Ricevimento 25-11-21

FARE ESERCIZI COMPITI D'ESAME

$f: \mathbb{R}^2 \to \mathbb{R}^2 \quad f: \mathbb{R}^2 \to \mathbb{R}^3 \quad f: \mathbb{R}^3 \to \mathbb{R}^3, \dots$

$f: \mathbb{R}^{\overset{\text{colonne}}{n}} \to \mathbb{R}^{m}$ righe $\quad n = m$ opp. $n \neq m$

$$M(f)_{m \times n} = \begin{pmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

Esercizi per casa su Applicaz. lineari

3.1

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{cases} f(1,-1,0) = (1,h,0) \\ f(0,-1,0) = (h,1,0) \\ f(0,0,2) = (0,0,2h-1) \end{cases}$$

$\rightsquigarrow \{ \underbrace{f(1e_1 - 1e_2 + \cancel{0e_3})}_{\text{I membro}} = f(e_1 - e_2) = f(e_1) - f(e_2)$

$f(0,-1,0) = f(\cancel{0e_1} - 1e_2 + \cancel{0e_3}) = f(-e_2)$

Studiare Imf, Ker$f$ e le equaz. cartesiane di Imf e Ker$f$

Risoluzione

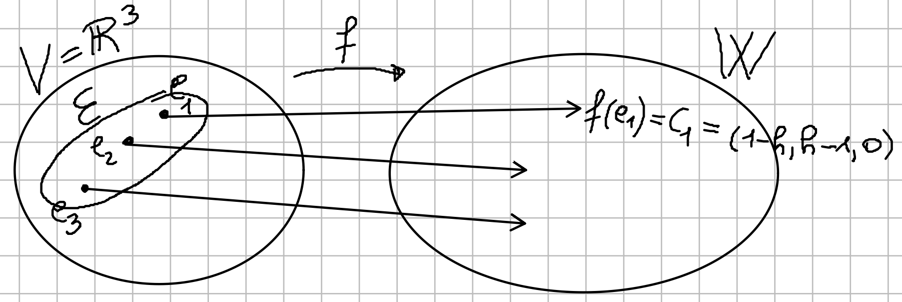
Metodo "standard" rientra nella 2° tipologia di sistemattina

$$\begin{cases} f(e_1) - f(e_2) = (1,h,0) \\ -f(e_2) = (h,1,0) \\ 2f(e_3) = (0,0,2h-1) \end{cases} \longrightarrow \begin{cases} f(e_1) - (-h,-1,0) = (1,h,0) \\ f(e_2) = (-h,-1,0) \\ f(e_3) = \left(0,0,\frac{2h-1}{2}\right) \end{cases} \longrightarrow$$

$$\begin{cases} f(e_1) = (1,h,0) + (-h,-1,0) = (1-h, h-1, 0) & C_1 \\ f(e_2) = \_\_\_ & C_2 \\ f(e_3) = \_\_\_ & C_3 \end{cases}$$

$$A = \begin{pmatrix} 1-h & -h & 0 \\ h-1 & -1 & 0 \\ 0 & 0 & \frac{2h-1}{2} \end{pmatrix}$$

$$\begin{cases} f(\underbrace{2,0,-2}) = (1,h,0) \Rightarrow f(2e_1 + \cancel{0e_2} - 2e_3) = f(2e_1 - 2e_3) = f(2e_1) - f(2e_3) = 2f(e_1) - 2f(e_3) \\ f(-3,1,0) = (1,h,0) \Rightarrow f(-3e_1 + 1e_2 + \cancel{0e_3}) = f(-3e_1 + e_2) = -3f(e_1) + f(e_2) \\ \text{etc.} \dots \end{cases}$$



$$A = \begin{pmatrix} 1-h & -h & 0 \\ h-1 & -1 & 0 \\ 0 & 0 & \frac{2h-1}{2} \end{pmatrix}$$

Studio il rango ($\rho = \dim \operatorname{Im} f$)

$$\det A = (1-h)(-1)\left(\frac{2h-1}{2}\right) - \left(\frac{2h-1}{2}\right)(-h)(h-1) =$$

$$= \frac{2h-1}{2}\left[-1+h+h^2+h\right] = \frac{2h-1}{2}(h^2+2h-1) \neq 0 \quad ****$$

Se è $\neq 0$ IL RANGO È 3 $\Rightarrow 2h-1 \neq 0 \quad h \neq \frac{1}{2}$

$h^2+2h-1 \neq 0 \quad h \neq -1 \pm \sqrt{1+1} = -1 \pm \sqrt{2}$

Se $\boxed{h \neq \frac{1}{2}, -1-\sqrt{2}, -1+\sqrt{2}}$ $\quad \rho = 3 \Rightarrow \dim \operatorname{Im} f = 3 \quad \operatorname{Im} f \subseteq W$

