Price formation in financial markets: a game-theoretic perspective

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Abstract

We propose two novel frameworks to study the price formation of an asset negotiated in an order book. Specifically, we develop a game-theoretic model in many-person games and mean-field games, considering costs stemming from limited liquidity. We derive analytical formulas for the formed price in terms of the realized order flow. We also identify appropriate conditions that ensure the convergence of the price we find in the finite population game to that of its mean-field counterpart. We numerically assess our results with a large experiment using high-frequency data from ten stocks listed in the NASDAQ, a stock listed in B3 in Brazil, and a cryptocurrency listed in Binance.

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1. Introduction

Price formation of asset prices is a process constituted as a result of the interaction of traders through the limit order book (LOB), cf. Easley and O'Hara (1995). More precisely, traders decide on how to act according to the current state of the book, and this process results in the price of the asset. Following Lehalle (2013), we put it in distinction to the *price discovery* perspective, in which we assume that the true asset price exists, for fundamental reasons, and the agents inject their views on the fair price in the order book; assuming market efficiency, we would expect that the long-run "true" price is eventually *discovered* in this way. Following the terminology of that work, in the price formation problem, the construction of prices occurs in a forward way, whereas in price discovery frameworks, the price is unraveled in a backward mechanism.

From a fundamental perspective, the study of price formation aims to investigate questions such as the ones Kyle (1985) poses.[1] The development of numerous models resulted from these investigations, including those considering liquidity issues, of which we cite Mike and Farmer (2008), Daniels et al. (2003), Weber and Rosenow (2005), Roşu (2009), Cont et al. (2010), Avellaneda and Stoikov (2008), Lachapelle et al. (2016), Lehalle et al. (2011). In this connection, we refer to the survey on agent-based models Chakraborti et al. (2011a); see also Schinckus (2012), Mastromatteo et al. (2014). [2]

Furthermore, there is a vast literature taking the price discovery perspective. For instance, we mention the efforts Cohen et al. (1981), Garbade and Silber (1979), Ho and Stoll (1980, 1981, 1983) regarding market

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[1] "How quickly is new private information about the underlying value of a speculative commodity incorporated into market prices? How valuable is private information to an insider? How does noise trading affect the volatility of prices? What determines the liquidity of a speculative market?" (Kyle, 1985, Introduction)

[2] The flourishing of the statistical investigation of order books has led to many results, especially from researchers originally from the physics community, see Bouchaud et al. (2002), Potters and Bouchaud (2003), Chakraborti et al. (2011b), Lan and Tan (2007). A popular designation of the latter approach is *econophysics* — it represents a direction that has its foundations in analogies between financial markets and physics.

microstructure properties in such a framework; see also the survey Biais et al. (2005), and the references therein. Key econometric measures for price discovery analysis are treated in Hasbrouck (1995), Gonzalo and Granger (1995), Gonzalo and Ng (2001), and further advances include Booth et al. (1999), Chu et al. (1999), Dias et al. (2016). Following the line of Hasbrouck (1995), in Grammig et al. (2005), Lien and Shrestha (2009), De Jong and Schotman (2010), Dias et al. (2018), Fernandes and Scherrer (2018) these works investigate assets traded in many venues envisaging to identify which one leads the process of impound of novel information.

In financial practice, knowledge of price formation is quite valuable. In the work of Kyle (1985), the equilibrium configuration implied by the market clearing condition indicates that prices are formed proportionally to the order flow. Under our model, we might interpret the constant of proportionality — commonly designated as Kyle's lambda — as the book depth. This aspect is a relevant contribution linking the investigation of the causal effect of the order flow on the price to the study of *price impact*, i.e., understanding how the order flow perturbs the dynamics of the corresponding asset price: "(...) the correlation between an incoming order (to buy or to sell) and the subsequent price change", as put by (Bouchaud, 2009, Section 1).

Recently, electricity markets became a strong source of motivation for research in price formation. In effect, we cite Shrivats et al. (2020), which considers Solar Renewable Energy Certificate (SREC) markets. The agents decide on how to optimally behave according to their SREC generation and their trading activities. From a market clearing condition, the determination of the price occurs as a result of the optimal behavior of the players. The construction of the performance criteria of the agents relies on their views on their future inventory, as is typical in price formation settings. In Shrivats and Jaimungal (2020), there is a numerical assessment of the results of Shrivats et al. (2020). Using similar models, to a certain extent, there are various results regarding equilibrium prices in the papers Fujii and Takahashi (2020b,a), both for infinite and finite populations. In the latter framework, these authors also consider multiple populations in Fujii (2019). We also mention Féron et al. (2020b), which treats price formation in intraday electricity markets under intermittent renewable generation; in Féron et al. (2020a) they consider this problem in the presence of a leader.

Understanding the price formation is a matter of high relevance on its own. For instance, we refer to the efforts Brennan and Subrahmanyam (1995, 1996), Locke and Onayev (2007) in the econometrics literature. Conceptually speaking, the authors seek to explain price dynamics in terms of order flow. For such a study in the context of cryptocurrencies, see Ciaian et al. (2016).

Moreover, in many of the works we mentioned above, the formed price from the model does replicate many empirical features of the market. In general, we can say that any plausible explanation of the asset price from the price formation viewpoint yields a benchmark trade execution price. This explanation has applications in Transaction Cost Analysis (TCA). Indeed, given a trading rate, its effect on the market will be to perturb the realized order flow. From our formed price, in terms of the perturbed order flow, we can derive a benchmark for the price impact induced by the strategy under scrutiny.

To take liquidity issues into account, we build upon the phenomenological framework of Bertsimas and Lo (1998) and Almgren and Chriss (2001). There, price impacts are twofold, viz., they are either permanent or temporary. This distinction is relative to their persistence in time. Thus, the impact on the dynamics of the price is what we regard as permanent, whereas the impact incurred by a single agent as a consequence of having to consume some layers of the book to have their order completely executed is the temporary one. We call *order book resilience* the underlying assumption that the book replenishes immediately from the temporary impact. Further advances in optimal execution problems generalized the idea of temporary price impact to a transient type of this friction, see Gatheral et al. (2012). Also, Obizhaeva and Wang developed a model of a block-shaped book with finite resilience, see Obizhaeva and Wang (2013). In Kallsen and Muhle-Karbe (2014), the authors see the Almgren and Chriss model as a high resilience limit of that of Obizhaeva and Wang.

Although the final goal of the above models was to study optimal execution problems, they also serve as a basis for the analysis of many other problems, such as optimal hedging Almgren and Li (2016), Bank et al. (2017), Guéant and Pu (2017), management of accelerated share repurchase contracts Guéant et al. (2015), and even price formation Féron et al. (2020b,a). Here, we will develop our contributions to the price

formation problem with a related model. We will consider a temporary price impact in a standard way, but we will not assume a specific shape for the price impact on the asset price dynamics. In turn, we are willing to establish an explanation for the asset price in terms of its realized order flow.

At first, we will make a simplifying assumption that there are infinitely many players. In this manner, we will use the machinery of Mean-Field Game (MFG) models. MFGs comprise game-theoretic models serving to investigate equilibria of large and competitive populations of strategically behaving individuals. On many occasions, they are useful precisely because they provide analytic tractability. They were introduced in the mathematical community independently by J-M. Lasry and P-L. Lions, see Lasry and Lions (2006a,b, 2007), and by M. Huang, P. Caines and R. Malhamé, see Huang et al. (2007, 2006). In a second moment, we investigate the many-person game corresponding to the MFG one. We work on a framework of asymmetric information among decentralized homogeneous agents, allowing each agent to impact the price in a rather general way. Our goal is to interpret the results for the infinite population as limiting cases of some for the finite population counterpart. In the literature, the work Gjerstad and Dickhaut (1998) is another game-theoretic approach to the price formation problem, but in a context of double auctions.

In the MFG setting, we introduce the notion of competitive equilibria for our problem. Through this definition, we can analyze the interplay between the supply/demand resulting from the order flow and the corresponding asset prices. We assume continuity of the price and order flow, as well as that they are deterministic, working in an MFG framework akin to that of Gomes et al. (2020) (cf. Ashrafyan et al. (2021) and Gomes et al. (2021) for further developments on theoretical aspects). Although our assumptions are strong at this point, we will see that our results are satisfactory empirically. Moreover, our analysis of the finite population counterpart will shed light upon our explanation of the asset price endogenously resulting from our MFG model.

We resort to many-person games to assess sufficient conditions to ensure the convergence of the price we find as the population size grows to infinity. We allow for more flexibility in this setting, such as a heterogeneous informational structure as in Evangelista and Thamsten (2020). For analytic tractability, as well as for a clearer comparison to the previous framework, we keep the assumption that the parameters are homogeneous among the players. However, we consider a broader class of admissible prices when defining the competitive equilibria for the finite population games. Thus, we investigate some consequences of considering price impacts stemming from each individual agent – a "microstructural impact", or "micro-impact", for short. In terms of this micro-impact, alongside a further assumption on the limiting behavior of the uncertainty in the traders' inventories, we are able to formulate a limiting theorem, thus recovering the formula for the price we found for the MFG.

We organize the remainder of this paper as follows. In Section 2, we analyze competitive equilibria in the MFG setting. Then, in Section 3, we investigate the many-person game to gain further insight into the results we obtained for the infinite population framework. We conclude the paper numerically assessing our results with high-frequency data from NASDAQ and other two different financial markets. In Section 4 We empirically assess the results we derived in the previous sections. Our many-person approach indicates a regression covariate shown to be relevant empirically. Finally, in Section 5 we state our general findings and provide our concluding remarks.

2. Price formation with a MFG model

2.1. The model

The market. We consider a single venue in which infinitely many agents trade a single asset. By this, we mean that their trading in other assets do not influence their decisions in the specific one we are focusing on. We denote its reference price [3] by p. Our point of view is that p is the result of the interaction among the population through the order book. As a simplifying assumption, we stipulate that p is deterministic, as in Gomes et al. (2020). In Section 3 we will relax this restriction in the finite population setting, and we

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[3]We assume that the reference price is given by the asset's midprice, i.e., the average between the best bid and best ask quotes.

will provide an appropriate interpretation for the price we find in the present section as a limit when the number of agents goes to infinity.

Information structure. We fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leqslant t \leqslant T})$, with $\mathcal{F} = \mathcal{F}_T$, supporting a one-dimensional Brownian motion W. Therefore, we currently suppose symmetric information throughout the population.

The traders. Let us consider a representative trader beginning a trading program at time $t \in [0, T]$, targeting to execute a trade of size $|q|$ shares of the asset, where $q \in \mathbb{R}$. If $q > 0$ (resp., $q < 0$) the agent seeks to liquidate (resp., execute) q shares over $[t, T]$. The trader controls her turnover rate $\{\nu_u\}_{t \leqslant u \leqslant T} \in \mathcal{U}_t$, where

$$\mathcal{U}_t := \left\{ \nu = \{\nu_u\}_{t \leqslant u \leqslant T} : \nu \text{ is } \mathbb{F} - \text{progressively measurable, and } \mathbb{E}\left[\int_t^T \nu_u^2 \, du \right] < \infty \right\}$$

is the set of admissible controls. In this manner, her inventory process $\{q_u\}_{t \leqslant u \leqslant T}$ satisfies

$$\begin{cases} dq_u = \nu_u \, du + \sigma dW_u, \\ q_t = q. \end{cases} \tag{2.1}$$

Above, we allowed uncertainty in the traders' inventory. As put by Duffie and Zhu (2017), this is justified because it represents an effect observed in reality. If the trader is a broker, she will receive an order flow from her clients, something which we can model as (2.1), cf. (Carmona et al., 2015, Subsection 2.1). Another explanation for introducing uncertainty in the dynamics (2.1) of the inventory is that when the agent actually conducts her trades, at the so-called tactical layer of an execution algorithm, in the terminology of Guéant (2016), she is subject to execution uncertainty stemming from operational risks.

As the trader executes a volume $\nu_u \, du$ over the time interval $[u, u + du[$ she must pay or receive — depending on the sign of ν_u — the quantity $-p(u)\nu_u \, du$. If we assume a linear temporary price impact, then the indirect cost per share she incurs from this effect is $-\kappa\nu_u$; hence, this trade's total cost is $-\kappa\nu_u^2 \, du$. Gathering these two pieces together, we derive the following dynamics for the corresponding cash process $\{c_t\}_{0 \leqslant t \leqslant T}$:

$$\begin{cases} dc_u = -p(u)\nu_u \, du - \kappa\nu_u^2 \, du = - [p(u) + \kappa\nu_u] \, \nu_u \, du, \\ c_t = c. \end{cases} \tag{2.2}$$

In (2.2), we denoted by $c \in \mathbb{R}$ the initial cash amount that the trader holds. We remark that we assume κ to be the same among players. Since we suppose they are all trading the same asset in the same venue, this is not an unrealistic condition.

