28 novembre 2022

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$$f(x) = \frac{a(x)}{b(x)}$$

$gr \cdot a > gr \cdot b$

$\exists\, q, r : gr\, r < gr \cdot b \qquad a(x) = b(x)\, q(x) + r(x)$

$$f(x) = \underset{\substack{\uparrow \\ \text{polinomio}}}{q(x)} + \frac{r(x)}{b(x)} \leftarrow \text{propria}$$

FRATTI SEMPLICI

$$\underset{\substack{\uparrow \\ \text{li facciamo}}}{\frac{A}{(x-c)^m}} \qquad \underset{\substack{\uparrow \\ \text{li facciamo per } m = 1, 2}}{\frac{Bx + C}{\left[(x-p)^2 + q^2\right]^m}}$$

es. $b(x) = 0 \Rightarrow b(x) = x^2 (x-2) \left[(x-3)^2 + 4\right] \left[x^2 + 1\right]^3$

i fratti semplici saranno

$$\frac{A_1}{x} \quad \frac{A_2}{x^2} \quad \frac{A_3}{x-2} \quad \frac{B_1 x + C_1}{(x-3)^2 + 4} \quad \frac{B_2 x + C_2}{x^2+1} \quad \frac{B_3 x + C_3}{(x^2+1)^2} \quad \frac{B_4 x + C_4}{(x^2+1)^3}$$

$b(x) = (x-1)(x-2)^2 \qquad \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2} = \frac{- - -}{(x-1)(x-2)^2}$

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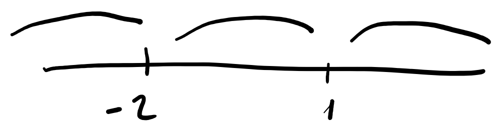
es. $\displaystyle\int \frac{3x+4}{x^2+x-2}\, dx$



$x^2 + x - 2 = 0 \qquad \frac{-1 \pm 3}{2} \begin{smallmatrix} \nearrow -2 \\ \searrow 1 \end{smallmatrix}$

$$\underbrace{\frac{3x+4}{x^2+x-2}} = \frac{A_1}{x+2} + \frac{A_2}{x-1} = \underbrace{\frac{A_1 x - A_1 + A_2 x + 2A_2}{(x+2)(x-1)}}$$

$(A_1 + A_2)x - A_1 + 2A_2 = 3x + 4 \qquad \begin{cases} A_1 + A_2 = 3 \\ -A_1 + 2A_2 = 4 \end{cases}$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = 3 \qquad A_1 = \frac{\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}}{3} = \frac{2}{3} \qquad A_2 = \frac{\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}}{3} = \frac{7}{3}$$

$$\int \frac{3x+4}{x^2+x-2}\,dx = \frac{2}{3}\int\frac{dx}{x+2} + \frac{7}{3}\int\frac{dx}{x-1} = \frac{2}{3}\log|x+2| + \frac{7}{3}\log|x-1| + k$$

INTEGRAZ. DEI FRATTI SEMPLICI

$$I = \int \frac{A}{(x-c)^m}\,dx = A\int\frac{dx}{(x-c)^m}$$

$m=1$ $\qquad I = A\log|x-c| + k$

$m>1$ $\qquad I = \int (x-c)^{-m}\,dx = \frac{1}{-m+1}(x-c)^{-m+1} + k$

es. $m=2$ $\qquad \int\frac{dx}{(x-4)^2} = -\frac{1}{x-4} + k$

$$I = \int\frac{Bx+C}{(x-p)^2+q^2}\,dx = B\underbrace{\int\frac{x}{\cdots}\,dx}_{(1)} + C\underbrace{\int\frac{dx}{\cdots}}_{(2)}$$

$(m=1)$

se $p \neq 0$

(1) $$\int\frac{x}{(x-p)^2+q^2}\,dx = \int\frac{(x-p)+p}{(x-p)^2+q^2}\,dx = \int\frac{x-p}{(x-p)^2+q^2}\,dx + p\underbrace{\int\frac{dx}{(x-p)^2+q^2}}_{(2)}$$

$$\int\frac{x-p}{(x-p)^2+q^2}\,dx = \left[\int\frac{t}{t^2+q^2}\,dt\right]_{t=x-p} \qquad \frac{t}{t^2+q^2} \quad x-p$$

quindi possiamo sempre supporre $p=0$

(1) $$\int\frac{x}{x^2+q^2}\,dx = \frac{1}{2}\int\frac{2x}{x^2+q^2}\,dx = \frac{1}{2}\log(x^2+q^2) + k$$

