Imperial College London

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M3R Research Project

Household Consumption-Portfolio Choice Problem

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Introduction

- Portfolio Choice Problems is a class of problems in Finance concerning how much and how an agent should invest their personal wealth.
- Considering the problem in a single time frame the most popular model of asset management is Mean-Variance theory which prescribes to invest money in a combination of a risky asset representing the Market and a risk-less asset like a US Treasury Bond.
- The quantity invested in each depends on the degree of risk aversion of the individual.

Introduction

- Mean-variance model assumes a single period. What happens if we have a continuous (potentially infinite) period ?
- This is the framework of the Intertemporal portfolio choice problem pioneered by the Nobel prize Robert Merton in 1969.
- Since the investment opportunities change (in terms of expectation and variance of returns), a result of Merton's solution is that the portfolio formula will have to include a term to hedge against these changes.

Goals

- Introduce Household Portfolio choice Problems
- Outline the theory and machinery necessary to solve the problem
- Exemplify the methods through coded financial application
- Solve the Portfolio-Consumption Problem through numerical schemes and Python implementation

Households Portfolio Choice Problem

- Suppose we have a household with wealth W_t at time t that wants to maximise the present value of the utility of their consumption C_t in a infinite time horizon.
- The household is subject to a constant income flow Y per unit time.
- The remaining part of their unconsumed budget is invested in a portfolio comprising a risk-less asset with return r and a risky asset with return dR_t.
- The utility of consumption is given by the economics Costant Relative Risk Aversion (CRRA) formula

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \tag{1}$$

Households Portfolio Choice Problem

- Define the instantaneous return of the portfolio altogether as

$$dR_{p,t} = (1 - \phi_t)rdt + \phi_t dR_t \tag{2}$$

- Where ϕ_t is fraction in the risky asset at time t.
- Both ϕ_t and C_t are controlled by the household.
- Therefore the wealth of the household changes according to

$$dW = W_{t+dt} = (W_t + (Y - C_t)dt)(1 + dR_{p,t}) \tag{3}$$

Households Portfolio Choice Problem

3 questions must be answered to solve the problem:

- How can we develop a framework to find the best controls for the given objective?
- How can we model the risky asset return dR_t?
  How can we operate on such an unpredictable variable?
- Once we get an equation for the optimal control,
  will it be possible to solve it for a close form solution?
  If not what should we use?

Households Portfolio Choice Problem

- Control theory (a.k.a. Dynamical programming)
- Stochastic Models and Stochastic Calculus
- Numerical methods: Finite Difference Schemes

Finite Difference Schemes

- Finite difference schemes are a numerical method to solve PDEs.
- They involve approximating the derivatives with finite difference formulas on a discrete mesh.

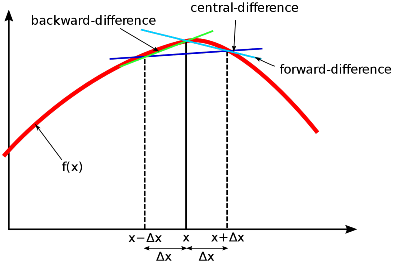

Figure 1: Finite Difference Formulas, image by Wikipedia

Finite Difference Schemes for a 2nd order PDE

- FTCS
  Explicit, unstable for small Δt and has error of order $o(\Delta t)$.
- BTCS
  Implicit, unconditionally stable and has error of order $o(\Delta t)$.
- Crank-Nicholson
  It is implicit, unconditionally stable and has error of order $o(\Delta t^2)$.

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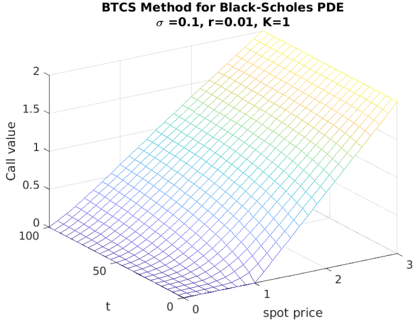
Black Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{4}$$

where $t \in (0, T]$ and $S \in [0, \infty)$. V is the price of the derivative, t is the current time, T is the expiry time, S and σ are the price and volatility of the underlying , r is the risk-free rate.



Stochastic Calculus

- The use of stochastic processes to model the stock price can be traced back to Louis Bachelier's 1900 thesis "Theory of Speculation" [DE06] which developed the diffusion processes framework in modelling options.
- After the seminal works of Black and Scholes [Tou11], this is the preferred framework in the theory of asset pricing.
- The general form for a diffusion process is

$$dx(t) = \mu(x(t), t)dt + \sigma(x(t), t)dz \tag{5}$$

where $\mu(x(t), t)$ and $\sigma(x(t), t)$ are time dependent but deterministic functions of Brownian motion $z(t)$.

