29 NOVEMBRE 2022

martedì 29 novembre 2022 14:02

Ripasso degli int. di funz. raz.

(1) $I = \int \frac{3x+1}{x^2+x}\,dx$ $\qquad x^2+x = x(x+1)$ $\qquad x=0,\ x=-1$

$$\frac{3x+1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax+A+Bx}{x(x+1)} \qquad \begin{cases} A+B=3 \\ A=1 \end{cases} \qquad \begin{cases} A=1 \\ B=2 \end{cases}$$

$$I = \int \frac{dx}{x} + \int \frac{2\,dx}{x+1} = \log|x| + 2\log|x+1| + k$$

(2) $I = \int \frac{3x+1}{x^2-10x+25}\,dx$ $\qquad \frac{3x+1}{x^2-10x+25} = \frac{A}{x-5} + \frac{B}{(x-5)^2} = \frac{Ax-5A+B}{(x-5)^2}$

$$\begin{cases} A=3 \\ -5A+B=1 \end{cases} \qquad \begin{matrix} A=3 \\ B=16 \end{matrix} \qquad I = \int \frac{3}{x-5}dx + \int \frac{16}{(x-5)^2}dx =$$

$$= 3\log|x-5| - \frac{16}{x-5} + k$$

(3) $I = \int \frac{3x+1}{x^2+x+3}\,dx = 3\int \frac{x+\frac{1}{3}}{-\,-}\,dx = \frac{3}{2}\int \frac{2x+\frac{2}{3}}{x^2+x+3}\,dx =$

$$= \frac{3}{2}\int \frac{2x+1-\frac{1}{3}}{-\,-\,-}\,dx = \frac{3}{2}\int \frac{2x+1}{x^2+x+3}\,dx - \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{11}{4}} =$$

$$= \frac{3}{2}\log(x^2+x+3) - \frac{1}{2}\frac{2}{\sqrt{11}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{11}}{2}} + k \qquad \left(a=\frac{\sqrt{11}}{2}\right)$$

Ripasso dell'integrazione per razionalizzazione

$$D(e^x) = e^x$$
$$D(\operatorname{tg} x) = 1+\operatorname{tg}^2 x$$

$$I = \int \frac{\operatorname{tg} x}{\operatorname{tg}^2 x - \operatorname{tg} x - 2}\,dx = \int (1+\operatorname{tg}^2 x)\frac{\operatorname{tg} x}{(1+\operatorname{tg}^2 x)(\operatorname{tg}^2 x - \operatorname{tg} x - 2)}\,dx = \Big[\underbrace{\int \frac{t}{(1+t^2)(t^2-t-2)}\,dt}_{J}\Big]_{t=\operatorname{tg} x}$$

$t^2-t-2=0 \qquad t=\frac{1\pm 3}{2} \begin{cases} 2 \\ -1 \end{cases} \qquad t^2-t-2=(t+1)(t-2)$

$$\frac{t}{(1+t^2)(t^2-t-2)} = \frac{At+B}{t^2+1} + \frac{C}{t+1} + \frac{D}{t-2} = \frac{(At+B)(t^2-t-2)+C(t^2+1)(t-2)+D(t^2+1)(t+1)}{(t^2+1)(t+1)(t-2)}$$

$$At^3 - At^2 - 2At + Bt^2 - Bt - 2B + Ct^3 - 2Ct^2 + Ct - 2C + Dt^3 + Dt^2 + Dt + D = t$$

$$At^3 - At^2 - 2At + Bt^2 - Bt - 2B + Ct^2 - 2Ct + Ct - 2C + Dt \dots$$

$$\begin{cases} A + C + D = 0 \\ -A + B - 2C + D = 0 \\ -2A - B + C + D = 1 \\ -2B - 2C + D = 0 \end{cases} \qquad \begin{cases} A = -C - D \\ \cancel{C} + D - \cancel{C} + \frac{1}{2}D - 2C + D = 0 \\ 2C + 2D + C - \frac{1}{2}D + C + D = 1 \\ B = -C + \frac{1}{2}D \end{cases}$$

$$-2C + \frac{5}{2}D = 0$$

$$4C + \frac{5}{2}D = 1$$

$$\begin{cases} 4C - 5D = 0 \\ 8C + 5D = 2 \end{cases} \qquad 4C = 2 - 8C \Rightarrow C = \frac{1}{6}$$

