + \cdots$ converges uniformly on S, there exists a positive number N such that $|1 - u_{N+1}| + |1 - u_{N+2}| + \cdots < \frac{1}{2}$ on S, and hence, by Conclusion 4.128, for every positive integer n,

$$\begin{aligned} |u_{N+1} \cdots u_{N+n}| - 1 &= |u_{N+1} \cdots u_{N+n}| - |1| \\ &= \underbrace{|1 - (u_{N+1} \cdots u_{N+n})| \leq 2(|1 - u_{N+1}| + \cdots + |1 - u_{N+n}|)} \leq 2 \cdot \frac{1}{2} \end{aligned}$$

on S. It follows that for every positive integer n, $|u_{N+1} \cdots u_{N+n}| \leq 2$ on S. We have to show that $\left\{\prod_{n=1}^{N} u_n\right\}$ converges uniformly on S, that is

$$\{u_1, u_1u_2, \ldots, (u_1\cdots u_N), (u_1\cdots u_N)u_{N+1}, (u_1\cdots u_N)u_{N+1}u_{N+2}, \ldots\}$$

is Cauchy on S, that is

$$\{(u_1\cdots u_N)u_{N+1}, (u_1\cdots u_N)u_{N+1}u_{N+2}, \ldots\}$$

is Cauchy on S, that is

$$\{u_{N+1}, u_{N+1}u_{N+2}, \ldots\}$$

is Cauchy on S. Now, for every positive integers m, n satisfying $m<n$, we have

$$\begin{aligned}
&\left|(u_{N+1}\cdots u_{N+m})\left(u_{N+(m+1)}\cdots u_{N+n}\right) - (u_{N+1}\cdots u_{N+m})\right| \\
&\quad = |u_{N+1}\cdots u_{N+m}|\left|1 - u_{N+(m+1)}\cdots u_{N+n}\right| \le 2\left|1 - u_{N+(m+1)}\cdots u_{N+n}\right| \\
&\quad \le 2\cdot 2\left(\left|1 - u_{N+(m+1)}\right| + \cdots + |1 - u_{N+n}|\right)
\end{aligned}$$

on S. It follows that for every positive integers m, n satisfying $m<n$, we have

$$|(u_{N+1}\cdots u_{N+n}) - (u_{N+1}\cdots u_{N+m})| \le 4\left(\left|1 - u_{N+(m+1)}\right| + \cdots + |1 - u_{N+n}|\right)$$

on S. Now, since $|1 - u_1| + |1 - u_2| + \cdots$ converges uniformly on S, $\{u_{N+1}, u_{N+1}u_{N+2}, \ldots\}$ is Cauchy on S. ∎)

II. Problem 4.133 If $z \in S$, and each $u_n(z)$ is nonzero, then $\prod_{n=1}^{\infty} u_n(z)$ is nonzero.

(Solution Let us fix any $z \in S$. Let each $u_n(z)$ be nonzero. We have to show that $\prod_{n=1}^{\infty} u_n(z) \neq 0$. Since

$$|1 - u_1(z)| + |1 - u_2(z)| + \cdots$$

is convergent, there exists a positive number N such that

$$|1 - u_{N+1}(z)| + |1 - u_{N+2}(z)| + \cdots < \frac{1}{2},$$

and hence, by Conclusion 4.128, for every positive integer n,

$$|1 - (u_{N+1}(z)\cdots u_{N+n}(z))| \le 2(|1 - u_{N+1}(z)| + \cdots + |1 - u_{N+n}(z)|).$$

Now, by Problem 4.132,

$$\begin{aligned}
&\left|1 - \prod_{n=1}^{\infty} u_{N+n}(z)\right| \\
&\quad = \underbrace{\left|1 - \lim_{n\to\infty}(u_{N+1}(z)\cdots u_{N+n}(z))\right| \le 2(|1 - u_{N+1}(z)| + |1 - u_{N+2}(z)| + \cdots)} < 2\cdot\frac{1}{2} = 1,
\end{aligned}$$

so $\prod_{n=1}^{\infty} u_{N+n}(z) \neq 0$. Since $\prod_{n=1}^{\infty} u_{N+n}(z) \neq 0$, and each $u_n(z)$ is nonzero, we have $\prod_{n=1}^{\infty} u_n(z) \neq 0$. ∎)

Conclusion 4.134 Let S be a nonempty set. For every positive integer n, let $u_n : S \to \mathbb{C}$ be a function. Suppose that $\sum_{n=1}^{\infty} |1 - u_n|$ converges uniformly of S. Then,

I. $\left\{\prod_{n=1}^{N} u_n\right\}$ converges uniformly to $\prod_{n=1}^{\infty} u_n$ on S,
II. if $z \in S$, and each $u_n(z)$ is nonzero, then $\prod_{n=1}^{\infty} u_n(z)$ is nonzero.

Since $z \mapsto |z^2|$ is continuous,

$$\left(\sum_{n=1}^{\infty}\left|1-\left(1-\frac{z^2}{n^2}\right)\right|=\right)\sum_{n=1}^{\infty}\left|\frac{z^2}{n^2}\right|$$

converges uniformly on compact subsets of $\mathbb{C}$, and hence, by Conclusion 4.134, $\left\{\prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right)\right\}$ converges uniformly to $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$ on compact subsets of $\mathbb{C}$. It follows that $\left\{\pi z\prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right)\right\}$ converges uniformly to $\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$ on compact subsets of $\mathbb{C}$.

Notation If f is a differentiable function, then by $L(f)$ we shall mean $\frac{f'}{f}$ at points where f is nonzero.

Let Ω be a region. For every positive integer n, let $f_n \in H(\Omega)$. Let $f \in H(\Omega)$. Suppose that $\{f_n\}$ converges uniformly to f on compact subsets of Ω. Suppose that $f \neq 0$. Let K be a nonempty compact subset of Ω. Suppose that f has no zero in K.

Problem 4.135 $\left\{\frac{(f_n)'}{f_n}\right\}$ converges uniformly to $\frac{f'}{f}$ on K.

(**Solution** Since $f_n \in H(\Omega)$, $\{f_n\}$ converges uniformly to f on K, and f has no zero in K, $\{\log \circ f_n\}$ converges uniformly to $\log \circ f$ on K, and hence, by Conclusion 1.172, $\left\{\frac{(f_n)'}{f_n}\right\}$ converges uniformly to $\frac{f'}{f}$ on K. ∎)

Conclusion 4.136 Let Ω be a region. For every positive integer n, let $f_n \in H(\Omega)$. Let $f \in H(\Omega)$. Suppose that $\{f_n\}$ converges uniformly to f on compact subsets of Ω. Suppose that $f \neq 0$. Let K be a nonempty compact subset of Ω. Suppose that f has no zero in K. Then $\{L(f_n)\}$ converges uniformly to $L(f)$ on K.

Let Ω be a region. Let $f, g \in H(\Omega)$. Suppose that f has no zero in Ω, and g has no zero in Ω. Let $L(f) = L(g)$ on Ω. Since

$$\left(\frac{f'}{f}=\right)L(f)=L(g)\left(=\frac{g'}{g}\right),$$

we have

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = \frac{f}{g}\left(\frac{f'}{f} - \frac{g'}{g}\right) = 0,$$

and hence $\frac{f}{g} = $ constant.

Conclusion 4.137 Let Ω be a region. Let $f, g \in H(\Omega)$. Suppose that f has no zero in Ω, and g has no zero in Ω. Let $L(f) = L(g)$ on Ω. Then there exists a complex number c such that $f = cg$.

By Conclusion 4.125, for every $z \in (\mathbb{C} - \mathbb{Z})$,

$$\begin{aligned} L(\sin \pi z) = \pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty}\left(L\left(1 - \frac{z}{n}\right) + L\left(1 + \frac{z}{n}\right)\right) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} L\left(\left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right)\right) \\ &= L(z) + \sum_{n=1}^{\infty} L\left(1 - \frac{z^2}{n^2}\right) = L\left(z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\right), \end{aligned}$$

so, by Conclusion 4.137, there exists a complex number c such that

$$\sin \pi z = cz\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

It follows that

$$\pi = \underbrace{\lim_{z\to 0}\frac{\sin \pi z}{z} = \lim_{z\to 0} c\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)} = c\prod_{n=1}^{\infty}\left(1 - \frac{0^2}{n^2}\right) = c,$$

and hence, for every $z \in (\mathbb{C} - \mathbb{Z})$, we have $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$.

Conclusion 4.138 For every complex number z, $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$.

This formula, known as **Euler's infinite product of sine function**, is due to L. Euler (15.04.1707–18.09.1783).

4.9 Gauss's Formula

Note 4.139 Observe that $\{a_n\}$ is a sequence of nonzero complex numbers where $a_1 = -1, a_2 = -2, a_3 = -3, \ldots$. Here, $\lim_{n\to\infty}|a_n| = \infty$. Observe that $\{p_n\}$ is a sequence of nonnegative integers where, each $p_n = 1$. Next, for every $r > 0$,

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+p_n} = \sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{1+1} = r^2\left(\frac{1}{|a_1|^2}+\frac{1}{|a_2|^2}+\frac{1}{|a_3|^2}+\frac{1}{|a_4|^2}+\cdots\right)$$
$$= r^2\left(\frac{1}{|-1|^2}+\frac{1}{|-2|^2}+\frac{1}{|-3|^2}+\cdots\right) = r^2\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) < \infty.$$

It follows, by Conclusion 2.44, that

1. $$\underbrace{P : z \mapsto \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)} = \prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}$$
$$= \prod_{n=1}^{\infty}\left(1-\frac{z}{-n}\right)e^{\frac{z}{-n}} = \frac{\left(1+\frac{z}{1}\right)}{e^{\frac{z}{1}}}\frac{\left(1+\frac{z}{2}\right)}{e^{\frac{z}{2}}}\cdots$$

   is an entire function,

2. $-1, -2, -3, \ldots$ are the only zeros of P, and each zero of P is simple.

   It follows that

$$\underbrace{h_1 : z \mapsto \frac{e^{\frac{z}{1}}}{\left(1+\frac{z}{1}\right)}\frac{e^{\frac{z}{2}}}{\left(1+\frac{z}{2}\right)}\frac{e^{\frac{z}{3}}}{\left(1+\frac{z}{3}\right)}\cdots} = \prod_{n=1}^{\infty}\frac{e^{\frac{z}{n}}}{\left(1+\frac{z}{n}\right)}$$

is a function meromorphic in $\mathbb{C}$ with simple poles at $-1, -2, -3, \ldots$, and hence

$$\underbrace{h : z \mapsto \frac{1}{z}\frac{e^{\frac{z}{1}}}{\left(1+\frac{z}{1}\right)}\frac{e^{\frac{z}{2}}}{\left(1+\frac{z}{2}\right)}\frac{e^{\frac{z}{3}}}{\left(1+\frac{z}{3}\right)}\cdots} = \frac{1}{z}\prod_{n=1}^{\infty}\frac{e^{\frac{z}{n}}}{\left(1+\frac{z}{n}\right)}$$

is a function meromorphic in $\mathbb{C}$ with simple poles at $0, -1, -2, -3, \ldots$. For every positive integer N, let

$$h_N : z \mapsto \frac{1}{z}\prod_{n=1}^{N}\frac{e^{\frac{z}{n}}}{\left(1+\frac{z}{n}\right)}.$$

It follows that

$$
\begin{aligned}
h_N(z+1) &= \frac{1}{z+1}\prod_{n=1}^{N}\frac{e^{\frac{z+1}{n}}}{\left(1+\frac{z+1}{n}\right)} = \frac{1}{z+1}\prod_{n=1}^{N}\frac{e^{\frac{z}{n}}e^{\frac{1}{n}}}{\left(1+\frac{z+1}{n}\right)}\\
&= \frac{1}{z+1}e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\cdot\prod_{n=1}^{N}e^{\frac{z}{n}}\cdot\prod_{n=1}^{N}\frac{n}{n+z+1}\\
&= \frac{1}{z+1}e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\cdot\prod_{n=1}^{N}e^{\frac{z}{n}}\cdot\left(\frac{1}{z+2}\frac{2}{z+3}\cdots\frac{N-1}{z+N}\frac{N}{z+(N+1)}\right)\\
&= \frac{1}{z+1}e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\cdot\prod_{n=1}^{N}e^{\frac{z}{n}}\cdot\left(\frac{2}{z+2}\cdots\frac{N}{z+N}\right)\frac{1}{z+(N+1)}\\
&= e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\cdot\prod_{n=1}^{N}e^{\frac{z}{n}}\cdot\left(\frac{1}{z+1}\frac{2}{z+2}\cdots\frac{N}{z+N}\right)\frac{1}{z+(N+1)}\\
&= \frac{1}{z+(N+1)}e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\cdot\prod_{n=1}^{N}e^{\frac{z}{n}}\cdot\prod_{n=1}^{N}\frac{n}{z+n}\\
&= \frac{1}{z+(N+1)}e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\prod_{n=1}^{N}\frac{e^{\frac{z}{n}}}{\left(1+\frac{z}{n}\right)}\\
&= \frac{1}{z+(N+1)}e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\cdot zh_N(z) = z\frac{N}{z+(N+1)}\cdot\frac{1}{N}e^{1+\frac{1}{2}+\cdots+\frac{1}{N}}\cdot h_N(z)\\
&= z\frac{N}{z+(N+1)}e^{\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)-\ln N}\cdot h_N(z),
\end{aligned}
$$

and hence

$$
\begin{aligned}
h(z+1) &= \underbrace{\lim_{N\to\infty}h_N(z+1) = \lim_{N\to\infty}z\frac{N}{z+(N+1)}e^{\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)-\ln N}\cdot h_N(z)}\\
&= z\lim_{N\to\infty}\frac{1}{1+(z+1)\frac{1}{N}}e^{\lim_{N\to\infty}\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)-\ln N\right)}\cdot\lim_{N\to\infty}h_N(z)\\
&= z\frac{1}{1+0}e^{\gamma}h(z) = e^{\gamma}zh(z).
\end{aligned}
$$

Thus, $h(z+1) = e^{\gamma}zh(z)$. Clearly,

$$
\underbrace{g : z \mapsto \frac{e^{-\gamma z}}{z}\frac{e^{\frac{z}{1}}}{\left(1+\frac{z}{1}\right)}\frac{e^{\frac{z}{2}}}{\left(1+\frac{z}{2}\right)}\frac{e^{\frac{z}{3}}}{\left(1+\frac{z}{3}\right)}\cdots} = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\frac{e^{\frac{z}{n}}}{\left(1+\frac{z}{n}\right)} = e^{-\gamma z}h(z)
$$

is a function meromorphic in $\mathbb{C}$ with simple poles at $0, -1, -2, -3, \ldots$. Also, for every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$,

$$g(z+1) = e^{-\gamma(z+1)}h(z+1) = e^{-\gamma(z+1)}e^{\gamma}zh(z) = z \cdot e^{-\gamma z}h(z) = zg(z),$$

so, for every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\}), g(z+1) = zg(z)$. Clearly, g has no zero.

I. Problem 4.140 For every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$,

$$\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\frac{e^{\frac{z}{n}}}{\left(1+\frac{z}{n}\right)} = \underbrace{g(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}},$$

(Solution Let us take any $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$. We have to show that

$$\frac{e^{-\gamma z}}{z}\frac{e^{\frac{z}{1}}}{\left(1+\frac{z}{1}\right)}\frac{e^{\frac{z}{2}}}{\left(1+\frac{z}{2}\right)}\frac{e^{\frac{z}{3}}}{\left(1+\frac{z}{3}\right)}\cdots = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

$$\begin{aligned}
\text{LHS} &= \frac{e^{-\gamma z}}{z}\frac{e^{\frac{z}{1}}}{\left(1+\frac{z}{1}\right)}\frac{e^{\frac{z}{2}}}{\left(1+\frac{z}{2}\right)}\frac{e^{\frac{z}{3}}}{\left(1+\frac{z}{3}\right)}\cdots \\
&= \frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\prod_{k=1}^{n}\frac{e^{\frac{z}{k}}}{\left(1+\frac{z}{k}\right)} = \frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\left(\prod_{k=1}^{n}\frac{e^{\frac{z}{k}}}{\left(1+\frac{z}{k}\right)}\right) \\
&= \frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\left(\prod_{k=1}^{n}\frac{ke^{\frac{z}{k}}}{z+k}\right) = \frac{e^{-\gamma z}}{z}\lim_{n\to\infty}\left(\frac{1e^{\frac{z}{1}}}{z+1}\frac{2e^{\frac{z}{2}}}{z+2}\cdots\frac{ne^{\frac{z}{n}}}{z+n}\right) \\
&= e^{-\gamma z}\lim_{n\to\infty}\left(\frac{n!}{z(z+1)(z+2)\cdots(z+n)}e^{z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)}e^{(\ln n)z}\right) \\
&= e^{-\gamma z}e^{z\lim\limits_{n\to\infty}\left(\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\ln n\right)}\lim_{n\to\infty}\frac{n!e^{(\ln n)z}}{z(z+1)(z+2)\cdots(z+n)} \\
&= e^{-\gamma z}e^{z\gamma}\lim_{n\to\infty}\left(\frac{n!e^{(\ln n)z}}{z(z+1)(z+2)\cdots(z+n)}\right) \\
&= \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)} = \text{RHS}.
\end{aligned}$$

∎)

II. Clearly,

$$\begin{aligned}
g(1) &= \lim_{n\to\infty}\frac{n!n^1}{1(1+1)\cdots(1+n)} = \lim_{n\to\infty}\frac{n}{(1+n)} = 1,\ g(2) = 1\cdot g(1) \\
&= g(1) = 1 = 1!, g(3) = 2\cdot g(2) = 2\cdot(1!) = 2!, \text{etc.}
\end{aligned}$$

Next,

$$\begin{aligned}\operatorname{Res}(g;-1) &= \lim_{z\to-1}(z—1)g(z) = \lim_{z\to-1}(z+1)g(z)\\ &= \lim_{z\to-1}\frac{(z+1)(zg(z))}{z} = \lim_{z\to-1}\frac{(z+1)g(z+1)}{z}\\ &= \lim_{z\to-1}\frac{g(z+2)}{z} = \frac{g(-1+2)}{-1} = -g(1) = -1,\end{aligned}$$

so $\operatorname{Res}(g;-1) = -1$. Also,

$$\begin{aligned}\operatorname{Res}(g;-2) &= \lim_{z\to-2}(z—2)g(z) = \lim_{z\to-2}(z+2)g(z)\\ &= \lim_{z\to-2}\frac{(z+2)((z+1)zg(z))}{(z+1)z} = \lim_{z\to-2}\frac{(z+2)g(z+2)}{(z+1)z}\\ &= \lim_{z\to-2}\frac{g(z+3)}{(z+1)z} = \frac{g(-2+3)}{(-2+1)(-2)} = \frac{1}{2!}g(1) = \frac{1}{2!}\cdot 1 = \frac{1}{2!},\end{aligned}$$

so $\operatorname{Res}(g;-2) = \frac{1}{2!}$. Similarly, $\operatorname{Res}(g;-3) = -\frac{1}{3!}$, etc. Further,

$$\operatorname{Res}(g;0) = \lim_{z\to0}(z-0)g(z) = \lim_{z\to0} g(z+1) = g(0+1) = g(1) = 1.$$

III. Thus, for every $n \in \{0,1,2,\ldots\}, \operatorname{Res}(g;-n) = (-1)^n\frac{1}{n!}$.

IV. Problem 4.141 $\left\{\left(1+\frac{z}{n}\right)^n\right\}$ converges to e^z in $H(\mathbb{C})$.

(Solution Let K be a nonempty compact subset of $\mathbb{C}$. By Conclusion 3.147, it suffices to show that,

$$\left\{n\log\left(1+\frac{z}{n}\right)\right\} = \underbrace{\left\{\log\left(\left(1+\frac{z}{n}\right)^n\right)\right\}}$$

converges uniformly to z for every z in K. It follows that K is bounded, and hence there exists a positive integer n_0 such that for every $z \in K$, we have $|z| < n_0$. For every $n > n_0$, and for every $z \in K$, we have

$$\begin{aligned}\left|n\log\left(1+\frac{z}{n}\right) - z\right| &= \left|n\left(-\frac{\left(\frac{z}{n}\right)^2}{2!} + \frac{\left(\frac{z}{n}\right)^3}{3!} - \cdots\right)\right| = |z|\left|\frac{\frac{z}{n}}{2!} - \frac{\left(\frac{z}{n}\right)^2}{3!} + \cdots\right|\\ &\le |z|\left(\left|\frac{\frac{z}{n}}{2!}\right| + \left|\frac{\left(\frac{z}{n}\right)^2}{3!}\right| + \cdots\right) \le |z|\left(\frac{|z|}{n} + \left(\frac{|z|}{n}\right)^2 + \cdots\right)\\ &= |z|\frac{\frac{|z|}{n}}{1-\frac{|z|}{n}} = |z|^2\frac{1}{n-|z|} \le (n_0)^2\frac{1}{n-|z|}\\ &\le (n_0)^2\frac{1}{n-n_0} \to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence, $\left\{\log\left(\left(1+\frac{z}{n}\right)^n\right)\right\}$ converges uniformly to z for z in K. ∎)

V. Problem 4.142 For every $t \in [0, \infty), t \le n \Rightarrow \left(1 - \frac{t}{n}\right)^n \le e^{-t}$.

(Solution Let us fix any $t_0 \in [0, \infty)$. Next, let us take any integer $n \ge t_0$. We have to show that

$$\begin{aligned} - t_0 - n\left(\frac{\left(\frac{t_0}{n}\right)^2}{2} + \frac{\left(\frac{t_0}{n}\right)^3}{3} + \cdots\right) &= n\left(\frac{-t_0}{n} - \frac{\left(-\frac{t_0}{n}\right)^2}{2} + \frac{\left(-\frac{t_0}{n}\right)^3}{3} - \cdots\right) \\ &= n\ln\left(1 - \frac{t_0}{n}\right) = \underbrace{\ln\left(\left(1 - \frac{t_0}{n}\right)^n\right) \le - t_0,} \end{aligned}$$

that is

$$0 \le n\left(\frac{\left(\frac{t_0}{n}\right)^2}{2} + \frac{\left(\frac{t_0}{n}\right)^3}{3} + \cdots\right).$$

This is trivially true. ∎)

VI. Problem 4.143 For every complex number z satisfying $\mathrm{Re}(z) > 0$, $\Gamma(z) = g(z)$.

(Solution Since g is a function meromorphic in $\mathbb{C}$ with simple poles at $0, -1, -2, -3, \ldots$, g is holomorphic on $\{z : \mathrm{Re}(z) > 0\}$. By Conclusion 4.108, Γ is meromorphic in $\mathbb{C}$ with simple poles at $0, -1, -2, -3, \ldots$, so Γ is holomorphic on $\{z : \mathrm{Re}(z) > 0\}$. By Problem 4.140, for every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$,

$$g(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)},$$

and, by Conclusion 4.95, for every $x \in (0, \infty)$,

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)},$$

so, for every $x \in (0, \infty)$, $\Gamma(x) = g(x)$. Since 1 is a limit point of $\{z : \Gamma(z) = g(z)\}$ $(\supset (0, \infty))$, and $1 \in \{z : \mathrm{Re}(z) > 0\}$, by Theorem 1.135, for every $z \in \{w : \mathrm{Re}(w) > 0\}$, $\Gamma(z) = g(z)$. ∎)

Thus, for every complex number z satisfying $\mathrm{Re}(z) > 0$,

$$\int\limits_0^\infty e^{-t}t^{z-1}\mathrm{d}t = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\frac{e^{\frac{z}{n}}}{\left(1 + \frac{z}{n}\right)} = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)} \quad (*).$$

VII. Problem 1.144 For every complex number $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$, $\Gamma(z) = g(z)$.

(Solution Since g is a function meromorphic in $\mathbb{C}$ with simple poles at $0, -1, -2, -3, \ldots$, g is holomorphic on $(\mathbb{C} - \{0, -1, -2, \ldots\})$. By Conclusion 4.108, Γ is meromorphic in $\mathbb{C}$ with simple poles at $0, -1, -2, -3, \ldots$, so Γ is holomorphic on $(\mathbb{C} - \{0, -1, -2, \ldots\})$. By Problem 4.140, for every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$,

$$g(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)},$$

and by Conclusion 4.95, for every $x \in (0, \infty)$,

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)},$$

so, for every $x \in (0, \infty)$, $\Gamma(x) = g(x)$. Since 1 is a limit point of $\{z : \Gamma(z) = g(z)\} (\supset (0, \infty))$, and $1 \in (\mathbb{C} - \{0, -1, -2, \ldots\})$, by Theorem 4.135, for every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$, $\Gamma(z) = g(z)$. ∎)

Thus, for every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$, we have

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{\frac{z}{n}}}{\left(1 + \frac{z}{n}\right)} = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)} \quad (**).$$

Here, the second equality known as the **Gauss's formula**, is due to J. C. F. Gauss (30.04.1777–23.02.1855).

VIII. Let $z \in (\mathbb{C} - \mathbb{Z})$.

Problem 4.145 $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ $(\neq 0)$. And hence, Γ has no zero in $(\mathbb{C} - \{0, -1, -2, \ldots\})$.

(Solution

$$\begin{aligned}
\text{LHS} &= \Gamma(z)\Gamma(1-z) = \Gamma(z)\Gamma((-z)+1) = \Gamma(z)\cdot(-\mathrm{z})\Gamma(-z) \\
&= (-\mathrm{z})\cdot\Gamma(z)\cdot\Gamma(-z) = (-z)\cdot\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\frac{e^{\frac{z}{n}}}{\left(1+\frac{z}{n}\right)}\cdot\frac{e^{-\gamma(-z)}}{(-z)}\prod_{n=1}^{\infty}\frac{e^{\frac{-z}{n}}}{\left(1+\frac{-z}{n}\right)} \\
&= \frac{1}{z}\prod_{n=1}^{\infty}\frac{1}{\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n}\right)} = \frac{1}{z}\prod_{n=1}^{\infty}\frac{1}{1-\frac{z^2}{n^2}} = \frac{1}{z}\frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} \\
&= \frac{\pi}{\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} = \frac{\pi}{\sin \pi z} = \text{RHS}.
\end{aligned}$$

∎)

IX. Problem 4.146 For every $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$, $\Gamma(z+1) = z\Gamma(z)$.

(Solution Let us fix any $z \in (\mathbb{C} - \{0, -1, -2, \ldots\})$. We have to show that $\Gamma(z+1) = z\Gamma(z)$. We have seen that $g(z+1) = zg(z)$. Now, by (VII), $\Gamma(z+1) = z\Gamma(z)$. ∎)

X. Conclusion 4.147 $\Gamma : (\mathbb{C} - \{0, -1, -2, \ldots\}) \to (\mathbb{C} - \{0\})$ is a function.

4.10 Preparation for the Prime Number Theorem

Note 4.148

Notation In the discussion of the prime number theorem, it is customary to denote a complex variable by s, $\mathrm{Re}(s)$ by σ, and $\mathrm{Im}(s)$ by t.

I. Problem 4.149 There are infinite-many prime numbers.

(Solution If not, otherwise suppose that there are finite-many prime numbers. We have to arrive at a contradiction.

Let $\{p_1, \ldots, p_k\}$ be the set of all distinct prime numbers. By the fundamental theorem of arithmetic, there exists a prime p such that $p|((p_1 \cdots p_k) + 1)$. Since p is a prime, and $\{p_1, \ldots, p_k\}$ is the set of all distinct prime numbers, $p \in \{p_1, \ldots, p_k\}$, and hence there exists $j \in \{1, \ldots, k\}$ such that $p = p_j$. Since $p = p_j$, and $p|((p_1 \cdots p_k) + 1)$, we have $p_j|((p_1 \cdots p_k) + 1)$ where $j \in \{1, \ldots, k\}$. This contradicts the fact that for every $j \in \{1, \ldots, k\}, p_j$ does not divide $(p_1 \cdots p_k) + 1$. ∎)

II. Problem 4.150 Suppose that p_n is the n-th prime (in increasing order: $2, 3, 5, 7, 11, \ldots$). Then $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots$ is divergent.

(Solution If not, otherwise, let

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots$$

be convergent. We have to arrive at a contradiction. Since

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \cdots$$

is convergent, there exists a positive integer N such that

$$(0 <)\ \frac{1}{p_{N+1}} + \frac{1}{p_{N+2}} + \frac{1}{p_{N+3}} + \cdots < \frac{1}{2},$$

and hence

$$\left(\frac{1}{p_{N+1}}+\frac{1}{p_{N+2}}+\frac{1}{p_{N+3}}+\cdots\right)+\left(\frac{1}{p_{N+1}}+\frac{1}{p_{N+2}}+\frac{1}{p_{N+3}}+\cdots\right)^2 + \left(\frac{1}{p_{N+1}}+\frac{1}{p_{N+2}}+\frac{1}{p_{N+3}}+\cdots\right)^3+\cdots<\infty \quad (*).$$

Observe that for every positive integer r,

$$p_1\nmid(r(p_1\cdots p_N)+1),\ldots,p_N\nmid(r(p_1\cdots p_N)+1).$$

It follows that for every positive integer r, there exists k_r primes in $\{p_{N+1},p_{N+2},\ldots\}$ whose product is $r(p_1\cdots p_N)+1$, and hence for every positive integer r,

$$\frac{1}{r(p_1\cdots p_N)+1}$$

is a summand in the expansion of

$$\left(\frac{1}{p_{N+1}}+\frac{1}{p_{N+2}}+\frac{1}{p_{N+3}}+\cdots\right)^{k_r}.$$

Now, from $(*)$,

$$\frac{1}{p_1\cdots p_N}\sum_{r=1}^{\infty}\frac{1}{r+1}=\sum_{r=1}^{\infty}\frac{1}{r(p_1\cdots p_N)+(p_1\cdots p_N)}<\underbrace{\sum_{r=1}^{\infty}\frac{1}{r(p_1\cdots p_N)+1}}$$

is convergent, and hence $\sum_{r=1}^{\infty}\frac{1}{r}$ is convergent. This contradicts the fact that $\sum_{r=1}^{\infty}\frac{1}{r}$ is divergent. ∎)

Definition By $\mu(1)$, we mean 1. If $p_1,\ldots,p_k$ are prime numbers, and $a_1,\ldots,a_k$ are positive integers, then

$$\mu((p_1)^{a_1}\cdots(p_k)^{a_k})=\begin{cases}(-1)^k & \text{if } a_1=\cdots=a_k=1\\ 0 & \text{otherwise.}\end{cases}$$

Thus, $\mu:\mathbb{N}\to\{-1,0,1\}$ $(\subset\mathbb{C})$ is an arithmetic function, and is called the ***Möbius function***.

Here,

$$\mu(5\cdot 7)=(-1)^2,\mu\left(5\cdot 7^3\right)=0,\mu(2\cdot 5\cdot 7)=(-1)^3,\text{etc.}$$

III. Problem 4.151 For every positive integer N, $\sum_{d|N} \mu(d) = \begin{cases} 1 & \text{if } N = 1 \\ 0 & \text{if } N > 1. \end{cases}$ In other words, for every positive integer N, $\sum_{d|N} \mu(d) = \left[\frac{1}{N}\right]$, where $[x]$ denotes the greatest integer $\leq x$.

(Solution For $N = 1, \sum_{d|N} \mu(d) = \mu(1) = 1$. So, it remains to show that for every $N > 1, \sum_{d|N} \mu(d) = 0$.

For this purpose, let us take any positive integer $N > 1$. Let $(p_1)^{a_1} \cdots (p_k)^{a_k}$ be the prime factorization of N, where $p_1, \ldots, p_k$ are k distinct primes. We have to show that

$$\sum_{d|((p_1)^{a_1} \cdots (p_k)^{a_k})} \mu(d) = 0.$$

By the definition of μ,

$$\sum_{d|(p_1)^{a_1} \cdots (p_k)^{a_k}} \mu(d) = \mu(1) + \left(\underbrace{\mu(p_1) + \cdots + \mu(p_k)}_{k \text{ terms}} \right)$$
$$+ \left(\underbrace{\mu(p_1p_2) + \mu(p_1p_3) + \cdots + \mu(p_{k-1}p_k)}_{\binom{k}{2} \text{terms}} \right) + \cdots$$
$$= 1 + \left(\underbrace{(-1)^1 + \cdots + (-1)^1}_{k \text{ terms}} \right) + \left(\underbrace{(-1)^2 + (-1)^2 + \cdots + (-1)^2}_{\binom{k}{2} \text{terms}} \right)$$
$$+ \cdots = 1 + \binom{k}{1}(-1)^1 + \binom{k}{2}(-1)^2 + \cdots = (1 + (-1))^k = 0$$

Thus,

$$\sum_{d|((p_1)^{a_1} \cdots (p_k)^{a_k})} \mu(d) = 0.$$

■)

Definition For every positive integer n, by $\varphi(n)$ we mean the number of elements in the set $\{k : k : 1 \leq k \leq n, \text{ and gcd of } k \text{ and } n \text{ is } 1\}$. In other words, for every positive integer n,

$$\varphi(n) \equiv |\{k : 1 \leq k \leq n, \text{ and } (k, n) = 1\}|.$$

Thus, $\varphi : \mathbb{N} \to \mathbb{N}\ (\subset \mathbb{C})$ is an arithmetic function, and is called the ***Euler's totient function***. Here,

$$\left\{k : k : 1 \leq k \leq 2 \cdot 3^2, \text{ and } \left(k, 2 \cdot 3^2\right) = 1\right\} = \{1, 5, 7, 11, 13, 17\},$$

so

$$\varphi\left(2 \cdot 3^2\right) = |\{1, 5, 7, 11, 13, 17\}| = 6, \text{etc.}$$

IV. Problem 4.152 For every positive integer N, $\sum_{d|N} \varphi(d) = N$.

(Solution Let us fix any positive integer N. We have to show that

$$\sum_{d|N} \varphi(d) = N.$$

Let $S \equiv \{1, \ldots, N\}$. Observe that for every positive integer d satisfying $d|N$,

$$d \in \{k : k \in S, \text{and } (k, N) = d\}.$$

Thus, the collection

$$\{\{k : k \in S, \text{and } (k, N) = d\} : d|N\}$$

is a partition of S, and hence

$$(N =)\ |S| = \sum_{d|N} |\{k : k \in S, \text{and } (k, N) = d\}|.$$

Next, by the definition of φ,

$$\underbrace{\sum_{d|N} \varphi(d) = \sum_{d|N} |\{k : 1 \leq k \leq d, \text{ and } (k, d) = 1\}|}$$

$$= \sum_{d|N} \left| \left\{k : 1 \leq k \leq \frac{N}{d}, \text{ and } \left(k, \frac{N}{d}\right) = 1\right\} \right|,$$

so

$$\sum_{d|N} \varphi(d) = \sum_{d|N} \left| \left\{k : 1 \leq k \leq \frac{N}{d}, \text{and} \left(k, \frac{N}{d}\right) = 1\right\} \right|.$$

Thus, it suffices to show that

$$\sum_{d|N}\left|\left\{k : 1 \le k \le \frac{N}{d}, \text{and}\left(k, \frac{N}{d}\right) = 1\right\}\right| = \sum_{d|N} |\{k : k \in S, \text{and } (k, N) = d\}|.$$

Again, it suffices to show that for every positive integer d satisfying $d|N$,

$$\left\{m : 1 \le m \le \frac{N}{d}, \text{and}\left(m, \frac{N}{d}\right) = 1\right\}$$

is in 1-1 correspondence with $|\{k : k \in S, \text{and}(k, N) = d\}|$. For this purpose, let us fix any positive integer d satisfying $d|N$. Observe that if $k \in S$ $(= \{1, \ldots, N\})$, and $(k, N) = d$, then $1 \le \frac{k}{d} \le \frac{N}{d}$, and $\left(\frac{k}{d}, \frac{N}{d}\right) = 1$, and hence

$$\frac{k}{d} \in \left\{m : 1 \le m \le \frac{N}{d}, \text{and } \left(m, \frac{N}{d}\right) = 1\right\}.$$

Thus, for every positive integer d satisfying $d|N$, $\Lambda_d : k \mapsto \frac{k}{d}$ is a function from $\{k : k \in S, \text{and } (k, N) = d\}$ to

$$\left\{m : 1 \le m \le \frac{N}{d}, \text{and } \left(m, \frac{N}{d}\right) = 1\right\}.$$

Clearly, each Λ_d is 1-1. It remains to show that

$$\Lambda_d : \{k : k \in S, \text{and } (k, N) = d\} \to \left\{m : 1 \le m \le \frac{N}{d}, \text{and } \left(m, \frac{N}{d}\right) = 1\right\}$$

is onto. For this purpose, let us take any positive integer m satisfying $1 \le m \le \frac{N}{d}, \text{and}\left(m, \frac{N}{d}\right) = 1$. It follows that $(1 \le)\ d \le md \le N$, and $\left(m \cdot d, \frac{N}{d} \cdot d\right) = 1 \cdot d$, and hence

$$md \in \{k : k \in S, \text{and } (k, N) = d\}.$$

Next, $\Lambda_d(md) = \frac{md}{d} = m$. Hence, Λ_d is onto. ∎)

V. Problem 4.153 For every positive integer $N \ge 2$,

$$\sum_{d|N} \frac{\mu(d)}{d} = \prod_{\substack{cp|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right).$$

(Solution Since

$$\sum_{d|2} \frac{\mu(d)}{d} = \frac{\mu(1)}{1} + \frac{\mu(2)}{2} = \frac{1}{1} + \frac{-1}{2} = \frac{1}{2},$$

and

$$\prod_{\substack{cp|2 \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{2}\right) = \frac{1}{2},$$

$\sum_{d|N} \frac{\mu(d)}{d} = \prod_{\substack{p|N \\ p \text{ isprime}}} \left(1 - \frac{1}{p}\right)$ holds for $N = 2$. Now, let us take any positive integer $N > 2$. It suffices to show that

$$\sum_{d|N} \frac{\mu(d)}{d} = \prod_{\substack{p|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right).$$

Let $(p_1)^{a_1} \cdots (p_k)^{a_k}$ be the prime factorization of N, where $p_1, \ldots, p_k$ are k distinct primes. Now,

$$\text{RHS} = \prod_{\substack{p|N \\ p \text{ isprime}}} \left(1 - \frac{1}{p}\right) = \prod_{\substack{p|((p_1)^{a_1} \cdots (p_k)^{a_k}) \\ p \text{ isprime}}} \left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

$$= 1 + \left(\underbrace{\frac{(-1)^1}{p_1} + \cdots + \frac{(-1)^1}{p_k}}_{\binom{k}{1} \text{terms}} \right) + \left(\underbrace{\frac{(-1)^2}{p_1 p_2} + \frac{(-1)^2}{p_1 p_3} + \cdots + \frac{(-1)^2}{p_{k-1} p_k}}_{\binom{k}{2} \text{terms}} \right) + \cdots$$

$$= \frac{\mu(1)}{1} + \left(\underbrace{\frac{\mu(p_1)}{p_1} + \cdots + \frac{\mu(p_k)}{p_k}}_{\binom{k}{1} \text{terms}} \right) + \left(\underbrace{\frac{\mu(p_1 p_2)}{p_1 p_2} + \frac{\mu(p_1 p_3)}{p_1 p_3} + \cdots + \frac{\mu(p_{k-1} p_k)}{p_{k-1} p_k}}_{\binom{k}{2} \text{terms}} \right)$$

$$+ \cdots = \sum_{d|N} \frac{\mu(d)}{d} = \text{LHS}.$$

∎)

Definition Let $f : \mathbb{N} \to \mathbb{C}$, and $g : \mathbb{N} \to \mathbb{C}$ be arithmetic functions. The arithmetic function

$$(f * g) : N \mapsto \sum_{d|N} f(d) g\left(\frac{N}{d}\right)$$

from $\mathbb{N}$ to $\mathbb{C}$ is called the ***Dirichlet product*** *of f and g*.

VI. Problem 4.154 For every arithmetic function f and g, $f * g = g * h$.

(Solution Let us take any positive integer N. We have to show that

$$(f * g)(N) = (g * f)(N).$$

$$\begin{aligned} \text{LHS} = (f * g)(N) &= \sum_{a \cdot b = N} f(a) g(b) = \sum_{a \cdot b = N} g(b) f(a) \\ &= \sum_{b \cdot a = N} g(a) f(b) = \sum_{a \cdot b = N} g(a) f(b) = (g * f)(N) = \text{RHS}. \end{aligned}$$

∎)

VIIa. Problem 4.155 For every arithmetic function f, g and $(f * g) * h = f * (g * h)$.

(Solution Let us take any positive integer N. We have to show that

$$((f * g) * h)(N) = (f * (g * h))(N).$$

Since

$$\begin{aligned} ((f * g) * h)(N) &= \sum_{a \cdot b = N} (f * g)(a) h(b) \\ &= \sum_{a \cdot b = N} \left(\sum_{c \cdot d = a} f(c) g(d) \right) h(b) = \sum_{a \cdot b = N} \left(\sum_{c \cdot d = a} f(c) g(d) h(b) \right) \\ &= \sum_{(c \cdot d) \cdot b = N} f(c) g(d) h(b) = \sum_{a \cdot b \cdot c = N} f(a) g(b) h(c), \end{aligned}$$

we have

$$((f * g) * h)(N) = \sum_{a \cdot b \cdot c = N} f(a) g(b) h(c).$$

Similarly,

$$
\begin{aligned}
((f*(g*h))(N) &=)((g*h)*f)(N) \\
&= \sum_{a\cdot b\cdot c=N} g(a)h(b)f(c) = \sum_{a\cdot b\cdot c=N} f(c)g(a)h(b) \\
&= \sum_{b\cdot c\cdot a=N} f(a)g(b)h(c) = \sum_{a\cdot b\cdot c=N} f(a)g(b)h(c) \\
&= ((f*g)*h)(N),
\end{aligned}
$$

and hence

$$((f*g)*h)(N) = (f*(g*h))(N).$$

∎)

VIIb. Problem 4.156 For every arithmetic function f, $f*I=f$, where I denotes the arithmetic function

$$n \mapsto \left[\frac{1}{n}\right].$$

(Solution Let us take any positive integer N. We have to show that

$$(f*I)(N) = f(N).$$

$$
\begin{aligned}
\text{LHS} = (f*I)(N) = \sum_{d|N} f(d)I\left(\frac{N}{d}\right) &= f(N)I\left(\frac{N}{N}\right) \\
= f(N)I(1) = f(N)\cdot 1 = f(N) &= \text{RHS}.
\end{aligned}
$$

∎)

VIII. Problem 4.157 For every arithmetic function f for which $f(1)$ is nonzero, there exists an arithmetic function g for which $g(1)$ is nonzero, and $f*g=I$, where I denotes the arithmetic function $n \mapsto \left[\frac{1}{n}\right]$.

Notation Here, g is denoted by f^{-1}. Thus, $f*(f^{-1})=I$.

