Estimating the sources of M/EEG activity

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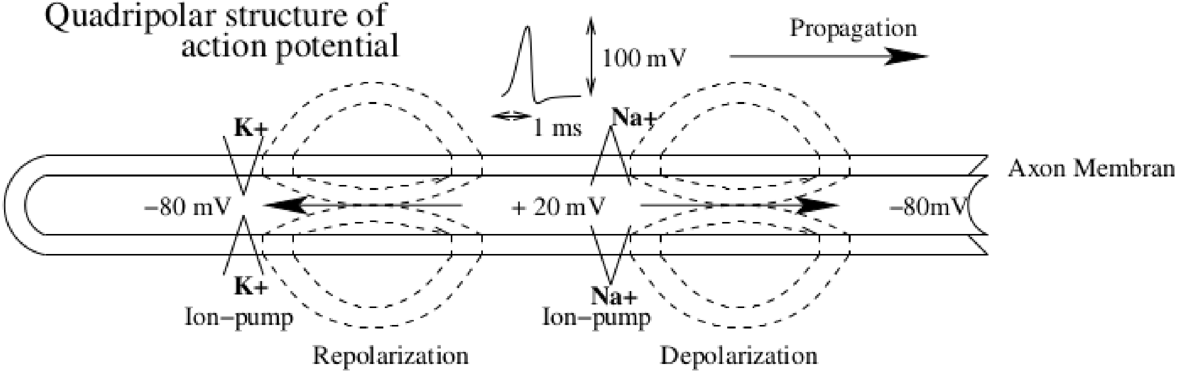
MVA / MSV

Imagerie fonctionnelle cérébrale et interface cerveau machine

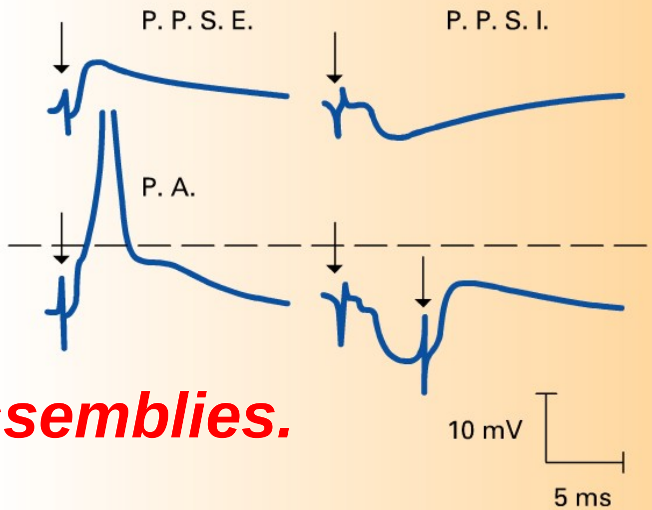
Neural current sources

- Action potentials (AP): strong and short, quadrupolar → faster decrease with distance ($1/r^3$).



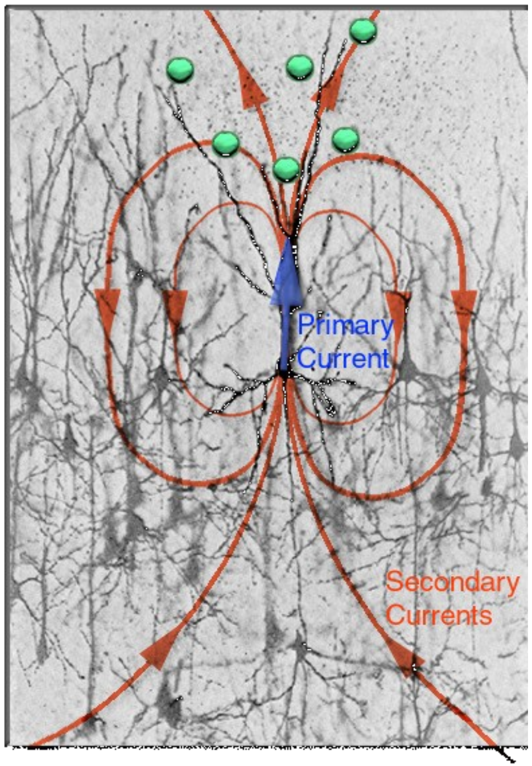
The ion exchanges corresponding to action potentials.

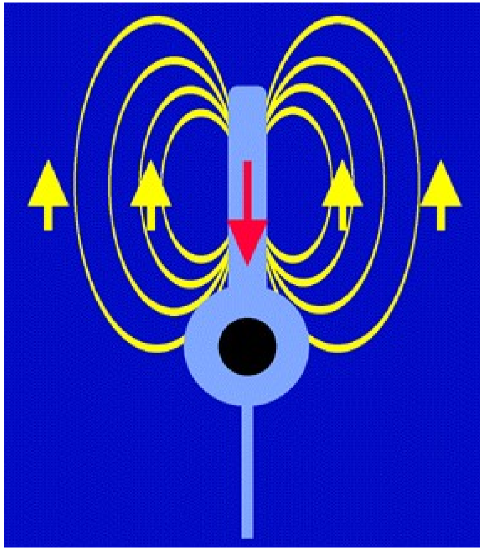
- Postsynaptic potentials (PSP):
  Weaker, but wider and slower and bipolar.
  →Superposition in synchronized neural assemblies.
  →Weaker decrease with distance ($1/r^2$).



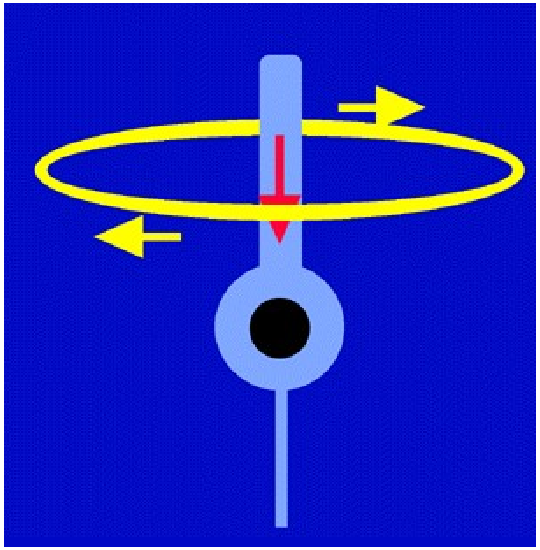
Neural current sources

- EEG/MEG directly measure PSP currents after propagation.
- Sources modeled as dipoles.





