Household Portfolio-Consumption Choice Problem

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June 2021

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Abstract:

Portfolio choice problems are the basis of modern asset pricing theory. This paper develops and solves an intertemporal household portfolio problem with constant wage income and stochastic market volatility given by the Heston model. The advanced mathematical methods required to describe the problem in a stochastic control framework are introduced to derive the associated PDE. Feynman-Kac theorem and a finite difference scheme are finally used to solve for the optimal controls. The numerical method is then implemented in Python to yield the portfolio solution.

JEL classification: G11; D15; Y4

Keywords: Intertemporal portfolio-consumption optimization; Control theory; Numerical methods; Stochastic Volatility.

Previously submitted in partial fulfilment of the requirements for the BSc. degree in Mathematics with Statistics for Finance of Imperial College London.

Introduction

Finance as a discipline is the study of money and investments. A fertile and important branch of the subject is dedicated to the study of the systems of asset allocation that can yield the best possible outcome for an investor. The fundamental question pertaining how to allocate limited personal resources to maximise a certain definition of utility or satisfaction through consumption is a problem that everyone faces daily in their lives. Time, food, sleep, or energy expenditure during the day are all ubiquitous types of decision making that everyone faces and tries to optimize for their general well-being. Furthermore, they are all examples of problems that do not arise and resolve in a limited time frame but carry on during our entire lives. On a broader scale, it could be argued that intertemporal asset management is part of those activities that define the life of any conscious human being: through career planning, raising a family or granting their own future welfare.

Therefore, the possibility of modelling the problem in a framework that grants consistent, rational, and optimal solutions would be of utmost importance. This is the problem that the Nobel prize winner Robert Merton decided to face in two articles published in 1969 and 1971 that introduced his celebrated Portfolio Choice solution Merton (1969), Merton (1971). The problem arose in trying to consider the best possible investment in continuous time rather than just several discrete dates like the mathematician Frank Ramsey had done in a pioneering 1920 paper Ramsey (1920). In the problem setting, the investor seeks to maximize through its wealth consumption and proportion of investment in a risky (i.e. not yielding deterministic returns) portfolio the present expected value of the utility of future consumption. Surprisingly, a closed-form optimal solution can be found for Merton's formulation through the powerful framework of stochastic control and Bellman optimal principle. Merton's original paper proved to be seminal and defined the field of intertemporal asset pricing.

Numerous variations of Merton's problem have been the object of study in finance and economics. Introducing different utility functions, bequest values or different cash flows, the framework can be used to represent more complex and diverse settings. Constantinides (1990) considered a time-varying utility in which agents get "used" to their utility and Campbell and Cochrane (1999) expanded this idea to take into account utility changing with the utility of other agents in a "Keeping up with the Joneses" fashion. Obstfeld (1994) developed a recursive utility function for which the utility of the household depends at any point in time on the expected value of future utilities. Bhamra and Uppal (2006) extended this model to stochastic investment opportunities and demonstrate that in this case, the allocation to the portfolio can depend on both risk aversion and the elasticity of intertemporal substitution.

The goal of this thesis will be to develop the mathematical framework necessary to solve portfolio choice problems. In particular, a portfolio choice problem will be solved similar to the one of Bhamra and Uppal (2006) with stochastic investment opportunities but under the simplifying assumption that elasticity of inter-temporal rate ϵ is equal to $\frac{1}{1-\gamma}$ thus returning a Constant Relative Risk Aversion (CRRA) utility function. To enhance the similarity to reality and make the model more interesting a source of stable income Y will be considered in the household budget. The resulting non-linear PDE will be solved through a numerical method to yield the optimal ratios of consumption and risky investment for a household.

The paper is divided in two parts. The first part introduces the mathematical framework of control problems both in discrete, continuous-time and continuous time with a stochastic state variable. The fundamental Hamilton-Jacobi-Bellman equation is derived there. Finally, the second part makes use of the theory previously described to solve the household portfolio-consumption choice problem with income and stochastic volatility. A numerical method to solve the non-linear PDE will be derived and implemented in the appendix to give the optimal controls.

This paper was originally submitted in fulfilment of the M3R Third Year Research Module of the BSc. Mathematics with Statistics for Finance at Imperial College London. The version here presented was abridged to respect the requirements of the IAES Best Undergraduate Paper competition.