Finally, we define the agent's wealth w as

$$w_u := c_u + q_u p(u) = c - \int_t^u [p(\tau) + \kappa\nu_\tau] \, \nu_\tau \, d\tau + q_u p(u) \qquad (t \leqslant u \leqslant T). \tag{2.3}$$

The value function. We intend to approach the price formation in the MFG model via the dynamic programming perspective. To conduct this endeavor, we define the value function V as

$$\begin{aligned} V(t, q) &:= \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[w_T - c - A q_T^2 - \phi \int_t^T q_u^2 \, du \, \middle| \, c_t = c, \, q_t = q \right] \\ &= \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[q_T \left(p(T) - Aq_T \right) - \int_t^T \left\{ \kappa \nu_u^2 + \phi \, q_u^2 + \nu_u p(u) \right\} \, du \, \middle| \, q_t = q \right]. \end{aligned} \tag{2.4}$$

Here, V is the supremum of performance criteria which are in turn formed by three parts. Firstly, they include the mean of the difference $w_T - c$ between the corresponding trader's terminal wealth and initial cash. We do not compare $w_T - c$ to the pre-trade price $q \, p(t)$ of the initial inventory, a common enterprise

in *Implementation Shortfall* algorithms, because we are not interested in taking differentials of the price. Secondly, the criteria also include a terminal penalization for finishing with non-vanishing inventory, i.e., $-A\,q_T^2$. Thirdly, there is the mean of an urgency term $-\phi \int_t^T q_u^2\,du$, inducing traders to accelerate their program. Analogously as authors in Cartea et al. (2015) explain, we emphasize that we do not interpret this as a financially meaningful penalization but rather as an extra risk management tool. However, as put by Duffie and Zhu (2017) in a related modeling context, even if we assume traders are risk-neutral, they can incur costs for holding non-zero inventory, either long or short, which advocates in part of taking ϕ to be the same for all agents.

Inventory density. We denote the density of the law of the players' inventories at time t, q_t, by $m(t,\cdot)$. Additionally, we consider the initial density m_0 as given, i.e., $m(0,\cdot) = m_0$.

Market clearing condition. If an agent with inventory level q trades at a rate $\nu(t,q)$, then we can define the aggregation rate μ as

$$\mu(t) = \int_{\mathbb{R}} \nu(t,q) m(t,q)\,dq.$$

Heuristically, being in equilibrium with a market demand/supply rate Λ should require the market clearing condition, cf. Gomes et al. (2020):

$$\mu(t) + \Lambda(t) = 0 \qquad (0 \leqslant t \leqslant T). \tag{2.5}$$

Therefore, if the market is bullish (respectively, bearish) on aggregate, then we would expect $\mu(t) > 0$ (respectively, $\mu(t) < 0$), i.e., traders are required to buy (respectively, sell) some shares of the asset; hence, $\Lambda(t)$ describes the opposite movement, which occurs in the inventory of the traders' counterparts as a whole. We thus see Λ as the interaction process of agents with the LOB itself (such as by adding or subtracting shares of an asset from there), or rather the resulting movement in the LOB due to the negotiations taking place there. We remark that we do not suppose that the players behaving optimally in our MFG trade among themselves — in fact, players trading aggressively (meaning via market orders) could be, say, all buying the asset simultaneously at a certain time instant. Thus, we do implicitly regard that there are exogenous liquidity providers accepting to serve as counterparts at price p to the mean-field of traders we are studying.

2.2. Competitive equilibria

We are now ready to provide the following definition of competitive equilibria for the MFG model.

Definition 2.1. *A competitive equilibrium for the MFG model is a tuple $(p, V, m, \nu^*, \Lambda)$, where $p, \Lambda : [0,T] \to \mathbb{R}$ and $V, m, \nu^* : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfy:*

(a) *For each square-integrable random variable ξ, the SDE $dq_u^* = \nu^*(u, q_u^*)\,du + \sigma dW_u$, $q_t^* = \xi$, has a strong solution;*

(b) *For each $(t,q) \in [0,T] \times \mathbb{R}$, if we consider a solution of the SDE above with initial condition $q_t^* = q$ and write $\nu^* := \{\nu_u^* := \nu^*(u, q_u^*)\}_{t \leqslant u \leqslant T}$, then the supremum in (2.4) is attained by the strategy ν^*, and it is equal to $V(t,q)$;*

(c) *If the density of $\mathrm{Law}(q_0^*)$ is $m(0,\cdot)$ and $\{q_t^*\}_{0 \leqslant t \leqslant T}$ is a solution of SDE in (a) with such initial condition, then the function m is the inventory distribution density of q^*, i.e. $m(t,\cdot)$ is the density of $\mathrm{Law}(q_t^*)$, for $0 \leqslant t \leqslant T$;*

(d) *Equation (2.5) holds for ν^* and m:*

$$\int_{\mathbb{R}} \nu^*(t,q) m(t,q)\,dq + \Lambda(t) = 0 \qquad (0 \leqslant t \leqslant T). \tag{2.6}$$

On the one hand, in condition (b) of Definition 2.1 is akin to the "*Profit Maximization*" property of the equilibria (Kyle, 1985, Sections 2, 3, and 4). On the other hand, we do not propose "market efficiency" in the context of our equilibria configurations. Rather, we leave a degree of freedom, viz., the order flow of liquidity taking orders. In our condition (d), we assume market makers clear the market, but following a perspective similar to that of Kyle (1985),[4] i.e., we do not delve into the (interesting) question of their optimizing behavior in doing so.

2.3. Some properties of the competitive equilibria

We now turn to the analysis of competitive equilibria in terms of Definition 2.1. The first step is to assume that the price component p of the tuple $(p, V, m, \nu^*, \Lambda)$ is given and continuous and from this derive the well-posedness for (V, m). Particularly, the linear-quadratic shape of the criteria in (2.4) will allow us to show that V is itself quadratic in the inventory variable. These comprise the content of the subsequent result.

Proposition 2.2. *Let us consider a competitive equilibrium* $(p, V, m, \nu^*, \Lambda)$, *with* $p \in C([0,T])$. *Then,* (V, m) *must solve the system*

$$\begin{cases} \partial_t V(t,q) + \frac{\sigma^2}{2}\partial_q^2 V(t,q) + \frac{1}{4\kappa}\left[\partial_q V(t,q) - p(t)\right]^2 - \phi q^2 = 0, \\ \partial_t m(t,q) - \frac{\sigma^2}{2}\partial_q^2 m(t,q) + \partial_q\left[\frac{(\partial_q V(t,q) - p(t))}{2\kappa} m(t,q)\right] = 0, \\ V(T,q) = q\left[p(T) - Aq\right] \text{ and } m(0,q) = m_0(q), \end{cases} \quad (2.7)$$

where we understand the first PDE above in the viscosity sense, and the second one in the distributional sense. Moreover, for each such p, the system (2.7) *admits a unique smooth solution* $(V,m) \in \left[C^{1,2}\right]^2$, *where*

$$V(t,q) = \theta_0(t) + \theta_1(t)q + \theta_2(t)q^2, \quad (2.8)$$

where

$$\theta_2(t) = \sqrt{\kappa\phi}\frac{1 - ce^{2\gamma(T-t)}}{1 + ce^{2\gamma(T-t)}}, \quad \text{for } c := \frac{\sqrt{\kappa\phi} + A}{\sqrt{\kappa\phi} - A} \text{ and } \gamma := \sqrt{\frac{\phi}{\kappa}}, \quad (2.9)$$

$$\theta_1(t) = p(T)e^{\frac{1}{\kappa}\int_t^T \theta_2(s)ds} - \frac{1}{\kappa}\int_t^T \theta_2(s)p(s)e^{\frac{1}{\kappa}\int_t^s \theta_2(\tau)d\tau}ds, \quad (2.10)$$

as well as

$$\theta_0(t) = \int_t^T\left[\frac{1}{4\kappa}\left(\theta_1(u) - p(u)\right)^2 + 2\sigma^2\theta_2(u)\right]du. \quad (2.11)$$

Proof. Given $p \in C([0,T])$, the value function V, which we define according to (2.4), solves in the viscosity sense the Hamilton-Jacobi-Bellman equation comprising the first PDE in (2.7), see (Pham, 2009, Theorems 4.3.1 and 4.3.2, Remarks 4.3.4 and 4.3.5) — the terminal condition is clear from (2.4). Now, given p and V, we check (arguing as in (Cardaliaguet, 2010, Lemma 3.3)) that the density of players' inventory m (weakly) solves the Kolmogorov-Fokker-Planck equation in (2.7), with initial data $m(0, \cdot) = \text{Law}(q_0)$. To prove the second part of the Proposition, we make the ansatz

$$V(t,q) = \theta_0(t) + \theta_1(t)q + \theta_2(t)q^2. \quad (2.12)$$

Thus, the first equation in (2.7) becomes

$$\begin{cases} \theta_2'(t) + \frac{1}{\kappa}\theta_2^2(t) - \phi = 0, \\ \theta_1'(t) + \frac{1}{\kappa}\theta_2(t)(\theta_1(t) - p(t)) = 0, \\ \theta_0'(t) + \frac{1}{4\kappa}(\theta_1(t) - p(t))^2 + 2\sigma^2\theta_2(t) = 0, \\ \theta_0(T) = 0, \theta_1(T) = p(T), \theta_2(T) = -A. \end{cases} \quad (2.13)$$

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[4] "Modelling how market makers can earn the positive frictional profits necessary to attract them into the business of market making is an interesting topic which takes us away from our main objective of studying how price formation is influenced by the optimizing behavior of an insider in a somewhat idealized setting.", see (Kyle, 1985, Section 2).

Solving the Ricatti equation determining θ_2 is straightforward; it also appears in the standard Almgren-Chriss model, with only one trader, see Cartea et al. (2015). With θ_2 at hand, we derive the formula for θ_1 by solving the second equation in (2.13), which is a linear ODE. Finally, we obtain θ_0 from the third one via direct integration. At this point, we can carry out standard verification arguments to validate (2.12). With such $V \in C^{1,2}$ at hand, showing that the Fokker-Plack-Kolmogorov equation admits a unique smooth solution m is standard, see Cardaliaguet (2010). $\qquad\square$

Corollary 2.3. *If $(p, V, m, \nu^*, \Lambda)$ is a competitive equilibrium, then the optimal control in feedback form is given by*

$$\nu^*(t, q) = \frac{\partial_q V(t, q) - p(t)}{2\kappa} = \frac{1}{2\kappa}\left(\theta_1(t) - p(t)\right) + \frac{\theta_2(t)}{\kappa}q. \tag{2.14}$$

Remark 2.4. *We notice that the market clearing condition (2.6) is not necessary to derive neither the characterization of (V, m) we provided in Proposition 2.2 nor relation (2.14) we presented above.*

The identity (2.14) is somewhat insightful, as it sheds light on how we can interpret the functions θ_1 and θ_2. In fact, on the one hand, since $\theta_2 < 0$, a consequence of this equation is that $\theta_2 q$ is a contribution to the trading rate of a player holding inventory q that represents a force driving her to clear these holdings. On the other hand, there is another term: $\theta_1 - p$. If $\theta_1 > p$ (respectively, $\theta_1 < p$) then this term leads the trader to buy (respectively, sell) shares. Therefore, we can interpret θ_1 as some indication of the future value of the asset — something that relation (2.10) itself suggests.

2.4. Comparison with Duffie and Zhu (2017)

Our goal is to understand how prices in competitive equilibria are determined by the interactions of traders taking place in the LOB. We will do so by expressing the price component p of a competitive equilibrium $(p, V, m, \nu^*, \Lambda)$ in terms of Λ. Prior to that, but already in this direction, let us fix such a tuple $(p, V, m, \nu^*, \Lambda)$. We observe that a full clearing for optimal traders is equivalent to require that $\Lambda \equiv 0$. In this case, from the market clearing condition (2.6), we would have

$$\int_{\mathbb{R}} \nu^*(t, q)m(t, q)dq = 0.$$

In view of (2.14), this implies that

$$p(t) = \theta_1(t) + 2\theta_2(t)E(t), \tag{2.15}$$

where we have written $E(t) := \int_q qm(t, q)dq$. In this way, we have recovered a formula which is much similar to those derived in (Duffie and Zhu, 2017, Eq. (9)), for the discrete double auction, and in (Duffie and Zhu, 2017, Eq. (A.99)), for the continuous case, even though they analyzed a framework considerably distinct from ours. Particularly, we once again observe from Eq. (2.15) the analogy between θ_1 and a future expected value of the asset's price. However, here we do not have to necessarily constrain ourselves to $\Lambda \equiv 0$. In fact, if we consider a given realized order flow Λ, not necessarily vanishing, we obtain

$$p(t) = 2\kappa\Lambda(t) + \theta_1(t) + 2\theta_2(t)E(t).$$

Nevertheless, at this moment, the influence of Λ on E is not completely clear. The main result of the present section will address this issue, and we will be able, to a certain extent, to parametrize competitive equilibria with respect to Λ. Before advancing in this direction, we infer some representations for the population's behavior in an equilibrium configuration.

Proposition 2.5. *Let us consider a competitive equilibrium $(p, V, m, \nu^*, \Lambda)$. In this setting, the optimal inventory holdings $\{q_t^*\}_{0 \leqslant t \leqslant T}$ of a trader beginning with $q_0 \in L^2(\Omega)$ shares is*

$$q_t^* = q_0 e^{\int_0^t \frac{\theta_2(t)}{\kappa}} + \frac{1}{2\kappa}\int_0^t (\theta_1(s) - p(s))e^{\int_s^t \frac{\theta_2(\tau)}{\kappa}d\tau}ds + \sigma\int_0^t e^{\int_s^t \frac{\theta_2(\tau)}{\kappa}d\tau}dW_s, \tag{2.16}$$

for $t \in [0, T]$. Moreover, the optimal trading rate process is

$$\nu_t^* = \frac{1}{2\kappa} \left(\theta_1(t) - p(t) \right) + \frac{\theta_2(t)}{\kappa} q_t^*, \tag{2.17}$$

where θ_1, θ_2 are given, respectively, in (2.9) and (2.10).

Proof. The representation (2.17) of the optimal strategy follows promptly from (2.14). From (2.1) and (2.14), we derive

$$dq_t^* = \frac{1}{2\kappa} (\theta_1(t) - p(t)) \, dt + \frac{\theta_2(t)}{\kappa} q_t^* \, dt + \sigma dW_t.$$

We easily see that the solution of this linear SDE is (2.16). $\qquad\square$

2.5. On the formed price in the MFG

We are now ready to state and prove the main theoretical result of the current section. Summing up, its content is a representation of the price in terms of the market supply/demand path in a competitive equilibrium, see (2.18) below. Firstly, we provide the formula as a necessary condition for a given tuple to be a competitive equilibrium. We argue that this is the most relevant part of the theorem. In effect, this relation allows us to jump out of the model, looking for its validation on data stemming from real world prices of assets, which will be done in Section 4.