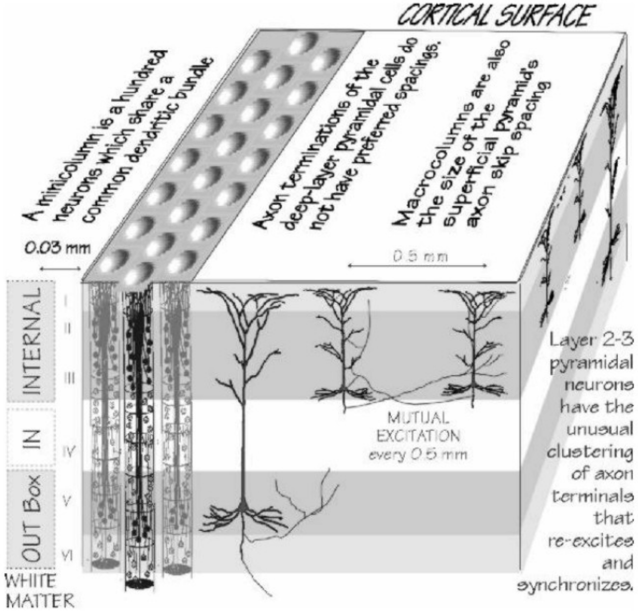
Electrical current flow lines

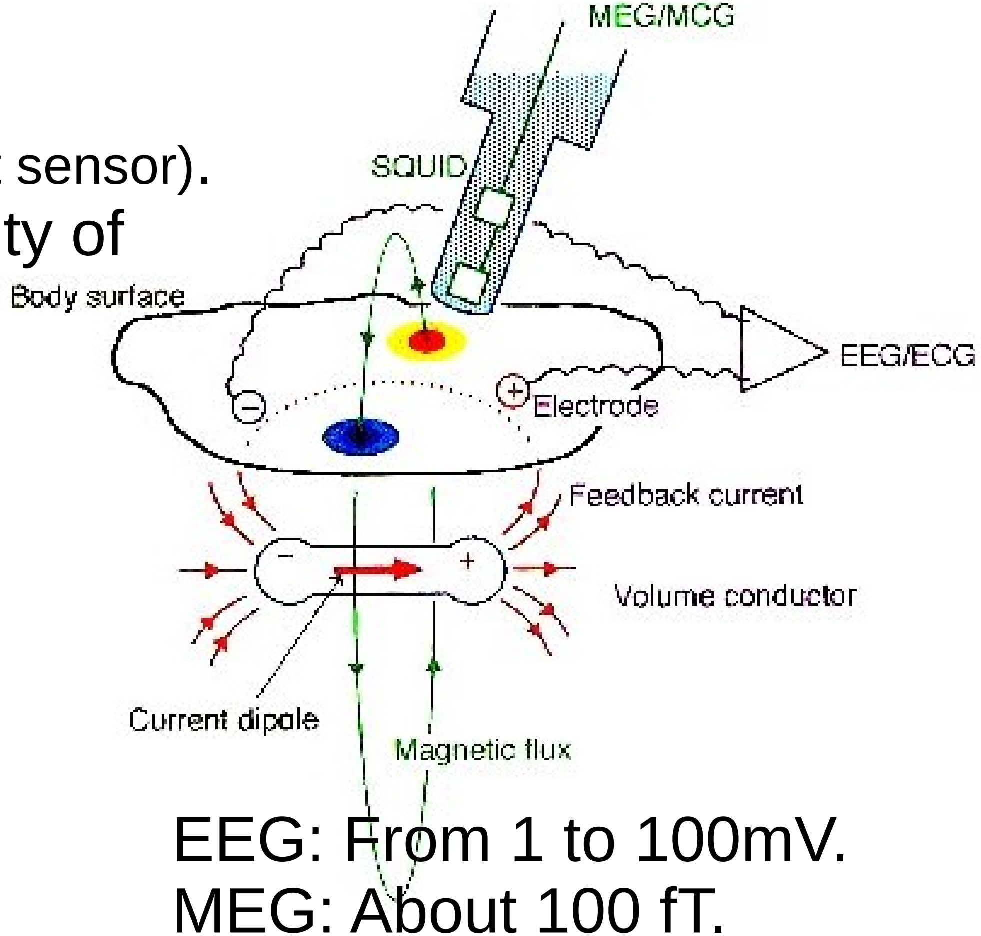


Magnetic field flow lines

EEG/MEG measurements

- A dipole is about 20fAm
  → too small to measure (10nAm at sensor).
- Synchronized and coherent activity of millions of pyramidal neurons.





EEG: From 1 to 100mV.
MEG: About 100 fT.

Goals

Source localisation: Compute sources from measurements.

To do that we will "compare" measurements with

simulated / predicted values for these measurements.

Forward model: Predict sensor values from sources…

This is called an **inverse problem**.

Also known as the inverse problem of source localization.

Forward model:

Predicting sensor values from sources….

- Start from physics.
- Establish computational models of electric/magnetic propagation on the head.
- Pinpoint some theoretical properties and difficulties:
  - Silent sources.
  - Nested sphere geometries.
- Various models of increasing complexity.
  - Nested closed surfaces.
    - Surfacic methods.
    - Volumic methods.

Some maths/physics...

Reminder on differential operators (1)

Let (x, y, z) denote the canonical basis of $\mathbb{R}^3$.

The *nabla* operator is $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$

This is just a notation.

The *gradient* of a scalar field $a(x, y, z)$ is $\nabla a = \begin{pmatrix} \partial a / \partial x \\ \partial a / \partial y \\ \partial a / \partial z \end{pmatrix}.$

The *divergence* of a vector field $\mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$ is the scalar field

$$\nabla \cdot \mathbf{b} = \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}$$

The *Laplacian* is $\Delta a = \nabla \cdot \nabla a = \dfrac{\partial^2 a}{\partial x^2} + \dfrac{\partial^2 a}{\partial y^2} + \dfrac{\partial^2 a}{\partial z^2}$

Reminder on differential operators (2)

The *curl* of vector field $\mathbf{b}$ is

$$\nabla \times \mathbf{b} = \begin{pmatrix} \partial a_y/\partial z - \partial a_z/\partial y \\ \partial a_z/\partial x - \partial a_x/\partial z \\ \partial a_x/\partial y - \partial a_y/\partial x \end{pmatrix}.$$

Product rule for the gradient

$$\nabla(a\,b) = a\,\nabla b + b\nabla a$$

Product of a scalar and a vector

$$\nabla \cdot (a\,\mathbf{b}) = a\,\nabla \cdot \mathbf{b} + \mathbf{b} \cdot \nabla a$$

$$\nabla \times (a\,\mathbf{b}) = a\,\nabla \times \mathbf{b} + \nabla a \times \mathbf{b}$$

Vector dot product

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

Vector cross product

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

Important properties:

$$\nabla \times \nabla a = 0$$

$$\nabla \cdot (\nabla \times \mathbf{b}) = 0$$

$$\nabla \times \nabla \times \mathbf{b} = \nabla(\nabla \cdot \mathbf{b}) - \Delta \mathbf{b}$$

Electrical current propagation

- Maxwell equations

Name	Differential form
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
Ampère's circuital law	$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$

- **E** electric field.
- **B** magnetic field.
- **J** electric current sources.
- ρ charge density.
- t time.

$\varepsilon_0 = 8.85\,10^{-12} kg^{-1} m^{-3} A^2 s^4$, $\mu_0 = 4\pi 10^{-7} kg\, m\, A^{-2} s^{-2}$, $\varepsilon_0 \mu_0 c^2 = 1$.
$\mathbf{E}$ is expressed in $V\, m^{-1}$, $\mathbf{B}$ in T (tesla), $\mathbf{J}$ in $A\, m^{-2}$ and ρ in $C\, m^{-3}$.

Electric current sources **J**

- Two components
  - Volumic ohmic currents $\sigma \mathbf{E}$.
  - Polarization currents $\frac{\partial \mathbf{P}}{\partial t}$.

$$\mathbf{J} = \sigma \mathbf{E} + \frac{\partial \mathbf{P}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right)$$

- **P** = $(\varepsilon - \varepsilon_0)$ **E** : polarization vector.
- ε: Permitivity of the medium.
- σ: Conductivity.

Electrical current propagation

- Quasistatic approximation
  → time derivatives can be neglected.

Name	Differential form
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Poisson equation

From Maxwell-Ampere $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ we deduce $\nabla \cdot \mathbf{J} = 0 \ . \quad (1)$

$$\nabla \times \mathbf{E} = 0 \implies \mathbf{E} = -\nabla \boxed{V} \ .$$

This is the so-called potential, which is defined up to a constant.V

We divide the current density $\mathbf{J}$ into two components:

$$\mathbf{J} = -\sigma \nabla V + \mathbf{J}^{\mathrm{p}} .$$

primary current

Ohmic or *return* current $\sigma \mathbf{E}$

Plugging this expression of J in (1) gives: $\nabla \cdot (\sigma \nabla V) = \nabla \cdot \mathbf{J}^{\mathrm{p}} \ .$

Green function of the Laplacian

The Green function G_L for the Laplacian in $\mathbb{R}^3$ is a solution in $\mathbb{R}^3$ of:

$$\Delta G_L(\mathbf{r}) = \delta_0(\mathbf{r}) \ ,$$

where δ_0 denotes the dirac mass positionned at the origin of the space.

Theorem A.1. *The Green function for the Laplacian in $\mathbb{R}^3$ with radial symmetry is:*

$$G_L(\mathbf{r}) = -\frac{1}{4\pi\|\mathbf{r}\|} \ .$$

Theorem A.2. *A solution of the equation $\Delta u = f$ is given by:*

$$u(\mathbf{r}) = (G_L * f)(\mathbf{r}) = \int_{\mathbb{R}^3} G_L(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') d\mathbf{r}' \ .$$

Biot-Savart Law

From Maxwell-Ampere $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, we can write a Poisson like equation

$$-\Delta \mathbf{B} = \mu_0 \nabla \times \mathbf{J}$$

Using the Green function of the Laplacian, we get:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla' \times \mathbf{J}(\mathbf{r}') \frac{1}{||\mathbf{r} - \mathbf{r}'||} \, d\mathbf{r}' .$$

It is easy to show that B does not depend on conductivity in an infinite and homogeneous medium.

Biot-Savart Law (alternate form).

Integrating by part the previous formula, we get:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \, d\mathbf{r}' \; .$$

Or:

$$\mathbf{B} = \mathbf{B}_0 - \frac{\mu_0}{4\pi} \int \sigma \nabla' V(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \, d\mathbf{r}' \; ,$$

$$\mathbf{B}_0 = \frac{\mu_0}{4\pi} \int \mathbf{J}^{\mathrm{p}}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \, d\mathbf{r}' \; .$$

Silent sources

There are configurations of non-null sources that give null measurements at sensors.

Examples are:

- $\nabla \cdot \mathbf{J}^{\mathrm{p}} = 0$, which can happen if $\mathbf{J}_{\mathbf{p}} = \nabla \times \mathbf{b}$ for any vector field $\mathbf{b}$.
  $\Longrightarrow$ The potential $\mathbf{V}$ is constant everywhere (but $\mathbf{B}$ varies).
- Radially oriented sources for MEG in spherical geometry (but there is EEG signal).
- Equally distributed sources on a closed smooth surface (both EEG and EEG are zero outside of the surface and constant inside it).

Forward models…

From equations to models

- Equations depend on some physical quantities: sources $\mathbf{J}^{\mathrm{p}}$ and conductivities σ.
- Conductivities depend on the tissues $\Longrightarrow$ geometry.
- Methods will depend on how we take into account of this geometry.
- *Realistic geometries are provided by MR images.*

Forward models

- **Analytic methods**
  - simple geometry (nested spheres),
  - very often used for MEG in practise.
- **Surface methods**
  - conductivity assumed piecewise constant.
- **Volume methods**
  - conductivity (scalar/tensor) defined at each voxel.

From previous equations.