The original structure had two additional chapters. In the first chapter, finite difference schemes (FTCS, BTCS and Crank-Nicholson) methods were introduced. To exemplify their utility in finance the BTCS scheme was used to solve Black-Scholes PDE for the value of a European call option through MATLAB code. The following chapter introduced the fundamental stochastic calculus machinery necessary to correctly model and operate on stochastic processes. The Ito lemma and Girsanov theorem were presented and proved to be later used in the portfolio solution. Feynman-Kac theorem was also introduced. The popular Heston model was described and calibrated through call options on the SP500 in Python code. The remaining chapters followed the current structure.

Control problems

Introduction

The central topic of this paper is the problem of optimizing the choices that an agent can make in several time steps to get the best possible result. This is a well-explored topic in applied mathematics literature where it takes the name of dynamical programming. Dynamical programming refers to a method of solving temporal problems by breaking them down into several time steps and recursively solving them. This is the famous "Bellman's Principle of Optimality" that lays the foundations of the Bellman equation and the Hamilton-Jacobi-Bellman (HJB) equation, both powerful tools to solve many discrete intertemporal optimization problems.

Dynamic programming framework

First of all, describing the problem requires a brief Dramatis Personae. Here will be defined the main 'actors' of the problem at hand:

The **objective function** prescribes the objective of the optimization problem over the time frame considered. This can be maximizing the utility of consumption, minimizing cost or other complex requirements.

- The **value function** $V(x)$ is the value of the optimized objective function.
- The **state variables** $S(x_t, c_t)$ keep track at every time the current situation.
- The **control variables** c_t permit the agent to influence or determine future states. Therefore, the control variables are the optimizers of the objective function.
- The **goal** of the problem is to find expressions for the control variables that satisfy the objective function and depend on the state variables at time t.

Bellman equation

The fundamental principle on which dynamic programming relies is the so-called "Bellman Principle of Optimality":

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy concerning the state resulting from the first decision". Bellman (1957) Chap. III.3.

The infinite time horizon dynamic programming framework is given by

$$V(x_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t P(x_t, c_t),$$

with $P(x_t, c_t)$ the payoff at time t of the current state and controls. A discount factor δ^t with $0 < \delta < 1$ is introduced to discount future values. The problem is subject to a constraint $c_t \in \Gamma(x_t)$ and evolution law $x_{t+1} = S(x_t, c_t)$ which determines the new state.

Following the principle of optimality and setting the first decision:

$$\max_{a_0} F(x_0, c_0) + \delta \left[\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta^{t-1} P(x_t, c_t) \right].$$

This will permit to rewrite the problem as

$$V(x_0) = \max_{c_0} P(x_0, c_0) + \delta v(x_1).$$

Since this framework can be started at any point in time, the time subscripts can be dropped to get Bellman Equation:

$$V(x) = \max_{c \in \Gamma(x)} P(x, c) + \delta v\big(S(x, c)\big).$$

Hamilton-Jacobi-Bellman equation

Hamilton-Jacobi-Bellman equation plays a crucial role in this thesis as it enables the treatment and solution of continuous-time stochastic control problems. The equation plays an analogous role to the Bellman equation in discrete time.

To derive the Hamilton-Jacobi-Bellman equation consider a continuous-time stochastic control problem of form

$$V_T(x(0),0) = \max_c \int_0^T P\,[x(t),u(t)]dt,$$

$$\text{subject to } \dot{x}(t) = S[x(t),c(t)].$$

Using Bellman's principle of optimality over an infinitesimal increase in time it becomes

$$V(x(t),t) = \max_c V(x(t+dt),t+dt) + \int_t^{t+dt} P\left(x(s),c(s)\right)ds,$$

whose value function can be Taylor expanded to give

$$V(x(dt+t),t+dt) = V(x(t),t) + \partial_t V(x,t)dt + S\big(x(t),c(t)\big)\,\partial_x V(x,t)dt + o(dt).$$

Now, subtracting $V(x(t),t)$ from both sides and dividing by dt

$$0 = \max_c \partial_t V(x,t) + S\big(x(t),c(t)\big)\,\partial_x V(x,t) + o(dt)/dt + \frac{1}{dt}\int_t^{t+dt} P\left(x(s),c(s)\right)ds.$$

