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Assignment 4 - Group 17

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0 Variance-covariance method for VaR & ES in linear portfolio: a simple example of data mining

At the end of the 20[th] of February 2020, we were asked to compute daily $\text{VaR}_{99\%}$ and $\text{ES}_{99\%}$ for an equally weighted equity linear portfolio (with notional €15MIO) with Adidas, Allianz, Munich Re and L'Oréal, a 5 years estimation, via t-student parametric approach.

We started by filling the dataset via the Python function `ffill`, adding previous days values in case they resulted missing due to different trading days of the different stocks. We isolated the values of the stocks for the firms and the period considered and, once computed the logreturns' matrix for the stocks, we got the portfolio mean (μ_{ptf}) and variance (σ^2_{ptf}), that we needed to rescale portfolio VaR and ES. We computed the losses and verified that the portfolio's loss distribution respected the t-Student distribution: we plotted the histogram of the losses against the t-distribution (with mean (μ_{ptf}) and variance (σ^2_{ptf})) probability density function. The resulting graph is shown in Figure 1: it can be observed that the histogram of losses follows well the t-Student PDF, especially along the tails.

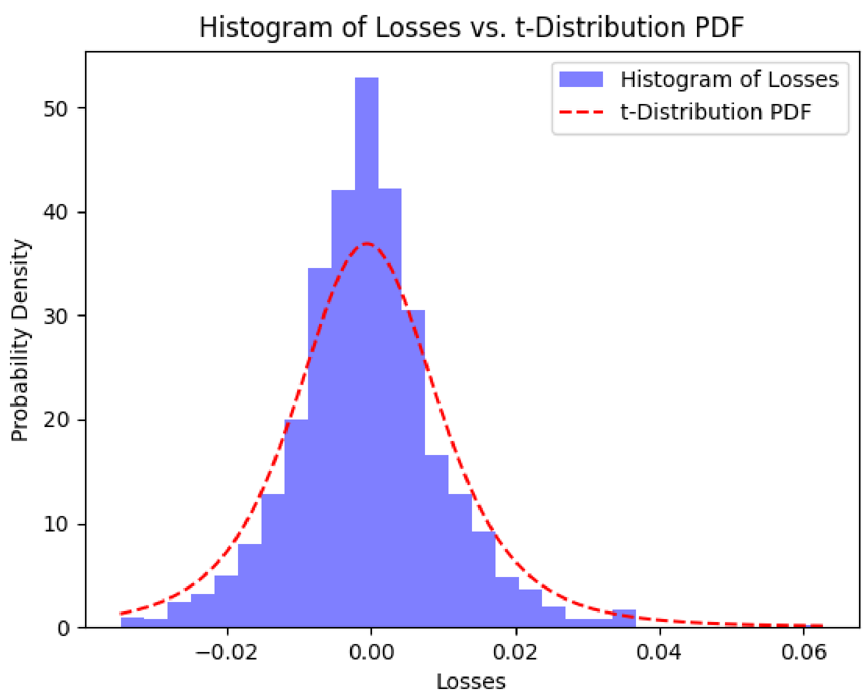

Figure 1: Graph representing the losses VS the t-Student ditribution (with mean and variance equal to the portfolio mean and variance values and 4 degrees of freedom)

We verified this hypothesis also by doing a statistical test by computing a t-statistic and the corresponding p-value: the test failed to reject the null hypothesis with a 95% significance, therefore we can assume that the data are consistent with t-distribution.

After calculating standard VaR ($VaR^{std}_{99\%}$) and ES ($ES^{std}_{99\%}$) for the t-Student, we computed the required quantities, obtaining the following results:

$$Var_{99\%} = \Delta\mu_{ptf} + \sqrt{\Delta\sigma^2_{ptf}}VaR^{std}_{99\%} = 563223.316726$$

$$ES_{99\%} = \Delta\mu_{ptf} + \sqrt{\Delta\sigma^2_{ptf}}ES^{std}_{99\%} = 787977.234225$$

where Δ is the time interval of the risk measure (time scaling factor).

We can notice that the ES is grater than the VaR, as we expected from the theory, and both quantities are two orders of magnitude lower than the notional value of the portfolio.