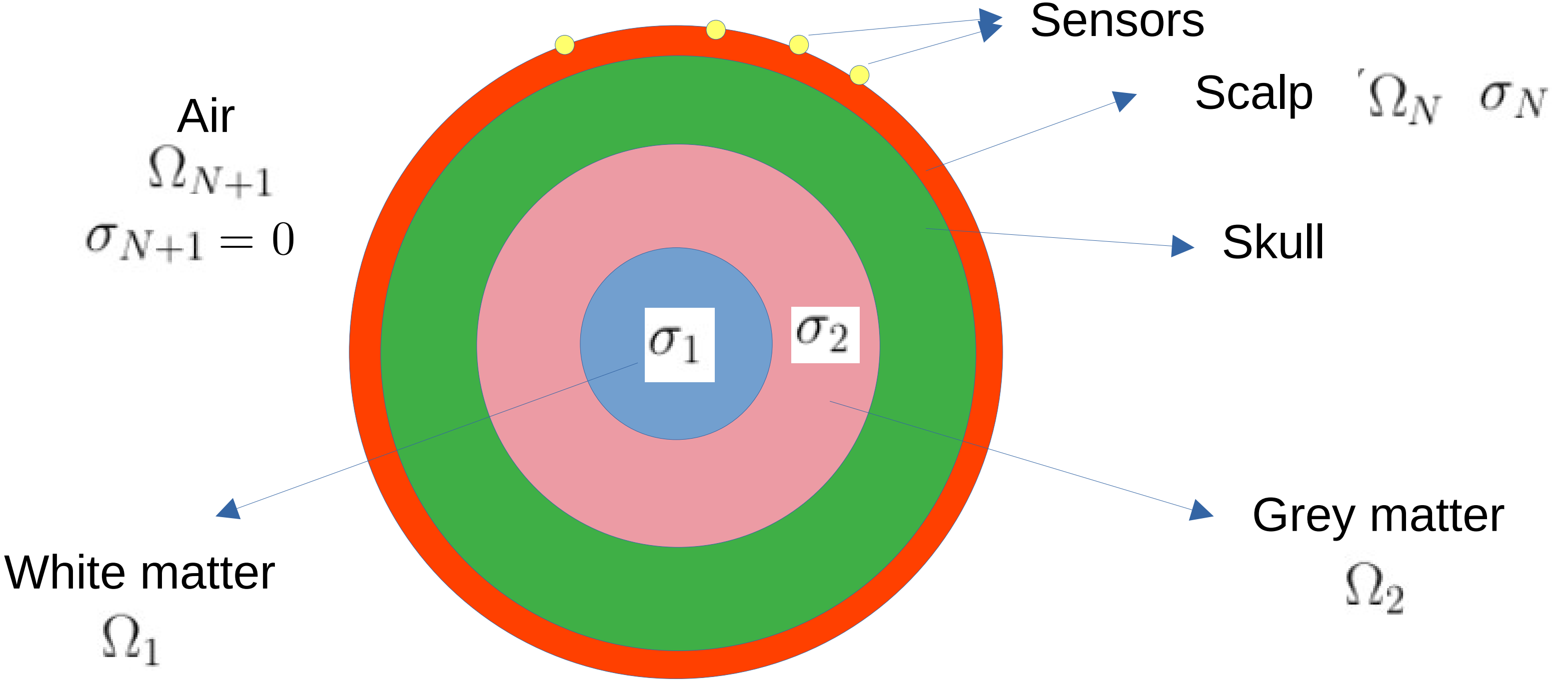
- **Discretization:** BEM or FEM, P0 or P1, . . .

$$A\,X = B$$

Structure	Unknowns	Source term
Δ_σ	$V, \partial_{\mathbf{n}} V$	$f(\mathbf{J}^{\mathrm{p}})$

Spherical models

The head is modeled as a set of concentric spheres with homogeneous tissue between any two spheres.



Each tissue has an homogeneous constant and isotropic conductivity.

Parameters are the radii of the spheres and the conductivities.

$\Downarrow$

With these parameters, it is possible to compute values at sensors analytically.

Spherical models

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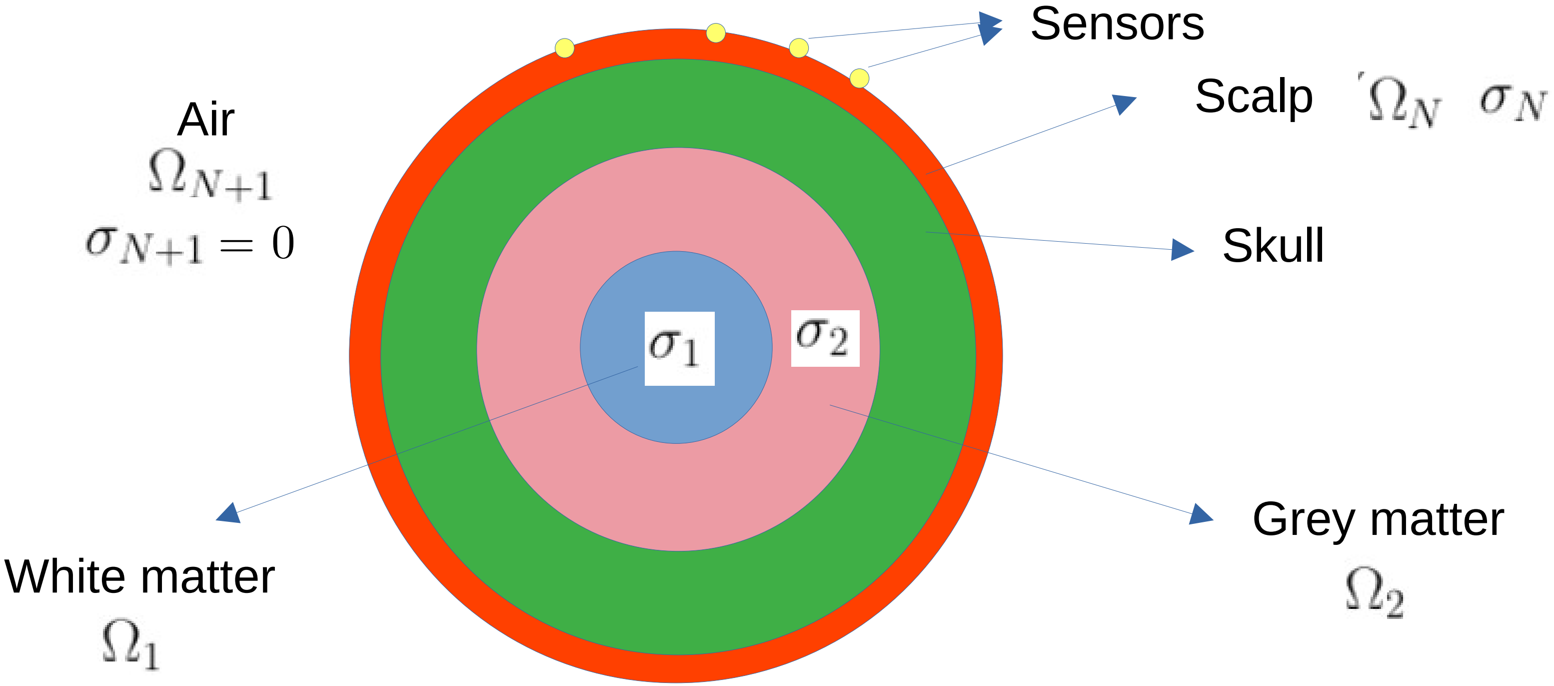
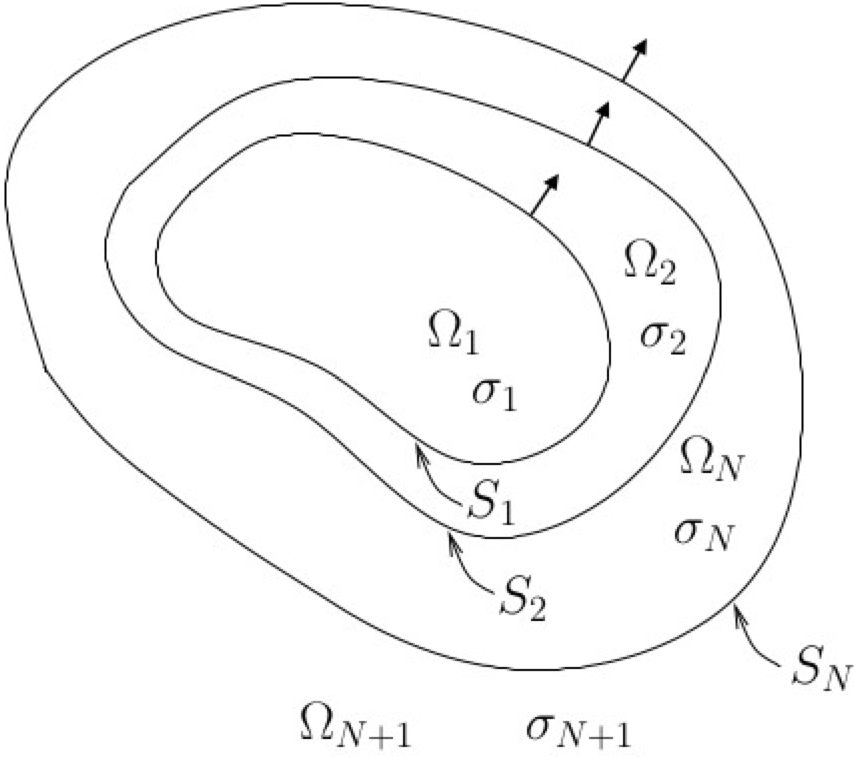


With these specific models, it is possible to show that the magnetic field $\mathbf{B}$ does not depend on conductivity (only $\mathbf{B}_0$ is non-zero) and on radial components of sources.

Semi-realistic model



A generalization of concentric spheres which allow to change the shape of the interfaces between tissues.

Conductivities are still constant homogeneous and isotropic within each tissue.

