12 DICEMBRE 2022

lunedì 12 dicembre 2022 09:00

Esercizi sugli int. indef.

(1) $I = \int \frac{3}{(x+2)^2} \log(x-4)\, dx =$ $\quad D\left(\frac{1}{x+2}\right) = \frac{-1}{(x+2)^2}$

$-3 \int \frac{-1}{(x+2)^2} \log(x-4)\, dx = -3 \frac{1}{x+2} \log(x-4) + 3 \underbrace{\int \frac{1}{(x+2)(x-4)} dx}_{J}$

FD

$\frac{1}{(x+2)(x-4)} = \frac{A}{x+2} + \frac{B}{x-4}$ da completare per eserc.

(2) $I = \int \frac{\log x + 3}{x(\log^3 x + \log^2 x)} dx =$ $\quad \frac{1}{x} = D(\log x)$

$= \left[\underbrace{\int \frac{t+3}{t^3+t^2} dt}_{J}\right]_{t = \log x}$

$\frac{t+3}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1} = \frac{At(t+1) + B(t+1) + C\, t^2}{t^2(t+1)} = \frac{At^2 + At + Bt + B + Ct^2}{t^2(t+1)}$

$\begin{cases} A + C = 0 \\ A + B = 1 \\ B = 3 \end{cases} \qquad \begin{cases} C = 2 \\ A = -2 \\ B = 3 \end{cases}$

$J = -2 \int \frac{dt}{t} + 3 \int \frac{dt}{t^2} + 2 \int \frac{dt}{t+1} = -2 \log|t| - 3\frac{1}{t} + 2 \log|t+1| + k$

$I = -2 \log|\log x| - \frac{3}{\log x} + 2 \log|\log x + 1| + k$

(3) $I = \int \frac{\operatorname{tg} x}{\operatorname{tg}^2 x - \operatorname{tg} x - 2} dx =$ RAZIONALIZZ

$= \int \frac{\operatorname{tg} x}{(\operatorname{tg}^2 x - \operatorname{tg} x - 2)(1 + \operatorname{tg}^2 x)} (1 + \operatorname{tg}^2 x)\, dx = \left[\underbrace{\int \frac{t}{(t^2 - t - 2)(1 + t^2)} dt}_{J}\right]_{t = \operatorname{tg} x}$

$\frac{1 \pm 3}{2} \begin{matrix} 2 \\ -1 \end{matrix}$ $\qquad \frac{t}{(t-2)(t+1)(t^2+1)} = \frac{A}{t-2} + \frac{B}{t+1} + \frac{Ct + D}{t^2+1}$ da compl

④ $$\int \frac{dx}{e^{2x}+2e^{x}-8} = \int \frac{e^{x}}{e^{x}(e^{2x}+2e^{x}-8)}\,dx = \left[\int \underbrace{\frac{dt}{t(t^2+2t-8)}}_{J}\right]_{t=e^{x}}$$

$$\frac{-1\mp 3}{2} \begin{matrix} \nearrow 1 \\ \searrow -2 \end{matrix} \qquad \frac{1}{t(t^2+2t-8)} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{t+2}$$ da compl

⑤ $$\int \frac{x^2+5}{x-1}\,dx = \int \frac{x^2-1+6}{x-1}\,dx =$$

$x^2-1=(x-1)(x+1)$

$$= \int \frac{x^2-1}{x-1}\,dx + \int \frac{6}{x-1}\,dx = \int (x+1)\,dx + 6\int \frac{dx}{x-1} =$$

$$= \frac{x^2}{2} + x + 6\log|x-1| + k$$

⑥ $$I = \int \frac{x^2+2x+7}{x-3}\,dx = \int \frac{x^2-9+2x+16}{x-3}\,dx =$$

$$= \int \frac{x^2-9}{x-3}\,dx + 2\int \frac{x+8}{x-3}\,dx = \int (x+3)\,dx + 2\int \frac{x-3+11}{x-3}\,dx =$$

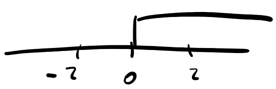
$$= \underbrace{\frac{x^2}{2}+3x+2x}\, + 22\log|x-3| + k$$