Finally, taking the limit as dt tends to zero, the Hamilton-Jacobi-Bellman equation can be derived:

$$\partial_t V(x,t) + \max_c S(x,c)\,\partial_x V(x,t) + P(x,c) = 0.$$

Stochastic control problems

A natural kind of extension for dynamic programming is the case where state variables change according to some stochastic law. The framework and the principle developed by Bellman are still applicable with only minor changes required. As in the last example, we have a continuous-time dynamic problem with objective function:

$$V_T(x(0),0) = \sup_{c} E_0 \int_0^T e^{-\rho t}\, U\big(x(t),u(t)\big)dt,$$

evolving according to the law

$$dx(t) = \mu\big(x(t),c(t)\big)dt + \sigma\big(x(t),c(t)\big)dz.$$

The discount factor for impatience $e^{-\rho t}$ was introduced due to its utility in economic applications. For consistency with the following chapter, denote $U(x(t),t)$ as the payoff function.

Similarly to the previous case, the Hamilton-Jacobi-Bellman equation can be derived to solve the problem

$$0 = \max_{c(t)} P\big(s(t),c(t)\big) - \rho V(s(t),t) + V'^{(s(t),t)} f\big(s(t),c(t)\big) + \frac{1}{2} V''^{\big(s(t)\big)\big(\sigma^2(s(t),c(t))\big)}.$$

First derive the Bellman equation using the principle of optimality

$$V(x(t),t) = \sup_{c(t)_{t\in[t,t+dt)}} E_t\left[E_t\left[\int_t^{t+dt} e^{-\rho(u-t)}\, U\big(s(u),c(u)\big)du\right] + e^{-\rho dt} V(s(t+dt),t)\right].$$

Applying first-order Taylor expansion to the terms yields:

$$E_t\left[\int_t^{t+dt} e^{-\rho(u-t)}\, U\big(s(u),c(u)\big)du\right] = U\big(s(t),c(t)\big)dt + o(t),$$

$$e^{-\rho dt} = 1 - \rho dt + o(t),$$

$$dtE_t[V(s(t+dt),t)] = dtV\big(s(t)\big) + o(t).$$

Finally, recombining get Bellman's equation

$$V(s(t),t) = \sup_{c(u)_{u\in[t,t+dt)}} U\big(s(t),c(t)\big)dt + E_t V\big(s(t+dt)\big) - \rho dt V\big(s(t)\big) + o(dt).$$

For ease of notation let $V\big(s(t)\big) \equiv V(s(t),t)$. Now Ito's lemma for the function states:

$$V\big(s(t+dt)\big) = V\big(s(t)\big) + dV\big(s(t)\big),$$

$$E_t\big[V\big(s(t+dt)\big)\big] = V\big(s(t)\big) + V_s\big(s(t)\big)\mu\big(s(t),c(t)\big)dt + \frac{1}{2}\partial_{ss}^2 V\big(s(t)\big)\sigma^2\big(s(t),c(t)\big)dt + o(dt).$$

Reinserting this last result back into the Bellman's equation and taking the limit as $dt \rightarrow 0$ we get the HJB equation for the problem:

$$0 = \sup_{c(t)} U\big(s(t),c(t)\big) - \rho V\big(s(t)\big) + V'^{(s(t))}\mu\big(s(t),c(t)\big) + \frac{1}{2}V''^{(s(t))\sigma^2(s(t),c(t))}.$$

Notice how in the stochastic version the second-order term persists. This, like in Ito's lemma, is due to the quadratic variation of the Brownian motion.

Now to solve the stochastic control problem it is just necessary to find the solutions to the first-order conditions for the controls. Reinserting them in the HJB equation yields

$$0 = U\big(s(t),c^*(t)\big) - \rho V\big(s(t)\big) + V'^{(s(t))}\mu\big(s(t),c^*(t)\big) + \frac{1}{2}V''^{(s(t))\sigma^2(s(t),c^*(t))}.$$

The resulting equation is a non-linear PDE which in general does not have a closed-form solution and in general, must be solved by numerical methods. Finite difference schemes are a very popular choice in the field. However, in the following portfolio-consumption solution a more sophisticated method will be used. The solution to the non-linear PDE does not have boundary conditions. Therefore, it does not satisfy the usual solution to a differential equation and it requires a new concept solution called viscosity solution Tourin (2011). Since a practical approach through numerical methods will be used, a detailed discussion of the properties of these solutions is beyond the scope of this paper. However, Barles (2013) book on the topic can be consulted for further details.