There is no longer an analytic solution, but there are continuous surface equations that need to be solved.

Discretization gives a linear system
$\Longrightarrow$ Surface methods
(Boundary Element Methods - BEM).

Semi-realistic model

Magnetic field

Idea:

- Start from Biot-Savart (second form).
- Split the integral along the homogeneous domains.
- On each domain, get the conductivity out of the integral.
- Apply Stockes to convert volume integral to boundary ones (1 boundary = 2 surfaces). Use the identity

$$\nabla \times (V \, \nabla g) = \nabla V \times \nabla g$$

- Re-arrange integrals along surfaces.

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0(\mathbf{r}) + \frac{\mu_0}{4\pi} \sum_{i=1}^{N} (\sigma_i - \sigma_{i+1}) \int_{S_i} V(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \times \mathbf{n}' \, ds(\mathbf{r}')$$

Semi-realistic model

Magnetic field for concentric spheres:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0(\mathbf{r}) + \frac{\mu_0}{4\pi}\sum_{i=1}^{N}(\sigma_i - \sigma_{i+1})\int_{S_i} V(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{\|\mathbf{r}-\mathbf{r}'\|^3} \times \mathbf{n}'\, ds(\mathbf{r}')$$

- First show that $\mathbf{B}(\mathbf{r}) \cdot \mathbf{r} = \mathbf{B}_0(\mathbf{r}) \cdot \mathbf{r}$
- Outside of the head, introduce a magnetic potential U: possible because we have $\nabla \times \mathbf{B} = 0$.
- Express U as an integral of $\mathbf{B}(\mathbf{r}).\mathbf{e}_r$, where is a radial unit vector.
- All components of B are independent of conductivities.

Semi-realistic model

Magnetic field for concentric spheres: **Sarvas formula.**
Valid outside of the outer sphere.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_o}{4\pi F^2}\left(F\mathbf{q}\times\mathbf{r}_0 - (\mathbf{q},\mathbf{r}_0,\mathbf{r})\nabla F\right) \ ,$$

$\mathbf{B}(\mathbf{r}) = \mathbf{0}$ for radial dipoles, outside of the head.

with

$$\nabla F = \left(a + 2r + \frac{\mathbf{a}\cdot\mathbf{r}}{a}\right)\mathbf{a} + \left(\frac{a^2}{r} + a\right)\mathbf{r} \ ,$$

$$\nabla F = \left(\frac{a^2}{r} + \frac{\mathbf{a}\cdot\mathbf{r}}{a} + 2(a+r)\right)\mathbf{r} - \left(a + 2r + \frac{\mathbf{a}\cdot\mathbf{r}}{a}\right)\mathbf{r}_0 \ .$$

$\mathbf{a} = \mathbf{r} - \mathbf{r}_0,\ a = \|\mathbf{a}\|,\ r = \|\mathbf{r}\|,\ F = a(ar + \mathbf{a}\cdot\mathbf{r}).$

Semi-realistic model

Electric field

Idea:

- Consider the potential V^∞ created by a source in an infinite homogeneous model (in domain i).
- Consider the function $u = \sigma V - \sigma_i V^\infty$ (other choices are possible).
- Show that u is harmonic.
- Apply the representation theorem of harmonic functions.

$$\frac{\sigma_j + \sigma_{j+1}}{2} V_j - \sum_{k=1}^{N} (\sigma_k - \sigma_{k+1})\, \mathcal{D}_{jk}\, V_k = \sigma_i V_j^\infty$$

Gesselovitz formula (1967).

Representation theorem for Harmonic functions

Any harmonic function $u(\mathbf{r})$ can be expressed as a function of some quantities defined at some surfaces:

$$u(\mathbf{r}) \quad = \mathcal{D}\left[u\right] - \mathcal{S}\left[\partial_{\mathbf{n}} u\right] \qquad \text{for } \mathbf{r} \in \Omega$$

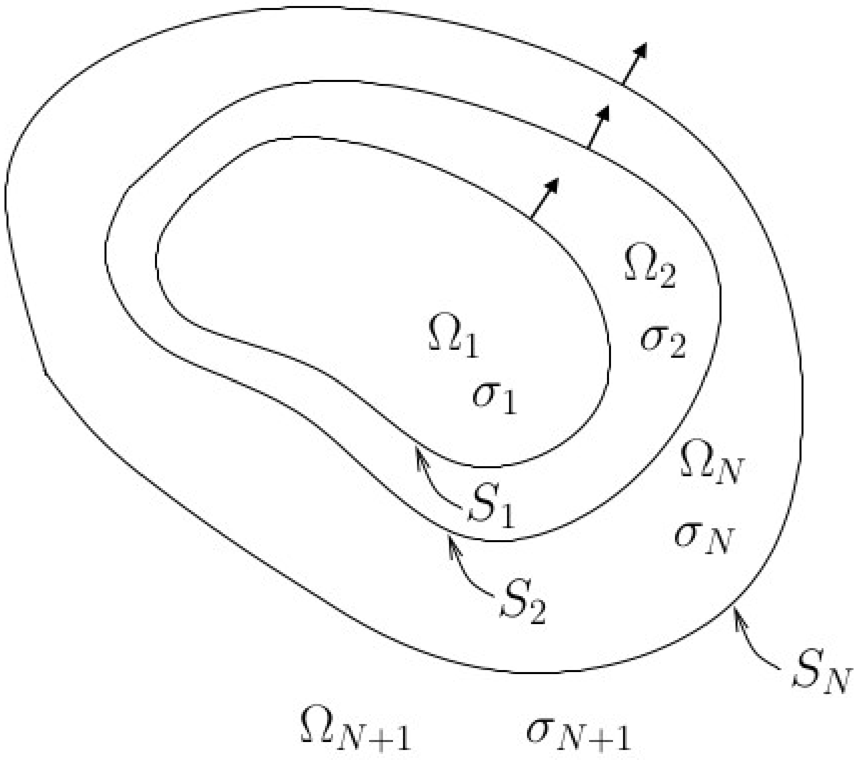
$$u^{\mp}(\mathbf{r}) \quad = \left(\pm\frac{I}{2} + \mathcal{D}\right)\left[u\right] - \mathcal{S}\left[\partial_{\mathbf{n}} u\right] \qquad \text{for } \mathbf{r} \in \partial\Omega$$

With:

$$\big(\mathcal{D}f\big)(\mathbf{r}) \quad = \int_{\partial\Omega} \partial_{\mathbf{n}'} G_L(\mathbf{r}-\mathbf{r}') f(\mathbf{r}')\,\mathrm{d}s(\mathbf{r}')$$

$$\big(\mathcal{S}f\big)(\mathbf{r}) \quad = \int_{\partial\Omega} G_L(\mathbf{r}-\mathbf{r}') f(\mathbf{r}')\,\mathrm{d}s(\mathbf{r}')\ .$$

Realistic model



A generalization of concentric spheres which allow to change the shape of the interfaces between tissues.

Conductivities are no longer constant homogeneous and isotropic within each tissue (they can change at each point).

There is no longer an analytic solution, but there are continuous volumic equations that need to be solved.

Discretization gives a linear system
$\Longrightarrow$ Volumic methods
(Finite Element Methods - FEM).

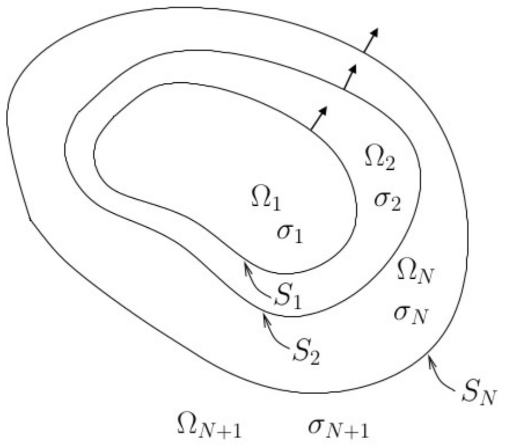
Realistic model

Electric field: Variational formulation

Equivalence of the following formulations:

① $V \in H^2(\Omega)$ is solution of:

$$\begin{cases} \nabla\cdot(\sigma\nabla V) &= f \quad \text{in} \quad \Omega \\ \sigma\dfrac{\partial V}{\partial \mathbf{n}} = \sigma\nabla V\cdot\mathbf{n} &= g \quad \text{on} \quad S=\partial\Omega. \end{cases}$$



Everything remains similar with an anisotropic conductivity Σ.

② $V \in H^1(\Omega)$ is such that

$$\forall w \in H^1(\Omega) \quad \int_\Omega \sigma(\mathbf{r})\nabla V(\mathbf{r})\cdot\nabla w(\mathbf{r})\,d\mathbf{r}+\int_\Omega f(\mathbf{r})w(\mathbf{r})\,d\mathbf{r}-\int_S g(\mathbf{r})w(\mathbf{r})\,ds=0\;.$$

Multiply by w, integrate and apply integration by parts.