(Solution Since we seek g such that

$$f(1)g(1) = \sum_{a\cdot b=1} f(a)g(b) = \underbrace{(f*g)(1) = I(1)} = 1,$$

we have $g(1) = \frac{1}{f(1)}$. Since we seek g such that

$$f(1)g(2) + f(2)g(1) = \sum_{a\cdot b=2} f(a)g(b) = \underbrace{(f*g)(2) = I(2)} = 0,$$

we have $g(2) = -\frac{f(2)g(1)}{f(1)}$. Since we seek g such that

$$f(1)g(3) + f(3)g(1) = \sum_{a\cdot b=3} f(a)g(b) = \underbrace{(f * g)(3) = I(3)} = 0,$$

we have $g(3) = -\frac{f(3)g(1)}{f(1)}$. Since we seek g such that

$$f(1)g(4) + f(4)g(1) + f(2)g(2) = \sum_{a\cdot b=4} f(a)g(b) = \underbrace{(f * g)(4) = I(4)} = 0,$$

we have $g(4) = -\frac{f(4)g(1)+f(2)g(2)}{f(1)}$, etc. Thus, an arithmetic function g is uniquely defined for which $g(1)$ is nonzero, and $f * g = I$. ■)

IX. Clearly, the collection of all those arithmetic functions f for which $f(1)$ is nonzero constitute an abelian group under Dirichlet multiplication $*$ as binary operation, and the arithmetic function $I : n \mapsto \left[\frac{1}{n}\right]$ serves the purpose of the identity element.

Definition The arithmetic function $u : n \mapsto 1$ is called the ***unit function***.

X. Problem 4.158 $\mu^{-1} = u$.

(Solution Since for every positive integer N,

$$(u * \mu)(N) = \sum_{a\cdot b=N} u(a)\mu(b) = \sum_{a\cdot b=N} 1 \cdot \mu(b) = \sum_{b|N} \mu(b) = I(N),$$

we have $u * \mu = I$, and hence $\mu^{-1} = u$. ■)

XI. Problem 4.159 Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. Let f be the arithmetic function $N \mapsto \sum_{d|N} g(d)$. Then, for every positive integer N, $g(N) = \sum_{d|N} f(d) \cdot \mu\left(\frac{N}{d}\right)$.

(Solution For every positive integer N,

$$\underbrace{f(N) = \sum_{d|N} g(d)} = \sum_{d|N} g(d) \cdot 1 = \sum_{d|N} g(d) \cdot u\left(\frac{N}{d}\right) = (g * u)(N),$$

we have $f = g * u$, and hence

$$\underbrace{f * \mu = (g * u) * \mu} = g * (u * \mu) = g * I = g.$$

Thus, $g = f * \mu$. Now, for every positive integer N,

$$g(N) = (f * \mu)(N) = \sum_{d|N} f(d) \cdot \mu\left(\frac{N}{d}\right).$$

■)

XII. Problem 4.160 For every positive integer $N \geq 2$, $\varphi(N) = \sum_{d|N} \mu(d) \frac{N}{d}$.

(Solution By Problem 4.152, for every positive integer $N \geq 2$,

$$\mathrm{Id}(N) = N = \underbrace{\sum_{d|N} \varphi(d)}_{},$$

so, by Problem 4.159, for every positive integer N,

$$\underbrace{\varphi(N) = \sum_{d|N} \mathrm{Id}(d) \cdot \mu\left(\frac{N}{d}\right)}_{} = \sum_{d|N} \mathrm{Id}\left(\frac{N}{d}\right) \cdot \mu(d) = \sum_{d|N} \frac{N}{d} \mu(d) = \sum_{d|N} \mu(d) \frac{N}{d},$$

and hence for every positive integer $N \geq 2$, $\varphi(N) = \sum_{d|N} \mu(d) \frac{N}{d}$. ■)

XIII. Problem 4.161 For every positive integer $N \geq 2$, $\varphi(N) = N \prod_{\substack{p|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right)$.

(Solution By Problem 4.160, for every positive integer $N \geq 2$,

$$\underbrace{\varphi(N) = \sum_{d|N} \mu(d) \frac{N}{d}}_{} = N \sum_{d|N} \frac{\mu(d)}{d},$$

and, by Problem 4.153, for every positive integer $N \geq 2$, $\sum_{d|N} \frac{\mu(d)}{d} = \prod_{\substack{p|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right)$, so, for every positive integer $N \geq 2$,

$$\varphi(N) = N \prod_{\substack{p|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right).$$

■)

Definition Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. If for every positive integers m, n satisfying $(m, n) = 1$, $f(mn) = f(m)f(n)$, then we say that f *is* ***multiplicative***.

XIV. Problem 4.162 The Euler's totient function φ is multiplicative.

(Solution For this purpose, let us take any positive integers M, N satisfying $(M, N) = 1$, and $M \geq 2, N \geq 2$. We have to show that $\varphi(MN) = \varphi(M)\varphi(N)$. On using Problem 4.161,

$$\begin{aligned}
\text{LHS} = \varphi(MN) &= MN \prod_{\substack{p|MN \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right) \\
&= MN \left(\prod_{\substack{p|M \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right) \right) \left(\prod_{\substack{p|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right) \right) \\
&= \left(M \prod_{\substack{p|M \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right) \right) \left(N \prod_{\substack{p|N \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right) \right) \\
&= \varphi(M)\varphi(N) = \text{RHS}.
\end{aligned}$$

∎)

XV. Problem 4.163 Let a, b be positive real numbers such that $a < b$. Let $f : [a, b] \to \mathbb{C}$ be a continuously differentiable function. Then

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\mathrm{d}t + \int_a^b (t - [t])f'(t)\mathrm{d}t + ([b] - b)f(b) - ([a] - a)f(a).$$

(Solution Here,

$$\begin{aligned}
\text{RHS} &= \int_a^b f(t)\mathrm{d}t + \int_a^b (t - [t])f'(t)\mathrm{d}t + ([b] - b)f(b) - ([a] - a)f(a) \\
&= \left((f(t) \cdot t)\Big|_{t=a}^{t=b} - \int_a^b f'(t) \cdot t\mathrm{d}t \right) + \int_a^b (t - [t])f'(t)\mathrm{d}t
\end{aligned}$$

$$+([b]-b)f(b)-([a]-a)f(a)=(f(b)\cdot b-f(a)\cdot a)$$
$$-\int_a^b f'(t)\cdot t\mathrm{d}t+\left(\int_a^b t\cdot f'(t)\mathrm{d}t-\int_a^b [t]\cdot f'(t)\mathrm{d}t\right)$$
$$+([b]f(b)-bf(b))-([a]f(a)-af(a))$$
$$=-\int_a^b [t]\cdot f'(t)\mathrm{d}t+[b]f(b)-[a]f(a),$$

so it suffices to show that

$$\int_a^b [t]\cdot f'(t)\mathrm{d}t=-\sum_{a<n\leq b} f(n)+[b]f(b)-[a]f(a).$$

Next,

$$\mathrm{LHS}=\int_a^b [t]\cdot f'(t)\mathrm{d}t=\int_{[a]}^b [t]\cdot f'(t)\mathrm{d}t-\int_{[a]}^a [t]\cdot f'(t)\mathrm{d}t=\int_{[a]}^b [t]\cdot f'(t)\mathrm{d}t$$
$$-\int_{[a]}^a [a]\cdot f'(t)\mathrm{d}t=\int_{[a]}^b [t]\cdot f'(t)\mathrm{d}t-[a]\int_{[a]}^a f'(t)\mathrm{d}t=\int_{[a]}^b [t]\cdot f'(t)\mathrm{d}t$$
$$-[a](f(a)-f([a]))=\left(\int_{[a]}^{[b]} [t]\cdot f'(t)\mathrm{d}t+\int_{[b]}^b [t]\cdot f'(t)\mathrm{d}t\right)$$
$$-[a](f(a)-f([a]))=\left(\int_{[a]}^{[b]} [t]\cdot f'(t)\mathrm{d}t+\int_{[b]}^b [b]\cdot f'(t)\mathrm{d}t\right)$$
$$-[a](f(a)-f([a]))=\int_{[a]}^{[b]} [t]\cdot f'(t)\mathrm{d}t+[b](f(b)-f([b]))$$
$$-[a](f(a)-f([a]))=\left(\int_{[a]}^{[a]+1} [t]\cdot f'(t)\mathrm{d}t+\int_{[a]+1}^{[a]+2} [t]\cdot f'(t)\mathrm{d}t\right.$$
$$\left.+\cdots+\int_{[b]-1}^{[b]} [t]\cdot f'(t)\mathrm{d}t\right)+[b](f(b)-f([b]))-[a](f(a)-f([a]))$$
$$=\left(\int_{[a]}^{[a]+1} [a]\cdot f'(t)\mathrm{d}t+\int_{[a]+1}^{[a]+2} ([a]+1)\cdot f'(t)\mathrm{d}t+\cdots+\int_{[b]-1}^{[b]} ([b]-1)\cdot f'(t)\mathrm{d}t\right)$$

$$
\begin{aligned}
&+[b](f(b)-f([b]))-[a](f(a)-f([a]))\\
&=([a](f([a]+1)-f([a]))+([a]+1)(f([a]+2)-f([a]+1))\\
&\quad+\cdots+([b]-1)(f([b])-f([b]-1)))\\
&\quad+[b](f(b)-f([b]))-[a](f(a)-f([a]))\\
&=((([a]+1)f([a]+1)-[a]f([a])-f([a]+1))\\
&\quad+(([a]+2)f([a]+2)-([a]+1)f([a]+1)-f([a]+2))\\
&\quad+(([a]+3)f([a]+3)-([a]+2)f([a]+2)-f([a]+3))\\
&\quad+\cdots+([b]f([b])-([b]-1)f([b]-1)-f([b])))\\
&\quad+[b](f(b)-f([b]))-[a](f(a)-f([a]))\\
&=-[a]f([a])+[b]f([b])-(f([a]+1)+f([a]+2)+\cdots+f([b]))\\
&\quad+[b](f(b)-f([b]))-[a](f(a)-f([a]))\\
&=-(f([a]+1)+f([a]+2)+\cdots+f([b]))+[b]f(b)-[a]f(a)\\
&=-\sum_{a<n\le b}f(n)+[b]f(b)-[a]f(a)=\text{RHS}.
\end{aligned}
$$

∎)

XVI. Conclusion 4.164 If $0<a<b$, and $f:[a,b]\to\mathbb{C}$ is a continuously differentiable function, then $\sum_{a<n\le b}f(n)=\int_a^b f(t)\mathrm{d}t+\int_a^b(t-[t])f'(t)\mathrm{d}t+([b]-b)f(b)-([a]-a)f(a)$.

4.11 Abscissa of Convergence of a Dirichlet Series

Note 4.165

Definition Let $f:\mathbb{N}\to\mathbb{C}$ be an arithmetic function. Let $s(\equiv\sigma+it,$ where σ,t are real numbers$)$ be a complex number. The series

$$1+\frac{f(2)}{2^s}+\frac{f(3)}{3^s}+\cdots$$

is called a ***Dirichlet series*** *with coefficients* $f(n)$.

I. Suppose that the Dirichlet series $\sum_{n=1}^{\infty}\frac{f(n)}{n^{a+ib}}$ converges absolutely, where a,b are real numbers.

Problem 4.166 $\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ converges absolutely in the 'half-plane' $\{s:a\le\sigma\}$.

(Solution First of all, we note that if $a\le\sigma$, then for every positive integer n, $|n^s|=n^{\sigma}\ge n^a$, and hence for every positive integer n,

$$\left|\frac{f(n)}{n^s}\right| = \frac{|f(n)|}{|n^s|} \le \frac{|f(n)|}{n^a}.$$

Now, since

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{a+ib}}\right| \left(= \sum_{n=1}^{\infty}\frac{|f(n)|}{n^a}\right)$$

is convergent, $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ is convergent. Thus, $\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ converges absolutely in the 'half-plane' $\{s : a \le \sigma\}$. ■)

Conclusion 4.167 Suppose that the Dirichlet series $\sum_{n=1}^{\infty}\frac{f(n)}{n^{a+ib}}$ converges absolutely, where a, b are real numbers. Then $\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ converges absolutely in the 'half-plane' $\{s : a \le \sigma\}$.

II. Suppose that $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ does not converge for all complex number s, and, $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ does not diverge for all complex number s. Let D be the set of all real numbers σ for which $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ is divergent.

Since $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ does not converge for all complex number s, D is a nonempty set of real numbers. Since $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ does not diverge for all complex number s, there exists a complex number $a+ib$, where $a, b \in \mathbb{R}$, such that $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{a+ib}}\right|$ is convergent.

Problem 4.168 a is an upper bound of D.

(Solution If not, otherwise suppose that there exists a complex number $c+id$, where $c, d \in \mathbb{R}$ such that $a < c$ and $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{c+id}}\right|$ diverges. Since $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{a+ib}}\right|$ is convergent, and $a < c$, by (I), $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{c+id}}\right|$ is convergent. This is a contradiction. ■)

Since D is a nonempty bounded above set of real numbers, $\sup D$ exists. Thus, $\sup D \in \mathbb{R}$. Also, $\sup D \le a$.

Case I: when $\sigma < \sup D$. Here, there exists a complex number $c+id$, where $c, d \in \mathbb{R}$ such that $c \in D$, $\sigma < c$ and $\left(\sum_{n=1}^{\infty}\frac{|f(n)|}{n^c} =\right) \sum_{n=1}^{\infty}\left|\frac{f(n)}{n^{c+id}}\right|$ is divergent. Since $\sigma < c$, we have, for every positive integer n, $(\ln n)\sigma \le (\ln n)c$, and hence, for every positive integer n, $n^\sigma = \underbrace{e^{(\ln n)\sigma} \le e^{(\ln n)c}} = n^c$. Thus, for every positive integer n, $\frac{|f(n)|}{n^c} \le \frac{|f(n)|}{n^\sigma}$. Now, since $\sum_{n=1}^{\infty}\frac{|f(n)|}{n^c}$ is divergent, $\left(\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right| =\right) \sum_{n=1}^{\infty}\frac{|f(n)|}{n^\sigma}$ is divergent.

Case II: when $\sup D < \sigma$. It follows that $\sigma \notin D$, and hence $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ is convergent.

Conclusion 4.169 Suppose that $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ does not converge for all s, and $\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$ does not diverge for all s. Then there exists a real number σ_a such that

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\begin{cases}\text{converges absolutely} & \text{if } \sigma_a<\sigma\\ \text{does not converge absolutely} & \text{if } \sigma<\sigma_a.\end{cases}$$

Here, σ_a is called the ***abscissa of absolute convergence***.

III. Let N be a positive real number. Let $\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ be a Dirichlet series whose abscissa of absolute convergence is σ_a. Let c be a real number such that $\sigma_a<c$.

Problem 4.170 For every σ satisfying $c\le\sigma$,

$$\left|\sum_{n=N}^{\infty}\frac{f(n)}{n^s}\right|\le\frac{1}{N^{\sigma-c}}\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c}.$$

(**Solution** Let us take any complex number $s=\sigma+it$, where $c\le\sigma$. Now,

$$\left|\sum_{n=N}^{\infty}\frac{f(n)}{n^s}\right|\le\sum_{n=N}^{\infty}\left|\frac{f(n)}{n^s}\right|=\sum_{n=N}^{\infty}\frac{|f(n)|}{|n^s|}=\sum_{n=N}^{\infty}\frac{|f(n)|}{n^\sigma}=\sum_{n=N}^{\infty}|f(n)|\frac{1}{n^c}n^{c-\sigma}$$
$$=\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c}\frac{1}{n^{\sigma-c}}\le\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c}\frac{1}{N^{\sigma-c}}=\frac{1}{N^{\sigma-c}}\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c},$$

so

$$\left|\sum_{n=N}^{\infty}\frac{f(n)}{n^s}\right|\le\frac{1}{N^{\sigma-c}}\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c}.$$

∎)

Conclusion 4.171 Let N be a positive real number. Let $\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$ be a Dirichlet series whose abscissa of absolute convergence is σ_a. Let c be a real number such that $\sigma_a<c$. Then, for every σ satisfying $c\le\sigma$,

$$\left|\sum_{n=N}^{\infty}\frac{f(n)}{n^s}\right|\le\frac{1}{N^{\sigma-c}}\sum_{n=N}^{\infty}\frac{|f(n)|}{n^c}.$$

IV. Let $F:s\mapsto\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\,(\sigma_a<\sigma)$ be the 'sum' function. Thus, for every s in $\{s=\sigma+it:\sigma_a<\sigma\}$,

$$F(s) = f(1) + \sum_{n=2}^{\infty} \frac{f(n)}{n^s}.$$

Problem 4.172 $\lim_{\sigma\to\infty}\left(\sum_{n=2}^{\infty}\frac{f(n)}{n^s}\right) = 0$ uniformly for $t \in \mathbb{R}$.

(Solution Let us take any c satisfying $\sigma_a < c$. For every σ satisfying $c \le \sigma$, by Conclusion 4.171, we have

$$\underbrace{\left|\sum_{n=2}^{\infty}\frac{f(n)}{n^s}\right| \le \frac{1}{2^{\sigma-c}}\sum_{n=2}^{\infty}\frac{|f(n)|}{n^c}} = \left(2^c\sum_{n=2}^{\infty}\frac{|f(n)|}{n^c}\right)\frac{1}{2^\sigma},$$

and hence $\left|\sum_{n=2}^{\infty}\frac{f(n)}{n^s}\right| \le M\frac{1}{2^\sigma}$, where $M\left(\equiv 2^c\sum_{n=2}^{\infty}\frac{|f(n)|}{n^c}\right)$ is independent of σ and t. Now, since $\frac{1}{2^\sigma} \to 0$ as $\sigma \to \infty$, $\lim_{\sigma\to\infty}\left(\sum_{n=2}^{\infty}\frac{f(n)}{n^{\sigma+it}}\right) = 0$ uniformly for $t \in \mathbb{R}$. ■)

Conclusion 4.173 Let

$$F : s \mapsto \sum_{n=1}^{\infty}\frac{f(n)}{n^s}\,(\sigma_a < \sigma)$$

be the sum function. Then $\lim_{\sigma\to\infty} F(\sigma + it) = f(1)$ uniformly for $t \in \mathbb{R}$.

V. Let

$$F : s \mapsto \sum_{n=1}^{\infty}\frac{f(n)}{n^s}\,(\sigma_a < \sigma), \text{and } G : s \mapsto \sum_{n=1}^{\infty}\frac{g(n)}{n^s}\,(\sigma_a < \sigma)$$

be two sum functions. Let $\{s_k\}$ be a sequence in $\{s = \sigma + it : \sigma_a < \sigma\}$ such that $\lim_{k\to\infty}\sigma_k = \infty$. Suppose that for every positive integer k, $F(s_k) = G(s_k)$.

Problem 4.174 For every positive integer n, $f(n) = g(n)$.

(Solution If not, otherwise suppose that there exists a positive integer n_0 such that $f(n_0) \ne g(n_0)$. We have to arrive at a contradiction.

There exists a positive integer N such that $f(N) \ne g(N), f(N-1) = g(N-1), \ldots, f(1) = g(1)$. Let $h \equiv f - g$, and $H \equiv F - G$. Here, $h(N) \ne 0, h(N-1) = 0, \ldots, h(1) = 0$, and each $H(s_k) = 0$. Since

$$F : s \mapsto \sum_{n=1}^{\infty}\frac{f(n)}{n^s} \text{ and } G : s \mapsto \sum_{n=1}^{\infty}\frac{g(n)}{n^s}$$

are absolutely convergent in $\{s=\sigma+it : \sigma_a<\sigma\}$, $(H \equiv)(F-G) : s \mapsto \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is absolutely convergent in $\{s=\sigma+it : \sigma_a<\sigma\}$, and hence the abscissa of absolute convergence for $H : s \mapsto \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is $\leq \sigma_a$.

Let us fix any c satisfying $\sigma_a<c$. Next, let us take any $\varepsilon>0$. Since $\frac{N}{N+1} \in (0,1)$, and $\lim_{k\to\infty} \sigma_k=\infty$, we have $\lim_{k\to\infty}\left(\frac{N}{N+1}\right)^{\sigma_k}=0$. It follows that there exists a positive integer K_1 such that $k \geq K_1 \Rightarrow \left(\frac{N}{N+1}\right)^{\sigma_k}<\varepsilon$. Since $\lim_{k\to\infty} \sigma_k=\infty$, there exists a positive integer K_2 such that $k \geq K_2 \Rightarrow c \leq \sigma_k$. Put $K \equiv \max\{K_1, K_2\}$. Thus,

$$k \geq K \Rightarrow \left(c \leq \sigma_k \text{ and} \left(\frac{N}{N+1}\right)^{\sigma_k}<\varepsilon\right).$$

Next, for every positive integer $k \geq K$, we have

$$\begin{aligned}
|h(N)| &= \left|N^{s_k}\frac{h(N)}{N^{s_k}}\right| = N^{\sigma_k}\left|\frac{h(N)}{N^{s_k}}\right| \\
&= N^{\sigma_k}\left|\sum_{n=1}^{\infty}\frac{h(n)}{n^{s_k}} - \left(\frac{h(1)}{1^{s_k}}+\cdots+\frac{h(N-1)}{(N-1)^{s_k}}\right) - \sum_{n=N+1}^{\infty}\frac{h(n)}{n^{s_k}}\right| \\
&= N^{\sigma_k}\left|\sum_{n=1}^{\infty}\frac{h(n)}{n^{s_k}} - \left(\frac{0}{1^{s_k}}+\cdots+\frac{0}{(N-1)^{s_k}}\right) - \sum_{n=N+1}^{\infty}\frac{h(n)}{n^{s_k}}\right| \\
&= N^{\sigma_k}\left|H(s_k) - \sum_{n=N+1}^{\infty}\frac{h(n)}{n^{s_k}}\right| = N^{\sigma_k}\left|0 - \sum_{n=N+1}^{\infty}\frac{h(n)}{n^{s_k}}\right| \\
&= N^{\sigma_k}\left|\sum_{n=N+1}^{\infty}\frac{h(n)}{n^{s_k}}\right| \leq N^{\sigma_k}\cdot\frac{1}{(N+1)^{\sigma_k-c}}\sum_{n=N+1}^{\infty}\frac{|h(n)|}{n^c} \leq \left(\frac{N}{N+1}\right)^{\sigma_k}\cdot A,
\end{aligned}$$

where $A \equiv (N+1)^c \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^c}$. Thus,

$$k \geq K \Rightarrow 0 \leq |h(N)| \leq A \cdot \left(\frac{N}{N+1}\right)^{\sigma_k} < A\varepsilon,$$

where A is independent of k Hence, $h(N)=0$. This is a contradiction. ■)

Conclusion 4.175 Let

$$F : s \mapsto \sum_{n=1}^{\infty}\frac{f(n)}{n^s}\,(\sigma_a<\sigma), \text{and } G : s \mapsto \sum_{n=1}^{\infty}\frac{g(n)}{n^s}\,(\sigma_a<\sigma)$$

be two sum functions. Let $\{s_k\}$ be a sequence in $\{s = \sigma + it : \sigma_a < \sigma\}$ such that $\lim_{k\to\infty} \sigma_k = \infty$. Suppose that for every positive integer k, $F(s_k) = G(s_k)$. Then for every positive integer n, $f(n) = g(n)$.

VI. Let $F : s \mapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \, (\sigma_a < \sigma)$ be a sum function. Suppose that $F(s)$ is nonzero for some s satisfying $\sigma > \sigma_a$.

Problem 4.176 There exists $c > \sigma_a$ such that F is nonzero at every point of the half-plane

$$\{s = \sigma + it : c \leq \sigma\}.$$

(Solution If not, otherwise suppose that, for every $c > \sigma_a$, there exists $s \equiv \sigma + it$ such that $c \leq \sigma$, and $\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = 0$. We have to arrive at a contradiction.

It follows that for every positive integer k, there exists $s_k \equiv \sigma_k + it_k$ such that $k < \sigma_k$ and $\sum_{n=1}^{\infty} \frac{f(n)}{n^{s_k}} = 0$, and hence $\lim_{k\to\infty} \sigma_k = \infty$. Now, by Conclusion 4.175, for every positive integer n, $f(n) = 0$, and hence $F(s)$ is zero for every s satisfying $\sigma > \sigma_a$. This contradicts the assumption. ■)

Conclusion 4.177 Let $F : s \mapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \, (\sigma_a < \sigma)$ be a sum function. Suppose that $F(s)$ is nonzero for some s satisfying $\sigma > \sigma_a$. Then there exists $c > \sigma_a$ such that F is nonzero at every point of the half-plane $\{s = \sigma + it : c \leq \sigma\}$.

VII. Let $F : s \mapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \, (a < \sigma)$ and $G : s \mapsto \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \, (b < \sigma)$ be sum functions.

Problem 4.178 For every s for which both series converge absolutely,

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f * g)(n)}{n^s}.$$

(Solution For every s for which both series converge absolutely, their Cauchy product

$$\frac{f(1)}{1^s}\frac{g(1)}{1^s} + \left(\frac{f(1)}{1^s}\frac{g(2)}{2^s} + \frac{f(2)}{2^s}\frac{g(1)}{1^s}\right) + \cdots$$

converges absolutely, and

$$\begin{aligned}(F(s)G(s) =)&\left(\frac{f(1)}{1^s} + \frac{f(2)}{2^s} + \cdots\right)\left(\frac{g(1)}{1^s} + \frac{g(2)}{2^s} + \cdots\right)\\ =&\frac{f(1)}{1^s}\frac{g(1)}{1^s} + \left(\frac{f(1)}{1^s}\frac{g(2)}{2^s} + \frac{f(2)}{2^s}\frac{g(1)}{1^s}\right) + \cdots.\end{aligned}$$

Since

$$\frac{f(1)}{1^s}\frac{g(1)}{1^s}+\left(\frac{f(1)}{1^s}\frac{g(2)}{2^s}+\frac{f(2)}{2^s}\frac{g(1)}{1^s}\right)+\cdots$$

converges absolutely, we can rearrange its terms in any way without altering the sum, and hence

$$\begin{aligned}
\text{LHS} = F(s)G(s) &= \frac{f(1)}{1^s}\frac{g(1)}{1^s}+\left(\frac{f(1)}{1^s}\frac{g(2)}{2^s}+\frac{f(2)}{2^s}\frac{g(1)}{1^s}\right)\\
&\quad+\left(\frac{f(1)}{1^s}\frac{g(3)}{3^s}+\frac{f(3)}{3^s}\frac{g(1)}{1^s}\right)+\left(\frac{f(1)}{1^s}\frac{g(4)}{4^s}+\frac{f(2)}{2^s}\frac{g(2)}{2^s}+\frac{f(4)}{4^s}\frac{g(1)}{1^s}\right)+\cdots\\
&=\sum_{k=1}^{\infty}\left(\sum_{m\cdot n=k}\frac{f(m)g(n)}{(mn)^s}\right)=\sum_{k=1}^{\infty}\left(\sum_{m|k}\frac{f(m)g\left(\frac{k}{m}\right)}{k^s}\right)\\
&=\sum_{k=1}^{\infty}\left(\frac{1}{k^s}\sum_{m|k}f(m)g\left(\frac{k}{m}\right)\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n^s}\sum_{m|n}f(m)g\left(\frac{n}{m}\right)\right)\\
&=\sum_{n=1}^{\infty}\left(\frac{1}{n^s}(f*g)(n)\right)=\sum_{n=1}^{\infty}\frac{(f*g)(n)}{n^s}=\text{RHS}.
\end{aligned}$$

∎)

Conclusion 4.179 Let $F : s \mapsto \sum_{n=1}^{\infty}\frac{f(n)}{n^s}\,(a<\sigma)$ and $G : s \mapsto \sum_{n=1}^{\infty}\frac{g(n)}{n^s}\,(b<\sigma)$ be sum functions. Then, for every s for which both series converge absolutely, $F(s)G(s)=\sum_{n=1}^{\infty}\frac{(f*g)(n)}{n^s}$.

VIII. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function. Let $\sum_{n=1}^{\infty} f(n)$ be absolutely convergent.

Since $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent,

$$\sum_{k=0}^{\infty}\left|f\left(p^k\right)\right| \leq \underbrace{\sum_{n=1}^{\infty}|f(n)|<\infty}_{},$$

and hence for every prime p,

$$1+|f(p)|+\left|f\left(p^2\right)\right|+\cdots<\infty.$$

Thus, for every prime p, $1+f(p)+f\left(p^2\right)+\cdots$ is convergent. Now, since for every $x\geq 2$, $\{p : p\,\text{is prime, and}\,p\leq x\}$ is a nonempty finite set, for every $x \geq 2$,

$$\prod_{\substack{p\,\text{is prime}\\ p\leq x}}\left(1+f(p)+f\left(p^2\right)+\cdots\right)$$

is a product of finite-many absolutely convergent series. It follows that we can multiply the series and arrange the terms in any manner without altering the sum. For every $x \geq 2$, a typical term of

$$\prod_{\substack{p \text{ is prime} \\ p \leq x}} \left(1 + f(p) + f\left(p^2\right) + \cdots\right)$$

is of the form

$$f\left((p_1)^{a_1}\right) f\left((p_2)^{a_2}\right) \cdots f\left((p_r)^{a_r}\right),$$

where $\{p_1, p_2, \ldots, p_r\}$ is the set of all distinct primes $\leq x$, and $a_1, a_2, \ldots, a_r$ are any nonnegative integers. Since $p_1, p_2, \ldots, p_r$ are distinct primes, and $f : \mathbb{N} \to \mathbb{C}$ is a multiplicative arithmetic function, we have

$$f\left((p_1)^{a_1}\right) f\left((p_2)^{a_2}\right) \cdots f\left((p_r)^{a_r}\right) = f\left((p_1)^{a_1} (p_2)^{a_2} \cdots (p_r)^{a_r}\right).$$

Hence, for every $x \geq 2$,

$$\prod_{\substack{p \text{ is prime} \\ p \leq x}} \left(1 + f(p) + f\left(p^2\right) + \cdots\right) = \sum_{n \in A} f(n),$$

where A denotes the set of all positive integers n whose every prime factor is $\leq x$. Let B be the set of all positive integers n such that at least one prime factor of n is $> x$. It follows that for every $x \geq 2$,

$$\sum_{n=1}^{\infty} f(n) - \sum_{n \in A} f(n) = \sum_{n \in B} f(n),$$

and hence for every $x \geq 2$,

$$\left| \sum_{n=1}^{\infty} f(n) - \sum_{n \in A} f(n) \right| = \left| \sum_{n \in B} f(n) \right| \leq \sum_{n \in B} |f(n)| \leq \sum_{n > x} |f(n)|.$$

Thus, for every $x \geq 2$,

$$\left| \sum_{n=1}^{\infty} f(n) - \prod_{\substack{p \text{ is prime} \\ p \leq x}} \left(1 + f(p) + f\left(p^2\right) + \cdots\right) \right|$$

$$= \left| \sum_{n=1}^{\infty} f(n) - \sum_{n \in A} f(n) \right| \leq \sum_{n > x} |f(n)|.$$

Since $\sum_{n=1}^{\infty}|f(n)|$ is convergent, $\lim_{x\to\infty}\sum_{n>x}|f(n)|=0$. Now, since for every $x\geq 2$,

$$\left|\sum_{n=1}^{\infty}f(n)-\prod_{\substack{p \text{ is prime}\\ p\leq x}}\left(1+f(p)+f\left(p^2\right)+\cdots\right)\right|\leq\sum_{n>x}|f(n)|,$$

we have

$$\begin{aligned}&\prod_{p \text{ is a prime}}\left(1+f(p)+f\left(p^2\right)+\cdots\right)\\&=\lim_{x\to\infty}\left(\prod_{\substack{p \text{ is prime}\\ p\leq x}}\left(1+f(p)+f\left(p^2\right)+\cdots\right)\right)=\sum_{n=1}^{\infty}f(n).\end{aligned}$$

Conclusion 4.180 Let $f:\mathbb{N}\to\mathbb{C}$ be a multiplicative arithmetic function. Let $\sum_{n=1}^{\infty}f(n)$ be absolutely convergent. Then $\prod_{p \text{ is a prime}}(1+f(p)+f(p^2)+\cdots)$ $=\sum_{n=1}^{\infty}f(n)$.

Definition Here, we say that $\prod_{p \text{ isaprime}}(1+f(p)+f(p^2)+\cdots)$ is the ***Euler product*** *of the series* $\sum_{n=1}^{\infty}f(n)$.

Problem 4.181 $\prod_{p \text{ is a prime}}(1+f(p)+f(p^2)+\cdots)$ is absolutely convergent.

(Solution By Conclusion 3.192, it suffices to show that $\sum_{p \text{ is a prime}}|f(p)+f(p^2)$ $+\cdots|<\infty$, that is for every $x\geq 2$,

$$\sum_{\substack{p \text{ is prime}\\ p\leq x}}\left|f(p)+f\left(p^2\right)+\cdots\right|<\infty.$$

For this purpose, let us fix any $x\geq 2$. We have to show that

$$\sum_{\substack{p \text{ is prime}\\ p\leq x}}\left|f(p)+f\left(p^2\right)+\cdots\right|<\infty.$$

Since

$$\sum_{\substack{p \text{ is prime} \\ p \le x}} \left|f(p) + f(p^2) + \cdots\right| \le \sum_{\substack{p \text{ is prime} \\ p \le x}} \left(|f(p)| + \left|f(p^2)\right| + \cdots\right)$$
$$\le \sum_{n=2}^{\infty} |f(n)| < \infty,$$

we have

$$\sum_{\substack{p \text{ is prime} \\ p \le x}} \left|f(p) + f(p^2) + \cdots\right| < \infty.$$

∎)

Conclusion 4.182 Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function. Let $\sum_{n=1}^{\infty} f(n)$ be absolutely convergent. Then the Euler product of the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent.

Definition Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. If for every positive integers m, n, $f(mn) = f(m)f(n)$, then we say that f *is* ***completely multiplicative***.

Problem 4.183 Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function. Let $\sum_{n=1}^{\infty} f(n)$ be absolutely convergent. Further, if f is completely multiplicative, then

$$\prod_{p \text{ is a prime}} \frac{1}{1 - f(p)} = \sum_{n=1}^{\infty} f(n).$$

(Definition Here, we say that $\prod_{p \text{ is a prime}} \frac{1}{1-f(p)}$ is the ***Euler product*** *of the series* $\sum_{n=1}^{\infty} f(n)$.)

(Solution Since f is completely multiplicative, and, for every prime p, $1 + |f(p)| + \left|f(p^2)\right| + \cdots < \infty$, we have, for every prime p,

$$1 + |f(p)| + |f(p)|^2 + \cdots = \underbrace{1 + |f(p)| + |f(p)f(p)| + \cdots < \infty},$$

and hence for every prime p, $\lim_{n \to \infty} |f(p)|^n = 0$. This shows that for every prime p, $|f(p)| < 1$. By Conclusion 4.180,

$$\begin{aligned}\sum_{n=1}^{\infty} f(n) &= \prod_{p \text{ is a prime}} \left(1 + f(p) + f\left(p^2\right) + \cdots\right) \\ &= \prod_{p \text{ is a prime}} \left(1 + f(p) + f(p \cdot p) + \cdots\right) \\ &= \prod_{p \text{ is a prime}} \left(1 + f(p) + f(p)f(p) + \cdots\right) \\ &= \prod_{p \text{ is a prime}} \frac{1}{1 - f(p)},\end{aligned}$$

so

$$\prod_{p \text{ is a prime}} \frac{1}{1 - f(p)} = \sum_{n=1}^{\infty} f(n).$$

∎)

Conclusion 4.184 Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function. Let $\sum_{n=1}^{\infty} f(n)$ be absolutely convergent. Then $\prod_{p \text{ is a prime}} \left(1 + f(p) + f\left(p^2\right) + \cdots\right)$ is absolutely convergent, and $\prod_{p \text{ is a prime}} \left(1 + f(p) + f\left(p^2\right) + \cdots\right) = \sum_{n=1}^{\infty} f(n)$. Further, if f is completely multiplicative, then $\prod_{p \text{ is a prime}} \frac{1}{1 - f(p)} = \sum_{n=1}^{\infty} f(n)$.

On using Conclusion 4.180, we get the following

Conclusion 4.185 Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative arithmetic function. (It follows that $n \mapsto \frac{f(n)}{n^s}$ is also multiplicative.) Suppose that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely for $\{s = \sigma + it : \sigma_a < \sigma\}$. Then,

$$\prod_{p \text{ isaprime}} \left(1 + \frac{f(p)}{p^s} + \frac{f\left(p^2\right)}{p^{2s}} + \cdots\right) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

on $\{s = \sigma + it : \sigma_a < \sigma\}$.

Further, if f is completely multiplicative, then

$$\prod_{p \text{ is a prime}} \frac{1}{1 - \frac{f(p)}{p^s}} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

on $\{s = \sigma + it : \sigma_a < \sigma\}$. By Conclusion 4.185,

$$\underbrace{\prod_{p \text{ is a prime}} \frac{1}{1 - \frac{1}{p^s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}} = \zeta(s)$$

on $\{s : 1 < \sigma\}$. In short, we get the following

Conclusion 4.186 $\zeta(s) = \prod_p \frac{1}{1-\frac{1}{p^s}}\ (1<\sigma)$.

IX. Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. Let $A : x \mapsto \begin{cases} \sum_{n \le x} a(n) & \text{if } 1 \le x \\ 0 & \text{if } x < 1 \end{cases}$ be the 'sum function'. Let α, β be any positive real numbers such that $\alpha + 1 < \beta$. Let $f : [\alpha, \beta] \to \mathbb{C}$ be a continuously differentiable function.

Problem 4.187 $\sum_{\alpha<n\le\beta} a(n)f(n) = -\int_\alpha^\beta A(t)f'(t)\mathrm{d}t + A(\beta)f(\beta) - A(\alpha)f(\alpha)$.

(Solution Here,

$$
\begin{aligned}
\text{LHS} &= \sum_{\alpha<n\le\beta} a(n)f(n) = \sum_{n=[\alpha]+1}^{[\beta]} a(n)f(n) \\
&= \sum_{n=[\alpha]+1}^{[\beta]} (A(n) - A(n-1))f(n) \\
&= \sum_{n=[\alpha]+1}^{[\beta]} (A(n)f(n) - A(n-1)f(n)) \\
&= \sum_{n=[\alpha]+1}^{[\beta]} A(n)f(n) - \sum_{n=[\alpha]+1}^{[\beta]} A(n-1)f(n) \\
&= \sum_{n=[\alpha]+1}^{[\beta]} A(n)f(n) - \sum_{n=[\alpha]}^{[\beta]-1} A(n)f(n+1) \\
&= (A([\alpha]+1)f([\alpha]+1) + A([\alpha]+2)f([\alpha]+2) + \cdots \\
&\quad + A([\beta]-1)f([\beta]-1) + A([\beta])f([\beta])) \\
&\quad - (A([\alpha])f([\alpha]+1) + A([\alpha]+1)f([\alpha]+2) + \cdots \\
&\quad + A([\beta]-2)f([\beta]-1) + A([\beta]-1)f([\beta])) \\
&= (A([\alpha]+1)(f([\alpha]+1) - f([\alpha]+2)) + \cdots \\
&\quad + A([\beta]-1)(f([\beta]-1) - f([\beta]))) \\
&\quad - A([\alpha])f([\alpha]+1) + A([\beta])f([\beta]) \\
&= \left(-A([\alpha]+1) \int_{[\alpha]+1}^{[\alpha]+2} f'(t)\mathrm{d}t - \cdots - A([\beta]-1) \int_{[\beta]-1}^{[\beta]} f'(t)\mathrm{d}t \right) \\
&\quad - A([\alpha])f([\alpha]+1) + A([\beta])f([\beta]) \\
&= -\sum_{n=[\alpha]+1}^{[\beta]-1} \left(A(n) \int_n^{n+1} f'(t)\mathrm{d}t \right) - A([\alpha])f([\alpha]+1) + A([\beta])f([\beta])
\end{aligned}
$$

$$= -\sum_{n=[\alpha]+1}^{[\beta]-1}\left(\int\limits_{n}^{n+1} A(n)f'(t)\mathrm{d}t\right) - A([\alpha])f([\alpha]+1) + A([\beta])f([\beta])$$

$$= -\int\limits_{[\alpha]+1}^{[\beta]} A(t)f'(t)\mathrm{d}t - A([\alpha])f([\alpha]+1) + A([\beta])f([\beta])$$

and

$$\begin{aligned}
\text{RHS} &= -\int\limits_{\alpha}^{\beta} A(t)f'(t)\mathrm{d}t + A(\beta)f(\beta) - A(\alpha)f(\alpha) \\
&= -\left(\int\limits_{\alpha}^{[\alpha]+1} A(t)f'(t)\mathrm{d}t + \int\limits_{[\alpha]+1}^{[\beta]} A(t)f'(t)\mathrm{d}t\right. \\
&\quad \left. + \int\limits_{[\beta]}^{\beta} A(t)f'(t)\mathrm{d}t\right) + A(\beta)f(\beta) - A(\alpha)f(\alpha) \\
&= -\left(\int\limits_{\alpha}^{[\alpha]+1} A([\alpha])f'(t)\mathrm{d}t + \int\limits_{[\alpha]+1}^{[\beta]} A(t)f'(t)\mathrm{d}t\right. \\
&\quad \left. + \int\limits_{[\beta]}^{\beta} A([\beta])f'(t)\mathrm{d}t\right) + A(\beta)f(\beta) - A(\alpha)f(\alpha) \\
&= -\left(A([\alpha])(f([\alpha]+1) - f(\alpha)) + \int\limits_{[\alpha]+1}^{[\beta]} A(t)f'(t)\mathrm{d}t\right. \\
&\quad \left. + A([\beta])(f(\beta) - f([\beta]))\right) + A(\beta)f(\beta) - A(\alpha)f(\alpha) \\
&\quad - \left(A([\alpha])(f([\alpha]+1) - f(\alpha)) + \int\limits_{[\alpha]+1}^{[\beta]} A(t)f'(t)\mathrm{d}t\right. \\
&\quad \left. + A([\beta])(f(\beta) - f([\beta]))\right) + A([\beta])f(\beta) - A([\alpha])f(\alpha) \\
&= -A([\alpha])f([\alpha]+1) - \int\limits_{[\alpha]+1}^{[\beta]} A(t)f'(t)\mathrm{d}t + A([\beta])f([\beta]),
\end{aligned}$$

so LHS = RHS. ■)

Conclusion 4.188 Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. Let

$$A : x \mapsto \begin{cases} \sum_{n \leq x} a(n) & \text{if } 1 \leq x \\ 0 & \text{if } x < 1 \end{cases}$$

be the 'sum function'. Let α, β be any positive real numbers such that $\alpha + 1 < \beta$. Let $f : [\alpha, \beta] \to \mathbb{C}$ be a continuously differentiable function. Then

$$\sum_{\alpha < n \leq \beta} a(n) f(n) = -\int_{\alpha}^{\beta} A(t) f'(t) \mathrm{d}t + A(\beta) f(\beta) - A(\alpha) f(\alpha).$$

X. Let $s_0 = \sigma_0 + it_0$, where σ_0, t_0 are real numbers. Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^{s_0}}$ has bounded partial sums, that is there exists $M > 0$ such that, for every real $x \geq 1$,

$$\left| \sum_{n \leq x} \frac{f(n)}{n^{s_0}} \right| \leq M.$$

Let α, β be any positive real numbers such that $\alpha + 1 < \beta$. Let s be a complex number such that $\sigma_0 < \sigma$.