Stochastic Calculus Toolbox

- Ito's lemma
  It can be seen as the stochastic equivalent of the Taylor expansion.

$$dF = \left[\partial_x F \mu(x(t),t) + \partial_t F + \frac{1}{2}\partial^2_{xx}\sigma(x(t),t)\right]dt + \partial_x \sigma(x(t),t)dz \tag{6}$$

- Girsanov Theorem
  It enables us to calculate how the expectation of a process changes under a new probability measure.

$$E_t^{\mathrm{P}'}[dx(t)] = E_t[dx(t)] + E_t[\frac{dM_t}{M_t}dx(t)] \tag{7}$$

Stochastic Toolbox: Feynman-Kac Theorem

Consider the partial differential equation

$$\partial_t U(x,t) + \mu(x,t)\partial_x U(x,t) + \frac{1}{2}\sigma(x,t)^2 \partial_{xx}^2 U(x,t) \tag{8}$$

$$-v(x,t)U(x,t) + f(x,t) = 0 \tag{9}$$

subject to terminal condition $U(x,t) = \Psi(x)$ and
$U : \mathbb{R} \times [0, T] \to \mathbb{R}$
then Feynman-Kac theorem states that the solution to the above PDE is given by the following expectation of a stochastic process

$$U(x,t) = E^{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^r v(X_\tau,\tau)d\tau} f(X_r, r)dr \tag{10}$$

$$+\int_t^T e^{-\int_t^T v(X_\tau,\tau)d\tau} \Psi(X_T) | X_t = x\Big] \tag{11}$$

Heston Model

- developed by Steven Heston in 1993. [Hes93]
- tries to overcome some of the problematic assumptions of Black-Scholes like constant volatility.
- Price S_i and volatility v_t of asset i are described through two stochastic processes:

$$dS_i = \mu S_t dt + \sqrt{v_t} S_t dW_t^S \tag{12}$$

$$dv_t = k(\theta - v_t)dt + \xi \sqrt{v_t} dW_t^v \tag{13}$$

$$E(dW_t^S dW_t^v) = \rho \tag{14}$$

Heston Model Calibration

Table 1: Estimated model parameters SP500 call options data

Parameters	June 21 Raw Data	June 21 Preprocessed	16/17 Data
θ	0.000015	1.751837	0.122251
κ	10.072485	0.037471	4.996804
ξ	0.010340	0.67546	0.849266
ρ	-0.916820	-0.988886	-0.637706
v_0	0.001545	0.00562 1	0.079484
error	135805.64	95.36	2.87608

Control problem

- The goal of the problem is to find expressions for the control variables that satisfy the objective function and depend on the state variables at time t.

$$V_T(x(0), 0) = \max_c \left\{ \int_0^T P[x(t), u(t)]dt \right\} \tag{15}$$

subject to

$$\dot{x}(t) = S[x(t), c(t)] \tag{16}$$

.

- This can be easily adapted to a stochastic version for our problem

- Dynamical programming refers to a method of solving temporal problems. It relies on "Bellman Principle of Optimality" which solves the problem by breaking it down in several time steps and solving in a recursive manner.
- In continuous time it gives the powerful Hamilton-Jacobi-Bellman equation for solving the control problem:

$$\partial_t V(x,t) + \max_c \left\{ S(x,c)\partial_x V(x,t) + P(x,c) \right\} = 0 \quad (17)$$

Household Portfolio Choice Problem

In accordance with the Mutual Fund Theorem, it will be assumed that the household can invest in a combination of a risk-less asset and a market ETF. The return of the ETF will modeled by

$$dR_t = (r + \lambda v_t)dt + \sqrt{v_t}dZ_t \tag{18}$$

and

$$dv_t = \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t}dZ_{v,t} \tag{19}$$

which is the previously described Heston Model.

State variables

Substituting for dR_t in (3) we get the following continuous stochastic law also called the dynamic inter-temporal budget constraint which is our first state variable

$$dW_t = (Y - C_t)dt + W_t[rdt + \phi_t(\lambda v_t dt + \sqrt{v_t} dZ_t)] \tag{20}$$

which is subject to the two controls C_t and ϕ_t. The other state variable is given by

$$dv_t = \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t} dZ_{v,t} \tag{21}$$

Transformation

$$V_t = W_t + \frac{Y}{r} \tag{22}$$

The presence of an income enables the households to scale up their investment in the ETF by a factor

$$\hat{\phi}_t = \frac{V_t}{W_t}\phi_t \tag{23}$$

Since the income has no risk it acts exactly like a bond. Therefore, the household will want to allocate more money to the stock. This relation will permit us to find optimal controls through the income-less problem.

Objective function

$$J_t = \sup_{(C_s)_{s\geq t},(\phi_s)_{s\geq t}} E_t \int_0^{\infty} e^{-\delta(s-t)} u(C_s) ds \qquad (24)$$

subject to the incomeless state variables evolution outlined above. This can be expressed in terms of the Hamilton-Jacobi-Bellman Equation

$$0 = \sup_{C_t} u(C_t) - \delta J_t + \sup_{\phi_t} E \left[\frac{dJ_t}{dt} \right]$$

(25)

ODE derivation

Using Ito's lemma

$0 = \sup_{C_t} \Big[u(C_t) - \delta J_t + \sup_{\phi_t} \big[\partial_t J + \partial_v J[\kappa(\theta - v_t)] + \partial_W J[W_t r - C_t + W_t \phi_t \lambda v_t] + \frac{1}{2}\partial^2_{vv} J(\epsilon^2 v_t) + \frac{1}{2}\partial^2_{WW} J(W_t^2 \phi_t^2 v_t) - \partial^2_{Wv} J(\rho\epsilon\phi_t W_t v_t) \big] \Big]$

FOC:

$$C^{-\gamma} = \partial_W J \tag{26}$$

This tells us that consumption is inversely proportional to wealth growth

$$\lambda \partial_W J + \phi_t \partial^2_{WW} J W_t - \rho\epsilon\partial_{Wv} = 0 \tag{27}$$

This can be broke down in terms of risk premium, risk and hedging demand contributions.