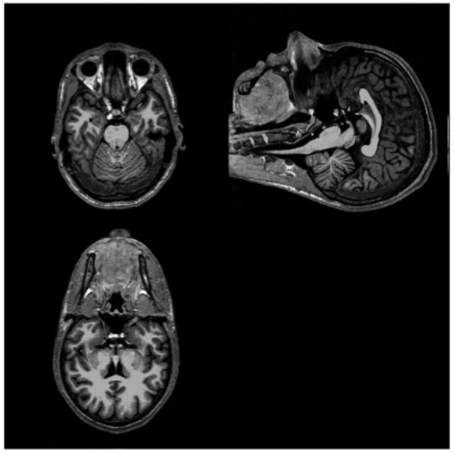
③ $V = arg\min_{\phi\in H^1(\Omega)} E(\phi)$ with:

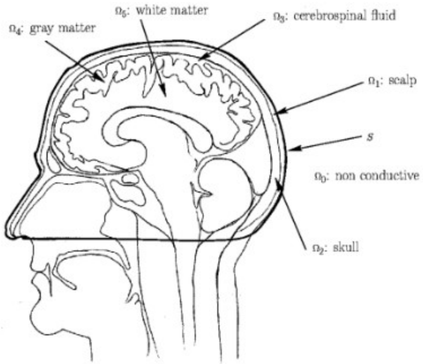
$$E(\phi)=\frac{1}{2}\int_\Omega \sigma(\mathbf{r})\|\nabla\phi(\mathbf{r})\|^2\,d\mathbf{r}+\int_\Omega f(\mathbf{r})\phi(\mathbf{r})\,d\mathbf{r}-\int_S g(\mathbf{r})\phi(\mathbf{r})\,ds\;.$$

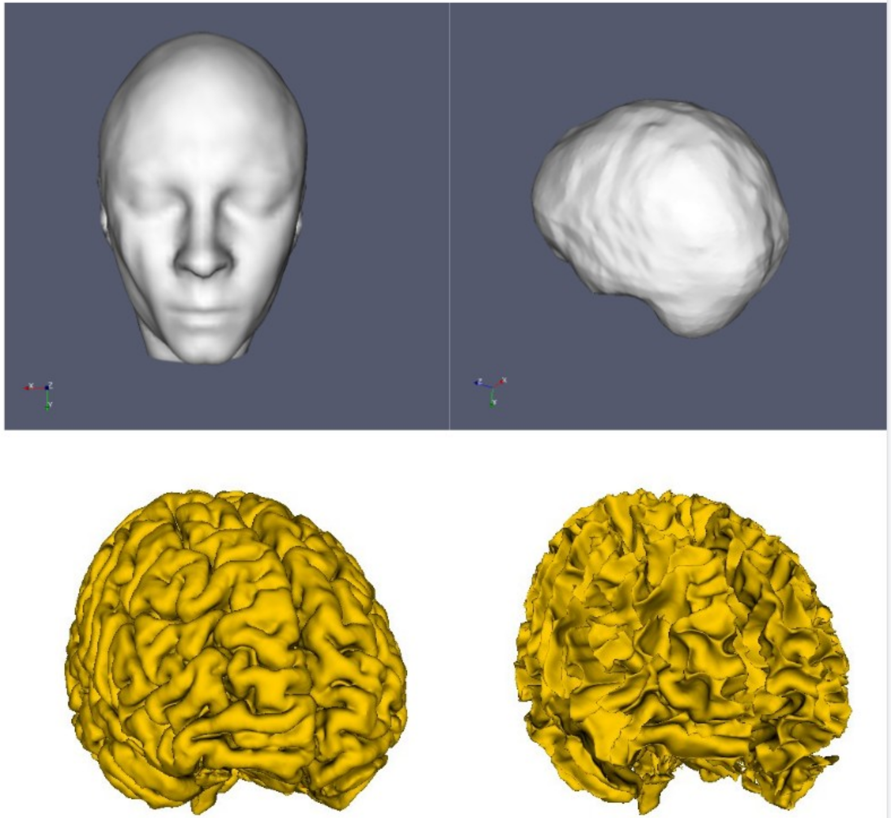
Solve this problem. Notice the decreased derivatives.

Tissue modelling

Needed for semi-realistic and realistic methods





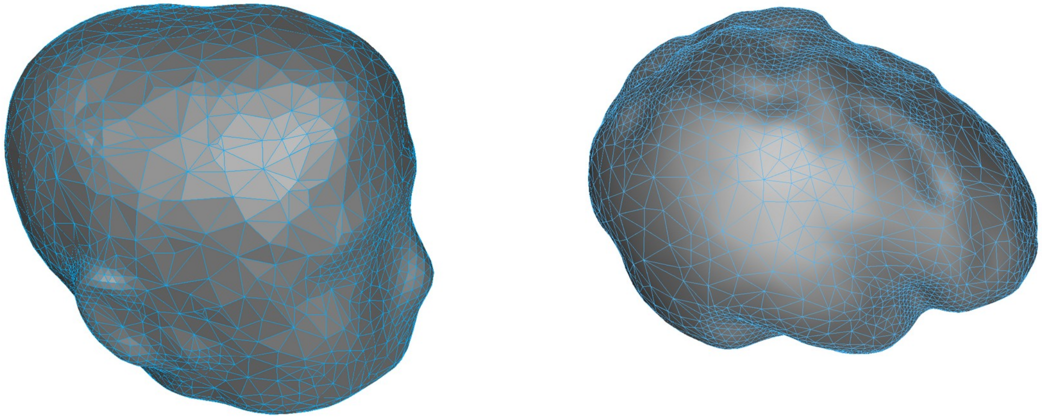


Discretization

Continuous models involve continuous quantities:

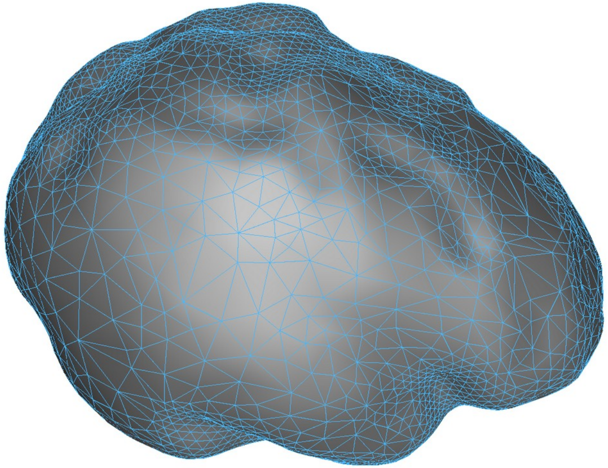
- Geometric models.
- Physical quantities (e.g. potential or magnetic field).

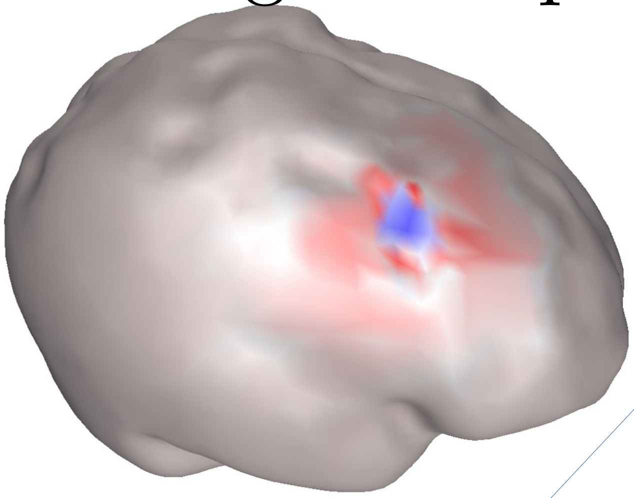
Those quantities need to be discretized in order to get a computational model $\Longrightarrow$ meshes (surfacic or volumic).



Physical quantities

- Discretization inherited from geometry.
- Discrete $\to$ Continuous through interpolation.



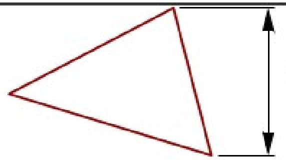
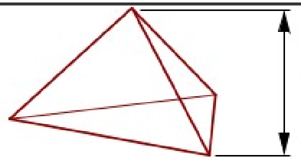
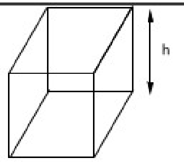
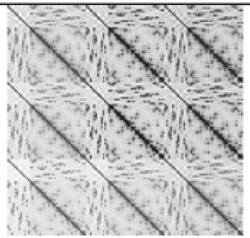
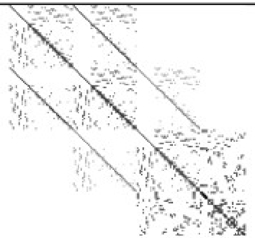
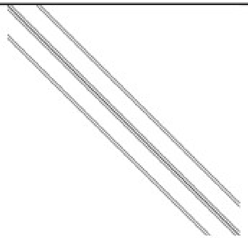


Interpolation formula.

Basis functions (with local support).

$$H_h^1(\Omega_h) = \left\{ \phi_h, \exists(\phi_1, \ldots, \phi_{N_V}) \in \mathbb{R}^{N_V}, \quad \phi_h(\mathbf{r}) = \sum_{i=1} \phi_i w^i(\mathbf{r}) \right\}$$

Discretization

Boundary Elements	**Finite Elements**	**Implicit Finite Elements**
piecewise constant	arbitrary	arbitrary
surface	volume	volume
		
$N = O(h^{-2}), \quad 10^5 \sim 10^6$	$N = O(h^{-3}), \quad 10^7 \sim 10^8$	$N = O(h^{-3}), \quad 10^7 \sim 10^8$
 126 $\times$ 126, full	 156 $\times$ 156, **sparse** (8%)	 **sparse,banded** (1.5%)
symmetric	symmetric positive	symmetric positive
GMRES, QMR, ...	conjugate gradient	conjugate gradient
FMM, multiscale	multilevel	multigrid

Computing the Forward model

Realistic and semi-realistic models.