Let $a : n \to \frac{f(n)}{n^{s_0}}$ be an arithmetic function, and

$$A : x \mapsto \begin{cases} \sum_{n \leq x} a(n) \left(= \sum_{n \leq x} \frac{f(n)}{n^{s_0}} = \sum_{n=1}^{[x]} \frac{f(n)}{n^{s_0}} \right) & \text{if } 1 \leq x \\ 0 & \text{if } x < 1 \end{cases}$$

be the 'sum function'. Let $g : x \mapsto x^{s_0 - s}$ be a function defined on $[\alpha, \beta]$. Thus, for every real x, $|A(x)| \leq M$. By Conclusion 4.188,

$$\begin{aligned} \sum_{\alpha < n \leq \beta} \frac{f(n)}{n^s} &= \sum_{\alpha < n \leq \beta} \frac{f(n)}{n^{s_0}} \cdot n^{s_0 - s} \\ &= \underbrace{\sum_{\alpha < n \leq \beta} a(n) \cdot g(n) = -\int_{\alpha}^{\beta} A(t) g'(t) \mathrm{d}t + A(\beta) g(\beta) - A(\alpha) g(\alpha)} \\ &= -\int_{\alpha}^{\beta} A(t) \cdot (s_0 - s) t^{s_0 - s - 1} \mathrm{d}t + A(\beta) g(\beta) - A(\alpha) g(\alpha) \\ &= A(\beta) g(\beta) - A(\alpha) g(\alpha) - (s_0 - s) \int_{\alpha}^{\beta} A(t) t^{s_0 - s - 1} \mathrm{d}t, \end{aligned}$$

so

$$
\begin{aligned}
\left|\sum_{\alpha<n\leq\beta}\frac{f(n)}{n^s}\right| &\leq |A(\beta)||g(\beta)|+|A(\alpha)||g(\alpha)|+|s_0-s|\left|\int\limits_{\alpha}^{\beta}A(t)t^{s_0-s-1}\mathrm{d}t\right|\\
&\leq M|g(\beta)|+M|g(\alpha)|+|s_0-s|\left|\int\limits_{\alpha}^{\beta}A(t)t^{s_0-s-1}\mathrm{d}t\right|\\
&\leq M|g(\beta)|+M|g(\alpha)|+|s_0-s|\int\limits_{\alpha}^{\beta}\left|A(t)t^{s_0-s-1}\right|\mathrm{d}t\\
&= M|g(\beta)|+M|g(\alpha)|+|s_0-s|\int\limits_{\alpha}^{\beta}|A(t)|\left|t^{s_0-s-1}\right|\mathrm{d}t\\
&\leq M|g(\beta)|+M|g(\alpha)|+|s_0-s|\int\limits_{\alpha}^{\beta}M\left|t^{s_0-s-1}\right|\mathrm{d}t\\
&= M|g(\beta)|+M|g(\alpha)|+|s_0-s|M\int\limits_{\alpha}^{\beta}t^{\sigma_0-\sigma-1}\mathrm{d}t\\
&= M|g(\beta)|+M|g(\alpha)|+|s_0-s|M\frac{\beta^{\sigma_0-\sigma}-\alpha^{\sigma_0-\sigma}}{\sigma_0-\sigma}\\
&= M|\beta^{s_0-s}|+M|\alpha^{s_0-s}|+|s_0-s|M\frac{\beta^{\sigma_0-\sigma}-\alpha^{\sigma_0-\sigma}}{\sigma_0-\sigma}\\
&= M\frac{1}{\beta^{\sigma-\sigma_0}}+M\frac{1}{\alpha^{\sigma-\sigma_0}}+|s_0-s|M\frac{\beta^{\sigma_0-\sigma}-\alpha^{\sigma_0-\sigma}}{\sigma_0-\sigma}\\
&\leq M\frac{1}{\alpha^{\sigma-\sigma_0}}+M\frac{1}{\alpha^{\sigma-\sigma_0}}+|s_0-s|M\frac{\beta^{\sigma_0-\sigma}-\alpha^{\sigma_0-\sigma}}{\sigma_0-\sigma}\\
&= 2M\frac{1}{\alpha^{\sigma-\sigma_0}}+M\frac{|s_0-s|}{\sigma-\sigma_0}\left(\frac{1}{\alpha^{\sigma-\sigma_0}}-\frac{1}{\beta^{\sigma-\sigma_0}}\right)\\
&\leq 2M\frac{1}{\alpha^{\sigma-\sigma_0}}+M\frac{|s_0-s|}{\sigma-\sigma_0}\left(\frac{1}{\alpha^{\sigma-\sigma_0}}+\frac{1}{\alpha^{\sigma-\sigma_0}}\right)\\
&= 2M\frac{1}{\alpha^{\sigma-\sigma_0}}+2M\frac{|s_0-s|}{\sigma-\sigma_0}\frac{1}{\alpha^{\sigma-\sigma_0}}=\frac{2M}{\alpha^{\sigma-\sigma_0}}\left(1+\frac{|s_0-s|}{\sigma-\sigma_0}\right),
\end{aligned}
$$

and hence

$$
\left|\sum_{\alpha<n\leq\beta}\frac{f(n)}{n^s}\right|\leq\frac{2M}{\alpha^{\sigma-\sigma_0}}\left(1+\frac{|s_0-s|}{\sigma-\sigma_0}\right).
$$

Conclusion 4.189 Let $s_0 = \sigma_0 + it_0$ where σ_0, t_0 are real numbers. Suppose that M is a positive real number such that, for every real $x \geq 1$, $\left|\sum_{n \leq x} \frac{f(n)}{n^{s_0}}\right| \leq M$. Let α, β be any positive real numbers such that $\alpha + 1 < \beta$. Let s be a complex number such that $\sigma_0 < \sigma$. Then,

$$\left|\sum_{\alpha < n \leq \beta} \frac{f(n)}{n^s}\right| \leq \frac{2M}{\alpha^{\sigma - \sigma_0}}\left(1 + \frac{|s - s_0|}{\sigma - \sigma_0}\right).$$

XI. Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ converges, where σ_0, t_0 are real numbers.

Problem 4.190 $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges in the 'half-plane' $\{s : \sigma_0 < \sigma\}$.

(**Solution** Let us fix any s such that $\sigma_0 < \sigma$. We have to show that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is convergent. It suffices to show that the Cauchy criteria for $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is satisfied.

Since $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ is convergent, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ has bounded partial sums. That is, there exists $M > 0$ such that, for every real $x \geq 1$, $\left|\sum_{n \leq x} \frac{f(n)}{n^{\sigma_0 + it_0}}\right| \leq M$. Now, by Conclusion 4.189, for every positive real numbers α, β satisfying $\alpha + 1 < \beta$,

$$\left|\sum_{\alpha < n \leq \beta} \frac{f(n)}{n^s}\right| \leq \left(2M\left(1 + \frac{|s - s_0|}{\sigma - \sigma_0}\right)\right)\frac{1}{\alpha^{\sigma - \sigma_0}}.$$

On letting $\alpha \to \infty$,

$$\left(2M\left(1 + \frac{|s - s_0|}{\sigma - \sigma_0}\right)\right)\frac{1}{\alpha^{\sigma - \sigma_0}} \to 0,$$

and hence, the Cauchy criteria for $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is satisfied. ■)

Conclusion 4.191 Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ converges, where σ_0, t_0 are real numbers. Then, $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges in the 'half-plane' $\{s : \sigma_0 < \sigma\}$.

XII. Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ diverges, where σ_0, t_0 are real numbers.

Problem 4.192 $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ diverges in the 'half-plane' $\{s : \sigma < \sigma_0\}$.

(**Solution** Let us fix any s such that $\sigma < \sigma_0$. We have to show that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is divergent. If not, otherwise let $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ be convergent. We have to arrive at a contradiction. Since

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma+it}} = \underbrace{\sum_{n=1}^{\infty} \frac{f(n)}{n^s}}$$

is convergent, and $\sigma < \sigma_0$, by Conclusion 4.191, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0+it_0}}$ is convergent. This is a contradiction. ■)

Conclusion 4.193 Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0+it_0}}$ diverges, where σ_0, t_0 are real numbers. Then $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ diverges in the 'half-plane' $\{s : \sigma < \sigma_0\}$.

XIII. Suppose that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ does not converge for all complex number s, and $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ does not diverge for all complex number s. Let D be the set of all real numbers σ for which $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is divergent.

Since $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ does not converge for all complex number s, D is a nonempty set of real numbers. Since $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ does not diverge for all complex number s, there exists a complex number $a+ib$, where $a, b \in \mathbb{R}$, such that $\sum_{n=1}^{\infty} \frac{f(n)}{n^{a+ib}}$ is convergent.

Problem 4.194 a is an upper bound of D.

(Solution If not, otherwise suppose that there exists a complex number $c+id$, where $c, d \in \mathbb{R}$, such that $a < c$ and $\sum_{n=1}^{\infty} \frac{f(n)}{n^{c+id}}$ diverges. Since $\sum_{n=1}^{\infty} \frac{f(n)}{n^{a+ib}}$ is convergent, and $a < c$, by Conclusion 4.191, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{c+id}}$ is convergent. This is a contradiction. ■)

Since D is a nonempty bounded above set of real numbers, $\sup D$ exists. Thus, $\sup D \in \mathbb{R}$. Also, $\sup D \leq a$.

Case I: when $\sigma < \sup D$. Here, there exists a complex number $c+id$, where $c, d \in \mathbb{R}$, such that $c \in D$, $\sigma < c$ and $\sum_{n=1}^{\infty} \frac{f(n)}{n^{c+id}}$ is divergent. Now, since $\sigma < c$, by Conclusion 4.193, $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is divergent.
Case II: when $\sup D < \sigma$. It follows that $\sigma \notin D$, and hence $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is convergent.

Conclusion 4.195 Suppose that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ does not converge for all s, and $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ does not diverge for all s. Then there exists a real number σ_c such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \begin{cases} \text{converges} & \text{if } \sigma_c < \sigma \\ \text{diverges} & \text{if } \sigma < \sigma_c. \end{cases}$$

(Definition Here, σ_c is called the ***abscissa of convergence***.)

XIV. Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0+it_0}}$ converges, where σ_0, t_0 are real numbers.

Problem 4.196 $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely in the 'half-plane' $\{s : \sigma_0 + 1 < \sigma\}$.

(Solution Let us fix any complex number $s \equiv \sigma + it$ such that $\sigma_0 + 1 < \sigma$. We have to show that $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma}$ is convergent.

Since $\sigma_0 + 1 < \sigma$, we have $\sum_{n=1}^{\infty} \frac{1}{n^{\sigma - \sigma_0}}$ is convergent. Since $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ converges, we have

$$\lim_{n\to\infty} \frac{|f(n)|}{n^{\sigma_0}} = \underbrace{\lim_{n\to\infty} \left| \frac{f(n)}{n^{\sigma_0 + it_0}} \right| = 0},$$

and hence $\left\{ \frac{|f(n)|}{n^{\sigma_0}} \right\}$ is a bounded sequence. It follows that there exists $A > 0$ such that for every positive integer n, $\frac{|f(n)|}{n^{\sigma_0}} \leq A$. Now, since for every positive integer n,

$$\frac{|f(n)|}{n^\sigma} = \frac{|f(n)|}{n^{\sigma_0}} \frac{1}{n^{\sigma - \sigma_0}} \leq A \frac{1}{n^{\sigma - \sigma_0}},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{\sigma - \sigma_0}}$ is convergent, by comparison test, $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma}$ is convergent. ■)

Conclusion 4.197 Suppose that the Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ converges, where σ_0, t_0 are real numbers. Then $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely in the 'half-plane' $\{s : \sigma_0 + 1 < \sigma\}$.

XV. Problem 4.198 (i) $\sigma_c \leq \sigma_a$, (ii) $\sigma_a - \sigma_c \leq 1$.

(Solution For (i): If not, otherwise let $\sigma_a < \sigma_c$. We have to arrive at a contradiction.

Since $\sigma_a < \sigma_c$, there exists a complex number $\sigma + it$, where σ, t are real numbers such that $\sigma_a < \sigma < \sigma_c$. Since $\sigma_a < \sigma$, by Problem 4.166, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma+it}}$ is absolutely convergent, and hence $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma+it}}$ is convergent. Since $\sigma < \sigma_c$, by Conclusion 4.195, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma+it}}$ is divergent. This is a contradiction.

For (ii): If not, otherwise let $1 < \sigma_a - \sigma_c$. We have to arrive at a contradiction.

It follows that there exist complex numbers $\sigma_1 + it_1$ and $\sigma_2 + it_2$, where $\sigma_1, t_1, \sigma_2, t_2$ are real numbers such that $\sigma_c < \sigma_1 < \sigma_c + 1 < \sigma_1 + 1 < \sigma_2 < \sigma_a$. Since $\sigma_c < \sigma_1$, by Conclusion 4.195, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_1 + it_1}}$ is convergent. Now, since $\sigma_1 + 1 < \sigma_2$, by Conclusion 4.197, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_2 + it_2}}$ converges absolutely. Since $\sigma_2 < \sigma_a$, by Conclusion 4.169, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_2 + it_2}}$ does not converge absolutely. This is a contradiction. ■)

Conclusion 4.199 (i) $\sigma_c \le \sigma_a$, (ii) $\sigma_a - \sigma_c \le 1$.

XVI. Problem 4.200 The Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges uniformly on compact subsets of the 'half-plane' $\{s = \sigma + it : \sigma_c < \sigma\}$.

(Solution For this purpose, let us take a nonempty compact subset K of $\{s = \sigma + it : \sigma_c < \sigma\}$. We have to show that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges uniformly on K.

Since K is compact, K is bounded. Since $(\sigma + it) \mapsto \sigma$ is a continuous function, and K is compact, there exists $\alpha + it_0 \in K$ such that, for every $\sigma + it \in K$, $\alpha \le \sigma$. Now, since K is bounded, there exist real numbers β, c, d such that $\alpha < \beta, c < d$, and $K \subset [\alpha, \beta] \times [c, d]$. Since

$$\alpha + it_0 \in K \subset \{s = \sigma + it : \sigma_c < \sigma\},$$

we have $\sigma_c < \alpha$, and hence

$$(K \subset)[\alpha, \beta] \times [c, d] \subset \{s = \sigma + it : \sigma_c < \sigma\}.$$

It suffices to show that $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges uniformly on compact set $[\alpha, \beta] \times [c, d]$, where $\sigma_c < \alpha$.

Let us take any $s_0 = \sigma_0 + it_0$, where σ_0, t_0 are real numbers, and $\sigma_c < \sigma_0$.

Now, by Conclusion 4.195, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0 + it_0}}$ is convergent, and hence there exists $M > 0$ such that, for every real $x \ge 1$, $\left|\sum_{n \le x} \frac{f(n)}{n^{s_0}}\right| \le M$. Now, by Conclusion 4.189, for every positive real numbers α_1, β_1 satisfying $\alpha_1 + 1 < \beta_1$, and, for every complex number $s = \sigma + it$ satisfying $\sigma_0 < \sigma$,

$$\left|\sum_{\alpha_1 < n \le \beta_1} \frac{f(n)}{n^s}\right| \le \frac{2M}{\alpha^{\sigma - \sigma_0}} \left(1 + \frac{|s - s_0|}{\sigma - \sigma_0}\right)$$

For every $s \equiv \sigma + it$ in $[\alpha, \beta] \times [c, d]$, we have $\left(\frac{\sigma_c + \alpha}{2} <\right) \alpha \le \sigma$. Since $\sigma_c < \alpha$, by Conclusion 4.195, $\sum_{n=1}^{\infty} \frac{f(n)}{n^{\frac{\sigma_c + \alpha}{2}}}$ is convergent. Now, by Conclusion 4.189, for every positive real numbers α_1, β_1 satisfying $\alpha_1 + 1 < \beta_1$, and for every complex number $s = \sigma + it$ in $[\alpha, \beta] \times [c, d]$,

$$\begin{aligned}\left|\sum_{\alpha_1 < n \le \beta_1} \frac{f(n)}{n^s}\right| &\le \frac{2M}{(\alpha_1)^{\sigma - \frac{\sigma_c + \alpha}{2}}} \left(1 + \frac{\left|s - \frac{\sigma_c + \alpha}{2}\right|}{\sigma - \frac{\sigma_c + \alpha}{2}}\right) \le \frac{2M}{(\alpha_1)^{\alpha - \frac{\sigma_c + \alpha}{2}}} \left(1 + \frac{\left|s - \frac{\sigma_c + \alpha}{2}\right|}{\sigma - \frac{\sigma_c + \alpha}{2}}\right) \\ &\le \frac{2M}{(\alpha_1)^{\frac{\alpha - \sigma_c}{2}}} \left(1 + \frac{\left|s - \frac{\sigma_c + \alpha}{2}\right|}{\alpha - \frac{\sigma_c + \alpha}{2}}\right) = \frac{2M}{(\alpha_1)^{\frac{\alpha - \sigma_c}{2}}} \left(1 + \frac{\left|s - \frac{\sigma_c + \alpha}{2}\right|}{\frac{\alpha - \sigma_c}{2}}\right) \\ &\le 2M\left(1 + \frac{\max\left\{\left|s - \frac{\sigma_c + \alpha}{2}\right| : s \in [\alpha, \beta] \times [c, d]\right\}}{\frac{\alpha - \sigma_c}{2}}\right) \cdot \frac{1}{(\alpha_1)^{\frac{\alpha - \sigma_c}{2}}},\end{aligned}$$

and hence for every positive real numbers α_1, β_1 satisfying $\alpha_1 + 1 < \beta_1$, and for every complex number $s = \sigma + it \in [\alpha, \beta] \times [c, d]$,

$$\left| \sum_{\alpha_1 < n \leq \beta_1} \frac{f(n)}{n^s} \right| \leq B \frac{1}{(\alpha_1)^{\frac{\alpha - \sigma_c}{2}}},$$

where

$$B \equiv 2M\left(1 + \frac{\max\left\{\left|s - \frac{\sigma_c + \alpha}{2}\right| : s \in [\alpha, \beta] \times [c, d]\right\}}{\frac{\alpha - \sigma_c}{2}}\right).$$

Now, since B is independent of s and α_1, and $\frac{1}{(\alpha_1)^{\frac{\alpha-\sigma_c}{2}}} \to 0$ as $\alpha_1 \to \infty$, the Cauchy criteria for uniform convergence is satisfied, and hence $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges uniformly on compact set $[\alpha, \beta] \times [c, d]$. ■)

Conclusion 4.201 The Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges uniformly on compact subsets of the half-plane $\{s = \sigma + it : \sigma_c < \sigma\}$.

XVII. Problem 4.202 The sum function

$$F : s \mapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \left(= \sum_{n=1}^{\infty} f(n) e^{(\ln n)(-s)} \right)$$

is holomorphic in the half-plane $\{s = \sigma + it : \sigma_c < \sigma\}$. Also,

$$F' : s \mapsto -\sum_{n=1}^{\infty} \frac{f(n) \ln n}{n^s}$$

is holomorphic in $\{s = \sigma + it : \sigma_c < \sigma\}$,

$$F'' : s \mapsto -\sum_{n=1}^{\infty} \frac{f(n)(\ln n)^2}{n^s}$$

is holomorphic in $\{s = \sigma + it : \sigma_c < \sigma\}$, etc.

(**Solution** Since each function $s \mapsto \frac{f(n)}{n^s}$ is holomorphic in the open set $\{s = \sigma + it : \sigma_c < \sigma\}$, each function $F_n : s \mapsto \sum_{k=1}^{n} \frac{f(k)}{k^s}$ is holomorphic in the open set $\{s = \sigma + it : \sigma_c < \sigma\}$. By Conclusion 4.201, $\{F_n\}$ converges to F uniformly on compact subsets of the half-plane $\{s = \sigma + it : \sigma_c < \sigma\}$. It follows, by Conclusion 4.172, that $F : s \mapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is holomorphic in the half-plane $\{s = \sigma + it : \sigma_c < \sigma\}$. Also,

$$\underbrace{F' : s \mapsto \sum_{n=1}^{\infty} f(n)\left(e^{(\ln n)(-s)}(-(\ln n))\right)}$$
$$= \sum_{n=1}^{\infty} f(n)\left(\frac{1}{n^s}(-(\ln n))\right) = -\sum_{n=1}^{\infty} \frac{f(n)\ln n}{n^s}$$

is holomorphic in the half-plane $\{s = \sigma + it : \sigma_c < \sigma\}$, etc. ■)

XVIII. Conclusion 4.203 The sum function $F : s \mapsto \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is holomorphic in the half-plane $\{s = \sigma + it : \sigma_c < \sigma\}$. Also, $F' : s \mapsto -\sum_{n=1}^{\infty} \frac{f(n)\ln n}{n^s}$ is holomorphic in $\{s = \sigma + it : \sigma_c < \sigma\}$, $F'' : s \mapsto -\sum_{n=1}^{\infty} \frac{f(n)(\ln n)^2}{n^s}$ is holomorphic in $\{s = \sigma + it : \sigma_c < \sigma\}$, etc.

4.12 Riemann Zeta Function Has a Simple Pole at 1

Note 4.204

Definition Let $a \in (0, 1]$. Let $s \equiv \sigma + it$, where σ, t are real numbers, be a complex number such that $1 < \sigma$. The series

$$\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$$

is denoted by $\zeta(s, a)$. It is customary to say that $\zeta(s, a)$ *is a* ***Hurwitz zeta function*** *defined for* $1 < \sigma$.

Definition Let $s \equiv \sigma + it$, where σ, t are real numbers, be a complex number such that $1 < \sigma$. The series

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

is denoted by $\zeta(s)$. It is customary to say that $\zeta(s)$ *is a* ***Riemann zeta function*** *defined for* $1 < \sigma$. Clearly, for every $\sigma > 1$, $\zeta(s) = \zeta(s, 1)$.

I. Let $s \equiv \sigma + it$, where σ, t are real numbers, be a complex number such that $1 < \sigma$.

Problem 4.205 $\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$ is absolutely convergent.

(**Solution:** We must show that

$$\left|\frac{1}{a^s}\right| + \left|\frac{1}{(1+a)^s}\right| + \left|\frac{1}{(2+a)^s}\right| + \cdots$$

is convergent. Observe that

$$\begin{aligned}
&\left|\frac{1}{a^s}\right| + \left|\frac{1}{(1+a)^s}\right| + \left|\frac{1}{(2+a)^s}\right| + \cdots \\
&\quad = \frac{1}{|a^s|} + \frac{1}{|(1+a)^s|} + \frac{1}{|(2+a)^s|} + \cdots \\
&\quad = \frac{1}{|e^{(\ln a)s}|} + \frac{1}{|e^{(\ln(1+a))s}|} + \frac{1}{|e^{(\ln(2+a))s}|} + \cdots \\
&\quad = \frac{1}{|e^{(\ln a)(\sigma+it)}|} + \frac{1}{|e^{(\ln(1+a))(\sigma+it)}|} + \frac{1}{|e^{(\ln(2+a))(\sigma+it)}|} + \cdots \\
&\quad = \frac{1}{|e^{(\ln a)\sigma}e^{i(\ln a)t}|} + \frac{1}{|e^{(\ln(1+a))\sigma}e^{i(\ln(1+a))t}|} + \frac{1}{|e^{(\ln(2+a))\sigma}e^{i(\ln(2+a))t}|} + \cdots \\
&\quad = \frac{1}{e^{(\ln a)\sigma}} + \frac{1}{e^{(\ln(1+a))\sigma}} + \frac{1}{e^{(\ln(2+a))\sigma}} + \cdots \\
&\quad = \frac{1}{a^\sigma} + \frac{1}{(1+a)^\sigma} + \frac{1}{(2+a)^\sigma} + \cdots.
\end{aligned}$$

So, it suffices to show that

$$\frac{1}{a^\sigma} + \frac{1}{(1+a)^\sigma} + \frac{1}{(2+a)^\sigma} + \cdots$$

is convergent. Since

$$\frac{1}{a^\sigma} + \left(\frac{1}{(1+a)^\sigma} + \frac{1}{(2+a)^\sigma} + \cdots\right) \le \frac{1}{a^\sigma} + \left(\frac{1}{1^\sigma} + \frac{1}{2^\sigma} + \cdots\right),$$

and

$$\frac{1}{1^\sigma} + \frac{1}{2^\sigma} + \cdots$$

is known to be convergent, $\frac{1}{a^\sigma} + \frac{1}{(1+a)^\sigma} + \frac{1}{(2+a)^\sigma} + \cdots$ is convergent. ■)

Conclusion 4.206 Let $s \equiv \sigma + it$, where σ, t are real numbers, be a complex number such that $1 < \sigma$. Then $\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$ is absolutely convergent.

II. Problem 4.207 Let $\delta > 0$. Then $\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$ converges uniformly in $\{s = \sigma + it : (1+\delta) \le \sigma\}$.

(Solution Since for every $s \equiv \sigma + it$ satisfying $(1+\delta) \le \sigma$, $\left|\frac{1}{(n+a)^s}\right| = \frac{1}{(n+a)^\sigma} \le \frac{1}{(n+a)^{1+\delta}}$, and $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty$, by the Weierstrass M-test, $\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$ converges uniformly in $\{s = \sigma + it : (1+\delta) \le \sigma\}$. ■)

Conclusion 4.208 For every $\delta > 0$, $\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$ converges uniformly in $\{s = \sigma + it : (1+\delta) \le \sigma\}$.

III. Problem 4.209 For every $a \in (0,1]$,

$$\zeta_a : s \mapsto \frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$$

converges uniformly on compact subsets of the half-plane $\{s = \sigma + it : 1 < \sigma\}$.

(Solution For this purpose, let us take any nonempty compact subset K of $\{s = \sigma + it : 1 < \sigma\}$. We have to show that

$$\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$$

converges uniformly on K.

Since K is compact, K is bounded. Since $(\sigma + it) \mapsto \sigma$ is a continuous function, and K is compact, there exists $\alpha + it_0 \in K$ such that, for every $\sigma + it \in K$, $\alpha \le \sigma$. Since $\alpha + it_0 \in K \subset \{s = \sigma + it : 1 < \sigma\}$, we have $1 < \alpha$. Put $\delta \equiv \alpha - 1(> 0)$. Now, since for every $\sigma + it \in K$, $(1 + \delta =)\alpha \le \sigma$, we have $K \subset \{s = \sigma + it : (1+\delta) \le \sigma\}$. By Conclusion 4.208,

$$\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$$

converges uniformly in $\{s = \sigma + it : (1+\delta) \le \sigma\}(\supset K)$, so, $\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$ converges uniformly on K. ■)

Conclusion 4.210 For every $a \in (0,1], \zeta_a : s \mapsto \frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots$ converges uniformly on compact subsets of the half-plane $\{s = \sigma + it : 1 < \sigma\}$.

IV. Problem 4.211 For every $a \in (0,1]$, the sum function

$$F : s \mapsto \left(\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots\right)$$

is holomorphic in the half-plane $\{s = \sigma + it : 1 < \sigma\}$.

(Solution Since each function $s \mapsto \frac{1}{(n+a)^s}$ is holomorphic in the open set $\{s = \sigma + it : 1 < \sigma\}$, each function $F_n : s \mapsto \sum_{k=0}^{n} \frac{1}{(k+a)^s}$ is holomorphic in the open set $\{s = \sigma + it : 1 < \sigma\}$. By Conclusion 4.210, $\{F_n\}$ converges to F uniformly on compact subsets of the half-plane $\{s = \sigma + it : 1 < \sigma\}$. It follows, by Conclusion 4.172, that $F : s \mapsto \sum_{n=0}^{\infty} \frac{1}{(k+a)^s}$ is holomorphic in the half-plane $\{s = \sigma + it : 1 < \sigma\}$. ∎)

Conclusion 4.212 For every $a \in (0,1]$, the sum function $F : s \mapsto \left(\frac{1}{a^s} + \frac{1}{(1+a)^s} + \frac{1}{(2+a)^s} + \cdots\right)$ is holomorphic in the half-plane $\{s = \sigma + it : 1 < \sigma\}$.

V. Problem 4.213 Let $1 < \sigma$. Then,

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1} \mathrm{d}t.$$

(Solution First of all, we shall prove the following three minor results:

Problem 7.214 For every $t > 0$, $\sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1}$.

(Solution Let us fix any $t > 0$. We have to show that $\sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1}$. Since $t > 0$, we have $e^{-t} \in (0,1)$, and hence

$$\sum_{n=1}^{\infty} e^{-nt} = \sum_{n=1}^{\infty} (e^{-t})^n = \frac{e^{-t}}{1 - e^{-t}} = \frac{1}{e^t - 1}.$$

∎)

Problem 7.215 $\sum_{n=1}^{\infty} e^{-nt}$ converges uniformly on compact subsets of $(0, \infty)$.

(Solution For this purpose, let us take any nonempty compact subset K of $(0, \infty)$. We have to show that $\sum_{n=1}^{\infty} e^{-nt}$ converges uniformly on K.

Since K is a compact subset of $(0, \infty)$, and $t \mapsto t$ is a continuous function, there exists $m \in K (\subset (0, \infty))$ such that for every $t \in K$, $m \le t$, and hence for every $t \in K$, $\frac{1}{e^t} \le \frac{1}{e^m}$. Thus, for every positive integer n, and for every $t \in K$, we have

$$|e^{-nt}| = \frac{1}{e^{nt}} = \left(\frac{1}{e^t}\right)^n \le \left(\frac{1}{e^m}\right)^n.$$

Since $m \in (0, \infty)$, we have $\frac{1}{e^m} \in (0,1)$, and hence the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{e^m}\right)^n$ is convergent. Now, since for every positive integer n, and for every $t \in K$, $|e^{-nt}| \le \left(\frac{1}{e^m}\right)^n$, by the Weierstrass M-test, $\sum_{n=1}^{\infty} e^{-nt}$ converges uniformly on K. ∎)

Problem 7.216 For every $s = \sigma + it$ satisfying $1 < \sigma$, $\int_0^\infty \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t < \infty$.

(Solution Let us fix any $s = \sigma + it$ satisfying $1 < \sigma$. We have to show that $\int_0^\infty \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t < \infty$. Since

$$\int\limits_0^1 \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t = \int\limits_0^1 \frac{t^{\sigma-1}}{t+\frac{1}{2!}t^2+\cdots}\mathrm{d}t \le \int\limits_0^1 \frac{t^{\sigma-1}}{t}\mathrm{d}t = \int\limits_0^1 t^{\sigma-2}\mathrm{d}t = \frac{1}{\sigma-1}t^{\sigma-1}\Big|_{t=0}^1$$
$$= \frac{1}{\sigma-1}(1-0) = \frac{1}{\sigma-1},$$

we have $\int_0^1 \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t < \infty$. It suffices to show that $\int_1^\infty \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t < \infty$. Since

$$\lim_{t\to\infty} \frac{e^t}{e^t-1} = \lim_{t\to\infty} \frac{1}{1-e^{-t}} = \frac{1}{1-0} = 1,$$

we have $\lim_{t\to\infty} \frac{e^t}{e^t-1} = 1 < 2$, and hence there exists $M > 1$ such that for every $t \ge M$, $\frac{e^t}{e^t-1} < 2$. It follows that, for every $t \ge M$, $\frac{1}{e^t-1} < \frac{2}{e^t}$. By Conclusion 4.111, for every $\varepsilon > 0$, there exists a real number $\kappa > M$ such that, for every $\beta \in (\kappa, \infty)$,

$$\frac{1}{2}\int\limits_\kappa^\beta \frac{1}{e^t-1}t^{\sigma-1}\mathrm{d}t \le \underbrace{\int\limits_\kappa^\beta e^{-t}t^{\sigma-1}\mathrm{d}t < \frac{\varepsilon}{2}}_{},$$

and hence $\int_\kappa^\infty \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t = 0$. It suffices to show that $\int_1^\kappa \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t < \infty$. Since $t \mapsto \frac{t^{\sigma-1}}{e^t-1}$ is continuous on $[1, \kappa]$, we have $\int_1^\kappa \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t < \infty$. ∎)

Now, we want to apply Theorem 1.136. Observe that for every $t > 0$,

$$\lim_{n\to\infty}\left(\sum_{k=1}^n t^{s-1}e^{-kt}\right) = \lim_{n\to\infty}\left(t^{s-1}\sum_{k=1}^n e^{-kt}\right) = t^{s-1}\lim_{n\to\infty}\left(\sum_{k=1}^n e^{-kt}\right)$$
$$= t^{s-1}\sum_{k=1}^\infty e^{-kt} = t^{s-1}\frac{1}{e^t-1} = \frac{t^{s-1}}{e^t-1},$$

and hence for every $t > 0$, $\lim_{n\to\infty}\left(\sum_{k=1}^n t^{s-1}e^{-kt}\right) = \frac{t^{s-1}}{e^t-1}$. Next, for every positive integer n, and for every $t > 0$,

$$\left|\sum_{k=1}^n t^{s-1}e^{-kt}\right| \le \sum_{k=1}^n \left|t^{s-1}e^{-kt}\right| = \sum_{k=1}^n t^{\sigma-1}e^{-kt}$$
$$= t^{\sigma-1}\sum_{k=1}^n e^{-kt} = t^{\sigma-1}\frac{1}{e^t-1} = \frac{t^{\sigma-1}}{e^t-1}.$$

Thus, for every positive integer n, and for every $t > 0$, $\left|\sum_{k=1}^{n} t^{s-1}e^{-kt}\right| \leq \frac{t^{\sigma-1}}{e^t-1}$. Also, for every $s = \sigma + it$ satisfying $1 < \sigma$, $\int_0^\infty \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t < \infty$. Now, by Theorem 1.136, Vol. 1,

$$\begin{aligned}
\Gamma(s)\zeta(s) &= \left(\int\limits_{t=0}^{t=\infty} t^{s-1}e^{-t}\mathrm{d}t\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n^s}\int\limits_{t=0}^{t=\infty} t^{s-1}e^{-t}\mathrm{d}t\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{n}\int\limits_{t=0}^{t=\infty}\left(\frac{nt}{n}\right)^{s-1}e^{-nt}\mathrm{d}(nt)\right) = \sum_{n=1}^{\infty}\left(\int\limits_{0}^{\infty} t^{s-1}e^{-nt}\mathrm{d}t\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\left(\int\limits_{0}^{\infty} t^{s-1}e^{-kt}\mathrm{d}t\right)\right) = \lim_{n\to\infty}\left(\int\limits_{0}^{\infty}\left(\sum_{k=1}^{n} t^{s-1}e^{-kt}\right)\mathrm{d}t\right) = \int\limits_{0}^{\infty}\frac{t^{s-1}}{e^t-1}\mathrm{d}t,
\end{aligned}$$

so $\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t-1}\mathrm{d}t$. ■)

Conclusion 4.217 $\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t-1}\mathrm{d}t \quad (1 < \sigma)$.

Let C be a closed curve (or, say, contour) around the negative real axis, which consists of three parts, C_1, C_2, C_3. Here, C_2 is a positively oriented circle of radius c, where $0 < c < 2\pi$; C_1 is the 'lower edge' of a cut in the z-plane along the negative real axis, traversed towards the origin; and C_3 is the 'upper edge' of a cut in the z-plane along the negative real axis, traversed away from the origin.

On C_1, we shall use the parametrization $r \mapsto \left(e^{i(-\pi)}\right)r$ from $r = \infty$ to $r = c$. On C_3, we shall use the parametrization $r \mapsto (e^{i\pi})r$ from $r = c$ to $r = \infty$. On C_2, we shall use the parametrization $\theta \mapsto ce^{i\theta}$ from $\theta = -\pi$ to $\theta = \pi$.

Problem 4.218 $s \mapsto \frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z$ is an entire function, that is

$$s \mapsto \frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z$$

is holomorphic in $\mathbb{C}$.

(Solution On C_1, z^s means

$$\underbrace{\left(e^{-i\pi s}\right)r^s} = e^{-i\pi s}e^{(\ln r)s} = e^{-i\pi s + (\ln r)s} = e^{(\ln r - i\pi)s},$$

and on C_3, z^s means

$$\underbrace{\left(e^{i\pi s}\right)r^s} = e^{i\pi s}e^{(\ln r)s} = e^{i\pi s + (\ln r)s} = e^{(\ln r + i\pi)s}.$$

Let us take any positive integer M. Let us take any $s = \sigma + it$ such that $|s| \leq M$. It suffices to show that

1. $\int_{C_1} \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z$ converges uniformly on $\{s : |s| < M\}$,
2. $\int_{C_3} \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z$ converges uniformly on $\{s : |s| < M\}$

For 1: On C_1, for every $r \geq 1 (= \ln e > \ln 2)$,

$$\begin{aligned}
\left|\frac{e^z}{1-e^z} z^{s-1}\right| &= \left|\frac{e^z}{1-e^z}\right| \left|z^{s-1}\right| = \left|\frac{e^z}{1-e^z}\right| \left|\left(e^{-i\pi(s-1)}\right) r^{s-1}\right| \\
&= \left|\frac{e^z}{1-e^z}\right| \left|e^{-i\pi(s-1)}\right| \left|r^{s-1}\right| = \left|\frac{e^z}{1-e^z}\right| \left|e^{-i\pi((\sigma-1)+it)}\right| r^{\sigma-1} \\
&= \left|\frac{e^z}{1-e^z}\right| \left|e^{-i\pi(\sigma-1)} e^{\pi t}\right| r^{\sigma-1} = \left|\frac{e^z}{1-e^z}\right| e^{\pi t} r^{\sigma-1} \\
&\leq \left|\frac{e^z}{1-e^z}\right| e^{\pi t} r^{M-1} \leq \left|\frac{e^z}{1-e^z}\right| e^{\pi M} r^{M-1} = \left|\frac{e^{-r}}{1-e^{-r}}\right| e^{\pi M} r^{M-1} \\
&= \frac{1}{1-e^{-r}} e^{\pi M} \left(r^{M-1} e^{-r}\right) \leq \frac{e^r}{e^r - 1} e^{\pi M} \left(r^{M-1} e^{-r}\right) \\
&= \left(1 + \frac{1}{e^r - 1}\right) e^{\pi M} \left(r^{M-1} e^{-r}\right) \leq \left(1 + \frac{1}{e^{\ln 2} - 1}\right) e^{\pi M} \left(r^{M-1} e^{-r}\right) \\
&= (1+1) e^{\pi M} \left(r^{M-1} e^{-r}\right) = 2 e^{\pi M} \left(r^{M-1} e^{-r}\right).
\end{aligned}$$

Thus, on C_1, for every $r \geq 1$, we have

$$\left|\frac{e^z}{1-e^z} z^{s-1}\right| \leq 2 e^{\pi M} \left(r^{M-1} e^{-r}\right).$$

Now, since $2e^{\pi M}(r^{M-1}e^{-r})$ is independent of s, and

$$\begin{aligned}
\int_c^\infty r^{M-1} e^{-r} \mathrm{d}r &= r^{M-1} \frac{e^{-r}}{-1}\bigg|_{r=c}^{r=\infty} - \int_c^\infty (M-1) r^{M-2} \frac{e^{-r}}{-1} \mathrm{d}r \\
&= -r^{M-1}(0 - e^{-c}) + (M-1) \int_c^\infty r^{M-2} e^{-r} \mathrm{d}r \\
&= r^{M-1} e^{-c} + (\cdots) \int_c^\infty e^{-r} \mathrm{d}r = r^{M-1} e^{-c} + (\cdots) \frac{e^{-r}}{-1}\bigg|_{r=c}^{r=\infty} \\
&= r^{M-1} e^{-c} - (\cdots)(0 - e^{-c}) < \infty,
\end{aligned}$$

$\int_{C_1} \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z$ converges uniformly on $\{s : |s| < M\}$.

For 2: On C_3, for every $r \geq 1 (= \ln e > \ln 2)$,

$$\begin{aligned}
\left|\frac{e^z}{1-e^z} z^{s-1}\right| &= \left|\frac{e^z}{1-e^z}\right| \left|z^{s-1}\right| = \left|\frac{e^z}{1-e^z}\right| \left|\left(e^{i\pi(s-1)}\right) r^{s-1}\right| \\
&= \left|\frac{e^z}{1-e^z}\right| \left|e^{i\pi(s-1)}\right| \left|r^{s-1}\right| = \left|\frac{e^z}{1-e^z}\right| \left|e^{i\pi((\sigma-1)+it)}\right| r^{\sigma-1} \\
&= \left|\frac{e^z}{1-e^z}\right| \left|e^{i\pi(\sigma-1)} e^{-\pi t}\right| r^{\sigma-1} = \left|\frac{e^z}{1-e^z}\right| e^{-\pi t} r^{\sigma-1} \\
&\leq \left|\frac{e^z}{1-e^z}\right| e^{\pi(-t)} r^{M-1} \leq \left|\frac{e^z}{1-e^z}\right| e^{\pi M} r^{M-1} \\
&= \left|\frac{e^{-r}}{1-e^{-r}}\right| e^{\pi M} r^{M-1} = \frac{1}{1-e^{-r}} e^{\pi M}\left(r^{M-1} e^{-r}\right) \\
&\leq \frac{e^r}{e^r-1} e^{\pi M}\left(r^{M-1} e^{-r}\right) = \left(1 + \frac{1}{e^r-1}\right) e^{\pi M}\left(r^{M-1} e^{-r}\right) \\
&\leq \left(1 + \frac{1}{e^{\ln 2}-1}\right) e^{\pi M}\left(r^{M-1} e^{-r}\right) = (1+1) e^{\pi M}\left(r^{M-1} e^{-r}\right) \\
&= 2e^{\pi M}\left(r^{M-1} e^{-r}\right).
\end{aligned}$$

Thus, on C_3, for every $r \geq 1$, we have

$$\left|\frac{e^z}{1-e^z} z^{s-1}\right| \leq 2e^{\pi M}\left(r^{M-1} e^{-r}\right).$$

Now, since $2e^{\pi M}(r^{M-1} e^{-r})$ is independent of s, and

$$\begin{aligned}
\int_c^\infty r^{M-1} e^{-r} \mathrm{d}r &= r^{M-1} \frac{e^{-r}}{-1}\bigg|_{r=c}^{r=\infty} - \int_c^\infty (M-1) r^{M-2} \frac{e^{-r}}{-1} \mathrm{d}r \\
&= -r^{M-1}(0 - e^{-c}) + (M-1) \int_c^\infty r^{M-2} e^{-r} \mathrm{d}r \\
&= r^{M-1} e^{-c} + (\cdots) \int_c^\infty e^{-r} \mathrm{d}r = r^{M-1} e^{-c} + (\cdots) \frac{e^{-r}}{-1}\bigg|_{r=c}^{r=\infty} \\
&= r^{M-1} e^{-c} - (\cdots)(0 - e^{-c}) < \infty,
\end{aligned}$$

$\int_{C_3} \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z$ converges uniformly on $\{s : |s| < M\}$. This shows that $s \mapsto \frac{1}{2\pi i} \int_C \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z$ is holomorphic in $\mathbb{C}$. ■)

Problem 4.219 $\zeta(s) = \Gamma(1-s)\left(\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z\right)$ $(\sigma > 1)$.