(2) anche qui, eventualmente integrando per sostit., possiamo supporre $p=0$

$$\int\frac{dx}{x^2+q^2} = \int\frac{1}{q^2}\frac{dx}{\left(\frac{x}{q}\right)^2+1} = \frac{1}{q}\int\underbrace{\frac{1}{q}}_{D\left(\frac{x}{q}\right)}\frac{dx}{\left(\frac{x}{q}\right)^2+1} = \qquad \frac{1}{t^2+1} \quad \frac{x}{q}$$

$$= \frac{1}{q}\operatorname{arctg}\frac{x}{q} + k$$

$$\int\frac{dx}{x^2+4} = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + k \qquad \int\frac{dx}{x^2+7} = \frac{1}{\sqrt7}\operatorname{arctg}\frac{x}{\sqrt7} + k$$

in definitiva

$$\int \frac{Bx+C}{(x-p)^2+q^2}\,dx = \text{un logaritmo} + \text{un arctg}$$

es. $$\int \frac{x-3}{(x+2)^2+9}\,dx = \frac{1}{2}\int \frac{2x-6}{---}\,dx = \frac{1}{2}\int \frac{2x+4-10}{---}\,dx =$$

$D\left[(x+2)^2+9\right] = 2x+4$

$$= \frac{1}{2}\int \frac{2x+4}{(x+2)^2+9}\,dx - 5\int \frac{dx}{(x+2)^2+9} =$$

$$= \frac{1}{2}\log\left((x+2)^2+9\right) - \frac{5}{3}\,\text{arctg}\,\frac{x+2}{3} + k$$

$\underline{m=2}$ $$\int \frac{Bx+C}{\left[(x-p)^2+q^2\right]^2}\,dx = B\underbrace{\int \frac{x}{\left[(x-p)^2+q^2\right]^2}\,dx}_{(1)} + C\underbrace{\int \frac{dx}{\left[(x-p)^2+q^2\right]^2}}_{(2)}$$

$$(1) = \frac{1}{2}\int \frac{2x}{---}\,dx = \frac{1}{2}\int \frac{2x-2p+2p}{---}\,dx =$$

$$= \frac{1}{2}\int \frac{2x-2p}{\left[(x-p)^2+q^2\right]^2}\,dx + p\underbrace{\int \frac{dx}{\left[(x-p)^2+q^2\right]^2}}_{(2)}$$

$$\int \frac{2x-2p}{\left[(x-p)^2+q^2\right]^2}\,dx = 2\left[\int \frac{t}{\left[t^2+q^2\right]^2}\,dt\right]_{t=x-p}$$

$$\int \frac{t}{(t^2+q^2)^2}\,dt = \frac{1}{2}\int \frac{2t}{(t^2+q^2)^2}\,dt = \frac{1}{2}\left[\int \frac{dy}{y^2}\right]_{y=t^2+q^2} =$$

$$= -\frac{1}{2}\,\frac{1}{(x-p)^2+q^2} + k$$ (aggiustare le costanti)

(2) $$\int \frac{dx}{\left((x-p)^2+q^2\right)^2} = \left[\int \frac{dt}{(t^2+q^2)^2}\right]_{t=x-p}$$ l'abbiamo già fatta per parti

CASO DI MAGGIOR INTERESSE

$$f(x) = \frac{ax+b}{x^2+px+q}$$

x^2+px+q $\quad \Delta = p^2-4q$

$\Delta > 0$ $\quad x^2+px+q = (x-c_1)(x-c_2)$ $\quad c_1, c_2 \in \mathbb{R}$

$x^2+px+q \qquad \Delta = p^2-4q$

I caso $\Delta>0 \qquad x^2+px+q=(x-c_1)(x-c_2)$ $\qquad c_1, c_2 \in \mathbb{R}$, $c_1 \neq c_2$

$$\frac{ax+b}{x^2+px+q} = \frac{A}{x-c_1} + \frac{B}{x-c_2} = \frac{Ax-Ac_2+Bx-Bc_1}{(x-c_1)(x-c_2)}$$

$$\begin{cases} A+B=a \\ -c_2A-c_1B=b \end{cases} \qquad \begin{vmatrix} 1 & 1 \\ -c_2 & -c_1 \end{vmatrix} = c_2-c_1 \neq 0 \Rightarrow$$ il sist. ha una unica sol.

