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Energy Finance

Project C: FWD Calibration and Monte Carlo simulation (Arithmetic)

Computational Finance - A.A. 2024-2025

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Introduction

Commodity markets, such as the energy sector, are characterized by specific price behaviors driven by seasonality, mean reversion, and jumps. Capturing these dynamics is essential for pricing derivatives and managing financial risk. This report focuses on modeling French electricity futures using stochastic processes, aiming to provide a robust framework for calibration and simulation.
In our report, we tested two different models:

- *Arithmetic Model with Ornstein-Uhlenbeck Processes and Inverse Gaussian Jumps (OU-IG)*: This model captures jumps and mean reversion with seasonal adjustments.
- *Arithmetic Model - Gaussian OU Model (OU-Normal)*: A streamlined version that reduces complexity while preserving key dynamics.

The choice of these models reflects their flexibility in handling diverse price behaviors while balancing complexity and interpretability.

1 Spot price

In order to write the spot price we follow the guidelines of an Arithmetic process which is described by the following dynamics:

$$\begin{cases} S_t = \Lambda_t + \sum_{i=1}^{m} X_i(t) + \sum_{j=0}^{n} Y_j(t) \\ dX_i = [\mu_i(t) + \alpha_i(t) \cdot X_i(t)] \cdot dt + \sum_{k=1}^{p} \sigma_{i,k} dB_k & \text{for } i = 1, ..., m \\ dY_j = [\sigma_j(t) + \beta_i(t) \cdot Y_j(t)] \cdot dt + \eta_j dI_j(t) & \text{for } j = 1, ..., n \end{cases} \tag{1}$$

Respectively **m** represents the number of Gaussian processes $X_i(t)$ that we take into consideration, **p** represents the number of independent Brownian motions that lie inside the Gaussian process and **n** represents the number of Independent Increments jump processes $Y_j(t)$ that drive the spot price.
Moreover, if we delve into a more detailed view of the processes X_t and Y_t we can describe them as an Ornstein-Uhlenbeck (**OU**) processes with β, α mean reversion speeds and δ, μ mean of the processes which are set to zero in order to have the seasonality function as mean price. Moreover, η and σ are the variances of the driving OU.

In our assignment, we are asked to consider an Arithmetic model with $m = p = 0$, $n = 2$, the functions $\delta_1 = \delta_2 = 0$, β_1, β_2, η_1, η_2 are assumed to be constant and $Y_0 = 1$. Therefore, we have a spot price that follows the dynamics of an Arithmetic model where there are no Gaussian components but two OU processes with jumps described by an Inverse Gaussian. Hence the expression above can be simplified as:

$$S_t = \Lambda(t) + Y_{1,t} + Y_{2,t} \tag{2}$$

$$\text{where for } j = 1, 2 \quad Y_{j,t} = y_0 \cdot e^{-\beta_j \cdot t} + \eta_j \cdot \int_0^t e^{-\beta_j \cdot (t-u)}\, dI_j \quad \text{with} \quad \beta_j, \eta_j > 0$$

It should be noted that this model belongs to a class proposed by [Benth, Kallsen and Meyerrandis(2007)], which ensures positive values of the spot price (with probability 1) and thus positive prices, by choosing a suitable seasonality function. This property arises because the model satisfies all the hypotheses of the class, including having no diffusion term ($m = 0$) and relying on an Inverse Gaussian subordinator. The subordinator guarantees positive jumps, while the compensator measures are concentrated on the positive real line, ensuring the positivity of the stochastic part of the price process.

1.1 Seasonality

Seasonality represents predictable, periodic fluctuations in prices driven by factors like weather, demand, and supply. In the energy market, it is critical due to recurring patterns, such as an increase in electricity demand in winter for heating or in summer for cooling.
The seasonality-function $\Lambda(t)$ is defined as:

$$\Lambda(t) = A \cdot \sin(2 \cdot \pi \cdot t) + B + C \cdot t \tag{3}$$

where:

- A: Amplitude of the sinusoidal component.
- $\sin(2 \cdot \pi \cdot t)$: Periodic component.

- B: Initial value.
- $C \cdot t$: Linear growth or decay.

Thus, the seasonality function models cyclical price variations with a sinusoidal term and long-term trends with a linear component, providing a practical and interpretable approach for energy market analysis.

1.2 Inverse Gaussian

An Inverse Gaussian I_j with Lévy density described as follows, is a key component of the process since it is responsible of the jumps that characterize the spot price.

$$\nu(x) = \frac{1}{\sqrt{2 \cdot \pi \cdot k}} \frac{c \cdot e^{-\frac{x}{2 \cdot k}}}{x^{\frac{3}{2}}} \cdot \mathbb{1}_{x>0} \tag{4}$$

The parameters that describe an Inverse Gaussian are:

- $\mu > 0$ is the mean of the distribution.
- $\lambda > 0$ is the shape parameter, controlling the "spread" of the distribution.