1 Historical (HS & WHS) Simulation, Bootstrap and PCA for VaR & ES in a linear portfolio

We were asked to compute risk measures for different portfolios at the end of the 20$^{\text{th}}$ of March 2019 following Historical Simulation approach, Bootstrap method, Weighted Historical Simulation approach and Gaussian parametric Principal Components Analysis.

Moreover, for each portfolio, we were asked to check the results' order of magnitude via a Plausibility check: in order to do that, we started, for each portfolio, by computing covariance matrix (C) and upper and lower 5% quantiles on portfolio losses. We then used those quantiles to calculate the signed VaR (sVaR), which ultimately gave us the following rule of thumb:

$$VaR = \sqrt{\Delta}V_t\sqrt{(sVaR \cdot C)sVaR}$$

where Δ is the time interval in days from the starting date to the corresponding VaR computation date and V_t is the value of the portfolio.

1.1 A: Historical Simulation and Bootstrap

Portfolio 1 is composed by 25K shares of Total, 20K shares of AXA, 20K shares of Sanofi and 10K shares of Volkswagen and we were asked to compute daily $\text{VaR}_{95\%}$ and $\text{ES}_{95\%}$ with a 5y estimation using the dataset provided via a Historical Simulation approach and a Bootstrap method with 200 simulations.

Firstly, we computed the portfolio weights ($\underline{w}_t$) from shares and stock values at settlement time, and we calculated, considering a 5 years window, the losses under the Frozen Portfolio assumption via $L = -V_t(\underline{w}_t \cdot \underline{x}_{t+\Delta})$, where $\underline{x}_t$ are the logReturns at time t.

Once sorted in descending order the N losses, we were able to get the $\text{VaR}_{95\%}$ as the $\lfloor N(1-\alpha)\rfloor^{th}$

[1] greatest loss, and $\text{ES}_{95\%}$ as the mean of the losses smaller or equal to $\lfloor N(1-\alpha) \rfloor$: in the Python code, given that arrays' positions start from 0, there was taken the value in position $\lfloor N(1-\alpha) \rfloor - 1$ in the ordered vector of losses to estimate the VaR and the ES was computed accordingly.

Concerning the bootstrap Method we randomly picked 200 of the losses previously computed, and then we computed, as done in the Historical Simulation approach, $\text{VaR}_{95\%}$ and $\text{ES}_{95\%}$ as the $\lfloor N(1-\alpha) \rfloor^{th}$ greatest loss among those that we sampled. The obtained results are shown in table 1:

	$VaR_{95\%}$	$ES_{95\%}$	Computational time
H-S	96039.466152	143630.829079	0.000996
Bootstrap	90502.702556	148922.222702	0.000873
Plausibility Check	92035.626807		

Table 1: $\text{VaR}_{95\%}$ and $\text{ES}_{95\%}$ computed for portfolio I via Historical Simulation and Statistical bootstrap method, and plausibility check for the VaR

We can observe that the H-S and the Bootstrap method give us fairly similar results, and the Plausibility Check confirmes their validity as well.

1.2 B: Weighted Historical Simulation

Portfolio 2 has equally weighted equity with shares from Adidas, Airbus, BBVA, BMW and Deutsche Telekom, and we were asked to compute daily $\text{VaR}_{95\%}$ and $\text{ES}_{95\%}$ with a 5y estimation via a Weighted Historical Simulation approach with $\lambda = 0.95$. The notional of the portfolio is equal to 1.

We started by calculating the weights of the portfolio at settlement time, and we computed the decreasing weights of the losses as $\underline{w}_s = C\lambda^{t-s}$, using the normalization factor $C = \frac{1-\lambda}{1-\lambda^n}$.

We then proceeded to calculate and order the losses as done for the H-S approach. Starting from the greatest to the smallest one, we computed the cumulative weight of the losses, until we found the $\text{VaR}_{95\%}$ as the largest loss[2] having cumulative weight equal or smaller than 0.05: we found the $i^{*\ th}$ largest one via the `numpy` function `searchsorted`, and then we take the value in position $i^* - 1$ in the vector of the ordered losses, since Python arrays' positions start from 0.

