Cosmology Coursework (Hubble drift)

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1 Answer to question 1

In this question it is required the use of multiple Taylor series to calculate $d_L = d_L(z)$ (the luminosity distance[1]) up to and including z^2.

a) Taylor series of $a(t)$ around $t = t_0$:

$$a(t) = 1 + \tau H_0 - \frac{q_0 H_0^2}{2}\tau^2 + \mathcal{O}(\tau^3) \tag{1.0.1}$$

where $\tau = t - t_0$.
To write H_0 and q_0 in terms of the scale parameters, one can Taylor expand $a(t)$ around $t = t_0$ and compare with (1.0.1). Therefore:

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a}(t_0)(t - t_0)^2 + ...$$

Now divide both sides by $a(t_0)(= a_0)$;

$$\frac{a(t)}{a(t_0)} = 1 + \frac{\dot{a}(t_0)}{a(t_0)}\tau + \frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)}\tau^2 + \mathcal{O}(\tau^3) \tag{1.0.2}$$

If we set $a_0 = 1$ and then compare each terms of eq. (1.0.1) and eq. (1.0.2) we obtain the following:

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \dot{a}(t_0) \tag{1.0.3}$$

We recognise the above, eq. (1.0.3) is known as the Hubble constant. Now by comparing the 3^{rd} term of of eq. (1.0.1) and eq. (1.0.2) we have that:

$$-\frac{q_0 H_0^2}{2} = \frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)}$$

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)H_0^2} = -\frac{\ddot{a}(t_0)}{H_0^2} \tag{1.0.4}$$

where q_0 is known as the deceleration parameter.

b) Finding the comoving distance χ given by:

$$\chi = c\int_{t'=t}^{t'=t_0} \frac{dt'}{a(t')}$$

where c is the speed of light in a vacuum.
Before evaluating the above integral, one can expand $1/a(t)$ at $t = t_0$:

$$\begin{aligned}\frac{1}{a(t)} &= \frac{1}{a(t_0)} - \frac{\dot{a}(t_0)}{a^2(t_0)}\tau - \left[\frac{\ddot{a}(t_0)}{a^2(t_0)} - 2\frac{\dot{a}^2(t_0)}{a^3(t_0)}\right]\frac{\tau^2}{2} + ... \\ &= 1 - \dot{a}(t_0)\tau - \left[\ddot{a}(t_0) - 2\dot{a}^2(t_0)\right]\frac{\tau^2}{2} + ...\end{aligned}$$

[1] Luminosity is the energy per unit time emitted, for example from a star. Hence, luminosity distance is a measure that quantifies the distance of an astronomical object based on the amount of light emitted.

Thus by comparing the components of the above expansion, with the findings of the previous part (*in particular see equations (1.0.3) and (1.0.4)*), one obtains:

$$\frac{1}{a(t)} = 1 - H_0\tau - H_0^2(q_0 - 2)\frac{\tau^2}{2} + \mathcal{O}(\tau^3) \tag{1.0.5}$$

Gathering all the above information, in particular by plugging (1.0.5) into the given integral we have the following:

$$\begin{aligned}
\chi &= c\int_{t'=t}^{t'=t_0} dt' 1 - H_0(t'-t) - H_0^2(q_0-2)\frac{(t'-t_0)^2}{2} + ... \\
&= c\left[t' - H_0(\frac{t'^2}{2} - t_0t'\right) - \frac{H_0^2}{6}(q_0-2)(t'-t_0)^3 + ...\Bigg]_{t'=t}^{t'=t_0} \\
&= c\left(t_0 + H_0\frac{t_0^2}{2} + ... - t + H_0\left(\frac{t^2}{2} - t_0t+\right) + ...\right) \\
&= c\left(-(t-t_0) + \frac{H_0}{2}(t^2 - 2t_0t + t_0^2) + ...\right) \\
&= c\left(-\tau + \frac{H_0}{2}\tau^2 + \mathcal{O}(\tau^3)\right)
\end{aligned}$$

Hence we have obtained the comoving distance,

$$\chi = -c\tau\left(1 - \frac{H_0}{2}\tau + \mathcal{O}(\tau^2)\right) \tag{1.0.6}$$

c) Using the given relation, i.e. $\tau = \frac{1}{H_0}(-z + (2+q_0)\frac{z^2}{2})$, one can rewrite eq.(1.0.6) as a function of z.

