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Assignment 2 - Group 17

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1 Bootstrap for Euribor 3M Interbank curve

Via the function `bootstrap`, we realized the bootsrap for the Discount Factors' curve on all the settlement date and expiries of quoted underlyings.
To build the curve we selected the most liquid assets avaliable. We used the Depos up to 1 month (whose expiry coincides with the first future settlement), in which we computed the discounts from the quoted Libor rates by reversing the formula. Then we used the rates of the first seven futures, which are the most liquid assets: we interpolated or extrapolated the discounts from the initial date to the delivered underlying's settlement date $B(t_0, t_i)$ (we always interpolated by passing to the zero rates and doing a linear interpolation on the zero rates, with flat extrapolation), we determined the forward discount at the initial date between the settlement and the expiry date $B(t_0, t_i, t_{i+1})$ by reversing the formula and, finally, we multiplied these two discounts to get the discount factor from the initial date to the expiry $B(t_0, t_{i+1})$. Finally, from the quoted swap rates we computed the discount factors from the initial dates to the Swap expiries, including all swap rates except for the first one which was interpolated from the discounts obtained from the futures.
From the Discount Factors obtained, we computed the zero rates curve. We plotted the results and we obtained the curves respresented in Figure 1, while the numerical results obtained are reported in Table 1.

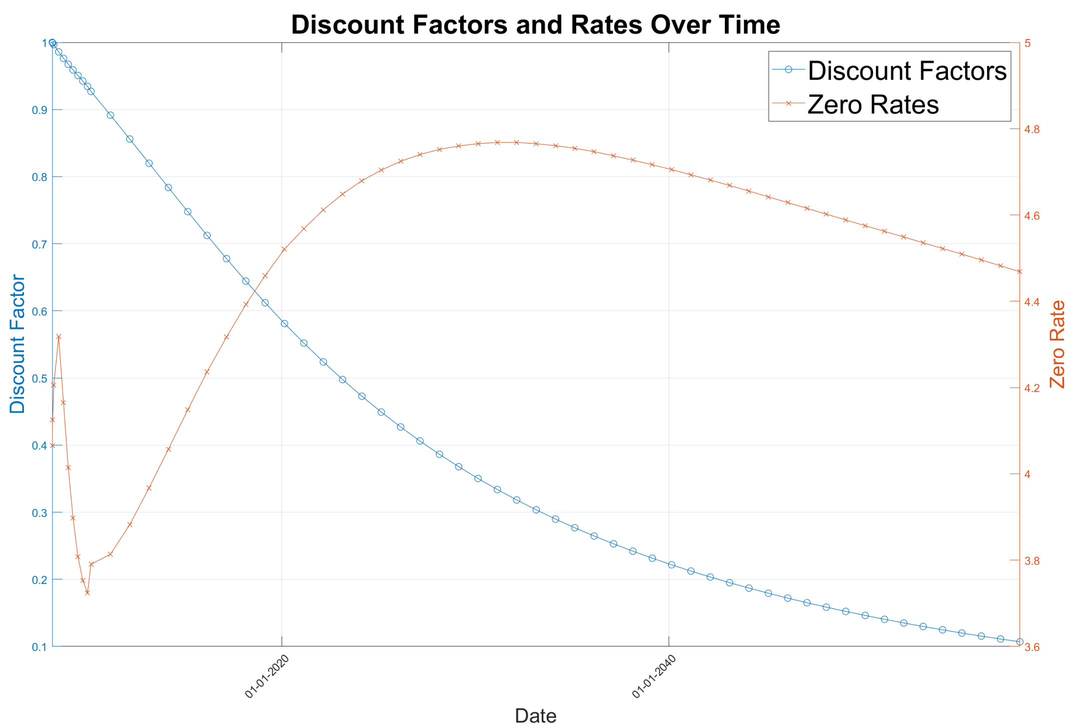

Figure 1: Graphical representation of the Discount Factors' (blue line) and the Zero Rates' (orange line) curves

