POLITECNICO MILANO 1863

Financial Engineering 2023/24

Assignment 6 - Group 20

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Contents

Note

We noticed that the holiday dates in the given function `eurCalendar` arrive only at the year 2065, while our bootstrap results arrive at the year 2074; however, we decided to not modify the function since we noticed that there is not any holiday that falls in the considered dates in the last years.

a Bootstrap of the market discounts

In this assignment we were tasked to consider a structured bond issued by an hypothetical bank on February the 16th, 2024 at 10:45 C.E.T. in a single-rate curve interest rate modeling setting and neglecting counterparty risk.
The first point asked us to bootstrap the market discounts from quoted financial instruments, respectevely depos, futures and swaps. Firstly, as the swap dates were missing, we computed each settlement date with a modified following convention, using the two Matlab functions `isbusdate, nextbusdate`.
Secondly, in order to build a complete curve of swap rates we computed a spline interpolation on the mid rates of quoted swap rates, by the Matlab function `interp1`. In this way we managed to obtain a complete set of rates onto which we were finally able to execute the bootstrap.
All the passages described above are executed in our function `interpolateAndLaunchBootsrap`, while the results are reported in Table1

Dates	Discounts
20-feb-24	1
21-feb-24	0.999891372912237
27-feb-24	0.999240299805398
20-mar-24	0.996883382560450
22-apr-24	0.993368931260857
24-jun-24	0.986679696103947
23-sep-24	0.977913258182718
20-dec-24	0.970395776062312
20-mar-25	0.963524640966420
23-jun-25	0.956906215171774
22-sep-25	0.950975712763411
19-dec-25	0.945449386512830
20-feb-26	0.941694839272557
22-feb-27	0.919578164311307
21-feb-28	0.897938246601254
20-feb-29	0.876192083548183
20-feb-30	0.854184201034599
20-feb-31	0.832167545017148
20-feb-32	0.809930590592249
21-feb-33	0.787654668231042
20-feb-34	0.765245415760480
20-feb-35	0.742941779576611
20-feb-36	0.720936907228110
20-feb-37	0.699517283362406
22-feb-38	0.678831339969539
21-feb-39	0.659373629863272
20-feb-40	0.641163642636969
20-feb-41	0.624162535260720
20-feb-42	0.608348381499272
20-feb-43	0.593622751307294
22-feb-44	0.579788591904779
20-feb-45	0.566942554975866
20-feb-46	0.554768007361503
20-feb-47	0.543270533618380
20-feb-48	0.532368261703120
22-feb-49	0.521926248525446
21-feb-50	0.511981221387193
20-feb-51	0.502406720573732
20-feb-52	0.493138551889210
20-feb-53	0.484173578876776
20-feb-54	0.475452266107078
22-feb-55	0.466903657550634
21-feb-56	0.458632460983387
20-feb-57	0.450571982259470
20-feb-58	0.442680318188496
20-feb-59	0.434988443898900
20-feb-60	0.427497117790627
21-feb-61	0.420197024778268
20-feb-62	0.413134477460604
20-feb-63	0.406263448356180
20-feb-64	0.399609906063082
20-feb-65	0.393189454355595
22-feb-66	0.386960091959339
21-feb-67	0.381024278525737
20-feb-68	0.375339873377761
20-feb-69	0.369912367369057
20-feb-70	0.364751833226180
20-feb-71	0.359880120874641
22-feb-72	0.355286292429030
20-feb-73	0.351068897458959
20-feb-74	0.347148305149298

Table 1: Dates and discounts obtained from the bootstrap

b Pricing: determine the upfront X%

The request of the second point was to compute the X% of the principal amount that needed to be paid by Party B at starting date, according to the Swap Termsheet. The computation had to be repeated twice, both using flat and spot volatilities. The idea for computing this value was to impose the Net Present Value of the contract equal to 0, as the upfront was the only data missing from the Termsheet.