Portfolio Choice

Introduction

A question is ever-present in finance: "How and how much to invest?" In a single period, one of the most popular frameworks for answering this question is the Mean-Variance model developed by Markowitz (1952). The model prescribes investing in a combination of a (virtually) risk-less asset like a US treasury bond and a portfolio of risky assets. The quantity to be invested in each is dependent on the mean-variance tradeoff of the investment opportunities currently offered and the risk aversion of the investor.

Mean-variance theory gives a popular portfolio framework when a single period is considered. If multiple periods or a continuous period are considered the question can be nicely framed in a stochastic control problem. The framework introduced in Merton (1969) is a particularly powerful one and incredibly flexible. In his paper, Merton used stochastic control and HJB equation to solve the inter-temporal asset allocation problem by maximizing utility through optimal consumption and optimal investment in a risky and risk-less portfolio. Quite strikingly the model and the HJB equation can be solved to yield a closed-form solution. This was the basis for his Intertemporal CAPM model which divides portfolio allocation into a myopic (based on constant investment opportunities) and intertemporal part (which hedges against changes in investment opportunities).

The framework is particularly powerful because it permits multiple adjustments to take into account income, more complex stock returns, pensions, bequest and other financial flows. The model also found theoretical confirmation in Cox et. al. (1985) who started from more primitive assumptions and developing their economic model derived results that agreed with Merton’s.

Following the fruitful literature of expanding the stochastic control framework to more realistic settings, it will be here solved the portfolio-consumption problem for a household receiving a stable income and investing in an Exchange Traded Fund (ETF) described by two stochastic processes.

The household problem

Consider the following stochastic optimal control problem in which a household is trying to optimize the utility of their consumption by investing in a portfolio comprising a risk-less and risky asset. The wealth of the household at time t is given by W_t and it evolves according to

$$W_{t+dt} = (W_t + (Y - C_t)dt)\left(1 + dR_{p,t}\right)$$

where C_t denotes the consumption of the household at time t, Y denotes the income and $dR_{p,t} = (1 - \phi_t)rdt + \phi_t dR_t$ the instantaneous return of the portfolio. The consumption is a control variable and therefore decided by the household at each time.

Following the Mutual Fund theorem, it will be assumed that the household can invest in a combination of a riskless asset and a market ETF. The return and volatility of the ETF will be modelled by

$$dR_t = (r + \lambda v_t)dt + \sqrt{v_t}dZ_t$$

and

$$dv_t = \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t}dZ_{v,t},$$

subject to the two Brownian motions W_t^S and W_t^v with correlation ρ. This is the popular model developed by Heston (1993). Compared to Black-Scholes it has the significant advantage of not assuming constant volatility.

If ϕ_t denotes the control variable of the ratio invested in the risky asset then the following continuous stochastic law describes the evolution of state variable W_t. This is called the dynamic inter-temporal budget constraint

$$dW_t = (Y - C_t)dt + W_t\left[rdt + \phi_t\left(\lambda v_t dt + \sqrt{v_t}dZ_t\right)\right],$$

which is subject to the two controls C_t and ϕ_t.

This can be expressed in terms of the total stock of wealth at time t by the following transformation which considers the present value of future labour income flows:

$$V_t = W_t + \left(1 - e^{-r(T-t)}\right)\frac{Y}{r}.$$

Note that if an infinite time frame $T = \infty$ is considered, it becomes:

$$V_t = W_t + \frac{Y}{r}.$$

Therefore, the presence of an income enables the households to scale up their investment in the ETF by a factor

$$\widehat{\phi_t} = \frac{V_t}{W_t}\phi_t.$$

This relation enables to ignore the effect of the income for the sake of the control problem and calculate the optimal controls as scaled version of the controls for the incomeless problem.

