14 dicembre 2022

mercoledì 14 dicembre 2022 08:03

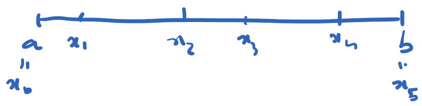
RIPASSO

$f: [a,b] \to \mathbb{R}$ cont.

$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$

$D = \{x_0; \dots ; x_n\}$

$|D| = \max\limits_{i=1,\dots,n} (x_i - x_{i-1})$



$\exists\, y_i \in [x_{i-1}, x_i]: \; f(y_i) = \min\limits_{[x_{i-1}, x_i]} f$

$\exists\, z_i \in$ " $\quad f(z_i) = \max\limits_{[\dots]} f$

$s(f, D) = \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}) \qquad \underline{S}$

$S(f, D) = \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1}) \qquad \overline{S}$

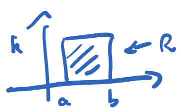
$s(f, D_1) \le S(f, D_2) \quad \forall D_1, D_2$

TEOR. $\underline{S}$ e $\overline{S}$ sono contigui

$\forall \varepsilon > 0 \quad \exists D: \quad S(f, D) - s(f, D) < \varepsilon$

$\sup \underline{S} = \int_a^b f(x)\,dx = \inf \overline{S}$

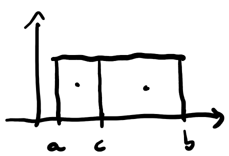
es. $f(x) = k > 0$



$\int_a^b k\,dx = k(b-a) = \text{area } R$

Proprietà dell'int. di Riemann

ADDITIVA se $c \in]a,b[$ $\quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



DISTRIBUTIVA $\quad \int_a^b (h f(x) + k g(x))\,dx = h \int_a^b f(x)\,dx + k \int_a^b g(x)\,dx$

DELLA MEDIA (I forma) $\quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$m = \min\limits_{[a,b]} f \qquad M = \max\limits_{[a,b]} f$

DIM $\quad D = \{a; b\} \qquad s(f, D) = m(b-a) \le \int_a^b f(x)\,dx \le$

[a,b]   [a,b]

DIM. $\quad D = \{a, b\} \quad s(f, D) = m(b-a) \leq \int_a^b f(x)\,dx \leq$
$\leq S(f, D) = M(b-a)$

II forma del teorema della media

$$\exists\ c \in [a,b] : \quad \int_a^b f(x)\,dx = f(c)(b-a)$$

DIM.

I forma $\Rightarrow$ II forma $\qquad m \leq \dfrac{\int_a^b f(x)dx}{b-a} \leq M \Rightarrow$ per il teor.

di esistenza dei valori intermedi $\exists\, c : f(c) = \dfrac{\int}{b-a} \Rightarrow$ TS.

II forma $\Rightarrow$ I forma perché $\quad m(b-a) \leq f(c)(b-a) \leq M(b-a)$

CONSEGUENZE:

I PROPR. DI MONOTONIA

$$f(x) \geq 0 \quad \forall x \in [a,b] \Rightarrow \int_a^b f(x)\,dx \geq 0$$

$= 0 \Leftrightarrow f(x) = 0 \ \forall x$

DIM. Basta osservare che $m \geq 0$. Non dim. che
$\int = 0 \Leftrightarrow f(x) = 0 \quad \forall x$

II PROPR. DI MONOTONIA

$f(x) \leq g(x) \ \forall x \in [a,b] \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

DIM. $\quad g(x) - f(x) \geq 0 \Rightarrow \int_a^b (g(x) - f(x))\,dx \geq 0$

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0 \Rightarrow \text{TS}$$

Ricordiamo che abbiamo esteso il concetto di int. definito

$f : (\alpha, \beta) \to \mathbb{R} \quad a, b \in (\alpha, \beta)$

$$\int_a^b f(x)\,dx = \begin{cases} \int_a^b f(x)\,dx \ \text{(di Riemann)} & \text{se } a < b \\ 0 & \text{se } a = b \\ -\int_b^a f(x)\,dx \quad \text{"} & \text{se } a > b \end{cases}$$

il teor. della media vale SOLO per l'int. di Riemann

la propr. distrib. vale per qualsiasi int. def.

" additiva " " "

PROPR. ADDITIVA!

