Hypercomplex number systems

Matteo Mario Di Venti

Imperial College London

Introduction

Complex numbers have been developed in the 16th century to solve cubic equations. Quaternions instead were discovered by Hamilton in the 19th century as he was trying to develop a three dimensional space equivalent to the two-dimensional space of complex numbers. It is well known the anecdote of his frustration for not being able to divide "triplets" and how in a sudden revelation came to discover the multiplication rules for quaternions then carved in an act of mathematical vandalism under a bridge in Dublin.
An interesting remark is that quaternions are not commutative as the complex numbers. Further generalization brings the Octonions ($\mathbb{O}$) which are not associative and Sedenions ($\mathbb{S}$) which don't have the alternative property. A famous theorem by Hurwitz proves that only $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are normed division algebras [1].
The purpose of this poster is to introduce the concept of hypercomplex number systems which are unital algebras over the reals of which all the aforementioned are examples. Furthermore this poster will investigate other lesser known examples of 4-dimensional hypercomplex numbers systems (like split-quaternions or hyperbolic quaternions) and their relevance to pure and applied mathematics. A special mention will be made to Clifford algebras and Grassman numbers.

Model

We will start developing the concept of hypercomplex numbers from the same construction that gives the complex numbers [2]. Consider the set C of expressions $(a+bi)$ with $a, b \in \mathbb{R}$ and $i \notin \mathbb{R}$.
We define addition naturally as for the complex numbers. Multiplication however is arbitrary. We impose it to be distributive.This almost completely determines the multiplication except for i^2. Imposing $i^2 \in C$ we can reduce any choice to three systems:

- Complex numbers ($i^2 = -1$)
- Dual numbers ($i^2 = 0$)
- Double numbers ($i^2 = 1$)

It's easy to see that both dual and double numbers are not division systems since

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\frac{c-di}{c-di} = \frac{ac+(bc-ad)i+(-bd)i^2}{c^2-d^2i^2}. \tag{1}$$

We can clearly see that for dual numbers there is no quotient for numbers with no real part and the same happens for double numbers with the same absolute value for the real and 'imaginary part'.
While complex numbers have been the subject of a thorough study over the centuries, dual and double numbers have acquired more importance just recently. In particular dual numbers are used in quantum physics and automatic differentiation [3].

Hypercomplex Number Systems

Now from the first example we can define a general hypercomplex number system [2].
An hypercomplex number system of dimension $n+1$ (H) is a set of all the expressions $a_0 + a_1e_1 + ... + a_ne_n$ with $a_i \in \mathbb{R}$ and $e_j \notin \mathbb{R}$ with two operations defined on it:

- addition $(a_0 + a_1e_1 + ... + a_ne_n) + (b_0 + b_1e_1 + ... + b_ne_n) = (a_0 + b_0) + (a_1 + b_1)e_1 + ... + (a_n + b_n)e_n$
- multiplication that satisfies distributivity and it is completely determined by the products $e_je_k \in H$

Compare the definiton of an hypercomplexnumber system of dimension $n+1$ with an algebra A of the same dimension which is the set of all expression of form $a_0e_0 + a_1e_1 + ... + a_ne_n$ with addtion and a distributive multiplication defined on it. It can be clearly seen that any hypercomplex number system is a particular case of an algebra with an element (in this case $e_0 = 1$) $e_0 * a = a * e_0 = a$ for $a \in A$. An algebra satisfying this property is named unital algebra.
Consider the hypercomplex system of expressions of form $a+bi+cj+dk$ and the algebra of $ai+bj+ck$ with two different multiplications defined by the following tables:

	i	**j**	**k**
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

	i	**j**	**k**
i	0	k	-j
j	-k	0	i
k	j	-i	0

The first one describes the hypercomplex number system of the quaternions discovered by Hamilton. As $ij \neq ji$ the system is not commutative but it is still associative and division is well defined. The second one encodes the cross product in three dimesions and it is not an unital algebra.

Clifford Algebras and Grassman Numbers

Through the analysis of hypercomplex numbers systems further theories have developed. In particular we will just mention Clifford Algebras and Grassman Numbers for the attention they are receiving physics. Clifford Algebras over the reals are hypercomplex associative number systems whose bases are the span of a basis $1, e_1, e_2, ..., e_n$ with $e_ie_j = -e_je_i$. The algebras are classified over the number p of $e_i^2 = +1$ and q of $e_j^2 = -1$ $(Cl_{p,q}(\mathbf{R}))$. For example $\mathbb{H}$ is $Cl_{0,2}(\mathbf{R})$ and the double numbers are $Cl_{1,0}(\mathbf{R})$. Octonions are not Clifford Algebras as they do not satisfy associativity. It can be trivially checked that the spanned basis is a group.
On the contrary, Grassmann numbers are characterised by the fact that $\forall i, e_i^2 = 0$. Dual numbers are the simplest example of such algebras. Grassman numbers seem to play a role in quantum physics encoding through their nilpotent properties pauli exclusion principle for wave functions.

Quaternionic Algebras

As it has been seen for complex numbers, different definitions of the multiplication table yield different systems. A few examples of hypercomplex systems with three imaginary variables are the ones defined by the following tables:

	i	**j**	**k**
i	1	k	-j
j	-k	1	i
k	j	-i	1

	i	**j**	**k**
i	-1	k	-j
j	k	1	i
k	-j	i	-1

	i	**j**	**k**
i	-1	k	-j
j	-k	1	-i
k	j	i	1

The first table represents the hyperbolic quaternions. Although it satisfies the anticommutation property $e_ie_j = -e_je_i$, note $(jk)k = ik = -j \neq -k = ji = j(kk)$, the system is not associative or alternative and thus not a Clifford Algebra. Considering only the expressions with $j, k = 0$, it contains the system of double numbers as a subalgebra. This is system is called hyperbolic after its connection with hyperboloids [4]. The second table represents bicomplex numbers also called tessarines. From the table it can be evinced that the system is commutative and that it contains both the complex and the double numbers as a subalgebra. Applications of this algebra are found in digital signal processing [5] The third table represents split-complex quaternions. It is an anti-commutative, associative system and a clifford algebra $Cl_{2,0}(\mathbf{R})$. Again contains both double numbers and complex numbers as sub algebras and thus contains zero divisors. The bijective map $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftrightarrow \frac{(a+d)+(c-b)i+(b+c)j+(a-d)k}{2}$ connects them to $2x2$ matrices.

Conclusions

Altough largely overlooked during the 20th century, hypercomplex numbers have been recentely revaluated due to their ability to encode particular calculations in a smaller number of terms and for their connection to non-euclidean geometries, the theory of algebras and matrices. For recent updates on their connection to arithmetic geometry see [6]. To conclude the exposition, it can be rightfully claimed that hypercomplex numbers are a truly versatile concept with many fruitful applications.

References

[1] Conway, J.H. ; Smith, D.A. On quaternions and Octonions. A.K. Peters, 2003.

[2] Kantor, I.L. ; Solodovnikov, A.S. Hypercomplex Numbers. New York: Springer-Verlag, 1989.

[3] Berland, H. Automatic differentiation. NTNU, 2006. available from: http://www.pvv.ntnu.no/ berland/resources/autodiff-triallecture.pdf accessed 13/06/2019

[4] Macfarlane, A. Hyperbolic Quaternions. Proceedings of the Royal Society of Edinburgh. 1902; 23: 169-180,

[5] Alfsmann, D. ; Göckler, H. G. On Hyperbolic Complex LTI Digital Systems EURASIP. (2007);

[6] Voight, J. Quaternion algebras. Dartmouth, May 2019.