$\Rightarrow \dim \operatorname{Ker} f = \dim V - \dim \operatorname{Im} f = 3-3=0 \Rightarrow \operatorname{Ker} f = \{\underline{0}\} \Rightarrow$

$\Rightarrow f$ è iniettiva

Inoltre poiché $\overset{\rho}{\dim \operatorname{Im} f} = \dim \overset{(\mathbb{R}^3)}{W} = 3 \Rightarrow f$ è suriettiva

QUINDI f è ISOMORFISMO

BASE $\operatorname{Im} f = \{3 \text{ colonne}\} = \{C_1, C_2, C_3\} = \{(1-h, h-1, 0), (-h, -1, 0), (0, 0, \frac{2h-1}{2})\}$

PER IL NUCLEO $\nexists$ BASE DATO CHE $\dim \operatorname{Ker} f = 0$

EQUAZ. CARTESIANA DI $\operatorname{Im} f$ É BANALE

IN QUANTO $\operatorname{Im} f \equiv \mathbb{R}^3$ $\quad \boxed{\forall x, \forall y, \forall z}$

EQUAZ. CARTESIANA DI $\operatorname{Ker} f$ É $\boxed{x=0 \quad y=0 \quad z=0}$

IN QUANTO $\operatorname{Ker} f = \{\underline{0}\}$

$h \neq \frac{1}{2}, -1-\sqrt{2}, -1+\sqrt{2}$

LEVO LA CONDIZIONE : $\boxed{h = \frac{1}{2}}$

$$A = \begin{pmatrix} 1-h & -h & 0 \\ h-1 & -1 & 0 \\ 0 & 0 & \frac{2h-1}{2} \end{pmatrix} = \begin{pmatrix} 1/2 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho < 3$$

$\neq 0$

$\boxed{\rho = 2}$

$\Rightarrow \dim \operatorname{Im} f = 2 \rightarrow$ NON É SURIETTIVA ; BASE $\operatorname{Im} f = \{C_1, C_2\} =$

$= \{(1/2, -\frac{1}{2}, 0), (-\frac{1}{2}, -1, 0)\}$

$\dim \operatorname{Ker} f = 3 - 2 = 1$ $\quad A\underline{x} = \underline{0}$ $\quad (\operatorname{Ker} f = \{ v \in V \mid \underbrace{f(v)}_{} = \underline{0} \})$

$$\begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\underbrace{f(x,y,z)} = \underline{0}$

$A \cdot \underline{x} = \underline{0}$

$$\begin{cases} \frac{1}{2}x - \frac{1}{2}y = 0 \\ -\frac{1}{2}x - y = 0 \end{cases} \quad \begin{cases} x = y \\ \frac{1}{2}x + y = 0 \end{cases} \quad \begin{cases} x = 0 \\ \frac{1}{2}y + y = 0 \quad y + 2y = 0 \quad \cancel{3}y = 0 \quad y = 0 \end{cases} \quad \forall z$$

$\operatorname{Ker} f = \{(0,0,z)\}$ una incognita libera (z)

BASE $\operatorname{Ker} f = \{(0,0,1)\}$

Equazioni cartesiane di Imf :

$$\begin{pmatrix} 1° \text{ vett. base} \\ 2° \text{ vett. base} \\ --- \\ x \quad y \quad z \quad t \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & 0 \\ x & y & z \end{pmatrix}$$

3x3

dato che è quadrata si calcola il determinante e si pone uguale a zero

$\boxed{\text{DET} = 0}$

EQU. CART. DI Imf

$$-\frac{1}{2}z - z\left(\frac{1}{4}\right) = \frac{-2z - z}{4} = -\frac{\cancel{3}z}{\cancel{4}} = \frac{0}{3} \qquad \boxed{z = 0}$$

Equazioni cartesiane di Kerf :

sono quelle del sistema lin. omogeneo

$A\underline{x} = \underline{0}$

$\cancel{\frac{3}{2}} \cdot x) = 0 \quad x = 0$

Kerf

$$\begin{cases} x = 0 & \forall z \\ y + 2y = 0 \quad \cancel{3}y = 0 \quad y = 0 \end{cases}$$

$\operatorname{Ker} f = \{(x,y,z) \in \mathbb{R}^3 \mid \underline{x = 0 \quad y = 0}\}$