es. $I=\int \frac{3x+5}{x^2+6x+8}\,dx \qquad \frac{\Delta}{4}=1 \qquad -3\pm1 \begin{cases} -2 \\ -4 \end{cases}$

$$\frac{3x+5}{x^2+6x+8} = \frac{A}{x+2} + \frac{B}{x+4} = \frac{(A+B)x+4A+2B}{(x+2)(x+4)} \qquad \begin{cases} A+B=3 \\ 4A+2B=5 \end{cases}$$

$$A = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{-2} = -\frac{1}{2} \qquad B = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix}}{-2} = \frac{7}{2}$$

$$I = -\frac{1}{2}\int\frac{dx}{x+2} + \frac{7}{2}\int\frac{dx}{x+4} = -\frac{1}{2}\log|x+2| + \frac{7}{2}\log|x+4| + k$$

II caso $\Delta=0 \qquad x^2+px+q=(x-c)^2$

$$\frac{ax+b}{x^2+px+q} = \frac{A}{x-c} + \frac{B}{(x-c)^2} = \frac{Ax-cA+B}{(x-c)^2} \qquad \begin{cases} A=a \\ -cA+B=b \end{cases} \qquad \begin{cases} A=a \\ B=b+ca \end{cases}$$

es. $I=\int\frac{4x+1}{x^2-6x+9}\,dx =$

$$x^2-6x+9=(x-3)^2 \qquad \frac{4x+1}{x^2-6x+9} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2}$$

$$\begin{cases} A=4 \\ -3A+B=1 \end{cases} \Rightarrow \begin{cases} A=4 \\ B=13 \end{cases}$$

$$I = 4\int\frac{1}{x-3}\,dx + 13\int\frac{dx}{(x-3)^2} = 4\log|x-3| - \frac{13}{x-3} + k$$

III caso $\Delta<0 \qquad x^2+px+q=0$ ha due sol. $b\pm ic$ immag. coniug.

$$x^2+px+q = [x-(b+ic)][x-(b-ic)] = [(x-b)-ic][(x-b)+ic] =$$
$$= (x-b)^2+c^2$$

per trovare b e c si usa il metodo del completamento dei quadrati

$x^2+2x+6 = (x+1)^2+5$

$\int \frac{2x+1}{x^2+2x+6}\,dx =$ I passo: al num. la deriv. del denom.

$$= \int \frac{2x+2-1}{x^2+2x+6}\,dx = \int \frac{2x+2}{x^2+2x+6}\,dx - \int \frac{dx}{x^2+2x+6} =$$

$$= \log(x^2+2x+6) - \int \frac{dx}{(x+1)^2+5} = \qquad (q=\sqrt{5})$$

$$= \log(x^2+2x+6) - \frac{1}{\sqrt{5}} \operatorname{arctg} \frac{x+1}{\sqrt{5}} + c$$

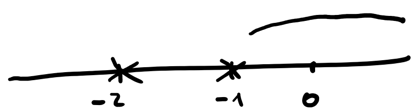
Esempio

$I = \int \frac{x-6}{x^2+3x+2}\,dx$ $\qquad x^2+3x+2=0$ per $x = \frac{-3\pm1}{2} \begin{cases} -2 \\ -1 \end{cases}$

$$\frac{x-6}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{(A+B)x+2A+B}{(x+1)(x+2)} \qquad \begin{cases} A+B=1 \\ 2A+B=-6 \end{cases}$$

$$A = -\begin{bmatrix} 1 & 1 \\ -6 & 1 \end{bmatrix} = -7 \qquad B = -\begin{bmatrix} 1 & 1 \\ 2 & -6 \end{bmatrix} = 8$$

$I = -7\log|x+1| + 8\log|x+2| + c$



trovare una prim. f di $\frac{x-6}{x^2+3x+2}$ su tale che $f(0)=4$

$f(x) = -7\log(x+1) + 8\log(x+2) + c$

$f(0)=4 \qquad -7\log 1 + 8\log 2 + c = 4 \quad \Rightarrow \quad c = 4 - 8\log 2$

$f(x) = -7\log(x+1) + 8\log(x+2) + 4 - 8\log 2 \qquad x \in]-1,+\infty[$

Trovare f prim di $\frac{2x-3}{x^2+4x+4}$ tale che $f(1)=1$

$x^2+4x+4 = (x+2)^2 \qquad \frac{2x-3}{x^2+4x+4} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{Ax+2A+B}{(x+2)^2}$

$$\begin{cases} A=2 \\ 2A+B=-3 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=-7 \end{cases}$$

$$\int \frac{2x-3}{x^2+4x+4}\,dx = 2\int \frac{dx}{x+2} - 7\int \frac{dx}{(x+2)^2} = 2\log|x+2| + \frac{7}{x+2} + c$$

$$\int \frac{\cdots}{x^2+4x+4}\,dx = 2\int \frac{\cdots}{x+2} \cdots + \int \frac{\cdots}{(x+2)^2} \cdots \qquad x+2$$