OPPURE:

$$\begin{array}{r|l} x^2+2x+7 & x-3 \\ \cline{2-2} -x^2+3x & x+5 \\ \hline 5x+7 & \\ -5x+15 & \\ \hline 22 & \end{array} \qquad \frac{x^2+2x+7}{x-3} = x+5+\frac{22}{x-3}$$

⑦ $f(x) = |x^2-4| - 2x + 3$ $\quad F$ prim. in $[0,+\infty[$ e tale che $F(1)=0$

$$f(x) = \begin{cases} 4-x^2-2x+3 = 7-2x-x^2 & \text{in } [0,2] \\ x^2-4-2x+3 = x^2-2x-1 & \text{in } [2,+\infty[\end{cases}$$



f è continua

una prim ~~...~~ sarà del tipo $F(x) = \begin{cases} 7x - x^2 - \frac{1}{3}x^3 + c & \text{in } [0,2] \\ \frac{1}{3}x^3 - x^2 - x + k & \text{in }]2,+\infty[\end{cases}$

imponiamo la continuità $\lim_{x\to 2^-} F(x) = \lim_{x\to 2^+} F(x)$

$$14 - \cancel{4} - \frac{8}{3} + c = \frac{8}{3} - \cancel{4} - 2 + k$$

$$16 - \frac{16}{3} + c = k \qquad k = c + \frac{32}{3}$$

le prim. sono $F(x) = \begin{cases} 7x - x^2 - \frac{1}{3}x^3 + c & \text{in } [0,2] \leftarrow \\ \frac{1}{3}x^3 - x^2 - x + \underbrace{\left(c+\frac{32}{3}\right)}_{k} & \text{in } [2,+\infty[\end{cases}$

imponiamo che $f(1)=0$ $\qquad 7-1-\frac{1}{3}+c=0 \Rightarrow c=-\frac{17}{3}$

la funz. cercata è $\qquad f(x)=\begin{cases} 7x-x^2-\frac{1}{3}x^3-\frac{17}{3} & \text{in } [0,2] \\ \frac{1}{3}x^3-x^2-x+5 & \text{in } [2,+\infty[\end{cases}$

(8) det. f prim in $]-\infty,+\infty[$

$$f(x)=\begin{cases} \cos x & x<0 \\ x^2+1 & x\geq 0 \end{cases}$$

e tale che $f\left(-\frac{\pi}{2}\right)=3$

$\lim_{x\to 0^-} f(x)=1$

$\lim_{x\to 0^+} f(x)=1$ $\qquad \Rightarrow f$ è cont.

una prim. sarà del tipo $\qquad f(x)=\begin{cases} \sin x+c & x<0 \\ \frac{x^3}{3}+x+k & x\geq 0 \end{cases}$

imponiamo la continuità $\qquad \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$

$c=k$

$$f(x)=\begin{cases} \sin x+c & x\leq 0 \quad \leftarrow \\ \frac{x^3}{3}+x+c & x\geq 0 \end{cases}$$

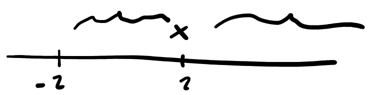
imponiamo che $f\left(-\frac{\pi}{2}\right)=3 \Rightarrow \qquad -1+c=3 \Rightarrow c=4$

la funz cercata è $\qquad f(x)=\begin{cases} \sin x+4 & x\leq 0 \\ \frac{x^3}{3}+x+4 & x\geq 0 \end{cases}$

(9) det. f prim di $f(x)=\frac{\sqrt{x+2}}{x-2}$ nel suo ins. di def.
e tale che $f(7)=6$

$x\geq -2$

$x\neq 2$



$(a,b)=]2,+\infty[$

$\sqrt{x+2}=t \qquad t\geq 0$

$x+2=t^2$

$x=t^2-2=g(t) \qquad t^2-2>2 \qquad t^2>4 \qquad t>2 \lor t<-2$

$(c,d)=]2,+\infty[\qquad g'(t)=2t>0 \ \forall t\in]2,+\infty[\Rightarrow g$ invert.