The $\text{ES}_{95\%}$ was then computed as the weighted mean of the losses preceding, and comprising, the one picked as VaR:

$$\text{ES}_{95\%} = \frac{\Sigma_{i=1}^{i^*} w_i L^i}{\Sigma_{i=1}^{i*} w_i}$$

we remark that, also in this case, the indices in the Python code go from 0 to $i^* - 1$.

[1] $\lfloor N(1-\alpha) \rfloor$ is the largest integer smaller than the $N(1-)$

[2] the $i^{*\ th}$ largest one

The obtained results are shown in Table 2.

	$\mathrm{VaR}_{95\%}$	$\mathrm{ES}_{95\%}$
W-H-S	0.015977	0.021544
Plausibility Check	0.019219	

Table 2: $\mathrm{VaR}_{95\%}$ and $\mathrm{ES}_{95\%}$ computed for portfolio II via Weighted Historical Simulation and plausibility check for the VaR

We can observe that the WHS result for he VaR is very closed to the plausibility check, therefore the rule of thumb is already a good approximation in this case.

1.3 C: Gaussian Parametric PCA

Portfolio 3 is an equally weighted equity portfolio with shares of 18 companies, for which we had to do compute 10 days $\mathrm{VaR}_{95\%}$ and $\mathrm{ES}_{95\%}$ via Gaussian parametric PCA for a number n of principal components ranging from 1 to 5. We started by finding eigenvalues and eigenvectors for the yearly covariance matrix, previously computed, using Python function `eig` from `numpy` library. Once sorted the eigenvalues' array, we identified the n principal components as the n largest eigenvalues. We proceeded by computing mean and covariance of the reduced portfolio, which (multiplied respectively by $t = 10$ and $\sqrt{t} = \sqrt{10}$, according to the time scaling rule) were used to rescale the standard daily $VaR^{std}_{95\%}$ and $ES^{std}_{95\%}$ obtained considering a Gaussian distribution. We also computed the plausibility check for the VaR. The obtained results are shown in the following table:

n	$\mathrm{VaR}_{95\%}$	$\mathrm{ES}_{95\%}$
1	0.054537	0.068815
2	0.054319	0.068613
3	0.054340	0.068634
4	0.054412	0.068710
5	0.054397	0.068695
Plausibility check for VaR	0.054632	

Table 3: 95% 10 days VaR and the ES via Gaussian parametric approach (PCA) and plausibility check for the VaR

Computing the risk measures for different parameters, we observed that the biggest difference in $\mathrm{VaR}_{95\%}$ and $\mathrm{ES}_{95\%}$ values occurs when passing from 1 to 2 principal components: indeed, the difference between the VaR computed with $n = 2$ and the VaR with $n = 1$ is of the order of $o(10^{-4})$, while the other differences are of an order of $o(10^{-5})$ (and the same holds for the ES). From this result we can suppose that the first principal component already explains a big proportion of the dataset's variance, and by taking $n = 2$ we would already obtain a good approximation of the correct result.

To further investigate this assumption, we decided to compute the explained variance ratio for all the 18 components, and then to plot it. The graph obtained is represented in Figure 2.

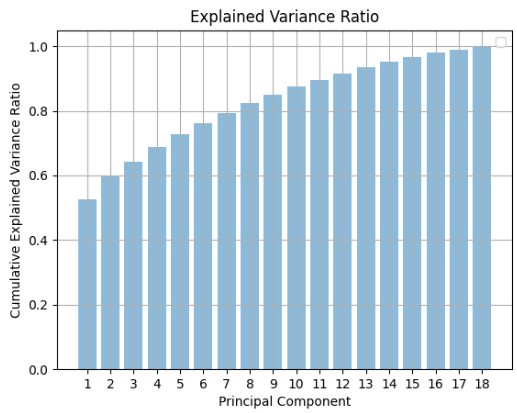

Figure 2: Cumulative explained variance ratio by considering the first n principal components for $n = 1, ..., 18$

As expected, the first principal component already explains more than 50% of the dataset variance, while the contribution of the other components is significantly lower. Afterwards, we continued our analysis by plotting $\mathrm{VaR}_{95\%}$ and $\mathrm{ES}_{95\%}$ values computed with the PCA method for an increasing number of principal components n and we compared them with the values of the risk measures computed analytically via Gaussian parametric approach: we plotted the results in Figure 3 and in Figure 4, where we can observe that, as n increases, the values of $\mathrm{VaR}_{95\%}$ and $\mathrm{ES}_{95\%}$ tend to the ones that we computed analytically.