The parameter k, equal to the variance of the subordinator at time 1, can be defined as: $k = \frac{\mu^2}{\lambda}$

1.3 Constraints

By setting the initial conditions, we obtain a constraint that helps us find the value of the spot price today:

$$S_0 = S(0) = \Lambda(0) + Y_{1,0} + Y_{2,0}$$
$$S_0 = B + 2$$

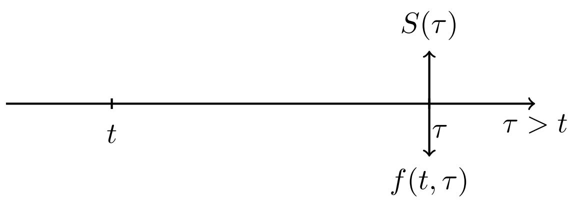
The seasonality parameters are free to move in $\mathbb{R}$, given that B is such that we have a positive Spot price today. While β_j, η_j, for the validity of the mean reversion property of the OU, are bound to be positive.
Another key aspect of the description of an Arithmetic process is the so called **condition A**, according to which there exists a constant $c_j > 0$ such that

$$\int_0^\tau \int_{|z|\geq 1} |z|^{c_j} \, l_j(du, dz) < +\infty, \; j = 1, 2 \tag{5}$$

Condition A represents a moment condition on the jump size distribution of the Lévy process that drives the components $Y_{1,t}$ and $Y_{2,t}$. This condition ensures that the jump intensity is finite for sufficiently large jump sizes. The condition is satisfied for $c_j = 1$ for example. This can be shown in the computations that we use to calculate γ in the next section.
This type of condition is often required in the context of processes with infinite activity but finite variation. By ensuring that this condition holds for $j = 1, 2$, it guarantees that the driving jump process I_j remains manageable in terms of its contribution to the spot price S_t.

2 Forward price

A forward contract is an agreement to buy or sell an asset at a specified future time for a predetermined price. Forward prices reflect the market's expectations of future spot prices adjusted for risk and costs.



The general formulation is given by:

$$f(t,\tau) = \Lambda(\tau) + \Theta(t,\tau;\theta) + \sum_{i=1}^{m} \int_t^\tau \mu_i(u) e^{-\int_u^\tau \alpha_i(v)dv} du + \sum_{j=1}^{n} \int_t^\tau \delta_j(u) e^{-\int_u^\tau \beta_j(v)dv} du + $$
$$+ \sum_{i=1}^{m} X_i^{\mathbb{Q}}(t) e^{-\int_t^\tau \alpha_i(s)ds} + \sum_{j=1}^{n} Y_j^{\mathbb{Q}}(t) e^{-\int_t^\tau \beta_j(s)ds} \tag{6}$$

for $0 \leq t \leq \tau$, where

$$\Theta(t,\tau;\theta) = \sum_{k=1}^{p} \sum_{i=1}^{m} \int_t^\tau \sigma_{ik}(u) \hat{\theta}_k(u) e^{-\int_u^\tau \alpha_i(v)dv} du + \sum_{j=1}^{n} \int_t^\tau \eta_j(u) e^{-\int_u^\tau \beta_j(v)dv} d\gamma_j(u) \tag{7}$$
$$+ \sum_{j=1}^{n} \int_t^\tau \int_{\mathbb{R}} \eta_j(u) e^{-\int_u^\tau \beta_j(v)dv} z \left(e^{\hat{\theta}_j(u)z} - \mathbb{1}_{|z|<1} \right) \ell_j(dz, du)$$

Now let us explicitly find the forward price $f(t,\tau)$ of a contract written on the previous spot price, under the hypothesis given in the previous section:

$$f(t,\tau) = \mathbb{E}_{\mathbb{Q}}\left[S(\tau)|\mathcal{F}_t\right] = \Lambda(t) + \Theta(t,\tau) + \sum_{j=1}^{n} Y_j^{\mathbb{Q}}(t)e^{-\beta_j(\tau-t)} \tag{8}$$

We proceed by retrieving the explicit expressions of $\Theta(t,\tau)$, $\gamma_j(t)$ and $Y_j^{\mathbb{Q}}(t)$.

2.1 Computation of $\gamma_j(t)$

The parameter $\gamma_j(t)$ adjusts forward prices to incorporate the contribution of jumps in the spot price dynamics under the risk-neutral measure. Derived from the Lévy density of the Inverse Gaussian process, it quantifies the expected impact of jumps over time. In this model, we have $\gamma_j(t) = \gamma_j \cdot t$, where γ_j is defined as:

$$\begin{aligned}
\gamma &= \int_{-1}^{1} x\,\nu(dx) = \int_0^1 \frac{x}{\sqrt{2\pi k}} \cdot \frac{e^{-\frac{x}{2k}}}{x^{\frac{3}{2}}}\,dx = \frac{1}{\sqrt{2\pi k}}\int_0^1 e^{-\frac{x}{2k}} \cdot \frac{1}{2\sqrt{x}}\,dx = \\
&= \left\{\text{we perform the change of variable: } \sqrt{x} = y \;\;\Rightarrow\;\; \frac{1}{2\sqrt{x}}dx = dy\right\} = \\
&= \frac{2}{\sqrt{2\pi k}}\int_0^1 e^{-\frac{y^2}{2k}}\,dy = \sqrt{\frac{2}{\pi k}}\int_0^1 e^{-\frac{y^2}{2k}}\,dy = \sqrt{\frac{2}{\pi k}}\int_0^1 e^{-\frac{1}{2}\left(\frac{y}{\sqrt{k}}\right)^2}\,dy = \\
&= \left\{\text{we perform the change of variable: } \frac{y}{\sqrt{k}} = z \;\;\Rightarrow\;\; \frac{dy}{\sqrt{k}} = dz\right\} = \\
&= 2\int_0^{\frac{1}{\sqrt{k}}} \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz = 2\cdot\Phi\left(\sqrt{\frac{1}{k}}\right) - 1
\end{aligned} \tag{9}$$