$g^{-1}(x)=\sqrt{x+2}$

$$J=\int \frac{\sqrt{x+2}}{x-2}\,dx=\left[\underbrace{\int \frac{t}{t^2-2}\,2t\,dt}_{J}\right]_{t=\sqrt{x+2}}$$

$$J=\int \frac{2t^2}{t^2-2}\,dt = 2\int \frac{t^2-2+2}{t^2-2}\,dt = 2\int\left(1+\frac{2}{t^2-2}\right)dt =$$

$$=2t+4\underbrace{\int\frac{dt}{t^2-2}}_{J'}$$

$t^2-2=(t-\sqrt{2})(t+\sqrt{2})$

$\frac{1}{\ldots}=\frac{A}{\ldots}+\frac{B}{t+\sqrt{2}}=\frac{At+\sqrt{2}A+Bt-\sqrt{2}B}{\ldots}$

J' $\qquad$ $t^2-2=(t-\sqrt{2})(t+\sqrt{2})$

$$\frac{1}{t^2-2}=\frac{A}{t-\sqrt{2}}+\frac{B}{t+\sqrt{2}}=\frac{A\,t+\sqrt{2}A+Bt-\sqrt{2}B}{(t-\sqrt{2})(t+\sqrt{2})}$$

$$\begin{cases}A+B=0\\ \sqrt{2}A-\sqrt{2}B=1\end{cases}\qquad \begin{cases}B=-A\\ 2\sqrt{2}A=1\end{cases}\qquad \begin{matrix}B=-A\\ A=\frac{1}{2\sqrt{2}}\end{matrix}$$

$$\frac{4}{2\sqrt{2}}$$

$$J'=\frac{1}{2\sqrt{2}}\log|t-\sqrt{2}|-\frac{1}{2\sqrt{2}}\log|t+\sqrt{2}|+c$$

$$J=2t+\sqrt{2}\log|t-\sqrt{2}|-\sqrt{2}\log|t+\sqrt{2}|+c$$

$$I=2\sqrt{x+2}+\sqrt{2}\log\left|\frac{\sqrt{x+2}-\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right|+c$$

imponiamo che $f(7)=6$

$$6+\sqrt{2}\log\frac{3-\sqrt{2}}{3+\sqrt{2}}+c=6\qquad c=-\sqrt{2}\log\frac{3-\sqrt{2}}{3+\sqrt{2}}$$

$$f(x)=2\sqrt{x+2}+\sqrt{2}\log|\ldots|-\sqrt{2}\log\frac{3-\sqrt{2}}{3+\sqrt{2}}$$

"per casa" $\qquad \int x^2\cosh x\,dx$

$$\int\frac{dx}{tg^2x}$$

$$\int(x-2)\log\frac{x+3}{x}\,dx$$

Integrale definito secondo Riemann

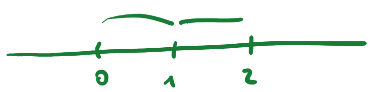
PREMESSA

$A, B\subseteq\mathbb{R}$ separati se $a\le b\quad\forall a\in A,\ \forall b\in B$

es. $\quad A=[0,1]\qquad B=[4,5]$

$\qquad A=[0,1]\qquad B=[1,2]$

A, B separati $\iff \sup A\le\inf B$



se $\sup A=\inf B$ $\quad A$ e B CONTIGUI

elem. di separazione = tutti i numeri x : $\sup A\le x\le\inf B$

A, B contigui $\iff$ hanno un unico elem. di separaz.