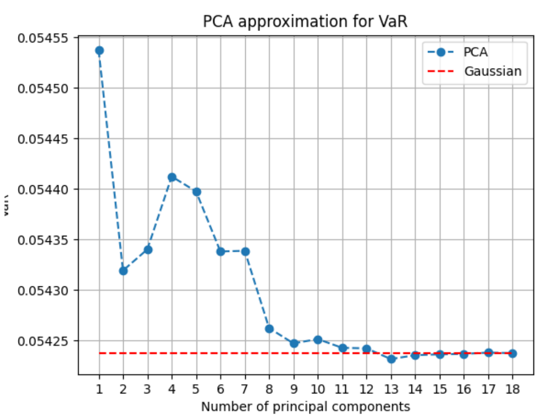

Figure 3: Graph representing the VaR computed via a Gaussian parametric PCA approximation for different numbers of principal components n (blue line) against the VaR computed via Variance-Covariance Gaussian parametric approach (red line)

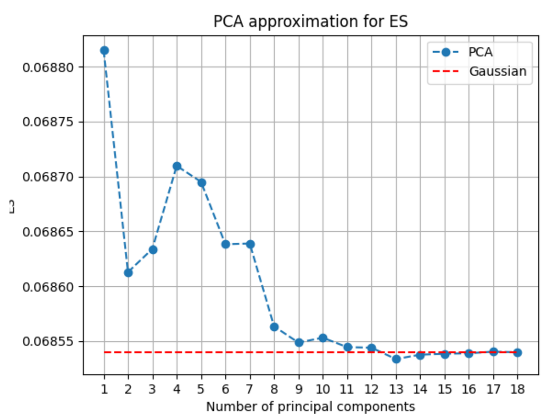

Figure 4: Graph representing the ES computed via a Gaussian parametric PCA approximation for different numbers of principal components n (blue line) against the ES computed via Variance-Covariance Gaussian parametric approach (red line)

From this results we can observe that by considering 9 principal components we are already very close to the risk measures' values computed via Variance-Covariance method (since the error among the two is of the order of 10^{-6}). Moreover, if we compare only the cases in which $n = 1, ..., 5$ (as requested in the assignment) we can immediately observe that the best approximation of the Variance-Covariance method's VaR is obtained for $n = 2$, confirming the supposition made above.

2 Full Monte-Carlo and Delta normal VaR

At the end of the 16^{th} of January 2017 we are given a non linear portfolio, made of long position in BMW shares for 1,186,680 € and a short position on the same numbers of call options (with expiry on the 18^{th} of April 2017, with strike 25 Euro volatility equal to 15.4%, dividend yield of 3.1%, and fixed interest rate of 0.5% for the period), written on BMW as well. This portfolio consists in a covered call strategy.

We started to evaluate the $\text{VaR}_{95\%}^{10\ days}$ via a Full Monte-Carlo approach with a 2y Weighted Historical Simulations with $\lambda = 0.95$.
In this framework, the first step is to simulate the risk factors for the the risk measure in the time interval. For each simulation, we sampled 10 indices (one for each day of the period) from a weighted distribution, assigning a higher probability for recent log-returns to be picked: this was made since by simply taking a uniform distribution the relevance given by the weights to most recent returns would have been lost.
We then computed the overall weight of each simulation by taking the mean of the weights of the chosen daily returns (again, if a uniform selection of the samples had been made, the corresponding mean-weight given to the simulation would have been very close to the maximum weight divided by a factor of 10); afterwards, we normalized the weights by dividing each one of them by their total sum, such that they summed up to 1.
For each simulated risk factor we computed the values of the assets composing the portfolio: in our case, this computation is done via a closed formula, but for exotic derivatives, whose price can be only computed numerically, a nested MonteCarlo is required and this would cause a sky-rocketing increase in computational time.
Once the values of the assets are found for each simulation, we proceeded with the losses computation and we finally estimated the VaR from the simulations as described in the WHS framework.