2.2 Computation of $\Theta(t,\tau;0)$

Now we decompose $\Theta(t,\tau;0)$ to simplify the expression obtained. Θ comes from the Esscher transform, a change of measure that guarantees the distributional properties of the Jumps by using a predetermined set of parameters, in our case we set $\theta = 0$.

$$\begin{aligned}
\Theta(t,\tau;\theta) &= \sum_{j=1}^{n}\int_t^\tau \eta_j e^{-\beta_j(\tau-u)}\,d\gamma_j(u) + \int_t^\tau\int_R \eta_j e^{-\beta_j(\tau-u)} z\left(e^{\theta_j(u)z} - \mathbb{1}_{|z|<1}\right) l_j(dz,du) \\
&= \sum_{j=1}^{n}\int_t^\tau \eta_j e^{-\beta_j(\tau-u)}\,\gamma_j du + \int_t^\tau\int_1^\infty \eta_j e^{-\beta_j(\tau-u)} z\cdot l_j(dz)du \\
&= \sum_{j=1}^{n}\eta_j\cdot\frac{1-e^{-\beta_j(\tau-t)}}{\beta_j}\cdot\gamma_j + \int_t^\tau\int_1^\infty \eta_j e^{-\beta_j(\tau-u)} z\cdot l_j(dz)du \\
&= \sum_{j=1}^{n}\eta_j\cdot\frac{1-e^{-\beta_j(\tau-t)}}{\beta_j}\cdot\gamma_j + \eta_j\int_t^\tau e^{-\beta_j(\tau-u)}du\int_1^\infty z\cdot l_j(z)dz \\
&= \sum_{j=1}^{n}\eta_j\cdot\frac{1-e^{-\beta_j(\tau-t)}}{\beta_j}\cdot\gamma_j + \eta_j\frac{1-e^{-\beta_j(\tau-t)}}{\beta_j}2\cdot\left(1-\Phi\left(\sqrt{\frac{1}{k}}\right)\right) \\
&= \sum_{j=1}^{n}\eta_j\cdot\frac{1-e^{-\beta_j(\tau-t)}}{\beta_j}
\end{aligned} \tag{10}$$

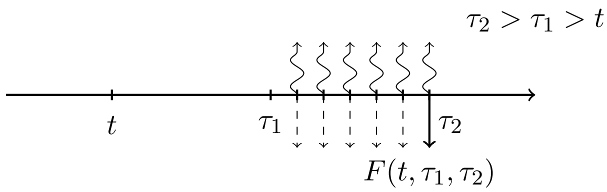
2.3 $\mathbb{Q}$-measure

We can translate $Y_j(t)$ to the $\mathbb{Q}$-measure thanks to **Proposition 4.4** of **Stochastic Modelling of electricity and related markets** from which, using also the other computations, the expression below follows:

$$Y_j^{\mathbb{Q}} = Y_j(t) - \frac{\eta_j}{\beta_j}(1-e^{-\beta_j t})\cdot\left(\gamma + \int_{\mathbb{R}}(1-\mathbb{1}_{|z|<1})y\nu(dy)\right) = Y_j(t) - \frac{\eta_j}{\beta_j}(1-e^{-\beta_j t}) \tag{11}$$

3 Swap price

A swap contract represents an agreement to exchange cash flows based on the difference between the spot price and a predetermined strike price over a set period. In the energy market, swaps are used to hedge against price volatility.
Moreover, in the commodity market framework, Futures and Swaps are practically the same, in fact we will proceed on calculating the price of the Future contract.



The formulation of the future price $F(t, \tau_1, \tau_2)$ leverages the forward price dynamics. Taking the expectation under the $\mathbb{Q}$-measure of the discounted difference between the spot price and the future price, conditioned on the filtration $\mathcal{F}_t$, it can be shown that the future price is expressed as the integral of forward prices weighted by $\omega(u, \tau_1, \tau_2)$. This highlights the connection between forward and future prices, with ω capturing the discounting and weighting over the settlement period.

$$F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \omega(u, \tau_1, \tau_2) \cdot f(t, u)\, du \qquad \text{where} \qquad \omega(u, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1}.$$

We denote that $\omega(u, \tau_1, \tau_2)$ is written in the specific case of the settlement at the end of the monitoring period. The explicit expression for the future price $F(t, \tau_1, \tau_2)$ in the general form is given by:

$$\begin{aligned} F(t, \tau_1, \tau_2) = & \int_{\tau_1}^{\tau_2} \omega(u, \tau_1, \tau_2)\Lambda(u)\, du + \Theta(t, \tau_1, \tau_2; \theta(\cdot)) + \sum_{i=1}^{m} \int_t^{\tau_2} \int_{\max(v,\tau_1)}^{\tau_2} \omega(u, \tau_1, \tau_2)\mu_i(v) e^{-\int_v^u \alpha_i(s)\, ds}\, du\, dv + \\ & + \sum_{i=1}^{m} X_i(t) \int_{\tau_1}^{\tau_2} \omega(u, \tau_1, \tau_2) e^{-\int_t^u \alpha_i(s)\, ds}\, du + \sum_{j=1}^{n} \int_t^{\tau_2} \int_{\max(v,\tau_1)}^{\tau_2} \omega(u, \tau_1, \tau_2)\eta_j(v) e^{-\int_v^u \beta_j(s)\, ds}\, du\, dv + \\ & + \sum_{j=1}^{n} Y_j(t) \int_{\tau_1}^{\tau_2} \omega(u, \tau_1, \tau_2) e^{-\int_t^u \beta_j(s)\, ds}\, du \end{aligned}$$