(Solution Put $g(z) \equiv \frac{e^z}{1-e^z}$. Now,

$$\begin{aligned}
\mathrm{RHS} &= \Gamma(1-s)\left(\frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z\right)\\
&= \Gamma(1-s)\frac{1}{2\pi i}\left(\int\limits_{C_1} g(z)z^{s-1}\mathrm{d}z + \int\limits_{C_2} g(z)z^{s-1}\mathrm{d}z + \int\limits_{C_3} g(z)z^{s-1}\mathrm{d}z\right)\\
&= \Gamma(1-s)\frac{1}{2\pi i}\left(\int\limits_{\infty}^{c} g(-r)r^{s-1}e^{-i\pi(s-1)}\mathrm{d}(-r)\right.\\
&\quad \left. + \int\limits_{-\pi}^{\pi} g\left(ce^{i\theta}\right)c^{s-1}e^{i\theta(s-1)}\mathrm{d}\left(ce^{i\theta}\right) + \int\limits_{c}^{\infty} g(-r)r^{s-1}e^{i\pi(s-1)}\mathrm{d}(-r)\right)\\
&= \Gamma(1-s)\frac{1}{2\pi i}\left(\int\limits_{\infty}^{c} g(-r)r^{s-1}e^{-i\pi s}\mathrm{d}r\right.\\
&\quad \left. + \int\limits_{-\pi}^{\pi} g\left(ce^{i\theta}\right)c^{s-1}e^{i\theta s}ci\mathrm{d}\theta + \int\limits_{c}^{\infty} g(-r)r^{s-1}e^{i\pi s}\mathrm{d}r\right)\\
&= \Gamma(1-s)\frac{1}{2\pi i}\left(\int\limits_{\infty}^{c} g(-r)r^{s-1}e^{-i\pi s}\mathrm{d}r\right.\\
&\quad \left. + i\int\limits_{-\pi}^{\pi} g\left(ce^{i\theta}\right)c^{s}e^{i\theta s}\mathrm{d}\theta + \int\limits_{c}^{\infty} g(-r)r^{s-1}e^{i\pi s}\mathrm{d}r\right)\\
&= \Gamma(1-s)\frac{1}{2\pi i}\left(e^{-i\pi s}\int\limits_{\infty}^{c} g(-r)r^{s-1}\mathrm{d}r\right.\\
&\quad \left. + ic^s\int\limits_{-\pi}^{\pi} g\left(ce^{i\theta}\right)e^{i\theta s}\mathrm{d}\theta + e^{i\pi s}\int\limits_{c}^{\infty} g(-r)r^{s-1}\mathrm{d}r\right)\\
&= \Gamma(1-s)\frac{1}{2\pi i}\left(\left(e^{i\pi s} - e^{-i\pi s}\right)\int\limits_{c}^{\infty} g(-r)r^{s-1}\mathrm{d}r + ic^s\int\limits_{-\pi}^{\pi} g\left(ce^{i\theta}\right)e^{i\theta s}\mathrm{d}\theta\right)\\
&= \Gamma(1-s)\frac{1}{2\pi i}\left((2i\sin\pi s)\int\limits_{c}^{\infty} g(-r)r^{s-1}\mathrm{d}r + ic^s\int\limits_{-\pi}^{\pi} g\left(ce^{i\theta}\right)e^{i\theta s}\mathrm{d}\theta\right)\\
&= \Gamma(1-s)\frac{1}{2\pi}\left((2\sin\pi s)I_1(s,c) + I_2(s,c)\right),
\end{aligned}$$

where

$$\underbrace{I_1(s,c) \equiv \int_c^\infty g(-r)r^{s-1}\mathrm{d}r} = \int_c^\infty \frac{e^{-r}}{1-e^{-r}} r^{s-1}\mathrm{d}r = \int_c^\infty \frac{1}{e^r - 1} r^{s-1}\mathrm{d}r,$$

and

$$I_2(s,c) \equiv c^s \int_{-\pi}^{\pi} g\left(ce^{i\theta}\right)e^{i\theta s}\mathrm{d}\theta.$$

Observe that

$$\lim_{c\to 0} I_1(s,c) = \lim_{c\to 0}\int_c^\infty \frac{1}{e^r-1}r^{s-1}\mathrm{d}r = \int_0^\infty \frac{1}{e^r-1}r^{s-1}\mathrm{d}r = \Gamma(s)\zeta(s).$$

So,

$$\lim_{c\to 0} I_1(s,c) = \Gamma(s)\zeta(s).$$

Problem 7.220 $\lim_{c\to 0} I_2(s,c) = 0$.

(Solution Since $g : z \mapsto \frac{e^z}{1-e^z}$ is holomorphic in $(\{z : |z| < 2\pi\} - \{0\})$, and

$$\begin{aligned}\underbrace{g : z \mapsto \frac{e^z}{1-e^z}} &= -1 - \frac{1}{e^z - 1} = -1 - \frac{1}{z + \frac{z^2}{2!} + \cdots} = -1 - \frac{1}{z}\frac{1}{1 + \frac{z}{2} + \cdots} \\ &= -1 - \frac{1}{z}\left(1 - \frac{z}{2} + \cdots\right) = \frac{-1}{z} - \frac{1}{2} + \cdots\end{aligned}$$

has a simple pole at $z = 0$ with residue (-1), $z \mapsto zg(z)$ is holomorphic in $D(0; 2\pi)$, and hence this function is bounded on the compact set C_2. It follows that there exists $A > 0$ such that, for every $z \in C_2$, $|z||g(z)| = \underbrace{|zg(z)| \le A}$. Now,

$$\begin{aligned}0 \le |I_2(s,c)| &= \left|c^s \int_{-\pi}^{\pi} g\left(ce^{i\theta}\right)e^{i\theta s}\mathrm{d}\theta\right| = c^\sigma\left|\int_{-\pi}^{\pi} g\left(ce^{i\theta}\right)e^{i\theta s}\mathrm{d}\theta\right| \\ &\le c^\sigma \int_{-\pi}^{\pi} \left|g\left(ce^{i\theta}\right)e^{i\theta s}\right|\mathrm{d}\theta = c^\sigma \int_{-\pi}^{\pi} \left|g\left(ce^{i\theta}\right)\right|\left|e^{i\theta s}\right|\mathrm{d}\theta\end{aligned}$$

$$\leq c^{\sigma} \int_{-\pi}^{\pi} \frac{A}{|ce^{i\theta}|} \left|e^{i\theta s}\right| \mathrm{d}\theta = c^{\sigma} \int_{-\pi}^{\pi} \frac{A}{c} \left|e^{i\theta s}\right| \mathrm{d}\theta = c^{\sigma-1} A \int_{-\pi}^{\pi} \left|e^{i\theta s}\right| \mathrm{d}\theta$$

$$= c^{\sigma-1} A \int_{-\pi}^{\pi} \left|e^{i\theta(\sigma+it)}\right| \mathrm{d}\theta = c^{\sigma-1} A \int_{-\pi}^{\pi} \left|e^{i\theta\sigma} e^{-\theta t}\right| \mathrm{d}\theta$$

$$= c^{\sigma-1} A \int_{-\pi}^{\pi} e^{-\theta t} \mathrm{d}\theta \leq c^{\sigma-1} A \left(e^{\pi|t|} 2\pi\right) \to 0$$

as $c \to 0$, so $\lim_{c\to 0} I_2(s,c) = 0$. ■)

Hence, on using Problem 4.146,

$$\begin{aligned}
\text{RHS} &= \Gamma(1-s) \left(\frac{1}{2\pi i} \int_C \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z \right) \\
&= \Gamma(1-s) \frac{1}{2\pi} \left((2 \sin \pi s) \cdot \Gamma(s) \zeta(s) + 0 \right) \\
&= \underbrace{\frac{1}{\pi} \sin \pi s \zeta(s) (\Gamma(s)\Gamma(1-s)) = \frac{1}{\pi} \sin \pi s \zeta(s) \frac{\pi}{\sin \pi s}} = \zeta(s) = \text{LHS}.
\end{aligned}$$

■)

Since

$$s \mapsto \frac{1}{2\pi i} \int_C \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z$$

is holomorphic in $\mathbb{C}$, and $s \mapsto \Gamma(1-s)$ is holomorphic in $\mathbb{C} - \{1, 2, 3, \ldots\}$,

$$s \mapsto \Gamma(1-s) \left(\frac{1}{2\pi i} \int_C \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z \right)$$

is holomorphic in $\{s = \sigma + it : \sigma \leq 1\} - \{1\}$. It follows that in order to extend holomorphically the domain of ζ from $\{s = \sigma + it : 1 < \sigma\}$ to $\mathbb{C} - \{1\}$ we can use the equation

$$\zeta(s) = \Gamma(1-s) \left(\frac{1}{2\pi i} \int_C \frac{e^z}{1-e^z} z^{s-1} \mathrm{d}z \right). \quad (*)$$

The equation

$$\zeta(s) = \Gamma(1-s)\left(\frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z\right) \quad (\sigma \le 1, \text{and}\, s \neq 1) \quad (**)$$

gives a holomorphic extension of the Riemann zeta function defined for $\sigma > 1$ to $\mathbb{C} - \{1\}$. Thus,

$$\zeta : s \mapsto \begin{cases} \Gamma(1-s)\left(\frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z\right) & \text{if}\, s \notin \{1,2,3,4,\ldots\} \\ 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots & \text{if}\, s \in \{2,3,4,\ldots\} \end{cases}$$

is a holomorphic function from $(\mathbb{C} - \{1\})$ to $\mathbb{C}$. Since

$$s \mapsto \frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z$$

is holomorphic in $\mathbb{C}$, and $s \mapsto \Gamma(1-s)$ is holomorphic in $\mathbb{C} - \{1,2,3,\ldots\}$,

$$s \mapsto \Gamma(1-s)\left(\frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z\right) (= \zeta(s))$$

is holomorphic in $\mathbb{C} - \{1,2,3,\ldots\}$, and hence ζ is holomorphic in $\mathbb{C} - \{1,2,3,\ldots\}$.

We have seen that ζ is holomorphic in $\{s = \sigma + it : 1 < \sigma\}$, so ζ is holomorphic in $(\mathbb{C} - \{1,2,3,\ldots\}) \cup \{s = \sigma + it : 1 < \sigma\} (= \mathbb{C} - \{1\})$. Thus, $\mathrm{dom}(\zeta) = \mathbb{C} - \{1\}$. It follows that $s = 1$ is the only possible pole of ζ. Since

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z} z^{1-1}\mathrm{d}z &= \frac{1}{2\pi i}\int\limits_C \frac{e^z}{1-e^z}\mathrm{d}z \\ &= \frac{1}{2\pi i}\left(\int\limits_{C_1} \frac{e^z}{1-e^z}\mathrm{d}z + \int\limits_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z + \int\limits_{C_3} \frac{e^z}{1-e^z}\mathrm{d}z\right) \\ &= \frac{1}{2\pi i}\left(\int\limits_{\infty}^{c} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r) + \int\limits_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z + \int\limits_{c}^{\infty} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r)\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\left(-\int_c^{\infty} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r) + \int_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z + \int_c^{\infty} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r) \right) \\
&= \frac{1}{2\pi i}\int_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z = \mathrm{Res}\left(z \mapsto \frac{e^z}{1-e^z}; 0 \right) \\
&= \lim_{z\to 0} z\frac{e^z}{1-e^z} = \left(\lim_{z\to 0} \frac{z}{1-e^z} \right)\left(\lim_{z\to 0} e^z \right) \\
&= \left(\lim_{z\to 0} \frac{1}{-e^z} \right)\left(\lim_{z\to 0} e^z \right) = \frac{1}{-1}\cdot 1 = -1,
\end{aligned}
$$

we have

$$
\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{1-1}\mathrm{d}z = -1.
$$

Next,

$$
\begin{aligned}
\mathrm{Res}(\zeta;1) &= \lim_{s\to 1}(s-1)\zeta(s) \\
&= \lim_{s\to 1}(s-1)\left(\Gamma(1-s)\left(\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z \right) \right) \\
&= \lim_{s\to 1}((s-1)\Gamma(1-s))\left(\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z \right) \\
&= -\lim_{s\to 1}((1-s)\Gamma(1-s))\left(\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z \right) \\
&= -\lim_{s\to 1}\Gamma((1-s)+1)\left(\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z \right) \\
&= -\lim_{s\to 1}\Gamma(2-s)\left(\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{s-1}\mathrm{d}z \right) \\
&= -\Gamma(2-1)\cdot \frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z} z^{1-1}\mathrm{d}z = -\Gamma(1)\cdot \frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z}\mathrm{d}z \\
&= -1\cdot \frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z}\mathrm{d}z = -\frac{1}{2\pi i}\int_C \frac{e^z}{1-e^z}\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2\pi i}\left(\int\limits_{C_1} \frac{e^z}{1-e^z}\mathrm{d}z + \int\limits_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z + \int\limits_{C_3} \frac{e^z}{1-e^z}\mathrm{d}z\right) \\
&= -\frac{1}{2\pi i}\left(\int\limits_{\infty}^{c} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r) + \int\limits_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z + \int\limits_{c}^{\infty} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r)\right) \\
&= -\frac{1}{2\pi i}\left(-\int\limits_{c}^{\infty} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r) + \int\limits_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z + \int\limits_{c}^{\infty} \frac{e^{-r}}{1-e^{-r}}\mathrm{d}(-r)\right) \\
&= \frac{1}{2\pi i}\int\limits_{C_2} \frac{e^z}{1-e^z}\mathrm{d}z = -\mathrm{Res}\left(z \mapsto \frac{e^z}{1-e^z}; 0\right) \\
&= -\lim_{z\to 0} z\frac{e^z}{1-e^z} = -\left(\lim_{z\to 0}\frac{z}{1-e^z}\right)\left(\lim_{z\to 0} e^z\right) \\
&= -\left(\lim_{z\to 0}\frac{1}{-e^z}\right)\left(\lim_{z\to 0} e^z\right) = -\frac{1}{-1}\cdot 1 = 1.
\end{aligned}
$$

Conclusion 4.221 The Riemann zeta function $\zeta \in H(\mathbb{C} - \{1\})$ has a simple pole at $s = 1$ with residue 1, where

$$
\zeta : s \mapsto \begin{cases} \Gamma(1-s)\left(\frac{1}{2\pi i}\int\limits_{C}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right) & \text{if } s \notin \{1,2,3,4,\ldots\} \\ 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots & \text{if } s \in \{2,3,4,\ldots\} \end{cases}
$$

is a function from $(\mathbb{C} - \{1\})$ to $\mathbb{C}$.

4.13 Riemann Hypothesis

Note 4.222 Let

$$
g : z \mapsto \frac{e^z}{1-e^z}
$$

be a function from $(\mathbb{C} - \{0, 2\pi i, -2\pi i, 4\pi i, -4\pi i, \ldots\})$ to $(\mathbb{C} - \{0\})$.

Problem 4.223 g is bounded over

$$\{z = x + iy : 1 \le |x|\}(= \{z = x + iy : x \le -1\} \cup \{z = x + iy : 1 \le x\}).$$

(Solution Case I: when $x \le -1$. Here

$$\begin{aligned}|g(z)| &= \left|\frac{e^z}{1 - e^z}\right| = \left|\frac{e^{x+iy}}{1 - e^{x+iy}}\right| = \frac{|e^{x+iy}|}{|1 - e^{x+iy}|} = \frac{e^x}{|1 - e^{x+iy}|}\\ &= e^x \frac{1}{|1 - e^{x+iy}|} \le e^x \frac{1}{||1| - |e^{x+iy}||} = e^x \frac{1}{|1 - e^x|}\\ &= e^x \frac{1}{1 - e^x} = \frac{1}{e^{-x} - 1} \le \frac{1}{e^1 - 1} = \frac{1}{e - 1},\end{aligned}$$

so g is bounded over $\{z = x + iy : x \le -1\}$.

Case II: when $1 \le x$. Here

$$\begin{aligned}|g(z)| &= \left|\frac{e^z}{1 - e^z}\right| = \left|\frac{e^{x+iy}}{1 - e^{x+iy}}\right| = \frac{|e^{x+iy}|}{|1 - e^{x+iy}|} = \frac{e^x}{|1 - e^{x+iy}|}\\ &= e^x \frac{1}{|1 - e^{x+iy}|} \le e^x \frac{1}{||1| - |e^{x+iy}||} = e^x \frac{1}{|1 - e^x|}\\ &= e^x \frac{1}{e^x - 1} = 1 + \frac{1}{e^x - 1} \le 1 + \frac{1}{e^1 - 1} = \frac{e}{e - 1},\end{aligned}$$

so g is bounded over $\{z = x + iy : 1 \le x\}$. ■)

Problem 4.224 Let $r \in (0, \pi)$. Then g is bounded over

$$\begin{aligned}&\{z = x + iy : |x| \le 1\}\\ &\quad - (D(0; r) \cup D(2\pi i; r) \cup D(-2\pi i; r) \cup D(4\pi i; r) \cup D(-4\pi i; r) \cup \cdots).\end{aligned}$$

(Solution Since for every z in $(\mathbb{C} - \{0, 2\pi i, -2\pi i, 4\pi i, -4\pi i, \ldots\})$,

$$g(z + 2\pi i) = \frac{e^{z+2\pi i}}{1 - e^{z+2\pi i}} = \frac{e^z e^{2\pi i}}{1 - e^z e^{2\pi i}} = \frac{e^z \cdot 1}{1 - e^z \cdot 1} = \frac{e^z}{1 - e^z} = g(z),$$

we have, for every z in $(\mathbb{C} - \{0, 2\pi i, -2\pi i, 4\pi i, -4\pi i, \ldots\})$,

$$g(z + 2\pi i) = g(z),$$

and hence it suffices to show that g is bounded over $(\{z = x + iy : |x| \le 1, |y| \le \pi\} - D(0; r))(= [-1, 1] \times [-\pi, \pi] - D(0; r))$.

Since g is a continuous function, and $([-1, 1] \times [-\pi, \pi] - D(0; r))$ is a compact set, $g([-1, 1] \times [-\pi, \pi] - D(0; r))$ is compact, and hence $g([-1, 1] \times [-\pi, \pi] - D(0; r))$ is bounded. Thus, g is bounded over $(\{z = x + iy : |x| \le 1, |y| \le \pi\} - D(0; r))$. ■)

Conclusion 4.225 Let $g : z \mapsto \frac{e^z}{1-e^z}$ be a function from $(\mathbb{C} - \{0, 2\pi i, -2\pi i, 4\pi i, -4\pi i, \ldots\})$ to $(\mathbb{C} - \{0\})$. Let $r \in (0, \pi)$. Then g is bounded over $\mathbb{C} - (D(0; r) \cup D(2\pi i; r) \cup D(-2\pi i; r) \cup D(4\pi i; r) \cup D(-4\pi i; r) \cup \cdots)$.

Let $C(1)$ be a closed curve (see Fig. 4.1), which consists of four parts, $C_{11}, C_2, C_{13}, C_{14}$. Here, C_2 is a positively oriented circle with center at the origin O and radius $c \in (0, \pi)$. Next, C_{14} is a negatively oriented circle with center at the origin O and radius $3\pi, C_{11}$ is the 'lower edge' of a cut in the z-plane along the negative real axis, traversed towards the origin, and C_{13} is the 'upper edge' of a cut in the z-plane along the negative real axis, traversed away from the origin.

On C_{11}, we shall use the parametrization $r \mapsto \left(e^{i(-\pi)}\right)r$ from $r = 3\pi$ to $r = c$. On C_{13}, we shall use the parametrization $r \mapsto (e^{i\pi})r$ from $r = c$ to $r = 3\pi$. On C_2, we shall use the parametrization $\theta \mapsto ce^{i\theta}$ from $\theta = -\pi$ to $\theta = \pi$. And on C_{14}, we shall use the parametrization $\theta \mapsto (3\pi)e^{i\theta}$ from $\theta = \pi$ to $\theta = -\pi$.
Similarly, let $C(2)$ be a closed curve (see Fig. 4.1), which consists of four parts, $C_{21}, C_2, C_{23}, C_{24}$. Here, C_2 is a positively oriented circle with center at the origin O and radius $c \in (0, \pi)$. Next, C_{24} is a negatively oriented circle with center at the origin O and radius $5\pi, C_{21}$ is the 'lower edge' of a cut in the z-plane along the negative real axis, traversed towards the origin, and C_{23} is the 'upper edge' of a cut in the z-plane along the negative real axis, traversed away from the origin.

On C_{21}, we shall use the parametrization $r \mapsto \left(e^{i(-\pi)}\right)r$ from $r = 5\pi$ to $r = c$. On C_{23}, we shall use the parametrization $r \mapsto (e^{i\pi})r$ from $r = c$ to $r = 5\pi$. On C_2, we shall use the parametrization $\theta \mapsto ce^{i\theta}$ from $\theta = -\pi$ to $\theta = \pi$. And on C_{24}, we shall use the parametrization $\theta \mapsto (5\pi)e^{i\theta}$ from $\theta = \pi$ to $\theta = -\pi$, etc.

Let C be the contour as described in Problem 4.218.

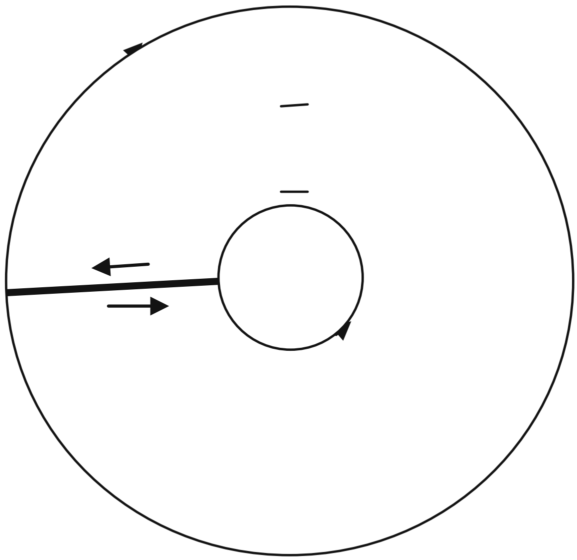

Fig. 4.1 Contour as described above

Problem 4.226 For every $s=\sigma+it$ satisfying $\sigma<0$,

$$\lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C(N)}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right)=\frac{1}{2\pi i}\int\limits_{C}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z,$$

that is

$$\begin{aligned}&\lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C_{N4}}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right)\\&=\underbrace{\lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C(N)}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z-\frac{1}{2\pi i}\int\limits_{C}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right)=0},\end{aligned}$$

that is

$$\lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C_{N4}}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right)=0.$$

(**Solution** Observe that for every positive integer N, C_{N4} is contained in $(\mathbb{C}-\{0,2\pi i,-2\pi i,4\pi i,-4\pi i,\ldots\})$, so, by Conclusion 4.225, there exists $A>0$ such that for every $z\in C_{N4}$, $\left|\frac{e^z}{1-e^z}\right|\le A$.

For every positive integer N, and for every $z\in C_{N4}$,

$$\begin{aligned}\left|\frac{e^z}{1-e^z}z^{s-1}\right|&=\left|\frac{e^z}{1-e^z}\right|\left|z^{s-1}\right|\le A\left|z^{s-1}\right|\\&=A\left|((2N+1)\pi)^{s-1}e^{i\theta(s-1)}\right|\\&=A\left|((2N+1)\pi)^{s-1}\right|\left|e^{i\theta(s-1)}\right|\\&=A((2N+1)\pi)^{\sigma-1}\left|e^{i\theta((\sigma-1)+it)}\right|\\&=A((2N+1)\pi)^{\sigma-1}e^{-\theta t}\le A((2N+1)\pi)^{\sigma-1}e^{\pi|t|},\end{aligned}$$

so for every positive integer N, and for every $z\in C_{N4}$,

$$\left|\frac{e^z}{1-e^z}z^{s-1}\right|\le A((2N+1)\pi)^{\sigma-1}e^{\pi|t|}.$$

It follows that for every positive integer N,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int\limits_{C_{N4}}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right| &= \frac{1}{2\pi}\left|\int\limits_{C_{N4}}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right| \\ &\leq \frac{1}{2\pi}\Big(A((2N+1)\pi)^{\sigma-1}e^{\pi|t|}\cdot 2\pi((2N+1)\pi)\Big) \\ &= A((2N+1)\pi)^{\sigma}e^{\pi|t|}.\end{aligned}$$

Thus, for every positive integer N,

$$\left|\frac{1}{2\pi i}\int\limits_{C_{N4}}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right| \leq Ae^{\pi|t|}\cdot((2N+1)\pi)^{\sigma}.$$

Now, since $\sigma < 0$, we have

$$\lim_{N\to\infty}(((2N+1)\pi)^{\sigma}) = 0,$$

and hence

$$\lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C_{N4}}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right) = 0.$$

∎)

Conclusion 4.227 For every $s = \sigma + it$ satisfying $\sigma < 0$,

$$\lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C(N)}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z\right) = \frac{1}{2\pi i}\int\limits_{C}\frac{e^z}{1-e^z}z^{s-1}\mathrm{d}z.$$

Let $s = \sigma + it$ satisfying $\sigma > 1$. It follows that $(1-\sigma) < 0$, and $(1-s) \notin \{1, 2, 3, 4, \ldots\}$, and hence, by Conclusion 4.221,

$$\zeta(1-s) = \Gamma(1-(1-s))\left(\frac{1}{2\pi i}\int\limits_{C}\frac{e^z}{1-e^z}z^{(1-s)-1}\mathrm{d}z\right).$$

Next, by Conclusion 4.227,

$$\lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C(N)} \frac{e^z}{1-e^z}z^{(1-s)-1}\mathrm{d}z\right) = \frac{1}{2\pi i}\int\limits_{C} \frac{e^z}{1-e^z}z^{(1-s)-1}\mathrm{d}z\left(= \frac{\zeta(1-s)}{\Gamma(s)}\right),$$

and

$$\begin{aligned}
&-\lim_{N\to\infty}\sum_{n=1}^{N}\left(\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; 2n\pi i\right)+\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; -2n\pi i\right)\right)\\
&= \lim_{N\to\infty}\left((-1)\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; 2\pi i\right)+(-1)\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; -2\pi i\right)\right.\\
&\quad\left. + \cdots + (-1)\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; 2N\pi i\right)+(-1)\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; -2N\pi i\right)\right)\\
&= \lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C(N)} \frac{e^z}{1-e^z}z^{-s}\mathrm{d}z\right) = \lim_{N\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C(N)} \frac{e^z}{1-e^z}z^{(1-s)-1}\mathrm{d}z\right),
\end{aligned}$$

so

$$\zeta(1-s) = -\Gamma(s)\left(\lim_{N\to\infty}\sum_{n=1}^{N}\left(\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; 2n\pi i\right)+\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; -2n\pi i\right)\right)\right).$$

Now, since for every positive integer n,

$$\begin{aligned}
&\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; 2n\pi i\right)+\mathrm{Res}\left(z\mapsto \frac{e^z}{1-e^z}z^{-s}; -2n\pi i\right)\\
&= \lim_{z\to 2n\pi i}(z-2n\pi i)\left(\frac{e^z}{1-e^z}z^{-s}\right) + \lim_{z\to -2n\pi i}(z-(-2n\pi i))\left(\frac{e^z}{1-e^z}z^{-s}\right)\\
&= \lim_{z\to 2n\pi i}\frac{z-2n\pi i}{1-e^z}\cdot \lim_{z\to 2n\pi i}\frac{e^z}{z^s} + \lim_{z\to -2n\pi i}\frac{z+2n\pi i}{1-e^z}\cdot \lim_{z\to -2n\pi i}\frac{e^z}{z^s}\\
&= \left(\lim_{z\to 2n\pi i}\frac{z-2n\pi i}{1-e^{z-2n\pi i}e^{2n\pi i}}\right)\frac{e^{2n\pi i}}{(2n\pi i)^s} + \left(\lim_{z\to -2n\pi i}\frac{z+2n\pi i}{1-e^{z+2n\pi i}e^{-2n\pi i}}\right)\frac{e^{-2n\pi i}}{(-2n\pi i)^s}\\
&= \left(\lim_{z\to 2n\pi i}\frac{z-2n\pi i}{1-e^{z-2n\pi i}\mapsto 1}\right)\frac{1}{(2n\pi i)^s} + \left(\lim_{z\to -2n\pi i}\frac{z+2n\pi i}{1-e^{z+2n\pi i}\mapsto 1}\right)\frac{1}{(-2n\pi i)^s}
\end{aligned}$$

$$\begin{aligned}
&= \left(\lim_{w\to 0}\frac{w}{1-e^w}\right)\frac{1}{(2n\pi i)^s} + \left(\lim_{w\to 0}\frac{w}{1-e^w}\right)\frac{1}{(-2n\pi i)^s} \\
&= \left(\lim_{w\to 0}\frac{w}{1-e^w}\right)\left(\frac{1}{(2n\pi i)^s} + \frac{1}{(-2n\pi i)^s}\right) \\
&= (-1)\left(\frac{1}{\left(2n\pi e^{i\frac{\pi}{2}}\right)^s} + \frac{1}{\left(2n\pi e^{-i\frac{\pi}{2}}\right)^s}\right) = -\left(\left(2n\pi e^{i\frac{\pi}{2}}\right)^{-s} + \left(2n\pi e^{-i\frac{\pi}{2}}\right)^{-s}\right) \\
&= -\left((2n\pi)^{-s}e^{i\frac{\pi}{2}(-s)} + (2n\pi)^{-s}e^{-i\frac{\pi}{2}(-s)}\right) = -(2n\pi)^{-s}\left(e^{i\frac{\pi s}{2}} + e^{-i\frac{\pi s}{2}}\right) \\
&= -(2n\pi)^{-s}\left(2\cos\left(\frac{\pi s}{2}\right)\right) = -(2\pi)^{-s}n^{-s}\left(2\cos\left(\frac{\pi s}{2}\right)\right) \\
&= \frac{-1}{(2\pi)^s}\left(2\cos\left(\frac{\pi s}{2}\right)\right)\frac{1}{n^s},
\end{aligned}$$

we have

$$\begin{aligned}
&\underbrace{\zeta(1-s) = -\Gamma(s)\left(\lim_{N\to\infty}\sum_{n=1}^{N}\frac{-1}{(2\pi)^s}\left(2\cos\left(\frac{\pi s}{2}\right)\right)\frac{1}{n^s}\right)} \\
&\quad = \Gamma(s)\sum_{n=1}^{\infty}\frac{1}{(2\pi)^s}\left(2\cos\left(\frac{\pi s}{2}\right)\right)\frac{1}{n^s} \\
&\quad = \Gamma(s)\frac{1}{(2\pi)^s}\left(2\cos\left(\frac{\pi s}{2}\right)\right)\sum_{n=1}^{\infty}\frac{1}{n^s} \\
&\quad = \Gamma(s)\frac{1}{(2\pi)^s}\left(2\cos\left(\frac{\pi s}{2}\right)\right)\zeta(s).
\end{aligned}$$

Thus, for every $s = \sigma + it$ satisfying $\sigma > 1$, we have

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

Conclusion 4.228 For every $s = \sigma + it$ satisfying $\sigma > 1$, $\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)$ $\cos\left(\frac{\pi s}{2}\right)\zeta(s)$.

Since $s \mapsto \Gamma(s)$ is holomorphic in $\mathbb{C} - \{0, -1, -2, \ldots\}$, $s \mapsto \zeta(s)$ is holomorphic in $\mathbb{C} - \{1\}$, and $s \mapsto 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)$ is holomorphic in $\mathbb{C}$, their product

$$f : s \mapsto 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)(= \text{RHS})$$

is holomorphic in $\mathbb{C} - \{1, 0, -1, -2, \ldots\}$. Since $s \mapsto \zeta(s)$ is holomorphic in $\mathbb{C} - \{1\}$, $g : s \mapsto \zeta(1-s)(= \text{LHS})$ is holomorphic in $\mathbb{C} - \{0\}$. By Conclusion 4.228, $f = g$ on $\{s = \sigma + it : \sigma > 1\}$, and $\{s = \sigma + it : \sigma > 1\}$ has a limit point in the common domain $\mathbb{C} - \{1, 0, -1, -2, \ldots\}$ of f and g, so, by Theorem 4.135, $f = g$ on $\mathbb{C} - \{1, 0, -1, -2, \ldots\}$. Thus, for every $s \in (\mathbb{C} - \{1, 0, -1, -2, \ldots\})$,

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

It follows that for every $s \in (\mathbb{C} - \{0, 1, 2, 3, \ldots\})$,

$$\underbrace{\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\cos\left(\frac{\pi(1-s)}{2}\right)\zeta(1-s)}$$
$$= 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s).$$

Conclusion 4.229 If $\zeta(1-s)$, and $2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$ are meaningful for s, that is, if $s \in (\mathbb{C} - \{1, 0, -1, -2, \ldots\})$, then

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \quad (*).$$

In other words, if $s \in (\mathbb{C} - \{0, 1, 2, 3, \ldots\})$, then

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s) \quad (**).$$

Here, equation $(**)$ is known as the ***functional equation*** *for the Riemann zeta function*.

By Problem 4.145, Γ has no zero. Now, from $(**)$, it follows that in $(\mathbb{C} - \{0, 1, 2, 3, \ldots\})$,

$$\zeta(s) = 0 \Leftrightarrow \sin\left(\frac{\pi s}{2}\right)\zeta(1-s) = 0 \Leftrightarrow \left(\sin\left(\frac{\pi s}{2}\right) = 0 \,\text{or}\, \zeta(1-s) = 0\right).$$

Thus, $s = -2, -4, -6, \ldots$ are some of the zeros of ζ.

Definition $-2, -4, -6, \ldots$ are known as the ***trivial zeros*** *of* ζ.

The assertion:

$$\text{all non-trivial zeros of } \zeta \text{ have real part } \frac{1}{2}$$

has been neither proved nor disproved to date. This assertion is known as the ***Riemann hypothesis***. It is interesting to note that the Clay Mathematical Institute of Cambridge, Massachusetts, USA, has offered one million US dollars as prize money for solving the Riemann hypothesis.

4.14 Chebyshev's Functions

Note 4.230

Definition Let N be a positive integer. We define

$$\Lambda(N) \equiv \begin{cases} \ln p & \text{if } N \text{ is of the form } p^a \text{ for some prime } p, \text{ and for some positive integer } a \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\Lambda : \mathbb{N} \to [0, \infty)(\subset \mathbb{C})$ is an arithmetic function, and is called the ***Mangoldt function***.

Here, $\Lambda(5 \cdot 7) = 0$, $\Lambda(7^3) = \ln 7$, $\Lambda(7^2) = \ln 7$, etc.

Problem 4.231 For every positive integer N, $\sum_{d|N} \Lambda(d) = \ln N$.

(Solution For $N = 1$, $\sum_{d|N} \Lambda(d) = \Lambda(1) = 0 = \ln 1 = \ln N$. So, it remains to show that for every $N > 1$, $\sum_{d|N} \Lambda(d) = \ln N$. For this purpose, let us take any positive integer $N > 1$. Let $(p_1)^{a_1} \cdots (p_k)^{a_k}$ be the prime factorization of N, where $p_1, \ldots, p_k$ are k distinct primes. We have to show that

$$\underbrace{\sum_{d|((p_1)^{a_1} \cdots (p_k)^{a_k})} \Lambda(d) = \ln((p_1)^{a_1} \cdots (p_k)^{a_k})} = a_1 \ln p_1 + \cdots + a_k \ln p_k.$$

By the definition of Λ,

$$\begin{aligned} \sum_{d|(p_1)^{a_1} \cdots (p_k)^{a_k}} \Lambda(d) &= \Lambda(1) + \left(\underbrace{\Lambda\Big((p_1)^1\Big) + \Lambda\Big((p_1)^2\Big) + \cdots + \Lambda((p_1)^{a_1})}_{a_1 \text{ terms}} \right) \\ &\quad + \cdots + \left(\underbrace{\Lambda\Big((p_k)^1\Big) + \Lambda\Big((p_k)^2\Big) + \cdots + \Lambda((p_k)^{a_k})}_{a_k \text{ terms}} \right) \\ &= 0 + \left(\underbrace{\ln p_1 + \cdots + \ln p_1}_{a_1 \text{ terms}} \right) + \cdots + \left(\underbrace{\ln p_k + \cdots + \ln p_k}_{a_k \text{ terms}} \right) \\ &= a_1 \ln p_1 + \cdots + a_k \ln p_k. \end{aligned}$$

Thus,

$$\sum_{d|((p_1)^{a_1}\cdots(p_k)^{a_k})} \Lambda(d) = \ln((p_1)^{a_1}\cdots(p_k)^{a_k}).$$

∎)

Conclusion 4.232 For every positive integer N, $\sum_{d|N} \Lambda(d) = \ln N$.

Problem 4.233 For every positive integer N,

$$\Lambda(N) = \sum_{d|N} \left(\ln \frac{N}{d}\right) \cdot \mu(d) = -\sum_{d|N} (\ln d) \cdot \mu(d).$$

(Solution Let us fix any positive integer N. We have to show that $\Lambda(N) = \sum_{d|N} \left(\ln \frac{N}{d}\right) \cdot \mu(d)$. By Problem 4.272, $\ln N = \sum_{d|N} \Lambda(d)$, and hence, by Problems 4.179, and 4.158,

$$\begin{aligned}
\Lambda(N) &= \sum_{d|N} \ln(d) \cdot \mu\left(\frac{N}{d}\right) = \sum_{d|N} \ln\left(\frac{N}{d}\right) \cdot \mu(d) \\
&= \sum_{d|N} (\ln N - \ln d)\mu(d) = (\ln N) \sum_{d|N} \mu(d) - \sum_{d|N} (\ln d)\mu(d) \\
&= -\sum_{d|N} (\ln d)\mu(d) + \begin{cases} (\ln N) \cdot 0 & \text{if } N \in \{2, 3, 4, \ldots\} \\ 0 \cdot \sum_{d|N} \mu(d) & \text{if } N = 1 \end{cases} \\
&= -\sum_{d|N} (\ln d)\mu(d) + 0 = -\sum_{d|N} (\ln d)\mu(d).
\end{aligned}$$

Thus,

$$\Lambda(N) = \sum_{d|N} \left(\ln \frac{N}{d}\right) \cdot \mu(d) = -\sum_{d|N} (\ln d) \cdot \mu(d).$$

∎)

Conclusion 4.234 For every positive integer N, $\Lambda(N) = \sum_{d|N} \left(\ln \frac{N}{d}\right) \cdot \mu(d) = -\sum_{d|N} (\ln d) \cdot \mu(d)$.

Definition The function $\psi : x \mapsto \begin{cases} \sum_{n \le x} \Lambda(n) & \text{if } 1 \le x \\ 0 & \text{if } x < 1 \end{cases}$ from $(0,\infty)$ to $[0,\infty)$ is called the ***Chebyshev's*** ψ*-function*.

Thus, for every $x > 4$, we have

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{\substack{n \le x \\ n = p^a \\ p \,\text{isprime} \\ a \in N}} \ln p = \sum_{\substack{p^a \le x \\ p \,\text{isprime} \\ a \in N}} \ln p = \sum_{m=1}^{\infty} \left(\sum_{p^m \le x} \ln p \right)$$

$$= \sum_{p \le x} \ln p + \sum_{p^2 \le x} \ln p + \sum_{p^3 \le x} \ln p + \cdots = \sum_{p \le x} \ln p + \sum_{p \le x^{\frac{1}{2}}} \ln p + \sum_{p \le x^{\frac{1}{3}}} \ln p + \cdots$$

$$= \sum_{2 \le x^{\frac{1}{m}}} \left(\sum_{p \le x^{\frac{1}{m}}} \ln p \right) = \sum_{1 \le m \le \log_2 x} \left(\sum_{p \le x^{\frac{1}{m}}} \ln p \right),$$

and hence, for every $x > 4$,

$$\psi(x) = \sum_{1 \le m \le \log_2 x} \left(\sum_{p \le x^{\frac{1}{m}}} \ln p \right) \quad (*).$$

Definition The function $\vartheta : x \mapsto \sum_{\substack{p \le x \\ p \,\text{prime}}} \ln p$ from $(0,\infty)$ to $[0,\infty)$ is called the ***Chebyshev's*** ϑ*-function*.

From $(*)$, for every $x > 4$, we have

$$\psi(x) = \sum_{1 \le m \le \log_2 x} \vartheta\left(x^{\frac{1}{m}}\right) \left(= \vartheta\left(x^{\frac{1}{1}}\right) + \sum_{2 \le m \le \log_2 x} \vartheta\left(x^{\frac{1}{m}}\right) = \vartheta(x) + \sum_{2 \le m \le \log_2 x} \vartheta\left(x^{\frac{1}{m}}\right) \right).$$

It follows that, for every $x > 4$,

$$\begin{aligned}
0 \le \psi(x) - \vartheta(x) &= \sum_{2 \le m \le \log_2 x} \vartheta\left(x^{\frac{1}{m}}\right) = \sum_{2 \le m \le \log_2 x} \left(\sum_{p \le x^{\frac{1}{m}}} \ln p \right) \\
&\le \sum_{2 \le m \le \log_2 x} \left(\sum_{p \le x^{\frac{1}{m}}} \ln\left(x^{\frac{1}{m}}\right) \right) \le \sum_{2 \le m \le \log_2 x} \left(x^{\frac{1}{m}} \ln\left(x^{\frac{1}{m}}\right) \right) \\
&= \sum_{2 \le m \le \log_2 x} \left(x^{\frac{1}{m}} \frac{1}{m} \ln x \right) = (\ln x) \sum_{2 \le m \le \log_2 x} \left(x^{\frac{1}{m}} \frac{1}{m} \right) \\
&\le (\ln x) \sum_{2 \le m \le \log_2 x} \left(x^{\frac{1}{2}} \frac{1}{m} \right) = (\ln x) x^{\frac{1}{2}} \sum_{2 \le m \le \log_2 x} \left(\frac{1}{m} \right) \\
&\le (\ln x) x^{\frac{1}{2}} \sum_{2 \le m \le \log_2 x} \left(\frac{1}{2} \right) \le (\ln x) x^{\frac{1}{2}} \left(\frac{1}{2} \cdot \log_2 x \right) = \frac{1}{\ln 4} (\ln x)^2 \sqrt{x},
\end{aligned}$$

and hence, for every $x > 4$, $0 \le \frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \le \frac{1}{\ln 4} \frac{(\ln x)^2}{\sqrt{x}}$. Now, since

$$\begin{aligned}
\lim_{x \to \infty} \frac{(\ln x)^2}{\sqrt{x}} &= \lim_{x \to \infty} \frac{2(\ln x)\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 4 \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \\
&= 4 \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 8 \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0,
\end{aligned}$$

we have

$$\lim_{x \to \infty} \left(\frac{\psi(x)}{x} - \frac{\vartheta(x)}{x} \right) = 0.$$

Conclusion 4.235 If one of $\lim_{x \to \infty} \frac{\psi(x)}{x}$ or $\lim_{x \to \infty} \frac{\vartheta(x)}{x}$ exists, then the other also exists and they have the same value.