$$\begin{aligned}
\chi &= -c\tau\left(1 - \frac{H_0}{2}\tau + \mathcal{O}(\tau^2)\right) \\
&= \frac{c}{H_0}\left((-z + (2+q_0)\frac{z^2}{2}\right)\left(1 + \frac{z}{2} - (2+q_0)\frac{z^2}{4} + ...\right) \\
&= \frac{c}{H_0}\left(z - (2+q_0)\frac{z^2}{2} + \frac{z^2}{2} + ...\right)
\end{aligned}$$

Therefore:

$$\chi(z) = \frac{c}{H_0}\left(z - (1+q_0)\frac{z^2}{2} + \mathcal{O}(z^3)\right) \tag{1.0.7}$$

d) **Luminosity distance**, denoted by d_L, it can be written as $d_L = r(\chi)(1+z)$. One can can recall the *Robertson-Walker* metric (RW), given by:

$$ds^2 = c^2 dt^2 - a^2(t)[d\chi^2 + r^2 d\Omega^2] \tag{1.0.8}$$

where $d\chi^2 = \frac{dr^2}{1-kr^2}$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

For a "flat" universe $k = 0$, i.e. $d\chi = dr \implies \chi = r$. Hence we know that, for the above case, $r(\chi) = \chi$. We previously derived $\chi(z)$, so one can rewrite r as a function of z, namely $r(z) = \frac{c}{H_0}\left(z - (1+q_0)\frac{z^2}{2} + \mathcal{O}(z^3)\right)$.

By substituting $r(z)$ in $d_L = r(1+z)$ one obtains:

$$d_L = \frac{c}{H_0}\left(z - (1+q_0)\frac{z^2}{2} + ...\right)(1+z)$$

$$= \frac{c}{H_0}\left(z + z^2 - (1+q_0)\frac{z^2}{2} + ...\right)$$

Therefore,

$$d_L = \frac{c}{H_0}\left(z + (1-q_0)\frac{z^2}{2} + ...\right) \tag{1.0.9}$$

as required!

2 Answer to question 2

Here we were asked to work with a set of data containing information about supernova. We are given a file `sn-data-CW.csv`, this files is divided into `"Name"`, `"Redshift"` and `"Distance modulus"`. All the codes for the graphs in this sections have been written on Python.

a) We are given luminosity distance, $d_L = 10^{\frac{\mu}{5}+1}$ pc, this is related to the distance modulus μ, and pc is **parsecs** units.

It is required to produce a plot with label d_L/ℓ as a function of z, with $\ell = ch/H_0$, with:$H_0 = 100h$ km s^{-1} Mpc^{-}1, where c is the speed of light in a vacuum and h is a constant that will be determined in part (b). Furthermore Mpc is **mega parsecs** i.e. 1Mpc $= 10^6$pc.

Before plotting d_L/ℓ, one first need to ensure that everything is on the same units. For simplicity we are going to use S.I. units, hence H_0 becomes: $H_0 = 0.1\text{ms}^{-1}$ pc^{-1}.

The modified code produces the following plot:

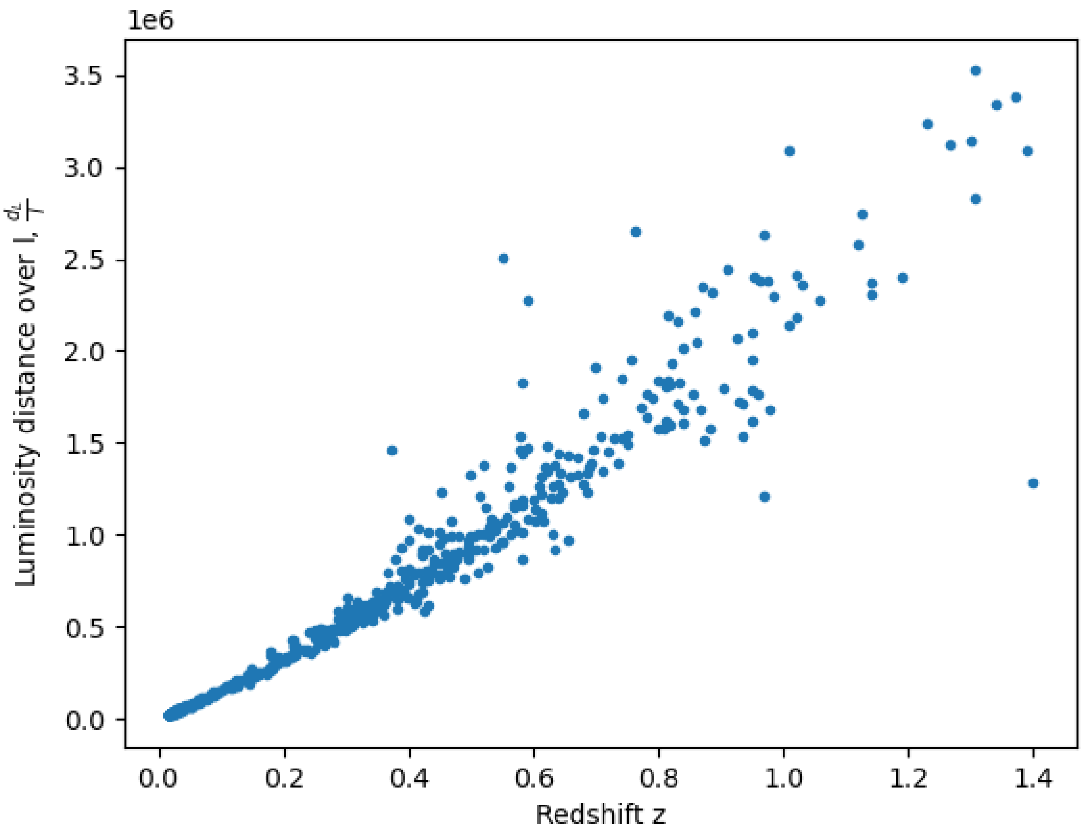

Figure 1: Plot q2 a, shows the relationship between the luminosity distance over the parameter ℓ, d_L/ℓ, and the redshift z. Plot obtained using Python.

b) **Linear regression**
in this part, we are asked to use linear regression to estimate the value of h in 1. To do so, we are given that d_L/ℓ on z can be written as:

$$\frac{d_L}{\ell} = c_0 + c_1 z. \tag{2.0.1}$$

Where the coefficient c_j, $j = 0, 1$, can be estimated by solving the following linear system:

$$\begin{bmatrix} \sum_i z_i^0 & \sum_i z_i^1 \\ \sum_i z_i^1 & \sum_i z_i^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_i d_i z_i^0/\ell \\ \sum_i d_i z_i^1/\ell \end{bmatrix} \tag{2.0.2}$$

where a data point for $(z, d_L/\ell)$ is referred as $(z_i, d_i/\ell)$.
Further more the linear regression model is only expected to be accurate for small values of z, i.e. $z << 1$, in particular we need to determine the value c_1 and plot the value of h as a fucntion of z_{max}, where $z_{max} \in [0.05, 0.2]$.

Using the provided hint neglect unwanted values of z, in the code we have used the following:

- define lower and upper bound as $z_{lower} = 0.05$ and $z_{upper} = 0.2$.
- then encode the hint as `df.loc[(df["Redshift"] < zupper) & (df["Redshift"] > zlower)]["Redshift"]` for the redshift values, and `df.loc[(df["Redshift"] < zupper) & (df["Redshift"] > zlower)]["Distance Modulus"]` for the modulus distace values.

Before writing the required code to determine c_1 and plot the values of h, we found the inverse of the matrix in (2.0.2), and then applied it to both sides.
Matrix inverse:

$$\frac{1}{\sum_i z_i^0 \sum_i z_i^2 - \left(\sum_i z_i^1\right)^2} \begin{bmatrix} \sum_i z_i^2 & -\sum_i z_i^1 \\ -\sum_i z_i^1 & \sum_i z_i^0 \end{bmatrix} \tag{2.0.3}$$

using the formula for inverse 2x2 matrices.

Applying (2.0.3) to both sides of eq. (2.0.2) one obtains:

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \frac{1}{\sum_i z_i^0 \sum_i z_i^2 - \left(\sum_i z_i^1\right)^2} \begin{bmatrix} \sum_i z_i^2 & -\sum_i z_i^1 \\ -\sum_i z_i^1 & \sum_i z_i^0 \end{bmatrix} \begin{bmatrix} \sum_i d_i z_i^0/\ell \\ \sum_i d_i z_i^1/\ell \end{bmatrix} \tag{2.0.4}$$

By coding the data on supernova, and the above equation (2.0.4), the output values for c_0 and c_1 are the following:

- $c_0 = -0.01489540266501915$
- $c_1 = 1.6989327076922247$

To verify the accuracy of the above findings, the package `sklearn.linear_model` was adopted and a separate code that performs linear regression in Python was used obtaining the following results as, $c_0 = -0.014895402665018986$ and $c_1 = 1.6989327076922212$ we can see the results agree up to 14 decimal places. As the values of c_0 and c_1 have been obtained, we can now determine a range of values for h and plot them, to do so compare eq. (2.0.1) to eq. (1.0.9) divided by ℓ:

$$\frac{d_L}{\ell} = \frac{c}{H_0} \times \frac{H_0}{ch} \left(z + (1 - q_0)\frac{z^2}{2} + ... \right) = c_0 + c_1 z$$

by neglecting terms of order z^2 and higher we are left with (as we are considering small values of z):

$$\frac{1}{h} z \approx c_0 + c_1 z$$

hence,

$$h \approx \frac{z}{c_0 + c_1 z} \tag{2.0.5}$$

Now one can plot eq. (2.0.5) for z_{max} values to obtain the following plot:

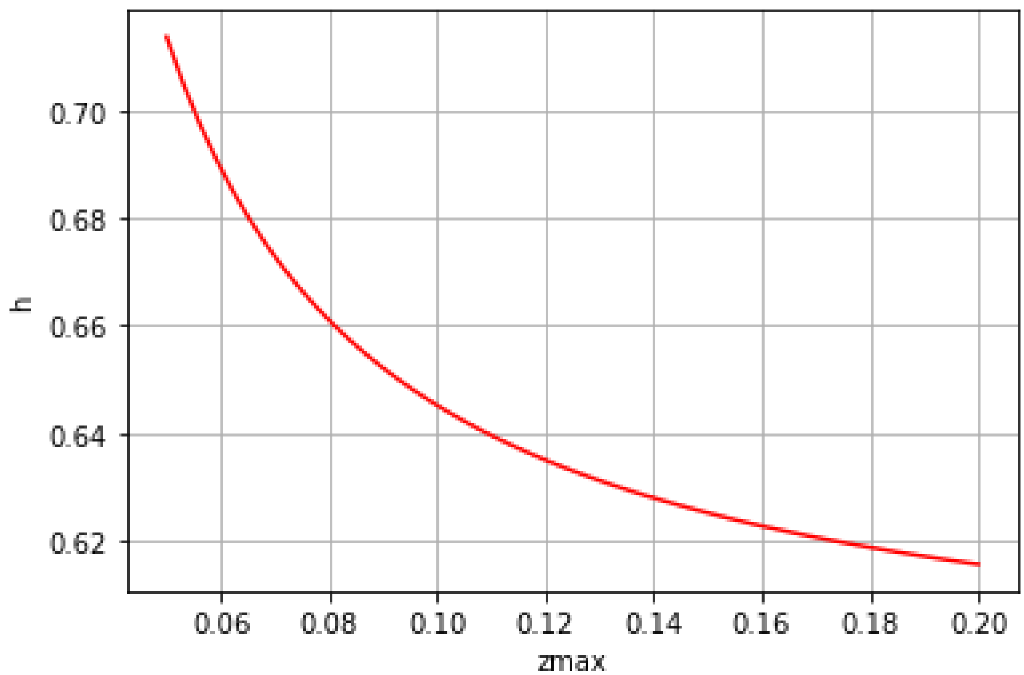

Figure 2: Plot obtained using Python, where the expression (2.0.5) was implemented in the code.

If we now compare the value of h in the literature, $h_{literature} \approx 0.7$, to the values of our findings, $h_{findings} \in [0.616, 0.714]$, it is quite clear that the value of h found in the literature falls into the values of $h_{findings}$.

c) **Deceleration Parameter**
In this part of the question we are asked to estimate the deceleration parameter q_0 by performing another linear regression, but now we are given that:

$$\frac{hd_L}{z\ell} = \tilde{c}_0 + \tilde{c}_1 z$$

Rearranging the above to make d_L/ℓ the subject:

$$\frac{d_L}{\ell} = \frac{1}{h}(\tilde{c}_0 z + \tilde{c}_1 z^2) \tag{2.0.6}$$

It is required to choose a suitable value for h. For this particular case we set $h = 0.7$. Using eq. (2.0.6) one can adapt the linear regression model (2.0.2), to find $\tilde{c}_0$ and $\tilde{c}_1$. The adapted regression[1] model is:

$$\begin{bmatrix} \sum_i z_i^1 & \sum_i z_i^2 \\ \\ \sum_i z_i^2 & \sum_i z_i^3 \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \\ \tilde{c}_1 \end{bmatrix} = \begin{bmatrix} \sum_i d_i z_i^0/\ell \\ \\ \sum_i d_i z_i^1/\ell \end{bmatrix} \tag{2.0.7}$$