Theorem 2.6. *Given a competitive equilibrium $(p, V, m, \nu^*, \Lambda)$ for the MFG, the formed price is*

$$p(t) = p(0) + 2\phi E_0 t - 2\phi \int_0^t (t-u)\Lambda(u) du + 2\kappa(\Lambda(t) - \Lambda(0)). \tag{2.18}$$

Proof. Let us assume $(p, V, m, \nu^*, \Lambda)$ is a competitive equilibrium. Remember that the law of q_t^* is given by $m(t, \cdot)$. We define:

$$\begin{cases} \Pi(t) := \mathbb{E}[\partial_q V(t, q_t^*)] = \int_{\mathbb{R}} \partial_q V(t, q) m(t, q) \, dq; \\ E(t) := \mathbb{E}[q_t^*] = \int_{\mathbb{R}} q \, m(t, q) \, dq. \end{cases} \tag{2.19}$$

Differentiating the first equation in (2.7) with respect to q yields

$$\partial_q \partial_t V(t, q) + \frac{\sigma^2}{2} \partial_q^3 V(t, q) + \frac{(\partial_q V(t, q) - p(t))}{2\kappa} \partial_q V(t, q) - 2\phi q = 0. \tag{2.20}$$

We emphasize that this differentiation is licit since V is a quadratic polynomial with respect to q, with coefficients that are differentiable in time. Therefore, we observe that

$$\begin{aligned} \Pi'(t) &= \int_{\mathbb{R}} \partial_q \partial_t V(t, q) m(t, q) \, dq + \int_{\mathbb{R}} \partial_q V(t, q) \partial_t m(t, q) \, dq \\ &= \int_{\mathbb{R}} \left\{ -\frac{\sigma^2}{2} \partial_q^3 V(t, q) - \frac{(\partial_q V(t, q) - p(t))}{2\kappa} \partial_q^2 V(t, q) + \phi q \right\} m(t, q) \, dq \\ &\qquad - \int_{\mathbb{R}} \frac{1}{2\kappa} \partial_q V(t, q) \partial_q \left[(\partial_q V(t, q) - p(t)) m(t, q) \right] dq. \end{aligned} \tag{2.21}$$

Next, we notice that carrying out integration by parts in identity (2.21), we have

$$\Pi'(t) = 2\phi E(t). \tag{2.22}$$

Employing (2.12) and (2.14) in (2.1), we infer

$$q_t^* = q_0 + \int_0^t \frac{\partial_q V(s, q_s^*) - p(s)}{2\kappa} ds + \sigma W_t,$$

in such a way that

$$E(t) = \mathbb{E}[q_0] + \int_0^t \mathbb{E}\left[\frac{\partial_q V(s, q_s^*) - p(s)}{2\kappa}\right] ds. \tag{2.23}$$

From (2.23) and (2.6), we deduce

$$E'(t) = \mathbb{E}\left[\frac{\partial_q V(t, q_t^*) - p(t)}{2\kappa}\right] = -\Lambda(t), \tag{2.24}$$

and additionally, by the definition of Π, we find

$$E'(t) = \mathbb{E}\left[\frac{\partial_q V(t, q_t^*) - p(t)}{2\kappa}\right] = \frac{\Pi(t) - p(t)}{2\kappa}. \tag{2.25}$$

Collecting (2.22), (2.24) and (2.25), we obtain

$$\begin{cases} \Pi'(t) & = 2\phi E(t), \\ \\ E'(t) & = \dfrac{\Pi(t)}{2\kappa} - \dfrac{p(t)}{2\kappa} = -\Lambda(t). \end{cases} \tag{2.26}$$

On the one hand, from the second equation in system (2.26), we have

$$E(t) = E_0 - \int_0^t \Lambda(s) ds.$$

On the other hand, the first one gives us

$$\Pi(t) = \Pi(0) + 2\phi \int_0^t E(s) ds$$

$$= \Pi(0) + 2\phi E_0 t - 2\phi \int_0^t \int_0^s \Lambda(u)\, du\, ds$$

$$= \Pi(0) + 2\phi E_0 t - 2\phi \int_0^t \int_u^t \Lambda(u)\, ds\, du$$

$$= \Pi(0) + 2\phi E_0 t - 2\phi \int_0^t (t - u)\Lambda(u)\, du,$$

as well as, again from the second equation,

$$p(t) = \Pi(t) + 2\kappa\Lambda(t)$$

$$= \Pi(0) + 2\phi E_0 t - 2\phi \int_0^t (t - u)\Lambda(u) du + 2\kappa\Lambda(t). \tag{2.27}$$

In particular,

$$\Pi(0) = p(0) - 2\kappa\Lambda(0). \tag{2.28}$$

In conclusion, (2.27) and (2.28) allow us to infer

$$p(t) = p(0) + 2\phi E_0 t - 2\phi \int_0^t (t - u)\Lambda(u) du + 2\kappa(\Lambda(t) - \Lambda(0)). \tag{2.29}$$

$\square$

Remark 2.7. *The expression for the formed price (2.18) implies the dynamics*

$$dp(t) = 2\phi\left(E_0 - \int_0^t \Lambda(u) du\right) dt + 2\kappa d\Lambda(t). \tag{2.30}$$

Notice the drift of p depends on the path of Λ through its integral. Moreover, one might use this dynamics to conclude the following correspondence of quadratic variations: $\langle p \rangle_t = 4\kappa^2 \langle \Lambda \rangle_t$.

2.6. Discussion on the formed price

Let us discuss the main aspects of the formula (2.18).

We first consider $2\kappa(\Lambda(t) - \Lambda(0))$ — it corresponds to the contribution of the current instantaneous market supply/demand $\Lambda(t)$ rate to the formed price (at time t). We remark that it is proportional to the difference $\Lambda(t) - \Lambda(0)$, the slope being 2κ. This is the component of the price which results from the fact that traders are willing to pay costs for trading the asset. In effect, for trading a volume of $|\Lambda(t) - \Lambda(0)|$, they will pay a total of $\kappa |\Lambda(t) - \Lambda(0)|^2$ in transaction costs due to this market friction. The resulting influence of this movement on the formed price is $2\kappa (\Lambda(t) - \Lambda(0))$.

A similar consideration is valid for $2\phi E_0 t$. Namely, if the average initial holdings (or, equivalently, the initial targets) of players, E_0, is positive (resp., negative), then our modeling implies they are willing to sell (resp., buy). However, this term $2\phi E_0 t$ indicates a positive (resp., negative) drift on the price, as long as $\phi > 0$, which grows proportionally to time. Thus, if players are not quick enough to execute their schedules, we would expect prices to be moving favorably to their current holdings, with this effect being more pronounced for larger times. In a sense, this term represents an inertia component on the asset's price. Here, the inertia we refer to is distinct from what was studied in Bayraktar et al. (2007) (see also references therein). In our setting, this terminology only indicates that $2\phi E_0 t$ exert a sort of resistance on trading rates moving the price.

Finally, we comment on $-2\phi \int_0^t (t-u)\Lambda(u)du$, which is undoubtedly the most interesting term. We notice we can naturally interpret it as a *memory* term. Indeed, the weight $(t - u)$ of the value $\Lambda(u)$ appearing within the integral is larger for older market supply/demand rate values. In a certain sense, as markets "digest" past information, they will appear more significantly in the formed price formula.

We finish the present section by carrying out some explicit comparative statics to highlight the influence of the parameters on the formed price (2.18).

Sensitivity relative to the temporary price impact. We observe that p is linear in κ, with slope $\partial_\kappa p(t) = 2(\Lambda(t) - \Lambda(0))$. As we discussed above, this is the term in the price representing the trader's willingness to pay costs to negotiate the asset.

Sensitivity relative to the average initial target. We notice that p is also linear with respect to E_0. In this case, we have slope $\partial_{E_0} p(t) = 2\phi t$, which always has the same sign as ϕ. Thus, if traders are risk averse, meaning $\phi > 0$ (resp., risk loving, meaning $\phi < 0$), then p is an increasing (resp., decreasing) function of E_0. Let us assume for concreteness that $\phi > 0$. Then, if the market has a larger target, *ceteris paribus* — particularly, while Λ remains still — we can only explain this by the price being larger as well. In other words, if we consider that, when behaving according to the same criteria, players still trade with all else remaining the same, especially demanding on average the same Λ, then the formed price must be larger.

Sensitivity relative to the trading urgency. The price p is also linear with respect to ϕ. The slope here is $\partial_\phi p(t) = 2E_0 t - 2\int_0^t (t - u)\Lambda(u)\, du$. Thus, the conflict between the memory term, representing the detrimental pressure traders undergo due to market frictions, and one of the inertia terms, comprising the average initial holdings/targets of players, governs how prices change proportionally to the urgency parameter.

Sensitivity relative to the order flow. Finally, we analyze how prices depend on Λ. They are linear on it, in a functional sense. Namely, given a time $t \in [0, T]$, as a function of the path $\{\Lambda(u)\}_{0 \leqslant u \leqslant t}$, the formed price $p(t)$ evaluated at time t is affine. We identify the analogous element to the slope here (which is now a whole path) if we perturb the order flow Λ by trading at a rate $\{\nu_u\}_{0 \leq u \leq t}$, and this yields the difference between the two resulting prices:

$$2\kappa\nu_t - 2\phi \int_0^t (t - u)\nu_u \, du.$$

2.7. Existence of Equilibrium

In this section we discuss two settings. Basically, among the realized order flow Λ and the formed price p, there is only one degree of freedom. Namely, assuming the price p is given, with suitable regularity, we can derive the order flow in terms of the traders' optimal turnover rate ν^* and their inventory distribution density m — this is the content of our next result. Conversely, if we assume that the order flow is given, and that it satisfies suitable compatibility conditions (which we will describe later, see (2.31)), then we can assert that p, determined by the formula (2.18), is *the* price formed by this order flow. In the latter framework, the necessity clause stems precisely from (2.18), whereas sufficiency is what demands the compatibility relation we mentioned above. We proceed to discuss these results in the sequel.

Before doing so, let us exemplify the case when p is given considering that $p(t) \equiv p_0$ is a constant, with $p_0 > 0$. Then, we can derive from (2.30) that Λ solves

$$\begin{cases} \kappa \Lambda''(t) = \phi \Lambda(t), \\ \Lambda(0) = \Lambda_0, \ \Lambda'(0) = \phi E_0 / \kappa. \end{cases}$$

If we assume that the model parameters satisfy

$$\Lambda_0 \sqrt{\kappa} - \sqrt{\phi} E_0 \neq 0,$$

then we can derive from the above second-order ODE the closed-form solution for Λ :

$$\Lambda(t) = \Lambda_0 e^{\int_0^t g(u)\,du},$$

where

$$g(u) = \gamma \left(\frac{e^{2\gamma u} \beta - 1}{e^{2\gamma u} \beta + 1} \right), u \geqslant 0,$$

the constants γ and β being given by

$$\gamma = \sqrt{\frac{\phi}{\kappa}} \text{ and } \beta = \frac{\Lambda_0 \sqrt{\kappa} + \sqrt{\phi} E_0}{\Lambda_0 \sqrt{\kappa} - \sqrt{\phi} E_0}.$$

The more general corollary below follows from Proposition 2.2.

Corollary 2.8. *Let* $p \in C([0,T])$, (V, m) *be solution to* (2.7), *and* ν^* *be given in feedback form by* (2.14). *Then, by setting* $\Lambda(t) = -\int_q \nu^*(t,q) m(t,q) dq$, *the tuple* $(p, V, m, \nu^*, \Lambda)$ *is a competitive equilibrium. In particular, competitive equilibria exist.*

We now consider the other situation where Λ is given and derive a condition for the price given by (2.18) that gives us a competitive equilibrium.

Proposition 2.9. *Given* Λ, *let us define* p *as in* (2.18), (V, m) *as the corresponding solution of* (2.7), *and* ν^* *as in* (2.14). *Then, the compatibility condition*[5]

$$\theta_1(t) + 2\theta_2(t) \int_{\mathbb{R}} q m(t,q) dq = p(0) + 2\phi E_0 t - 2\phi \int_0^t (t-u) \Lambda(u) du - 2\kappa \Lambda(0) \tag{2.31}$$

implies that $(p, V, m, \nu^*, \Lambda)$ *is a competitive equilibrium.*

Proof. Let us assume Λ is given, that we define p as (2.18), that (V, m) is the solution of (2.7) corresponding to p, and that ν^* satisfies (2.14). At this point, to show that $(p, V, m, \nu^*, \Lambda)$ is a competitive equilibrium,

---

[5]Notice that θ_i depends on p for $i = 1, 2$.

we need only to verify the market clearing condition (2.6). To do this, we integrate (2.14) against m (cf. Remark 2.4), whence we derive

$$\int_{\mathbb{R}} \nu^*(t,q) m(t,q) dq = \frac{1}{2\kappa}(\theta_1(t) - p(t)) + \frac{\theta_2(t)}{\kappa} \int_{\mathbb{R}} q m(t,q) dq$$

$$= \frac{\theta_1(t)}{2\kappa} + \frac{\theta_2(t)}{\kappa} \int_{\mathbb{R}} q m(t,q) dq - \frac{1}{2\kappa} \left[p(0) + 2\phi E_0 t - 2\phi \int_0^t (t-u)\Lambda(u) du + 2\kappa(\Lambda(t) - \Lambda(0)) \right]$$

$$= \frac{\theta_1(t)}{2\kappa} + \frac{\theta_2(t)}{\kappa} \int_{\mathbb{R}} q m(t,q) dq - \frac{1}{2\kappa} \left[p(0) + 2\phi E_0 t - 2\phi \int_0^t (t-u)\Lambda(u) du - 2\kappa\Lambda(0) \right] - \Lambda(t).$$

$$= -\Lambda(t),$$

where we used (2.31) in the last equality. $\qquad \square$

The "compatibility condition" (2.31) ensures that, if we *define* the price according to the necessary relation, then the corresponding tuple (consisting of the appropriate objects, viz., price, value function, density of inventories, strategy in feedback form, and market supply/demand path) is indeed a competitive equilibrium.