Let $a : \mathbb{N} \to \mathbb{C}$ be any arithmetic function. Let

$$A : x \mapsto \begin{cases} \sum\limits_{n \le x} a(n) & \text{if } 1 \le x \\ 0 & \text{if } x < 1 \end{cases}$$

be any function from $(0, \infty)$ to $\mathbb{C}$.

Problem 4.236 For every $x > 1$, $\sum_{n \le x}(x-n)a(n) = \int_1^x A(t)\mathrm{d}t$.

(Solution Let us fix any $x > 1$. We have to show that

$$xA(x) - \sum_{n \le x} n \cdot a(n) = x\sum_{n \le x} a(n) - \sum_{n \le x} n \cdot a(n)$$
$$= \underbrace{\sum_{n \le x}(x-n)a(n) = \int_1^x A(t)\mathrm{d}t,}$$

that is

$$\sum_{n \le x} n \cdot a(n) = xA(x) - \int_1^x A(t)\mathrm{d}t.$$

There exists $\alpha \in (0,1)$ such that $\alpha + 1 < x$. Now, by Conclusion 4.188, we have

$$\sum_{1 \le n \le x} a(n) \cdot n = \sum_{\alpha < n \le x} a(n) \cdot n = -\int_\alpha^x A(t) \cdot 1\mathrm{d}t + A(x) \cdot x - A(\alpha) \cdot \alpha$$
$$= -\int_\alpha^x A(t)\mathrm{d}t + A(x) \cdot x - 0 \cdot \alpha = -\int_\alpha^x A(t)\mathrm{d}t + A(x) \cdot x$$
$$= -\left(\int_\alpha^1 A(t)\mathrm{d}t + \int_1^x A(t)\mathrm{d}t\right) + A(x) \cdot x$$
$$= -\left(\int_\alpha^1 0\mathrm{d}t + \int_1^x A(t)\mathrm{d}t\right) + A(x) \cdot x = -\int_1^x A(t)\mathrm{d}t + A(x) \cdot x.$$

Thus,

$$\sum_{n \le x} n \cdot a(n) = xA(x) - \int_1^x A(t)\mathrm{d}t.$$

∎)

Conclusion 4.237 Let $a : \mathbb{N} \to \mathbb{C}$ be any arithmetic function. Let $A : x \mapsto \begin{cases} \sum_{n \le x} a(n) & \text{if } 1 \le x \\ 0 & \text{if } x < 1 \end{cases}$ be any function from $(0, \infty)$ to $\mathbb{C}$. Then, for every $x > 1$, $\sum_{n \le x}(x-n)a(n) = \int_1^x A(t)\mathrm{d}t$.

Let $a:\mathbb{N}\to[0,\infty)$ be any arithmetic function. Let

$$A : x \mapsto \begin{cases} \sum\limits_{n\le x} a(n) & \text{if } 1\le x \\ 0 & \text{if } x<1 \end{cases}$$

be any function from $(0,\infty)$ to $[0,\infty)$. Let $c>0$, and $L>0$. Suppose that

$$\int_1^x A(t)\mathrm{d}t \sim Lx^c \quad \text{as } x\to\infty.$$

Problem 4.238

1. For every $x>1$, and for every $\beta>1$,

$$\frac{A(x)}{x^{c-1}}(\beta-1) \le \left(\frac{\int_1^{\beta x} A(t)\mathrm{d}t}{(\beta x)^c}\beta^c - \frac{\int_1^x A(t)\mathrm{d}t}{x^c}\right),$$

2. $\left(\limsup_{x\to\infty}\frac{A(x)}{x^{c-1}}\right)\le Lc.$

(Solution

1: Let us fix any $x>1$, and, $\beta>1$. We have to show that

$$A(x)\cdot(\beta-1)x \le \int_1^{\beta x} A(t)\mathrm{d}t - \int_1^x A(t)\mathrm{d}t.$$

Since

$$\int_1^{\beta x} A(t)\mathrm{d}t - \int_1^x A(t)\mathrm{d}t = \left(\int_1^x A(t)\mathrm{d}t + \int_x^{\beta x} A(t)\mathrm{d}t\right) - \int_1^x A(t)\mathrm{d}t$$
$$= \int_x^{\beta x} A(t)\mathrm{d}t \ge \int_x^{\beta x} A(x)\mathrm{d}t = A(x)(\beta x - x) = A(x)\cdot(\beta-1)x,$$

we have

$$A(x)\cdot(\beta-1)x \le \int_1^{\beta x} A(t)\mathrm{d}t - \int_1^x A(t)\mathrm{d}t.$$

2: It follows that for every $\beta > 1$,

$$\begin{aligned}\limsup_{x\to\infty}\frac{A(x)}{x^{c-1}} &\le \limsup_{x\to\infty}\frac{1}{\beta-1}\left(\frac{\int_1^{\beta x}A(t)\mathrm{d}t}{(\beta x)^c}\beta^c - \frac{\int_1^{x}A(t)\mathrm{d}t}{x^c}\right)\\ &= \frac{1}{\beta-1}\limsup_{x\to\infty}\left(\beta^c\frac{\int_1^{\beta x}A(t)\mathrm{d}t}{(\beta x)^c} - \frac{\int_1^{x}A(t)\mathrm{d}t}{x^c}\right)\\ &= \frac{1}{\beta-1}\lim_{x\to\infty}\left(\beta^c\frac{\int_1^{\beta x}A(t)\mathrm{d}t}{(\beta x)^c} - \frac{\int_1^{x}A(t)\mathrm{d}t}{x^c}\right)\\ &= \frac{1}{\beta-1}(\beta^c L - L) = L\frac{\beta^c-1}{\beta-1},\end{aligned}$$

and hence for every $\beta > 1$,

$$\limsup_{x\to\infty}\frac{A(x)}{x^{c-1}} \le L\frac{\beta^c-1}{\beta-1}.$$

Now,

$$\underbrace{\left(\limsup_{x\to\infty}\frac{A(x)}{x^{c-1}}\right) \le \lim_{\beta\to 1^+} L\frac{\beta^c-1}{\beta-1}} = L\lim_{\beta\to 1^+}\frac{\beta^c-1^c}{\beta-1} = L\frac{d(x^c)}{\mathrm{d}x}\bigg|_{x=1} = Lc.$$

Thus,

$$\left(\limsup_{x\to\infty}\frac{A(x)}{x^{c-1}}\right) \le Lc.$$

∎)

Problem 4.239 For every $x > 1$, and for every $\alpha \in (0,1)$,

$$\left(\frac{\int_1^{\alpha x}A(t)\mathrm{d}t}{(\alpha x)^c}\alpha^c - \frac{\int_1^{x}A(t)\mathrm{d}t}{x^c}\right) \le \frac{A(x)}{x^{c-1}}(\alpha-1).$$

(Solution Let us fix any $x > 1$, and $\alpha \in (0,1)$. We have to show that

$$\int_1^{\alpha x}A(t)\mathrm{d}t - \int_1^{x}A(t)\mathrm{d}t \le A(x)\cdot(\alpha-1)x.$$

Since

$$\int_1^{\alpha x} A(t)\mathrm{d}t - \int_1^{x} A(t)\mathrm{d}t = \left(\int_1^{x} A(t)\mathrm{d}t + \int_x^{\alpha x} A(t)\mathrm{d}t\right) - \int_1^{x} A(t)\mathrm{d}t$$
$$= \int_x^{\alpha x} A(t)\mathrm{d}t \le \int_x^{\alpha x} A(x)\mathrm{d}t$$
$$= A(x)(\alpha x - x) = A(x)\cdot(\alpha - 1)x,$$

we have

$$\int_1^{\alpha x} A(t)\mathrm{d}t - \int_1^{x} A(t)\mathrm{d}t \le A(x)\cdot(\alpha - 1)x.$$

∎)

It follows that for every $\alpha \in (0,1)$,

$$(\alpha - 1)\liminf_{x\to\infty}\frac{A(x)}{x^{c-1}} = \liminf_{x\to\infty}\left(\frac{A(x)}{x^{c-1}}(\alpha - 1)\right) = \liminf_{x\to\infty}\frac{A(x)}{x^{c-1}}(\alpha - 1)$$
$$\le \liminf_{x\to\infty}\left(\frac{\int_1^{\alpha x} A(t)\mathrm{d}t}{(\alpha x)^c}\alpha^c - \frac{\int_1^{x} A(t)\mathrm{d}t}{x^c}\right)$$
$$= \lim_{x\to\infty}\left(\alpha^c\frac{\int_1^{\alpha x} A(t)\mathrm{d}t}{(\alpha x)^c} - \frac{\int_1^{x} A(t)\mathrm{d}t}{x^c}\right) = (\alpha^c L - L) = L(\alpha^c - 1),$$

and hence for every $\alpha \in (0,1)$,

$$L\frac{\alpha^c - 1}{\alpha - 1} \le \liminf_{x\to\infty}\frac{A(x)}{x^{c-1}}.$$

Now,

$$(Lc =)\lim_{\alpha\to 1^-} L\frac{\alpha^c - 1}{\alpha - 1} \le \liminf_{x\to\infty}\frac{A(x)}{x^{c-1}}.$$

Thus,

$$\underbrace{Lc \le \left(\liminf_{x\to\infty}\frac{A(x)}{x^{c-1}}\right)} \le \limsup_{x\to\infty}\frac{A(x)}{x^{c-1}} \le Lc,$$

and hence

$$\liminf_{x\to\infty}\frac{A(x)}{x^{c-1}} = \limsup_{x\to\infty}\frac{A(x)}{x^{c-1}} = Lc.$$

This shows that $\lim_{x\to\infty}\frac{A(x)}{x^{c-1}} = Lc$. In other words, $A(x) \sim L\cdot cx^{c-1}$ as $x\to\infty$.

Conclusion 4.240 Let $a:\mathbb{N}\to[0,\infty)$ be any arithmetic function. Let

$$A : x \mapsto \begin{cases} \sum\limits_{n\le x} a(n) & \text{if } 1\le x \\ 0 & \text{if } x<1 \end{cases}$$

be any function from $(0,\infty)$ to $[0,\infty)$. Let $c>0$, and $L>0$. Suppose that

$$\int_1^x A(t)\mathrm{d}t \sim Lx^c \quad \text{as } x\to\infty.$$

Then, $A(x)\sim L\cdot cx^{c-1}$ as $x\to\infty$.

Let $c>0$, and $u\in(0,1]$. Let k be a positive integer. Let R be a real number satisfying $(2k<)2k+c<R$. Let $C(R)$ be a closed curve, which consists of two parts, $C_1(R), C_2(R)$. Here, $C_2(R)$ is a positively oriented circular arc with center at the origin O and radius R, and $C_1(R)$ is a line segment of line the $\mathrm{Re}(z)=c$ directed along the positive direction of the imaginary axis.

Problem 4.241 $\lim_{R\to\infty}\left(\int_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)=0.$

(Solution Observe that for every $x+iy\equiv z\in C_2(R)$,

$$\begin{aligned}\left|\frac{u^{-z}}{z(z+1)\cdots(z+k)}\right| &= \frac{|u^{-z}|}{|z(z+1)\cdots(z+k)|} = \frac{u^{-x}}{|z(z+1)\cdots(z+k)|}\\
&\le \frac{1}{|z(z+1)\cdots(z+k)|}\left(\frac{1}{u}\right)^x \le \frac{1}{|z(z+1)\cdots(z+k)|}\left(\frac{1}{u}\right)^c\\
&= \frac{u^{-c}}{|z||z+1|\cdots|z+k|} = u^{-c}\frac{1}{|z|}\frac{1}{|z+1|}\cdots\frac{1}{|z+k|}\\
&= u^{-c}\frac{1}{R}\frac{1}{|z-1)}\frac{1}{|z-2)}\cdots\frac{1}{|z-k)} \le u^{-c}\frac{1}{R}\frac{1}{||z|-1|}\frac{1}{|z-2)}\cdots\frac{1}{|z-k)}\\
&\le u^{-c}\frac{1}{R}\frac{1}{||z|-1|}\frac{1}{||z|-2|}\cdots\frac{1}{||z|-k|} = u^{-c}\frac{1}{R}\frac{1}{|R-1|}\frac{1}{|R-2|}\cdots\frac{1}{|R-k|}\\
&= u^{-c}\frac{1}{R}\frac{1}{R-1}\frac{1}{R-2}\cdots\frac{1}{R-k} \le u^{-c}\frac{1}{R}\underbrace{\frac{1}{R-k}\frac{1}{R-k}\cdots\frac{1}{R-k}}_{k\text{ factors}}\\
&= u^{-c}\frac{1}{R}\frac{1}{(R-k)^k} \le u^{-c}\frac{1}{R}\frac{1}{\left(R-\frac{R}{2}\right)^k} = u^{-c}\frac{1}{R}\left(\frac{2}{R}\right)^k,\end{aligned}$$

so for every $x+iy \equiv z \in C_2(R)$,

$$\left|\frac{u^{-z}}{z(z+1)\cdots(z+k)}\right| \le u^{-c}\frac{1}{R}\left(\frac{2}{R}\right)^k,$$

and hence

$$\left|\int\limits_{C_2(R)} \frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right| \le \int\limits_{C_2(R)} \left|\frac{u^{-z}}{z(z+1)\cdots(z+k)}\right|\mathrm{d}z$$
$$\le \left(u^{-c}\frac{1}{R}\left(\frac{2}{R}\right)^k\right)(2\pi R) = \left(u^{-c}2\pi 2^k\right)\frac{1}{R^k} \to 0 \text{ as } R \to \infty.$$

Thus,

$$\lim_{R\to\infty}\left(\int\limits_{C_2(R)} \frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right) = 0.$$

∎)

Here, the poles of the function

$$z \mapsto \frac{u^{-z}}{z(z+1)\cdots(z+k)}$$

which lie inside $C(R)$, are $z = 0, -1, -2, \ldots, -k$. By Problem 4.146, for every $z \in C(R)$,

$$\underbrace{\frac{u^{-z}}{z(z+1)\cdots(z+k)} = \frac{u^{-z}}{\frac{\Gamma(z+1)}{\Gamma(z)}\frac{\Gamma(z+2)}{\Gamma(z+1)}\cdots\frac{\Gamma(z+k+1)}{\Gamma(z+k)}}} = \frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)},$$

so, by Conclusion 1.225,

$$\frac{1}{2\pi i}\int\limits_{C(R)} \frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z = \mathrm{Res}\left(z \mapsto \frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)}; 0\right)$$
$$+\mathrm{Res}\left(z \mapsto \frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)}; -1\right) + \cdots + \mathrm{Res}\left(z \mapsto \frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)}; -k\right).$$

Now, since

$$\begin{aligned}
&\lim_{R\to\infty}\left(\int\limits_{C(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)\\
&=\lim_{R\to\infty}\left(\int\limits_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z+\int\limits_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)\\
&=\lim_{R\to\infty}\left(\int\limits_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)+\lim_{R\to\infty}\left(\int\limits_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)\\
&=\lim_{R\to\infty}\left(\int\limits_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)+0=\lim_{R\to\infty}\left(\int\limits_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Res}\left(z\mapsto\frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)};0\right)+\mathrm{Res}\left(z\mapsto\frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)};-1\right)+\cdots\\
&\quad+\mathrm{Res}\left(z\mapsto\frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)};-k\right)=\lim_{z\to 0}(z-0)\frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)}\\
&\quad+\lim_{z\to -1}(z-1)\frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)}+\cdots+\lim_{z\to -k}(z-k)\frac{\Gamma(z)u^{-z}}{\Gamma(z+k+1)}\\
&=\lim_{z\to 0}z\Gamma(z)\cdot\lim_{z\to 0}\frac{u^{-z}}{\Gamma(z+k+1)}+\lim_{z\to -1}(z-1)\Gamma(z)\cdot\lim_{z\to -1}\frac{u^{-z}}{\Gamma(z+k+1)}+\cdots\\
&\quad+\lim_{z\to -k}(z-k)\Gamma(z)\cdot\lim_{z\to -k}\frac{u^{-z}}{\Gamma(z+k+1)}=\lim_{z\to 0}z\Gamma(z)\cdot\frac{u^{-0}}{\Gamma(0+k+1)}\\
&\quad+\lim_{z\to -1}(z-1)\Gamma(z)\cdot\frac{u^{1}}{\Gamma(-1+k+1)}+\cdots+\lim_{z\to -k}(z-k)\Gamma(z)\cdot\frac{u^{k}}{\Gamma(-k+k+1)}\\
&=\mathrm{Res}(\Gamma;0)\frac{1}{\Gamma(k+1)}+\mathrm{Res}(\Gamma;-1)\frac{u}{\Gamma(k)}+\mathrm{Res}(\Gamma;-2)\frac{u^2}{\Gamma(k-1)}+\cdots\\
&\quad+\mathrm{Res}(\Gamma;-k)\frac{u^k}{\Gamma(1)}=\mathrm{Res}(\Gamma;0)\frac{1}{k!}+\mathrm{Res}(\Gamma;-1)\frac{u}{(k-1)!}\\
&\quad+\mathrm{Res}(\Gamma;-2)\frac{u^2}{(k-2)!}+\cdots+\mathrm{Res}(\Gamma;-k)\frac{u^k}{0!},
\end{aligned}$$

we have

$$\left(\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)=\lim_{R\to\infty}\left(\frac{1}{2\pi i}\int\limits_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right)$$
$$=\lim_{R\to\infty}\left(\mathrm{Res}(\Gamma;0)\frac{1}{k!}+\mathrm{Res}(\Gamma;-1)\frac{u}{(k-1)!}\right.$$
$$\left.+\mathrm{Res}(\Gamma;-2)\frac{u^2}{(k-2)!}+\cdots+\mathrm{Res}(\Gamma;-k)\frac{u^k}{0!}\right).$$

Now, by Problems 4.140 and 4.144,

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z$$
$$=\lim_{R\to\infty}\left((-1)^0\frac{1}{0!}\frac{1}{k!}+(-1)^1\frac{1}{1!}\frac{u}{(k-1)!}+(-1)^2\frac{1}{2!}\frac{u^2}{(k-2)!}+\cdots+(-1)^k\frac{1}{k!}\frac{u^k}{0!}\right)$$
$$=(-1)^0\frac{1}{0!}\frac{1}{k!}+(-1)^1\frac{1}{1!}\frac{u}{(k-1)!}+(-1)^2\frac{1}{2!}\frac{u^2}{(k-2)!}+\cdots+(-1)^k\frac{1}{k!}\frac{u^k}{0!}$$
$$=\frac{1}{k!}+\frac{1}{1!(k-1)!}(-u)^1+\frac{1}{2!(k-2)!}(-u)^2+\cdots+\frac{1}{k!0!}(-u)^k$$
$$=\frac{1}{k!}\left(1+\frac{k!}{1!(k-1)!}(-u)^1+\frac{k!}{2!(k-2)!}(-u)^2+\cdots+\frac{k!}{k!0!}(-u)^k\right)$$
$$=\frac{1}{k!}\left(1+\binom{k}{1}(-u)^1+\binom{k}{2}(-u)^2+\cdots+\binom{k}{k}(-u)^k\right)$$
$$=\frac{1}{k!}(1+(-u))^k=\frac{1}{k!}(1-u)^k,$$

and hence

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z=\frac{1}{k!}(1-u)^k.$$

Conclusion 4.242 Let $c>0$, and $u>0$. Let k be a positive integer. Then

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z=\begin{cases}\frac{1}{k!}(1-u)^k & \text{if } u\in(0,1]\\ 0 & \text{if } u\in(1,\infty).\end{cases}$$

Proof of the remaining part It remains to show that for every $u > 1$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z = 0.$$

Let R be a real number satisfying $(2k<)2k+c<R$. Let $C(R)$ be a closed curve, which consists of two parts, $C_1(R), C_2(R)$. Here, $C_2(R)$ is a positively oriented circular arc with center at the origin O and radius R, and $C_1(R)$ is a line segment of line the $\mathrm{Re}(z) = c$ directed along the negative direction of the imaginary axis.

Here, the poles of the function

$$z \mapsto \frac{u^{-z}}{z(z+1)\cdots(z+k)}$$

are $z = 0, -1, -2, \ldots, -k$, all of which lie outside $C(R)$, so the function

$$z \mapsto \frac{u^{-z}}{z(z+1)\cdots(z+k)}$$

is holomorphic in an open set, which contains $C(R)$, and hence, by Theorem 1.110,

$$\left(\int_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z + \int_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z =\right)\int_{C(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z = 0.$$

Now,

$$\begin{aligned}
&-\int_{c-i\infty}^{c+i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z + \lim_{R\to\infty}\left(\int_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z\right)\\
&= \int_{c+i\infty}^{c-i\infty}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z + \lim_{R\to\infty}\left(\int_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z\right)\\
&= \lim_{R\to\infty}\left(\int_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z\right) + \lim_{R\to\infty}\left(\int_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z\right)\\
&= \lim_{R\to\infty}\left(\int_{C_1(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z + \int_{C_2(R)}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,\mathrm{d}z\right) = 0.
\end{aligned}$$

It suffices to show that

$$\lim_{R\to\infty}\left(\int\limits_{C_2(R)} \frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}z\right) = 0.$$

For every $z = (x+iy) \in C_2(R)$, we have

$$\begin{aligned}\left|\frac{u^{-z}}{z(z+1)\cdots(z+k)}\right| &= \frac{|u^{-z}|}{|z||z-1)}\cdots|z-k) = \frac{u^{-x}}{|z||z-1)}\cdots|z-k) \\ &\leq \frac{u^{-x}}{|z|||z|-|-1||\cdots||z|-|-k||} = \frac{u^{-x}}{R|R-1|\cdots|R-k|} \\ &= \frac{u^{-x}}{R(R-1)\cdots(R-k)},\end{aligned}$$

so

$$\begin{aligned}\left|\int\limits_{C_2(R)} \frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}s\right| &\leq \frac{u^{-x}}{R(R-1)\cdots(R-k)}\cdot 2\pi R = 2\pi\frac{u^{-x}}{(R-1)\cdots(R-k)} \\ &\leq 2\pi\frac{u^{-x}}{\underbrace{(R-k)\cdots(R-k)}_{k\text{ factors}}} = 2\pi\frac{1}{(R-k)^k}\frac{1}{u^x} \leq 2\pi\frac{1}{(R-k)^k}\frac{1}{u^c} \\ &= \frac{2\pi}{u^c}\frac{1}{(R-k)^k} \to 0 \text{ as } R\to\infty.\end{aligned}$$

Thus,

$$\lim_{R\to\infty}\left(\int\limits_{C_2(R)} \frac{u^{-z}}{z(z+1)\cdots(z+k)}\mathrm{d}s\right) = 0.$$

∎)

Problem 4.243 For every $x \geq 1$,

1. $\int_1^x \psi(t)\mathrm{d}t = \sum_{n\leq x}(x-n)\Lambda(n)$,
2. $\frac{1}{x}\int_1^x \psi(t)\mathrm{d}t = \sum_{n\leq x}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\left(\frac{x}{n}\right)^s}{s(s+1)}\mathrm{d}s\right)\Lambda(n)$.

(Solution

1: Let us fix any $x \geq 1$. We have to show that

$$\int_1^x \psi(t)\mathrm{d}t = \sum_{n \leq x}(x-n)\Lambda(n).$$

By Conclusion 4.237,

$$\sum_{n \leq x}(x-n)\Lambda(n) = \int_1^x \left(\sum_{n \leq t}\Lambda(n)\right)\mathrm{d}t \left(= \int_1^x \psi(t)\mathrm{d}t\right),$$

so

$$\int_1^x \psi(t)\mathrm{d}t = \sum_{n \leq x}(x-n)\Lambda(n).$$

2: Let $c > 1$, and $x \geq 1$ $\left(\text{and hence, } \frac{n}{x} \in (0,1] \text{for every positive integer} \, n \leq x\right)$. By Problem 4.243,

$$\frac{1}{x}\int_1^x \psi(t)\mathrm{d}t = \sum_{n \leq x}\left(1 - \frac{n}{x}\right)\Lambda(n).$$

By Conclusion 4.242, for every positive integer $n \leq x$,

$$\underbrace{\frac{1}{1!}\left(1-\frac{n}{x}\right)^1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{n}{x}\right)^{-z}}{z(z+1)}\mathrm{d}z} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{n}\right)^{s}}{s(s+1)}\mathrm{d}s.$$

It follows that

$$\frac{1}{x}\int_1^x \psi(t)\mathrm{d}t = \sum_{n \leq x}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{n}\right)^{s}}{s(s+1)}\mathrm{d}s\right)\Lambda(n).$$

∎)

Problem 4.244 Let $c > 1$, and $x \geq 1$. For every positive integer $n > x$, $\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{n}\right)^{s}}{s(s+1)}\mathrm{d}s = 0$.

(Solution Let us fix any positive integer $N > x$. Let us denote $s \equiv \sigma + it$. We have to show that

$$\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s = 0.$$

Let R be a real number satisfying $c<R$. Let $C(R)$ be a closed curve, which consists of two parts, $C_1(R), C_2(R)$. Here, $C_2(R)$ is a positively oriented circular arc with center at the origin O and radius R, and $C_1(R)$ is a line segment of line the $\mathrm{Re}(s) = c$ directed along the negative direction of the imaginary axis.

Here, the poles of the function $s \mapsto \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}$ are $s = 0, -1, -2$, all of which lie outside $C(R)$, so the function $s \mapsto \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}$ is holomorphic in an open set, which contains $C(R)$, and hence, by Theorem 1.110,

$$\int_{C_1(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s + \int_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s = \underbrace{\int_{C(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s = 0}$$

It follows that

$$\begin{aligned}
&-\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s + \lim_{R\to\infty}\left(\int_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s\right)\\
&= \int_{c+i\infty}^{c-i\infty} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s + \lim_{R\to\infty}\left(\int_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s\right)\\
&= \lim_{R\to\infty}\left(\int_{C_1(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s\right) + \lim_{R\to\infty}\left(\int_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s\right)\\
&= \lim_{R\to\infty}\left(\int_{C_1(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s + \int_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s\right) = 0.
\end{aligned}$$

It suffices to show that

$$\lim_{R\to\infty}\left(\int_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\,\mathrm{d}s\right) = 0.$$

For every $s = (\sigma + it) \in C_2(R)$,

$$\left|\frac{\left(\frac{x}{N}\right)^s}{s(s+1)}\right| = \frac{\left|\left(\frac{x}{N}\right)^s\right|}{|s||s-(-1)|} = \frac{\left(\frac{x}{N}\right)^\sigma}{|s||s-(-1)|} \leq \frac{\left(\frac{x}{N}\right)^\sigma}{|s|||s|-|-1||} = \frac{\left(\frac{x}{N}\right)^\sigma}{R|R-1|}$$
$$= \frac{\left(\frac{x}{N}\right)^\sigma}{R(R-1)},$$

so

$$\left|\int\limits_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)} \mathrm{d}s\right| \leq \frac{\left(\frac{x}{N}\right)^\sigma}{R(R-1)} \cdot 2\pi R = 2\pi \frac{1}{R-1} \cdot \left(\frac{x}{N}\right)^\sigma$$
$$\leq 2\pi \frac{1}{R-1} \cdot \left(\frac{x}{N}\right)^c = \left(2\pi\left(\frac{x}{N}\right)^c\right)\frac{1}{R-1} \to 0 \text{ as } R \to \infty.$$

Thus,

$$\lim_{R\to\infty}\left(\int\limits_{C_2(R)} \frac{\left(\frac{x}{N}\right)^s}{s(s+1)} \mathrm{d}s\right) = 0.$$

∎)

Problem 4.245 For every real $s \in (1, \infty)$, $\ln(\zeta(s)) = \sum_{n=2}^{\infty}\left(\frac{\Lambda(n)}{\ln n}(n^{-s})\right)$.

(Solution By Problem 4.183, for every real $s \in (1, \infty)$,

$$\ln(\zeta(s)) = -\sum_p \left(\ln\left(1 - \frac{1}{p^s}\right)\right) = -\sum_p \left(\sum_{m=1}^{\infty} \frac{\left(-\frac{1}{p^s}\right)^m}{m}\right) = \sum_p \left(\sum_{m=1}^{\infty} \frac{\frac{1}{p^{ms}}}{m}\right)$$
$$= \sum_p \left(\sum_{m=1}^{\infty}\left((p^m)^{-s} \cdot \frac{1}{m}\right)\right) = \sum_p \left(\sum_{m=1}^{\infty}((p^m)^{-s} \cdot \Lambda_1(p^m))\right),$$

where

$$\Lambda_1 : n \mapsto \begin{cases} \frac{1}{m} & \text{if } n = p^m \text{ for some prime } p \text{ and for some positive integer } m \\ 0 & \text{otherwise.} \end{cases}$$

Now, by the definition of Λ_1, for every real $s \in (1, \infty)$,

$$\ln(\zeta(s)) = \sum_{n=1}^{\infty}(n^{-s} \cdot \Lambda_1(n)).$$

If $n = p^m$ for some prime p, and for some positive integer m, then

$$\Lambda_1(n) = \frac{1}{m} = \frac{\ln p}{m(\ln p)} = \frac{\ln p}{\ln(p^m)} = \frac{\ln p}{\ln n} = \frac{\Lambda(p^m)}{\ln n} = \frac{\Lambda(n)}{\ln n}.$$

Thus, for every positive integer $n \geq 2$, $\Lambda_1(n) = \frac{\Lambda(n)}{\ln n}$. It follows that for every real $s \in (1, \infty)$,

$$\ln(\zeta(s)) = \sum_{n=2}^{\infty} \left(\frac{\Lambda(n)}{\ln n} (n^{-s}) \right).$$

■)

Problem 4.246 For every real $s = \sigma + it$ satisfying $1 < \sigma$,

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

(Solution By Problem 4.245, for every real $s \in (1, \infty)$,

$$\zeta(s) = \exp\left(\sum_{n=2}^{\infty} \left(\frac{\Lambda(n)}{\ln n} (n^{-s}) \right) \right).$$

Now since $s \mapsto \zeta(s)$, and

$$s \mapsto \exp\left(\sum_{n=2}^{\infty} \left(\frac{\Lambda(n)}{\ln n} (n^{-s}) \right) \right)$$

are holomorphic on $\{s = \sigma + it : 1 < \sigma\}$, we have

$$\zeta(s) = \exp\left(\sum_{n=2}^{\infty} \left(\frac{\Lambda(n)}{\ln n} (n^{-s}) \right) \right)$$

on $\{s = \sigma + it : 1 < \sigma\}$. On differentiation, we get

$$\zeta'(s) = \zeta(s) \left(\sum_{n=2}^{\infty} \left(\frac{\Lambda(n)}{\ln n} (n^{-s}(\ln n)(-1)) \right) \right) = -\zeta(s) \left(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \right),$$

and hence, for every real $s = \sigma + it$ satisfying $1 < \sigma$, $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$. ■)

Let $c > 1$, and $x \geq 1$.

By Problems 4.243 and 4.244, for every $x \geq 1$, we have

$$\frac{1}{x}\int_{1}^{x}\psi(t)\mathrm{d}t = \sum_{n=1}^{\infty}\left(\int_{c-i\infty}^{c+i\infty}\frac{1}{2\pi i}\frac{\Lambda(n)\left(\frac{n}{x}\right)^{-s}}{s(s+1)}\mathrm{d}s\right)$$

So, for every $x \geq 1$, we have

$$\frac{1}{x}\int_{1}^{x}\psi(t)\mathrm{d}t = \sum_{n=1}^{\infty}\left(\int_{c-i\infty}^{c+i\infty}f_n(s)\mathrm{d}s\right),$$

where, for every positive integer n,

$$f_n : s \mapsto \frac{1}{2\pi i}\frac{\Lambda(n)\left(\frac{n}{x}\right)^{-s}}{s(s+1)}.$$

Next, we wish to interchange the summation and integral in

$$\sum_{n=1}^{\infty}\left(\int_{c-i\infty}^{c+i\infty}f_n(s)\mathrm{d}s\right).$$

In view of Lemma 1.150, Vol. 1, it suffices to show that

$$\sum_{n=1}^{\infty}\left(\int_{c-i\infty}^{c+i\infty}|f_n(s)|\mathrm{d}s\right)$$

is convergent, that is

$$\sum_{n=1}^{\infty}\left(\Lambda(n)\int_{-\infty}^{\infty}\frac{\left(\frac{x}{n}\right)^{c}}{|c+it||c+it+1|}\mathrm{d}t\right)$$

is convergent, that is

$$\sum_{n=1}^{\infty}\left(\frac{\Lambda(n)}{n^c}\int_{-\infty}^{\infty}\frac{x^c}{|c+it||c+it+1|}\mathrm{d}t\right)$$

is convergent, that is $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^c}$ is convergent, which is true because of Problem 4.246. Thus, for every $x \geq 1$, we have

$$\frac{1}{x}\int_1^x \psi(t)\mathrm{d}t = \int_{c-i\infty}^{c+i\infty}\left(\sum_{n=1}^{\infty}\frac{1}{2\pi i}\frac{\Lambda(n)\left(\frac{n}{x}\right)^{-s}}{s(s+1)}\right)\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s(s+1)}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\mathrm{d}s,$$

by Problem 4.246.

Conclusion 4.247 Let $c > 1$, and $x \geq 1$. Then $\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\mathrm{d}s$.

Let $c > 1$, and $x \geq 1$. By Conclusion 4.242, for every $c > 0$,

$$\frac{1}{2!}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\left(\frac{1}{x}\right)^{-z}}{z(z+1)(z+2)}\mathrm{d}z$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s(s+1)(s+2)}\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{(c+1)-i\infty}^{(c+1)+i\infty}\frac{x^{(s-1)}}{(s-1)((s-1)+1)((s-1)+2)}\mathrm{d}(s-1)$$
$$= \frac{1}{2\pi i}\int_{(c+1)-i\infty}^{(c+1)+i\infty}\frac{x^{(s-1)}}{(s-1)s(s+1)}\mathrm{d}s,$$

so for every $c > 1$,

$$\frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{(s-1)s(s+1)}\mathrm{d}s.$$

Now, by Conclusion 4.247,

$$\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) - \frac{x^{s-1}}{(s-1)s(s+1)}\right)\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(x^{s-1}\cdot\frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)\right)\mathrm{d}s.$$

Conclusion 4.248 Let $c > 1$, and $x \geq 1$. Then

$$\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1}h(s)\mathrm{d}s$$

where $h : s \mapsto \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$.

4.15 Prime Number Theorem

Note 4.249 Let N be a positive integer. Let $s \equiv \sigma + it$ be any complex number satisfying $0 < \sigma$.

Problem 4.250 $\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x$.

(Solution Case I: when $\sigma > 1$. By Problem 4.163, for every integer $x > N$, we have

$$\sum_{N<n\leq x}\frac{1}{n^s} = \int_N^x \frac{1}{t^s}\mathrm{d}t + \int_N^x (t-[t])\frac{\mathrm{d}\left(\frac{1}{t^s}\right)}{\mathrm{d}t}\mathrm{d}t + ([x]-x)\frac{1}{x^s} - ([N]-N)\frac{1}{N^s}$$
$$= \int_N^x \frac{1}{t^s}\mathrm{d}t + \int_N^x (t-[t])\left(-st^{-s-1}\right)\mathrm{d}t + ([x]-x)\frac{1}{x^s} - ([N]-N)\frac{1}{N^s}$$
$$= \int_N^x \frac{1}{t^s}\mathrm{d}t - s\int_N^x (t-[t])t^{-s-1}\mathrm{d}t + (x-x)\frac{1}{x^s} - (N-N)\frac{1}{N^s}$$
$$= \int_N^x \frac{1}{t^s}\mathrm{d}t - s\int_N^x \frac{t-[t]}{t^{s+1}}\mathrm{d}t,$$

and hence for every integer $x > N$,

$$\sum_{N<n\le x} \frac{1}{n^s} = \int_N^x \frac{1}{t^s}\,\mathrm{d}t - s\int_N^x \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t \quad (*).$$

On letting $x \to \infty$, we get

$$\begin{aligned}
\zeta(s) - \sum_{n=1}^{N}\frac{1}{n^s} = \sum_{n=N+1}^{\infty}\frac{1}{n^s} &= \int_N^\infty \frac{1}{t^s}\,\mathrm{d}t - s\int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t \\
&= \frac{1}{-s+1}\frac{1}{t^{s-1}}\bigg|_{t=N}^{t=\infty} - s\int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t \\
&= \frac{1}{-s+1}\left(0 - \frac{1}{N^{s-1}}\right) - s\int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t \\
&= \frac{1}{s-1}\frac{1}{N^{s-1}} - s\int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t = \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t,
\end{aligned}$$

and hence

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t.$$

Case II: when $\sigma \in (0,1]$. Let us take any $\delta \in (0,\sigma]$. Since for every positive integer n,

$$\begin{aligned}
\left|\int_n^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t\right| &\le \int_n^\infty \left|\frac{t-[t]}{t^{s+1}}\right|\mathrm{d}t = \int_n^\infty \frac{|t-[t]|}{|t^{s+1}|}\,\mathrm{d}t = \int_n^\infty \frac{t-[t]}{t^{\sigma+1}}\,\mathrm{d}t \le \int_n^\infty \frac{1}{t^{\sigma+1}}\,\mathrm{d}t \\
&\le \int_n^\infty \frac{1}{t^{\delta+1}}\,\mathrm{d}t = \frac{1}{-\delta}\frac{1}{t^\delta}\bigg|_{t=n}^{t=\infty} = \frac{1}{-\delta}\left(0-\frac{1}{n^\delta}\right) = \frac{1}{\delta}\left(\frac{1}{n^\delta}\right) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

$s \mapsto \int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t$ converges uniformly for $\sigma \ge \delta$, and hence $s \mapsto \int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t$ is holomorphic in the half-plane $0<\sigma$. By Case I,

$$\int_N^\infty \frac{t-[t]}{t^{s+1}}\,\mathrm{d}t = \frac{1}{s}\left(\sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \zeta(s)\right)$$

for $1<\sigma$, and, since

$$s \mapsto \frac{1}{s}\left(\sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \zeta(s)\right)$$

is holomorphic in $\{s : s \neq 1 \text{ and } \sigma > 0\}$, we have

$$\int_N^\infty \frac{t-[t]}{t^{s+1}}\mathrm{d}t = \frac{1}{s}\left(\sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \zeta(s)\right)$$

on $\{s : s \neq 1 \text{ and } \sigma > 0\}$. ■)

Conclusion 4.251 Let N be a positive integer. Let $s \equiv \sigma + it$ be any complex number satisfying $0 < \sigma$. Then

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x.$$

Let A be a positive real number. Let

$$\Omega \equiv \left\{s \equiv \sigma + it : \sigma \in \left[\frac{1}{2}, \infty\right), t \in [e, \infty), \text{and } 1 - \sigma < \frac{A}{\ln t}\right\}.$$

Case I: when $s \equiv \sigma + it \in \Omega$, and $2 \leq \sigma$. Observe that

$$\begin{aligned}
|\zeta(s)| &= \left|1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right| \leq |1| + \left|\frac{1}{2^s}\right| + \left|\frac{1}{3^s}\right| + \cdots \\
&= 1 + \frac{1}{2^\sigma} + \frac{1}{3^\sigma} + \cdots \leq 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \\
&= \zeta(2) = \zeta(2)\cdot(\ln e) \leq \zeta(2)\cdot(\ln t),
\end{aligned}$$

so $|\zeta(s)| \leq M \cdot \ln t$, where $M \equiv \zeta(2)$.

Case II: when $s \equiv \sigma + it \in \Omega$, and $\left(\frac{1}{2} \leq\right)\sigma < 2(< e \leq t)$. By Conclusion 4.251, we have

$$\begin{aligned}
|\zeta(s)| &= \left|\sum_{n=1}^{[t]}\frac{1}{n^s} + \frac{[t]^{1-s}}{s-1} - s\int_{[t]}^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x\right| \\
&\leq \left|\sum_{n=1}^{[t]}\frac{1}{n^s}\right| + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{[t]} \left|\frac{1}{n^s}\right| + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&= \sum_{n=1}^{[t]} \frac{1}{n^\sigma} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&= \sum_{n=1}^{[t]} \frac{n^{1-\sigma}}{n} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&= \sum_{n=1}^{[t]} \frac{e^{(\ln n)(1-\sigma)}}{n} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&\le \sum_{n=1}^{[t]} \frac{e^{(\ln n)\frac{A}{\ln t}}}{n} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&\le \sum_{n=1}^{[t]} \frac{e^{(\ln n)\frac{A}{\ln[t]}}}{n} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&= \sum_{n=1}^{[t]} \frac{e^{A\cdot\frac{\ln n}{\ln[t]}}}{n} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&\le \sum_{n=1}^{[t]} \frac{e^{A\cdot 1}}{n} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \left|\frac{[t]^{1-s}}{s-1}\right| + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right| \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{[t]^{1-\sigma}}{|s-1|} + \left|s\int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}}\,\mathrm{d}x\right|
\end{aligned}
$$

$$
\begin{aligned}
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{[t]^{1-\sigma}}{\sqrt{(\sigma-1)^2+t^2}} + \left| s \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{[t]^{1-\sigma}}{t} + \left| s \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} \cdot \frac{[t]}{t} + \left| s \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} \cdot 1 + \left| s \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} + |\sigma+it| \left| \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} + (|\sigma|+|t|) \left| \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} + (\sigma+t) \left| \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} + (t+t) \left| \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right|
\end{aligned}
$$

$$
\begin{aligned}
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} + 2t \left| \int_{[t]}^{\infty} \frac{x-[x]}{x^{s+1}} \mathrm{d}x \right| \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} + 2t \int_{[t]}^{\infty} \left| \frac{x-[x]}{x^{s+1}} \right| \mathrm{d}x \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^{\sigma}} + 2t \int_{[t]}^{\infty} \frac{|x-[x]|}{x^{\sigma+1}} \mathrm{d}x
\end{aligned}
$$

$$\begin{aligned}
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^\sigma} + 2t \int_{[t]}^{\infty} \frac{1}{x^{\sigma+1}}\,\mathrm{d}x \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^\sigma} + 2t \frac{1}{-\sigma} \frac{1}{x^\sigma}\bigg|_{x=[t]}^{x=\infty} \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^\sigma} + 2t \frac{1}{-\sigma}\left(0 - \frac{1}{[t]^\sigma}\right) \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^\sigma} + 2t \cdot \frac{1}{\sigma} \cdot \frac{1}{[t]^\sigma} \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^\sigma} + 2t \cdot 2 \cdot \frac{1}{[t]^\sigma} \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{1}{[t]^\sigma} + 2([t]+1) \cdot 2 \cdot \frac{1}{[t]^\sigma} \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + (4[t]+5)\,\frac{[t]^{1-\sigma}}{[t]} \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \left(4 + 5\frac{1}{[t]}\right)[t]^{1-\sigma} \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \left(4 + 5 \cdot \frac{1}{[e]}\right)[t]^{1-\sigma} \\
&\le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \left(4 + 5 \cdot \frac{1}{2}\right)e^{(\ln[t])(1-\sigma)} \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{13}{2} e^{(\ln[t])(1-\sigma)} \le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{13}{2} e^{(\ln[t])\frac{A}{\ln t}} \\
&= e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{13}{2} e^{A\cdot\frac{\ln[t]}{\ln t}} \le e^A \sum_{n=1}^{[t]} \frac{1}{n} + \frac{13}{2} e^{A\cdot 1} \\
&= \frac{13}{2} e^A \left(\frac{2}{13}\sum_{n=1}^{[t]} \frac{1}{n} + 1\right) \le \frac{13}{2} e^A \left(\frac{2}{13}\sum_{n=1}^{[t]} \frac{1}{n} + \sum_{n=1}^{[e]} \frac{1}{n}\right) \\
&\le \frac{13}{2} e^A \left(\frac{2}{13}\sum_{n=1}^{[t]} \frac{1}{n} + \sum_{n=1}^{[t]} \frac{1}{n}\right) = \frac{15}{2} e^A \sum_{n=1}^{[t]} \frac{1}{n},
\end{aligned}$$

so

$$|\zeta(s)| \leq \frac{15}{2} e^A \cdot \sum_{n=1}^{[t]} \frac{1}{n}.$$

Now, by Conclusion 4.117,

$$0 \leq \left(\sum_{n=1}^{[t]} \frac{1}{n} - \ln[t] \right) - \gamma \leq \frac{1}{[t]} \left(\leq \frac{1}{[e]} = \frac{1}{2} < 1 \right),$$

so

$$\underbrace{\sum_{n=1}^{[t]} \frac{1}{n} \leq (1+\gamma) + \ln[t]} \leq (1+\gamma)(\ln e)$$
$$+ \ln t \leq (1+\gamma)(\ln t) + \ln t = (2+\gamma)(\ln t),$$

and hence

$$|\zeta(s)| \leq \frac{15}{2} e^A \cdot (2+\gamma)(\ln t) = M(\ln t),$$

where $M \equiv \frac{15}{2} e^A \cdot (2+\gamma)$.