$$D = \frac{2}{15}$$

$$A = -C - D = -\frac{1}{6} - \frac{2}{15} = -\frac{27}{30} = -\frac{9}{10}$$

$$B = -C + \frac{1}{2}D = -\frac{1}{6} + \frac{1}{15} = \frac{-3}{30} = -\frac{1}{10}$$

$$J = \int \frac{-\frac{9}{10}t - \frac{1}{10}}{t^2+1}\,dt + \int \frac{\frac{1}{6}}{t+1}\,dt + \int \frac{\frac{2}{15}}{t-2}\,dt =$$

$$= -\frac{9}{10}\int \frac{t + \frac{1}{9}}{t^2+1}\,dt + \frac{1}{6}\log|t+1| + \frac{2}{15}\log|t-2| =$$

$$= -\frac{9}{20}\int \frac{2t}{t^2+1}\,dt - \frac{1}{10}\int \frac{dt}{t^2+1} \quad - - - - - \quad =$$

$$= -\frac{9}{20}\log(t^2+1) - \frac{1}{10}\,\mathrm{arctg}\, t + \frac{1}{6}\log|t+1| + \frac{2}{15}\log|t-2| + k$$

$$I = -\frac{9}{20}\log(\mathrm{tg}^2 x + 1) - \frac{1}{10}x + \frac{1}{6}\log|\mathrm{tg}\, x + 1| + \frac{2}{15}\log|\mathrm{tg}\, x - 2| + k$$

$$\frac{1}{6} - \frac{9}{20} = \frac{10-27}{60}$$

trovare f; $f\left(\frac{\pi}{4}\right) = 0$

$$-\frac{9}{20}\log 2 - \frac{\pi}{40} + \frac{1}{6}\log 2 + k = 0 \Rightarrow k = \frac{17}{60}\log 2 + \frac{\pi}{40}$$

Trovare f prim. di $\dfrac{e^x + 1}{e^{3x} - e^{2x} + e^x - 1}$ tale che $f(0) = \frac{\pi}{4}$

$$I = \int \frac{e^x + 1}{e^{3x} - e^{2x} + e^x - 1}\,\frac{e^x}{e^x}\,dx = \Bigg[\underbrace{\int \frac{t+1}{t(t-1)(t^2+1)}\,dt}_{J}\Bigg]_{t = e^x}$$

$$e^{2x}(e^x - 1) + e^x - 1$$

$$\frac{t+1}{t(t-1)(t^2+1)} = \frac{A}{t} + \frac{B}{t-1} + \frac{Ct + D}{t^2+1} = \frac{(At - A)(t^2+1) + Bt(t^2+1) + (Ct + D)(t^2 - t)}{t(t-1)(t^2+1)}$$

$$At^3 + At - At^2 - A + Bt^3 + Bt + Ct^3 - Ct^2 + Dt^2 - Dt = t + 1$$

$$\begin{cases} A + B + C = 0 \\ -A - C + D = 0 \\ A + B - D = 1 \\ -A = 1 \end{cases} \qquad \begin{cases} A = -1 \\ B + C = 1 \\ C - D = 1 \\ B - D = 2 \end{cases}$$

$$\begin{cases} A = -1 \\ B = 1 - C \\ C - D = 1 \\ C + D = -1 \end{cases} \qquad \begin{cases} C = D + 1 \\ D + 1 + D = -1 \end{cases} \qquad \begin{cases} A = -1 \\ B = 1 \\ C = 0 \\ D = -1 \end{cases}$$

$$J = -\int \frac{1}{t}\,dt + \int \frac{1}{t-1}\,dt - \int \frac{1}{t^2+1}\,dt = -\log|t| + \log|t-1| - \operatorname{arctg} t + k$$

$$I = -x + \log|e^x - 1| - \operatorname{arctg} e^x + k$$

$$f(1) = \frac{\pi}{4} \qquad -1 + \log(e-1) - \operatorname{arctg} e + k = \frac{\pi}{4} \Rightarrow k = \frac{\pi}{4} + 1 - \log(e-1) + \operatorname{arctg} e$$

Funz. raz. fratte non proprie

- $$\int \frac{x+3}{x+5}\,dx = \int \frac{x+5-2}{x+5}\,dx = \int \frac{x+5}{x+5}\,dx - 2\int \frac{dx}{x+5} =$$
$$= x - 2\log|x+5| + k$$