3. Price formation with a finite population model

In the price discovery framework, a relevant research question is to investigate the convergence of the short-term price to the "true" long-term one. In effect, as Biais et al. (2005) points out, in reference to Walras (1896), the tradition L. Walras initiated has roots on this question.[6] We focus on a transposed version of this question in the price formation framework in the current section. More precisely, we inquire under which conditions we can ensure that the price which is formed as a result of the interaction among traders of a finite population converges to the one that we have identified in the mean-field setting, i.e., the one we found in (2.18).

3.1. The model

The market. We consider a market consisting of a single venue. There are $N \geqslant 1$ traders present there, trading a single asset, in similar terms as in Section 2. We index players using labels $i \in \mathcal{N} := \{1, \dots, N\}$. We denote the mid-price of the asset by $p = \{p_t\}_{0 \leqslant t \leqslant T}$. Here, we allow p to be stochastic. We will discuss our assumptions on p below.

Before doing this, we describe the information structure we assume throughout the present section.

Information structure. We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we consider a $N+1$-dimensional Brownian motion $W = \left\{ (W_t^0, W_t^1, \dots, W_t^N) \right\}_{0 \leqslant t \leqslant T}$. Additionally, we denote the filtration generated by W^0 and W^i by $\mathbb{F}^{0i} = \left\{ \mathcal{F}_t^{0i} \right\}_{0 \leqslant t \leqslant T}$, which satisfies the usual conditions and models the gathering of information of player i. We write $\mathbb{F}^0 := \left\{ \mathcal{F}_t^0 \right\}_{0 \leqslant t \leqslant T}$ for the common information shared by all the players, which in our case is the filtration generated by W^0.

---

[6] "In perfect markets, Walrasian equilibrium prices reflect the competitive demand curves of all potential investors. While determining these fundamental equilibrium valuations is the focus of (most) asset pricing, market microstructure studies how, in the short term, transaction prices converge to (or deviate from) long-term equilibrium values. Walras himself was concerned about the convergence to equilibrium prices, through a tâtonnement process. One of the first descriptions of the microstructure of a financial market can be found in the Elements d'Economie Politique Pure (1874), where he describes the workings of the Paris Bourse. Walras's field observations contributed to the genesis of his formalization of how supply and demand are expressed and markets clear. Market microstructure offers a unique opportunity to confront directly microeconomic theory with the actual workings of markets. This facilitates both tests of economic models and the development of policy prescriptions."

Admissible prices. In the finite population setting, we look for prices that are sensitive to the movements of each particular individual. That is, we consider possible prices p that are mappings depending on the strategies of each trader. To carry out this endeavor, we will first specify the class of admissible strategies. Henceforth, for each $i \in \mathcal{N} \cup \{0\}$, the set $\mathcal{A}_i$ consists of the $\mathbb{F}^{0i}$−progressively measurable processes $\{\nu_t\}_{0 \leqslant t \leqslant T}$ such that $\mathbb{E}\left[\int_0^T \nu_t^2\, dt\right] < \infty$. Also, we put

$$\mathcal{A} := \Pi_{i=1}^N \mathcal{A}_i, \text{ and } \mathcal{A}_{-i} := \Pi_{j \neq i} \mathcal{A}_j,$$

For each $\boldsymbol{\nu} = \left(\nu^1, \ldots, \nu^N\right)^\intercal \in \mathcal{A}$ and $i \in \mathcal{N}$, we will write $\boldsymbol{\nu}^{-i} := \left(\nu^1, \ldots, \nu^{i-1}, \nu^{i+1}, \ldots, \nu^N\right)^\intercal$, as well as $\left(\nu^i; \boldsymbol{\nu}^{-i}\right) := \boldsymbol{\nu}$.

Then, we focus on the class of prices $\mathfrak{P}$ comprising the mappings $\boldsymbol{\nu} \in \mathcal{A} \mapsto p(\boldsymbol{\nu}) \in \mathcal{A}_0$ satisfying the subsequent conditions:

(C1) For each $t \in [0, T]$, $i \in \mathcal{N}$ and $\boldsymbol{\nu} \in \mathcal{A}$, there exists a process $\left\{\xi_{t,u}^{i,N}(\boldsymbol{\nu})\right\}_{0 \leqslant u \leqslant t} \in \mathcal{A}_i$ such that

$$\langle D_i p_t(\boldsymbol{\nu}), w^i \rangle := \lim_{\epsilon \to 0} \frac{p_t(\nu^i + \epsilon w^i; \boldsymbol{\nu}^{-i}) - p_t(\nu^i; \boldsymbol{\nu}^{-i})}{\epsilon} = \mathbb{E}\left[\left. \int_0^t \xi_{t,u}^{i,N}(\boldsymbol{\nu}) w_u^i\, du \right| \mathcal{F}_t^0\right] \qquad (w^i \in \mathcal{A}_i),$$

the above limit holding strongly in the $L^2(\Omega)$−topology.

(C2) For each $t \in [0, T]$, $i \in \mathcal{N}$ and $\boldsymbol{\nu} \in \mathcal{A}$, we have $\xi_{T,t}^{i,N}(\boldsymbol{\nu}) \in L^2(\Omega, \mathcal{F}_T^0)$.

Due to the structure of the influence of an individual trading rate on the asset price, we call $\xi_{t,u}^{i,N}$ the microstructural kernel impact of player i on the asset price.

Next, we provide an example of prices that belong to the class $\mathfrak{P}$, whence it is non-empty.

Example 3.1. *If p is given by*

$$p_t(\boldsymbol{\nu}) = \widetilde{p}_t + \alpha \mathbb{E}\left[\left. \frac{1}{N} \sum_{i=1}^N q_t^i \right| \mathcal{F}_t^0\right], \tag{3.1}$$

for some $\alpha > 0$, and some process $\{\widetilde{p}_t\}_{0 \leqslant t \leqslant T} \in \mathcal{A}_0$, then $p \in \mathfrak{P}$. In this case, we have $\xi_{t,u}^{i,N}(\boldsymbol{\nu}) \equiv \alpha/N$.

Note that Example 3.1 corresponds to the case in which the reference price is linearly affected by the traders' turnover rates – it is a generalized formulation of a linear permanent price impact.

We elaborate further discussions on $\mathfrak{P}$ in Appendix C.

The traders. Let us fix $p \in \mathfrak{P}$. For each $i \in \mathcal{N}$, the trader i controls her turnover rate $\{\nu_t^i\}_{0 \leqslant t \leqslant T}$, selecting it out of the set $\mathcal{A}_i$. Consequently, her inventory process, which we denote by $\{q_t^i\}_{0 \leqslant t \leqslant T}$, has the dynamics

$$\begin{cases} dq_t^i = \nu_t^i dt + \sigma^i dW_t^i, \\ q_0^i \text{ given,} \end{cases} \tag{3.2}$$

where $\sigma^i \geq 0$ for each $i \in \mathcal{N}$. With a similar construction as in (2.2), we stipulate that her cash process $\{c_t^i\}_{0 \leqslant t \leqslant T}$ evolves according to

$$\begin{cases} dc_t^i = -\left[p_t(\boldsymbol{\nu}) + \kappa \nu_t^i\right] \nu_t^i\, dt, \\ c_0^i \text{ given.} \end{cases} \tag{3.3}$$

Furthermore, by denoting the wealth process of this trader by $\{w_t^i := c_t^i + q_t^i p_t(\boldsymbol{\nu})\}_{0 \leqslant t \leqslant T}$, we have, as in (2.3),

$$w_t^i = c_0^i - \int_0^t \left[p_u(\boldsymbol{\nu}) + \kappa \nu_u^i\right] \nu_u^i du + q_t^i p_t(\boldsymbol{\nu}), \tag{3.4}$$

for each $t \in [0, T]$.

The performance criteria. Following a similar reasoning as in the construction of (2.4), we build the performance criteria for player i. Assuming agent i uses strategy $\nu^i \in \mathcal{A}_i$, and that the remaining ones trade with a profile $\boldsymbol{\nu}^{-i} \in \mathcal{A}_{-i}$, we define the payoff $J_i(\nu^i; \boldsymbol{\nu}^{-i})$ of player i as

$$
\begin{aligned}
J^i(\nu^i; \boldsymbol{\nu}^{-i}) &:= \mathbb{E}\left[w_T^i - c_0^i - A(q_T^i)^2 - \phi \int_0^T (q_u^i)^2 du \right] \\
&= \mathbb{E}\left[-\int_0^T \left[\kappa(\nu_u^i)^2 + \phi(q_u^i)^2 + \nu_u^i p_u(\boldsymbol{\nu}) \right] du + q_T^i(p_T - Aq_T^i) \right].
\end{aligned}
\tag{3.5}
$$

3.2. Competitive equilibria

In this section, we seek a definition of competitive equilibria for the N−person game analogous to the one we made for the MFG, i.e., Definition 2.1. Regarding the optimizing behavior of the traders in the population, it is natural to expect that they will accommodate to a Nash equilibrium, since we want them to be competing and rational. A precise definition, in the present context, is the content of the subsequent definition.

Definition 3.2. *We say that an admissible strategy profile $\boldsymbol{\nu}^* = (\nu^{*1}, \cdots, \nu^{*N})^\mathsf{T} \in \mathcal{A}$ is a Nash equilibrium for the N−person game if, for each $i \in \mathcal{N}$ and $\nu^i \in \mathcal{A}_i$,*

$$
J^i(\boldsymbol{\nu}^*) \geqslant J^i\left(\nu^i; \boldsymbol{\nu}^{*-i}\right)
\tag{3.6}
$$

The next element we seek is a counterpart of the market clearing condition (2.6) to the present setting. Thus, we consider a stochastic process $\Lambda = \{\Lambda_t\}_{0 \leqslant t \leqslant T}$ representing the market supply rate. In this case, if player $i \in \mathcal{N}$ considers strategy $\nu^i \in \mathcal{A}_i$, the following market clearing condition must be valid:

$$
\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N \nu_t^i \,\Big|\, \mathcal{F}_t^0 \right] + \Lambda_t = 0 \qquad (0 \leqslant t \leqslant T).
\tag{3.7}
$$

We are ready to propose a definition of a competitive equilibrium for the N−person game.

Definition 3.3. *A triplet $(p, \boldsymbol{\nu}^*, \Lambda) \in \mathfrak{P} \times \mathcal{A} \times \mathcal{A}_0$ is a competitive equilibrium under the performance criteria $\{J^i\}_{i=1}^N$ if:*

(i) Given p, the process $\boldsymbol{\nu}^*$ is a Nash equilibrium under the criteria $\{J^i\}_{i=1}^N$;

(ii) The processes $\boldsymbol{\nu}^*$ and Λ together solve (3.7).

3.3. Some properties of the competitive equilibria

Using variational techniques, as in Casgrain and Jaimungal (2020) and Evangelista and Thamsten (2020), we can derive the subsequent necessary conditions for a Nash equilibrium. For its proof, see Appendix B.

Proposition 3.4. *Assume that, for each $i \in \mathcal{N}$ and $\boldsymbol{\nu}^{-i}$, the functional $\nu^i \mapsto J^i(\nu^i; \boldsymbol{\nu}^{-i})$ admits a unique maximizer. Then, a Nash equilibrium $\boldsymbol{\nu}^*$ for the criteria $\{J^1, ..., J^N\}$ must solve the FBSDE*

$$
\begin{cases}
q_t^{*i} = q^i + \int_0^t \nu_u^{*i}\, du + \sigma^i W_t^i, \\
2\kappa\, \nu_t^{*i} = -p_t(\nu^{*i}; \boldsymbol{\nu}^{*-i}) + 2\phi \int_0^t q_u^{*i}\, du \\
\qquad\quad + \mathbb{E}\left[\xi_{T,t}^{i,N}(\nu^{*i}; \boldsymbol{\nu}^{*-i}) q_T^{*i} - \int_t^T \xi_{u,t}^{i,N}(\nu^{*i}; \boldsymbol{\nu}^{*-i}) \nu_u^{*i}\, du \,\Big|\, \mathcal{F}_t^{0i} \right] + M_t^{*i},
\end{cases}
\tag{3.8}
$$

for suitable square-integrable martingales $\{M_t^{*i}, \mathcal{F}_t^{0i}\}_{0 \leqslant t \leqslant T}$ which satisfy

$$
M_T^{*i} = -2Aq_T^{*i} + p_T(\nu^{*i}; \boldsymbol{\nu}^{*-i}) - 2\phi \int_0^T q_t^{*i}\, dt,
$$

$\mathbb{P}$−*almost surely.*

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Remark 3.5. *The common information shared by all the players appears, for instance, into the martingale M^{*i}. Under the Brownian assumption, one could use the Martingale Representation Theorem to write this term as a combination of stochastic integrals with respect to W^i and W^0.*

It is illustrative to consider Proposition 3.4 in the case of permanent price impact. We comment on it in the subsequent example.