Hamilton-Jacobi-Bellman equation

In the framework of the stochastic control problem the household seeks to maximize through consumption and risky asset investment ratio the objective function

$$\sup_{(C_s)_{s\geq t},(\phi_s)_{s\geq t}} E_t \int_0^{\infty} e^{-\delta(s-t)}\, u(C_s)ds,$$

which is the present expected utility of the future consumption, where $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ is the Constant Relative Risk Aversion (CRRA) utility function. The dynamic budget constraint is then given by

$$dW_t = -C_t dt + W_t\big[rdt + \phi_t\big(\lambda v_t dt + \sqrt{v_t}dZ_t\big)\big].$$

Through the control theory fundamentals presented in the previous chapter, the Hamilton-Jacobi-Bellman equation for the problem can be derived as:

$$0 = \sup_{C_t} u(C_t) - \delta J_t + \sup_{\phi_t} E\left[\frac{dJ_t}{J_t}\right]$$

The HJB equation provides a useful tool for solving the inter-temporal problem. However, it is not a unique method of solution. Pontryagin maximum principle and Hamiltonians also yield the solution to the problem. The two methods are connected and can be reciprocally derived Bhamra (2015).

A significant difference is that Pontryagin does not assume time consistency (i.e. following the same strategy until end time) which is helpful when the agent might change strategy or management like corporations or central banks.

Using Ito's lemma the HJB becomes:

$$0 = \sup_{C_t}[u(C_t) - \delta J_t + \sup_{\phi_t}[\partial_t J + \partial_v J[\kappa(\theta - v_t)] + \partial_W J[W_t r - C_t + W_t \phi_t \lambda v_t] + \frac{1}{2}\partial^2_{vv} J(\epsilon^2 v_t) + \frac{1}{2}\partial^2_{WW} J(W_t^2 \phi_t^2 v_t) - \partial^2_{Wv} J(\rho\epsilon\phi_t W_t v_t)]].$$

To find the optimal controls, First Order Conditions must be checked:

$$C^{-\gamma} = \partial_W J$$

and

$$\lambda\, \partial_W J + \phi_t\, \partial^2_{WW} J W_t - \rho\epsilon\, \partial_{Wv} \mathrm{J} = 0.$$

In terms of economics, this tells us that consumption is inversely proportional to wealth growth and that the optimal ratio can be broken down in terms of risk premium, risk, and hedging demand contributions.

Similarly for Second-Order Conditions

$$-\gamma C^{-\gamma-1} < 0, \forall \gamma, \forall C \geq 0$$

and

$$\partial^2_{WW} J W_t \leq 0 \Longrightarrow \partial^2_{WW} J \leq 0.$$

This means that the conditions are satisfied for optimal C independently of J while for ϕ_t a condition must be imposed on our proposed solution.

With ansatz $J_t = H(v_t)^\gamma U(W_t)$ the optimal controls are found as:

$$C^* = (\partial_W J_t)^{-1/\gamma} = (\frac{H(v_t)^\gamma}{W_t^\gamma})^{-1/\gamma} = \frac{W_t}{H(v_t)}$$

and

$$\phi_t^* = \frac{-\lambda\left(\frac{H(v_t)}{W_t}\right)^\gamma + \rho\epsilon\gamma H(v_t)^{\gamma-1} W^\gamma H'^{(v_t)}}{-\gamma W_t^{-\gamma-1} H(v_t)^\gamma W_t} = \frac{\lambda}{\gamma} - \rho\epsilon \frac{H'^{(v_t)}}{H(v_t)}.$$

Finally, reinserting the optimal controls and through transformation, an ODE for the unknown function $H(v_t)$ is derived:

$$0 = 1 - k(v)H(v) + \mu_v'(v)H'^{(V)} + \frac{1}{2}\sigma_v^2(v)H''^{(v)},$$

with

$$k(v) = \frac{\delta}{\gamma} + \left(1 - \frac{1}{\gamma}\right)(r + \frac{1}{2}\gamma v(\lambda^2 - \gamma^2\epsilon^2(1-\rho^2)(\frac{H'^{(v)}}{H(v)})^2));$$

$$\mu_v'(v) = k'^{(\theta' - v)} \text{ and } \sigma_v^2(v) = \epsilon^2 v.$$

A scheme for numerical solutions

To solve the ODE a recursive scheme will be formulated for $H(v_t)$ as in Bhamra (2021). Here the recursive scheme method will be outlined.