Conclusion 4.252 Let A be a positive real number. Let

$$\Omega \equiv \left\{ s \equiv \sigma + it : \sigma \in \left[\frac{1}{2}, \infty \right), t \in [e, \infty), \text{and } 1 - \sigma < \frac{A}{\ln t} \right\}.$$

Then, there exists a real number M such that for every $s \equiv \sigma + it \in \Omega$, $|\zeta(s)| \leq M(\ln t)$.

Let N be a positive integer. Let $s \equiv \sigma + it$ be any complex number satisfying $0 < \sigma$. By Conclusion 4.251,

$$\begin{aligned}
\zeta'(s) &= \frac{\mathrm{d}}{\mathrm{d}s} \left(\sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_N^\infty \frac{x - [x]}{x^{s+1}} \mathrm{d}x \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \left(\sum_{n=1}^{N} \frac{1}{n^s} \right) + \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{N^{1-s}}{s-1} \right) - \frac{\mathrm{d}}{\mathrm{d}s} \left(s \int_N^\infty \frac{x - [x]}{x^{s+1}} \mathrm{d}x \right) \\
&= \sum_{n=1}^{N} \left(\frac{1}{n^s} \left(\ln \frac{1}{n} \right) \right) + \frac{(s-1)(N^{1-s}(\ln N)(-1)) - 1N^{1-s}}{(s-1)^2}
\end{aligned}$$

$$
\begin{aligned}
&-\left(1\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x + s\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x\right)\right)\\
&=\sum_{n=1}^{N}\left(\frac{1}{n^s}\left(\ln\frac{1}{n}\right)\right)-\frac{N^{1-s}(\ln N)}{s-1}-\frac{N^{1-s}}{(s-1)^2}\\
&-\left(\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x + s\int_N^\infty \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x-[x]}{x^{s+1}}\right)\mathrm{d}x\right)\\
&=-\sum_{n=1}^{N}\left(\frac{1}{n^s}(\ln n)\right)-\frac{N^{1-s}(\ln N)}{s-1}-\frac{N^{1-s}}{(s-1)^2}\\
&-\left(\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x + s\int_N^\infty (x-[x])\left(\frac{1}{x^{s+1}}\ln\left(\frac{1}{x}\right)\right)\mathrm{d}x\right)\\
&=-\sum_{n=1}^{N}\left(\frac{1}{n^s}(\ln n)\right)-\frac{N^{1-s}(\ln N)}{s-1}-\frac{N^{1-s}}{(s-1)^2}\\
&-\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x + s\int_N^\infty (x-[x])\left(\frac{1}{x^{s+1}}\ln x\right)\mathrm{d}x,
\end{aligned}
$$

so

$$
\begin{aligned}
\zeta'(s) = &-\sum_{n=1}^{N}\frac{\ln n}{n^s}-\frac{N^{1-s}(\ln N)}{s-1}-\frac{N^{1-s}}{(s-1)^2}\\
&-\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x + s\int_N^\infty \frac{(x-[x])\ln x}{x^{s+1}}\mathrm{d}x.
\end{aligned}
$$

Conclusion 4.253 Let N be a positive integer. Let $s\equiv\sigma+it$ be any complex number satisfying $0<\sigma$. Then

$$
\begin{aligned}
\zeta'(s) = &-\sum_{n=1}^{N}\frac{\ln n}{n^s}-\frac{N^{1-s}(\ln N)}{s-1}-\frac{N^{1-s}}{(s-1)^2}\\
&-\int_N^\infty \frac{x-[x]}{x^{s+1}}\mathrm{d}x + s\int_N^\infty \frac{(x-[x])\ln x}{x^{s+1}}\mathrm{d}x.
\end{aligned}
$$

Let A be a positive real number. Let $\Omega\equiv\left\{s\equiv\sigma+it : \sigma\in\left[\frac{1}{2},\infty\right),\right.$ $t\in[e,\infty)$, and $\left.1-\sigma<\frac{A}{\ln t}\right\}$.

Case I: when $s \equiv \sigma + it \in \Omega$, and $2 \le \sigma$. Observe that

$$\begin{aligned}|\zeta'(s)| &= \left|\frac{\mathrm{d}}{\mathrm{d}s}\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)\right| = \left|0+\frac{1}{2^s}(-\ln 2)+\frac{1}{3^s}(-\ln 3)+\cdots\right| \\ &\le \left|\frac{1}{2^s}\right|\ln 2+\left|\frac{1}{3^s}\right|\ln 3+\cdots = \frac{\ln 2}{2^\sigma}+\frac{\ln 3}{3^\sigma}+\cdots \le \frac{\ln 2}{2^2}+\frac{\ln 3}{3^2}+\cdots,\end{aligned}$$

and

$$\begin{aligned}|\zeta'(2)| &= \left|\left(\frac{\mathrm{d}}{\mathrm{d}s}\left(1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\right)\right)_{s=2}\right| \\ &= \left|\left(0+\frac{1}{2^s}(-\ln 2)+\frac{1}{3^s}(-\ln 3)+\cdots\right)_{s=2}\right| \\ &= \left|\left(\frac{1}{2^s}(\ln 2)+\frac{1}{3^s}(\ln 3)+\cdots\right)_{s=2}\right| = \left|\frac{1}{2^2}(\ln 2)+\frac{1}{3^2}(\ln 3)+\cdots\right| \\ &= \frac{1}{2^2}(\ln 2)+\frac{1}{3^2}(\ln 3)+\cdots,\end{aligned}$$

so

$$\underbrace{|\zeta'(s)| \le |\zeta'(2)|} = |\zeta'(2)|(\ln e)^2 \le |\zeta'(2)|(\ln t)^2,$$

and hence $|\zeta'(s)| \le M \cdot (\ln t)^2$, where $M \equiv |\zeta'(2)|$.

Case II: when $s \equiv \sigma + it \in \Omega$, and $\left(\frac{1}{2} \le\right)\sigma < 2(< e \le t)$. By Conclusion 4.253, we have

$$\begin{aligned}|\zeta'(s)| = &\left|-\sum_{n=1}^{[t]}\frac{\ln n}{n^s} - \frac{[t]^{1-s}(\ln[t])}{s-1} - \frac{[t]^{1-s}}{(s-1)^2}\right. \\ &\left. -\int_{[t]}^{\infty}\frac{x-[x]}{x^{s+1}}\mathrm{d}x + s\int_{[t]}^{\infty}\frac{(x-[x])\ln x}{x^{s+1}}\mathrm{d}x\right| \\ \le &\left|\sum_{n=1}^{[t]}\frac{\ln n}{n^s}\right| + \left|\frac{[t]^{1-s}(\ln[t])}{s-1}\right| + \left|\frac{[t]^{1-s}}{(s-1)^2}\right| \\ &+\left|\int_{[t]}^{\infty}\int_{[t]}^{\infty}\frac{x-[x]}{x^{s+1}}\mathrm{d}x\right| + \left|s\int_{[t]}^{\infty}\frac{(x-[x])\ln x}{x^{s+1}}\mathrm{d}x\right| \\ \le &\sum_{n=1}^{[t]}\left|\frac{\ln n}{n^s}\right| + + \left|\frac{[t]^{1-s}(\ln[t])}{s-1}\right| + \left|\frac{[t]^{1-s}}{(s-1)^2}\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{[t]}^{\infty}\frac{x-[x]}{x^{s+1}}dx\right|+\left|s\int_{[t]}^{\infty}\frac{(x-[x])\ln x}{x^{s+1}}\mathrm{d}x\right| \\
&=\sum_{n=1}^{[t]}\frac{\ln n}{n^{\sigma}}++\frac{[t]^{1-\sigma}(\ln[t])}{\sqrt{(\sigma-1)^2+t^2}}+\frac{[t]^{1-\sigma}}{(\sigma-1)^2+t^2} \\
&+\left|\int_{[t]}^{\infty}\frac{x-[x]}{x^{s+1}}\mathrm{d}x\right|+|s|\left|\int_{[t]}^{\infty}\frac{(x-[x])\ln x}{x^{s+1}}\mathrm{d}x\right| \\
&\le\sum_{n=1}^{[t]}\frac{\ln n}{n^{\sigma}}++\frac{[t]^{1-\sigma}(\ln[t])}{t}+\frac{[t]^{1-\sigma}}{t^2} \\
&+\int_{[t]}^{\infty}\frac{x-[x]}{x^{\sigma+1}}\mathrm{d}x+(\sigma+t)\int_{[t]}^{\infty}\frac{(x-[x])\ln x}{x^{\sigma+1}}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\le\sum_{n=1}^{[t]}\frac{\ln n}{n^{\sigma}}++\frac{[t]^{1-\sigma}(\ln[t])}{t}+\frac{[t]^{1-\sigma}}{t^2} \\
&+\int_{[t]}^{\infty}\frac{1}{x^{\sigma+1}}\mathrm{d}x+(\sigma+t)\int_{[t]}^{\infty}\frac{1\cdot\ln x}{x^{\sigma+1}}\mathrm{d}x \\
&\le\sum_{n=1}^{[t]}\frac{\ln n}{n^{\sigma}}+\frac{[t]^{1-\sigma}(\ln[t])}{t}+\frac{[t]^{1-\sigma}}{t^2} \\
&+\int_{[t]}^{\infty}\frac{1}{x^{\sigma+1}}\mathrm{d}x+(t+t)\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\mathrm{d}x \\
&=\sum_{n=1}^{[t]}\frac{\ln n}{n}e^{(\ln n)(1-\sigma)}+\frac{[t]^{1-\sigma}(\ln[t])}{t} \\
&+\frac{[t]^{1-\sigma}}{t^2}+\int_{[t]}^{\infty}\frac{1}{x^{\sigma+1}}\mathrm{d}x+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\mathrm{d}x \\
&=\sum_{n=1}^{[t]}\frac{\ln n}{n}e^{(\ln n)\frac{A}{\ln t}}+\frac{[t]^{1-\sigma}(\ln[t])}{t}
\end{aligned}
$$

$$+\frac{[t]^{1-\sigma}}{t^2}+\int_{[t]}^{\infty}\frac{1}{x^{\sigma+1}}\,\mathrm{d}x+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x$$

$$=\sum_{n=1}^{[t]}\frac{\ln n}{n}e^{A\cdot\frac{\ln n}{\ln[t]}}+\frac{[t]^{1-\sigma}(\ln[t])}{t}$$

$$+\frac{[t]^{1-\sigma}}{t^2}+\int_{[t]}^{\infty}\frac{1}{x^{\sigma+1}}\,\mathrm{d}x+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x$$

$$\leq\sum_{n=1}^{[t]}\frac{\ln n}{n}e^{A\cdot 1}+\frac{[t]^{1-\sigma}(\ln[t])}{t}+\frac{[t]^{1-\sigma}}{t^2}$$

$$+\int_{[t]}^{\infty}\frac{1}{x^{\sigma+1}}\,\mathrm{d}x+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x$$

$$=e^A\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{[t]^{1-\sigma}(\ln[t])}{t}+\frac{[t]^{1-\sigma}}{t^2}$$

$$+\int_{[t]}^{\infty}\frac{1}{x^{\sigma+1}}\,\mathrm{d}x+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x$$

$$=e^A\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{e^{(\ln[t])(1-\sigma)}(\ln[t])}{t}+\frac{e^{(\ln[t])(1-\sigma)}}{t^2}$$

$$+\frac{1}{-\sigma}\left(0-\frac{1}{[t]^{\sigma}}\right)+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x$$

$$=e^A\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{e^{(\ln[t])(1-\sigma)}(\ln[t])}{t}+\frac{e^{(\ln[t])(1-\sigma)}}{t^2}$$

$$+\frac{1}{[t]\sigma}e^{(\ln[t])(1-\sigma)}+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x$$

$$\leq e^A\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{e^{(\ln[t])(1-\sigma)}(\ln[t])}{[t]}+\frac{e^{(\ln[t])(1-\sigma)}}{[t]^2}$$

$$+\frac{1}{[t]\sigma}e^{(\ln[t])(1-\sigma)}+2t\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x$$

$$\leq e^A\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{e^{(\ln[t])\left(\frac{A}{\ln t}\right)}(\ln[t])}{[t]}+\frac{e^{(\ln[t])\left(\frac{A}{\ln t}\right)}}{[t]^2}$$

$$
\begin{aligned}
&+\frac{1}{[t]\sigma}e^{(\ln[t])\left(\frac{A}{\ln t}\right)}+2t\int\limits_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x\\
&=e^{A}\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{e^{A\left(\frac{\ln[t]}{\ln t}\right)}(\ln[t])}{[t]}+\frac{e^{A\left(\frac{\ln[t]}{\ln t}\right)}}{[t]^{2}}\\
&+\frac{1}{[t]\sigma}e^{A\left(\frac{\ln[t]}{\ln t}\right)}+2t\int\limits_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x\\
&\le e^{A}\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{e^{A\cdot 1}(\ln[t])}{[t]}+\frac{e^{A\cdot 1}}{[t]^{2}}\\
&+\frac{1}{[t]\sigma}e^{A\cdot 1}+2t\int\limits_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x\\
&\le e^{A}\sum_{n=1}^{[t]}\frac{\ln n}{n}+\frac{e^{A}(\ln[t])}{[t]}+\frac{e^{A}}{[t]^{2}}\\
&+\frac{1}{[t]}\cdot 2\cdot e^{A}+2t\int\limits_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x\le e^{A}\sum_{n=1}^{[t]}\frac{\ln n}{n}\\
&+\frac{e^{A}(\ln[t])}{[e]}+\frac{e^{A}}{[e]^{2}}+\frac{1}{[e]}\cdot 2\cdot e^{A}+2t\int\limits_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x\\
&\le e^{A}\sum_{n=1}^{[t]}\frac{\ln n}{n}+e^{A}\frac{\ln[t]}{2}+\frac{e^{A}}{2^{2}}+\frac{1}{2}\cdot 2\cdot e^{A}\\
&+2([t]+1)\int\limits_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x=e^{A}\sum_{n=1}^{[t]}\frac{\ln n}{n}+e^{A}\frac{\ln[t]}{2}\\
&+\frac{5e^{A}}{4}+2([t]+1)\int\limits_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\,\mathrm{d}x.
\end{aligned}
$$

By Conclusion 4.117, there exists a real number γ_1 such that

$$
\underbrace{0\le\left(\sum_{n=1}^{[t]}\frac{\ln n}{n}-\frac{1}{2}(\ln[t])^{2}\right)-\gamma_1}\le\frac{\ln[t]}{[t]}\le\frac{\ln[t]}{[e]}=\frac{\ln[t]}{2},
$$

so

$$\sum_{n=1}^{[t]} \frac{\ln n}{n} \le \left(\frac{\ln[t]}{2} + \gamma_1\right) + \frac{1}{2}(\ln[t])^2 \le \left(\frac{\ln t}{2} + \gamma_1\right) + \frac{1}{2}(\ln t)^2$$
$$\le (\ln t)^2\left(\frac{1}{2} + \frac{1}{2\ln t} + \frac{\gamma_1}{(\ln t)^2}\right) \le (\ln t)^2\left(\frac{1}{2} + \frac{1}{2\ln e} + \frac{\gamma_1}{(\ln e)^2}\right)$$
$$= (1+\gamma_1)(\ln t)^2.$$

Thus,

$$|\zeta'(s)| \le e^A\cdot(1+\gamma_1)(\ln t)^2 + e^A\frac{\ln[t]}{2} + \frac{5e^A}{4} + 2([t]+1)\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\mathrm{d}x$$
$$\le e^A\cdot(1+\gamma_1)(\ln t)^2 + \frac{e^A}{2}\ln t + \frac{5}{4}e^A + 2([t]+1)\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\mathrm{d}x.$$

Now, since

$$2([t]+1)\int_{[t]}^{\infty}\frac{\ln x}{x^{\sigma+1}}\mathrm{d}x = 2([t]+1)\int_{[t]}^{\infty}(\ln x)x^{-\sigma-1}\mathrm{d}x$$
$$= 2([t]+1)\left((\ln x)\frac{x^{-\sigma}}{-\sigma}\bigg|_{x=[t]}^{\infty} - \int_{[t]}^{\infty}\frac{1}{x}\frac{x^{-\sigma}}{-\sigma}\mathrm{d}x\right)$$
$$= 2([t]+1)\left(\frac{1}{\sigma}\left(\frac{\ln[t]}{[t]^\sigma} - \lim_{x\to\infty}\frac{\ln x}{x^\sigma}\right) + \frac{1}{\sigma}\int_{[t]}^{\infty}x^{-\sigma-1}\mathrm{d}x\right)$$
$$= 2([t]+1)\left(\frac{1}{\sigma}\left(\frac{\ln[t]}{[t]^\sigma} - \lim_{x\to\infty}\frac{\frac{1}{x}}{\sigma x^{\sigma-1}}\right) + \frac{1}{\sigma^2}\left(\frac{1}{[t]^\sigma} - \lim_{x\to\infty}\frac{1}{x^\sigma}\right)\right)$$
$$= 2([t]+1)\left(\frac{1}{\sigma}\left(\frac{\ln[t]}{[t]^\sigma} - 0\right) + \frac{1}{\sigma^2}\left(\frac{1}{[t]^\sigma} - 0\right)\right)$$
$$= 2([t]+1)\left(\frac{1}{\sigma}\frac{\ln[t]}{[t]^\sigma} + \frac{1}{\sigma^2}\frac{1}{[t]^\sigma}\right)$$
$$\le 2([t]+1)\left(2\cdot\frac{\ln[t]}{[t]^\sigma} + 2^2\cdot\frac{1}{[t]^\sigma}\right) = 4([t]+1)(\ln[t]+2)\frac{1}{[t]}[t]^{1-\sigma}$$
$$= 4\left(1+\frac{1}{[t]}\right)(\ln[t]+2)[t]^{1-\sigma} \le 4\left(1+\frac{1}{[e]}\right)(\ln[t]+2)[t]^{1-\sigma}$$
$$= 4\left(1+\frac{1}{2}\right)(\ln[t]+2)e^{(\ln[t])(1-\sigma)} \le 6(\ln[t]+2)e^{(\ln[t])\left(\frac{A}{\ln t}\right)}$$
$$\le 6(\ln[t]+2)e^A \le 6e^A(\ln t+2)$$

we have

$$|\zeta'(s)| \leq e^A \cdot (1+\gamma_1)(\ln t)^2 + \frac{e^A}{2}\ln t + \frac{5}{4}e^A + 6e^A(\ln t + 2).$$

Hence,

$$|\zeta'(s)| \leq (\ln t)^2 e^A \left((1+\gamma_1) + \frac{13}{2}\frac{1}{\ln t} + \frac{53}{4}\frac{1}{(\ln t)^2} \right)$$
$$< e^A \left(1+\gamma_1 + \frac{13}{2} + \frac{53}{4} \right)(\ln t)^2 = \left(\left(\gamma_1 + \frac{83}{4} \right) e^A \right)(\ln t)^2.$$

Conclusion 4.254 Let A be a positive real number. Let

$$\Omega \equiv \left\{ s \equiv \sigma + it : \sigma \in \left[\frac{1}{2}, \infty\right), t \in [e, \infty), \text{and } 1 - \sigma < \frac{A}{\ln t} \right\}.$$

Then there exists a real number M such that for every $s \equiv \sigma + it \in \Omega$, $|\zeta'(s)| \leq M(\ln t)^2$.

Problem 4.255 The arithmetic function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is multiplicative.

(Solution For this purpose, let us take any positive integers m, n such that $(m, n) = 1$. We have to show that $\mu(m \cdot n) = \mu(m) \cdot \mu(n)$. If either m or n has a prime-square factor, then $m \cdot n$ has a prime-square factor, and hence if either m or n has a prime-square factor, then $\mu(m \cdot n) = 0 = \mu(m) \cdot \mu(n)$. So, it suffices to prove that

$$(m, n) = 1 \cdot \mu(m \cdot n) = \mu(m) \cdot \mu(n)$$

in the case when neither m nor n has a prime-square factor. For this purpose, let $m \equiv \underbrace{p_1 \cdots p_j}_{j\text{ distinct primes}}$, and let $n \equiv \underbrace{q_1 \cdots q_k}_{k\text{ distinct primes}}$. Next, let $(m, n) = 1$. We have to show that $\mu(m \cdot n) = \mu(m) \cdot \mu(n)$.

Since $(m, n) = 1$, the prime factorization of $m \cdot n$ is

$$\underbrace{p_1 \cdots p_j q_1 \cdots q_k}_{(j+k)\text{distinct primes}},$$

and hence

$$\text{LHS} = \mu(m \cdot n) = (-1)^{j+k} = (-1)^j \cdot (-1)^k = \mu(m) \cdot \mu(n) = \text{RHS}.$$

∎)

Let us fix any complex number s satisfying $1 < \sigma$. Since $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is multiplicative, $n \mapsto \frac{\mu(n)}{n^s}$ is a multiplicative arithmetic function. Further,

$$\sum_{n=1}^{\infty}\left|\frac{\mu(n)}{n^s}\right| \leq \sum_{n=1}^{\infty}\frac{|\mu(n)|}{|n^s|} = \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{\sigma}} \leq \sum_{n=1}^{\infty}\frac{1}{n^{\sigma}} < \infty,$$

so $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ is absolutely convergent. By Conclusion 4.180,

1. $\left(\prod_{p \text{ is a prime}}\left(1-\frac{1}{p^s}\right) = \prod_{p \text{ is a prime}}\left(1+\frac{(-1)^1}{p^s}+\frac{0}{p^{2s}}+\frac{0}{p^{3s}}+\cdots\right) =\right)$ $\prod_{p \text{ is a prime}}\left(1+\frac{\mu(p)}{p^s}+\frac{\mu(p^2)}{p^{2s}}+\frac{\mu(p^3)}{p^{3s}}+\cdots\right)$ is absolutely convergent,
2. $\prod_{p \text{ is a prime}}\left(1-\frac{1}{p^s}\right) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$.

It follows that $F : s \mapsto \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ converges absolutely for $1<\sigma$. Now, since $\zeta : s \mapsto \sum_{n=1}^{\infty}\frac{1}{n^s}$ converges absolutely for $1<\sigma$, by Conclusion 4.177 and Problem 4.151, for every $s \equiv \sigma + it$ satisfying $1<\sigma$, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right)\zeta(s) = F(s)\zeta(s) &= \sum_{n=1}^{\infty}\frac{(F*\zeta)(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\sum_{d|N}\mu(d)\cdot 1}{n^s}\\ &= \sum_{n=1}^{\infty}\frac{\sum_{d|N}\mu(d)}{n^s} = \frac{\sum_{d|1}\mu(d)}{1^s}+\frac{\sum_{d|2}\mu(d)}{2^s}+\frac{\sum_{d|3}\mu(d)}{3^s}+\cdots\\ &= \frac{1}{1^s}+\frac{0}{2^s}+\frac{0}{3^s}+\cdots = 1,\end{aligned}$$

and hence for every $s \equiv \sigma + it$ satisfying $1<\sigma$,

$$\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right)\zeta(s) = 1.$$

It follow that for every $s \equiv \sigma + it$ satisfying $1<\sigma$, $\zeta(s) \neq 0$,

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \neq 0, \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Conclusion 4.256 For every $s \equiv \sigma + it$ satisfying $1<\sigma$, $\zeta(s) \neq 0$, and $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$. Also, $\prod_{p \text{ is a prime}}\left(1-\frac{1}{p^s}\right)$ concerges absolutely in $1<\sigma$, and $\prod_{p \text{ is a prime}}\left(1-\frac{1}{p^s}\right) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\,(1<\sigma)$.

Problem 4.257 Let $s \equiv \sigma + it$ be any complex number satisfying $1 < \sigma$.

$$1 \leq (\zeta(\sigma))^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)|^2$$

(Solution By Conclusion 4.256, we have to show that

$$\underbrace{1 \leq \left(\frac{1}{\prod_p \left(1 - \frac{1}{p^\sigma}\right)} \right)^3 \left| \frac{1}{\prod_p \left(1 - \frac{1}{p^{\sigma+it}}\right)} \right|^4 \left| \frac{1}{\prod_p \left(1 - \frac{1}{p^{\sigma+2it}}\right)} \right|^2}$$
$$= \frac{1}{\left(\prod_p \left(1 - \frac{1}{p^\sigma}\right)\right)^3} \frac{1}{\left|\prod_p \left(1 - \frac{1}{p^{\sigma+it}}\right)\right|^4} \frac{1}{\left|\prod_p \left(1 - \frac{1}{p^{\sigma+2it}}\right)\right|^2}$$
$$= \frac{1}{\left(\prod_p \left(1 - \frac{1}{p^\sigma}\right)\right)^3} \frac{1}{\left(\prod_p \left|1 - \frac{1}{p^{\sigma+it}}\right|\right)^4} \frac{1}{\prod_p \left|1 - \frac{1}{p^{\sigma+2it}}\right|^2}$$
$$= \frac{1}{\prod_p \left(\left(1 - \frac{1}{p^\sigma}\right)^3 \left|1 - \frac{1}{p^{\sigma+it}}\right|^4 \left|1 - \frac{1}{p^{\sigma+2it}}\right|^2 \right)},$$

that is

$$\prod_p \left(\left(1 - \frac{1}{p^\sigma}\right)^3 \left|1 - \frac{1}{p^{\sigma+it}}\right|^4 \left|1 - \frac{1}{p^{\sigma+2it}}\right|^2 \right) \leq 1.$$

It suffices to show that for every prime p,

$$\left(1 - \frac{1}{p^\sigma}\right)^3 \left|1 - \frac{1}{p^{\sigma+it}}\right|^4 \left|1 - \frac{1}{p^{\sigma+2it}}\right|^2 \leq 1.$$

For this purpose, let us fix any prime p. We have to show that

$$\left(1 - \frac{1}{p^\sigma}\right)^3 \left|1 - \frac{1}{p^\sigma} e^{(\ln p)(-it)}\right|^4 \left|1 - \frac{1}{p^\sigma} e^{(\ln p)(-2it)}\right|^2$$
$$= \left(1 - \frac{1}{p^\sigma}\right)^3 \left|1 - \frac{1}{p^{\sigma+it}}\right|^4 \left|1 - \frac{1}{p^{\sigma+2it}}\right|^2 \leq 1,$$

that is

$$(1 - r)^3 \left|1 - re^{-i\theta}\right|^4 \left|1 - re^{-2i\theta}\right|^2 \leq 1,$$

where $r \equiv \frac{1}{p^\sigma} (\in (0, 1))$, and $\theta \equiv (\ln p)t$. Since

$$\begin{aligned}(1-r)^3\left|1-re^{-i\theta}\right|^4\left|1-re^{-2i\theta}\right|^2 &= (1-r)^3|(1-r\cos\theta)+ir\sin\theta|^4 \\ |(1-r\cos 2\theta)+ir\sin 2\theta|^2 &= (1-r)^3\left((1-r\cos\theta)^2+(r\sin\theta)^2\right)^2 \\ \left((1-r\cos 2\theta)^2+(r\sin 2\theta)^2\right) &= (1-r)^3\left(1+r^2-2r\cos\theta\right)^2\left(1+r^2-2r\cos 2\theta\right),\end{aligned}$$

it suffices to show that

$$\left(1+r^2-2r\cos\theta\right)^2\left(1+r^2-2r\cos 2\theta\right) \le \frac{1}{(1-r)^3},$$

that is

$$\left(\left(1+r^2-2r\cos\theta\right)^2\left(1+r^2-2r\cos 2\theta\right)\right)^{\frac{1}{3}} \le \frac{1}{1-r}\left(=1+r+r^2+r^3+\cdots\right).$$

Thus, by Conclusion 2.10(b), Vol.1, it is enough to show that

$$\frac{2(1+r^2-2r\cos\theta)+(1+r^2-2r\cos 2\theta)}{3} \le 1+r+r^2+r^3+\cdots.$$

Observe that

$$\begin{aligned}\frac{2(1+r^2-2r\cos\theta)+(1+r^2-2r\cos 2\theta)}{3} &= \frac{1}{3}\left(3+3r^2-4r\cos\theta-2r\left(2\cos^2\theta-1\right)\right) \\ &= \frac{1}{3}\left(3+2r+3r^2-4r\left(\cos^2\theta+\cos\theta\right)\right) = \frac{1}{3}\left(3+2r+3r^2-4r\left(\cos\theta+\frac{1}{2}\right)^2+r\right) \\ &= 1+r+r^2-\frac{4}{3}r\left(\cos\theta+\frac{1}{2}\right)^2 \le 1+r+r^2 \le 1+r+r^2+r^3+\cdots,\end{aligned}$$

so

$$\frac{2(1+r^2-2r\cos\theta)+(1+r^2-2r\cos 2\theta)}{3} \le 1+r+r^2+r^3+\cdots.$$

∎)

Conclusion 4.258 For every complex number $s \equiv \sigma + it$ satisfying $1 < \sigma$,

$$1 \le (\zeta(\sigma))^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|^2.$$

Problem 4.259 Let $s \equiv \sigma + it$ be any complex number satisfying $1 \le \sigma$ and $s \ne 1$. Then $\zeta(\sigma+it) \ne 0$.

(Solution If not, otherwise suppose that, there exists a complex number $s_0 \equiv \sigma_0 + it_0$ such that $1 \leq \sigma_0$, $s_0 \neq 1$, and $\zeta(\sigma_0 + it_0) = 0$. We have to arrive at a contradiction.

Case I: when $1 < \sigma_0$. Here, by Conclusion 4.256, $\zeta(s_0) \neq 0$. This is a contradiction.
Case II: when $\sigma_0 = 1$. Here, $\zeta(1 + it_0) = 0$. By Conclusion 4.257, for every $\varepsilon > 0$,

$$1 \leq (\zeta(1+\varepsilon))^3 |\zeta((1+\varepsilon) + it_0)|^4 |\zeta((1+\varepsilon) + 2it_0)|^2,$$

and hence, for every $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{\varepsilon} &\leq (\varepsilon \cdot \zeta(1+\varepsilon))^3 \left| \frac{\zeta((1+\varepsilon) + it_0) - 0}{\varepsilon} \right|^4 |\zeta((1+\varepsilon) + 2it_0)|^2 \\ &= (\varepsilon \cdot \zeta(1+\varepsilon))^3 \left| \frac{\zeta((1+\varepsilon) + it_0) - \zeta(1 + it_0)}{\varepsilon} \right|^4 |\zeta((1+\varepsilon) + 2it_0)|^2. \end{aligned}$$

Thus, for every $\varepsilon > 0$,

$$\frac{1}{\varepsilon} \leq (\varepsilon \cdot \zeta(1+\varepsilon))^3 \left| \frac{\zeta((1+\varepsilon) + it_0) - \zeta(1 + it_0)}{\varepsilon} \right|^4 |\zeta((1+\varepsilon) + 2it_0)|^2.$$

By Conclusion 4.221,

$$\begin{gathered} \lim_{\varepsilon \to 0^+} \varepsilon \cdot \zeta(1+\varepsilon) = \underbrace{\operatorname{Res}(\zeta; 1) = 1}, \\ \lim_{\varepsilon \to 0^+} \frac{\zeta((1+\varepsilon) + it_0) - \zeta(1 + it_0)}{\varepsilon} = \zeta'(1 + it_0), \end{gathered}$$

and

$$\lim_{\varepsilon \to 0^+} \zeta((1+\varepsilon) + 2it_0) = \zeta((1+0) + 2it_0) = \zeta(1 + 2it_0).$$

Thus,

$$\begin{aligned} &\lim_{\varepsilon \to 0^+} \left((\varepsilon \cdot \zeta(1+\varepsilon))^3 \cdot \left| \frac{\zeta((1+\varepsilon) + it_0) - \zeta(1 + it_0)}{\varepsilon} \right|^4 \cdot |\zeta((1+\varepsilon) + 2it_0)|^2 \right) \\ &= 1^3 \cdot |\zeta'(1 + it_0)|^4 \cdot |\zeta(1 + 2it_0)|^2 = |\zeta'(1 + it_0)|^4 \cdot |\zeta(1 + 2it_0)|^2 \in \mathbb{R}. \end{aligned}$$

Hence,

$$\lim_{\varepsilon \to 0^+} \left((\varepsilon \cdot \zeta(1+\varepsilon))^3 \cdot \left| \frac{\zeta((1+\varepsilon) + it_0) - \zeta(1 + it_0)}{\varepsilon} \right|^4 \cdot |\zeta((1+\varepsilon) + 2it_0)|^2 \right)$$

is a real number. Since for every $\varepsilon > 0$,

$$\frac{1}{\varepsilon} \leq (\varepsilon \cdot \zeta(1+\varepsilon))^3 \left| \frac{\zeta((1+\varepsilon)+it_0) - \zeta(1+it_0)}{\varepsilon} \right|^4 |\zeta((1+\varepsilon)+2it_0)|^2,$$

and $\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} = \infty$, we have

$$\lim_{\varepsilon\to 0^+} \left((\varepsilon \cdot \zeta(1+\varepsilon))^3 \cdot \left| \frac{\zeta((1+\varepsilon)+it_0) - \zeta(1+it_0)}{\varepsilon} \right|^4 \cdot |\zeta((1+\varepsilon)+2it_0)|^2 \right) = \infty.$$

This is a contradiction.

Hence, in all cases, we get a contradiction. ■)

Conclusion 4.260 For every complex number $s \equiv \sigma + it$ satisfying $1 \leq \sigma$ and $s \neq 1$, $\zeta(\sigma + it) \neq 0$. In short, $1 \leq \sigma$ is a zero-free region for $\zeta(s)$.

Let $s \equiv \sigma + it$ be any complex number satisfying $2 \leq \sigma$, and $e \leq t$. By Conclusion 4.256,

$$\begin{aligned} \left| \frac{1}{\zeta(s)} \right| &= \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \left| \frac{\mu(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{|n^s|} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \cdot |\mu(n)| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \cdot 1 = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) \\ &= (\zeta(2))(\ln e)^8 \leq (\zeta(2))(\ln t)^8, \end{aligned}$$

so

$$\left| \frac{1}{\zeta(s)} \right| \leq (\zeta(2))(\ln t)^8.$$

Since $2 \leq \sigma$, we have $(1-\sigma) \leq -1 < 0 \leq \frac{1}{\ln t}$, and hence $1 - \sigma < \frac{1}{\ln t}$. It follows that

$$\begin{aligned} &\{s \equiv \sigma + it : \sigma \in [2, \infty), t \in [e, \infty)\} \\ &\quad \subset \left\{ s \equiv \sigma + it : \sigma \in \left[\frac{1}{2}, \infty\right), t \in [e, \infty), \text{and } 1 - \sigma < \frac{1}{\ln t} \right\}, \end{aligned}$$

and hence, by Conclusion 4.254, there exists a $M > 0$ such that, for every $s_1 \equiv \sigma_1 + it_1$ satisfying $2 \leq \sigma_1$, and $e \leq t_1$, we have $|\zeta'(s_1)| \leq M(\ln t_1)^2$. Hence,

$$\begin{aligned} \left| \frac{\zeta'(s)}{\zeta(s)} \right| &= \left| \frac{1}{\zeta(s)} \right| \cdot |\zeta'(s)| \leq (\zeta(2))(\ln t)^8 \cdot |\zeta'(s)| \\ &\leq (\zeta(2))(\ln t)^8 \cdot M(\ln t)^2 = (M \cdot \zeta(2))(\ln t)^{10}. \end{aligned}$$

Conclusion 4.261 There exists $M > 0$ such that, for every $s \equiv \sigma + it$ satisfying $2 \le \sigma$, and $e \le t$,

$$\left|\frac{1}{\zeta(s)}\right| \le M(\ln t)^8, \quad \text{and} \quad \left|\frac{\zeta'(s)}{\zeta(s)}\right| \le M(\ln t)^{10}.$$

Problem 4.262 For every complex number $s \equiv \sigma + it$ satisfying $\sigma \in (1, 2]$, and for every $t \in [e, \infty)$,

$$\frac{1}{|\zeta(\sigma + it)|} \le (\zeta(\sigma))^{\frac{3}{4}} |\zeta(\sigma + 2it)|^{\frac{1}{2}}.$$

(**Solution:** Here, by Conclusion 4.257,

$$1 \le (\zeta(\sigma))^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)|^2.$$

Now, by Conclusion 4.258,

$$\frac{1}{|\zeta(\sigma + it)|^4} \le (\zeta(\sigma))^3 |\zeta(\sigma + 2it)|^2,$$

and hence

$$\frac{1}{|\zeta(\sigma + it)|} \le (\zeta(\sigma))^{\frac{3}{4}} |\zeta(\sigma + 2it)|^{\frac{1}{2}}.$$

■)

Problem 4.263 There exists $B > 0$ such that, for every $s \equiv \sigma + it$ satisfying $\sigma \in [1, 2]$, and $t \in [e, \infty)$, we have

$$B \frac{(\sigma - 1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}} \le |\zeta(\sigma + it)|.$$

(Solution By Conclusion 4.221,

$$f : z \mapsto \begin{cases} (z - 1)\zeta(z) & \text{if } z \neq 1 \\ 1 & \text{if } z = 1 \end{cases}$$

is a holomorphic function, and hence f is continuous on the compact set $[1, 2]$. It follows that $f([1, 2])$ is compact, and hence $f([1, 2])$ is bounded. Thus, there exists $M > 0$ such that for every $x \in (1, 2]$, $(x - 1)\zeta(x) < M$.

By Conclusion 4.252, there exists $M_1 > 0$ such that, for every $s \equiv \sigma + it$ satisfying $\sigma \in (1,2]$, and, $t \in [e, \infty)$, we have

$$|\zeta(\sigma + 2it)| \le M_1(\ln 2t) \le M_1(\ln(e \cdot t)) \le M_1(\ln(t \cdot t)) = (2M_1)\ln t.$$

Thus, by Problem 4.263, for every $s \equiv \sigma + it$ satisfying $\sigma \in (1,2]$, and $t \in [e, \infty)$, we have

$$\frac{1}{|\zeta(\sigma+it)|} \le \underbrace{(\zeta(\sigma))^{\frac{3}{4}}|\zeta(\sigma+2it)|^{\frac{1}{2}} \le (\zeta(\sigma))^{\frac{3}{4}}((2M_1)\ln t)^{\frac{1}{2}}} \le \left(\frac{M}{(\sigma-1)}\right)^{\frac{3}{4}}((2M_1)\ln t)^{\frac{1}{2}},$$

and hence for every $s \equiv \sigma + it$ satisfying $\sigma \in (1,2]$, and $t \in [e, \infty)$, we have

$$\frac{(\sigma-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}} \le M^{\frac{3}{4}}(2M_1)^{\frac{1}{2}}|\zeta(\sigma+it)|.$$

It follows that there exists $B > 0$ such that, for every $s \equiv \sigma + it$ satisfying $\sigma \in [1,2]$ and $t \in [e, \infty)$, we have

$$B\frac{(\sigma-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}} \le |\zeta(\sigma+it)|.$$

∎)

Let $\alpha \in (1,2)$, $t \in [e, \infty)$, and $\sigma \in (1,2)(= (1,\alpha) \cup [\alpha, 2))$.