Optimal Controls

With Ansatz $J_t = H(v_t)^\gamma U(W_t)$ we find the optimal controls:

$$C^* = (\partial_W J_t)^{-1/\gamma} = \left(\frac{H(v_t)^\gamma}{W_t^\gamma}\right)^{-1/\gamma} = \frac{W_t}{H(v_t)}$$

and

$$\phi_t^* = \frac{-\lambda(\frac{H(v_t)}{W_t})^\gamma - \rho\epsilon\gamma H(v_t)^{\gamma-1} W^\gamma H'(v_t)}{-\gamma W_t^{-\gamma-1} H(v_t)^\gamma W_t} = \frac{\lambda}{\gamma} - \rho\epsilon\frac{H'(v_t)}{H(v_t)} \quad (28)$$

ODE

Reinserting the optimal controls and with manipulation we get an ODE for the unknown function $H(v_t)$

$$0 = 1 - k(v)H(v) + \mu_v^{'}(v)H'(v) + \frac{1}{2}\sigma_v^2(v)H''(v) \tag{29}$$

where

$$k(v) = \frac{\delta}{\gamma} + (1 - \frac{1}{\gamma})\Big(r + \frac{1}{2}\gamma v\Big(\lambda^2 - \gamma^2\epsilon^2(1-\rho^2)\Big(\frac{H'(v)}{H(v)}\Big)^2\Big)\Big) \tag{30}$$

$$\mu_v^{'}(v) = k'(\theta' - v) \tag{31}$$

$$\sigma_v^2(v) = \epsilon^2 v \tag{32}$$

How to solve it?

A numerical method using finite difference schemes will used. Here is an overview of the passages

- Apply Feynman-Kac theorem to express the ODE in terms of the expectation under a risk-neutral measure
- Calculate the changes imposed by the new measure by applying Girsanov theorem
- Through first order approximation find a recursive scheme for $H(v_t)$ where we approximate $H'(v_t)$ by finite differences
- This can be efficiently coded in Python or MATLAB

Results

$$\phi_t = \left[\frac{\lambda}{\gamma} - \rho\epsilon \frac{H'(v_t)}{H(v_t)}\right] \tag{33}$$

$$C_t = \frac{W_t}{H(v_t)} \tag{34}$$

Giving the optimal controls for the income version as:

$$\phi_t = \frac{W_t + \frac{Y}{r}}{W_t}\left[\frac{\lambda}{\gamma} - \rho\epsilon \frac{H'(v_t)}{H(v_t)}\right] \tag{35}$$

$$C_t = \frac{1}{H(v_t)}\left(W_t + \frac{Y}{r}\right) \tag{36}$$

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Code results

$H(v_t)$ and optimal risky asset allocation ratio ϕ_t as a function of $\sqrt{vt}$, the x-axis spans from 0.001 to $\sqrt{1.5}$, risk-aversion $=2$

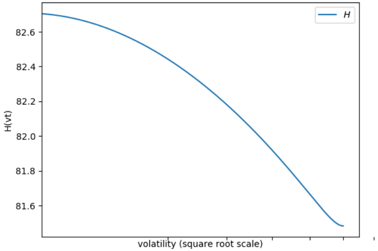

Figure 3: $H(v_t)$

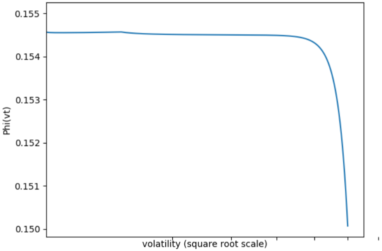

Figure 4: $\phi(v_t)$

Results

- The investor should scale the quantity invested in the risky asset based on the current state of volatility of the market.
- The Chicago Board Options Volatility Index (VIX) gives a measure of the percentage of volatility expected. Therefore, its value can be taken to be $\sqrt{v_t}$ in our model.
- The investor should check the VIX ($\sqrt{v_t}$) and adjust his portfolio according to the expression found for $\phi(v_t)$.
- As mean return λ increases more shall be invested. The only investor specific variable risk aversion γ which scales down ϕ_t.
- ρ determines the sign of the hedging demand part. We have taken it to be always positive because if it was negative it would imply that more return can be collected from low risk breaking the return-risk mechanism.

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Further possible developments

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Thank you for the attention

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[Hes93] Steven L Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, The Review of financial studies **6** (1993), no. 2, 327–343 (eng).

[Tou11] Agnes Tourin, *An introduction to finite diffference methods for pdes in finance*.