EQUAZ. CARTESIANE

LEVO LA 2° CONDIZIONE : $h = -1 - \sqrt{2}$ $\qquad -2 - 2\sqrt{2} - 1$

$$A = \begin{pmatrix} 1-h & -h & 0 \\ h-1 & -1 & 0 \\ 0 & 0 & \frac{2h-1}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 2+\sqrt{2} & 1+\sqrt{2} & 0 \\ -2-\sqrt{2} & -1 & 0 \\ 0 & 0 & \frac{-3-2\sqrt{2}}{2} \end{pmatrix}$$

det = 0

$\neq 0$

$\rho = 2$

$\dim \operatorname{Im} f = 2$ $\quad$ Base $\operatorname{Im} f = \{ C_2, C_3 \}$

2° e 3° colonna

Base $\operatorname{Im} f = \left\{ (1+\sqrt{2}, -1, 0), \left(0, 0, \frac{-3-2\sqrt{2}}{2}\right) \right\}$

Equaz. cartesiana di Imf
da completare a casa

25/1/21

$f: \mathbb{R}^3 \to \mathbb{R}^3 \qquad f(x,y,z) = (hy,\ (h+1)x + hy - z,\ -y - hz)$

$h \in \mathbb{R}$

1) Studiare Imf e Kerf e le equ. cartesiane.

Troviamo la matrice A

$f(e_1) = f(\underset{x}{1}, \underset{y}{0}, \underset{z}{0}) = (0,\ (h+1)\cdot 1 + \cancel{h\cdot 0} - \cancel{0},\ -0 - \cancel{h\cdot 0})$

$= (0,\ h+1,\ 0) = C_1$

$f(e_2) = f(\underset{x}{0}, \underset{y}{1}, \underset{z}{0}) = (h,\ h,\ -1) = C_2$

$f(e_3) = f(0,0,1) = (0,\ -1,\ -h) = C_3$

$$A = \begin{pmatrix} 0 & h & 0 \\ h+1 & h & -1 \\ 0 & -1 & -h \end{pmatrix}$$

CONDIZIONI

STUDIAMO IL RANGO $\quad \det A \neq 0$

$\det A = +h^2(h+1) \neq 0 \qquad h^2 \neq 0 \to h \neq 0$

$h+1 \neq 0 \qquad h \neq -1$

SE $h \neq 0, -1 \Rightarrow \rho = 3 \Rightarrow \dim \mathrm{Im} f = 3 = \mathbb{R}^3$

(SURIETTIVA) $\Rightarrow \dim \mathrm{Ker} f = 3 - 3 = 0 \qquad \mathrm{Ker} f = \{\underline{0}\}$

(INIETTIVA)

f ISOMORFISMO

Equ. cartesiane di Imf : $\forall x, \forall y, \forall z$

di Kerf : $x = y = z = 0$

LEVIAMO LA 1° CONDIZIONE : $\boxed{h = 0}$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \underset{\neq 0}{\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$\rho = 2 \to \dim \mathrm{Im} f = 2$

BASE $\mathrm{Im} f = \{C_1, C_2\} =$

$= \{(0,1,0), (0,0,-1)\}$

Equaz. cartesiane di Imf :

$$\underset{3\times 3}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ x & y & z \end{pmatrix}} \qquad \boxed{\det = 0}$$

$-x = 0 \Rightarrow \boxed{x = 0}$

$\dim \mathrm{Ker} f = 3 - 2 = 1 \qquad A\underline{X} = \underline{0} \qquad \begin{cases} 0 = 0 \\ x - z = 0 \\ -y = 0 \end{cases} \qquad \begin{cases} x = z \\ y = 0 \end{cases}$

$\mathrm{Ker} f = \{(z,0,z)\}$; BASE $\mathrm{Ker} f = \{(1,0,1)\}$

Equaz. cartesiane di $\mathrm{Ker} f$: $x = z \quad y = 0$

$\mathrm{Ker} f = \{(x,y,z) \in \mathbb{R}^3 \;/\; x - z = y = 0\}$

Leviamo la 2° condizione : $\boxed{h = -1}$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$\rho = 2 \quad \dim \mathrm{Im} f = 2$ Base $\mathrm{Im} f = \{C_2, C_3\}$

$\dim \mathrm{Ker} f = 1$

$A \underline{x} = \underline{0}$

$$\begin{cases} -y = 0 \\ -y - z = 0 \\ -y + z = 0 \end{cases} \quad \begin{cases} y = 0 \\ z = 0 \\ \forall x \end{cases}$$