with $0 \le t \le \tau_1 < \tau_2$, where:

$$\begin{aligned} \Theta(t, \tau_1, \tau_2; \theta) = & \sum_{k=1}^{p} \sum_{i=1}^{m} \int_t^{\tau_2} \int_{\max(v,\tau_1)}^{\tau_2} \omega(u, \tau_1, \tau_2)\sigma_{ik}(v)\hat{\theta}_i(v) e^{-\int_v^u \alpha_i(s)\, ds}\, du\, dv + \\ & \sum_{j=1}^{n} \int_t^{\tau_2} \int_{\max(v,\tau_1)}^{\tau_2} \omega(u, \tau_1, \tau_2)\eta_j(v) e^{-\int_v^u \beta_j(s)\, ds}\, du\, d\gamma_j(v) + \\ & \sum_{j=1}^{n} \int_t^{\tau_2} \int_{\mathbb{R}} \int_{\max(v,\tau_1)}^{\tau_2} \omega(u, \tau_1, \tau_2)\eta_j(v) e^{-\int_v^u \beta_j(s)\, ds} z \left(e^{\tilde{\theta}_j(v) z} - \mathbb{1}_{|z|<1}\right) du\, \ell_j(dz, dv) \end{aligned}$$

Which in our case simplifies in

$$F(t, \tau_1, \tau_2) = \int_t^{\tau_2} \omega(u, \tau_1, \tau_2)\Lambda(u)\, du + \Theta(t, \tau_1, \tau_2; 0) + \sum_{j=1}^{2} Y_j^{\mathbb{Q}}(t) \cdot \int_{\tau_1}^{\tau_2} \omega(u, \tau_1, \tau_2) e^{-\beta_j(u-t)} du \tag{12}$$

$$\begin{aligned} \Theta(t, \tau_1, \tau_2; 0) = & \sum_{j=1}^{2} \int_t^{\tau_2} \int_{\max(v,\tau_1)}^{\tau_2} \omega(u, \tau_1, \tau_2) e^{-\beta_j(u-v)} \gamma\, du\, dv \quad + \\ & \sum_{j=1}^{2} \int_t^{\tau_2} \int_R \int_{\max(v,\tau_1)}^{\tau_2} \omega(u, \tau_1, \tau_2) e^{-\beta_j(u-v)} z \left(1 - \mathbb{1}_{|z|<1}\right) l_j(dz, dv) du \end{aligned} \tag{13}$$

In the following, we solve the integrals to obtain the explicit expression of $\Theta(t, \tau_1, \tau_2; 0)$, using the formulation of $\omega(u, \tau_1, \tau_2)$ defined above.

3.1 Computation of $\Theta(t, \tau_1, \tau_2)$

The term $\Theta(t, \tau_1, \tau_2)$ also captures in this case the Esscher transform change of measure and comes from the integral of $\Theta(t, u)$ from τ_1 to τ_2. With some calculations, we can develop a more straightforward expression to speed up and improve the robustness of the calibration procedure.

$$
\begin{aligned}
&(\tau_2-\tau_1)\cdot\Theta(t,\tau_1,\tau_2)=\\
&=\sum_{j=1}^{2}\int_t^{\tau_2}\int_{max(v,\tau_1)}^{\tau_2}\eta_j e^{-\beta_j\cdot(u-v)}\,du\cdot d\gamma_j(v)+\int_t^{\tau_2}\int_{\mathbb{R}}\int_{max(v,\tau_1)}^{\tau_2}\eta_j\cdot e^{-\beta_j\cdot(u-v)}\cdot y\cdot(e^{\theta\cdot y}-\mathbb{1}_{|y|<1})\,du\cdot\nu(dy,dv)=\\
&=\sum_{j=1}^{2}\int_t^{\tau_2}\int_{max(v,\tau_1)}^{\tau_2}e^{-\beta_j\cdot u}\,du\cdot dv\cdot\left(\gamma+\int_{\mathbb{R}}y\cdot(1-\mathbb{1}_{|y|<1})\,\nu(y)dy\right)=\\
&=\left\{\text{since }\gamma+\int_{\mathbb{R}}y\cdot(1-\mathbb{1}_{|y|<1})\,\nu(y)dy=1\right\}=\\
&=\sum_{j=1}^{2}\eta_j\int_t^{\tau_2}e^{\beta_j\cdot v}\int_{max(v,\tau_1)}^{\tau_2}e^{-\beta_j\cdot u}\,du\cdot dv=\sum_{j=1}^{2}\frac{\eta_j}{\beta_j}\left[\int_t^{\tau_1}e^{\beta_j\cdot v}e^{-\beta_j\cdot\tau_1}e^{-\beta_j\cdot\tau_2}dv+\int_{\tau_1}^{\tau_2}1-e{-}\beta_j\cdot\tau_2 e^{\beta_j\cdot v}dv\right]=\\
&=\sum_{j=1}^{2}\frac{\eta_j}{\beta_j}\left[\frac{1}{\beta_j}\cdot\left(e^{-\beta_j\cdot\tau_1}e^{-\beta_j\cdot\tau_2}\right)\cdot\left(e^{-\beta_j\cdot\tau_1}e^{-\beta_j\cdot t}\right)+(\tau_2-\tau_1)-\frac{e^{-\beta_j\cdot\tau_2}}{\beta_j}\cdot\left(e^{-\beta_j\cdot\tau_2}-e^{-\beta_j\cdot\tau_1}\right)\right]=\\
&=\sum_{j=1}^{2}\frac{\eta_j}{\beta_j^2}\left[1-e^{-\beta_j\cdot(\tau_1-t)}-e^{-\beta_j\cdot(\tau_2-\tau_1)}+e^{-\beta_j\cdot(\tau_2-t)}+(\tau_2-\tau_1)\beta_j-1+e^{-\beta_j\cdot(\tau_2-\tau_1)}\right]=\\
&=\sum_{j=1}^{2}\frac{\eta_j}{\beta_j^2}\left(-e^{-\beta_j\cdot(\tau_1-t)}+e^{-\beta_j\cdot(\tau_2-t)}+(\tau_2-\tau_1)\beta_j\right)=\\
&=\sum_{j=1}^{2}\frac{\eta_j}{\beta_j^2}\left(-e^{-\beta_j\cdot(\tau_1-t)}+e^{-\beta_j\cdot(\tau_2-t)}+(\tau_2-\tau_1)\beta_j\right)
\end{aligned}
$$