- $$\int \frac{x^2 - x + 6}{x+1}\,dx = \int \frac{x^2 - 1 + 1 - x + 6}{x+1}\,dx =$$
$$= \int \frac{x^2-1}{x+1}\,dx - \int \frac{x-7}{x+1}\,dx = \int (x-1)\,dx - \int \frac{x+1-8}{x+1}\,dx = \ldots$$

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(1) Det. le primitive della funzione $f(x) = \begin{cases} x & \text{se } x \geq 0 \\ x^2 & \text{se } x < 0 \end{cases}$ in $]-\infty, +\infty[$

una primitiva sarà del tipo $\begin{cases} \frac{x^2}{2} + k & x \geq 0 \\ \frac{x^3}{3} + c & x < 0 \end{cases}$ f deve essere cont. in $\mathbb{R}$ in partic. per $x = 0$

$\lim f(x) = \lim f(x)$ $\quad (k = c) \Rightarrow f(x) = \begin{cases} \frac{x^2}{2} + k & x \geq 0 \end{cases}$

$\lfloor^{3}$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \qquad k = c \qquad \Rightarrow f(x) = \begin{cases} \frac{x^2}{2} + k & x \geq 0 \\ \frac{x^3}{3} + k & x < 0 \end{cases}$$

(2) Det. f prim. di $|x-3|$ in $[-1, 4]$ e tale che $f(0) = 2$

$$\rho(x) = \begin{cases} 3 - x & \text{in } [-1, 3[\\ x - 3 & \text{in } [3, 4] \end{cases}$$

$$f(x) = \begin{cases} 3x - \frac{1}{2}x^2 + k & \text{in } [-1, 3[\\ \frac{1}{2}x^2 - 3x + c & \text{in } [3, 4] \end{cases}$$

imponiamo la continuità

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\frac{9}{2} + k = -\frac{9}{2} + c \Rightarrow c = k + 9$$

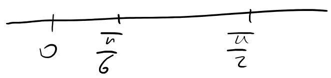
tutte le prim. sono

$$f(x) = \begin{cases} 3x - \frac{1}{2}x^2 + k & \text{in } [-1, 3[\\ \frac{1}{2}x^2 - 3x + k + 9 & \text{in } [3, 4] \end{cases} \longleftarrow \quad f(0) = 2 \iff k = 2$$

La funz. richiesta è $f(x) = \begin{cases} 3x - \frac{1}{2}x^2 + 2 & \text{in } [-1, 3[\\ \frac{1}{2}x^2 - 3x + 11 & \text{in } [3, 4] \end{cases}$

(3) Det. f prim. di $\underbrace{\left|\sin x - \frac{1}{2}\right|}_{\rho(x)}$ in $[0, \frac{\pi}{2}]$ e tale che $f(\frac{\pi}{4}) = \frac{\pi}{24}$

$$\rho(x) = \begin{cases} \frac{1}{2} - \sin x & \text{in } [0, \frac{\pi}{6}] \\ \sin x - \frac{1}{2} & \text{in }]\frac{\pi}{6}, \frac{\pi}{2}] \end{cases}$$



$$f(x) = \begin{cases} \frac{x}{2} + \cos x + k & \text{in } [0, \frac{\pi}{6}] \\ -\cos x - \frac{x}{2} + c & \text{in }]\frac{\pi}{6}, \frac{\pi}{2}] \end{cases}$$

imponiamo la continuità

$$\lim_{x\to (\frac{\pi}{6})^-} f(x) = \lim_{x\to (\frac{\pi}{6})^+} f(x)$$

$$\frac{\pi}{12} + \frac{\sqrt{3}}{2} + k = -\frac{\sqrt{3}}{2} - \frac{\pi}{12} + c \Rightarrow c = k + \frac{\pi}{6} + \sqrt{3}$$

$$f(x) = \begin{cases} \frac{x}{2} + \cos x + k & \text{in } [0, \frac{\pi}{6}] \\ -\cos x - \frac{x}{2} + k + \frac{\pi}{6} + \sqrt{3} & \text{in }]\frac{\pi}{6}, \frac{\pi}{2}] \end{cases} \longleftarrow$$

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{24}$$

$$-\frac{\sqrt{2}}{2} - \frac{\pi}{8} + k + \frac{\pi}{6} + \sqrt{3} = \frac{\pi}{24}$$

$$\bar{k} = \frac{\pi}{24} + \frac{\pi}{8} - \frac{\pi}{6} + \frac{\sqrt{2}}{2} - \sqrt{3}$$

