10 ottobre 2022

lunedì 10 ottobre 2022 08:36

RIPASSO

$f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ $P_0\ (x_0, y_0) \in D(X)$

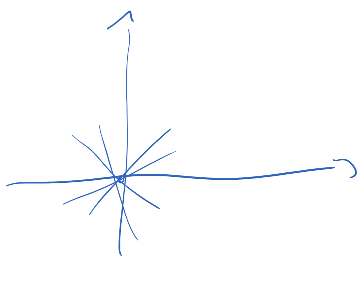
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \ (1) \quad \text{se}$$
$+\infty$
$-\infty \ (2)$

(1) $\forall\, \varepsilon > 0 \ \exists\, \delta > 0 :$ se $(x,y) \in X$, $(x,y) \neq (x_0,y_0)$, $(x,y) \in I_\delta(P_0)$
si ha $|f(x,y) - \ell| < \varepsilon$

(2) $\forall\, k > 0 \ \exists\, \delta > 0 :$. – – . – .
si ha $f(x,y) > k$
$< -k$

$f(\underbrace{g(x,y)}) \to \ell$ $f(t)$
$\downarrow$ $g(x,y)$
t_0
$\downarrow$
ℓ

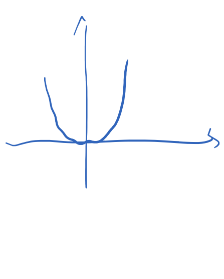
$g = f_{|Y}$



$E_m = \{ (x,y) \in \mathbb{R}^2 : y = mx \}$
$m \in \mathbb{R}$

$g_m(x) = f(x, mx)$ se $\lim_{x\to 0} g_m(x)$ dipende da m, f non è regol.

se " non dipende da m, allora
il limite POTREBBE essere quello



$0 \leq |f(x,y)| \leq$ . . . . $\Rightarrow \lim = 0$
$\downarrow$

$$0 \le |f(x,y)| \le \underset{\substack{\downarrow \\ 0}}{\dots\dots} \Rightarrow \lim = 0$$

f continua in (x_0, y_0) se $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

OPERAZIONI E COMPOSIZIONI DI FUNZ. CONT. SONO CONT.

T. Weierstrass f cont. in X chiuso e limitato $\Rightarrow$ f ha min e max assoluti

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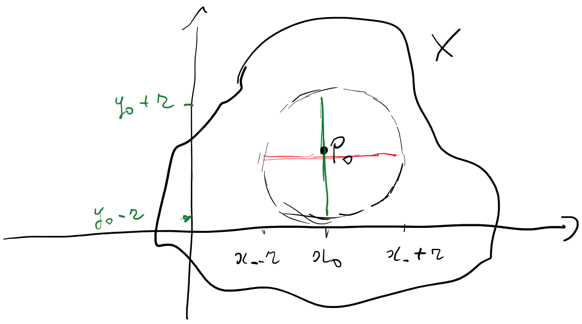
Derivate parziali

$f: X \to \mathbb{R}$ $\quad X \subseteq \mathbb{R}^2$ APERTO $\quad P_0(x_0, y_0) \in X \Rightarrow$

$\Rightarrow \exists\, r > 0 : B(P_0, r) \subseteq X$



se $x_0 - r < x < x_0 + r$

$P(x, y_0) \in B(P_0, r) \Rightarrow \in X$

infatti $d(P, P_0) = \sqrt{(x-x_0)^2 + (y_0-y_0)^2} = |x - x_0| < r$

cons. allora $g(x) = f(x, y_0)$ $\quad \forall x \in \,]x_0 - r, x_0 + r[$

se $\exists\, g'(x_0)$ si dice che f ammette derivata parziale rispetto alla x nel punto P_0

$$f_x(x_0, y_0) = \left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} = g'(x_0)$$

es. $f(x,y) = 2x^3y^2 - 2xy + 3x - 2y$ $\qquad f_x(1, 2)$

$g(x) = 8x^3 - 4x + 3x - 4$

$g'(x) = 24x^2 - 1$ $\qquad f_x(1,2) = 23$

$$f(x,y) = 6xy^2 - 3x + 4y^2 + x^3 y$$

$$f_x(x,y) = 6y^2 - 3 + 3x^2 y$$

Se $y \in]y_0 - r, y_0 + r[$ $\quad P(x_0, y) \in B(P_0, r) \Rightarrow P \in X$

cons. $h(y) = f(x_0, y) \quad \forall y \in]y_0 - r, y_0 + r[$

se $\exists\, h'(y_0)$ si dice che f è dotata di derivata parziale rispetto ad y nel punto P_0

$$f_y(x_0, y_0) = \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} = h'(y_0)$$