3.2 Simplified expression for the Swap price

By computing the remaining integrals in (12), the following simplified expression of the Swap price is obtained:

$$
\begin{aligned}
F(t,\tau_1,\tau_2)=&\frac{1}{\tau_2-\tau_1}\sum_{j=1}^{2}\frac{\eta_j}{\beta_j^2}\left(-e^{-\beta_j\cdot(\tau_1-t)}+e^{-\beta_j\cdot(\tau_2-t)}+(\tau_2-\tau_1)\beta_j\right)+\sum_{j=1}^{2}Y_j^{\mathbb{Q}}(t)\frac{e^{-\beta_j(\tau_1-t)}-e^{-\beta_j(\tau_2-t)}}{\beta_j(\tau_2-\tau_1)}\\
&+\frac{1}{\tau_2-\tau_1}\left(\frac{A}{2\pi}\big(\cos(2\pi\tau_1)-\cos(2\pi\tau_2)\big)+B(\tau_2-\tau_1)+\frac{C}{2}\big(\tau_2^2-\tau_1^2\big)\right)
\end{aligned}
\tag{14}
$$

The final expression for the swap price reveals no dependence on the variance parameter k of the subordinator. This simplification comes from the explicit evaluation of the integrals, which eliminates any dependence on k. In the subsequent sections, while simulating for the pricing of the Put Option, we will set $k = 1$, assuming that the jumps of the process have a unitary mean.

4 Calibration of OU-IG Model

A key step in the development of a model in finance is the calibration to set the parameters' values according to our available data. In this setting, we are going to perform a calibration over the price of 20 French power futures with different times to the start of delivery (τ_1) and three main categories of tenor: monthly, quarterly and yearly. As shown in the computations above, we have already managed to reduce the number of the parameters, so we will have to calibrate 7 parameters in total: A, B, and C for the seasonality and η_1, η_2, β_1, β_2 for what concerns the stochastic part of the model. Before implementing the calibration procedure we must notice that the number of parameters is pretty high with respect to the number of data points and moreover the prices are not from a very efficient market.

4.1 Calibration procedure

The calibration procedure is tested with two different approaches exploiting the MATLAB built-in functions *fmincon* and *lsqnonlin*.
In the calibration problem an important issue is the choice of the starting point of the algorithm that will

perform the minimization of the loss function, so we divided the procedure into two steps: the first one let the model learn just the seasonality parameters while keeping the others fixed, the second using as starting point the parameters found in the first step with all the parameters able to change freely. The measures of error of choice have been the Mean Square Error (MSE), the Mean Absolute Error (MAE) and the Cauchy Loss, although we have also tested other error measures that can be found in the MATLAB code.

4.2 Results and discussion

In Table 1 and Figure 1, we show the results of our two-step calibration. The obtained results are also confirmed running global optimizations without any clue of a starting point. By testing other error functions, some lower MSEs can be reached but the results of the parameters were completely far from any interpretation (e.g. obtaining negative values of the mean reversion parameters).
In this case, we can retrieve some useful information: the values of the seasonality parameters obtained via the two different MATLAB functions are very close. A high value of A indicates a predominant impact of the seasonality part in the model. The parameter B is strictly linked with the spot price of today, and therefore we can retrieve it by adding $\sum_{j=1}^{2} Y_j(0)$ to it and finding a positive value for $S(0) = 502.55$. To conclude the discussion on seasonality parameters, we observe a negative value of C, indicating that the model effectively recognizes the downward trend in future prices over time, which becomes the dominant factor in the yearly contracts, effectively overshadowing the sinusoidal effects of seasonality.

Concerning the jump part of the model, we observe that the two different calibration functions return different results. Calibrating through *fmincon*, the jump component is irrelevant, since the two η parameters, describing the intensity of jumps, are of the order of 10^{-6}, hence they can be assumed to be 0 and the model becomes deterministic.
In the second trial, using the *lsqnonlin* function, we obtained more meaningful estimates for the parameters of the OU processes. The values of β and η are notably high. A high β value indicates a faster mean reversion, a characteristic commonly observed in commodity markets with strong seasonal patterns, such as the electricity market. Similarly, the high η value suggests more frequent and significant jumps, reflecting the substantial volatility that typically drives these markets.

In conclusion of the two calibrations, we can notice that in both cases βs and ηs present he same values for any Y_j. This suggests that Y_2 is a redundant term and leveraging the paradigma of parsimony, a more parsimonious model should be preferred. Hence, we tried to run our code with just $n = 1$, obtaining consistent results.