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\forall\, a, b, c \in (\alpha, \beta)$$

es. c b a

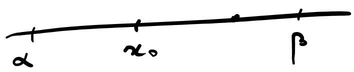
per le prop. add. dell'int. di Riemann in $[c, a]$

$$\int_c^a = \int_c^b + \int_b^a \Rightarrow -\int_b^a = -\int_c^a + \int_c^b \Rightarrow$$

$$\Rightarrow \int_a^b = \int_a^c + \int_c^b \Rightarrow \text{Ts.}$$

Funzione integrale

$f: (\alpha, \beta) \to \mathbb{R}$ continua $x_0 \in (\alpha, \beta)$



$\forall x \in (\alpha, \beta)$ $F(x) = \int_{x_0}^{x} f(t)\,dt$ $F: (\alpha, \beta) \to \mathbb{R}$ FUNZIONE INTEGRALE

$\int_x^{x_0} f(t)\,dt$ non è una funz. int. (è l'opposto di una funz. integrale)

$x_0, x_1 \in (\alpha, \beta)$

con $F(x) = \int_{x_0}^{x} f(t)\,dt$

$G(x) = \int_{x_1}^{x} f(t)\,dt$ sono due funz. integrali

Osserviamo $G(x) = \int_{x_1}^{x} \dots = \underbrace{\int_{x_1}^{x_0} f(t)\,dt}_{\text{è un numero}} + \underbrace{\int_{x_0}^{x} f(t)\,dt}_{= F(x)}$

$\Rightarrow$ due funz. integrali differiscono per una costante.
Sono allora primitive di f???

TEOREMA DI DERIVAZIONE DELLA FUNZ. INTEGRALE

IP $f: (\alpha, \beta) \to \mathbb{R}$ cont. $x_0 \in (\alpha, \beta)$

$F(x) = \int_{x_0}^{x} f(t)\,dt$

TS $F'(x) = f(x)$ $\forall x \in (\alpha, \beta)$

DIM. Sia $c \in (\alpha, \beta)$ dim. che $F'(c) = f(c)$

$$r(x) = \frac{F(x) - F(c)}{x - c} = \frac{\int_{x_0}^{x} f(t)\,dt - \int_{x_0}^{c} f(t)\,dt}{x - c} = \frac{\int_c^x f(t)\,dt}{x - c}$$

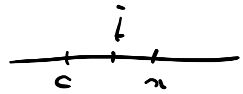
Se $x > c$ prop. della media $\Rightarrow \exists\, \bar{t} \in]c, x[: r(x) = f(\bar{t})$

Se $x < c$ $r(x) = \dfrac{-\int_x^c f(t)\,dt}{x - c} = \dfrac{\int_x^c f(t)\,dt}{c - x}$

prop. della media $\Rightarrow \exists\, \bar{t} \in]x, c[: r(x) = f(\bar{t})$

in ogni caso il rapporto increm. è uguale a $f(\bar{t})$ dove $\bar{t}$

è compreso fra c e x



Se $x \to c$ anche $\bar{t} \to c$. f è cont. $\Rightarrow$ $f(\bar{t}) \to f(c)$ $\Rightarrow$ TS.

TEOREMA FONDAMENTALE DEL CALCOLO INTEGRALE

Ogni funzione continua è dotata di primitive.

$$F(x) = \int_{x_0}^{x} f(t)\,dt \qquad F'(x) = f(x)$$

es. $$f(x) = \int_{1}^{x} \frac{\sin t}{t}\,dt \qquad f'(x) = \frac{\sin x}{x}$$

$$f(x) = \int_{x}^{1} \frac{\sin t}{t}\,dt = -\int_{1}^{x} \frac{\sin t}{t}\,dt \qquad f'(x) = -\frac{\sin x}{x}$$

$$f(x) = \int_{1}^{x^2+x} \frac{\sin t}{t}\,dt$$ è composta da $\int_{1}^{y} \frac{\sin t}{t}\,dt$

e da x^2+x

$$f'(x) = \frac{\sin(x^2+x)}{x^2+x}(2x+1)$$

$$f(x) = \int_{e^x}^{x^2+x} \frac{\sin t}{t}\,dt = \int_{e^x}^{1} \frac{\sin t}{t}\,dt + \int_{1}^{x^2+x} \frac{\sin t}{t}\,dt =$$

$$= -\int_{1}^{e^x} \frac{\sin t}{t}\,dt + \int_{1}^{x^2+x} \frac{\sin t}{t}\,dt$$

$$f'(x) = -\frac{\sin e^x}{e^x}e^x + \frac{\sin(x^2+x)}{x^2+x}(2x+1)$$

FORMULA FONDAMENTALE DEL CALCOLO INTEGRALE

IP $f: (\alpha, \beta) \to \mathbb{R}$ continua $a, b \in (\alpha, \beta)$ F prim. di f