First of all, applying Feynman-Kac theorem with the following identifications $V(x,t) \equiv k(v_t)$ and $\mu(x,t) \equiv \mu_v'^{(v_t)}$, the ODE can be transformed into an expectation depending on the risk-free measure $\mathbb{P}'$:

$$H(v_t) = E_t^{\mathbb{P}'}\left[\int_t^\infty e^{-\int_t^u k_s ds}\, du\right],$$

with volatility evolving according to the law:

$$dv_t = \mu_v'^{(v_t)} + \sigma_v(v_t)dZ_{v,t}^{\mathbb{P}'},$$

where $dZ_{v,t}^{\mathbb{P}'}$ is a standard Brownian motion under $\mathbb{P}$.

The change in probability measure had induced a change in the drift term which can be calculated by Girsanov theorem as follows. First, since the stochastic process for volatility must be equal under the two measures, and applying Girsanov theorem find

$$\mu'_v(v_t) = \mu_v(v_t)dt + E\left[\frac{dM'_t}{M'_t}dv_t\right].$$

The next step is identifying M'_t which is an exponential martingale under $\mathbb{P}$. Reinserting in the definition for the drift get

$$\kappa'^{(\theta'-v_t)} - \kappa(\theta - v_t)dt = E\left[\frac{dM'_t}{M'_t}dv_t\right].$$

Since $\kappa'\theta' = \kappa'\frac{\kappa}{\kappa'}\theta = \kappa$ it can be further simplified to $E\left[\frac{dM'_t}{M'_t}dv_t\right] = -(\kappa' - \kappa)v_t dt = \frac{(\gamma-1)\lambda\rho\epsilon}{\gamma}v_t dt.$

This way, define $\mathbb{P}'$ through M' and an event A that might or might not take place at date T.

$$E_t^{\mathbb{P}'}[I_A] = E_t^{\mathbb{P}}\left[\frac{M'_T}{M'_t}I_A\right].$$

Splitting the integral, recognizing the $H(v_{t+dt})$ form and considering the first-order approximations, it can be seen that

$$\begin{aligned}
H(v_t) \quad &= E_t^{\mathbb{P}'}[\int_t^{t+dt} e^{-\int_t^u k_s ds}\, du] + E_t^{\mathbb{P}'}[\int_{t+dt}^{\infty} e^{-\int_t^u k_s ds}\, du],\\
&= dt + E_t^{\mathbb{P}'}[\int_{t+dt}^{\infty} e^{-\int_t^u k_s ds}\, du] + o(dt),\\
&= dt + E_t^{\mathbb{P}'} e^{-\int_t^{t+dt} k_s ds} E_{t+dt}[\int_{t+dt}^{\infty} e^{-\int_{t+dt}^u k_s ds}\, du] + o(dt),\\
&= dt + E_t^{\mathbb{P}'} e^{-\int_t^{t+dt} k_s ds} H(v_{t+dt}) + o(dt),\\
&= dt + e^{-k_t dt} E_t^{\mathbb{P}'}[H(v_{t+dt})] + o(dt),\\
&= dt + (1 - k_t dt) E_t^{\mathbb{P}'}[H(v_{t+dt})] + o(dt),\\
&= E_t^{\mathbb{P}'}[dt + (1 - k_t dt)H(v_{t+dt})] + o(dt).
\end{aligned}$$

Finally, can derive a recursive scheme using the following formula

$$H(v_t) = dt + e^{-k_t dt} E_t^{\mathbb{P}'}[H(v_{t+dt})] = E_t^{\mathbb{P}'}[dt + (1 - k_t dt)H(v_{t+dt})].$$

From here a discretization of the state space in volatility can be set up $v_1, \dots, v_{N+1}$. This will be done using a continuous-time Markov chain to approximate volatility under the new measure.