Parameter	via *fmincon*	via *lsqnonlin*
A	343.45	343.71
B	500.55	499.86
C	-81.32	-81.63
β_1	0.9036	2.5166
β_2	0.9036	2.5166
η_1	$1.17 \cdot 10^{-6}$	2.4651
η_2	$1.17 \cdot 10^{-6}$	2.4651
MSE	21 333	21 346
MAE	118.47	118.57
Cauchy Loss	9.0256	9.0295
Computational time	$2.9045s$	$1.7810s$

Table 1: Calibration of the OU-IG Model Results and main metrics.

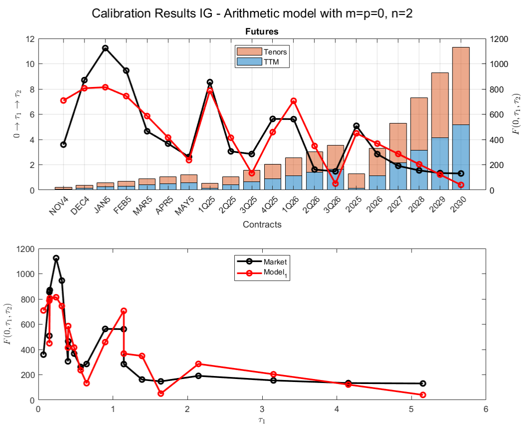


Figure 1: OU-IG Model Prices and Market Prices for the Futures used for calibration.

In the following sections, we adopt the model calibrated via *lsqnonlin*. Despite negligible differences in the cost functions compared to the other calibration, this approach preserves the stochastic behavior of the Arithmetic Model, ensuring consistency with its theoretical foundations. Moreover, also the computational time is reduced with this choice.

A final consideration can be that the high calibration errors highlight the model's challenges in fitting the observed data, largely attributable to the low liquidity typical of commodity markets. Limited trading activity and wide bid-ask spreads can result in prices that do not fully reflect market equilibrium values. This aligns with the less efficient nature of commodity markets compared to other financial markets. Despite these limitations, the model provides a useful representation of the overall dynamics, emphasizing the dominance of seasonality over the mean reversion and stochastic jump terms in the price structure.

5 Simulation of OU-IG Model

For this part of the exercise we go ahead and try to use a Monte-Carlo method to simulate the dynamics of the drivers of our model and to evaluate Put options on the 2028 future with strikes in the range 200/300 and maturity 2 years. The future has the settlement on 29 December 2028, and we decide to take as the start of delivery exactly one year before of the settlement, i.e. the 29 December 2027.
As observed in the graph, the at-the-money value shows a bifurcation between the put and call prices. This occurs because, in our model, seasonality is the dominant driver, leading to low variability. Consequently, most simulations are clustered around the mean value of the future contract, $F = 204.44$, resulting in a zero value for derivatives with out-of-the-money strikes. This is remarked by the fact that the 95% Confidence Interval for the Future is given by [204.4273, 204.4509].

Strike Prices	Put Prices
200	0
210	5.5609
220	15.561
230	25.561
240	35.561
250	45.561
260	55.561
270	65.561
280	75.561
290	85.561
300	95.561

Table 2: Option prices for various strike levels.

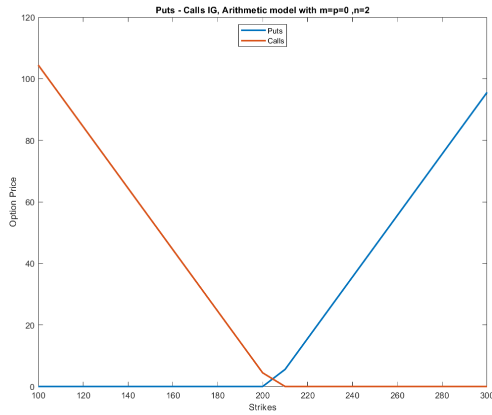


Figure 2: Put and Calls simulated via the Arithmetic Model with $m = p = 0$, $n = 2$.

To see the effect of the parameter k on the price, we try to repeat the simulation for different values of the variance of the subordinator. However, the effects of changing k are not visible because of the values of β and η. If we change them, we can appreciate the differences more clearly, but since the calibration opted for those values, we can conclude that this model relies mostly on the seasonality fit to price these contracts.

6 Arithmetic Model based on a Gaussian OU process

The objective of this section is to review the previous points discussed in the project, while introducing a substantial change in the modeling assumptions. Although the starting framework remains the Arithmetic Model, the primary change lies in considering an Arithmetic Model based on a single Gaussian OU process.

Starting from the general formulation of the Arithmetic Model reported in Equation (1), we simplify it by configuring the parameters according to the Gaussian OU process. Specifically, we set $m = p = 1$ and $n = 0$, resulting in the following simplified model:

$$\begin{cases} S_t = \Lambda_t + X(t) \\ dX = [\mu(t) - \alpha(t) \cdot X(t)]\, dt + \sigma(t) dB \end{cases} \tag{15}$$

As discussed previously, some additional assumptions can be made about the parameters considered. From a modeling perspective, it is also natural to choose $\mu(t) = 0$ since the OU process should ideally revert to zero to have the seasonality function $\Lambda(t)$ as the mean price level. Moreover, we consider the parameters α and σ to be constant in time.