es. $f(x,y) = 2x^2y^2 - 3xy^2 + x^2 y - 3x^4 + y$

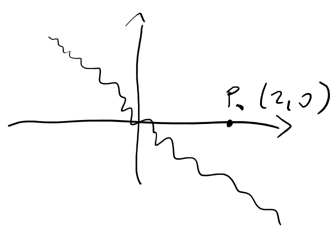
$$f_y(x,y) = 4x^2 y - 6xy + x^2 + 1$$

Osserviamo che $f_x(x_0, y_0) = \lim_{x \to x_0} \dfrac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$

$$f_y(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

se $\exists\, f_x(x_0, y_0)$ e $f_y(x_0, y_0)$ si cons. il vettore

$(\nabla f)(x_0, y_0) = (f_x(x_0, y_0),\ f_y(x_0, y_0))$ $\quad$ GRADIENTE DI f IN P_0



es. $f(x,y) = \dfrac{2x^2 y - 3x + y^2}{x + y}$

$$(\nabla f)(2,0) = \left(0, \frac{11}{2}\right)$$

$$f_x(x,y) = \frac{(4xy - 3)(x+y) - (2x^2 y - 3x + y^2)}{}$$

$$f_x(x,y) = \frac{(4xy-3)(x+y) - (2x^2y - 3x + y^2)}{(x+y)^2}$$

$$f_y(x,y) = \frac{(2x^2+2y)(x+y) - (2x^2y - 3x + y^2)}{(x+y)^2}$$

se $\exists\ f_x(x,y)\ \forall (x,y) \in X$ si dice che f è derivabile parzialm. rispetto a x in X e si considera la nuova funzione $(x,y) \in X \to f_x(x,y)$. Le sue derivate saranno le derivate seconde di f

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} \qquad f_{yx}(x,y) = \frac{\partial^2 f}{\partial y \partial x}$$

e analogamente

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}$$

f_{xx}, f_{yy} der. seconde PURE

f_{xy}, f_{yx} " MISTE

es. 1 $\quad f(x,y) = 2x^2y - 3xy^3$

$f_x(x,y) = 4xy - 3y^3 \qquad f_y(x,y) = 2x^2 - 9xy^2$

$f_{xy}(x,y) = 4x - 9y^2 \quad = \quad f_{yx}(x,y) = 4x - 9y^2$

es. 2 $\quad f_{x,y} = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$(x,y) \neq (0,0)$

$$f_x(x,y) = y\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{\cancel{2x^3} + 2xy^2 - \cancel{2x^3} + 2xy^2}{(x^2+y^2)^2} = y\,\frac{x^4 - y^4 + 4x^2y^2}{(x^2+y^2)^2}$$

$\exists\, f_x(0,0)$? $g(x) = f(x,0) = 0 \quad \forall x \Rightarrow f_x(0,0) = 0$

~~f_{xy}~~

~~$\exists f$~~

$\exists\, f_{xy}(0,0)$? $h(y) = f_x(0,y) = -y \Rightarrow \boxed{f_{xy}(0,0) = -1}$

$$f_y(x,y) = x\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{-2x^2y - \cancel{2y^3} - 2x^2y + \cancel{2y^3}}{(x^2+y^2)^2} = x\,\frac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2}$$

$\exists\, f_y(0,0)$? $h(y) = 0 \quad \forall y \Rightarrow f_y(0,0) = 0$

$\exists\, f_{yx}(0,0)$? $g(x) = f_y(x,0) = x \Rightarrow \boxed{f_{yx}(0,0) = 1}$

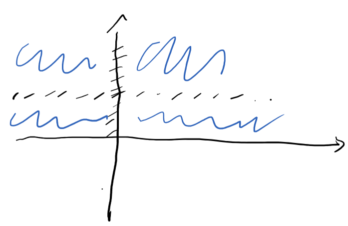
$-1 \neq 1$

LEMMA DI SCHWARZ

Sia f una funz. dotata in X (aperto di $\mathbb{R}^2$) di derivate prime e seconde. Se f_{xy} e f_{yx} sono continue in un punto $P_0 \in X$ allora $f_{xy}(P_0) = f_{yx}(P_0)$

ESERCIZI

1) $f(x,y) = \dfrac{2x^2y}{x\log y}$ $\nabla f(1,e)$ $\nabla f(2,e^2)$



$$f_x(x,y) = \frac{4x^2y\log y - 2x^2y\log y}{x^2\log^2 y} = \frac{2y}{\log y}$$

$$f_y(x,y) = \frac{2x^3\log y - 2x^2y\,\frac{x}{y}}{x^2\log^2 y} = \frac{2x^3\log y - 2x^3}{x^2\log^2 y} = \frac{2x\log y - 2x}{\log^2 y}$$