Dates:	Discount Factors	Zero rates
19 Feb 2008	1	-
20 Feb 2008	0.999888624	0.040654680
26 Feb 2008	0.999209237	0.041248958
19 Mar 2008	0.996664082	0.042056739
19 Jun 2008	0.985787289	0.043180644
18 Sep 2008	0.976098360	0.041651182
17 Dec 2008	0.967330318	0.040144261
17 Mar 2009	0.959004956	0.038975888
18 Jun 2009	0.950660593	0.038079039
17 Sep 2009	0.942498142	0.037527406
16 Dec 2009	0.934302355	0.037242700
19 Feb 2010	0.926896693	0.037904657
21 Feb 2011	0.891615154	0.038135745
20 Feb 2012	0.855991160	0.038820628
19 Feb 2013	0.819887102	0.039674247
19 Feb 2014	0.783753948	0.040572972
19 Feb 2015	0.747802679	0.041484118
19 Feb 2016	0.712370658	0.042365597
20 Feb 2017	0.677733837	0.043169727
19 Feb 2018	0.644300588	0.043922890
19 Feb 2019	0.612067369	0.044595126
19 Feb 2020	0.581075571	0.045208574
19 Feb 2021	0.551910197	0.045682254
21 Feb 2022	0.523911415	0.046119610
20 Feb 2023	0.497608738	0.046486958
19 Feb 2024	0.472734633	0.046794266
19 Feb 2025	0.449181319	0.047040250
19 Feb 2026	0.426979704	0.047242869
19 Feb 2027	0.406057937	0.047400532
21 Feb 2028	0.386237139	0.047519621
19 Feb 2029	0.367748915	0.047598687
19 Feb 2030	0.350230941	0.047653598
19 Feb 2031	0.333730112	0.047679951
19 Feb 2032	0.318190352	0.047680071
21 Feb 2033	0.303476422	0.047651056
20 Feb 2034	0.289725424	0.047606853
19 Feb 2035	0.276758055	0.047544437
19 Feb 2036	0.264486507	0.047466240
19 Feb 2037	0.252892525	0.047370773
19 Feb 2038	0.241917939	0.047270688
21 Feb 2039	0.231457444	0.047163455
20 Feb 2040	0.221615069	0.047051661
19 Feb 2041	0.212275864	0.046930649
19 Feb 2042	0.203383562	0.046809042
19 Feb 2043	0.194938018	0.046683501
19 Feb 2044	0.186912029	0.046554702

20 Feb 2045	0.179261339	0.046419233
19 Feb 2046	0.172016784	0.046286713
19 Feb 2047	0.165095523	0.046152989
19 Feb 2048	0.158494279	0.046019399
19 Feb 2049	0.152194100	0.045883313
21 Feb 2050	0.146143745	0.045750836
20 Feb 2051	0.140413850	0.045620030
19 Feb 2052	0.134942506	0.045489444
19 Feb 2053	0.129706969	0.045355252
19 Feb 2054	0.124707453	0.045223866
19 Feb 2055	0.119936320	0.045091742
21 Feb 2056	0.115359941	0.044957786
19 Feb 2057	0.111044213	0.044821030
19 Feb 2058	0.106903534	0.044684738

Table 1: Discount Factors and zero rates at the corresponding dates obtained via the bootstrap

In Figure 1 we can observe that the curves obtained from our results follows the usual trend of discount factors and zero rates curve.

As highlighted by Uri Ron[1], "the swap market can be used as benchmark for evaluating the performance of other fixed-income markets, and as reference rates for forecasting": in fact, "[...] the swap market's liquidity, depth, and high correlation with fixed-income products, other than plain vanilla government bonds, render its derived term structure a fundamental pricing mechanism for these products and a relevant benchmark for measuring the relative value of different fixed-income products". The swap term structure has several advantages over government curves: it is characterized by inter-market comparability due to the (almost) absence of government regulations; it is a market with high liquidity and of large size; the swaps have similar credit-risk properties across countries.

There are several ways, beyond the bootstrap, to fit the term structure of interest rates, such as the Nelson-Siegel function, the Svensson function, the Principal Component Analysis, or the Heath-Jarrow-Morton model. Some methods, as the Nelson-Siegel, focus on intuitive model interpretability and aim to get a smoother curve of easier use, others, as the PCA, focus on the identification of trends to explain market movements.

Among these methods the Nelson and Siegel function and the Svensson function tend to fit the market data very loosely, constructing smooth curves: this is appropriate for extracting expectations or comparative analysis across countries, but is not appropriate for market pricing, since over-smoothing the yield curve might cause the elimination of valuable market pricing information. The bootstrap method, instead, preserves variations in market observations, preserving important pricing information. Hence, the bootstrap model reflects the current market conditions and

[1]see Ron (2000) for the guide to swap curve constuction.

dynamics accurately.

Bootstrap presents other advantages: the fact that the discount factors are adjusted on those very liquid traded instruments assures market consistency, and that the discount factors obtained satisfy the no-arbitrage condition. Morover, curve flexibility to non standard rates variations is assured by this method; the bootstrap is also a powerful tool to manage credit risk, since it provides insights into the variability associated with risk estimates and models.

On the other hand, there can be some critical issues. Firstly, it is a quite computationally demanding method, even if less than others such as the HJM, and, requiring the use of interpolation and extrapolation techniques, it relies heavily on data quality (i.e. on high liquidity of the observed traded instruments). In addition, linear interpolation can introduce inaccuracies when there is significant curvature in the term structure, or sparse or noisy data, while cubic spline interpolation (the most common alternative method to linear interpolation) may produce inconsistent or implausible forward term structures.