$\mathrm{Ker} f = \{(x,0,0)\}$

BASE $\mathrm{Ker} f = \{(1,0,0)\}$

Equaz. cartesiane di $\mathrm{Ker} f$: $y = 0 \quad z = 0$

Equaz. cartesiane di $\mathrm{Im} f$ sono da trovare :

$$\begin{array}{l} C_2 \rightarrow \\ C_3 \rightarrow \\ \\ \end{array} \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 1 \\ x & y & z \end{pmatrix}$$

$z - x - x + y = 0$

$-2x + y + z = 0$

$\boxed{2x - y - z = 0}$

2) CALCOLARE LA CONTROIMMAGINE $f^{-1}(1,0,3)$

Risoluzione

$f^{-1}(1,0,3) = \{ v \in V \;/\; f(v) = (1,0,3) \}$

$f(x,y,z) = (1,0,3)$

$A \cdot \underline{x} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

SISTEMA LINEARE DOVE $B = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 0 & h & 0 \\ h+1 & h & -1 \\ 0 & -1 & -h \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{cases} hy = 1 \\ (h+1)x + hy - z = 0 \\ -y - hz = 3 \end{cases}$$

$y = \frac{1}{h}$ $\quad h \neq 0$

$$(A, B) = \left(\begin{array}{ccc|c} 0 & h & 0 & 1 \\ h+1 & h & -1 & 0 \\ 0 & -1 & -h & 3 \end{array}\right)$$

A quadrata

MATRICE CHE DETERMINA LA CONTROIMMAGINE DI $(1,0,3)$

Applico Cramer

SIST. DETERMINATO $\Longleftrightarrow$ DET A $\neq 0$ POI LA FORMULA

$$\begin{cases} x = \dfrac{\text{DET } B_1}{\text{DET } A} \\[2ex] y = \dfrac{\text{DET } B_2}{\text{DET } A} \\[2ex] z = \dfrac{\text{DET } B_3}{\text{DET } A} \end{cases}$$

$\text{DET } A = h^2(h+1) \neq 0$

$h \neq 0, -1$ IL SIST. É DETERMINATO

$$x = \frac{\begin{vmatrix} 1 & h & 0 \\ 0 & h & -1 \\ 3 & -1 & -h \end{vmatrix}}{h^2(h+1)} = \frac{-h^2-3h-1}{h^2(h+1)}$$

$$y = \frac{\begin{vmatrix} 0 & 1 & 0 \\ h+1 & 0 & -1 \\ 0 & 3 & -h \end{vmatrix}}{h^2(h+1)} = \frac{\cancel{h}\cancel{(h+1)}}{h^{\cancel{2}}\cancel{(h+1)}} = \frac{1}{h}$$

$$z = \frac{\begin{vmatrix} 0 & h & 1 \\ h+1 & h & 0 \\ 0 & -1 & 3 \end{vmatrix}}{h^2(h+1)} = \frac{-h-1-3h(h+1)}{h^2(h+1)} = \frac{-(h+1)-3h(h+1)}{h^2(h+1)} = \frac{\cancel{(h+1)}(-1-3h)}{h^2\cancel{(h+1)}}$$

Se $h \neq 0, -1$ $\quad f^{-1}(1,0,3) = \left\{ \left(\dfrac{-h^2-3h-1}{h^2(h+1)}, \dfrac{1}{h}, \dfrac{-1-3h}{h^2} \right) \right\}$

LEVO LA 1° CONDIZIONE

$$h=0 \qquad \begin{cases} hy = 1 \\ (h+1)x + hy - z = 0 \\ -y - hz = 3 \end{cases} \Rightarrow \begin{cases} 0 \cdot y = 1 \qquad 0=1 \qquad \emptyset \\ \\ \\ \end{cases}$$

$h=0 \qquad f^{-1}(1,0,3) = \emptyset$

LEVO LA 2° CONDIZIONE

$$h=-1 \qquad \begin{cases} hy = 1 \\ (h+1)x + hy - z = 0 \\ -y - hz = 3 \end{cases} \Rightarrow \begin{cases} -y = 1 \qquad \boxed{y=-1} \\ 0x - y - z = 0 \qquad y+z=0 \qquad \boxed{z=1} \\ +1-(-1)1 = 3 \qquad 2=3 \qquad \emptyset \end{cases}$$

$h=-1 \qquad f^{-1}(1,0,3) = \emptyset$

$$(A,B) = \left(\begin{array}{ccc|c} 0 & h & 0 & 1 \\ h+1 & h & -1 & 0 \\ 0 & -1 & -h & 3 \end{array}\right)$$

A quadrata

$h = 0$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 3 \end{array}\right) \qquad 0 = 1 \qquad \emptyset$$

$h = -1$:

$$(A,B) = \left(\begin{array}{ccc|c} 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 3 \end{array}\right)$$