La funz. richiesta è

$$f(x) = \begin{cases} \frac{x}{2} + \cos x + \bar{k} & \text{in } [0, \frac{\pi}{6}] \\ \ldots \end{cases}$$

$$f(x) = \begin{cases} \frac{x}{2} + \cos x + \bar{k} & \text{in } [0, \frac{\pi}{6}] \\ -\cos x - \frac{x}{2} + \bar{k} + \frac{\pi}{6} + \sqrt{3} & \text{in }]\frac{\pi}{6}, \frac{\pi}{2}] \end{cases}$$

Integrazione per sostituzione (seconda formula)

IP $\quad f: (a,b) \to \mathbb{R}$ dotata di primitive

$g: (c,d) \to (a,b)$ su tutto, derivabile, invertibile

TS $\quad \int f(x)\,dx = \left[\int f(g(t))\,g'(t)\,dt\right]_{t=g^{-1}(x)}$

DIM. $\quad$ I formula $\Rightarrow \quad \int f(g(t))\,g'(t)\,dt = \left[\int f(x)\,dx\right]_{x=g(t)}$

componiamo ambo i membri con $t = g^{-1}(x)$

$$\left[\int f(g(t))\,g'(t)\,dt\right]_{t=g^{-1}(x)} = \left[\int f(x)\,dx\right]_{x=g(g^{-1}(x))=x} = \int f(x)\,dx$$

$$\int \frac{e^x}{e^x+1}\,dx = \left[\int \frac{dt}{t+1}\right]_{t=e^x} \quad \text{si fa così}$$

II formula: $\quad x = \log t = g(t)$
$\quad g'(t) = \frac{1}{t}$

$$\Rightarrow \left[\int \frac{e^{\log t}}{e^{\log t}+1}\,\frac{1}{t}\,dt\right]_{t=e^x} =$$

$$= \left[\int \frac{\cancel{t}}{t+1}\,\frac{1}{\cancel{t}}\,dt\right]_{t=e^x}$$

NON VA BENE!

Occorre aggiungere: $\quad (a,b) = ?$
$(c,d) = ?$
g è su tutto, deriv, invert ?

$(a,b) =]-\infty, +\infty[\qquad (c,d) =]0, +\infty[\qquad g'(t) = \frac{1}{t} > 0 \Rightarrow$ è invert.

$g(t) = \log t :]0, +\infty[\to]-\infty, +\infty[$ su tutto

Esempi di applicazione della II formula

① $I = \int \frac{2x}{\sqrt{x-1}}\,dx \qquad (a,b) =]1, +\infty[$

$\sqrt{x-1} = t \qquad t > 0$

$x - 1 = t^2 \Rightarrow x = t^2 + 1 = g(t)$

$t>0 \Rightarrow t^2+1 \in \left]1,+\infty\right[\Rightarrow g$ è su tutto

$(c,d) = \left]0,+\infty\right[$

$g'(t) = 2t > 0 \Rightarrow g$ invert. $\quad g^{-1}(x) = \sqrt{x-1}$

$$I = \left[\int \frac{t^2+1}{\cancel{t}}\, 2\cancel{t}\, dt\right]_{t=\sqrt{x-1}} = \left[\frac{2}{3}t^3 + 2t + k\right]_{t=\sqrt{x-1}} = \frac{2}{3}(x-1)^{3/2} + 2\sqrt{x-1} + k$$

(2) $I = \displaystyle\int \frac{x+\sqrt{x}}{x+1}\, dx \qquad (a,b) = \left[0,+\infty\right[$

$\sqrt{x} = t \quad , t \geq 0 \qquad x = t^2 = g(t) \qquad (c,d) = \left[0,+\infty\right[$

g è su tutto

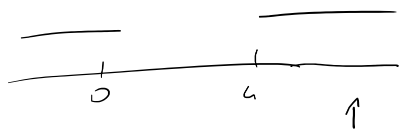
$g'(t) = 2t \Rightarrow g$ è invert. $\quad g^{-1}(x) = \sqrt{x}$

$$I = 2\Bigg[\underbrace{\int \frac{t^3+t^2}{t^2+1}\, dt}_{J}\Bigg]_{t=\sqrt{x}}$$

$$\frac{t^3+t^2}{t^2+1} = \frac{t^3+t-t+t^2}{t^2+1} = \frac{t(t^2+1)}{t^2+1} + \frac{t^2+1-1-t}{t^2+1} = t+1-\frac{t+1}{t^2+1}$$

$$J = \int\left(t+1 - \frac{1}{2}\frac{2t}{t^2+1} - \frac{1}{t^2+1}\right) dt = \frac{t^2}{2} + t - \frac{1}{2}\log(t^2+1) - \operatorname{arctg} t + k$$

$$I = \frac{x}{2} + \sqrt{x} - \frac{1}{2}\log(x+1) - \operatorname{arctg}\sqrt{x} + k$$