6.1 Spot price

To compute the Spot price $S(t)$, we need to solve the stochastic differential equation involving $X(t)$.

$$dX_t = -\alpha(t)X_t\,dt + \sigma(t)\,dB_t$$

From Stochastic Differential Equation theory, we know that its solution is of the form:

$$X_t = X_0 e^{-\int_0^t \alpha(u)\,du} + \int_0^t \sigma(u)e^{-\int_s^t \alpha(r)\,dr} dB_s$$
$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-u)}\,dB_u$$
$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} \underbrace{\int_0^t e^{\alpha u}\,dB_u}_{W_t}$$

In order to solve the stochastic integral W_t, we need to verify that $f = exp\{\alpha u\} \in M^2[0,t]$.
Since $f \in L^2_{loc}([0,\infty))$, $f|_{[0,t]} \in L^2([0,t])\ \forall t \geq 0$. Hence, $E\left[\int_0^t |f(s)|^2\,ds\right] < \infty\ \forall t\ \geq 0 \Leftrightarrow f \in M^2([0,t])\ \forall t \geq 0$.
We can compute the expected value and the variance of W_t

$$\mathbb{E}[W_t] = 0 \quad (f \text{ deterministic function})$$
$$\text{Var}(W_t) = \mathbb{E}[W_t^2] - \mathbb{E}[W_t]^2 = \mathbb{E}\left[\int_0^t e^{\alpha u} dB_u\right]^2 = \int_0^t e^{2\alpha u}\,du = \frac{1}{2\alpha}\left(e^{2\alpha t} - 1\right), \quad \alpha \neq 0$$

At this point, we are able to write the distribution of W_t

$$W_t \sim \mathcal{N}\left(0, \frac{1}{2\alpha}\left(e^{2\alpha t} - 1\right)\right) \sim \sqrt{\frac{1}{2\alpha}\left(e^{2\alpha t} - 1\right)} \cdot Z_t \quad \text{where} \quad Z_t \sim \mathcal{N}(0,1)$$

After these computations, X_t simplifies in:

$$X_t = X_0 e^{-\alpha t} \pm \sigma\sqrt{\frac{1}{2\alpha}\left(1 - e^{-2\alpha t}\right)} \cdot Z_t \tag{16}$$

Finally the Spot price is given by:

$$S_t = A \cdot sin(2\pi t) + B + C \cdot t + X_0 e^{-\alpha t} \pm \sigma\sqrt{\frac{1}{2\alpha}\left(1 - e^{-2\alpha t}\right)} \cdot Z_t \tag{17}$$

where $A, B, C \in \mathbb{R}$ and $\sigma, \alpha \in \mathbb{R}^+$

6.2 Forward Price

We proceed by computing the forward price under the risk neutral measure $\mathbb{Q}$. Under this measure, the dynamics of the Gaussian OU remains the same; thus, we have $X^{\mathbb{Q}}(t) = X(t)$. From **Proposition 4.10** of **Stochastic Modelling of electricity and related markets** , substituting the values of $\mu = 0$ and $\hat{\theta} = 0$, we know that

$$f(t,\tau) = \Lambda(\tau) + \Theta(t,\tau;\theta) + X_t\,e^{-\int_t^\tau \alpha\,ds}$$

for $0 \leq t \leq \tau$, where

$$\Theta(t,\tau;\theta) = \int_t^\tau \sigma\,\hat{\theta}\,e^{-\int_u^\tau \alpha\,dv}\,du = 0$$

Lastly, the forward price under the risk-neutral measure $\mathbb{Q}$ is:

$$f(t,\tau) = A \cdot sin(2\pi t) + B + C \cdot t + X_t\,e^{-\alpha(\tau - t)} \tag{18}$$

6.3 Future Price

We carry on by computing the Future Price $F(t,\tau_1,\tau_2)$ at time t for a contract that starts the delivery at τ_1 and has settlement time at τ_2. Following **Proposition 4.14** of **Stochastic Modelling of electricity and related markets** , we compute it as:

$$F(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \omega(u,\tau_1,\tau_2) \cdot \Lambda(u)du + \Theta(t,\tau_1,\tau_2,\theta) + X_t \int_{\tau_1}^{\tau_2} \omega(u,\tau_1,\tau_2) \cdot e^{-\int_t^u \alpha\,ds}\,du$$

where

$$\Theta(t,\tau_1,\tau_2;\theta) = \int_t^{\tau_2}\int_{max\{v,\tau_1\}}^{\tau_2} \omega(u,\tau_1,\tau_2)\cdot\sigma\,\hat{\theta}\,e^{-\int_v^u \alpha\,ds}\,du\,dv = 0$$

In conclusion, solving the integrals the future price is:

$$F(t,\tau_1,\tau_2) = -\frac{A}{2\pi(\tau_2-\tau_1)}\cdot(cos(2\pi\,\tau_2) - cos(2\pi\,\tau_1)) + B + \frac{C}{\tau_2-\tau_1}(\tau_2^2-\tau_1^2) - X_t\,\frac{e^{-\alpha\,(\tau_2-t)} - e^{-\alpha\,(\tau_1-t)}}{\alpha(\tau_2-\tau_1)}$$

7 Calibration of OU-Normal Model

Following a two-step calibration as implemented above, we calibrate the parameters of the Gaussian OU model. A notable observation can be made even before evaluating the results: this model exhibits greater parsimony by requiring the calibration of only four parameters in total. The first three correspond to the seasonality parameters, while the fourth, α, defines the mean reversion speed of the process. Furthermore, it is important to note that, since we are at $t = 0$, the dependency on σ disappears.
Given that no significant differences arise from the choice of the optimization function, we directly rely on *lsqnonlin* for calibration.