Case I: when $\sigma \in (1,\alpha)(\subset (1,2))$. By the fundamental theorem of calculus, (e.g., see Sects. 6.21 and 6.23 of [5]),

$$\int_{\sigma}^{\alpha} \zeta'(u+it)\mathrm{d}u = \zeta(\alpha+it) - \zeta(\sigma+it),$$

and hence

$$|\zeta(\alpha+it)| - |\zeta(\sigma+it)| \le \underbrace{|\zeta(\alpha+it) - \zeta(\sigma+it)| = \left|\int_{\sigma}^{\alpha} \zeta'(u+it)\mathrm{d}u\right|} \le \int_{\sigma}^{\alpha} |\zeta'(u+it)|\mathrm{d}u.$$

Thus,

$$|\zeta(\alpha+it)| \le |\zeta(\sigma+it)| + \int_{\sigma}^{\alpha} |\zeta'(u+it)|\mathrm{d}u.$$

By Problem 4.264, there exists $B>0$ such that, for every $s_1 \equiv \sigma_1 + it_1$ satisfying $\sigma_1 \in [1,2]$, and $t_1 \in [e,\infty)$, we have

$$B\frac{(\sigma_1-1)^{\frac{3}{4}}}{(\ln t_1)^{\frac{1}{2}}} \le |\zeta(\sigma_1+it_1)|.$$

It follows that

$$B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}} \le |\zeta(\sigma+it)| + \int_{\sigma}^{\alpha} |\zeta'(u+it)|\mathrm{d}u.$$

By Conclusion 4.254, there exists a $M>0$ such that, for every $s_1 \equiv \sigma_1 + it_1$ satisfying $\sigma_1 \in \left[\frac{1}{2},\infty\right), t_1 \in [e,\infty)$, and $1-\sigma_1 < \frac{1}{\ln t_1}$, we have $|\zeta'(s_1)| \le M(\ln t_1)^2$. It follows that for every $u \in [\sigma,\alpha](\subset (1,\alpha) \subset (1,2))$, we have $|\zeta'(u+it)| \le M(\ln t)^2$, and hence

$$B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}} - |\zeta(\sigma+it)| \le \underbrace{\int_{\sigma}^{\alpha} |\zeta'(u+it)|\mathrm{d}u \le \left(M(\ln t)^2\right)(\alpha-\sigma)} \le \left(M(\ln t)^2\right)(\alpha-1).$$

Thus,

$$B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}} - \left(M(\ln t)^2\right)(\alpha-1) \le |\zeta(\sigma+it)|.$$

Case II: when $\sigma \in [\alpha,2)(\subset [1,2])$. By Problem 4.264, there exists $B>0$ such that, for every $s_1 \equiv \sigma_1 + it_1$ satisfying $\sigma_1 \in [1,2]$, and $t_1 \in [e,\infty)$, we have

$$B\frac{(\sigma_1-1)^{\frac{3}{4}}}{(\ln t_1)^{\frac{1}{2}}} \le |\zeta(\sigma_1+it_1)|.$$

It follows that

$$B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}}-\left(M(\ln t)^2\right)(\alpha-1)\le B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}}\le \underbrace{B\frac{(\sigma-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}}\le|\zeta(\sigma+it)|}.$$

Thus,

$$B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}}-\left(M(\ln t)^2\right)(\alpha-1)\le|\zeta(\sigma+it)|.$$

Hence, for every $\alpha\in(1,2)$, for every $t\in[e,\infty)$, and for every $\sigma\in(1,2)$, we have

$$B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}}-\left(M(\ln t)^2\right)(\alpha-1)\le|\zeta(\sigma+it)|.$$

For every $\alpha\in(1,2)$, observe that

$$\begin{aligned}B\frac{(\alpha-1)^{\frac{3}{4}}}{(\ln t)^{\frac{1}{2}}}=2\cdot\left(M(\ln t)^2\right)(\alpha-1)\Leftrightarrow \frac{B}{(\ln t)^{\frac{1}{2}}}=2\cdot\left(M(\ln t)^2\right)(\alpha-1)^{\frac{1}{4}}\\ \Leftrightarrow\frac{B^4}{(\ln t)^2}=16M^4(\ln t)^8(\alpha-1)\Leftrightarrow\alpha=1+\frac{B^4}{16M^4(\ln t)^{10}}.\end{aligned}$$

Now, since

$$\lim_{t\to\infty}\left(1+\frac{B^4}{16M^4(\ln t)^{10}}\right)=1,$$

there exists $t_0>e$ such that

$$t\ge t_0\Rightarrow\left(1+\frac{B^4}{16M^4(\ln t)^{10}}\right)\in(1,2),$$

and hence $t\ge t_0$ implies that for every $\sigma\in(1,2)$,

$$\begin{aligned}2\cdot\left(M(\ln t)^2\right)\left(\left(1+\frac{B^4}{16M^4(\ln t)^{10}}\right)-1\right)\\ -\left(M(\ln t)^2\right)\left(\left(1+\frac{B^4}{16M^4(\ln t)^{10}}\right)-1\right)\le|\zeta(\sigma+it)|.\end{aligned}$$

Thus, for every $\sigma \in (1,2)$, and for every $t \in [t_0, \infty)$,

$$\frac{B^4}{16M^3(\ln t)^8} \leq |\zeta(\sigma+it)|.$$

Now, since ζ is continuous, for every $\sigma \in [1,2]$, and for every $t \in [t_0, \infty)$, we have

$$\frac{B^4}{16M^3(\ln t)^8} \leq |\zeta(\sigma+it)|.$$

Since

$$(\sigma, t) \mapsto |\zeta(\sigma+it)|(\ln t)^8$$

is a continuous function on the compact set $[1,2] \times [e, t_0]$, there exist $\sigma_1 \in [1,2]$, and $t_1 \in [e, t_0]$ such that, for every $(\sigma, t) \in [1,2] \times [e, t_0]$, we have

$$|\zeta(\sigma_1+it_1)|(\ln t_1)^8 \leq |\zeta(\sigma+it)|(\ln t)^8.$$

Since $\sigma_1 \in [1,2]$, by Conclusion 4.259, $\zeta(\sigma_1+it_1) \neq 0$, and hence

$$|\zeta(\sigma_1+it_1)|(\ln t_1)^8 > 0.$$

Thus, there exists $M > 0$ such that for every $\sigma \in [1,2]$, and for every $t \in [e, \infty)$,

$$\frac{M}{(\ln t)^8} \leq |\zeta(\sigma+it)|.$$

If we combine this result with Conclusion 4.261, we get the following

Conclusion 4.264 There exists $M > 0$ such that for every $\sigma \in [1, \infty)$, and for every $t \in [e, \infty)$,

1. $\frac{1}{|\zeta(\sigma+it)|} \leq M(\ln t)^8$,
2. $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq M(\ln t)^{10}$.

Proof of the remaining part (2) There exists $M > 0$ such that for every $\sigma \in [1, \infty)$, and for every $t \in [e, \infty)$,

$$\frac{1}{|\zeta(\sigma+it)|} \leq M(\ln t)^8.$$

It suffices to show that for every $\sigma \in [1, \infty)$, and for every $t \in [e, \infty)$, $|\zeta'(\sigma+it)| \leq M_1(\ln t)^2$ for some $M_1 > 0$. This is known to be true from Conclusion 4.254. ∎

Note 4.265

Problem 4.266 The function $s \mapsto \frac{\zeta'(s)}{\zeta(s)}$ has a simple pole at $s = 1$ with residue (-1). And hence

$$s \mapsto \left(\frac{\zeta'(s)}{\zeta(s)} - \frac{(-1)}{s-1}\right)$$

is holomorphic in some open neighborhood of 1. Also,

$$h : s \mapsto \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$$

is holomorphic on some open set D containing the half-plane $1 \le \sigma$.

(Solution It suffices to show that

1. $f : s \mapsto \frac{\zeta'(s)}{\zeta(s)}$ is holomorphic in $(\{s \equiv \sigma + it : 1 \le \sigma\} - \{1\})$,
2. there exists a function h holomorphic in some open neighborhood D of 1 such that for every $s \in D$,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{h(s)}{s-1}, h(1) \neq 0, \quad \text{and} \quad \lim_{s \to 1}(s-1) \cdot \frac{\zeta'(s)}{\zeta(s)} = -1.$$

For 1: By Conclusion 4.221, $\zeta \in H(\mathbb{C} - \{1\})$. It follows, by Lemma 1.117, that $\zeta' \in H(\mathbb{C} - \{1\})$. By Conclusion 4.259, for every complex number $s \equiv \sigma + it$ satisfying $1 \le \sigma$ and $s \neq 1$, we have $\zeta(\sigma + it) \neq 0$. It follows that $f : s \mapsto \frac{\zeta'(s)}{\zeta(s)}$ is holomorphic in $(\{s \equiv \sigma + it : 1 \le \sigma\} - \{1\})$.

For 2: By Conclusion 4.221, $\zeta \in H(\mathbb{C} - \{1\})$ has a simple pole at $s = 1$ with residue 1. It follows that there exists $g \in H(\mathbb{C})$ such that, for every $s \in (\mathbb{C} - \{1\})$, $\zeta(s) = \frac{g(s)}{s-1}$, $g(1) \neq 0$, and $\lim_{s \to 1}(s-1) \cdot \zeta(s) = 1$.

Since for every $s \in (\{s \equiv \sigma + it : 1 \le \sigma\} - \{1\})$, we have

$$\begin{aligned}\zeta'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}(\zeta(s)) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{g(s)}{s-1}\right) = \frac{g'(s) \cdot (s-1) - g(s)}{(s-1)^2} \\ &= \frac{g(s)}{s-1} \cdot \frac{1}{s-1}\left(\frac{g'(s)}{g(s)} \cdot (s-1) - 1\right) = \zeta(s) \cdot \frac{1}{s-1}\left(\frac{g'(s)}{g(s)} \cdot (s-1) - 1\right),\end{aligned}$$

we have, for every $s \in (\{s \equiv \sigma + it : 1 \le \sigma\} - \{1\})$,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1}\left(\frac{g'(s)}{g(s)} \cdot (s-1) - 1\right).$$

Now, since $g(1) \neq 0$, and $g \in H(\mathbb{C})$,

$$h : s \mapsto \left(\frac{g'(s)}{g(s)} \cdot (s-1) - 1\right)$$

is holomorphic in some open neighborhood D of 1. Also, $\frac{\zeta'(s)}{\zeta(s)} = \frac{h(s)}{s-1}$. Next, $h(1) = \frac{g'(1)}{g(1)} \cdot (1-1) - 1 = -1 \neq 0$. It remains to show that $\lim_{s\to 1}(s-1) \cdot \frac{\zeta'(s)}{\zeta(s)} = -1$.

$$\begin{aligned}\text{LHS} &= \lim_{s\to 1}(s-1) \cdot \frac{\zeta'(s)}{\zeta(s)} = \lim_{s\to 1} h(s) = h(1) \\ &= \frac{g'(1)}{g(1)} \cdot (1-1) - 1 = -1 = \text{RHS}.\end{aligned}$$

∎)

Conclusion 4.267 The function $s \mapsto \frac{\zeta'(s)}{\zeta(s)}$ has a simple pole at $s = 1$ with residue (-1). And hence $s \mapsto \left(\frac{\zeta'(s)}{\zeta(s)} - \frac{(-1)}{s-1}\right)$ is holomorphic in some open neighborhood of 1. Also, $h : s \mapsto \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$ is holomorphic on some open set D containing the half-plane $1 \leq \sigma$.

Let $1 \leq x$. By Conclusion 4.248,

$$\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^{s-1}h(s)\mathrm{d}s$$

where

$$h : s \mapsto \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right).$$

For every $T > 0$, let $R(T)$ be a rectangle-shaped closed curve, which consists of four oriented intervals $[2-iT, 2+iT], [2+iT, 1+iT], [1+iT, 1-iT]$, and $[1-iT, 2-iT]$.

By Conclusion 4.267,

$$h : s \mapsto \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$$

is holomorphic on some open set D containing the half-plane $1 \leq \sigma$. Now, since

$$s \mapsto x^{s-1} \left(= e^{(\ln x)(s-1)} \right)$$

is holomorphic on $\mathbb{C}$,

$$s \mapsto x^{s-1} \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right)$$

is holomorphic on D. So, by Theorem 1.110, for every $T > 0$,

$$\int_{R(T)} x^{s-1} \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) \mathrm{d}s = 0.$$

It follows that for every $T > 0$,

$$\begin{aligned}
&\int_{[2-iT,2+iT]} x^{s-1} \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) \mathrm{d}s \\
&+ \int_{[2+iT,1+iT]} x^{s-1} \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) \mathrm{d}s \\
&+ \int_{[1+iT,1-iT]} x^{s-1} \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) \mathrm{d}s \\
&+ \int_{[1-iT,2-iT]} x^{s-1} \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) \mathrm{d}s = 0,
\end{aligned}$$

that is

$$\begin{aligned}
&\int_{-T}^{T} x^{(2+iv)-1} \frac{1}{(2+iv)((2+iv)+1)} \left(-\frac{\zeta'(2+iv)}{\zeta(2+iv)} - \frac{1}{(2+iv)-1} \right) i\,\mathrm{d}v \\
&+ \int_{2}^{1} x^{(u+iT)-1} \frac{1}{(u+iT)((u+iT)+1)} \left(-\frac{\zeta'(u+iT)}{\zeta(u+iT)} - \frac{1}{(u+iT)-1} \right) \mathrm{d}u \\
&+ \int_{T}^{-T} x^{(1+iv)-1} \frac{1}{(1+iv)((1+iv)+1)} \left(-\frac{\zeta'(1+iv)}{\zeta(1+iv)} - \frac{1}{(1+iv)-1} \right) i\,\mathrm{d}v \\
&+ \int_{1}^{2} x^{(u-iT)-1} \frac{1}{(u-iT)((u-iT)+1)} \left(-\frac{\zeta'(u-iT)}{\zeta(u-iT)} - \frac{1}{(u-iT)-1} \right) \mathrm{d}u = 0.
\end{aligned}$$

Now, since

$$\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^{s-1}h(s)\mathrm{d}s$$

$$= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^{s-1}\frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}-\frac{1}{s-1}\right)\mathrm{d}s$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\left(\int_{2-iT}^{2+iT} x^{s-1}\frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}-\frac{1}{s-1}\right)\mathrm{d}s\right)$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\left(\int_{-T}^{T} x^{(2+iv)-1}\frac{1}{(2+iv)((2+iv)+1)}\left(-\frac{\zeta'(2+iv)}{\zeta(2+iv)}-\frac{1}{(2+iv)-1}\right)i\mathrm{d}v\right)$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\left(-\int_{2}^{1} x^{(u+iT)-1}\frac{1}{(u+iT)((u+iT)+1)}\left(-\frac{\zeta'(u+iT)}{\zeta(u+iT)}-\frac{1}{(u+iT)-1}\right)\mathrm{d}u\right.$$

$$-\int_{T}^{-T} x^{(1+iv)-1}\frac{1}{(1+iv)((1+iv)+1)}\left(-\frac{\zeta'(1+iv)}{\zeta(1+iv)}-\frac{1}{(1+iv)-1}\right)i\mathrm{d}v$$

$$\left.-\int_{1}^{2} x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(-\frac{\zeta'(u-iT)}{\zeta(u-iT)}-\frac{1}{(u-iT)-1}\right)\mathrm{d}u\right)$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{2}^{1} x^{(u+iT)-1}\frac{1}{(u+iT)((u+iT)+1)}\left(\frac{\zeta'(u+iT)}{\zeta(u+iT)}+\frac{1}{(u+iT)-1}\right)\mathrm{d}u$$

$$+\frac{1}{2\pi i}i\lim_{T\to\infty}\int_{T}^{-T} x^{(1+iv)-1}\frac{1}{(1+iv)((1+iv)+1)}\left(\frac{\zeta'(1+iv)}{\zeta(1+iv)}+\frac{1}{(1+iv)-1}\right)\mathrm{d}v$$

$$+\frac{1}{2\pi i}\lim_{T\to\infty}\int_{1}^{2} x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(\frac{\zeta'(u-iT)}{\zeta(u-iT)}+\frac{1}{(u-iT)-1}\right)\mathrm{d}u,$$

we have

$$\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1-\frac{1}{x}\right)^2$$

$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{2}^{1} x^{(u+iT)-1}\frac{1}{(u+iT)((u+iT)+1)}\left(\frac{\zeta'(u+iT)}{\zeta(u+iT)}+\frac{1}{(u+iT)-1}\right)\mathrm{d}u$$

$$+\frac{1}{2\pi i}i\lim_{T\to\infty}\int_T^{-T}x^{(1+iv)-1}\frac{1}{(1+iv)((1+iv)+1)}\left(\frac{\zeta'(1+iv)}{\zeta(1+iv)}+\frac{1}{(1+iv)-1}\right)\mathrm{d}v$$
$$+\frac{1}{2\pi i}\lim_{T\to\infty}\int_1^2 x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(\frac{\zeta'(u-iT)}{\zeta(u-iT)}+\frac{1}{(u-iT)-1}\right)\mathrm{d}u.$$

Problem 4.268 $\lim_{T\to\infty}\int_2^1 x^{(u+iT)-1}\frac{1}{(u+iT)((u+iT)+1)}\left(\frac{\zeta'(u+iT)}{\zeta(u+iT)}+\frac{1}{(u+iT)-1}\right)\mathrm{d}u=0.$ Similarly,

$$\lim_{T\to\infty}\int_1^2 x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(\frac{\zeta'(u-iT)}{\zeta(u-iT)}+\frac{1}{(u-iT)-1}\right)\mathrm{d}u=0.$$

(Solution Since

$$\begin{aligned}0&\le\left|\lim_{T\to\infty}\int_1^2 x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(\frac{\zeta'(u-iT)}{\zeta(u-iT)}+\frac{1}{(u-iT)-1}\right)\mathrm{d}u\right|\\
&=\lim_{T\to\infty}\left|\int_1^2 x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(\frac{\zeta'(u-iT)}{\zeta(u-iT)}+\frac{1}{(u-iT)-1}\right)\mathrm{d}u\right|\\
&\le\lim_{T\to\infty}\int_1^2\left|x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(\frac{\zeta'(u-iT)}{\zeta(u-iT)}+\frac{1}{(u-iT)-1}\right)\right|\mathrm{d}u\\
&=\lim_{T\to\infty}\int_1^2\left|\overline{\left(x^{(u-iT)-1}\frac{1}{(u-iT)((u-iT)+1)}\left(\frac{\zeta'(u-iT)}{\zeta(u-iT)}+\frac{1}{(u-iT)-1}\right)\right)}\right|\mathrm{d}u\\
&=\lim_{T\to\infty}\int_1^2\left|x^{(u+iT)-1}\frac{1}{(u+iT)((u+iT)+1)}\left(\frac{\zeta'(u+iT)}{\zeta(u+iT)}+\frac{1}{(u+iT)-1}\right)\right|\mathrm{d}u,\end{aligned}$$

and

$$\begin{aligned}0&\le\left|\lim_{T\to\infty}\int_2^1 x^{(u+iT)-1}\frac{1}{(u+iT)((u+iT)+1)}\left(\frac{\zeta'(u+iT)}{\zeta(u+iT)}+\frac{1}{(u+iT)-1}\right)\mathrm{d}u\right|\\
&=\left|-\lim_{T\to\infty}\int_1^2 x^{(u+iT)-1}\frac{1}{(u+iT)((u+iT)+1)}\left(\frac{\zeta'(u+iT)}{\zeta(u+iT)}+\frac{1}{(u+iT)-1}\right)\mathrm{d}u\right|\end{aligned}$$

$$= \left| \lim_{T\to\infty} \int_1^2 x^{(u+iT)-1} \frac{1}{(u+iT)((u+iT)+1)} \left(\frac{\zeta'(u+iT)}{\zeta(u+iT)} + \frac{1}{(u+iT)-1} \right) \mathrm{d}u \right|$$
$$= \lim_{T\to\infty} \int_1^2 \left| x^{(u+iT)-1} \frac{1}{(u+iT)((u+iT)+1)} \left(\frac{\zeta'(u+iT)}{\zeta(u+iT)} + \frac{1}{(u+iT)-1} \right) \right| \mathrm{d}u,$$

it suffices to show that

$$\lim_{T\to\infty} \int_1^2 \left| x^{(u+iT)-1} \frac{1}{(u+iT)((u+iT)+1)} \left(\frac{\zeta'(u+iT)}{\zeta(u+iT)} + \frac{1}{(u+iT)-1} \right) \right| \mathrm{d}u = 0.$$

Now, since

$$0 \le \lim_{T\to\infty} \int_1^2 \left| x^{(u+iT)-1} \frac{1}{(u+iT)((u+iT)+1)} \left(\frac{\zeta'(u+iT)}{\zeta(u+iT)} + \frac{1}{(u+iT)-1} \right) \right| \mathrm{d}u$$
$$= \lim_{T\to\infty} \int_1^2 \left| x^{(u+iT)-1} \right| \frac{1}{|u+iT||(u+iT)+1|} \left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} + \frac{1}{(u+iT)-1} \right| \mathrm{d}u$$
$$= \lim_{T\to\infty} \int_1^2 x^{u-1} \frac{1}{|u+iT||(u+iT)+1|} \left(\left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} \right| + \left| \frac{1}{(u+iT)-1} \right| \right) \mathrm{d}u$$
$$= \lim_{T\to\infty} \int_1^2 x^{u-1} \frac{1}{|u+iT||(u+iT)+1|} \left(\left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} \right| + \frac{1}{|(u+iT)-1|} \right) \mathrm{d}u$$
$$\le \lim_{T\to\infty} \int_1^2 x^{u-1} \frac{1}{T\cdot T} \left(\left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} \right| + \frac{1}{T} \right) \mathrm{d}u = \lim_{\substack{T\to\infty \\ T>1}} \int_1^2 x^{u-1} \frac{1}{T^2} \left(\left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} \right| + \frac{1}{T} \right) \mathrm{d}u$$
$$\le \lim_{\substack{T\to\infty \\ T>1}} \int_1^2 x^{u-1} \frac{1}{T^2} \left(\left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} \right| + \frac{1}{1} \right) \mathrm{d}u = \lim_{\substack{T\to\infty \\ T>e}} \int_1^2 x^{u-1} \frac{1}{T^2} \left(\left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} \right| + 1 \right) \mathrm{d}u,$$

it suffices to show that

$$\lim_{\substack{T\to\infty \\ T>e}} \int_1^2 x^{u-1} \frac{1}{T^2} \left(\left| \frac{\zeta'(u+iT)}{\zeta(u+iT)} \right| + 1 \right) \mathrm{d}u = 0.$$

By Conclusion 4.264, there exists $M > 0$ such that for every $\sigma \in [1, \infty)$, and for every $t \in [e, \infty)$,

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \le M(\ln t)^{10}.$$

Now, since

$$\begin{aligned}
0 \le & \lim_{\substack{T \to \infty \\ T > e}} \int_1^2 x^{u-1} \frac{1}{T^2}\left(\left|\frac{\zeta'(u+iT)}{\zeta(u+iT)}\right| + 1\right) \mathrm{d}u \\
\le & \lim_{\substack{T \to \infty \\ T > e}} \int_1^2 x^{u-1} \frac{1}{T^2}\left(M(\ln t)^{10} + 1\right) \mathrm{d}u \\
= & \lim_{\substack{T \to \infty \\ T \ge e}} \left(\frac{1}{T^2}\left(M(\ln T)^{10} + 1\right) \int_1^2 x^{u-1} \mathrm{d}u\right) \\
= & \lim_{\substack{T \to \infty \\ T \ge e}} \left(\frac{1}{T^2}\left(M(\ln T)^{10} + 1\right) \frac{1}{\ln x}\left(x^{2-1} - x^{1-1}\right)\right) \\
= & \frac{1}{\ln x}(x-1) \lim_{\substack{T \to \infty \\ T \ge e}} \left(\frac{1}{T^2}\left(M(\ln T)^{10} + 1\right)\right) \\
= & \frac{1}{\ln x}(x-1) \lim_{\substack{T \to \infty \\ T \ge e}} \frac{1}{T^2}\left(M(\ln T)^{10} + 0\right) \\
= & \frac{1}{\ln x}(x-1) M \lim_{\substack{T \to \infty \\ T \ge e}} \frac{(\ln T)^{10}}{T^2},
\end{aligned}$$

it suffices to show that

$$\lim_{\substack{T \to \infty \\ T \geq e}} \frac{(\ln T)^{10}}{T^2} = 0.$$

$$\text{LHS} = \lim_{\substack{T \to \infty \\ T \geq e}} \frac{(\ln T)^{10}}{T^2} = \lim_{\substack{T \to \infty \\ T \geq e}} \frac{10(\ln T)^9 \frac{1}{T}}{2T} = \frac{10}{2} \lim_{\substack{T \to \infty \\ T \geq e}} \frac{(\ln T)^9}{T^2}$$
$$= \frac{10 \cdot 9}{2^2} \lim_{\substack{T \to \infty \\ T \geq e}} \frac{(\ln T)^8}{T^2} = \cdots = \frac{10 \cdot 9 \cdots 1}{2^{10}} \lim_{\substack{T \to \infty \\ T \geq e}} \frac{1}{T^2} = 0 = \text{RHS}.$$

∎)

It follows that for every $x \geq 1$,

$$\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2$$
$$= 0 + \frac{1}{2\pi i} i \lim_{T\to\infty} \int_T^{-T} x^{(1+iv)-1} \frac{1}{(1+iv)((1+iv)+1)} \left(\frac{\zeta'(1+iv)}{\zeta(1+iv)} + \frac{1}{(1+iv)-1}\right)\mathrm{d}v + 0$$
$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{[1+iT,1-iT]} x^{s-1} \frac{1}{s(s+1)} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right)\mathrm{d}s$$
$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{[1+iT,1-iT]} x^{s-1}(-h(s))\mathrm{d}s$$
$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{-T}^{T} x^{(1+iv)-1} h(1+iv) \cdot i\mathrm{d}v$$
$$= \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} x^{it} h(1+it)\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(1+it) e^{(\ln x)it}\mathrm{d}t,$$

and hence for every $x \geq 1$,

$$\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{(\ln x)it}\mathrm{d}t.$$

Conclusion 4.269 Let

$$h : s \mapsto \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right).$$

Then, for every $x \geq 1$,

1. $\frac{1}{x^2}\int_1^x \psi(t)\mathrm{d}t - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{(\ln x)it}\mathrm{d}t$,
2. $$\begin{aligned}&\int_{-\infty}^{-e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t\\ &+\int_{-e}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t\\ &+\int_{e}^{\infty}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t\\ &=\underbrace{\int_{-\infty}^{\infty}|h(1+it)|\mathrm{d}t \in (-\infty,\infty)}.\end{aligned}$$

Proof of the remaining part (2) It suffices to show that

I. $\int_e^{\infty}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t \in (-\infty,\infty)$,

II. $\int_{-\infty}^{-e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t \in (-\infty,\infty)$,

III. $\int_{-e}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t \in (-\infty,\infty)$.

By Conclusion 4.249, there exists $M > 0$ such that for every $\sigma \in [1,\infty)$, and for every $t \in [e,\infty)$,

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq M(\ln t)^{10}.$$

For I: Since,

$$\int_e^\infty \left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$= \int_e^\infty \frac{1}{|1+it||(1+it)+1|}\left|\frac{\zeta'(1+it)}{\zeta(1+it)}+\frac{1}{(1+it)-1}\right|\mathrm{d}t$$

$$\le \int_e^\infty \frac{1}{|1+it||(1+it)+1|}\left(\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right|+\left|\frac{1}{(1+it)-1}\right|\right)\mathrm{d}t$$

$$\le \int_e^\infty \frac{1}{t\cdot t}\left(\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right|+\frac{1}{t}\right)\mathrm{d}t \le \int_e^\infty \frac{1}{t\cdot t}\left(\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right|+1\right)\mathrm{d}t$$

$$\le \int_e^\infty \frac{1}{t^2}\left(M(\ln t)^{10}+1\right)\mathrm{d}t = M\int_e^\infty \frac{1}{t^2}\cdot(\ln t)^{10}\mathrm{d}t + \int_e^\infty \frac{1}{t^2}\mathrm{d}t$$

$$= M\int_e^\infty \frac{1}{t^2}\cdot(\ln t)^{10}\mathrm{d}t - \frac{1}{t}\bigg|_e^\infty = M\int_e^\infty (\ln t)^{10}\cdot\frac{1}{t^2}\mathrm{d}t - \left(0-\frac{1}{e}\right),$$

it remains to show that

$$\int_e^\infty \frac{1}{t^2}\cdot(\ln t)^{10}\mathrm{d}t \in (-\infty,\infty).$$

Since

$$\int_e^\infty (\ln t)^{10}\cdot\frac{1}{t^2}\mathrm{d}t = -\frac{(\ln t)^{10}}{t}\bigg|_e^\infty + 10\int_e^\infty (\ln t)^9\cdot\frac{1}{t^2}\mathrm{d}t$$

$$= -\left(\lim_{t\to\infty}\frac{(\ln t)^{10}}{t}-\frac{1}{e}\right) + 10\int_e^\infty (\ln t)^9\cdot\frac{1}{t^2}\mathrm{d}t$$

$$= -\left(0-\frac{1}{e}\right) + 10\int_e^\infty (\ln t)^9\cdot\frac{1}{t^2}\mathrm{d}t,$$

it is enough to show that

$$\int_e^\infty (\ln t)^9 \cdot \frac{1}{t^2}\,\mathrm{d}t \in (-\infty, \infty).$$

Similarly, it is enough to show that

$$\int_e^\infty (\ln t)^8 \cdot \frac{1}{t^2}\,\mathrm{d}t \in (-\infty, \infty), \text{etc.}$$

Finally, it is enough to show that

$$\int_e^\infty \frac{1}{t^2}\,\mathrm{d}t \in (-\infty, \infty).$$

Since $\int_e^\infty \frac{1}{t^2}\,\mathrm{d}t = -\frac{1}{t}\Big|_e^\infty = -\left(0 - \frac{1}{e}\right) \in (-\infty, \infty)$, we have proved I.
For II: Here, by I,

$$\begin{aligned}
&\int_{-\infty}^{-e} \left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t \\
&\quad = \int_{\infty}^{e} \left|\frac{1}{(1-it)((1-it)+1)}\left(-\frac{\zeta'(1-it)}{\zeta(1-it)} - \frac{1}{(1-it)-1}\right)\right|\mathrm{d}t \\
&\quad = -\int_{e}^{\infty} \left|\frac{1}{(1-it)((1-it)+1)}\left(-\frac{\zeta'(1-it)}{\zeta(1-it)} - \frac{1}{(1-it)-1}\right)\right|\mathrm{d}t \\
&\quad = -\int_{e}^{\infty} \left|\left(\frac{1}{(1-it)((1-it)+1)}\left(-\frac{\zeta'(1-it)}{\zeta(1-it)} - \frac{1}{(1-it)-1}\right)\right)^{-}\right|\mathrm{d}t \\
&\quad = -\int_{e}^{\infty} \left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)} - \frac{1}{(1+it)-1}\right)\right|\mathrm{d}t \in (-\infty, \infty),
\end{aligned}$$

So II is proved.
For III:

$$\int_{-e}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$=\int_{-e}^{0}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$+\int_{0}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$=\int_{e}^{0}\left|\frac{1}{(1-it)((1-it)+1)}\left(-\frac{\zeta'(1-it)}{\zeta(1-it)}-\frac{1}{(1-it)-1}\right)\right|\mathrm{d}t$$

$$+\int_{0}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$=-\int_{0}^{e}\left|\frac{1}{(1-it)((1-it)+1)}\left(-\frac{\zeta'(1-it)}{\zeta(1-it)}-\frac{1}{(1-it)-1}\right)\right|\mathrm{d}t$$

$$+\int_{0}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$=-\int_{0}^{e}\left|\left(\frac{1}{(1-it)((1-it)+1)}\left(-\frac{\zeta'(1-it)}{\zeta(1-it)}-\frac{1}{(1-it)-1}\right)\right)^{-}\right|\mathrm{d}t$$

$$+\int_{0}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$=-\int_{0}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t$$

$$+\int_{0}^{e}\left|\frac{1}{(1+it)((1+it)+1)}\left(-\frac{\zeta'(1+it)}{\zeta(1+it)}-\frac{1}{(1+it)-1}\right)\right|\mathrm{d}t=0\in(-\infty,\infty).$$

So III is proved. ■

Problem 4.270 Let $f : \mathbb{R} \to \mathbb{C}$ be any Lebesgue measurable function. Let $\alpha, \beta \in \mathbb{R}$, and $\alpha \neq 0$. Then,

$$\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right) = 0.$$

(Solution Case I: when f is a characteristic function, say $\chi_{[a,b]}$, for some real numbers a, b satisfying $a<b$.

Since

$$\begin{aligned}
0 &\leq \left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\right| = \left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} \chi_{[a,b]}(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\right| \\
&= \left|\lim_{\alpha\to\infty}\left(\int_{a}^{b} 1\sin(\alpha t+\beta)\mathrm{d}t\right)\right| = \lim_{\alpha\to\infty}\left|\int_{a}^{b}\sin(\alpha t+\beta)\mathrm{d}t\right| \\
&= \lim_{\alpha\to\infty}\left|\frac{1}{\alpha}(\cos(\alpha a+\beta)-\cos(\alpha b+\beta))\right| = \lim_{\alpha\to\infty}\frac{1}{|\alpha|}|\cos(\alpha a+\beta)-\cos(\alpha b+\beta)| \\
&\leq \lim_{\alpha\to\infty}\frac{1}{|\alpha|}(|\cos(\alpha a+\beta)|+|\cos(\alpha b+\beta)|) \leq \lim_{\alpha\to\infty}\frac{1}{|\alpha|}(1+1) = 2\lim_{\alpha\to\infty}\frac{1}{|\alpha|} = 2\cdot 0 = 0,
\end{aligned}$$

we have

$$\left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\right| = 0,$$

and hence

$$\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right) = 0.$$

Case II: when f is a step function. There exist real numbers a, b satisfying $a<b$, a partition $\{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$, and real numbers $c_1, \ldots, c_n$ such that

$$f = c_1\chi_{[x_0,x_1)} + \cdots + c_n\chi_{[x_{n-1},x_n)}.$$

$$\begin{aligned}
\text{LHS} &= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\\
&= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} \left(c_1\chi_{[x_0,x_1)}+\cdots+c_n\chi_{[x_{n-1},x_n)}\right)(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\\
&= \lim_{\alpha\to\infty}\left(c_1\int_{-\infty}^{\infty} \chi_{[x_0,x_1)}(t)\sin(\alpha t+\beta)\mathrm{d}t+\cdots+c_n\int_{-\infty}^{\infty} \chi_{[x_{n-1},x_n)}(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\\
&= c_1\lim_{\alpha\to\infty}\int_{-\infty}^{\infty} \chi_{[x_0,x_1)}(t)\sin(\alpha t+\beta)\mathrm{d}t+\cdots+c_n\lim_{\alpha\to\infty}\int_{-\infty}^{\infty} \chi_{[x_{n-1},x_n)}(t)\sin(\alpha t+\beta)\mathrm{d}t\\
&= c_1\cdot 0+\cdots+c_n\cdot 0=0=\text{RHS}.
\end{aligned}$$

Case III: when f is any real-valued Lebesgue integrable function. Let us take any $\varepsilon>0$.

There exists a step function s such that $f-s$ is a Lebesgue integrable function, and $\int_{-\infty}^{\infty}|f(t)-s(t)|\mathrm{d}t<\varepsilon$. Now,

$$\begin{aligned}
0 &\le \left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\right|\\
&= \left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} ((f(t)-s(t))\sin(\alpha t+\beta)+s(t)\sin(\alpha t+\beta))\mathrm{d}t\right)\right|\\
&= \left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} (f(t)-s(t))\sin(\alpha t+\beta)\mathrm{d}t+\int_{-\infty}^{\infty} s(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\right|\\
&= \left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} (f(t)-s(t))\sin(\alpha t+\beta)\mathrm{d}t\right)+\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} s(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\right|\\
&= \left|\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} (f(t)-s(t))\sin(\alpha t+\beta)\mathrm{d}t\right)+0\right| = \lim_{\alpha\to\infty}\left|\int_{-\infty}^{\infty} (f(t)-s(t))\sin(\alpha t+\beta)\mathrm{d}t\right|\\
&\le \lim_{\alpha\to\infty}\int_{-\infty}^{\infty} |(f(t)-s(t))\sin(\alpha t+\beta)|\mathrm{d}t = \lim_{\alpha\to\infty}\int_{-\infty}^{\infty} |(f(t)-s(t))||\sin(\alpha t+\beta)|\mathrm{d}t\\
&\le \lim_{\alpha\to\infty}\int_{-\infty}^{\infty} |(f(t)-s(t))|\cdot 1\mathrm{d}t \le \lim_{\alpha\to\infty}\varepsilon=\varepsilon.
\end{aligned}$$

Since ε is arbitrary,

$$\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right)=0.$$

Case IV: when f is any complex-valued Lebesgue integrable function.
Let $f=u+iv$, where u,v are real-valued Lebesgue integrable functions.

$$\begin{aligned}
\text{LHS} &= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right)=\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty}(u(t)+iv(t))\sin(\alpha t+\beta)\mathrm{d}t\right)\\
&= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} u(t)\sin(\alpha t+\beta)\mathrm{d}t+i\int_{-\infty}^{\infty} v(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\\
&= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} u(t)\sin(\alpha t+\beta)\mathrm{d}t\right)+i\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} v(t)\sin(\alpha t+\beta)\mathrm{d}t\right)\\
&= 0+i\cdot 0=0=\text{RHS}.
\end{aligned}$$

∎)

Conclusion 4.271 Let $f:\mathbb{R}\to\mathbb{C}$ be any Lebesgue measurable function. Let $\alpha,\beta\in\mathbb{R}$, and $\alpha\neq 0$. Then

1. $\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(\alpha t+\beta)\mathrm{d}t\right)=0$,
2. $\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)e^{i\alpha t}\mathrm{d}t\right)=0$.

Proof of the remaining part (2)

$$\begin{aligned}
\text{LHS} &= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)e^{i\alpha t}\mathrm{d}t\right)=\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)(\cos\alpha t+i\sin\alpha t)\mathrm{d}t\right)\\
&= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,\mathrm{d}t+i\int_{-\infty}^{\infty} f(t)\sin\alpha t\,\mathrm{d}t\right)\\
&= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,\mathrm{d}t\right)+i\lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,\mathrm{d}t\right)\\
&= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,\mathrm{d}t\right)+i\cdot 0\\
&= \lim_{\alpha\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin\left(\alpha t+\frac{\pi}{2}\right)\mathrm{d}t\right)=0=\text{RHS}.
\end{aligned}$$

∎)

Note 4.272

Problem 4.273 $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$. In short, $\psi(x) \sim x$ as $x \to \infty$.

(Solution Let

$$h : s \mapsto \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right).$$

By Conclusion 4.265, for every $x \geq 1$,

$$\frac{\int_1^x \psi(t)\mathrm{d}t}{x^2} = \frac{1}{2}\left(1-\frac{1}{x}\right)^2 + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} h(1+it)e^{i(\ln x)t}\mathrm{d}t.$$

Next, since $\lim_{x\to\infty}\ln x = \infty$, by Conclusion 4.265,

$$\begin{aligned} 2\pi\left(\lim_{x\to\infty}\frac{\int_1^x \psi(t)\mathrm{d}t}{x^2} - \frac{1}{2}(1-0)^2\right) &= \lim_{x\to\infty} 2\pi\left(\frac{\int_1^x \psi(t)\mathrm{d}t}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2\right) \\ &= \underbrace{\lim_{x\to\infty}\left(\int_{-\infty}^{\infty} h(1+it)e^{i(\ln x)t}\mathrm{d}t\right) = 0}, \end{aligned}$$

we have

$$\frac{1}{2}\lim_{x\to\infty}\frac{\psi(x)}{x} = \lim_{x\to\infty}\frac{\psi(x)}{2x} = \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x \psi(t)\mathrm{d}t\right)}{\frac{\mathrm{d}}{\mathrm{d}x}(x^2)} = \lim_{x\to\infty}\frac{\int_1^x \psi(t)\mathrm{d}t}{x^2} = \frac{1}{2},$$

and hence $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$. ■)

Conclusion 4.274 $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$.

Definition Let

$$a : n \to \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

be an arithmetic function. Here,

$$\vartheta : x \mapsto \sum_{\substack{p \leq x \\ p\, prime}} \ln p \left(= \sum_{1 \leq n \leq x} (a(n))(\ln n) \right)$$

is the 'sum function' of $n \mapsto (a(n))(\ln n)$.

The sum function

$$x \mapsto \begin{cases} \sum_{n \leq x} a(n) & \text{if } 1 \leq x \\ 0 & \text{if } x < 1 \end{cases}$$

of $n \mapsto a(n)$ is denoted by π. Thus, $\pi(x)$ is the number of primes less than or equal to x

Example $\pi(15.3) = (\text{number of elements in } \{2, 3, 5, 7, 11, 13\}) = 6$.

By Conclusion 4.167, for sufficiently large x,

$$\begin{aligned}(\vartheta(x)) &= \sum_{1 < n \leq x} a(n) \cdot \ln n = -\int_1^x \pi(t) \frac{1}{t} \mathrm{d}t + \pi(x) \ln x - \pi(1) \ln 1 \\ &= -\int_1^x \frac{\pi(t)}{t} \mathrm{d}t + \pi(x) \ln x - 0 \cdot 0 \\ &= -\int_1^x \frac{\pi(t)}{t} \mathrm{d}t + \pi(x) \ln x = -\int_2^x \frac{\pi(t)}{t} \mathrm{d}t + \pi(x) \ln x,\end{aligned}$$

so, for sufficiently large x,

$$\vartheta(x) = -\int_2^x \frac{\pi(t)}{t} \mathrm{d}t + \pi(x) \ln x.$$

Since ϑ is the 'sum function' of

$$n \mapsto (a(n))(\ln n),$$

by Conclusion 4.167, for sufficiently large x, we have

$$\begin{aligned}\pi(x) &= \sum_{1<n\le x} a(n) = \sum_{1<n\le x}(a(n)\cdot \ln n)\frac{1}{\ln n} = \sum_{\frac{3}{2}<n\le x}(a(n)\cdot \ln n)\frac{1}{\ln n}\\ &= -\int_{\frac{3}{2}}^{x}\vartheta(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\ln t}\right)\right)\mathrm{d}t+\vartheta(x)\frac{1}{\ln x}-\vartheta\left(\frac{3}{2}\right)\frac{1}{\ln\frac{3}{2}}\\ &= -\int_{\frac{3}{2}}^{x}\vartheta(t)\frac{-1}{(\ln t)^2 t}\mathrm{d}t+\vartheta(x)\frac{1}{\ln x}-0\cdot\frac{1}{\ln\frac{3}{2}}\\ &= \int_{\frac{3}{2}}^{x}\frac{\vartheta(t)}{(\ln t)^2 t}\mathrm{d}t+\frac{\vartheta(x)}{\ln x} = \int_{2}^{x}\frac{\vartheta(t)}{(\ln t)^2 t}\mathrm{d}t+\frac{\vartheta(x)}{\ln x},\end{aligned}$$

and hence, for sufficiently large x,

$$\pi(x) = \int_{2}^{x}\frac{\vartheta(t)}{(\ln t)^2 t}\mathrm{d}t+\frac{\vartheta(x)}{\ln x}.$$

Conclusion 4.275 For sufficiently large x,

1. $\vartheta(x) = -\int_2^x \frac{\pi(t)}{t}\mathrm{d}t+\pi(x)\ln x$,
2. $\pi(x) = \int_2^x \frac{\vartheta(t)}{(\ln t)^2 t}\mathrm{d}t+\frac{\vartheta(x)}{\ln x}$.

Problem 4.276 $\lim_{x\to\infty}\frac{(\ln x)\pi(x)}{x} = 1$.