TS $$\int_a^b f(x)\,dx = F(b) - F(a) = \left[F(x)\right]_a^b$$

Se G è un'altra prim. il risultato è lo stesso perché $G(x) = F(x) + k$

quindi $\left[G(x)\right]_a^b = F(b) + k - (F(a) + k) = F(b) - F(a)$

DIM. F è una prim. Un'altra prim. è $G(x) = \int_a^x f(t)\,dt$

Per il teor. sulle prim. $\exists\, k \in \mathbb{R}$: $F(x) = G(x) + k \quad \forall x \in (\alpha, \beta)$

in partic. per $x = a$ $\quad F(a) = G(a) + k = \int_a^a \cdots + k = k$

Per le teorie della prim...

in partic. per $x=a$ $\quad F(a) = G(a) + k = \int_a^a \cdots + k = k$

" per $x=b$ $\quad f(b) = G(b) + k = \int_a^b f(t)\,dx + f(a)$ c.v.d.

ESEMPI

$$\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3 = \frac{9}{2} - \frac{1}{2} = 4$$

$$\int_4^1 \log x\,dx = \left[x\log x - x\right]_4^1 = \qquad \int \log x\,dx = x(\log x - 1) + k$$

$$= -1 - 4\log 4 + 4$$

$$\int_0^2 |x^2-1|\,dx \qquad |x^2-1| = \begin{cases} x^2-1 & \text{in } [1,2] \\ 1-x^2 & \text{in } [0,1] \end{cases}$$

I modo trovare una prim in $[0,2]$

$$f(x) = \begin{cases} x - \frac{1}{3}x^3 + c_1 & \text{in } [0,1] \\ \frac{1}{3}x^3 - x + c_2 & \text{in }]1,2] \end{cases}$$

imponiamo la continuità $\quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$$\frac{2}{3} + c_1 = -\frac{2}{3} + c_2 \Rightarrow c_2 = c_1 + \frac{4}{3}$$

$$f(x) = \begin{cases} x - \frac{1}{3}x^3 + c_1 & \text{in } [0,1] \\ \frac{1}{3}x^3 - x + c_1 + \frac{4}{3} & \text{in }]1,2] \end{cases} \qquad \text{cons quella con } c_1 = 0$$

$$\int_0^2 |x^2-1|\,dx = [f(x)]_0^2 = \left[x - \frac{1}{3}x^3\right]_0^1 + \left[\frac{1}{3}x^3 - x + \frac{4}{3}\right]_1^2 =$$

$$= \cancel{1} - \frac{1}{3} + \frac{8}{3} - \cancel{2} + \cancel{\frac{4}{3}} - \frac{1}{3} + \cancel{1} - \cancel{\frac{4}{3}} = 2$$

II modo (consigliato) appl la prop add.

$$\int_0^2 |x^2-1|\,dx = \int_0^1 (1-x^2)\,dx + \int_1^2 (x^2-1)\,dx =$$

$$= \left[x - \frac{1}{3}x^3\right]_0^1 + \left[\frac{1}{3}x^3 - x\right]_1^2 =$$

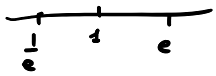
$$= \cancel{1} - \frac{1}{3} + \frac{8}{3} - \cancel{2} - \frac{1}{3} + \cancel{1} = 2$$

$$\int_0^{\frac{\pi}{2}} |2\sin x - 1|\,dx = \int_0^{\frac{\pi}{6}} (1 - 2\sin x)\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin x - 1)\,dx =$$

$$= \left[x + 2\cos x\right]_0^{\frac{\pi}{6}} + \left[-2\cos x - x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{6} + \sqrt{3} - 2 - \frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6}$$

$$I = \int_{\frac{1}{e}}^{e} x^2 |\log x| \, dx = \int_{\frac{1}{e}}^{1} -x^2 \log x \, dx + \int_{1}^{e} x^2 \log x \, dx$$



ERRORE $\int_1^e x^2 \log x = \left[\frac{x^3}{3} \log x - \int_1^e \frac{x^3}{3}\frac{1}{x} \, dx \right]_1^e$

Troviamo le primitive

$$\int \underset{\substack{\uparrow \\ \text{FD}}}{x^2} \log x \, dx = \frac{x^3}{3}\log x - \int \frac{x^3}{3}\frac{1}{x} \, dx = \frac{x^3}{3}\log x - \frac{x^3}{9} + k$$

$$I = \left[\frac{x^3}{9} - \frac{x^3}{3}\log x \right]_{\frac{1}{e}}^{1} + \left[\frac{x^3}{3}\log x - \frac{x^3}{9} \right]_1^e =$$

$$= \frac{1}{9} - \frac{1}{9e^3} - \frac{1}{3e^3} + \frac{e^3}{3} - \frac{e^3}{9} + \frac{1}{9}$$