- **Discretization:** BEM or FEM, P0 or P1, ...

$$AX = B$$

Structure Δ_σ | Unknowns $V, \partial_{\mathbf{n}} V$ | Source term $f(\mathbf{J}^{\mathrm{p}})$

- C : Number of EEG sensors.
- S: Size of the discretization of the source space.

We solve for $\mathbf{V}$ in all the volume of the head, and just keep values at sensors. Because the problem is linear in $\mathbf{J_p}$, we can do that for every source (or source component) independently.

For each source (or source component), we collect values at sensors in a matrix:

$$\mathbf{V}_s = \mathbf{G}\ \mathbf{J_p}$$

Vector of potential at sensors of size C.

Vector describing the sources of size S.

Leadfield (or gain) matrix C x S.

Computing the Forward model

Realistic and semi-realistic models.

- **Discretization:** BEM or FEM, P0 or P1, ...

$$AX = B$$

Structure	Unknowns	Source term
Δ_σ	$V, \partial_{\mathbf{n}} V$	$f(\mathbf{J}^{\mathrm{p}})$

- C' : Number of MEG sensors.
- S: Size of the discretization of the source space.

We solve for $\mathbf{B}$ in all the volume of the head, and just keep values at sensors. Because the problem is linear in $\mathbf{J_p}$, we can do that for every source (or source component) independently.

A very similar leadfield matrix can be constructed for the magnetic field.

$$\mathbf{B}_s = \mathbf{G_B}\ \mathbf{J_p}$$

Vector of magnetic field at sensors of size C'.

Vector describing the sources of size S.

Leadfield (or gain) matrix C' x S.

Computing the Forward model

Realistic and semi-realistic models.

For each source (or source component), we have two gain matrices:

Vector of potential at sensors of size C. $\leftarrow$ $\mathbf{V}_s = \mathbf{G}\ \mathbf{J_p}$ $\rightarrow$ Vector describing the sources of size S.

$\mathbf{G}$: $\boxed{\text{Leadfield (or gain) matrix C x S.}}$

Vector of magnetic field at sensors of size C'. $\leftarrow$ $\mathbf{B}_s = \mathbf{G_B}\ \mathbf{J_p}$ $\rightarrow$ Vector describing the sources of size S.

$\mathbf{G_B}$: $\boxed{\text{Leadfield (or gain) matrix C' x S.}}$

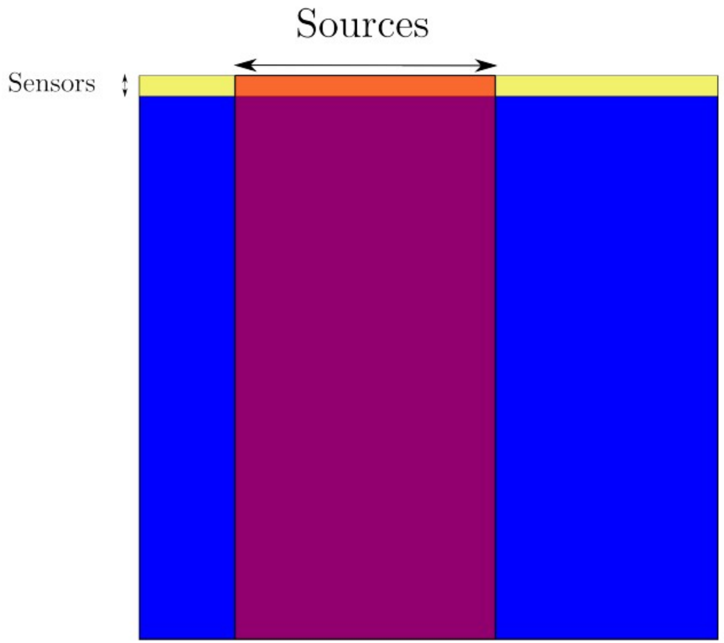
- C : Number of EEG sensors.
- C': Number of MEG sensors.
- S: Size of the discretization of the source space.

These equation are for a single time instant but can be easily extended to handle time windows (replace $\mathbf{V}_s$, $\mathbf{B}_s$ and $\mathbf{J_p}$ by matrices which second dimension is T (size of the time window). $\mathbf{G}$ and $\mathbf{G_B}$ are unchanged.

Adjoint method

Efficient leadfield computation

- Leadfield is the basis of inverse MEG/EEG problems.
- Computed from head and source models.



Leadfield
Inverse head matrix $\mathbf{H}^{-1}$

Trash (forward)
Trash (adjoint)

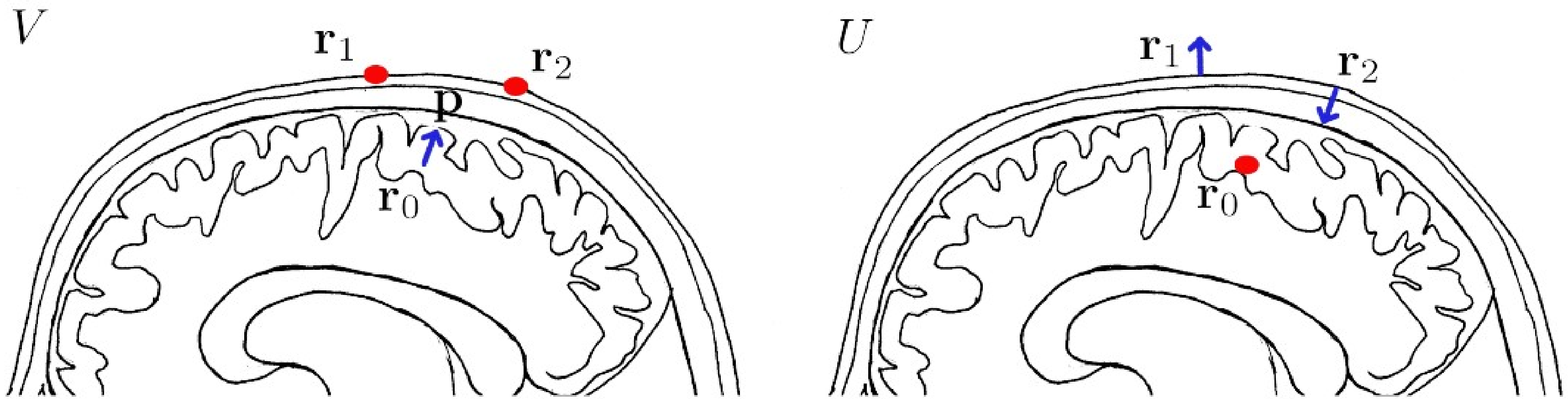
Source space $\gg$ Sensor space

Factor: $4 \leftrightarrow 100$

Standard way.

Adjoint method

Helmholtz reciprocity theorem (1853)...

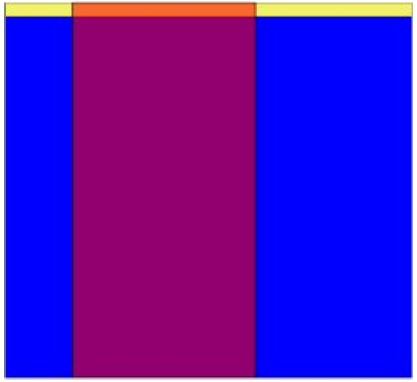


$$V(\mathbf{r}_2) - V(\mathbf{r}_1) = \nabla U(\mathbf{r}_0) \cdot \mathbf{p}$$

Adioint method

. . . or matrix manipulations.

$$\overset{\text{Head matrix}}{\mathbf{H}}\underset{\text{Potential}}{V} = \overset{\text{Projection operator}}{\mathbf{D}}\underset{\text{Sources}}{P} \qquad \overset{\text{Sensor potential}}{V_S} = \underset{\text{Selection}}{\mathbf{S}}V$$



$$\mathbf{L} = \frac{\partial V_S}{\partial P} = \mathbf{S}\mathbf{H}^{-1}\mathbf{D} = \mathbf{S}\blacksquare = \square\mathbf{D}$$

Standard	Adjoint
$\mathbf{H}\blacksquare = \mathbf{D}$	$\mathbf{H}^T\square^T = \mathbf{S}^T$
$\mathbf{L} = \mathbf{S}\blacksquare$	$\mathbf{L}^T = \mathbf{D}^T\square^T$

Inverse problems / Source localisation

Theoretical results / Practice

- Ill-posed problem
  - Non existence.
  - Non uniqueness $\rightarrow$ Silent sources.
  - Non continuity.
- Several cases where uniqueness can be proved.
  - Linear combination of isolated dipoles.
  - Surfacic distribution (up to a constant).
- This is with continuous measurements.
  In practice, we only have a finite number of them.

Measurement model

$$\mathbf{M} = \mathbf{G}\,\mathbf{J} + \boldsymbol{\varepsilon}$$

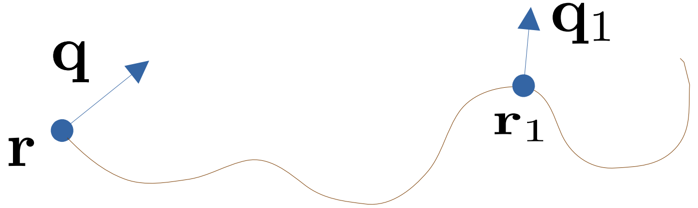
$$\mathbf{M} = \boldsymbol{\Sigma}\;\mathbf{G}(\mathbf{r}_{\mathrm{i}})\;\mathbf{J}_{\mathrm{i}} + \boldsymbol{\varepsilon}$$

Source models (J)

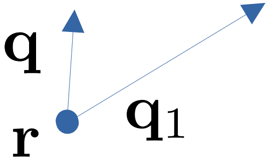
- Continuous vs isolated dipoles.
  We can model continuous distributions over a surface or a volume or just keep a finite number of single dipoles.
- Decrease number of parameters (often needed).
  - Known locations.
  - Cortical patches (poor spatial resolution of M/EEG $\Longrightarrow$ group dipoles of functional regions).
  - Constrain moments (in theory moments should be orthogonal to the cortex).