(**Solution** By Conclusion 4.274, $\lim_{x\to\infty}\frac{\psi(x)}{x} = 1$, and hence, by Conclusions 4.234 and 4.235,

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{(\ln x)\pi(x)}{x}-\frac{\ln x}{x}\int_2^x\frac{\vartheta(t)}{(\ln t)^2 t}\mathrm{d}t\right) &= \lim_{x\to\infty}\frac{\ln x}{x}\left(\pi(x)-\int_2^x\frac{\vartheta(t)}{(\ln t)^2 t}\mathrm{d}t\right)\\ &= \lim_{x\to\infty}\frac{\ln x}{x}\frac{\vartheta(x)}{\ln x} = \underbrace{\lim_{x\to\infty}\frac{\vartheta(x)}{x} = 1}.\end{aligned}$$

Now, it suffices to show that

$$\lim_{x\to\infty}\frac{(\ln x)}{x}\int_2^x\frac{\vartheta(t)}{(\ln t)^2 t}\mathrm{d}t = 0.$$

Since $\lim_{x\to\infty}\frac{\vartheta(x)}{x} = 1$, there exists $M_0 > 0$ such that for every $x \ge 2$, $(\vartheta(x) =)|\vartheta(x)| \le M_0 x$. Now,

$$
\begin{aligned}
0 &\le \lim_{x\to\infty}\frac{(\ln x)}{x}\int_2^x \frac{1}{(\ln t)^2 t}\cdot\vartheta(t)\mathrm{d}t \le \lim_{x\to\infty}\frac{(\ln x)}{x}\int_2^x \frac{1}{(\ln t)^2 t}\cdot M_0 t\mathrm{d}t \\
&= M_0\lim_{x\to\infty}\frac{(\ln x)}{x}\int_2^x \frac{1}{(\ln t)^2}\mathrm{d}t = M_0\lim_{x\to\infty}\frac{(\ln x)}{x}\left(\int_2^{\sqrt{x}}\frac{1}{(\ln t)^2}\mathrm{d}t + \int_{\sqrt{x}}^x \frac{1}{(\ln t)^2}\mathrm{d}t\right) \\
&= M_0\lim_{x\to\infty}\frac{(\ln x)}{x}\left(\frac{1}{(\ln 2)^2}\left(\sqrt{x}-2\right)+\frac{1}{(\ln\sqrt{x})^2}\left(x-\sqrt{x}\right)\right) \\
&\le M_0\lim_{x\to\infty}\left(\frac{(\ln x)}{x}\left(\frac{1}{(\ln 2)^2}\sqrt{x}+\frac{1}{(\ln\sqrt{x})^2}\left(x-\sqrt{x}\right)\right)\right) \\
&= M_0\left(\frac{1}{(\ln 2)^2}\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}+\lim_{x\to\infty}\frac{4}{x\ln x}\left(x-\sqrt{x}\right)\right) \\
&= M_0\left(\frac{1}{(\ln 2)^2}\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}+4\left(\lim_{x\to\infty}\frac{1}{\ln x}-\lim_{x\to\infty}\frac{1}{\sqrt{x}\ln x}\right)\right) \\
&= M_0\left(\frac{1}{(\ln 2)^2}\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}+4(0-0)\right) \\
&= \frac{M_0}{(\ln 2)^2}\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \frac{M_0}{(\ln 2)^2}\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \frac{2M_0}{(\ln 2)^2}\lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0,
\end{aligned}
$$

so $\lim_{x\to\infty}\frac{(\ln x)}{x}\int_2^x \frac{1}{(\ln t)^2 t}\cdot\vartheta(t)\mathrm{d}t = 0$. ■)

Conclusion 4.277 $\lim_{x\to\infty}\frac{(\ln x)\pi(x)}{x} = 1$. In other words, $\pi(x)\sim\frac{x}{\ln x}$ as $x\to\infty$.
This result is known as the **Prime Number Theorem** (PNT).

4.16 Picard's Theorem

Note 4.278 Let Ω be a region. Let Ω be simply connected. Let $f\in H(\Omega)$. Let $(-1)\notin f(\Omega)$, and $1\notin f(\Omega)$.

Let

$$h : z\mapsto 1-(f(z))^2(=(1+f(z))(1-f(z)))$$

be a function from Ω to $\mathbb{C}$. Now, since $f\in H(\Omega)$, we have $h\in H(\Omega)$. Since $(-1)\notin f(\Omega)$, and $1\notin f(\Omega)$, by the definition of h, $0\notin h(\Omega)$. Now, since $h\in H(\Omega)$, $\frac{1}{h}\in H(\Omega)$. It follows, by Theorem 2.188 ($(b)\Rightarrow(i)$), that there exists $g\in H(\Omega)$ such that $(1-f^2=)h=g^2$, and hence $((f+ig)(f-ig)=)f^2+g^2=$

$1(\neq 0)$. Thus, $(f+ig) \in H(\Omega)$, and $0 \notin (f+ig)(\Omega)$. Now, by Theorem 2.188 $((b) \Rightarrow (h))$, there exists $F \in H(\Omega)$ such that $(f+ig) = e^{iF}$. It follows that for every $z \in \Omega$,

$$\begin{aligned}\cos(F(z)) &= \frac{1}{2}\left(e^{iF(z)} + e^{-iF(z)}\right) = \frac{1}{2}\left(e^{iF(z)} + \frac{1}{e^{iF(z)}}\right)\\ &= \frac{1}{2}\left((f(z)+ig(z)) + \frac{1}{(f(z)+ig(z))}\right)\\ &= \frac{1}{2}((f(z)+ig(z)) + (f(z)-ig(z))) = f(z),\end{aligned}$$

and hence for every $z \in \Omega$, $f(z) = \cos(F(z))$.

Conclusion 4.279 Let Ω be a region. Let Ω be simply connected. Let $f \in H(\Omega)$. Let $(-1) \notin f(\Omega)$, and $1 \notin f(\Omega)$. Then there exists $F \in H(\Omega)$ such that for every $z \in \Omega$, $f(z) = \cos(F(z))$.

Let Ω be a region. Let Ω be simply connected. Let $f \in H(\Omega)$. Let $0 \notin f(\Omega)$, and $1 \notin f(\Omega)$. Let $g : z \mapsto (2f(z) - 1)$ be a function from Ω to $\mathbb{C}$.

Now, since $f \in H(\Omega)$, we have $g \in H(\Omega)$. Since $0 \notin f(\Omega)$, by the definition of g, $(-1) \notin g(\Omega)$. Since $1 \notin f(\Omega)$, by the definition of g, $1 \notin g(\Omega)$. Now, by Conclusion 4.279, there exists $F \in H(\Omega)$ such that for every $z \in \Omega$,

$$(2f(z) - 1 =)g(z) = \cos(\pi F(z)).$$

Thus, for every $z \in \Omega$,

$$f(z) = \frac{1}{2}(1 + \cos(\pi F(z))).$$

Since

$$(-1) \notin g(\Omega)(= \{g(z) : z \in \Omega\} = \{\cos(\pi F(z)) : z \in \Omega\}),$$

for every $z \in \Omega$, $\cos(F(z) \cdot \pi) \neq -1$, and hence for every $z \in \Omega$,

$$F(z) \notin \{\pm 1, \pm 3, \pm 5, \ldots\}.$$

Since

$$1 \notin g(\Omega)(= \{g(z) : z \in \Omega\} = \{\cos(\pi F(z)) : z \in \Omega\}),$$

for every $z \in \Omega$, $\cos(F(z) \cdot \pi) \neq 1$, and hence for every $z \in \Omega$,

$$F(z) \notin \{0, \pm 2, \pm 4, \pm 6, \ldots\}.$$

Thus, $F(\Omega)$ contains no integer.

Now, since $F \in H(\Omega)$, and $0 \notin F(\Omega)$, we have $\frac{1}{F} \in H(\Omega)$. Next, since $(-1) \notin F(\Omega)$, and $1 \notin F(\Omega)$, by Conclusion 4.279, there exists $k \in H(\Omega)$ such that for every $z \in \Omega$, $F(z) = \cos(\pi k(z))$. Now, for every $z \in \Omega$,

$$f(z) = \frac{1}{2}(1 + \cos(\pi \cos(\pi k(z)))).$$

Problem 4.280 $\left\{m + i\frac{1}{\pi}\ln\left(n + \sqrt{n^2 - 1}\right) : m \in \mathbb{Z} \text{ and } n \in \mathbb{N}\right\} \cap (k(\Omega)) = \emptyset.$

(Solution If not, otherwise suppose that there exist $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and $z \in \Omega$ such that

$$k(z) = m + i\frac{1}{\pi}\ln\left(n + \sqrt{n^2 - 1}\right).$$

We have to arrive at a contradiction. It follows that

$$\begin{aligned}
(\{0,1\} \not\ni) f(z) &= \frac{1}{2}\left(1 + \cos\left(\pi\cos\left(\pi\left(m + i\frac{1}{\pi}\ln\left(n + \sqrt{n^2-1}\right)\right)\right)\right)\right)\\
&= \frac{1}{2}\left(1 + \cos\left(\pi\cos\left(m\pi + i\ln\left(n + \sqrt{n^2-1}\right)\right)\right)\right)\\
&= \frac{1}{2}\left(1 + \cos\left(\pi\cdot(-1)^m\cos\left(i\ln\left(n + \sqrt{n^2-1}\right)\right)\right)\right)\\
&= \frac{1}{2}\left(1 + \cos\left(\pi\cdot\cos\left(i\ln\left(n + \sqrt{n^2-1}\right)\right)\right)\right)\\
&= \frac{1}{2}\left(1 + \cos\left(\pi\cdot\frac{1}{2}\left(\exp\left(i\cdot i\ln\left(n + \sqrt{n^2-1}\right)\right)\right.\right.\right.\\
&\quad \left.\left.\left. + \exp\left(-i\cdot i\ln\left(n + \sqrt{n^2-1}\right)\right)\right)\right)\right)\\
&= \frac{1}{2}\left(1 + \cos\left(\pi\cdot\frac{1}{2}\left(\exp\left(-\ln\left(n + \sqrt{n^2-1}\right)\right)\right.\right.\right.\\
&\quad \left.\left.\left. + \exp\left(\ln\left(n + \sqrt{n^2-1}\right)\right)\right)\right)\right)\\
&= \frac{1}{2}\left(1 + \cos\left(\pi\cdot\frac{1}{2}\left(\frac{1}{n + \sqrt{n^2-1}} + \left(n + \sqrt{n^2-1}\right)\right)\right)\right)\\
&= \frac{1}{2}\left(1 + \cos\left(\pi\cdot\frac{1}{2}\left(\left(n - \sqrt{n^2-1}\right) + \left(n + \sqrt{n^2-1}\right)\right)\right)\right)\\
&= \frac{1}{2}(1 + \cos(\pi\cdot n)) = \frac{1}{2}(1 + (-1)^n)(\in \{0,1\}).
\end{aligned}$$

This is a contradiction. ■)

Observe that points of

$$\left\{m+i\frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right) : m \in \mathbb{Z}\,\text{and}\,n \in \mathbb{N}\right\}$$

constitute a rectangular-shaped 'grid' type structure on the upper-half of the complex plane. Here, the horizontal length of each grid is 1. Since for fixed $n \in \mathbb{N}$,

$$\begin{aligned}
0 &\le \frac{1}{\pi}\ln\left((n+1)+\sqrt{(n+1)^2-1}\right)-\frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\\
&= \frac{1}{\pi}\ln\frac{(n+1)+\sqrt{(n+1)^2-1}}{n+\sqrt{n^2-1}} = \frac{1}{\pi}\ln\frac{(n+1)+\sqrt{n^2+2n}}{n+\sqrt{n^2-1}} = \frac{1}{\pi}\ln\frac{\left(1+\frac{1}{n}\right)+\sqrt{1+\frac{2}{n}}}{1+\sqrt{1-\frac{1}{n^2}}}\\
&\le \frac{1}{\pi}\ln\frac{\left(1+\frac{1}{1}\right)+\sqrt{1+\frac{2}{1}}}{1+\sqrt{1-\frac{1}{n^2}}} \le \frac{1}{\pi}\ln\frac{\left(1+\frac{1}{1}\right)+\sqrt{1+\frac{2}{1}}}{1} = \frac{1}{\pi}\ln\left(2+\sqrt{3}\right)\\
&< \frac{1}{\pi}\ln\left(2+\sqrt{4}\right) = \frac{2}{\pi}\cdot\ln 2 < 1\cdot\ln 2 < \ln e = 1,
\end{aligned}$$

the vertical length of each grid is strictly less than 1.

Problem 4.281 No open disk of radius 1 is contained in $k(\Omega)$.

(Solution If not, otherwise suppose that there exists $a \in \Omega$ such that $D(k(a);1) \subset k(\Omega)$. We have to arrive at a contradiction.

Since the horizontal length of each grid is 1, and the vertical length of each grid strictly less than 1, there exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $|\mathrm{Re}(k(a)) - m| \le \frac{1}{2}$, and

$$\left|\mathrm{Im}(k(a)) - \frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\right| < \frac{1}{2}.$$

It follows that

$$\begin{aligned}
&\left|k(a) - \left(m+i\frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\right)\right|\\
&\quad\le |\mathrm{Re}(k(a)) - m| + \left|\mathrm{Im}(k(a)) - \frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\right|\\
&\quad\le \frac{1}{2} + \left|\mathrm{Im}(k(a)) - \frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\right| < \frac{1}{2}+\frac{1}{2} = 1,
\end{aligned}$$

and hence

$$\left|k(a) - \left(m+i\frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\right)\right| < 1.$$

It follows that

$$\left(m+i\frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\right)\in D(k(a);1)(\subset k(\Omega)),$$

and hence

$$\left(m+i\frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right)\right)\in k(\Omega).$$

This shows that

$$\left\{m+i\frac{1}{\pi}\ln\left(n+\sqrt{n^2-1}\right) : m\in\mathbb{Z}\,\text{and}\,n\in\mathbb{N}\right\}\cap(k(\Omega))\neq\emptyset.$$

This contradicts Problem 3.280. ■)

Conclusion 4.282 Let Ω be a region. Let Ω be simply connected. Let $f\in H(\Omega)$. Let $0\notin f(\Omega)$, and $1\notin f(\Omega)$. Then there exists $k\in H(\Omega)$ such that

1. for every $z\in\Omega$, $f(z)=\frac{1}{2}(1+\cos(\pi\cos(\pi k(z))))$,
2. no open disk of radius 1 is contained in $k(\Omega)$.

Let $f\in H(D(0;1))$. Let $f(0)=0$, and $f'(0)=1$. Let M be a positive real number such that for every $z\in D(0;1)$, $|f(z)|\leq M$.

Problem 4.283 $1\leq M$, and $D\left(0;\frac{1}{6M}\right)\subset f(D(0;1))$.

(Solution Since $f\in H(D(0;1))$, by Conclusion 4.116, there exist complex numbers $c_0,c_1,c_2,\ldots$ such that for every $z\in D(0;1)$, $f(z)=c_0+c_1z+c_2z^2+\cdots$. It follows that $(0=)f(0)=c_0$, and hence, for every $z\in D(0;1)$,

$$f(z)=c_1z+c_2z^2+c_3z^3+\cdots.$$

Now, by Lemma 1.60,

$$c_1=\frac{f'(0)}{1!},c_2=\frac{f''(0)}{2!},c_3=\frac{f'''(0)}{3!},\ldots.$$

Since, $(1=)f'(0)=c_1$, for every $z\in D(0;1)$,

$$f(z)=z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+\cdots.$$

By Conclusion 1.166 $1=|1|=\underbrace{|f'(0)|\leq\frac{M(1!)}{1^1}}=M$, so, $1\leq M$. Again, by Conclusion 1.166,

$$|f''(0)| \leq \frac{M(2!)}{1^2}(= M(2!)), |f'''(0)| \leq \frac{M(3!)}{1^3}(= M(3!)), \text{etc.}$$

It remains to show that $D\left(0; \frac{1}{6M}\right) \subset f(D(0;1))$.

For this purpose, let us take any $w_0 \in D\left(0; \frac{1}{6M}\right)$, that is $|w_0| \leq \frac{1}{6M}$. We have to show that $w_0 \in f(D(0;1))$. It suffices to show that the function $g : z \mapsto (f(z) - w_0)$ from $D(0;1)$ to $\mathbb{C}$ has a zero in $D(0;1)$. Since $f(0) = 0$, f has a zero in $D\left(0; \frac{1}{4M}\right)(\subset D(0;1))$. It suffices to show that f and g have the same number of zeros in $D\left(0; \frac{1}{4M}\right)$.

By Theorem 4.229, it is enough to show that for every $a \in \left\{z : |z| = \frac{1}{4M}\right\}(\subset D(0;1))$,

1. $|f(a)| > 0$,
2. $|f(a) - g(a)| < |f(a)|$.

For this purpose, let us take any complex number a satisfying $|a| = \frac{1}{4M}$. We have to show that $|f(a)| > 0$, and

$$|f(a) - g(a)| < |f(a)|.$$

Since

$$f(a) = a + \frac{f''(0)}{2!}a^2 + \frac{f'''(0)}{3!}a^3 + \cdots,$$

we have

$$\begin{aligned}
|f(a)| &= \left|a - \left(-\left(\frac{f''(0)}{2!}a^2 + \frac{f'''(0)}{3!}a^3 + \cdots\right)\right)\right) \\
&\geq |a| - \left|\frac{f''(0)}{2!}a^2 + \frac{f'''(0)}{3!}a^3 + \cdots\right| \\
&= \frac{1}{4M} - \left|\frac{f''(0)}{2!}a^2 + \frac{f'''(0)}{3!}a^3 + \cdots\right| \\
&\geq \frac{1}{4M} - \left(\left|\frac{f''(0)}{2!}a^2\right| + \left|\frac{f'''(0)}{3!}a^3\right| + \cdots\right) \\
&= \frac{1}{4M} - \left(|f''(0)|\frac{1}{2!}|a|^2 + |f'''(0)|\frac{1}{3!}|a|^3 + \cdots\right)
\end{aligned}$$

$$= \frac{1}{4M} - \left(|f''(0)| \frac{1}{2!} \left(\frac{1}{4M} \right)^2 + |f'''(0)| \frac{1}{3!} \left(\frac{1}{4M} \right)^3 + \cdots \right)$$
$$\geq \frac{1}{4M} - \left(M(2!) \frac{1}{2!} \left(\frac{1}{4M} \right)^2 + M(3!) \frac{1}{3!} \left(\frac{1}{4M} \right)^3 + \cdots \right)$$
$$= \frac{1}{4M} - M \left(\left(\frac{1}{4M} \right)^2 + \left(\frac{1}{4M} \right)^3 + \cdots \right) = \frac{1}{4M} - M \cdot \frac{\left(\frac{1}{4M}\right)^2}{1 - \frac{1}{4M}}$$
$$= \frac{1}{4M} - \frac{1}{16M} \frac{1}{1 - \frac{1}{4M}} = \frac{1}{4M} - \frac{1}{4(4M-1)}$$
$$\geq \frac{1}{4M} - \frac{1}{4(4M-M)} = \frac{1}{4M} - \frac{1}{12M} = \frac{1}{6M} > 0,$$

and hence $|f(a)| > 0$. Also,

$$(|f(a) - g(a)| = |f(a) - (f(a) - w_0)| = |w_0| <) \frac{1}{6M} \leq |f(a)|.$$

This proves 1 and 2. ■)

Conclusion 4.284 Let $f \in H(D(0;1))$. Let $f(0) = 0$, and $f'(0) = 1$. Let M be a positive real number such that for every $z \in D(0;1)$, $|f(z)| \leq M$. Then $1 \leq M$, and $D\left(0; \frac{1}{6M}\right) \subset f(D(0;1))$.

Let $R > 0$. Let $g \in H(D(0;R))$. Let $g(0) = 0$, and $g'(0) \neq 0$. Let M be a positive real number such that, for every $z \in D(0;R)$, $|g(z)| \leq M$.

Problem 4.285 $R|g'(0)| \leq M$, and $D\left(0; \frac{R^2|g'(0)|^2}{6M}\right) \subset g(D(0;R))$.

(Solution Let

$$f : z \mapsto \frac{1}{Rg'(0)} g(Rz)$$

be a function from $D(0;1)$ to $\mathbb{C}$. Since $g(0) = 0$, we have $f(0) = 0$, Next,

$$f'(0) = \left(\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{Rg'(0)} g(Rz) \right) \right) \bigg|_{z=0} = \left(\frac{1}{Rg'(0)} g'(Rz) \cdot R \right) \bigg|_{z=0}$$
$$= \frac{1}{Rg'(0)} g'(0) \cdot R = 1,$$

so $f'(0) = 1$. Since $g \in H(D(0;R))$, by the definition of f, we have $f \in H(D(0;1))$. Next, for every $z \in D(0;1)$,

$$|f(z)| = \left|\frac{1}{Rg'(0)}g(Rz)\right| = \frac{1}{R|g'(0)|}|g(Rz)| \leq \frac{1}{R|g'(0)|}M,$$

so for every $z \in D(0;1)$,

$$|f(z)| \leq \frac{1}{R|g'(0)|}M.$$

Now, by Conclusion 4.284, $1 \leq \frac{1}{R|g'(0)|}M$, and

$$\begin{aligned}\left(D\left(0;\frac{R|g'(0)|}{6M}\right) =\right) D\left(0;\frac{1}{6\cdot\frac{1}{R|g'(0)|}M}\right) &\subset f(D(0;1))\\ = \{f(z) : z \in D(0;1)\} &= \left\{\frac{1}{Rg'(0)}g(Rz) : |z|<1\right\}\\ = \frac{1}{Rg'(0)}\{g(Rz) : |z|<1\} &= \frac{1}{Rg'(0)}\left\{g(z) : \left|\frac{z}{R}\right|<1\right\}\\ = \frac{1}{Rg'(0)}\{g(z) : |z|<R\} &= \frac{1}{Rg'(0)}g(D(0;R)),\end{aligned}$$

and hence,

$$D\left(0;\frac{R|g'(0)|}{6M}\right) \subset \frac{1}{Rg'(0)}g(D(0;R)).$$

It follows that

$$\begin{aligned}D\left(0;\frac{R^2|g'(0)|^2}{6M}\right) &= D\left(0;R|g'(0)|\cdot\frac{R|g'(0)|}{6M}\right) = D\left(0;|Rg'(0)|\cdot\frac{R|g'(0)|}{6M}\right)\\ &= (Rg'(0))\left(D\left(0;\frac{R|g'(0)|}{6M}\right)\right) \subset g(D(0;R)),\end{aligned}$$

and hence

$$D\left(0;\frac{R^2|g'(0)|^2}{6M}\right) \subset g(D(0;R)).$$

∎)

Conclusion 4.286 Let $R > 0$. Let $g \in H(D(0;R))$. Let $g(0) = 0$, and $g'(0) \neq 0$. Let M be a positive real number such that, for every $z \in D(0;R)$, $|g(z)| \leq M$. Then

$R|g'(0)| \leq M$, and $D\left(0;\frac{R^2|g'(0)|^2}{6M}\right) \subset g(D(0;R))$.

Let Ω be a bounded region. Let $f : \overline{\Omega} \to \mathbb{C}$. Let f be continuous, and let $f|_{\Omega}$ be holomorphic.

Since Ω is bounded, there exists $R > 0$ such that $\Omega \subset D(0;R)(\subset D[0;R])$, and hence $\overline{\Omega} \subset D[0;R]$. Now, since $\overline{\Omega}$ is a closed subset of the compact set $D[0;R]$, $\overline{\Omega}$ is compact. Next, since $f : \overline{\Omega} \to \mathbb{C}$ is continuous, $|f| : \overline{\Omega} \to [0,\infty)$ is continuous, and hence

$$\left(\max_{z\in\overline{\Omega}}|f(z)|\right) = \left(\sup_{z\in\overline{\Omega}}|f(z)|\right) \in \mathbb{R}.$$

Similarly,

$$\left(\max_{z\in(\partial\Omega)}|f(z)|\right) = \left(\sup_{z\in(\partial\Omega)}|f(z)|\right) \in \mathbb{R}.$$

Problem 4.287 $\max_{z\in\overline{\Omega}}|f(z)| = \max_{z\in(\partial\Omega)}|f(z)|$.

(Solution Since $(\partial\Omega) \subset \overline{\Omega}$, we have $\max_{z\in(\partial\Omega)}|f(z)| \le \max_{z\in\overline{\Omega}}|f(z)|$. It remains to show that

$$\max_{z\in\overline{\Omega}}|f(z)| \le \max_{z\in(\partial\Omega)}|f(z)|.$$

By Problem 4.281, there exists $z_0 \in (\partial\Omega)$ such that, for every z satisfying $z \in \overline{\Omega}$,

$$|f(z)| \le |f(z_0)|\left(\le \max_{z\in(\partial\Omega)}|f(z)|\right),$$

and hence

$$\max_{z\in\overline{\Omega}}|f(z)| \le \max_{z\in(\partial\Omega)}|f(z)|.$$

∎)

Conclusion 4.288 Let Ω be a bounded region. Let $f : \overline{\Omega} \to \mathbb{C}$. Let f be continuous, and $f|_{\Omega}$ is holomorphic. Then,

$$\max_{z\in\overline{\Omega}}|f(z)| = \max_{z\in(\partial\Omega)}|f(z)|.$$

Let Ω be a region. Let $D(0;2) \subset \Omega$. Let $f \in H(\Omega)$. Let $f(0) = 0$, and $f'(0) = 1$.

Since $f \in H(\Omega)$, by Lemma 1.117, $f' \in H(\Omega)$. Now, since $(\overline{D(0;1)}) = D[0;1] \subset \Omega$, by Conclusion 4.288, for every $r \in [0,1]$,

$$\max_{|z| \le r} |f'(z)| = \max_{|z|=r} |f'(z)|.$$

For every $r \in [0, 1]$, put

$$K(r) \equiv \max_{|z|=r} |f'(z)| \left(= \max_{|z| \le r} |f'(z)| \right).$$

Since $K : r \mapsto \max_{|z| \le r} |f'(z)|$ is an increasing function, $K : [0, 1] \to [0, \infty)$ is an increasing function. Clearly,

$$K(0) = \max_{|z|=0} |f'(z)| = |f'(0)| = |1| = 1,$$

so $K(0) = 1$. Observe that $K : [0, 1] \to [0, \infty)$ is continuous. Let $h : r \mapsto (1 - r)K(r)$ be a function from $[0, 1]$ to $[0, \infty)$. Since $K : [0, 1] \to [0, \infty)$ is continuous, $h : [0, 1] \to [0, \infty)$ is continuous. Next,

$$h(0) = (1 - 0)K(0) = (1 - 0) \cdot 1 = 1, \text{and}\, h(1) = (1 - 1)K(1) = 0.$$

Thus,

$$\left. \begin{array}{l} h(0) = 1 \\ h(1) = 0 \end{array} \right\}.$$

It follows that $0 \in h^{-1}(1)$, and hence $h^{-1}(1)$ ia nonempty subset of $[0, 1]$. Since $h : [0, 1] \to [0, \infty)$ is continuous, $h^{-1}(1)$ is a nonempty compact subset of $[0, 1]$, and hence

$$\left(\max\left(h^{-1}(1)\right)\right) = \left(\sup\left(h^{-1}(1)\right)\right) \in [0, 1].$$

Put $r_0 \equiv \max(h^{-1}(1))(= (\sup(h^{-1}(1))) \in [0, 1])$. Since $r_0 = \max(h^{-1}(1))$, we have $r_0 \in h^{-1}(1)$, and hence $h(r_0) = 1$. Since $h(r_0) = 1$, and $h(1) = 0$, we have $r_0 \neq 1$. Thus, $r_0 \in [0, 1)$. Further, $1 = h(0) = h(r_0)$.

Problem 4.289 $r \in (r_0, 1) \Rightarrow h(r) < 1$.

(Solution For this purpose, let us fix any $r \in (r_0, 1)$ (that is, $r_0 < r < 1$). We have to show that $h(r) < 1$. Since $r_0 = \max(h^{-1}(1))$, and $r_0 < r$, we have $r \notin h^{-1}(1)$, and hence $h(r) \neq 1$. It suffices to show that $1 \not< h(r)$.

If not, otherwise let $(h(1) = 0 <)1 < h(r)$. We have to arrive at a contradiction. Since $h(1) < 1 < h(r)$, and $h : [0, 1] \to [0, \infty)$ is continuous, by the intermediate value theorem, there exists $r^* \in (r, 1)$ such that $h(r^*) = 1$, and hence $r^* \in h^{-1}(1)$. It follows that $r^* \le \max(h^{-1}(1)) = r_0 < r$, and hence $r^* \notin (r, 1)$. This is a contradiction. ■)

Since for every $r \in [0,1]$, $K(r) = \max_{|z|=r} |f'(z)|$, and $r_0 \in [0,1]$, there exists a complex number a such that $|a| = r_0$, and

$$\frac{1}{1-r_0} = \frac{h(r_0)}{1-r_0} = \underbrace{K(r_0) = |f'(a)|}.$$

Thus, $|f'(a)| = \frac{1}{1-r_0} (\neq 0)$. It follows that $f'(a) \neq 0$. For every $z \in D\left(a; \frac{1}{2}(1-r_0)\right)$, we have

$$|z| - r_0 = |z| - |a| \leq \underbrace{|z-a| < \frac{1}{2}(1-r_0)},$$

and hence for every $z \in D\left(a; \frac{1}{2}(1-r_0)\right)$,

$$|z| < \left(\frac{1}{2}(1-r_0) + r_0\right) = \frac{1}{2}(1+r_0)\left(< \frac{1}{2}(1+1) = 1\right).$$

Thus, for every $z \in D\left(a; \frac{1}{2}(1-r_0)\right)$, we have $|z| < \frac{1}{2}(1+r_0)$, and hence

$$|z+a| \leq |z| + |a| = |z| + r_0 < \frac{1}{2}(1+r_0) + r_0 = \frac{1}{2} + \frac{3}{2}r_0 < \frac{1}{2} + \frac{3}{2} \cdot 1 = 2.$$

It follows that for every $z \in D\left(a; \frac{1}{2}(1-r_0)\right)$, $(z+a) \in D(0;2)(\subset \Omega)$, and $z \in D(0;1)(\subset \Omega)$. Since $\frac{1}{2}(1+r_0) \in (r_0, 1)$, by Problem 4.280,

$$h\left(\frac{1}{2}(1+r_0)\right) < 1.$$

Next, for every $z \in D\left(a; \frac{1}{2}(1-r_0)\right)$, $|z| \in [0,1]$, and hence, for every $z \in D\left(a; \frac{1}{2}(1-r_0)\right)$, we have

$$\begin{aligned}\frac{1}{\frac{1}{2}(1-r_0)} &= \frac{1}{\frac{1}{2}(1-r_0)} \cdot 1 > \frac{1}{\frac{1}{2}(1-r_0)} \cdot h\left(\frac{1}{2}(1+r_0)\right) \\ &= \frac{h\left(\frac{1}{2}(1+r_0)\right)}{1-\frac{1}{2}(1+r_0)} = K\left(\frac{1}{2}(1+r_0)\right) \geq \underbrace{K(|z|) = \left(\max_{|\zeta|=|z|} |f'(\zeta)|\right)} \geq |f'(z)|.\end{aligned}$$

Thus, for every $z \in D\left(a; \frac{1}{2}(1-r_0)\right)$, $|z| \in [0,1]$, and $|f'(z)| < \frac{1}{\frac{1}{2}(1-r_0)}$.

For every $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, we have $|z| < \frac{1}{6}(1-r_0)\left(\leq \frac{1}{6} < 1\right)$, and hence

$$|z+a| \leq |z| + |a| = |z| + r_0 < \frac{1}{6}(1-r_0) + r_0 = \frac{1}{6} + \frac{5}{6}r_0 < \frac{1}{6} + \frac{5}{6}\cdot 1 = 1.$$

Thus, for every $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, we have $z, (z+a) \in D(0;1)(\subset \Omega)$, and hence for every $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, $(f(z+a) - f(a)) \in \mathbb{C}$. Let

$$g : z \mapsto (f(z+a) - f(a))$$

be a function from $D\left(0; \frac{1}{6}(1-r_0)\right)$ to $\mathbb{C}$.

Problem 4.290 $g \in H\left(D\left(0; \frac{1}{6}(1-r_0)\right)\right)$.

Also, for every $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, the line segment $[a, a+z]$ lies in $D\left(a; \frac{1}{2}(1-r_0)\right)$.

(Solution For this purpose, let us take any $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, and $t \in [0,1]$. We have to show that

$$((1-t)a + t(a+z)) \in D\left(a; \frac{1}{2}(1-r_0)\right),$$

that is

$$(t|z| = |tz| =)|((1-t)a + t(a+z)) - a| < \frac{1}{2}(1-r_0),$$

that is $t|z| < \frac{1}{2}(1-r_0)$. This is clearly true, because $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, and $t \in [0,1]$. ■)

Hence, for every $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, and for every $\zeta \in [a, a+z]\left(\subset D\left(a; \frac{1}{2}(1-r_0)\right)\right)$, by Problem 4.281,

$$|f'(\zeta)| < \frac{1}{\frac{1}{2}(1-r_0)}.$$

So, for every $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$,

$$|g(z)| = |f(a+z) - f(a)| = \underbrace{\left| \int_{[a,a+z]} f'(\zeta)\mathrm{d}\zeta \right| \leq \frac{1}{\frac{1}{2}(1-r_0)} \cdot |(a+z) - a|}$$

$$= \frac{1}{\frac{1}{2}(1-r_0)} \cdot |z| < \frac{1}{\frac{1}{2}(1-r_0)} \cdot \frac{1}{6}(1-r_0) = \frac{1}{3}.$$

Thus, for every $z \in D\left(0; \frac{1}{6}(1-r_0)\right)$, $|g(z)| \le \frac{1}{3}$. Next, $g(0) = f(a+0) - f(a) = 0$, and $|g'(0)| = |f'(a+0) - 0| = |f'(a)| = \frac{1}{1-r_0} \neq 0$, and hence $g'(0) \neq 0$. Now, by Conclusion 4.286,

$$
\begin{aligned}
D\left(0; \frac{1}{72}\right) &= D\left(0; \frac{\left(\frac{1}{6}(1-r_0)\right)^2 \left(\frac{1}{1-r_0}\right)^2}{6 \cdot \frac{1}{3}}\right) \subset g\left(D\left(0; \frac{1}{6}(1-r_0)\right)\right) \\
&= \left\{ g(z) : |z| < \frac{1}{6}(1-r_0) \right\} = \left\{ f(z+a) - f(a) : |z| < \frac{1}{6}(1-r_0) \right\} \\
&= \left\{ f(z+a) : |z| < \frac{1}{6}(1-r_0) \right\} - f(a) = \left\{ f(z) : |z-a| < \frac{1}{6}(1-r_0) \right\} - f(a) \\
&= f\left(D\left(a; \frac{1}{6}(1-r_0)\right)\right) - f(a),
\end{aligned}
$$

so

$$
D\left(f(a); \frac{1}{72}\right) = f(a) + D\left(0; \frac{1}{72}\right) \subset f\left(D\left(a; \frac{1}{6}(1-r_0)\right)\right),
$$

and hence $f(S)$ contains an open disk of radius $\frac{1}{72}$, where $S \equiv D\left(a; \frac{1}{6}(1-r_0)\right)$.

Conclusion 4.291 Let Ω be a region. Let $D(0;2) \subset \Omega$. Let $f \in H(\Omega)$. Let $f(0) = 0$, and $f'(0) = 1$. Then, there exists an open disk S contained in Ω such that $f(S)$ contains an open disk of radius $\frac{1}{72}$.

This result, known as **Bloch's theorem**, is due to A. Bloch (20.11.1893–11.10.1948).

Let Ω be a region. Let $R > 0$. Suppose that $\left(\frac{R}{3} \cdot D[0;3] =\right) D[0;R] \subset \Omega$. Let $f \in H(\Omega)$. Let $f'(0) \neq 0$.

Let

$$
g : z \mapsto \frac{1}{\frac{R}{3} f'(0)} \left(f\left(\frac{R}{3} z\right) - f(0) \right)
$$

be a function from $(D(0;2) \subset) D(0;3)$ to $\mathbb{C}$. Clearly, $g \in H(D(0;3))$. Further,

$$
g(0) = \frac{1}{\frac{R}{3} f'(0)} \left(f\left(\frac{R}{3} \cdot 0\right) - f(0) \right) = 0.
$$

Next,

$$g'(0) = \frac{1}{\frac{R}{3}f'(0)}\left(\left(f'\left(\frac{R}{3}\cdot 0\right)\right)\cdot\frac{R}{3} - 0\right) = 1.$$

Now, by Conclusion 4.291, there exists an open disk S contained in $D(0;3)$ such that

$$\left(\frac{1}{\frac{R}{3}f'(0)}\left(f\left(\frac{R}{3}S\right) - f(0)\right) = \right)g(S)$$

contains an open disk of radius $\frac{1}{72}$, and hence $f\left(\frac{R}{3}S\right)$ contains an open disk of radius $\frac{1}{72}\cdot\frac{R}{3}|f'(0)|$. Since S is an open disk contained in $D(0;3)$, $\frac{R}{3}S$ is an open disk. Since S is contained in $D(0;3)$, $\frac{R}{3}S$ is contained in $\frac{R}{3}\cdot D(0;3) = (D(0;R))$, and hence $f\left(\frac{R}{3}S\right)$ is contained in $f(D(0;R))$.

Conclusion 4.292 Let Ω be a region. Let $R > 0$. Suppose that $D[0;R] \subset \Omega$. Let $f \in H(\Omega)$. Let $f'(0) \neq 0$. Then, $f(D(0;R))$ contains an open disk of radius $\frac{1}{72}\cdot\frac{R}{3}|f'(0)|$.

Let $f \in H(\mathbb{C})$. Let f be non-constant, and non-surjective (that is, not onto).

Problem 4.293 There exists a unique complex number a such that $a \notin f(\mathbb{C})$.

(Solution Existence: Since $f : \mathbb{C} \to \mathbb{C}$ is not surjective, there exists $a \in \mathbb{C}$ such that $a \notin f(\mathbb{C})$.

Uniqueness: If not, otherwise suppose that there exist complex numbers a and b such that $a \neq b$, $a \notin f(\mathbb{C})$, and $b \notin f(\mathbb{C})$. We have to arrive at a contradiction.

Let

$$g : z \mapsto \frac{f(z) - a}{b - a}$$

be a function from $\mathbb{C}$ to $\mathbb{C}$. Since $f \in H(\mathbb{C})$, by the definition of g, we have $g \in H(\mathbb{C})$. Since $a \notin f(\mathbb{C})$, for every $z \in \mathbb{C}$, $f(z) \neq a$, and hence for every $z \in \mathbb{C}$, $(g(z) =)\frac{f(z)-a}{b-a} \neq 0$. It follows that $0 \notin g(\mathbb{C})$. Since $b \notin f(\mathbb{C})$, for every $z \in \mathbb{C}$, $f(z) \neq b$, and hence for every $z \in \mathbb{C}$, $(f(z) - a) \neq (b - a)$. It follows that, for every $z \in \mathbb{C}$, $(g(z) =)\frac{f(z)-a}{b-a} \neq 1$, and hence, $1 \notin g(\mathbb{C})$.

Now, by Conclusion 4.282, there exists $h \in H(\mathbb{C})$ such that

1. for every $z \in \mathbb{C}$, $g(z) = \frac{1}{2}(1 + \cos(\pi\cos(\pi h(z))))$,
2. no open disk of radius 1 is contained in $h(\mathbb{C})$.

Since f is non-constant, $g : z \mapsto \frac{f(z)-a}{b-a}$ from $\mathbb{C}$ to $\mathbb{C}$ is non-constant, and hence, by 1, $h : \mathbb{C} \to \mathbb{C}$ is non-constant. Since $h : \mathbb{C} \to \mathbb{C}$ is non-constant, and $h \in H(\mathbb{C})$, by

Conclusion 1.116, and Lemma 1.60, h' cannot be identically 0, and hence there exists $z_0 \in \mathbb{C}$ such that $h'(z_0) \neq 0$.

Let

$$k : z \mapsto h(z + z_0)$$

be a function from $\mathbb{C}$ to $\mathbb{C}$. Since $h \in H(\mathbb{C})$, we have $k \in H(\mathbb{C})$. Next,

$$k'(0) = (h'(0 + z_0))(1 + 0) = h'(z_0) \neq 0,$$

so $k'(0) \neq 0$. Now, there exists $R > 0$ such that $1 < \frac{1}{72} \cdot \frac{R}{3} |k'(0)|$. By Conclusion 4.292,

$$(h(\mathbb{C}) \supset h(D(z_0; R)) = h(D(0; R) + z_0) =) k(D(0; R))$$

contains an open disk of radius $\frac{1}{72} \cdot \frac{R}{3} |f'(0)| (> 1)$, and hence $h(\mathbb{C})$ contains an open disk of radius 1. This contradicts 2. ■)

Conclusion 4.294 Let $f \in H(\mathbb{C})$. Let f be non-constant, and non-surjective. Then there exists a unique complex number a such that $a \notin f(\mathbb{C})$. In other words, a non-constant entire function omits at most one point.

This result, known as the **Picard's little theorem**, is due to E. Picard (14.07.1856–11.12.1941).

Exercises

4.1 Show that in the annulus $\{z : 2 < |z| < 3\}$,

$$\frac{1}{(z-2)(z-3)}$$

is representable by a series in both positive and negative powers of z.

4.2 Let $a < b$. Let $\gamma : [a, b] \to \mathbb{C}$ be any path. Let $\varphi : \mathrm{ran}(\gamma) \to \mathbb{C}$ be a continuous function. Let $f : (\mathbb{C} - \mathrm{ran}(\gamma)) \to \mathbb{C}$ be the Cauchy integral of φ over γ. Let $R \equiv \max\{|\zeta| : \zeta \in \mathrm{ran}(\gamma)\}$. Show that for every $z \in \{w : R < |w|\}$,

$$f(z) = \left(-\int_\gamma \varphi(\zeta) \mathrm{d}\zeta \right) \frac{1}{z} + \left(-\int_\gamma \zeta \cdot \varphi(\zeta) \mathrm{d}\zeta \right) \frac{1}{z^2} + \left(-\int_\gamma \zeta^2 \cdot \varphi(\zeta) \mathrm{d}\zeta \right) \frac{1}{z^3} + \cdots.$$

4.3 Let

$$f : z \mapsto \frac{z^3}{z-3}$$

be a function from $\mathbb{C} - \{3\}$ to $\mathbb{C}$. Show that ∞ is a pole of order 2 of f.

4.4 Show that

a. $\int_0^\infty \frac{1}{1+x^3}\,\mathrm{d}x$ is convergent,
b. $\int_0^\infty \frac{1}{1+x^3}\,\mathrm{d}x = \frac{2\pi}{3\sqrt{3}}$.

4.5 Let $f : D(0;1) \to \mathbb{C}$ be a meromorphic function. Show that there exists a subset A of $D(0;1)$ such that

a. A has no limit point in $D(0;1)$,
b. for every $z \in (D(0;1) - A)$, $f'(z)$ exists,
c. for every $\alpha \in A$, f has a pole at α,
d. there exist $g, h \in H(D(0;1))$ such that $f = \frac{g}{h}$ on $(D(0;1) - A)$.

4.6 Let $f : (0,\infty) \to (0,\infty)$ be a function satisfying the following conditions:

a. $f(1) = 1$,
b. for every $x \in (0,\infty)$, $f(x+1) = xf(x)$,
c. $(\ln \circ f) : (0,\infty) \to \mathbb{R}$ is convex.

Show that for every $x \in (0,\infty)$,

$$f(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}.$$

4.7 Show that Res $(\Gamma; 0) = 1$.

4.8 Let $z_1, z_2, \ldots$ be any nonzero complex numbers. Suppose that $|1-z_1| + |1-z_2| + \cdots$ is convergent. Show that $\prod_{n=1}^\infty z_n$ is a nonzero complex number.

4.9 Let $f : \mathbb{R} \to \mathbb{C}$ be any Lebesgue measurable function. Show that

$$\lim_{n\to\infty}\left(\int_{-\infty}^{\infty} f(t)\sin(nt)\mathrm{d}t\right) = 0.$$

4.10 Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Let f be non-constant and non-surjective. Show that there exists a unique complex number a such that $a \notin f(\mathbb{C})$.

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