The Markov chain is set up with a generator matrix

$$\begin{bmatrix} -p_{1,2} & p_{1,2} & 0 & \cdots & 0 & 0 \\ p_{2,1} & -(p_{1,2}+p_{2,3}) & p_{2,3} & 0 & \cdots & 0 \\ 0 & p_{3,2} & -(p_{3,2}+p_{3,4}) & p_{3,4} & \cdots & \cdots \\ \cdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_{N,N-1} & -(p_{N,N-1}+p_{N,N+1}) & p_{N,N+1} \\ 0 & 0 & \cdots & 0 & p_{N+1,N} & -p_{N+1,N} \end{bmatrix}$$

where

$$p_{n,n+1} = \frac{\mu'_v(v_n)}{\Delta v} I_{\mu'_v(v_n)>0} + \frac{\sigma_v^2(v_n)}{2(\delta)^2} \text{ and } p_{n,n+1} = -\frac{\mu'_v(v_n)}{\Delta v} I_{\mu'_v(v_n)<0} + \frac{\sigma_v^2(v_n)}{2(\delta)^2}.$$

Here the chain built moves along the volatility according to the measure $\mathbb{P}'$ with $p_{n,n+1}dt$ being the probability of volatility increasing by one discretization after time dt. The form that the generator takes is also typical of a birth-death process. In facts, the chain increases and decays by unit increments (dv) instantaneously. It follows that the transition probabilities matrix is given by e^{Sdt} and the recursive scheme

$$H = e^{Sdt}[1dt + (\mathbf{I} - \mathbf{K}dt)H],$$

where a non-linear operator (as it depends on the value taken by $K(H)$) is applied to H to give H. Therefore, H represents the fixed point of this operator and a recursive scheme can be set up to converge. However, it will first be necessary to discretize time and invert e^{Sdt}. Using a Taylor series expansion get

$$(e^{-S\Delta t}) = \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \frac{S^n(\Delta t)^n}{n!} \approx \mathbf{I} - \mathbf{S}\Delta t.$$

Giving the final form for the recursive scheme as

$$\bar{H} = \mathbf{I} - \mathbf{S}\Delta t[1dt + (\mathbf{I} - \mathbf{K}dt)H] + o(\Delta t).$$

Finally, let $H'_n = \frac{H(v_{n+1}) - H(v_n)}{\delta v}$ and

$$\mathrm{k}(v_n) = \frac{\delta}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\left(\mathrm{r} + \frac{1}{2}\gamma v_n \left(\lambda^2 - \gamma^2\epsilon^2(1-\rho^2)\left(\frac{H'_n}{H(v_n)} I_{\mu'_v(v_n)>0} + \frac{H'_{n-1}}{H(v_n)} I_{\mu'_v(v_n)<0}\right)^2\right)\right),$$

with $H(v_0), H(v_{N+1}) = 0$.

Starting from an initial guess for $\bar{H}$ and applying the recursive scheme the estimate will eventually converge to the true value of $\bar{H}$.

Now the optimal portfolio and expenditure for the income-less problem are

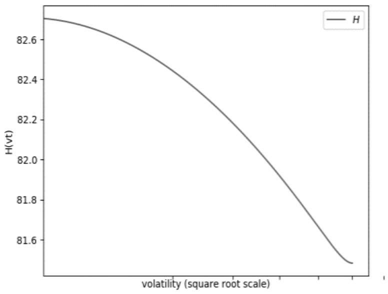
$$\phi_t = \left[\frac{\lambda}{\gamma} - \rho\epsilon\frac{H'(v_t)}{H(v_t)}\right] \text{ and } C_t = \frac{W_t}{H(v_t)}.$$

Giving the optimal controls for the income version as:

$$\phi_t = \frac{W_t + \frac{Y}{r}}{W_t}\left[\frac{\lambda}{\gamma} - \rho\epsilon\frac{H'(v_t)}{H(v_t)}\right],$$

$$C_t = \frac{1}{H(v_t)}\left(W_t + \frac{Y}{r}\right).$$

Implementing the method coded in the appendix, Figure 1 and 2 show the case of a low risk-averse investor ($\gamma = 2$) and market given by the Heston parameters calibrated on SP500 call options data. As can be seen, by the equations above the model suggest to the investor to calibrate his portfolio according to the volatility of the market. This can be done by employing the VIX index as our $\sqrt{v_t}$ and calibrating the portfolio according to $\phi(v_t)$ and the changes in the VIX index.