Example 3.6. *In the linear permanent price impact case, i.e., $p_t(\boldsymbol{\nu}) = \widetilde{p}_t + \frac{\alpha}{N}\sum_{i=1}^{N} q_t^i$, with $\alpha > 0$, $\widetilde{p} \in \mathcal{U}_0$ being a semimartingale, and $\widetilde{p}_0, \widetilde{p}_T \in L^2(\Omega)$, we have*

$$J^i(\boldsymbol{\nu}) = \mathbb{E}\left[\int_0^T \left\{\left[-\kappa\left(\nu_t^i\right)^2 - \phi\left(q_t^i\right)^2 + \frac{b}{N}q_t^i\sum_{j\neq i}\nu_t^j\right]dt + q_t^i\,d\widetilde{p}_t\right\} - \left(A - \frac{\alpha}{2N}\right)\left(q_T^i\right)^2 + q^i\widetilde{p}_0\right].$$

Thus, assuming $\kappa > 0$ and $A > \alpha/(2N)$, we conclude that the functional $\nu^i \in \mathcal{A}_i \mapsto J_i(\nu^i; \boldsymbol{\nu}^{-i}) \in \mathbb{R}$ is strictly concave and coercive, for each $i \in \mathcal{N}$ and $\boldsymbol{\nu}^{-i} \in \mathcal{A}_{-i}$; hence, it admits a unique maximizer. Conversely, employing the techniques we used in Evangelista and Thamsten (2020) (with suitable integrability assumptions on $\widetilde{p}$), we can prove that (3.8) admits a unique strong solution.

3.4. On the formed price in the finite population game

In Proposition 3.4, we have given necessary conditions for the Nash equilibrium resulting from a price process $p \in \mathfrak{P}$. In Example 3.6 above, we provided a class of prices for which we can, in fact, prove sufficient conditions, thus characterizing Nash equilibria in this circumstance. The next result we obtain is the main one in this section. We use this necessary conditions to establish a formula analogous to (2.18) in the current finite population framework in a competitive equilibrium. Some additional elements will appear due to the microstructural impact that we allow in prices, and the consequence of asymmetry in the information that results from the finite population of traders. We turn to the investigation of this question in what follows.

Theorem 3.7. *Let us consider a competitive equilibrium $(p^N, \boldsymbol{\nu}^*, \Lambda^N)$ for the $N-$person game. We assume that there exists a mapping $\Lambda \mapsto \left\{\xi_{t,u}^N(\Lambda)\right\}_{0\leqslant u\leqslant t\leqslant T}$[7] such that, for every $i \in \mathcal{N}$ and $\boldsymbol{\nu} \in \mathcal{A}$,*

$$\xi_{t,u}^{i,N}(\boldsymbol{\nu}) = \xi_{t,u}^N\left(-\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \nu_\cdot^i\middle|\mathcal{F}_\cdot^0\right]\right) \qquad (0 \leqslant u \leqslant t \leqslant T). \tag{3.9}$$

Then, the price $\{p_t^N = p_t(\boldsymbol{\nu}^*)\}_{0\leqslant t\leqslant T}$ is formed as

$$p_t^N = p_0^N + 2\kappa\left(\Lambda_t^N - \Lambda_0^N\right) + 2\phi\int_0^t E_{0,u}^N\,du - 2\phi\int_0^t (t-u)\Lambda_u^N\,du + \int_0^t \epsilon_u^N\,du + F_t^N + M_t^N, \tag{3.10}$$

where the process $\left\{F_t^N\right\}_{0\leqslant t\leqslant T}$ is given by

$$\begin{cases} F_t^N := f_t^N - f_0^N, \\ f_t^N := \mathbb{E}\left[\xi_{T,t}^N\left(\Lambda^N\right)\left(E_{0,T}^N - \int_0^T \Lambda_u^N\,du + \epsilon_T^N\right) + \int_t^T \xi_{u,t}^N\left(\Lambda^N\right)\Lambda_u^N\,du\middle|\mathcal{F}_t^0\right] & (0 \leqslant t \leqslant T), \end{cases} \tag{3.11}$$

whereas $\left\{\epsilon_t^N, \mathcal{F}_t^0\right\}_{0\leqslant t\leqslant T}$, $\left\{E_{0,t}^N, \mathcal{F}_t^0\right\}_{0\leqslant t\leqslant T}$ and $\left\{M_t^N, \mathcal{F}_t^0\right\}_{0\leqslant t\leqslant T}$ are square-integrable martingales given by

$$\epsilon_t^N := \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \sigma^i W_t^i\middle|\mathcal{F}_t^0\right] + \frac{1}{N}\sum_{i=1}^N \int_0^t \left\{\mathbb{E}\left[\nu_s^{*i}|\mathcal{F}_t^0\right] - \mathbb{E}\left[\nu_s^{*i}|\mathcal{F}_s^0\right]\right\}\,ds, \tag{3.12}$$

---

[7]Notice the slight abuse of notation since this process $\xi_{t,u}^N(\Lambda)$, which does not depend on i, is not the process $\left\{\xi_{t,u}^{i,N}(\boldsymbol{\nu})\right\}_{0\leqslant u\leqslant t}$ from condition (C1) above.

$$E_{0,t}^N := \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N q_0^i \,\middle|\, \mathcal{F}_t^0\right], \tag{3.13}$$

and

$$M_t^N := \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \left(M_t^{*i} - M_0^{*i}\right) \,\middle|\, \mathcal{F}_t^0\right] + \int_0^t \left\{\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N q_s^{*i} \,\middle|\, \mathcal{F}_t^0\right] - \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N q_s^{*i} \,\middle|\, \mathcal{F}_s^0\right]\right\} ds, \tag{3.14}$$

for the martingales $\left\{M_t^{*i}, \mathcal{F}_t^{0i}\right\}_{0\leqslant t\leqslant T}$, $i \in \mathcal{N}$, *figuring in the FBSDE (3.8).*

Proof. Throughout the present proof, we fix a competitive equilibrium $\left(p^N, \boldsymbol{\nu}^*, \Lambda^N\right)$. In particular, (3.7) holds, whence assumption (3.9) implies $\xi_{t,u}^{i,N}(\boldsymbol{\nu}^*) = \xi_{t,u}^N(\Lambda^N)$, for $0 \leqslant u \leqslant t \leqslant T$ and $i \in \mathcal{N}$.

From the inventory equation presented in (3.8), alongside (3.7), we have

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N q_t^{*i} \,\middle|\, \mathcal{F}_t^0\right] = E_{0,t}^N - \int_0^t \Lambda_u^N \, du + \epsilon_t^N. \tag{3.15}$$

Employing (3.7) and (3.15) in the average over $i \in \mathcal{N}$ of the BSDEs of (3.8) projected on $\mathbb{F}^0$, we deduce that

$$
\begin{aligned}
-2\kappa\Lambda_t^N &= -p_t^N + 2\phi\int_0^t\left(E_{0,\tau}^N - \int_0^\tau \Lambda_u^N\,du + \epsilon_\tau^N\right)d\tau + \widetilde{f}_t^N + M_t^N + \frac{1}{N}\sum_{i=1}^N M_0^{*i} \\
&= -p_t^N + 2\phi\left(\int_0^t E_{0,u}^N\,du - \int_0^t (t-u)\Lambda_u^N\,du + \int_0^t \epsilon_u^N\,du\right) + \widetilde{f}_t^N + M_t^N + \frac{1}{N}\sum_{i=1}^N M_0^{*i},
\end{aligned}
\tag{3.16}
$$

where we have written

$$\widetilde{f}_t^N = \mathbb{E}\left[\xi_{T,t}^N(\Lambda_N)\frac{1}{N}\sum_{i=1}^N q_T^i - \int_t^T \xi_{u,t}^N(\Lambda_N)\frac{1}{N}\sum_{i=1}^N \nu_u^{*i}\,du \,\middle|\, \mathcal{F}_t^0\right].$$

We claim $\widetilde{f}^N \equiv f^N$, where we describe the latter in (3.11). In effect, noticing that our assumption (3.9) implies that the random variable $\xi_{s,t}^N(\Lambda^N)$ is $\mathcal{F}_t^0$−measurable, for $t \leqslant s \leqslant T$, we use the conditional Fubini's Theorem, alongside the tower property of conditional expectations, to deduce the identities

$$
\begin{aligned}
\widetilde{f}_t^N &= \xi_{T,t}^N(\Lambda_N)\,\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N q_T^i \,\middle|\, \mathcal{F}_t^0\right] - \int_t^T \xi_{u,t}^N(\Lambda_N)\,\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \nu_u^{*i} \,\middle|\, \mathcal{F}_t^0\right]du \\
&= \xi_{T,t}^N(\Lambda_N)\,\mathbb{E}\left[\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N q_T^i \,\middle|\, \mathcal{F}_T^0\right]\middle|\, \mathcal{F}_t^0\right] - \int_t^T \xi_{u,t}^N(\Lambda_N)\,\mathbb{E}\left[\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \nu_u^{*i} \,\middle|\, \mathcal{F}_u^0\right]\middle|\, \mathcal{F}_t^0\right]du \\
&= \xi_{T,t}^N(\Lambda_N)\,\mathbb{E}\left[E_{0,T}^N - \int_0^T \Lambda_u^N\,du + \epsilon_T^N \,\middle|\, \mathcal{F}_t^0\right] + \int_t^T \xi_{u,t}^N(\Lambda_N)\,\mathbb{E}\left[\Lambda_u^N \,\middle|\, \mathcal{F}_t^0\right]du = f_t^N,
\end{aligned}
$$

as we wanted to show. Rearranging (3.16), we obtain

$$p_t^N = 2\kappa\Lambda_t^N + 2\phi\left(\int_0^t E_{0,u}^N\,du - \int_0^t (t-u)\Lambda_u^N\,du + \int_0^t \epsilon_u^N\,du\right) + f_t^N + M_t^N + \frac{1}{N}\sum_{i=1}^N M_0^{*i},$$

whence

$$p_t^N - p_0^N = 2\kappa\left(\Lambda_t^N - \Lambda_0^N\right) + 2\phi\left(\int_0^t E_{0,u}^N\,du - \int_0^t (t-u)\Lambda_u^N\,du + \int_0^t \epsilon_u^N\,du\right) + F_t^N + M_t^N.$$

The latter equation is clearly equivalent to (3.10). This finishes the proof. $\qquad\square$

Prior to our discussion of limiting consequences of Theorem 3.7 as $N \to +\infty$, we explore some particular cases of this result in the two subsequent corollaries. First, we consider the case in which the microstructural impact kernel, defined by $\xi_{t,u}^N$, does not depend on u. Moreover, this impact affects the price only through the aggregation of the N players' turnover rate. Next, we further assume that $\xi_t^N = \alpha/N$ and deduce that the formed price with this microstructural impact kernel leads to the permanent price impact model.

Corollary 3.8. *Let us maintain all the notations of the statement of Theorem 3.7. If $\xi_{t,u}^N(\Lambda^N)$ is independent of u, i.e., $\xi_{t,u}^N(\Lambda^N) = \xi_t^N(\Lambda^N)$, $0 \leqslant u \leqslant t$, then we can write p^N in (3.10) as follows:*

$$p_t^N = p_0^N + 2\phi \int_0^t E_{0,u}^N \, du - 2\phi \int_0^t (t-u)\Lambda_u^N \, du + 2\kappa \left(\Lambda_t^N - \Lambda_0^N \right) - \int_0^t \xi_u^N \left(\Lambda^N \right) \Lambda_u^N \, du + \int_0^t \epsilon_u^N \, du + \widetilde{M}_t^N, \quad (3.17)$$

for a square-integrable martingale $\left\{ \widetilde{M}_t^N, \mathcal{F}_t^0 \right\}_{0 \leqslant t \leqslant T}$.

Proof. Under the current hypothesis, we derive from (3.11) that

$$f_t^N = - \int_0^t \xi_u^N \left(\Lambda^N \right) \Lambda_u^N \, du + \widetilde{M}_t,$$

where $\left\{ \widetilde{M}_t, \mathcal{F}_t^0 \right\}_t$ is a square-integrable martingale, cf. (3.11). We can incorporate $\left\{ \widetilde{M}_t - \widetilde{M}_0, \mathcal{F}_t^0 \right\}_t$ into $\left\{ M_t^N, \mathcal{F}_t^0 \right\}_t$ to obtain the representation we asserted for p^N in (3.17). $\square$

Remark 3.9. *Notice that using (3.17), we can compute the quadratic variation of p^N in terms of the quadratic variations of Λ^N and $\widetilde{M}^N$. Indeed:*

$$\langle p^N \rangle_t = 4\kappa^2 \langle \Lambda^N \rangle_t + \langle \widetilde{M}^N \rangle_t + 2\kappa \langle \Lambda^N, \widetilde{M}^N \rangle_t.$$

One should compare this result with the one from MFG shown in Remark 2.7. Notice that the finite player setting materializes into the quadratic variation of $\widetilde{M}^N$ and the co-variation of $\widetilde{M}^N$ and Λ^N.

To state the second relevant particular case, we recall the following. Reference Evangelista and Thamsten (2020) contains the proof that in a linear price impacts model, with a slope of permanent price impact α, no inventory uncertainty ($\sigma^i = 0$), and with all the other parameters having the same meaning as in our current setting, the average of the inventories E_N of traders is deterministic and satisfies the second-order ODE:

$$\begin{cases} 2\kappa E_N'' + \alpha \left(1 - \frac{1}{N} \right) E_N' - 2\phi E_N = 0, \\ E_N(0) = E_{N,0} \text{ and } \kappa E_N'(T) + A E_N(T) = 0, \end{cases} \quad (3.18)$$

where $E_{N,0}$ is given. Here, E_N is defined as

$$E_N(t) = \frac{1}{N} \sum_{i=1}^N q_t^{*i},$$

where q^{*i} is defined as in Proposition 3.4. This case corresponds to the finite population counterpart of the MFG model introduced by Cardaliaguet and Lehalle (2018), in which the mean-field interactions are through the controls in a permanent (linear) price impact model. We can regard the next corollary as a sanity check.