$-2 \qquad 1$

in $]-2,+\infty[$

$$f(x) = 2\log(x+2) + \frac{7}{x+2} + k$$

$$f(1) = 1 \qquad 2\log 3 + \frac{7}{3} + k = 1 \quad \Rightarrow \quad k = -\frac{4}{3} - 2\log 3$$

$$f(x) = 2\log(x+2) + \frac{7}{x+2} - \frac{4}{3} - 2\log 3$$

Se fosse stato $f(-4) = 0$ $\qquad$ in $]-\infty,-2[$ $\quad f(x) = 2\log(-x-2) + \frac{7}{x+2} + k$

$$f(-4) = 0 \qquad 2\log 2 - \frac{7}{2} + k = 0 \Rightarrow k = \frac{7}{2} - 2\log 2$$

$$f(x) = 2\log(-x-2) + \frac{7}{x+2} + \frac{7}{2} - 2\log 2$$

in $]-\infty,-2[$

Trovare f prim di $\frac{x}{x^2+3x+4}$ tale che $f(0) = 2$

$$\Delta = -7 < 0$$

$$I = \int \frac{x}{x^2+3x+4}\,dx = \frac{1}{2}\int \frac{2x}{x^2+3x+4}\,dx = \frac{1}{2}\int \frac{2x+3-3}{x^2+3x+4}\,dx =$$

$$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+4}\,dx - \frac{3}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 + \frac{7}{4}} = \qquad \left(q = \frac{\sqrt{7}}{2}\right)$$

$$= \frac{1}{2}\log(x^2+3x+4) - \frac{3}{2}\frac{2}{\sqrt{7}}\operatorname{arctg}\frac{x+\frac{3}{2}}{\frac{\sqrt{7}}{2}} + C$$

$$f(0) = 2 \qquad \underset{\substack{\parallel \\ \log 2}}{\frac{1}{2}\log 4} - \frac{3}{\sqrt{7}}\operatorname{arctg}\frac{3}{2}\frac{2}{\sqrt{7}} + k = 2 \Rightarrow k = 2 - \log 2 + \frac{3}{\sqrt{7}}\operatorname{arctg}\frac{3}{\sqrt{7}}$$

$$f(x) = \frac{1}{2}\log(x^2+3x+4) - \frac{3}{\sqrt{7}}\operatorname{arctg}\frac{2}{\sqrt{7}}\left(x+\frac{3}{2}\right) + 2 - \log 2 + \frac{3}{\sqrt{7}}\operatorname{arctg}\frac{3}{\sqrt{7}}$$

Integrazione per razionalizzazione

$$I = \int \frac{2e^x+1}{e^x+3}\,dx = \int \underset{\substack{\uparrow \\ D(e^x)}}{e^x}\,\frac{2e^x+1}{e^x(e^x+3)}\,dx = \Big[\underbrace{\int \frac{2t+1}{t(t+3)}\,dt}_{J}\Big]_{t=e^x}$$

$$J \qquad \frac{2t+1}{t(t+3)} = \frac{A}{t} + \frac{B}{t+3} = \frac{(A+B)t+3A}{t(t+3)} \qquad \begin{cases} A+B = 2 \\ 3A = 1 \end{cases} \qquad \begin{cases} A = \frac{1}{3} \\ B = \frac{5}{3} \end{cases}$$

$$J = \frac{1}{3}\log|t| + \frac{5}{3}\log|t+3| + k$$

$$I = \frac{1}{3}\,x + \frac{5}{3}\log(e^{x}+3) + k$$

$$D(\mathrm{tg}\,x) = 1+\mathrm{tg}^2 x$$

$$I = \int \frac{\mathrm{tg}\,x}{\mathrm{tg}^2 x - \mathrm{tg}\,x + 2}\,dx =$$

$$= \int \underbrace{(1+\mathrm{tg}^2 x)}_{D(\mathrm{tg}\,x)} \frac{\mathrm{tg}\,x}{(1+\mathrm{tg}^2 x)(\mathrm{tg}^2 x - \mathrm{tg}\,x + 2)}\,dx = \Big[\underbrace{\int \frac{t}{(1+t^2)(t^2-t+2)}\,dt}_{J}\Big]_{t=\mathrm{tg}\,x}$$

$$\frac{t}{(t^2+1)(t^2-t+2)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2-t+2} = \frac{\qquad}{(t^2+1)(t^2-t+2)}$$