As the full MC procedure could be computationally expensive, tha VaR can be approximated with Delta Normal approach. We found two different ways to run the MonteCarlo: either to simulate the VaR for the risk period using the accordingly amount of risk factors daily samples or estimate the daily VaR and scale it by the risk period according to the square-root scaling rule: we tried to compute it in both ways, and the two methods gave an approximation of the same order of magnitude; in the end, we chose the first approach, to be consistent with the simulation done via the Full Monte-Carlo.

Once the risk factors were sampled, the losses were estimated using the sensitivities of the derivatives as follows:

$$L(\underline{X}_t) = -\sum_{i=0}^{d} sens_i(t) X_{i,t} \tag{1}$$

where $X_{i,t}$ are the risk factors in the risk period and $sens_i(t)$ is the first order greek (which in our case is the $Delta$), multiplied by the stock value.

The portfolio replicates a covered call strategy: since the call option is wide ITM, the exposure is very low w.r.t. the value of the underlying subscribed, and we expect to have limited losses. The VaRs obtained using the two approaches are shown below:

Approach	**10d VAR (€)**	**Computational time**
Full MC	1495.76	14.53
Delta Normal	627.04	3.14
Delta Gamma	627.04	3.14

Table 4: Summary of VaR for non linear portfolio (100000 simulations)

Despite the results could seem to be very different from each other, if we focus on the losses for each single asset we observe that it takes values of an higher order of magnitude: the Delta Neutral approach is correctly capturing the netting of the portfolio, with a significantly lower computational time.
In order to improve the accuracy of the approximation, the Delta Normal is enlarged to a Delta Gamma method including second order sensitivities of the derivatives. We tried to compute the VaR in this usually more precise framework but, as expected, we had no improvement in the estimate: as the call option is far in the money (i.e. $d_1 >> 1$) the Gamma is completely irrelevant so this method cannot give us a closer approximation. From our point of view, the discrepancy between the two risk measures could be due to the risk period we considered: since Delta method is nothing else than Taylor expansion, the results obtained for 1d VaRs would have been more similar. Another approach to seek for a better approximation could have been the Cornsih-Fisher approximation.

3 Case Study: Pricing in presence of counterparty risk

On the 15^{th} of February 2008 we are asked to price a Cliquet option with 7 years maturity, sold by ISP, written on an equity stock (with no dividends) and constant volatility $\sigma = 20\%$ and participation coefficient $L = 0.99$.

We started by computing the clean price of the derivative, namely the price without counterparty risk. By observing the yearly payoff of the option 2, we immediately recognized that the derivative consists in a strip of L Forward-start calls (**FS**) with strike rate $1/L$.

$$[L \cdot S(t_i) - S(t_{i-1})]^+ \tag{2}$$

We focused on a single FS with delivery date T_i and expiry T_{i+1}: its value in T_i will be:

$$\mathbf{FS}(T_i) = p(T_i, T_{i+1})\mathbb{E}^{T_{i+1}}\left[\left(S(T_{i+1}) - \frac{1}{L}S(T_i)\right)^+\right] = C\left(S(T_i), T_{i+1} - T_i, \frac{1}{L}S(T_i)\right) \tag{3}$$

where we used the forward measure given by retrieving the forward rate in T_{i+1} and the price of the call is obtained via the Geman-El Karoui-Rochet formula for the call option as described in Björk (1998). In order to simplify the obtained expression, we then use the same argument as in Musiela (1997):

$$\mathbf{FS}(T_i) = C\left(S(T_i), T_{i+1} - Ti, \frac{1}{L}S(T_i)\right) = S(T_i)C\left(1, T_{i+1} - T_i, \frac{1}{L}\right) \tag{4}$$

Now we have that the the call price is deterministic and does not depend explicitly from $S(T_i)$, so we can find the expectation at time t:

$$\mathbf{FS}(t) = p(t, T_i)\mathbb{E}^{T_i}\left[S(T_i)C\left(1, T_{i+1} - T_i, \frac{1}{L}\right)\right] = p(t, T_i)C\left(1, T_{i+1} - T_i, \frac{1}{L}\right)\mathbb{E}^{T_i}\left[S(T_i)\right] \tag{5}$$