$A=[0,1]\qquad B=[1,2]$

$A=[0,1[\qquad B=[1,2]$

$A=[0,1[\qquad B=]1,2]$

A, B contigui $\iff$

$\forall\varepsilon>0\ \exists\, a\in A,\ b\in B:$

$\qquad b-a<\varepsilon$

$f: [a,b] \to \mathbb{R}$ CONTINUA

$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

$D = \{x_0; x_1; \dots; x_n\}$

decomposizione di $[a,b]$

$$[a,b] = [\underset{\overset{\shortparallel}{a}}{x_0}, x_1] \cup [x_1, x_2] \cup \dots \cup [x_{n-1}, \underset{\overset{\shortparallel}{b}}{x_n}]$$

$$|D| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}$$

ampiezza della decomposizione

$\forall i = 1, \dots, n \quad \exists\, y_i, z_i \in [x_{i-1}, x_i]$:

$$f(y_i) = \min_{[x_{i-1}, x_i]} f \qquad f(z_i) = \max_{[x_{i-1}, x_i]} f$$

Con le somme

$s(f,D) = \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1})$ somma inferiore secondo RIEMANN

$S(f,D) = \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1})$ " superiore " "

$\underline{S}$ = ins. delle somme inferiori

$\bar{S}$ = " " superiori

Si può dim. che $\underline{S}$ ed $\bar{S}$ sono separati $(\sup \underline{S} \le \inf \bar{S})$

cioè $s(f, D_1) \le S(f, D_2) \quad \forall D_1, D_2$

TEOR. $\underline{S}$ ed $\bar{S}$ sono contigui

DIM. Basta dim. che $\forall \varepsilon > 0\ \exists\, D : S(f,D) - s(f,D) < \varepsilon$

Ricordiamo che data $f: (a,b) \to \mathbb{R}$ f è detta UNIFORMEMENTE CONTINUA se

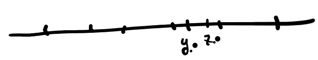
$\forall \varepsilon > 0 \quad \exists\, \delta > 0$: se $x, y \in (a,b)$, $|x-y| < \delta$ si ha $|f(x) - f(y)| < \varepsilon$

TEOREMA DI HEINE-CANTOR : una funz. continua in un intervallo chiuso e limitato è unif. cont.

DIM. fissato ε cons. $\frac{\varepsilon}{b-a}$ e dato che f è unif. cont.

$\exists\, \delta > 0$: se $|x-y| < \delta$ si ha $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Sia D : $|D| < \delta$ e cons.



$$S(f,D) - s(f,D) = \sum_{i=1}^{n} \big(f(z_i) - f(y_i)\big)(x_i - x_{i-1}) \qquad (*)$$

$y_i, z_i \in [x_{i-1}, x_i] \Rightarrow |y_i - z_i| < x_i - x_{i-1} < |D| < \delta$ Ne segue

$$(*) < \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) = \frac{\varepsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

$\Rightarrow$ TS.

Allora $\sup \underline{S} = \inf \overline{S}$

questo numero si chiama integrale definito secondo Riemann di f nell'intervallo $[a,b]$ e si denota con

$$\int_a^b f(x)\,dx$$

Si ha allora $\forall D_1, D_2$ $\quad s(f,D_1) \le \int_a^b f(x)\,dx \le S(f,D_2)$

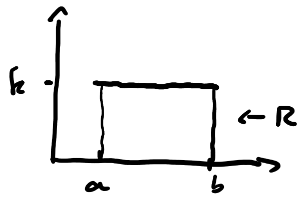
$$\int_a^b f(x)\,dx = \int_a^b f(z)\,dz = \int_a^b f(s)\,ds - - - -$$

Estensione dell'integrale definito

$f:(\alpha,\beta) \to \mathbb{R}$ continua $\quad a,b \in (\alpha,\beta)$

$$\int_a^b f(x)\,dx = \begin{cases} \text{int. di R. } \int_a^b f(x)\,dx & \text{se } a<b \\ \text{opposto dell'int di R. } -\int_b^a f(x)\,dx & \text{se } a>b \\ 0 & \text{se } a=b \end{cases}$$

es. $f(x) = k$



fissiamo D. $\quad s(f,D) = \sum_{i=1}^{n} k(x_i - x_{i-1}) = k(b-a)$

$S(f,D) = \quad k(b-a)$

$\overline{S} = \underline{S} = \{k(b-a)\} \Rightarrow \int_a^b k\,dx = k(b-a)$

area $R = k(b-a) = \int_a^b k\,dx$