First we considered party B, that paid the upfront quantity X at starting date, and quarterly the following coupons:

$$\begin{cases} 3\% & \text{for the first quarter} \\ min(€3m + 1.10\%, 4.30\%) & \text{up to } 5^{th} \text{ year,} \\ min(€3m + 1.10\%, 4.60\%) & \text{from the } 5^{th} \text{ to the } 10^{th} \text{ year} \\ min(€3m + 1.10\%, 5.10\%) & \text{from the } 10^{th} \text{ to the } 15^{th} \text{ year} \end{cases}$$

Party B received quarterly the €3m plus 2.00%.
In order to compute the NPV_B, we noticed that the coupons that B had to pay, could be rewritten in the following way (starting from the 6^{th} month, up to the 5^{th} year):

$$min(3m + 1.10\%, 4.30\%) = -max(-3m - 1.10\%, -4.30\%)$$
$$= 1.10\% + 3m - max(0, -3.20\% + 3m) = 1.10\% + 3m - (3m - 3.20\%)^+$$

We noticed that the expression $(€3m - 3.20\%)^+$ was exactly the payoff of a caplet with strike K = 3.20%, and the same consideration was repeated for the other two payoffs. After some simplifications, the expression of the cashflows from party B's point of view was:

$$\begin{cases} -X & \text{at starting date} \\ \text{€3m - 1\%} & \text{for the first quarter} \\ \text{Caplets}_{K=3.2\%} + 0.9\% & \text{up to } 5^{th} \text{ year} \\ \text{Caplets}_{K=3.5\%} + 0.9\% & \text{from the } 5^{th} \text{ to the } 10^{th} \text{ year} \\ \text{Caplets}_{K=4\%} + 0.9\% & \text{from the } 10^{th} \text{ to the } 15^{th} \text{ year} \end{cases}$$

The NPV_B could be then rewritten in the following way:

$$NPV_B = 0.9\% \sum_{i=2}^{60} B(0,t_i)\delta_i + 1 - B\left(0, \frac{1}{4}\right) - 1\%\, B\left(0, \frac{1}{4}\right) \delta_1 + Cap_{K=3.2\%} + Cap_{K=3.5\%} + Cap_{K=4\%} - X$$

where $Cap_{K=3.2\%} = \sum_{i=2}^{20} Caplet_{i,K=3.2\%}$ $\qquad Cap_{K=3.5\%} = \sum_{i=21}^{40} Caplet_{i,K=3.5\%}$

$$Cap_{K=4\%} = \sum_{i=41}^{60} Caplet_{i,K=4\%}$$

After imposing $NPV_B = 0$, we obtained the following expression for X:

$$X = 0.9\% \sum_{i=2}^{60} B(0,t_i)\delta_i + 1 - B\left(0, \frac{1}{4}\right) - 1\%\, B\left(0, \frac{1}{4}\right) \delta_1 + Cap_{K=3.2\%} + Cap_{K=3.5\%} + Cap_{K=4\%}$$

The only thing missing were then the caplet prices for the three different K. The market model that we chose is the Normal Libor Market Model, that allowed us to have negative Strikes and for which we knew the caplets'pricing formula, the Bachelier formula:

$caplet_i(T_0) = B(T_0, T_{i+1})\delta_i[L_i(T_0) - K]N(d) + \sigma_i\sqrt{T_i - T_0}\Phi(d) \qquad \text{with } d = \frac{L_i(T_0)-K}{\sigma\sqrt{T_i-T_0}}$

We were given the Market Cap volatilities, and had to compute then the caplets prices from those, both using directly the quoted values and then repeating the computation using the spot volatilities, obtained by boostrapping.

b.1 Prcing the upfront via flat volatilities

The first case was the easiest: from the theory we knew that the price of a Cap is computed supposing all the volatilities of the caplets as constant, so we plugged the five-year volatility into the Bachelier formula. To compute the payoff in from the 5^{th} to the 10^{th} year we considered both the Cap_{5Y} Cap_{10Y} for strike equal to 3.5 and the subtracting the first from the latter. We repeated the same procedure for the payoff between the 10^{th} to the 15^{th} year. In this case, we obtained:

$$X_{\text{flat}} = 18.872475\%$$

As our computation was done considering a Notional equal to 1, this quantity is intended to be multiplied by the Principal Amount of 50 MLN.

b.2 Pricing the upfront via spot volatilities

For the second case instead, the central thing was to obtain the spot volatility by bootstrapping. According to the theory, in the first year we considered the first three volatilities (the ones in 6, 9 and 12 months) as constant and equal to the flat volatility. We computed the second-year volatilities by following this algorithm:

- We computed ΔC, as the difference between the Cap price in 2y and the Cap price in 1 year:

$$\Delta C = Cap_{2Y} - Cap_{1Y}$$

- We then imposed the following relation:

$$\Delta C = \sum_{i=4}^{7} Caplet_i$$

- We obtained the missing relations by imposing the following linear constraint between the unknown volatilities $\sigma_4, \sigma_5, \sigma_6$ and σ_7:

$$\sigma_i = \sigma_\alpha + \frac{T_i - T_\alpha}{T_\beta - T_\alpha} * (\sigma_\beta - \sigma_\alpha)$$

where $i = 4, 5, 6, 7$ with $\alpha = 3$ (1 year) and $\beta = 7$ (2 years)

As we had 4 equations, 4 missing values, we were able to find the solution. Iterating this, for all years we managed to build the spot curve, for every strike K. For a visual representation, we managed to build the whole spot surface by performing a spline interpolation on the different Strikes and a linear interpolation on the different maturities - coherently with the calibration.
We obtained the following figure:

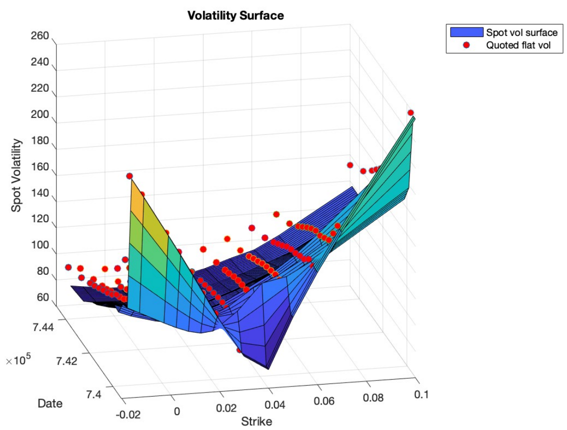

Figure 1: 3D plot of the spot surface vs flat volatilities

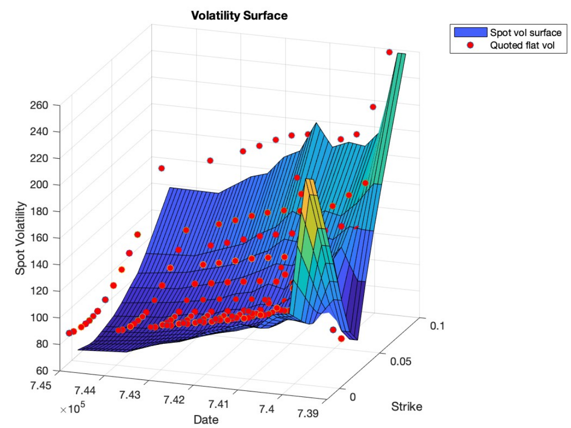

Figure 2: The same 3D plot, rotated

In order to obtain the ”traditional” representation volatility smile, we fixed the YTM respectevely equal to 5,10,15 years.

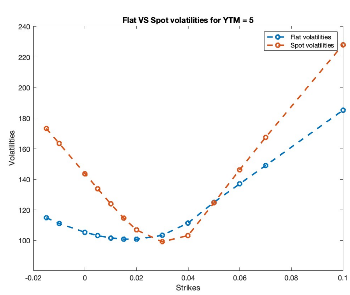

Figure 3: Flat vs Spot volatilities for a fixed YTM = 5

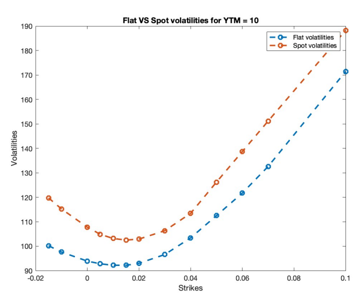

Figure 4: Flat vs Spot volatilities for a fixed YTM = 10

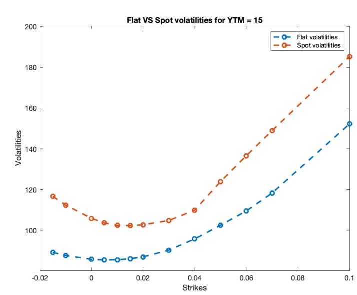

Figure 5: Flat vs Spot volatilities for a fixed YTM = 15

Having obtained this curve finally allowed us to compute the price of a Caplet: analogously as before, we plugged the spot volatility values into the Bachelier formula.