As the process $\frac{S(\tau)}{p(\tau, T_i)}$ is a martingale under the T_i forward measure we have that:

$$\mathbf{FS}(t) = C\left(1, T_{i+1} - T_i, \frac{1}{L}\right)S(t) = C\left(S(t), T_{i+1} - T_i, \frac{1}{L}S(t)\right) \tag{6}$$

As anticipated above, we used the Geman-El Karoui-Rochet formula with constant volatility to compute the prices of the call options

$$C\left(S(0), T_{i+1} - T_i, \frac{1}{L}S(0)\right) = S(0)N(d_1) - \frac{S(0)}{L}p(0, T_i, T_{i+1})N(d_2) \tag{7}$$

where:

$$d_2 = \frac{\ln\left(\frac{L}{p(0;T_i,T_{i+1})}\right) - \frac{1}{2}\sigma^2(T_{i+1} - T_i)}{\sqrt{T_{i+1} - T_i}\sigma}$$

$$d_1 = d_2 - \sigma\sqrt{T_{i+1} - T_i}$$

The price of the Forward starting options composing the strip are reported in Table 5:

Forward Start Maturity	Forward Start value (in %)
1Y	9.37
2Y	9.25
3Y	9.37
4Y	9.44
5Y	9.56
6Y	9.67
7Y	9.77

Table 5: Forward Start price via closed formula for different maturities

Finally, in order to get the price of the option, we multiply the unit price by the notional and the participation value, because of the assumption done at the beginning.
We then tried to price the derivative via MonteCarlo simulations and compare it with the analytical one and check the correctness of our arguments. By simulating the stock dynamics as a GBM, driven by the forward rates retrived from the bootstrap, we calculated and discounted the yearly payoffs. The next table 6 shows the prices obtained with the two methods described above:

Description	Price (€)
Cliquet clean price via closed formula	19 726 316.66
Cliquet clean price via Monte Carlo	19 809 764.12
Bid	19 707 353.75
Ask	19 912 174.48

Table 6: Summary of Cliquet clean prices (10 000 MC simulations)

Once we have found the clean prices, we have to consider the counterparty risk of ISP, the seller of the derivative. To include the risk of default in the price, we retrieved the survival probability of ISP by boostrapping the CDS quoted spreads and we made the assumption of independency from the underlying dynamics.
We simulated a uniform and by inverting the survival probability, we obtained the simulated default times τ. In case of default between T_i and T_{i+1}, we neglected the accrual term matured between T_i and τ and we assumed that the recovery term will be settled at the next reset date, so in T_{i+1}. The recovery term is composed by the recovery rate times the NPV of the next expected payments

Under the same assumptions for the recovery computations, we also computed the dirty price via the following closed formula:

$$L\cdot\text{Notional}\cdot\sum_{i=0}^{6}\left(S_0 P(0,T_i) FS\left(1, T_{i+1}-T_i, \frac{1}{L}\right) + R\sum_{j=i}^{6} S_0 FS\left(1, T_{j+1}-T_j, \frac{1}{L}\right)(1-P(T_j,T_{j+1}))\right)$$

where:

- $P(s,t)$ is the survival probability between s and t
- R is the recovery value
- L is the participation value.

Moreover, in order to have a check for the closed price, we performed a MonteCarlo simulation under the same assumptions mentioned before. The obtained prices are as follows

Description	**Price (€)**
Cliquet dirty price via closed formula	19 445 985.6
Cliquet dirty price via Monte Carlo	19 423 806.4
Bid	19 322 689.15
Ask	19 524 923.65

Table 7: Summary of Cliquet dirty prices (10000 MC simulations)

The derivative's cash flows consist in ISP payments, so the credit value adjustment should be considered only by the entity who buys the option. On the other hand, from an ISP perpective, the NPV of the trade until maturity is negative, and so the bank wants to sell the option at its clean price.

References

Björk, Tomas (1998). *Arbitrage theory in continuous time*. Oxford: Oxford university press.
Musiela, Marek (1997). *Martingale Methods in Financial Modelling*. Springer.