Source models (constraining moments)

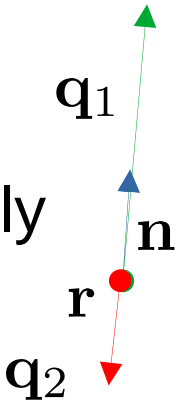
- Moving dipole: position and moment can change (6 parameters $\mathbf{r}$, $\mathbf{q}$).



- Rotation dipole: position is fixed only the moment can change (3 parameters).



- Fixed dipole: position and moment direction are fixed, only the strength of the moment can change (1 parameter).



$$\mathbf{q} = \lambda \mathbf{n}$$

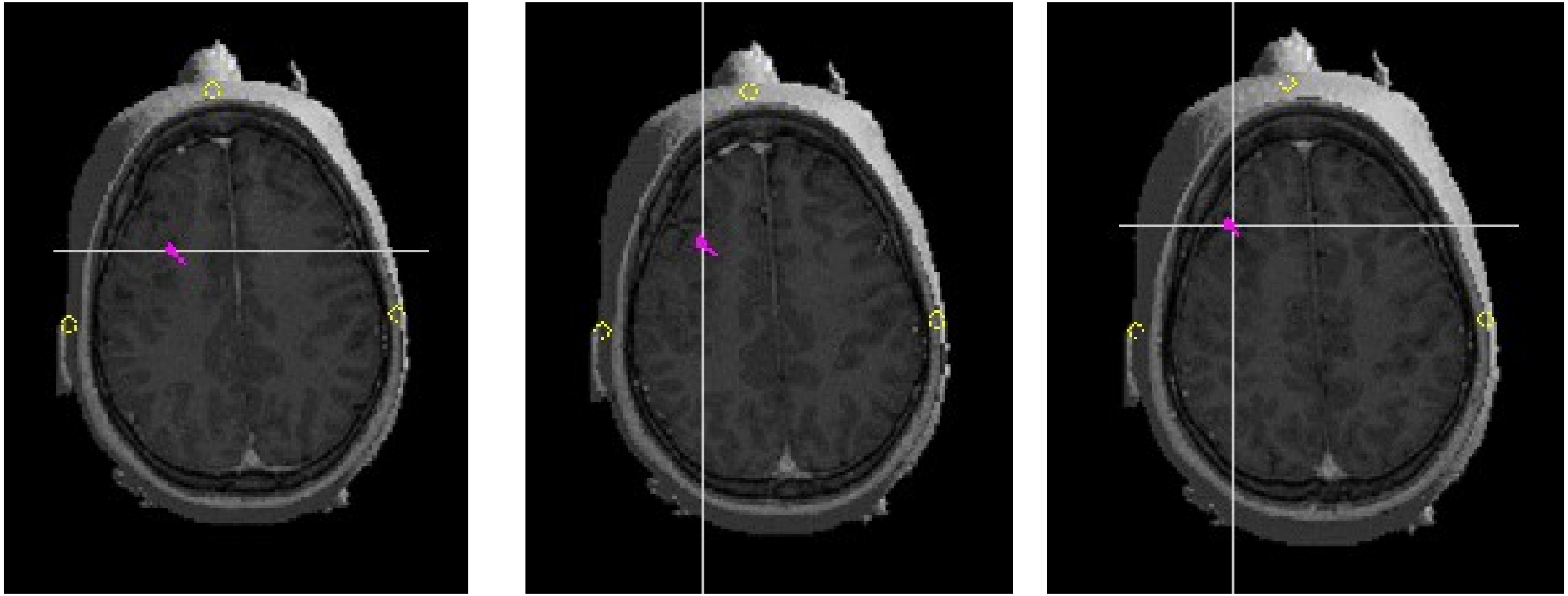
Dipole fit

- Find the dipole(s) position(s) and moment(s) that best fit the measurements.

$$\mathbf{J}_{\text{sol}} = \underset{\text{dipole(s) } \mathbf{J}}{\text{minimize}} \; \|\mathbf{M} - \mathbf{G}\,\mathbf{J}\|_F^2$$

- Works when the number of (isolated) dipoles is low.
- Non-linear problem (in position), linear (in moment)
  → Solved by gradient descent.

Dipole fit (example of solution)



$\sigma_{\mathrm{scalp}}/\sigma_{\mathrm{skull}} = 20$ $\sigma_{\mathrm{scalp}}/\sigma_{\mathrm{skull}} = 40$ $\sigma_{\mathrm{scalp}}/\sigma_{\mathrm{skull}} = 80$

Dipole fit

Advantages

- Very simple method.
- No assumption on dipole positions.

Drawbacks

- Depends on initialization.
- More complex when the number of dipole increases.
- Choice of the right number of dipoles ?
- Local minima.

Imaging method

- Opposite view of dipole fit:
  - Place dipole everywhere and evaluate their strengths.
- Very often used with “Fixed dipole paradigm”.
- Add regularization to remove “spurious” solutions.

Imaging method

Data attachment

$$C_\lambda(\mathbf{J}) = \|\mathbf{M} - \mathbf{G}\,\mathbf{J}\|^2 + \lambda\|\mathbf{J}\|^2 \ .$$

$$\mathbf{J}_{\mathrm{sol}} = \mathbf{J}_\lambda = \underset{\mathbf{J}}{\text{minimize}}\ \mathrm{C}_\lambda(\mathbf{J})$$

Smoothness / regularization.

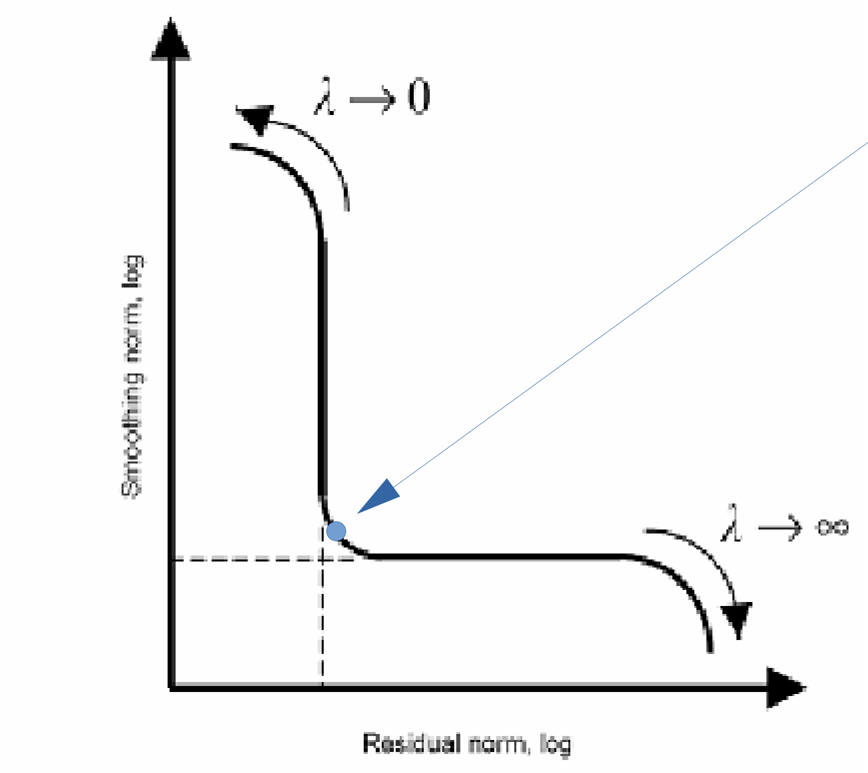
Solution:

$$\begin{aligned}\mathbf{J}_\lambda &= \left(\mathbf{G}^T\mathbf{G} + \lambda\,\mathbf{I}\right)^{-1}\mathbf{G}^T M \\ &= \mathbf{G}^T\left(\mathbf{G}\,\mathbf{G}^T + \lambda\,\mathbf{I}\right)^{-1} M\end{aligned}$$

More efficient.

Imaging method (L-curve)

How to find a proper value for λ.