This allowed us to finally obtain the value:

$$X = 18.866488\%$$

We can immediately notice that the X value computed with the spot volatilities is slightly bigger than the value obtained via the flat volatilities, but the results are still very close: we were expecting this result, since we calibrated the spot volatilities by imposing that the Caps' prices remained the same, therefore the value of the structured contract remains roughly the same.

c Delta-bucket sensitivities

We were then asked to compute delta-bucket sensitivities. At point b, we had already structured the function `priceX` to compute the upfront X directly from quoted rates. We decided to shift only the quoted rates and not the results from the bootstrap obtained in point a, to be more consistent with the market. Therefore, we only needed to iteratively bump up only one of the 28 quoted rates[1] by $1bp$ before computing the new upfront values, leaving the spot volatilities unchanged. This clarification is essential because the computed DV01 represents the partial derivative with respect to rates, and thus, we assumed that the rate shock does not immediately impact the volatilities. Once we got this new shifted values, we only needed do subtract from them X to obtain the Delta-Bucket sensitivities: the results are shown in Table2.

$\Delta DEPOS$	
1d	0.0000000000%
1w	0.0000000000%
1m	0.0005137837%
2m	0.0002505142%

$\Delta FUTURES$	
3m	0.0016507673%
6m	0.0011719840%
9m	0.0005103076%
12m	0.0001271664%
15m	-0.0000033292%
18m	-0.0000227227%
21m	-0.0000064469%

$\Delta SWAPS$	
1y	-0.0002843262%
2y	-0.0010956947%
3y	-0.0010281319%
4y	0.0007266471%
5y	-0.0010825440%
6y	-0.0013037272%
7y	-0.0009202866%
8y	-0.0014195188%
9y	0.0041611593%
10y	0.0015326764%
12y	-0.0028943330%
15y	0.0416962292%
20y	0.0001230985%
25y	-0.0000304562%
30	0.0000065143%
40y	-0.0000005702%
50y	0.0000000713%

Table 2: Delta-Bucket sensitivities

It is noticeable that, due to structured bond maturity, Delta-Bucket values for dates further than 15 years plummet rapidly to 0. We can also observe that the Swaps sensitivities have alternated signs: by increasing the quoted swap rate only at a fixed year, we are modifying the forward curve;

[1] 4 depo rates, 7 future rates, 17 swap rates

the caplet of the year corresponding to the shift will likely be more in the money, but for the year immediately after we will have a decrease in the rate, and in the end this shift can result in a negative shift in the X value.

We then computed the Total-Delta, obtaining the result:

$$\Delta_{TOTAL} = 0.0003257076\%$$

We compared the Total-Delta with the cumulated sum of Delta-Bucket values, and the graphical represenation of the comparison is shown in Figure6.

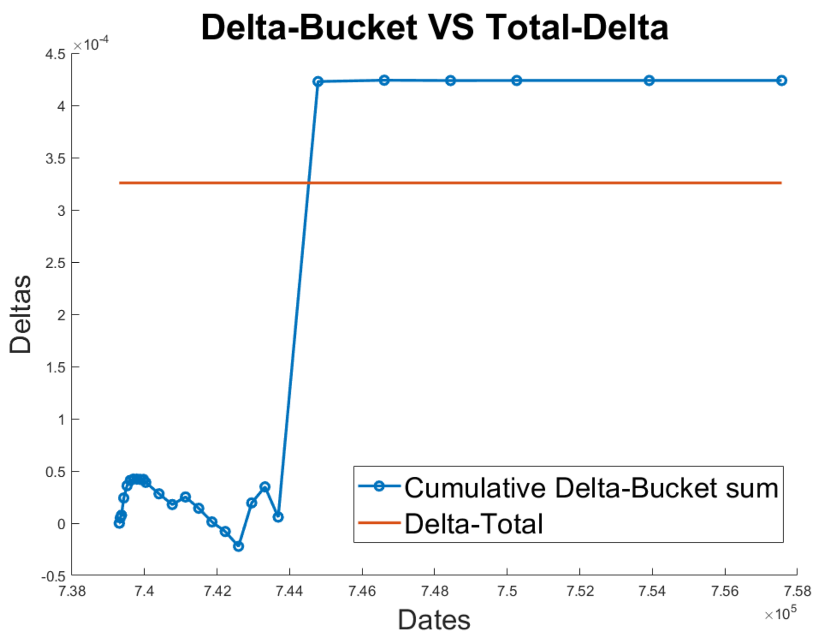

Figure 6: Cumulated sum of Delta-Bucket sensitivities compared to Delta-Total value