7.1 Results and discussion

The results align closely with the previous analysis, confirming that seasonality plays a predominant role in this type of model. The value of B is slightly higher, as expected, because it must now account for an additional component in the spot price, because it has only one stochastic contribution from the initial conditions. Consequently, the proposed spot price is $S(0) = B + X(0) = 502.28$.

Parameter	via *lsqnonlin*
A	343.49
B	501.28
C	−81.50
α	0.9074
MSE	21 339
MAE	118.52
Cauchy Loss	9.0274
Computational time	$0.8851s$

Table 3: Calibration of the Gaussian OU Model Results and main metrics.

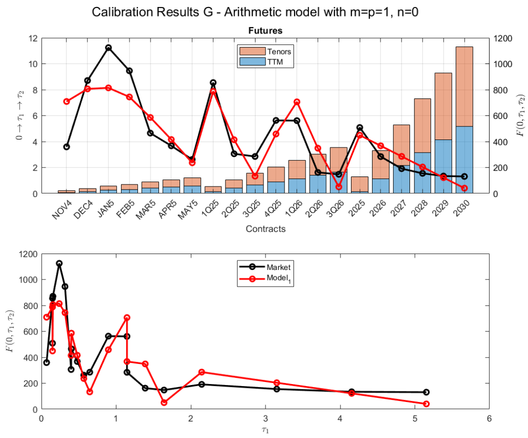


Figure 3: Calibration Results - Normal.

8 Simulation of OU-Normal Model

By simulating the model with a proposed value of $\sigma = 1$ we find similar results with a Future price of 204.4325, more specifically the price belongs to the CI $[204.4323, 204.4327]$, where the confidence level α is 0.05. Again we can see from the plot that the results make sense and are in agreement with the theory.

In this model can be interesting to see the change of the Option graph with respect to σ. As we can see from Figure 5, as we increase σ the slope of the option prices significantly changes indicating an increase in volatility. This can also be verified by the widening of the confidence interval of the *normfit* function applied to the simulated future prices. Hence, exploiting a greater variability also call and put with out-of-the-money strikes have not a null price.

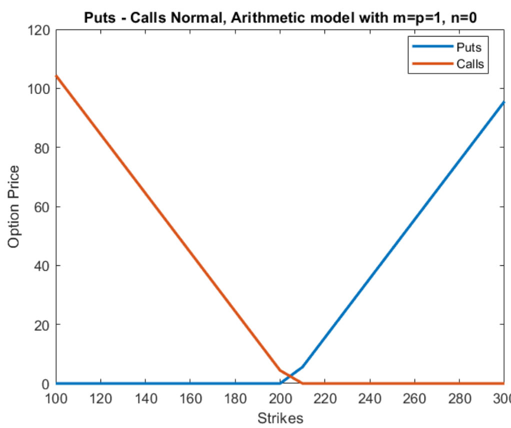


Figure 4: Put Calls simulated via the Arithmetic Model with $m = 1$, $n = 0$.

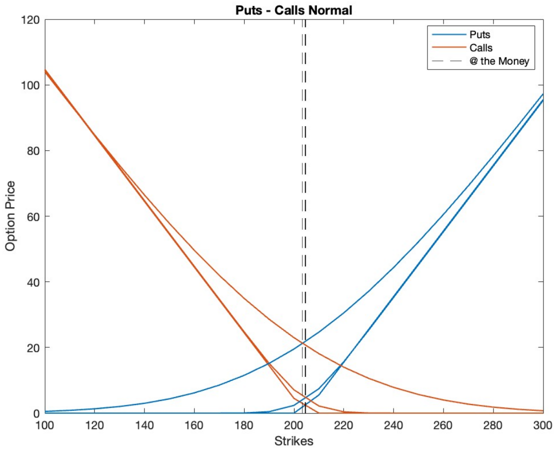


Figure 5: Put Calls simulated via the Arithmetic Model with $m = 1$, $n = 0$ for different values of volatility σ.

Strike Prices	Put Prices
200	0
210	5.5675
220	15.567
230	25.567
240	35.567
250	45.567
260	55.567
270	65.567
280	75.567
290	85.567
300	95.567

Table 4: Option prices for various strike levels.

9 Comparison & Conclusions

The Ornstein-Uhlenbeck model with Inverse Gaussian jumps (OU-IG) and the simplified Gaussian OU model (OU-Normal) offer different strengths based on the dataset and application. Both models capture the dominant seasonality of energy prices, obtaining very similar values for the parameters A, B, and C, where A reflects periodic variations, and C indicates a downward trend; the slight difference in the parameter B ensures that the Spot price $S(0)$ in the two models remains consistent, with a margin of error of 10^{-1}.

The OU-IG model introduces jumps through η, but calibration results show that these have a negligible impact on the observed market, rendering the added complexity unnecessary. Conversely, the OU-Normal model simplifies the dynamics by focusing on mean reversion (α) and seasonality, achieving comparable results with fewer parameters. Moreover, the Gaussian driven model performs better in terms of calibration speed due to its parsimony, converging in half of the time.

The choice of model depends on the market context. The OU-IG model provides theoretical flexibility for datasets with significant jumps, while the OU-Normal model is more parsimonious and practical when seasonality and mean reversion dominate. For this dataset, the OU-Normal model proves sufficient, highlighting the importance of aligning model complexity with market characteristics.

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