$\nabla f(1,e) = (2e, 0)$ $\nabla f(2,e^2) = (e^2, 1)$

$\nabla f(1,e) = (2e, 0)$ $\qquad$ $\nabla f(2, e^2) = (e^2, 1)$

2) $f(x,y) = 3x^2y^2 - \frac{2}{y} + 12x$ $\qquad$ $\nabla f(1,2)$ $\qquad$ $\nabla f(0,1)$

$\nabla f(x,y) = \left(6xy^2 + 12,\ 6x^2y + \frac{2}{y^2}\right)$

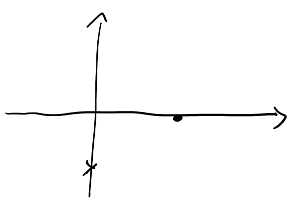
$\nabla f(1,2) = \left(36, \frac{25}{2}\right)$ $\qquad$ $\nabla f(0,1) = (12, 2)$

3) $f(x,y) = \sqrt{x^2+y^2}$

$\cancel{(x,y)}$ $x \neq 0$, $y \neq 0$ $\qquad$ $f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ $\qquad$ $f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$

$f_x(0,0)$? $\qquad$ $g(x) = f(x,0) = \sqrt{x^2} = |x|$ $\qquad$ $\nexists\ f_x(0,0)$

$f_x(a,0)$? $\qquad$ $g(x) = f(x,0) = |x|$ $\qquad$ $f_x(a,0) = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$

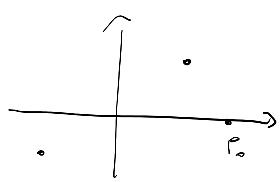


$f_y(a,0)$? $\qquad$ $h(y) = f(a,y) = \sqrt{a^2+y^2}$

$f_y(a,y) = \frac{y}{\sqrt{a^2+y^2}}$

$f_y(0,0)$? $\qquad$ $h(y) = f(0,y) = |y|$ $\qquad$ $\nexists\ f_y(0,0)$

4) $f(x,y) = |xy|$



$xy > 0$

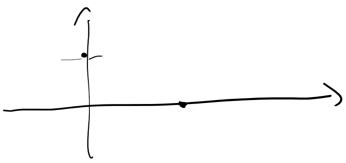
$f(x,y) = xy$ $\qquad$ $f_x(x,y) = y$ $\qquad$ $f_y(x,y) = x$

$xy < 0$

$f(x,y) = -xy$ $\qquad$ " $-y$ $\qquad$ " $-x$

$P_0(a,0)$ $\quad \exists f_x(a,0)$? $\quad g(x) = f(x,0) = 0 \ \forall x \Rightarrow f_x(a,0) = 0 \ \forall x$

$\exists f_y(a,0)$? $\quad h(y) = f(a,y) = |a|\,|y| \Rightarrow f_y(a,0) = \begin{cases} |a| \\ -|a| \end{cases}$



$a > 0$

$h(y) = a|y| = \begin{cases} ay & y > 0 \\ -ay & y < 0 \end{cases}$ $\quad \nexists f_y(a,y)$

$a < 0$ $\quad$ "

$\exists f_y(0,0)$? $\quad h(y) = f(0,y) = 0 \ \forall y \Rightarrow f_y(0,0) = 0$

$P_0(0,b)$ $\quad \exists f_x(0,b)$? $\quad g(x) = f(x,b) = |x|\,|b|$

$b > 0$ $\quad g(x) = b|x| = \begin{cases} bx & x > 0 \\ -bx & x < 0 \end{cases}$ $\quad \nexists f_x(0,b)$

$b < 0$ $\quad \nexists f_x(0,b)$

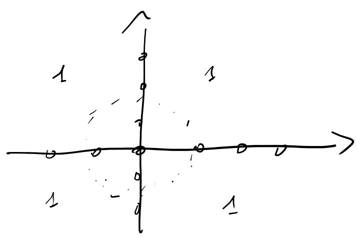
$$\boxed{f_x(0,0) = f_y(0,0) = 0}$$

CONT $\not\Rightarrow$ DER

DER $\not\Rightarrow$ CONT $\quad$ opp. DER $\Rightarrow$ CONT ?

vera

es.



$f(x,y) = \begin{cases} 0 & xy = 0 \\ 1 & xy \neq 0 \end{cases}$

in ogni intorno di $(0,0)$ ci sono infiniti punti in cui vale 1
e " " " 0

quindi non è continua in $(0,0)$

$\exists f_x(0,0)$? $\quad g(x) = f(x,0) = 0 \ \forall x \Rightarrow f_x(0,0) = 0$

analog. $f_y(0,0) = 0$