As expected, the Total Delta is similar to the sum of all Delta-Buckets values, as both results have the same order of magnitude ($o(10^{-4})$) and the difference is:

$$\Delta_{TOTAL} - \sum_i \Delta_{BUCKET,i} = -0.0000980807\%$$

d Total Vega

In order to compute the Total Vega, we bumped up all the quoted volailities by $1bp$ and computed with those shifted volatilities the new Upfront value using again the function `priceX`; by subtracting the result from the original upfront X, we obtained the Total Vega for a bp in the volatilities:

$$Vega_{TOTAL} = N \cdot (X_{\sigma \text{ shifted}} - X) = 55\,831.86€$$

where N is the notional value.
We observe that the Total Vega is positive: as the volatilities increase, the value of the upfront X of the contract increases as well; we expected this result from the theory, indeed payer B receives 3 Caps and since they are sum of call options on the Libor rate (i.e. caplets), the Vega of this product should be positive.

We computed also the closed formula for the vega of a Caplet by deriving Bachelier's formula, obtaining the following expression:

$$Vega_{caplet_i} = \frac{\partial caplet_i}{\partial \sigma} = B(T_0, T_{i+1})\, \delta_i \Bigg\{ [L_i(T_0) - K]\Phi(d) \left(-\frac{d}{\sigma_i^2} \right) + \sqrt{T_i - T_0}\Phi(d) + \sqrt{T_i - T_0}\Phi(d) \left(\frac{d^2}{\sigma_i} \right) \Bigg\} =$$

$$= B(T_0, T_{i+1})\, \delta_i \sqrt{T_i - T_0}\Phi(d) \Bigg\{ \left(-\frac{d^2}{\sigma_i} \right) + 1 + \left(\frac{d^2}{\sigma_i} \right) \Bigg\} =$$

$$= B(T_0, T_{i+1})\, \delta_i \sqrt{T_i - T_0}\Phi(d)$$

However, even in the cases where a closed formula is available, as a market practice, numerical computation obtained by finite difference is still preferred; for this reason and to limit the computational time we did not implement the code for the closed formula.

e Vega-bucket sensitivities

Aiming at computing the Vega-bucket sensitivities, we iteratively bumped up by $1bp$ the quoted volatilities for every one of the 12 quoted maturities. Afterwards, we computed with them new upfront values, from which we obtained Vega-Bucket sensitivities as:

$$Vega_{\text{BUCKET SENSITIVITY}} = N(X_{\sigma \text{ shifted}} - X)$$

The obtained results are shown in Table3. We remark that only the sensitivities for the buckets up to 15y are reported, since the contract expiry is at 15y, therefore all the following Vega-Bucket sensitivities are equal to zero.

VEGA-BUCKET SENSITIVITIES	
1y	89.47 €
2y	-20.97 €
3y	16.82 €
4y	-10.97 €
5y	957.23 €
6y	-14.93 €
7y	20.07 €
8y	-3.19 €
9y	-7.83 €
10y	3 430.29 €
12y	-0.00 €
15y	51 415.79 €

Table 3: Vega-Bucket sensitivities

By looking at the results, it is immediately evident that the Vega-bucket sensitivity at 15 years is by far the most relevant value. Moreover, an alternation of positive and negative signs can be noticed. At first glance, this result appears quite surprising, considering that caplets are typically vega-positive instruments. We attribute this unexpected negative Vega to the spline interpolation we employ on spot volatilities, where the smoothness varies as we increase the flat quoted values. To compare the Vega-bucket sensitivities with the Total Vega computed at the previous point, we plotted the Total Vega against the cumulative sum of the Vega-bucket as shown in Figure7.