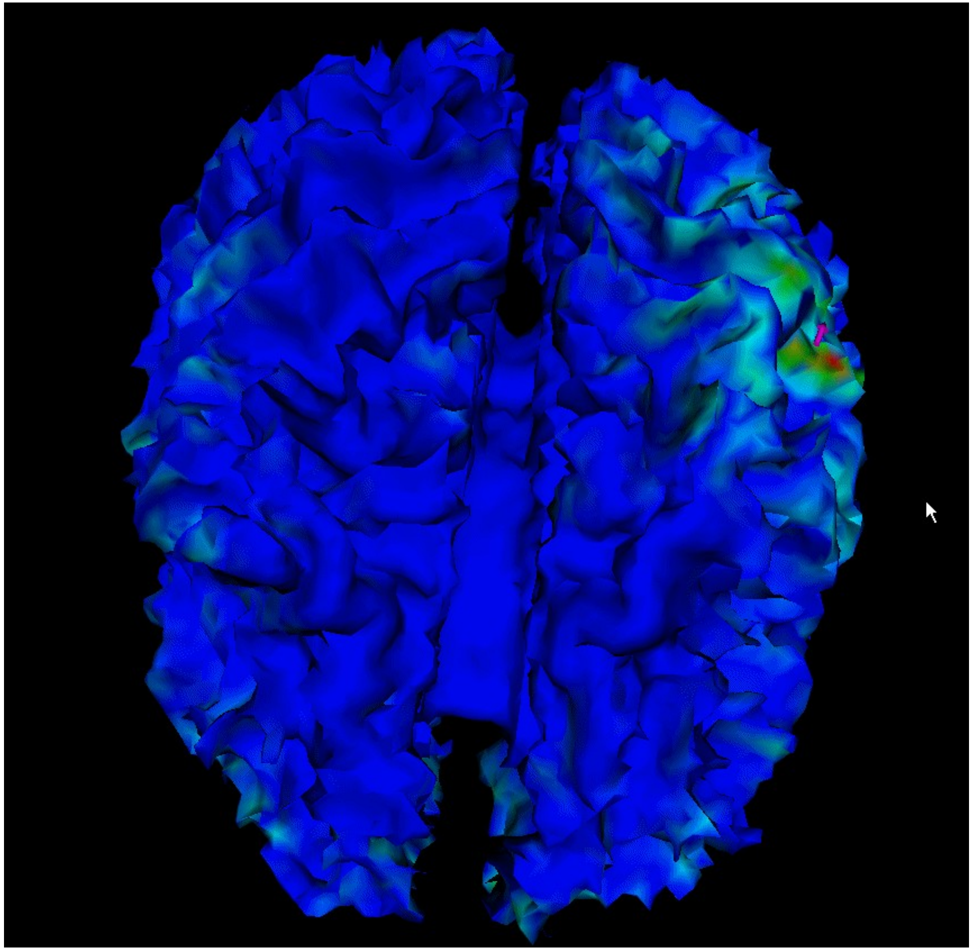
The best compromise between “smoothness” and “data attachment”.

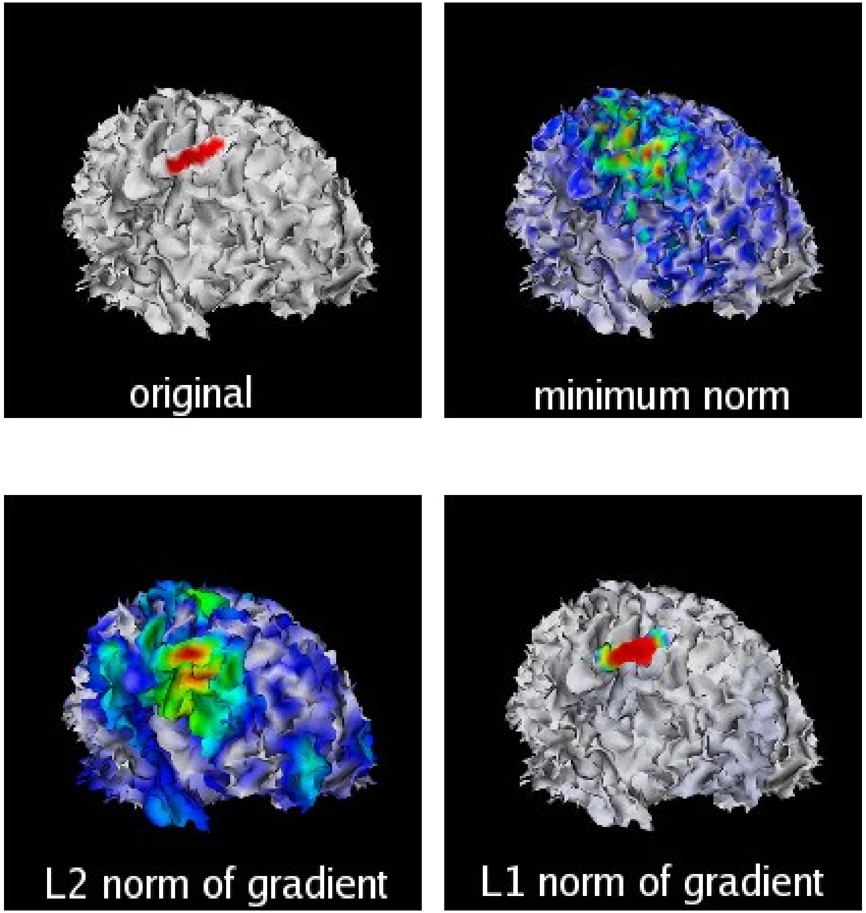
Imaging method (LOOCV)

Finding a proper value for λ with Leave One Out Cross-Validation.

- With multiple trials for the same task.
- Keep one sample as a test-set.
  Use the others for finding the solution $\mathbf{J}_\lambda$.
- Select the value of λ that minimizes the reconstruction error (data attachment) over all choices of the “leaved out” sample.

Imaging method (examples of solution)





Simulated data (10% of noise).

Imaging methods

Advantages

- Very simple method.
- Problem with unique solution.
- No need to choose a number of dipoles.
- Efficient (closed-form) solutions.

Drawbacks

- Depends on the regularization parameter.
- Complex solution which has to be interpreted by a human.
- Exploration of the solution.

Scanning method

Intermediate between “Moving dipole” and “Imaging methods”:

- As in moving dipole, assume a limited number of dipoles (choice of this number).
- As in imaging methods, possible dipole positions are fixed a priori.

<u>Selection of columns (positions) in the leadfield matrix.</u>

<u>Spatio-temporal methods.</u>

- Two families: MUSIC and beamformers/LCMV (Linear Constrained Minimum Variance).

Scanning method: MUSIC

MUltiple SIgnal Classification

- **Gain matrix assumption:** The $\mathbf{G}$ matrix for p dipoles is full rank (i.e. of rank r).

Source k

- **Asynchronous assumption:** The correlation matrix $R_{\mathbf{Q}} = E(\mathbf{Q}_k \mathbf{Q}_k^T)$ matrix for p dipoles is full rank (i.e. of rank r).

- **Noise whiteness assumption:** The noise is considered additive and temporally and spatially zero-mean white noise with variance σ^2. When a good noise model can be established, a prewhitening phase ensures that this is the case. Additionally the signal and noise are assumed to be uncorrelated.

Scanning method: MUSIC

Compute the matrix $\mathbf{F} = E(\mathbf{M}_k\,\mathbf{M}_k^T)$ or $\mathbf{F} \approx E(\mathbf{M}_k)\,E(\mathbf{M}_k)^T$.

Data for trial k.

Find the eigenvectors $\mathbf{U}$ of $\mathbf{F}$.

Split $\mathbf{U}$ between signal and noise spaces $\mathbf{U} = [\mathbf{U}_r, \mathbf{U}_{m-r}]$.

Signal space. Noise space.

$$C(x_i) = \frac{\|\mathbf{U}_{m-r}^T \mathbf{G}_i\|}{\|\mathbf{G}_i\|} = \frac{\|P_{\mathbf{U}_r}^{\perp} \mathbf{G}_i\|}{\|\mathbf{G}_i\|}$$

Find the position(s) x_i (corresponding to $\mathbf{G}_i$) that minimize(s) the projection $C(x_i)$ of the measurements on the noise space (i.e. maximize the contribution in the signal space).

Scanning method: MUSIC

Find the position(s) x_i (corresponding to $\mathbf{G}_i$) that minimize(s) the projection $C(x_i)$ of the measurements on the noise space (i.e. maximize the contribution in the signal space).

- Extract the first p maxima (Standard MUSIC).
  $\rightarrow$ Problem: close sources often explain the same signal.

- Greedy approach (RAP-MUSIC):
  1. Extract the biggest maximum.
  2. Remove the contribution of that source to the signal.
  3. Re-apply MUSIC on this new signal (p times) to succesive sources.

- Many other variants (TRAP-MUSIC)...

Scanning method: Beamformers

- the noise $\mathbf{N}$ is zero-mean, with covariance C_N.;
- the sources are decorrelated: if $i \neq k$, $E\left([J(x_i) - \overline{J(x_i)}]\,[J(x_k) - \overline{J(x_k)}]^T\right)$ is the 3×3 null matrix;
- the noise and the source amplitudes are decorrelated

Idea: find an approximate “inverse” matrix that inverts the leadfield for the source and minimizes the values on other sources. The selected source is the one that minimizes a variance criterion (using the optimal filter for that source).

Scanning method: Beamformers

Similar ideas as with MUSIC but for the criterion:

$$\widehat{VarJ(x_0)} = \frac{Tr\left((G(x_0)^T C_{\mathbf{M}}^{-1} G(x_0))^{-1}\right)}{Tr\left((G(x_0)^T C_{\mathbf{N}}^{-1} G(x_0))^{-1}\right)} \, .$$

$J(x_0)$ is the reconstructed source by applying a filter $W(x_0)$ to the data.

The concept behind beamforming is, for a given spatial position x_0, to apply a spatial filtering to the measurements, which filters out sources which do not come from a small volume around x_0. Let $W(x_0)$ be a $m \times 3$ matrix representing the spatial filter: the source amplitude in the vicinity of x_0 will be estimated by

$$S(x_0) = W(x_0)^T \mathbf{M} \, .$$

$W(x_0)$ is computed to minimize the strength of the reconstructed source under the constraint that $W(x_0)\mathbf{G}(x_0)^T = \mathbf{I}$.