Scrivere l'eq della tangente al grafico di $f(x) = \int_2^x e^t \, dt$ nel p. $x = 2$

$$\boxed{y = f(c) + f'(c)(x - c)}$$

$f(2) = 0$

calc. $f'(2)$

I modo $f(x) = \int_2^x e^t \, dt = \left[e^t \right]_2^x = e^x - e^2$

$f'(x) = e^x \qquad f'(2) = e^2 \qquad y = e^2(x-2)$

II modo $f'(x) = e^x \qquad f'(2) = e^2 \qquad$ ---

lo stesso eserc. con

$f(x) = \int_2^x e^{t^2} \, dt \qquad f(2) = 0$

$f'(x) = e^{x^2} \qquad f'(2) = e^4$

$y = e^4(x-2)$

eq. della tang. al gr. di $f(x) = \int_1^{x^2} \frac{\sin t}{t} \, dt \qquad x = 1$

$f(1) = 0$

$f'(x) = \frac{\sin x^2}{x^2} 2x = 2\frac{\sin x^2}{x}$

$f'(1) = 2 \sin 1$

eq. tang $y = 2(\sin 1)(x - 1)$

trovare f prim di $f(x) = e^{x^2}$ tale che $f(3) = 6$

$\int_3^x e^{t^2} \, dt + \cancel{k}\ 6$

. . . .

f prim. di $|x^2-4|$ tale che $f(3) = 1$

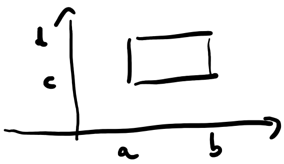
$$\int_3^x |t^2-4|\, dt + 1 = \dots\dots$$

^ si deve calc.

PER CASA   f prim di e^{-x^2} : $f(3)=2$

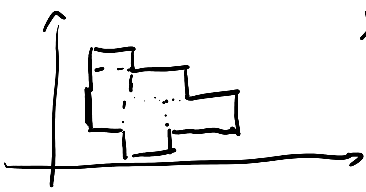
tang. al gr. di $f(x) = \int_1^{\frac{2x}{\pi}} \frac{\cos(\pi t)}{t^2}\, dt$ $x=1$

TEORIA DELLA MISURA SECONDO PEANO - JORDAN



$X = [a,b] \times [c,d]$

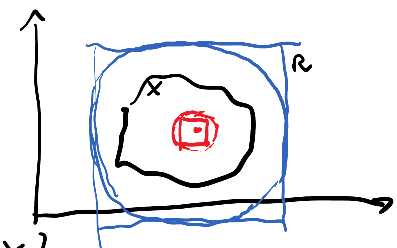
def. area $X = (b-a)(d-c)$



X plurirettangolo $X = X_1 \cup \dots \cup X_n$
$X_1, \dots, X_n$ rettang.

def area X = area $X_1 + \dots +$ area X_n

Studieremo la categoria di ins X limitati e dotati di punti interni



$\bar{A} = \{$aree dei plurirettangoli $R \supseteq X\}$

$\underline{A} = \{$ " " $R \subseteq X\}$

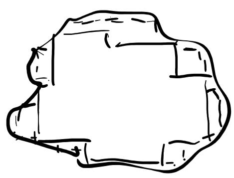
$\underline{A}$ e $\bar{A}$ sono separati

Se sono contigui si def.

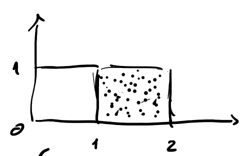
area $X = \sup \underline{A} = \inf \bar{A}$

e si dice che X è

MISURABILE



es. di ins. non misurabile



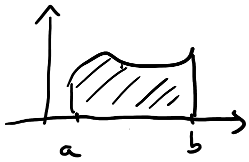
$$X = ([0,1]\times[0,1]) \cup \left(([1,2]\cap\mathbb{Q}) \times ([0,1]\cap\mathbb{Q})\right)$$

il più grande plurirettangolo $R \subseteq X$ è $R = [0,1]\times[0,1] \Rightarrow \sup \underline{A} = 1$

il più piccolo " " $R \supseteq X$ è $R = [0,2]\times[0,1] \Rightarrow \inf \bar{A} = 2$

il più preciso

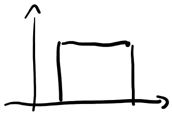
es. di ins. misurabile (interpretazione geometrica dell'integrale di Riemann)