Corollary 3.10. *Let us consider that all conditions we stipulated in Corollary 3.8 hold, and let us maintain its notation. If, moreover, $\xi_t^N \equiv \alpha/N$, where α is a non-negative constant, $\Lambda_t^N \equiv -E_N'(t)$, and $\sigma^i \equiv 0$, for all i, then*

$$p_t^N = p_0^N + \alpha \left(E_N(t) - E_N(0) \right) + \widetilde{M}_t^N.$$

Proof. Upon the use of (3.18), we notice that the current assumptions imply

$$p_t^N = p_0^N + 2\kappa \left(E_N'(0) - E_N'(t)\right) + 2\phi \int_0^t E_N(u)\, du + \frac{\alpha}{N} \left(E_N(t) - E_N(0)\right) + \widetilde{M}_t^N$$

$$= p_0^N + \alpha \left(1 - \frac{1}{N}\right) \left(E_N(t) - E_N(0)\right) + \frac{\alpha}{N} \left(E_N(t) - E_N(0)\right) + \int_0^t \epsilon_u\, du + \widetilde{M}_t^N$$

$$= p_0^N + \alpha \left(E_N(t) - E_N(0)\right) + \widetilde{M}_t^N.$$

□

3.5. A convergence result

The last result we present in this section comprises the conclusion of our initial goal. Namely, we provide conditions guaranteeing that the formed price in a competitive equilibrium (for the finite population game) converges to its MFG counterpart when N goes to infinity.

Theorem 3.11. *Let us consider a sequence $\left\{\Lambda^N\right\}_N$ of elements of $\mathcal{A}_0$. We keep all notations of the statement of Theorem 3.7. We suppose the following:*

(i) The sequence of initial prices $\left\{p_0^N\right\}_N$ satisfies $\mathbb{E}\left[p_0^N\right] \xrightarrow{N\to\infty} p_0$, for some $p_0 \in \mathbb{R}$;

(ii) For some $E_0 \in \mathbb{R}$, we have $\mathbb{E}\left[E_{0,0}^N\right] \xrightarrow{N\to\infty} E_0$;

(iii) The sequence $\left\{\Lambda^N\right\}_N$ is such that $\overline{\Lambda}(t) := \sup_{N \geqslant 1}\left|\mathbb{E}\left[\Lambda_t^N\right]\right|$ satisfies $\int_0^T \overline{\Lambda}(t)\, dt < \infty$;

(iv) For some $\Lambda : [0,T] \to \mathbb{R}$, we have $\mathbb{E}\left[\Lambda_t^N\right] \xrightarrow{N\to\infty} \Lambda(t)$, for each $t \in [0,T]$;

(v) The sequence $\left\{F^N\right\}_N$ (cf. (3.11)) satisfies $\mathbb{E}\left[F^N\right] \xrightarrow{N\to\infty} 0$.

Then, it follows that

$$\lim_{N\to\infty} \mathbb{E}\left[p_t^N\right] = p(t), \tag{3.19}$$

where p is the formed price in the MFG with supply rate function Λ, as in (2.18).

Proof. Taking expectations in (3.10), we derive

$$\mathbb{E}\left[p_t^N\right] = \mathbb{E}\left[p_0^N\right] + 2\kappa \left(\mathbb{E}\left[\Lambda_t^N\right] - \mathbb{E}\left[\Lambda_0^N\right]\right) + 2\phi \mathbb{E}\left[E_{0,0}^N\right] t - 2\phi \int_0^t (t - u)\mathbb{E}\left[\Lambda_u^N\right]\, du.$$

Using conditions $(i) - (v)$, with the aid of the Dominated Convergence Theorem, we can pass the above identity to the limit as $N \to \infty$ to conclude that

$$\mathbb{E}\left[p_t^N\right] \xrightarrow{N\to\infty} p_0 + 2\kappa \left(\Lambda(t) - \Lambda(0)\right) + 2\phi E_0 t - 2\phi \int_0^t (t - u)\Lambda(t)\, du,$$

which is readily equivalent to (3.19). □

Remark 3.12. *Out of the conditions we stipulated in Theorem 3.11, the one that probably deserves more attention is (v). We observe from (3.11) that, if $\xi_{t,u}^N\left(\Lambda^N\right) = O\left(\frac{1}{N}\right)$, as $N \to \infty$, then $(ii) - (iv)$, together with an appropriate integrability condition on $\left\{\epsilon^N\right\}_N$, ensure that this condition does hold. Demanding this of ξ^N is natural. Indeed, it follows if we assume $p^N(\boldsymbol{\nu}) = \widetilde{p}\left(-\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N \nu_\cdot^j \Big| \mathcal{F}_\cdot^0\right]\right)$, for an adequate mapping $\Lambda \mapsto \widetilde{p}(\Lambda)$, cf. Example 3.1.*

The general consideration about (v) is that we must ask that the microstructural impact coefficient ξ^N vanish, in an adequate sense, as the population size grows. We can relate it with opinion formation models in the economic literature, e.g., Golub and Jackson (2010). There, the authors characterize the phenomenon of the wisdom of the crowds, in which all opinions in a large population converge to the truth, as being equivalent to the weight of the most influential agent vanishing as the population size grows. Our result seems in line with the intuitive idea that a vanishing microstructural influence leads, in the limit, to consistency (in a certain sense) to the MFG price.

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Table 1: The data we use in this work is from November 2014 for ten NASDAQ listed stocks. It contains all trades and snapshots of the LOB for every 100ms of the trading day.

Stocks	Midprice	Spread	ADV
MU	33.4 (0.823)	0.01 (0.001)	3,325,174 (972,993)
FB	74.68 (0.893)	0.01 (0.002)	5,616,503 (1,381,900)
PCAR	66.57 (0.64)	0.02 (0.009)	373,722 (122,522)
AMZN	317.18 (13.903)	0.11 (0.049)	706,172 (214,172)
EBAY	54.11 (0.664)	0.01 (0.002)	1,224,661 (262,809)
SMH	52.52 (1.011)	0.01 (0.005)	263,909 (95,574)
INTC	34.5 (1.104)	0.01 (0.001)	4,717,047 (2,069,603)
VOD	34.57 (1.196)	0.01 (0.002)	850,241 (588,596)
GOOG	542.87 (6.036)	0.17 (0.08)	291,492 (73,852)
MSFT	48.4 (0.704)	0.01 (0.001)	5,190,672 (971,123)

4. Empirical assessments

In this section we assess the performance of the formed price formulas under the MFG and finite population settings using real high-frequency data. We first specify the data and the order flow measures we consider. Next, we write down the regressions of price against explanatory variables that arise from the theory we developed in the previous sections. We finally show the numerical results of these experiments.

4.1. The data

The data we use in this work can be found at http://sebastian.statistics.utoronto.ca/books/algo-and-hf-trading/data/. It is from November 2014 for the following tickers: AMZN, EBAY, FB, GOOG, INTC, MSFT, MU, PCAR, SMH, VOD. This data contains all trades and snapshots of the LOB for every 100ms of the trading day. Table 1 displays the mean and standard deviation in parenthesis for each stock, of the midprice, spread, and average daily volume (ADV).

4.2. Order flow measures

From the market clearing condition (2.6), we see that Λ should be taken as the additive opposite of a measure of order flow. Firstly, we consider the Trade Imbalance (TI), see Cont et al. (2014), as such a measure. We divide the trading day $[0, T]$ in 12 corresponding intervals of 30 minutes $[0 = T_0, T_1], \ldots, [T_{11}, T_{12} = 360]$, further dividing such slice into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. We then define the time series $\mathrm{TI}^i = \left\{\mathrm{TI}_k^i\right\}_{k=0}^{179}$ on $[T_i, T_{i+1}]$ as

$$\mathrm{TI}_k^i = \sum_{n=N_{i,k}+1}^{N_{i,k+1}} \left\{\mu_n^+ - \mu_n^-\right\}, \tag{4.1}$$

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where $N_{i,k}$ denotes the number of events that have occurred on $[T_i, t_{i,k}]$, and μ_n^+ (respectively, μ_n^-) is the volume (possibly zero) of buying (respectively, selling) market orders that are traded within $]t_{i,n-1}, t_{i,n}]$, for each $n \geqslant 1$.

In order to take the LOB queue sizes into account, we also utilize a metric called Order Flow Imbalance (OFI), introduced in (Cont et al., 2014, Subsection 2.1). The goal in using this is to consider information from the LOB first queues, since it comprises more information. We proceed to provide the definition of OFI, referring to their paper for an in-depth discussion on its design. Let us represent by $\mathfrak{p}_n^b$ and q_n^b (respectively, $\mathfrak{p}_n^a$ and q_n^a) the bid (respectively, ask) price and volume (at-the-touch), respectively, resulting from the $n-$th set of events (i.e., those occurring within the window $]t_{i,n-1}, t_{i,n}]$). We form the quantity

$$e_n := I_{\left\{\mathfrak{p}_n^b \geqslant \mathfrak{p}_{n-1}^b\right\}} q_n^b - I_{\left\{\mathfrak{p}_n^b \leqslant \mathfrak{p}_{n-1}^b\right\}} q_{n-1}^b - I_{\left\{\mathfrak{p}_n^a \leqslant \mathfrak{p}_{n-1}^a\right\}} q_n^a + I_{\left\{\mathfrak{p}_n^a \geqslant \mathfrak{p}_{n-1}^a\right\}} q_{n-1}^a, \tag{4.2}$$

and we define

$$\mathrm{OFI}_k := \sum_{n=N_{i,k}+1}^{N_{i,k+1}} e_n. \tag{4.3}$$

4.3. Regressions

Considering the formed price under the MFG setting (2.18), we undertake the subsequent regression over each time window $[T_i, T_{i+1}]$:

$$\mathfrak{p}_k^i = a_{i,0} S_k^{i,0} + a_{i,1} S_k^{i,1} + a_{i,2} S_k^{i,2} + a_{i,3} S_k^{i,3} + \epsilon_k^i, \tag{4.4}$$

where we denote the error term by ϵ_k^i, the observed midprice at time by $\mathfrak{p}_k^i$, both at time $t_{i,k}$, and the time series covariates:

$$\begin{cases} S^{i,0} := \left\{S_k^{i,0} := 1\right\}_{k=0}^{179}, \; S^{i,1} := \left\{S_k^{i,1} := t_{i,k}\right\}_{k=0}^{179}, \\ S^{i,2} := \left\{S_k^{i,2} := \int_{T_i}^{t_{i,k}} \Lambda(u)(t_{i,k} - u)\, du\right\}_{k=0}^{179}, \; S^{i,3} := \left\{S_k^{i,3} := \Lambda(t_{i,k})\right\}_{k=0}^{179}. \end{cases} \tag{4.5}$$

In $S^{i,2}$ we use the trapezoidal rule to approximate the integral. We assume parameters are constant in each of the time windows.

Furthermore, under the finite population setting, in particular considering the context and notation of Corollary 3.8, we define the variable

$$I_{\mathrm{micro}}(t) := \int_0^t \xi_u^N \left(\Lambda_u^N\right) \Lambda_u^N\, du.$$

We notice that I_{micro} figures in the price formation formula (3.17) for the finite population game. We call it I_{micro} since it appears due to market friction due to the microstructural impact suffered by an individual trader. If we assume, in particular, that this impact $\xi_u^N \equiv \xi^N$ is constant, then

$$I_{\mathrm{micro}}(t) = \xi^N \int_0^t \Lambda_u^N\, du.$$

In the case of a decentralized game, it is reasonable to expect that $\xi^N = O(1/N)$, as we previously discussed in Remark 3.12. However, if there is some persistence in the impact of some "special" traders, as in a case when there is a hierarchy among the population, cf. (Evangelista and Thamsten, 2020, Section 5), we could expect that such term would play a relevant role in the price formation process.

Inspired by this discussion, we consider the regression (4.4) (over the time slices $[T_i, T_{i+1}]$), adding the following explanatory variable:

$$S^{i,4} := \left\{S_k^{i,4} := \int_{T_i}^{t_{i,k}} \Lambda(u)\, du\right\}_{k=0}^{179}, \tag{4.6}$$

Table 2: Average R^2 results of GOOG stock on November 3rd, 2014.

Stock	R^2_{TI}	R^2_{OFI}	$R^2_{TI_{micro}}$	$R^2_{OFI_{micro}}$	$R^2_{TI_{bench}}$	$R^2_{OFI_{bench}}$
GOOG	0.505631	0.555541	0.703997	0.803111	0.17395	0.459976

for each window $[T_i, T_{i+1}]$. In this way, the regression we investigate in the current setting is

$$\mathfrak{p}_k^i = a_{i,0} S_k^{i,0} + a_{i,1} S_k^{i,1} + a_{i,2} S_k^{i,2} + a_{i,3} S_k^{i,3} + a_{i,0} S_k^{i,4} + \eta_k^i, \qquad (4.7)$$

where we denote the error term by η_k^i at time $t_{i,k}$.

4.4. Results

We first consider that the order flow measure Λ is given by the trade imbalance with $\Lambda(t_{i,k}) = -\mathrm{TI}_k^i$. As a benchmark, we take a model inspired by Kyle (1985), viz.,

$$\Delta\mathfrak{p}_k^i := \mathfrak{p}_k^i - \mathfrak{p}_{k-1}^i = b_{i,0} + b_{i,1}\mathrm{TI}_k^i + \widetilde{\epsilon}_k^i. \qquad (4.8)$$

Thus, in this benchmark, we simply assume that a price movement must correspond linearly to the market trade imbalance.