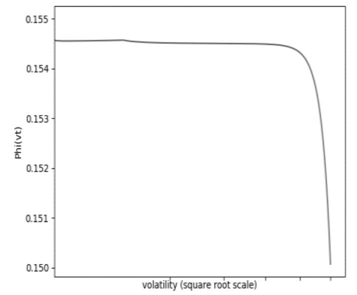


Figure 1 and 2: The Optimal Control. Plots for the unknown function of volatility $H(v_t)$ and optimal risky investment ratio $\phi(v_t)$ against volatility v_t expressed in square root scale. *Source: own calculations based on Heston parameters calibrated from Balaraman data (2017)*

Conclusion

This paper presents and builds the control theory necessary to solve a household portfolio-consumption problem with stable income and stochastic volatility returns. The final chapter sets up and solves for the optimal controls of the household problem through a finite difference method implemented in the python code in the appendix.

Solving the household problem, the main finding is that the quantity invested in the ETF increases with excess return λ , current wealth W_t and income Y while decreases with increasing risk aversion γ , interest rates r and volatility of volatility ϵ. This is consistent with real markets and the economy as high net-worth individuals tend to make larger use of financial products, and higher excess returns on the market encourage more investments. Note that ρ, the correlation between the Brownian motion of the stock returns and volatility, decides and scales the contribution of the second term. A negative ρ would break the risk-return mechanism, giving lower returns for a less volatile ETF. This can be seen also by the positive contribution the hedging demand part would take in that case.

The result from the optimal expenditure is consistent with the real world as from the results above it can be seen that $H(v_n)$ decreases with volatility and therefore the household "flights" to safety as the volatility of the market increases.

Finally, the optimal portfolio ratio formula can be analyzed in terms of the myopic $\frac{\lambda}{\gamma}$ and intertemporal part $-\rho\epsilon\frac{H'(v_t)}{H(v_t)}$. This breakdown determines the ratio that is allocated to the current best return of the market (the myopic term) and the part that is used to hedge changes in the state variable i.e. volatility.

The advice that the model gives to a hypothetical investor is to continuously re-calibrate the risky asset allocation ratio by checking the current state of the volatility of the market, which can be done through the VIX index.

Possible future developments

To conclude, the author would like to introduce an idea that he recently developed while learning about the topic of stochastic controls. In light of the various modifications of the utility function, he believes an interesting development to the theory would be considering utility that it is exogenous and depends on a competitive environment. Interest in the topic is motivated by the coursework project done for the module "Dynamics of Games" which employed No Regret learning to solve a Colonel Blotto fund allocation problem for the 2020 US elections candidates.

The framework envisioned would be the following:

- The agent tries to win a series of m zero-sum battles leading to winning or losing a war depending on winning a majority of the battles.
- Each battle can be modelled as a Colonel Blotto game with b equivalently valued battlefields. To win the battle it would be required to win most battlefields. For a start, the number of battlefields can be taken to be $b = 1$.
- The agent and the opponent have a starting predetermined discrete number of troops A_0 and O_0. However, only the agent also knows the starting number of troops of the opponent.
- At each battle, the two participants are called to decide how many troops "consume" by employing them in battle. The remaining would be invested in a portfolio. For an initial enquiry, the portfolio would be taken to be purely risk-free.
- The two players would aim to maximize the expected utility of their consumption, i.e. maximize the probability of winning.
- To make the game more interesting the opponent initial resources will be assumed to be more than the agent ones.

The first step would be to determine the evolution of the utility function through time, this would determine if Bellman's optimality principle is applicable. If it is, then the problem should be solved recursively and a corresponding Bellman equation should be set up and potentially solved through the methods outlined in this paper.

For this project, in addition to the theory here exposed, the papers in sequential games by Klumpp and Konrad (2018) and Solomon (2018) would be a good starting point for analyzing how the strategy changes without investment income.

For determining how the players would allocate resources in multiple battlefields Behnezhad et. al. (2016) gives an extremely efficient algorithm.

Acknowledgements

I would like to thank my supervisor, Professor Harjoat Bhamra, whose guidance and mentorship was invaluable. I would also like to thank my colleague Wasim Ahmed for reading through the first drafts of this paper.

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