$f: [a,b] \to]0,+\infty[$ continua

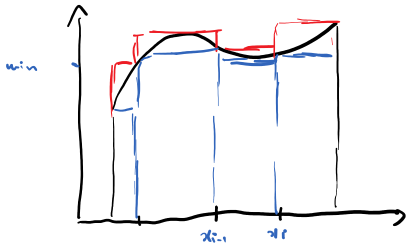
$R_f = \{(x,y) \in \mathbb{R}^2 : x \in [a,b],\ 0 \le y \le f(x)\}$

rettangoloide



TEOR. R_f è misurabile e

area $R_f = \int_a^b f(x)\,dx$



$D = \{x_0 = a; x_1; \dots; x_n = b\}$

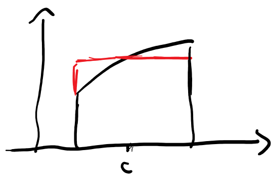
$s(f,D) = \sum_{i=1}^{n} \min_{[x_{i-1},x_i]} f\ (x_i - x_{i-1}) \in \underline{A}$

$S(f,D) \in \bar{A}$

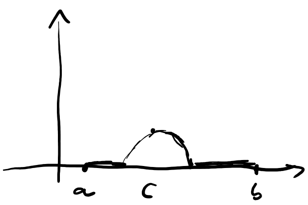
$s(f,D) \le \int_a^b f(x)\,dx \le S(f,D)$

($\underline{S}$ e $\bar{S}$ sono contigui) $\Rightarrow$ $\underline{A}$ e $\bar{A}$ sono contigui e l'elem. di sep. è $\int_a^b f(x)\,dx$

OSSERV. teor. della media $\Rightarrow$ $\exists\, c \in [a,b]$: $\int_a^b f(x)\,dx = f(c)(b-a)$

1. 

$f(c)$ = valore medio

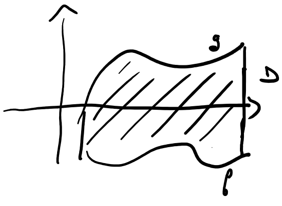
2. 

se $f \ge 0$

$\int_a^b = 0 \iff f$ ident. nulla

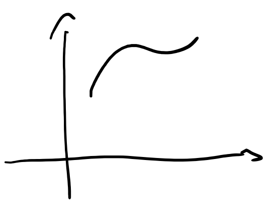
3. Se f, g : $f(x) \le g(x)$ si può dim. che $\int_a^b (g(x) - f(x))\,dx =$

$=$ area D



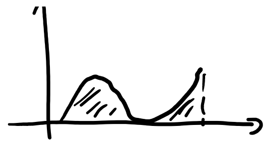
$D = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ f(x) \le y \le g(x)\}$

dominio normale risp. $\vec{x}$



Ne segue in partic. che se $f \ge 0$

area $R_f = \int_a^b f(x)\,dx$



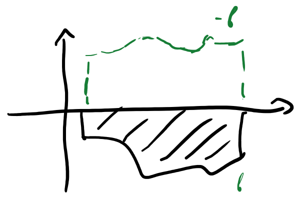
area $R_f = \int_a^b f(x)\,dx$

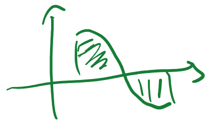
$\uparrow$

$\int_a^b (f - 0)$

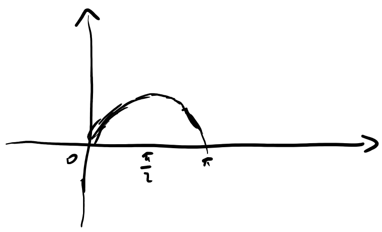
(4) Se $f(x) \le 0 \quad \forall x$

area $R_f = -\int_a^b f(x)\,dx$

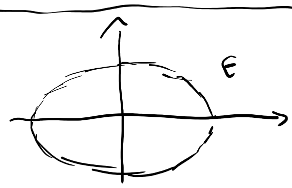




Eserc. Calc. l'area del rettang. rel alla funz $f(x) = \sin x$ e all'interv. $[0, \pi]$

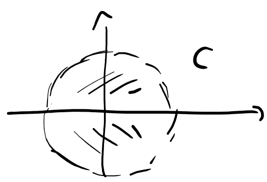


$$\text{area } R_f = \int_0^\pi \sin x \, dx = \Big[-\cos x\Big]_0^\pi = 1 + 1 = 2$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

area $E = \pi a b$



$$x^2 + y^2 = r^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

$$f(x) = \sqrt{r^2 - x^2}$$

$$g(x) = -\sqrt{r^2 - x^2}$$

area $C = \pi r^2$