Now one can repeat the same process adopted in answer 2 part (b), to determine $(\tilde{c}_0, \tilde{c}_1)$;

$$\begin{bmatrix} \tilde{c}_0 \\ \\ \tilde{c}_1 \end{bmatrix} = \frac{1}{\sum_i z_i^1 \sum_i z_i^3 - \left(\sum_i z_i^2\right)^2} \begin{bmatrix} \sum_i z_i^3 & -\sum_i z_i^2 \\ \\ -\sum_i z_i^2 & \sum_i z_i^1 \end{bmatrix} \begin{bmatrix} \sum_i d_i z_i^0/\ell \\ \\ \sum_i d_i z_i^1/\ell \end{bmatrix} \tag{2.0.8}$$

By implementing the above, (2.0.8), into Python code one obtains the following values for $(\tilde{c}_0, \tilde{c}_1)$:

- $\tilde{c}_0 = 1.0404559922445231$
- $\tilde{c}_1 = 0.6855414034883148$

Having calculated the values of $(\tilde{c}_0, \tilde{c}_1)$, we can now proceed to determine a range of values of q_0 and plot them as a function of $z_{max} \in [0.1, 0.7]$. Compare eq. (1.0.9) with eq. (2.0.6) divided by ℓ;

$$\frac{d_L}{\ell} = \frac{c}{H_0} \times \frac{H_0}{ch}\left(z + (1 - q_0)\frac{z^2}{2} + ...\right) = \frac{1}{h}(\tilde{c}_0 z + \tilde{c}_1 z^2)$$

Note h cancels out in the above equation, this time we will consider terms up and including terms z^2, to make q_0 the subject of the above equation.

$$z + \frac{z^2}{2}(1 - q_0) + ... = \tilde{c}_0 z + \tilde{c}_1 z^2$$

$$1 + \frac{z}{2}(1 - q_0) + ... = \tilde{c}_0 + \tilde{c}_1 z$$

$$1 - q_0 + ... = \frac{2}{z}(\tilde{c}_0 + \tilde{c}_1 z)$$

Therefore,

$$q_0 \approx 1 - \frac{2}{z}(\tilde{c}_0 + \tilde{c}_1 z) \tag{2.0.9}$$

[1] note this is not a linear regression, but rather a quadratic regression, due to the presence of z^2.

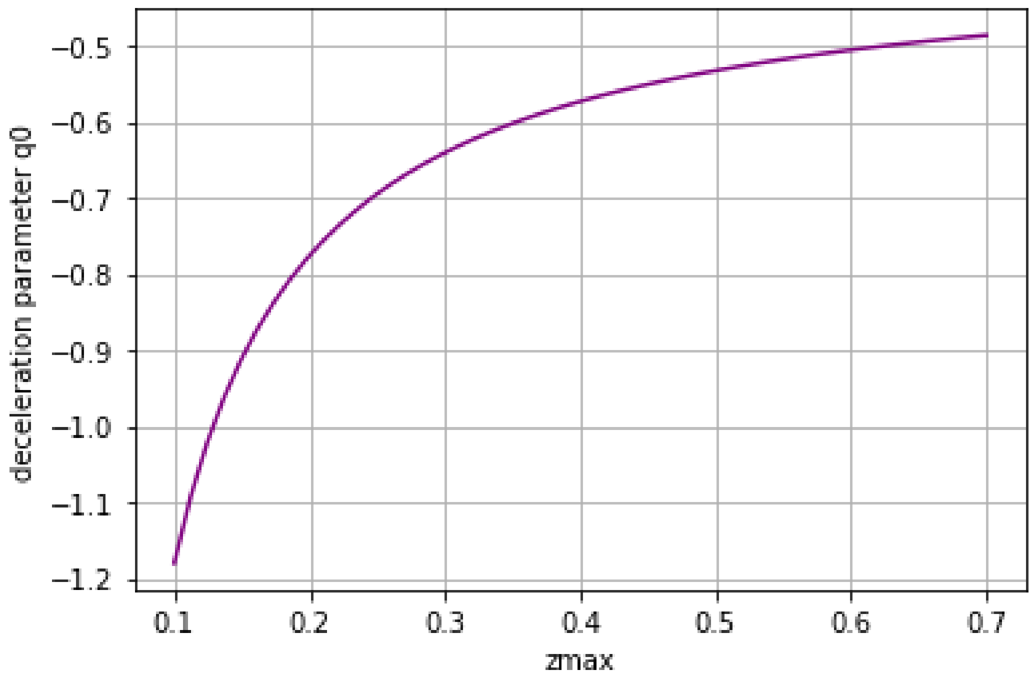

Figure 3: This is a plot of q_0 values as a function of $z_{max} \in [0.1, 0.7]$, using (2.0.9).

From 3 we see that the values of q_0 lie in the range $[-1.180, -0.487]$, the value found in the literature is $q_0 = -0.55$, which is included in the range of values worked out from the quadratic regression.