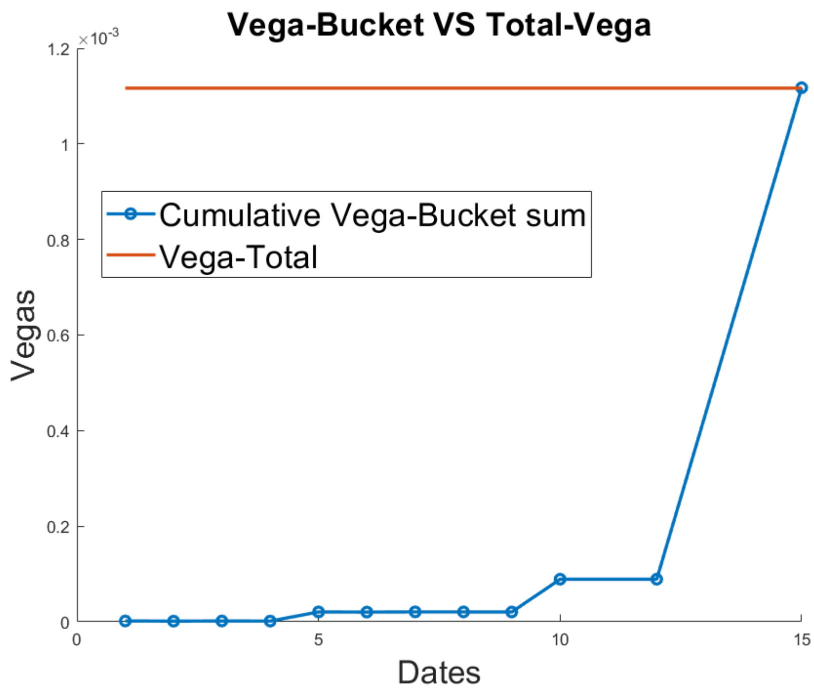

Figure 7: Cumulated sum of nominal Vega-Bucket sensitivities compared to nominal Vega-Total value

From the graph it can be easily appreciated that the cumulative sum skyrockets by adding the last term and in the end is very close to the Total Vega; we also computed the difference obtaining:

$$Vega_{TOTAL} - \sum_{i} Vega_{BUCKET,i} = -39.91€$$

f Hedging of the Delta risk with swaps

From the bucket Delta sensitivities computed in point c, we were asked to determine the coarse-grained Deltas and hedge them using particular swaps. Coarse-grained Deltas are frequently employed in risk management frameworks to depict exposure to a specific segment of the interest rates curve. The selected buckets have four peaks: 2, 5, 10 and 15 years. As it is possible to observe in Figure8 in order to determine the weights in the aggregated single shock Delta sum, we considered as start of each bucket the previous bucket's peak (the settlement for the first peak) and as end of the bucket the next one.