(3) $\displaystyle\int \sqrt{\frac{x-4}{x}}\, dx$

cerchiamo le primitive in $\left[4,+\infty\right[= (a,b)$

$\sqrt{\frac{x-4}{x}} = t \qquad t \geq 0 \qquad \frac{x-4}{x} = t^2 \Rightarrow x-4 = t^2 x \Rightarrow (1-t^2)x = 4$

$x = \frac{4}{1-t^2} = g(t)$

cerchiamo (c,d)

$\frac{4}{1-t^2} \geq 4 \qquad \frac{4}{1-t^2} - 4 \geq 0 \qquad \frac{4-4+4t^2}{1-t^2} \geq 0 \qquad -1<t<1$

$(c,d) = \left[0,1\right[$

Se avessimo scelto $\left]-\infty,0\right[$ avremmo dovuto verificare $g(t)<0$

$\frac{4}{1-t^2} < 0 \qquad t<-1 \ \vee\ t>1 \Rightarrow (c,d) = \left]1,+\infty\right[$

$g'(t) = \dfrac{8t}{(1-t^2)^2}$

$$I = \left[\int t\, \frac{8t}{(1-t^2)^2}\, dt\right]_{t=\sqrt{\frac{x-4}{x}}} = 8\Bigg[\underbrace{\int \frac{t^2}{(t^2-1)^2}\, dt}_{J}\Bigg]_{t=\sqrt{-}}$$

$$g'(t) = \frac{8t}{(1-t^2)^2} \qquad I = \left[\int t \, \frac{\ldots}{(1-t^2)^2}\, dt\right]_{t=\sqrt{\frac{x-4}{x}}} = 8\left[\underbrace{\int \frac{\ldots}{(t^2-1)^2}}_{J}\right]_{t=\sqrt{\ldots}}$$

$$\frac{t^2}{(t^2-1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2} = \frac{A(t-1)(t+1)^2 + B(t+1)^2 + C(t+1)(t-1)^2 + D(t-1)^2}{(t^2-1)^2}$$

$$At^3 + 2At^2 + At - At^2 - 2At - A + Bt^2 + 2Bt + B + Ct^3 - 2Ct^2 + Ct + Ct^2 - 2Ct + C + Dt^2 - 2Dt + D = t^2$$

$$\begin{cases} A + C = 0 \\ A + B - C + D = 1 \\ -A + 2B - C - 2D = 0 \\ -A + B + C + D = 0 \end{cases} \qquad \begin{cases} A = -C \\ B - 2C + D = 1 \\ B = D \\ B + 2C + D = 0 \end{cases} \qquad \begin{cases} A = -C \\ B = D \\ 2B - 2C = 1 \\ 2B + 2C = 0 \end{cases}$$

$$\begin{cases} A = -C \\ B = D \\ C = -B \\ B = \frac{1}{4} \end{cases} \qquad \begin{cases} A = \frac{1}{4} \\ D = \frac{1}{4} \\ C = -\frac{1}{4} \\ B = \frac{1}{4} \end{cases}$$

$$J = \frac{1}{4}\int \frac{dt}{t-1} + \frac{1}{4}\int \frac{dt}{(t-1)^2} - \frac{1}{4}\int \frac{dt}{t+1} + \frac{1}{4}\int \frac{dt}{(t+1)^2} =$$

$$= \frac{1}{4}\left(\log(1-t) - \frac{1}{t-1} - \log(t+1) - \frac{1}{t+1}\right) + k = \frac{1}{4}\left(\log \frac{1-t}{1+t} - \frac{2t}{t^2-1}\right) + k$$

$$I = \frac{1}{4}\left(\log \frac{1-\sqrt{\frac{x-4}{x}}}{1+\sqrt{\frac{x-4}{x}}} - \frac{2\sqrt{\frac{x-4}{x}}}{\frac{x-4}{x} - 1}\right) + k$$