In Figure 1, we provide an example of the performance of our MFG model with parameters estimated as we described above, i.e., via the regression (4.5), together with the benchmark model (4.8). In Table 2, we present the average over all windows of of November, 3rd 2014 for the GOOG stock. In Figure 1, in green we have the formed price of the MFG using the TI order flow. The average R^2 on this trading day is given in the first column of Table 2 (cf. column R^2_{TI}). We also present the average R^2 over the whole period considered for all the stocks considered in this experiment in Table A.1. We notice that the MFG model beats the benchmark model systematically over all stocks we considered. Moreover, we would point out that the R^2 achieved by the benchmark is comparable to the one observed in the literature; see Cont et al. (2014), for instance.

To verify the statistical significance of our regression study, we display the Augmented Dickey–Fuller (ADF) test (cf. Fuller (1976)) for the residuals $\mathfrak{p}_k^i - a_{i,0} S_k^{i,0} + a_{i,1} S_k^{i,1} + a_{i,2} S_k^{i,2} + a_{i,3} S_k^{i,3} = \epsilon_k^i$. The ADF statistic used in the test is a negative number. The more negative it is, the stronger the rejection of the hypothesis that there is a unit root at some level of confidence. The critical value of (5%) for the sample size of 250 points is -3.43. In Table A.2, the average ADF statistic is given under the column named ADF Resid.. Although the average result is not smaller than -3.43, we see that one standard deviation is enough to cover the critical level. As we will see later in this section, the formed price under the finite population model displays better ADF tests' averages.

In Table A.2, we also showcase the average values over all windows and all days of the parameters we estimate from the regression (4.8). We immediately interpret the parameter a_0 as the average initial price of the corresponding stock sampled at the beginning of each window.
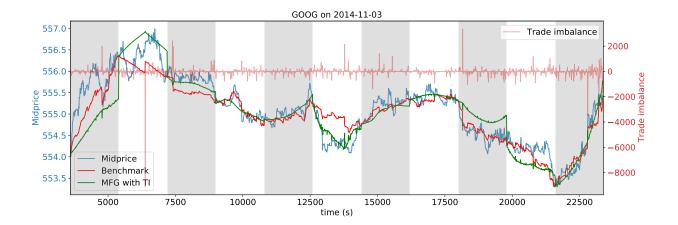
Figure 1: We use the Trade Imbalance, TI (see Cont et al. (2014)), as a measure of order flow (see (4.1)) and the MFG model coefficients of the formed price equation: $\mathfrak{p}_k^i = a_{i,0} S_k^{i,0} + a_{i,1} S_k^{i,1} + a_{i,2} S_k^{i,2} + a_{i,3} S_k^{i,3} + \epsilon_k^i$, where we denote the error term by ϵ_k^i and the observed price by $\mathfrak{p}_k^i$, both at time $t_{i,k}$. Such regression is performed in time intervals of 30 minutes $[T_i, T_{i+1}]$ (marked as gray and white stripes), where model parameters are kept constant. Moreover, we divide each window of 30 minutes into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. This regression corresponds to the MFG model in Theorem 2.6. As a benchmark, we take a model inspired by Kyle (1985), viz., $\Delta \mathfrak{p}_k^i := \mathfrak{p}_k^i - \mathfrak{p}_{k-1}^i = b_{i,0} + b_{i,1} TI_k^i + \tilde{\epsilon}_k^i$. This benchmark was also considered in Cont et al. (2014).

Furthermore, in Figure 2 we present the outcome of considering the regression (4.7) from the finite population model compared to the trade imbalance linear model. We show in Table 2 the average R^2 for the finite population model with the trade imbalance metric (cf. column $R^2_{TI_{micro}}$) on this trading day. For the R^2 over the whole analyzed period and for all the stocks, see Table A.1. We observe that the extra explanatory variable of the finite population game brings relevant information to the problem. Indeed, this novel model not only beats the benchmark considerably buy it also improves the MFG model for all assets. The significance of the novel explanatory variable (4.6) is an indication that there might be a persistent effect of certain players in the market. The average values over all windows and all days of the regression parameters for this model is shown in Table A.3.
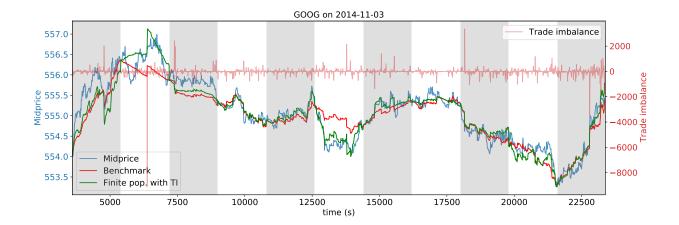
Figure 2: We use the Trade Imbalance, TI (see Cont et al. (2014)), as a measure of order flow (see (4.1)) and compute the finite population game model coefficients of the formed price equation: $\mathfrak{p}^i_k = a_{i,0}S^{i,0}_k + a_{i,1}S^{i,1}_k + a_{i,2}S^{i,2}_k + a_{i,3}S^{i,3}_k + a_{i,4}S^{i,4}_k + \eta^i_k$, where we denote the error term by η^i_k and the observed price by $\mathfrak{p}^i_k$, both at time $t_{i,k}$. Such regression is performed in time intervals of 30 minutes $[T_i, T_{i+1}]$, where model parameters are kept constant. Moreover, we divide each window of 30 minutes into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. This regression corresponds to the finite population price model in Corollary 3.8. As a benchmark, we take a model inspired by Kyle (1985), viz., $\Delta\mathfrak{p}^i_k := \mathfrak{p}^i_k - \mathfrak{p}^i_{k-1} = b_{i,0} + b_{i,1}\mathrm{TI}^i_k + \widetilde{\epsilon}^i_k$. This benchmark was also considered in Cont et al. (2014).

We now consider the order flow measure Λ given by the Order Flow Imbalance with $\Lambda(t_{i,k}) = -\mathrm{OFI}^i_k$. The benchmark we consider to compare our results for this metric is then

$$\Delta\mathfrak{p}^i_k = c_{i,0} + c_{i,1}\mathrm{OFI}^i_k + \bar{\epsilon}^i_k,$$

where the regression is performed over the slice $[T_i, T_{i+1}]$, which was studied in Cont et al. (2014).

The counterpart of Figure 2 in this setting is what we showcase in Figure 3. This plot indicates that using this metric leads to overall better results than the trade imbalance one. This is expected since the OFI metric leads to better results when compared to TI, as noted by Cont et al. (2014). A reason for the improvement can be due to the fact that more information is taken into account in the OFI metric, such as the size of the LOB queues.

From the results of Tables 2 and A.1, we notice that using the OFI metric leads to substantial improvement of the performance of the current model. Moreover, the model we derived also outperforms the benchmark. This highlights the importance of the memory term in the price formation process. However, the memory term does not seem to be the key of the performance boost relative to the pure linear in the OFI metric. Indeed, from Tables 2 and A.1, we see that when we employ $\Lambda = -\mathrm{OFI}$, the average R^2 of the MFG model is worse overall. Thus, the most valuable term from the perspective of the OFI metric seems to be the explanatory variable appearing from the microstructural impact, in the form of the time series (4.6). Finally, the average values over all windows and all days of the regression parameters for this model is shown in Table A.4.
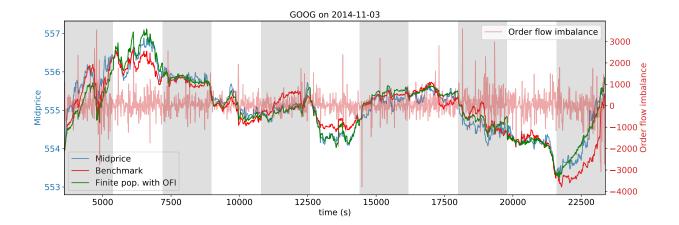
23

Figure 3: We use the Oder Flow Imbalance, OFI (see Cont et al. (2014)), as a measure of order flow (see (4.3)) and compute the finite population game model coefficients of the formed price equation: $\mathfrak{p}_k^i = a_{i,0} S_k^{i,0} + a_{i,1} S_k^{i,1} + a_{i,2} S_k^{i,2} + a_{i,3} S_k^{i,3} + a_{i,0} S_k^{i,4} + \eta_k^i$, where we denote the error term by η_k^i and the observed price by $\mathfrak{p}_k^i$, both at time $t_{i,k}$. Such regression is performed in time intervals of 30 minutes $[T_i, T_{i+1}]$, where model parameters are kept constant. Moreover, we divide each window of 30 minutes into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. This regression corresponds to the finite population game price model in Corollary 3.8. As a benchmark, we take the model introduced in Cont et al. (2014), viz. , $\Delta \mathfrak{p}_k^i = c_{i,0} + c_{i,1} \mathrm{OFI}_k^i + \bar{\epsilon}_k^i$, where OFI is given in (4.3).

In Figure 5 we illustrate the performance of all models we developed: MFG with TI, MFG with OFI, Finite population with TI, and Finite Population with OFI. This figure corresponds to November 3rd, 2014, which is the same day illustrated in Figures 1, 2, and 3.

To better visualize the differences of each formed price model, in Figure 5 we illustrate the squared relative difference to the midprice, i.e.,

$$\left(\frac{p_t^m - p_t}{p_t} \right)^2, \tag{4.9}$$

for $m \in \{$MFG with TI, MFG with OFI, Finite population with TI, and Finite Population with OFI$\}$, where p_t is the midprice at time t. As for the other figures, the regression is performed in time intervals of 30 minutes $[T_i, T_{i+1}]$, where model parameters are kept constant. Moreover, we divide each window of 30 minutes into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. Furthermore, in Table 3 we compute the sum of the quantity in (4.9) for the trading day of November 3rd, 2014, using the time discretization as mentioned above.

In order to show the robustness of our results, we also considered in Appendix D two assets of very different nature than the ones considered so far. Namely, Petrobras ordinary stock and Bitcoin traded on Binance, both quoted in Brazilian real (BRL).
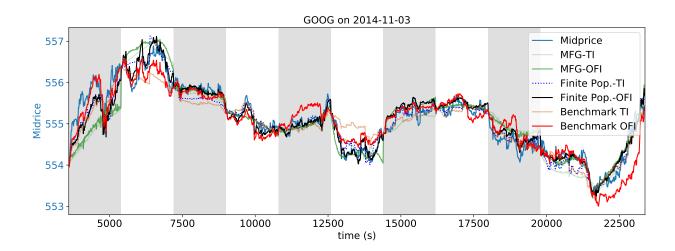
24

Figure 4: Comparison between the midprice and formed price models: MFG with TI, MFG with OFI, Finite population with TI, and Finite population with OFI.
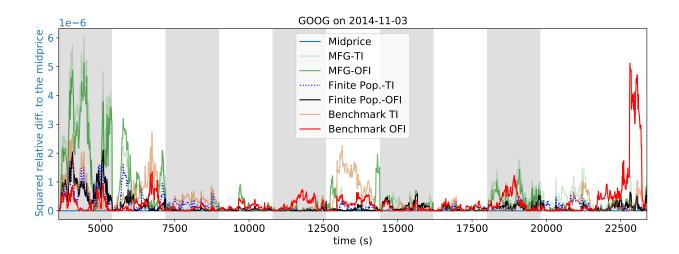
Figure 5: Squared relative difference with respect to the midprice of each formed price models: MFG with TI, MFG with OFI, Finite population with TI, and Finite population with OFI.

Table 3: Mean of the squared relative difference with respect to the midprice, given by (4.9), for each model: MFG with TI, MFG with OFI, Finite population with TI, Finite Population with OFI.

Stock	MFG_{TI}	MFG_{OFI}	Finite pop.$_{TI}$	Finite pop.$_{OFI}$	Bench$_{TI}$	$\mathrm{Bench}-OFI$
GOOG	4.64e-7	4.05e-7	1.62e-7	1.45e-7	2.89e-7	2.72e-7

5. Conclusions

We have proposed two frameworks of price formation. The first of which comprised a mean-field game model, where we determined the price assuming informed traders had objective functionals built such that

they sought to maximize their P&L, they did not want to terminate a given time horizon holding any position and an appropriate market clearing condition was in force. Moreover, we considered a penalization on non-vanishing inventory.

The second framework we developed was a finite population counterpart of the former. Here, we could assume that prices were stochastic processes, relaxing the simplification we made in the MFG, where we assumed prices to be deterministic. We once again obtained a necessary condition determining prices in terms of the realized order flow, where we considered assumptions similar to those of the MFG for each player's utility in the finite population. We also showed, under appropriate assumptions, that our finite population result converged, on average, the MFG one.

We provided empirical assessments for each of the models we proposed. In them, we considered two different types of order flow metrics, which also highlight the importance of such a choice in the impact of our results. In conclusion, our best model was derived from the finite population model, using an order flow metric that took limit order book queue size information into account. A novel regression covariate in the finite population model showed to be significant, and it pointed out in the direction of the possible persistence of the influence of individual players in the price formation process. This last term suggests that one of the assumptions ensuring convergence of the finite population model to the MFG one is not valid in real market data, namely, that single individuals' contribution to order flow have little to no effect on the formed price.

Acknowledgments

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Appendix A. Tables

Table A.1: Average R^2 over the whole period analyzed under each model (including benchmarks), each measure of order flow and each stock studied in the paper. We display the standard deviation of the computed R^2 in parenthesis.