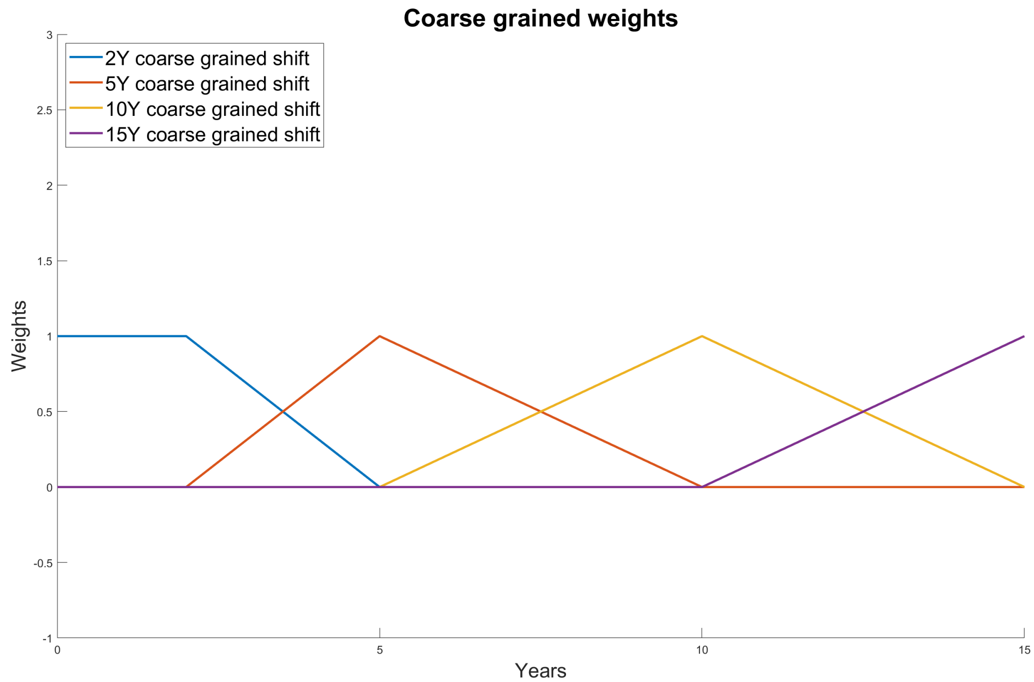

Figure 8: The plot represents the weights used in the coarse grained bucket delta

Except from the first coarse-grained bucket, for which we considered unitary weights up to peak, each bucket's weights evolved linearly from its start to the peak and it gradually decreased linearly until its end. The next table exhibits the aggregated sensitivities:

C-G BUCKET PEAK	DELTA
2Y	0.002837%
5Y	-0.002626%
10Y	0.001224%
15Y	0.040846%

Table 4: The table shows the deltas for the structured product aggregating the buckets in clusters with peaks of 2, 5, 10 and 15 years

For the same clustered shocks we computed the difference in the NPV for the swaps we have to use to hedge, whose maturities are the same as the buckets' peaks. We considered fixed payer swaps and we obtained the results in Table5.

	$\mathbf{DV01}_{\mathbf{2Y\,Swap}}$	$\mathbf{DV01}_{\mathbf{5Y\,Swap}}$	$\mathbf{DV01}_{\mathbf{10Y\,Swap}}$	$\mathbf{DV01}_{\mathbf{15Y\,Swap}}$
2Y CG Bucket	1.907×10^{-4}	6.823×10^{-16}	3.607×10^{-16}	-1.112×10^{-16}
5Y CG Bucket	0	4.601×10^{-4}	2.231×10^{-17}	-2.205×10^{-17}
10Y CG Bucket	0	0	8.649×10^{-4}	-1.332×10^{-16}
15Y CG Bucket	0	0	0	0.001215

Table 5: In each entry there is the coarse-grained DV01 for the relative Swap

As expected, all deltas associated with peaks exceeding the swap maturities are 0, as alterations in rates beyond those maturities have no impact on the Net Present Value of the swap. From a hedging perspective, our aim was to determine the notionals to align the deltas of the structured product with those of the portfolio of swaps for each bucket. The matching of deltas procedure involved starting by identifying the longest swap notional and hedging the farthest bucket, as that swap is the only instrument affected by the shift in the swaps portfolio. We then proceeded to the next longest peak and repeated the process. To accomplish this, we opted to solve a linear system where we utilized the swap sensitivities matrix, which was triangular due to the aforementioned properties, along with the known term of the structured product coarse-grained deltas. The obtained notionals are as follows:

Swap Maturity	Notional
2y	-7 436 762.14 €
5y	2 853 960.03 €
10y	-707 409.18 €
15y	-16 806 600.57 €

Table 6: Notional of swaps to hedge the coarse grained bucket delta

As the NPVs are referred to fixed payer swaps, the results are suggesting that Counterparty B needs to enter in receiver swaps for 2, 10 and 15 years maturities and enter in a payer one with the remaining maturity. The notional sign for each maturity swap in the linear system is influenced by the structured product coarse-grained deltas, with opposite signs. This occurs because the matrix in the linear system is nearly diagonal.

g Hedging the total Vega with an ATM 5y Cap

After we have computed the total Vega in point d, we were required to hedge it by using a 5 years Cap. As found in previous points the sign of the total Vega results to be positive for Counterparty B, as from its point of view it receives caplets in the contract window. We had to determine the notional of a Cap of 5y maturity, having the respective par swap rate as strike, in order to be Vega neutral. We started pricing the Cap in standard quoted condition and we computed the price after a shock in the flat volatilities, following the same procedure of the structured product. We obtained the result

$$Vega_{Cap_{5y}} = 0.025875€$$

as Vega for a unitary notional of the cap, and to determine the notional value we imposed the following equation:

$$Vega_{Cap} \cdot Notional_{Cap} + Vega_{StructuredProduct} \cdot Notional_{StructuredProduct} = 0$$

We determined that the notional of the 5-year Cap should be:

$$Notional_{Cap} = -220\,058\,019.08€$$

where the minus sign is indicating that we need to sell the cap. This result was expected, as we are already Vega positive, and in the hedging process, we are dealing with another Vega positive product.