In conclusion, despite not being computationally challenging and despite the possible presence of inaccuracies, the bootstrap fit an interest curve that, at the expense of smoothness, can well represent even unexpected minor variations in market prices.

2 DV01 for an IRS and modified duration for a coupon bond

We are asked to find the sensitivity measures for a 6 year swap with a not par rate, i.e. $S^f = 2.8173\%$.After calculating the BPV, as the summation of the discounted tenors up to maturity, we found the NPV for the fixed and the floating legs. In order to determine the $DV01$, we replicated the same calculations as above, after increasing by $1bps$ all quoted rates. The difference between the NPV obtained is the $DV01$, namely the variation in the portfolio value due to the increase of quoted rates. In the case of a fixed payer swap the sign of the $DV01$ is positive: the rise of rates causes a rise in the NPV of the floating leg. A simpler measure, obtained without performing the bootstrap twice, is the $DV01^{parallel}$. It is computed as the difference of the swaps NPV after a parallel shift of $1bps$ in the zero rate curve. Considering a notional of €10Mln the results are as follows:

BPV:	$DV01$	$DV01^{parallel}$
€52397880.34	€5014.17	€5204.97

Table 2: Sensitivity measures

We are then asked to compute the Macaulay Duration for a IB bond with maturity of 6 years: it is obtained as the mean of the payment dates weighted on the discounted cash flows. The duration is equal to:

$$MacD(t_0) = 5.5849$$

and, if multiplied by the bond price and the notional, it can be used as a proxy of $DV01^{parallel}$.

$$MacD(t_0) * BondPrice * Notional = €5201.67 \approx €5204.97$$

with an error of $o(Notional * 1bps)$.

3 Price of a 6y "I.B. coupon bond"

In order to price a 6 years fixed coupon bond, we just sum the discount cash flows:

$$BP = B(0, t_6) + \sum_{i=1}^{6} S^{7y}\delta(t_{i-1}, t_i)B(t_0, t_i)$$

where t_i is the i^{th} year. By adding and subtracting the term $S^{7y}\delta(t_6, t_7)B(t_0, t_7)$ we get:

$$\begin{aligned} BP = B(0, t_6) + \sum_{i=1}^{7} S^{7y}\delta(t_{i-1}, t_i)B(t_0, t_i) - S^{7y}\delta(t_6, t_7)B(t_0, t_7) = \\ = B(0, t_6) + 1 - B(t_0, t_7) - S^{7y}\delta(t_6, t_7)B(t_0, t_7) = \\ = B(0, t_6) + 1 - B(t_0, t_7)\left(1 + S^{7y}\delta(t_6, t_7)\right) \end{aligned}$$

where we recognize the summation term $\sum_{i=1}^{7} S^{7y}\delta(t_{i-1}, t_i)B(t_0, t_i)$ as the NPV of the fixed leg in a 7 years swap. Since we are referring to the par swap rate, we can substitute the summation with the simpler expression of the floating NPV: $1 - B(t_0, t_7)$.
The discount factors can be obtained from the bootstrap outputs and the price for the bond is:

$$BP = 100.4454$$

As we expected, since the 7 years swap rate is higher than the 6 years' one, the bond is over the par.

4 Garman–Kohlhagen formula for a European Call option

We want ti show that Garman-Kohlhagen formula for a European Call option holds for an underlying with interest rates, continuous dividends and volatilities deterministic functions of time. Being the underlying dynamics the following Geometric Brownian Motion:

$$dS_t = S_t[(\mu(t) - d(t))dt + \sigma(t)dW_t] \tag{1}$$

We proceed by applying Ito Formula for the natural logarithm of S_t:

$$dln(S_t) = [(\mu(t) - d(t))dt + \sigma(t)dW_t] - \frac{1}{2}\sigma(t)^2 dt \tag{2}$$

Integrating between t_0 and t we get:

$$\int_{t_0}^{t} dln(S_s) = \int_{t_0}^{t}[(\mu(s) - d(s)) - \frac{1}{2}\sigma(s)^2]ds + \int_{t_0}^{t}\sigma(s)dW_s \tag{3}$$

For semplicity, we define $V(t)^2 = \frac{1}{t-t_0}\int_{t_0}^{t}\sigma(s)^2 ds$.