After achieving a Vega-neutral portfolio, our focus shifted to hedging the delta for these products. Since we are already Vega neutral, we will employ a Vega-neutral product for hedging, such as a swap. Similarly to what was done for the Vega, we determined the full delta of the swap as the difference between the NPV after and before the shift, alternatively we could have set it as the BPV of the swap. We required that the delta of the portfolio made up by the structure product, the 5Y cap and the 5Y swap is equal to 0:

$$\Delta_{Cap} \cdot Notional_{Cap} + \Delta_{StructuredProduct} \cdot Notional_{StructuredProduct} + \Delta_{Swap_{5y}} \cdot Notional_{Swap_{5y}} = 0$$

By reversing this equation, we obtained the following result for the notional of the 5y swap:

$$Notional_{Swap_{5y}} = 64\,629\,959.61€$$

The negative notional value of the Cap made the portfolio Delta negative, so we need to enter a payer swap with the aforementioned notional to hedge the Delta risk.

h Hedging of the bucketed Vega with a 5y Cap and 15 year Cap

Considering the coarse-grained buckets for the Vega (0-5y and 5y-15y) we were asked to hedge the bucketed Vega with a 5y Cap and 15 year Cap.

We started imposing that the strikes of the two considered Caps were equal to the 5y Swap rate and 15y Swap rate respectively (as hinted in the previous point); we defined the weights for the two macro buckets via interpolation, as previously done in Section f for the coarse-grained Delta.

We proceeded by computing the Vega of the buckets of the Caps as the difference of their prices after and before the $1bp$ shifting of the flat volatilities, and we finally obtained the coarse-grained Vega of X (whose buckets had been already computed in Section e) and of the Caps as a weighted sum of the buckets' Vegas.

	$\mathbf{Vega_{5Y\,Cap}}$	$\mathbf{Vega_{15Y\,Cap}}$
5Y CG Bucket	2.535×10^{-4}	1.784×10^{-7}
15Y CG Bucket	0	0.001195

Table 7: In each entry there is the coarse-grained Vega for the relative Cap

We can observe that we obtained an upper triangular matrix and similar considerations as the ones done in Section h can be done.

To hedge, we imposed the total Vega of the portfolio equal to zero as in the following relationship:

$$Notional_{Cap5y} \cdot Vega_{Cap_{5y}} + Notional_{Cap_{15y}} \cdot Vega_{Cap_{15y}} + Notional_{StructProd} \cdot Vega_{StructProd} = 0$$

We obtained the following notional values for the Caps:

$$Notional_{Cap_{5y}} = -11\,131\,591.1788$$

$$Notional_{Cap_{15y}} = -44\,442\,441.0302$$

We observe that since the structured product has a positive Vega and also the Caps have positive values of coarse-grained Vega, we assume a short position on both the Caps to make the portfolio Vega-neutral, coherently with the Total-Vega hedging. One difference of the coarse-grained hedging from the total hedging seems to be that significantly lower amounts of the Caps need to be sold in the coarse-grained case: among the two methods a trade off between the cost of the fees and the hedging should be found in order to optimize the operation.

To conclude the last point of the assignment, we decided also to hedge the course-grained Delta of the portfolio composed by the structured products, the 5y and the 15y Caps, by using two Swaps: the first one with maturity in 5 years and the second one with maturity in 15 years.
By computing the course grained DV01 of the products and by imposing the total Delta of the portfolio equal to zero, we obtained the following notional values for the Swaps:

$$Notional_{Swap_{5y}} = 3\,404\,669.9454$$

$$Notional_{Swap_{15y}} = 5\,231\,997.1181$$

Coherently with the Delta hedging previously done, we need to assume long positions on both the Swaps.