Having deterministic interest rates and continuous dividends, we define also $M(t) = \frac{1}{t-t_0}\int_{t_0}^{t}\mu(s)ds$ and $D(t) = \frac{1}{t-t_0}\int_{t_0}^{t} d(s)ds$. Thus, by substitution in (3) we get:

$$ln(S_t) - ln(S_{t_0}) = (M(t) - D(t) - \frac{1}{2}V(t)^2)(t - t_0) + \int_{t_0}^{t}\sigma(s)dW_s \tag{4}$$

Being $\int_{t_0}^{t}\sigma(s)dW_s$ a stochastic integral with a deterministic integrand $\sigma(s)^2$, the stochastic integral is a Gaussian process; it has mean equal to zero and we use Ito Isometry to compute its variance:

$$Var(\int_{t_0}^{t}\sigma(s)dW_s) = \mathbb{E}[(\int_{t_0}^{t}\sigma(s)dW_s)^2] = \mathbb{E}[(\int_{t_0}^{t}\sigma(s)^2 ds)] = V(t)^2(t - t_0)$$

Then:

$$\int_{t_0}^{t}\sigma(s)dW_s \sim N(0, V(t)^2(t - t_0))$$

From (4) we can write:

$$S_t = S_0 \cdot exp\left[(t - t_0)(M(t) - D(t) - \frac{1}{2}V(t)^2) + \int_{t_0}^{t}\sigma(s)dW_s\right] \tag{5}$$

We get

$$S_t \sim S_0 \cdot exp[(t - t_0)(M(t) - D(t) - \frac{1}{2}V(t)^2) \pm \sqrt{t - t_0}V(t) \cdot h] \tag{6}$$

with $h \sim N(0, 1)$.

In order to price the option we need to compute:

$$C(t_0,t) = B(t_0,t)\mathbb{E}_0[(S_t - K)^+] = B(t_0,t)(\mathbb{E}_0[S_t \mathbb{1}_{S_t \geq K}] - K\mathbb{E}_0[\mathbb{1}_{S_t - K}]) \tag{7}$$

To proceed we compute the two means, starting with the computation of $\mathbb{E}_0[\mathbb{1}_{S_t-K}]$. We have that:

$$\begin{gathered} S_t \geq K \iff \\ S_0 \exp\left[(t-t_0)\left(M(t) - D(t) - \frac{1}{2}V(t)^2\right) - \sqrt{t-t_0}V(t)h\right] \geq K \iff \\ \left[(t-t_0)\left(M(t) - D(t) - \frac{1}{2}V(t)^2\right) - \sqrt{t-t_0}V(t)h\right] \geq \ln\left(\frac{K}{S_0}\right) \iff \\ h \leq \left[\ln\left(\frac{S_0}{K}\right) + (t-t_0)\left(M(t) - D(t) - \frac{1}{2}V(t)^2\right)\right]\frac{1}{V(t)\sqrt{t-t_0}} = d_2 \end{gathered}$$

Thus we get:

$$\mathbb{E}_0[\mathbb{1}_{S_t \geq K}] = \int_{-\infty}^{+\infty} \frac{\exp(-h^2/2)}{\sqrt{2\pi}} \mathbb{1}_{h \leq d_2} dh = N(d_2) \tag{8}$$

On the other hand, we have:

$$\mathbb{E}_0[S_t \mathbb{1}_{S_t \geq K}] = S_0 \mathbb{E}_0[exp[(t-t_0)(M(t) - D(t) - \frac{1}{2}V(t)^2) - \sqrt{t-t_0}V(t)h]\mathbb{1}_{h \leq d_2}] =$$

$$= S_0 exp[(t-t_0)(M(t) - D(t))] \int_{-\infty}^{d_2} exp\left[-\frac{1}{2}(t-t_0)V(t)^2 - \sqrt{t-t_0}V(t)h - \frac{h^2}{2}\right]\frac{dh}{\sqrt{2\pi}}$$

$$= S_0 exp[(t-t_0)(M(t) - D(t))] \int_{-\infty}^{d_2} exp\left[-\frac{1}{2}(h + \sqrt{t-t_0}V(t))^2\right]\frac{dh}{\sqrt{2\pi}}$$

By defining $h_1 = h + \sqrt{t-t_0}V(t)$, we get:

$$\mathbb{E}_0[S_t \mathbb{1}_{S_t \geq K}] = S_0 exp[(M(t) - D(t))(t-t_0)] \int_{-\infty}^{d_1} dh_1 \frac{1}{\sqrt{2\pi}} exp[-h_1^2/2]$$

Therefore:

$$\mathbb{E}_0[S_t \mathbb{1}_{S_t \geq K}] = S_0 exp[(M(t) - D(t))(t-t_0)]N(d_1) \tag{9}$$

with $d_1 = d_2 + V(t)\sqrt{t-t_0}$

Therefore we can conlude by substitution of (8) and (9) in the formula for the price in (7), and we obtain:

$$C(t_0,t) = B(t_0,t)[S_0 exp[(M(t) - D(t))(t - t_0)]N(d_1) - KN(d_2)] \tag{10}$$

with

$$d_{1,2} = \frac{1}{V(t)\sqrt{t - t_0}}[ln(\frac{S_0}{K}) + (M(t) - D(t))(t - t_0) \pm \frac{1}{2}V(t)^2(t - t_0)]$$