Stock	MFG model		Finite pop. game model		Benchmarks	
	R^2_{TI}	R^2_{OFI}	$R^2_{TI_{micro}}$	$R^2_{OFI_{micro}}$	$R^2_{TI_{bench}}$	$R^2_{OFI_{bench}}$
MU	0.5464292	0.5080833	0.7396288	0.9053048	0.1792843	0.6348812
	(0.2602291)	(0.2782342)	(0.2091207)	(0.0867001)	(0.100364)	(0.1116312)
FB	0.5627452	0.5499002	0.7927316	0.8879627	0.2700599	0.646802
	(0.2679504)	(0.2676806)	(0.1691284)	(0.1077574)	(0.1129601)	(0.1213636)
PCAR	0.538717	0.5227527	0.7177695	0.8825702	0.1302854	0.5943707
	(0.2709382)	(0.276842)	(0.2070019)	(0.122286)	(0.0977697)	(0.0919139)
AMZN	0.5794828	0.5752787	0.7357521	0.81932	0.1862541	0.497083
	(0.2529513)	(0.2578789)	(0.2089806)	(0.1666376)	(0.0902108)	(0.1019693)
EBAY	0.5287725	0.5161722	0.7524981	0.8985027	0.1888365	0.6237341
	(0.2692952)	(0.276937)	(0.1862722)	(0.0981883)	(0.1236811)	(0.1201012)
SMH	0.5671448	0.5273391	0.6386758	0.9045901	0.0383468	0.5366392
	(0.2329803)	(0.2483619)	(0.2168032)	(0.0835175)	(0.0489217)	(0.112495)
INTC	0.5680237	0.5356633	0.7669449	0.9272422	0.1755975	0.6298803
	(0.2503211)	(0.2681156)	(0.1882361)	(0.0654502)	(0.1053752)	(0.09795)
VOD	0.5972025	0.5577175	0.751673	0.936199	0.0768654	0.5950157
	(0.2682837)	(0.2822306)	(0.184862)	(0.0649528)	(0.0873495)	(0.1315352)
GOOG	0.5970308	0.5928375	0.7661087	0.8401094	0.1919996	0.4950987
	(0.2608654)	(0.2629105)	(0.1966308)	(0.1517892)	(0.1033645)	(0.0956888)
MSFT	0.5716435	0.5251484	0.7846665	0.9334051	0.2149702	0.6882648
	(0.2491445)	(0.2760493)	(0.17097)	(0.0655859)	(0.1124921)	(0.0949067)

Table A.2: This table displays the average model parameters and their standard deviation in parenthesis across all windows of 30 min and all trading days of November 2014 for each stock. Here, we use the Trade Imbalance, TI (see Cont et al. (2014)), as a measure of order flow (see (4.1)) and compute the MFG model coefficients of the formed price equation: $\mathfrak{p}_k^i = a_{i,0}S_k^{i,0} + a_{i,1}S_k^{i,1} + a_{i,2}S_k^{i,2} + a_{i,3}S_k^{i,3} + \epsilon_k^i$, where we denote the error term by ϵ_k^i and the observed price at time by $\mathfrak{p}_k^i$, both at time $t_{i,k}$. Such regression is performed in time intervals of 30 minutes $[T_i, T_{i+1}]$, where model parameters are kept constant. Moreover, we divide each window of 30 minutes into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. This regression corresponds to the MFG model in Theorem 2.6. As a benchmark, we take a model inspired by Kyle (1985), viz., $\Delta\mathfrak{p}_k^i := \mathfrak{p}_k^i - \mathfrak{p}_{k-1}^i = b_{i,0} + b_{i,1}TI_k^i + \tilde{\epsilon}_k^i$. This benchmark was also considered in Cont et al. (2014).

	MFG model coeff.					Benchmark model coeff.	
Stock	a_0	a_1	a_2	a_3	ADF Resid.	b_0	b_1
MU	33.3963422	0.0851441	0.001033	-7e-07	-2.8765641	-4.99e-05	2e-06
	(0.845783)	(0.5019527)	(0.0107715)	(1.3e-06)	(0.8859914)	(0.0004175)	(1.1e-06)
FB	74.6772701	0.0455335	0.0029012	-2.4e-06	-2.8127971	-0.0001932	3.5e-06
	(0.9212966)	(1.054565)	(0.0143527)	(1.9e-06)	(0.8257791)	(0.0007031)	(1.8e-06)
PCAR	66.5659824	-0.0138761	-0.0219083	1.9e-06	-2.8200954	8e-05	1.28e-05
	(0.6613602)	(0.4460253)	(0.1035331)	(1.09e-05)	(0.9382861)	(0.0005088)	(6.3e-06)
AMZN	317.1961939	-0.0435624	0.0485113	-1.78e-05	-2.7000014	-5.12e-05	7.28e-05
	(14.2859333)	(4.0944619)	(0.4366202)	(4.78e-05)	(0.8630214)	(0.0043365)	(3.32e-05)
EBAY	54.1099383	0.0237025	-0.0026724	-6e-07	-2.7107647	-2.22e-05	4.1e-06
	(0.6804064)	(0.3960183)	(0.0219799)	(2.5e-06)	(0.9637356)	(0.0004235)	(2.4e-06)
SMH	52.5159266	0.040239	-0.020156	-5e-07	-2.830251	7.09e-05	4e-06
	(1.0403845)	(0.4524628)	(0.1338062)	(1.19e-05)	(0.7951221)	(0.0004398)	(6.7e-06)
INTC	34.4902719	0.0760494	0.0003478	-3e-07	-2.9765764	-5.23e-05	1.2e-06
	(1.1351313)	(0.4756311)	(0.0059531)	(7e-07)	(0.8285871)	(0.000403)	(5e-07)
VOD	34.5713667	0.002355	-0.0038229	8e-07	-2.7983246	2.91e-05	1.7e-06
	(1.2332637)	(0.2978866)	(0.0250166)	(2.3e-06)	(0.8883831)	(0.0002807)	(1.6e-06)
GOOG	542.8725774	-0.5931504	0.027602	-2.31e-05	-2.7499078	-0.0008275	0.0001529
	(6.1632073)	(4.4657383)	(0.87042)	(0.000124)	(0.9356923)	(0.0049919)	(8.38e-05)
MSFT	48.3965698	0.0272846	0.000406	-2e-07	-2.8859986	-6.38e-05	1.3e-06
	(0.7240534)	(0.4672271)	(0.0064598)	(7e-07)	(0.8539199)	(0.0004826)	(6e-07)

Table A.3: This table displays the average model parameters and their standard deviation in parenthesis across all windows of 30 min and all trading days of November 2014 for each stock. Here, we use the Trade Imbalance TI (see Cont et al. (2014)) as a measure of order flow (see (4.1)) and compute the finite population game model coefficients of the formed price equation: $\mathfrak{p}_k^i = a_{i,0}S_k^{i,0} + a_{i,1}S_k^{i,1} + a_{i,2}S_k^{i,2} + a_{i,3}S_k^{i,3} + a_{i,4}S_k^{i,4}\eta_k^i$, where we denote the error term by η_k^i and the observed price at time by $\mathfrak{p}_k^i$, both at time $t_{i,k}$. Such regression is performed in time intervals of 30 minutes $[T_i, T_{i+1}]$, where model parameters are kept constant. Moreover, we divide each window of 30 minutes into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. This regression corresponds to the finite population price model in Corollary 3.8. As a benchmark, we take a model inspired by Kyle (1985), viz., $\Delta\mathfrak{p}_k^i := \mathfrak{p}_k^i - \mathfrak{p}_{k-1}^i = b_{i,0} + b_{i,1}\mathrm{TI}_k^i + \widetilde{\epsilon}_k^i$. This benchmark was also considered in Cont et al. (2014).

| Stock | Finite population model coeff. | | | | | | Benchmark model coeff. | |
	a_0	a_1	a_2	a_3	a_4	ADF Resid.	b_0	b_1
MU	33.3976412	-0.0237521	0.0007491	-8e-07	-0.0004818	-3.5569321	-4.99e-05	2e-06
	(0.8461137)	(0.3405124)	(0.0095541)	(1e-06)	(0.0006709)	(0.9644172)	(0.0004175)	(1.1e-06)
FB	74.6859923	-0.148864	0.0033705	-2.5e-06	-0.0006395	-3.3782803	-0.0001932	3.5e-06
	(0.91748)	(0.6879874)	(0.0115177)	(1.5e-06)	(0.00084)	(0.9240582)	(0.0007031)	(1.8e-06)
PCAR	66.5649544	0.0495253	-0.0189966	1.8e-06	-0.0037029	-3.4136948	8e-05	1.28e-05
	(0.6602739)	(0.4140773)	(0.0921828)	(8.8e-06)	(0.0049728)	(0.9976919)	(0.0005088)	(6.3e-06)
AMZN	317.2097833	-0.203643	0.0638642	-1.88e-05	-0.0151209	-3.1879314	-5.12e-05	7.28e-05
	(14.3062892)	(3.5137093)	(0.4144437)	(4.08e-05)	(0.0225343)	(0.8626377)	(0.0043365)	(3.32e-05)
EBAY	54.1087666	-0.0507869	-0.0035469	-7e-07	-0.0011191	-3.4126401	-2.22e-05	4.1e-06
	(0.6797354)	(0.3933079)	(0.0173387)	(2.1e-06)	(0.0015897)	(0.9577671)	(0.0004235)	(2.4e-06)
SMH	52.5160044	0.0453901	-0.0122174	-0.0	-0.0010844	-3.1661944	7.09e-05	4e-06
	(1.0403181)	(0.4029302)	(0.1250604)	(1.04e-05)	(0.004796)	(0.7650656)	(0.0004398)	(6.7e-06)
INTC	34.4926244	-0.0028086	-6.87e-05	-4e-07	-0.0002724	-3.626328	-5.23e-05	1.2e-06
	(1.1344234)	(0.3030998)	(0.0049006)	(6e-07)	(0.0003485)	(0.9473067)	(0.000403)	(5e-07)
VOD	34.5710566	-0.0225748	-0.0047916	6e-07	-0.0006803	-3.4371937	2.91e-05	1.7e-06
	(1.2329606)	(0.2898547)	(0.0234281)	(2e-06)	(0.0010536)	(0.9308903)	(0.0002807)	(1.6e-06)
GOOG	542.8578395	-0.7487758	0.0149796	-3e-05	-0.0355172	-3.351813	-0.0008275	0.0001529
	(6.1634995)	(3.4921034)	(0.7157249)	(9.66e-05)	(0.0528001)	(0.9196872)	(0.0049919)	(8.38e-05)
MSFT	48.3974368	-0.0307244	0.0002427	-3e-07	-0.0003152	-3.5282415	-6.38e-05	1.3e-06
	(0.7241274)	(0.2945148)	(0.0047474)	(5e-07)	(0.0004022)	(0.9376129)	(0.0004826)	(6e-07)

Table A.4: This table displays the average model parameters and their standard deviation in parenthesis across all windows of 30 min and all trading days of November 2014 for each stock. Here, we use the Oder Flow Imbalance, OFI (see Cont et al. (2014)), as a measure of order flow (see (4.3)) and compute the finite population game model coefficients of the formed price equation: $\mathfrak{p}_k^i = a_{i,0} S_k^{i,0} + a_{i,1} S_k^{i,1} + a_{i,2} S_k^{i,2} + a_{i,3} S_k^{i,3} + a_{i,0} S_k^{i,4} + \eta_k^i$, where we denote the error term by η_k^i and the observed price at time by $\mathfrak{p}_k^i$, both at time $t_{i,k}$. Such regression is performed in time intervals of 30 minutes $[T_i, T_{i+1}]$, where model parameters are kept constant. Moreover, we divide each window of 30 minutes into smaller intervals $T_i = t_{i,0} < \ldots < t_{i,179} = T_{i+1}$ of constant length $t_{i,k+1} - t_{i,k} = 10$ seconds. This regression corresponds to the finite population game price model in Corollary 3.8. As a benchmark, we take the model introduced in Cont et al. (2014), viz. , $\Delta \mathfrak{p}_k^i = c_{i,0} + c_{i,1} \text{OFI}_k^i + \bar{\epsilon}_k^i$, where OFI and e_n are given in (4.3) and (4.2) respectively.

| | Finite population model coeff. | | | | | | Benchmark model coeff. | |
Stock	a_0	a_1	a_2	a_3	a_4	ADF Resid.	c_0	c_1
MU	33.3976715	-0.0061925	0.0001617	-8e-07	-0.0002785	-3.7419097	-4.06e-05	1.6e-06
	(0.8449511)	(0.1907172)	(0.0022751)	(3e-07)	(0.0003134)	(1.2216516)	(0.0002898)	(5e-07)
FB	74.6837965	0.053123	0.001143	-1.6e-06	-0.0004546	-3.36312	0.0001599	2.8e-06
	(0.9150394)	(0.5698996)	(0.0037318)	(7e-07)	(0.0005303)	(0.9568442)	(0.0009176)	(1e-06)
PCAR	66.5654685	-0.0295009	-0.0021186	-6.5e-06	-0.001993	-3.4586142	-7.22e-05	1.15e-05
	(0.6580494)	(0.3106973)	(0.0200821)	(2.6e-06)	(0.0022566)	(1.0904161)	(0.0004809)	(3.3e-06)
AMZN	317.2187288	0.2560108	-0.0121746	-3.8e-05	-0.0111251	-3.15502	0.0018582	6.83e-05
	(14.3286014)	(3.204889)	(0.2088567)	(1.94e-05)	(0.0131955)	(0.8397061)	(0.0046468)	(1.88e-05)
EBAY	54.1095513	0.0074091	-0.0003064	-1.5e-06	-0.0005658	-3.5466687	4.68e-05	3e-06
	(0.6779297)	(0.211954)	(0.0048564)	(6e-07)	(0.0006356)	(0.9679349)	(0.00035)	(8e-07)
SMH	52.513595	0.0264258	-6.32e-05	-6e-07	-0.0001945	-3.662269	-9.6e-06	1.1e-06
	(1.042245)	(0.213225)	(0.0017799)	(2e-07)	(0.0002136)	(1.0288569)	(0.0003551)	(3e-07)
INTC	34.4921724	0.0070192	0.0001049	-4e-07	-0.00014	-3.8448383	1.25e-05	8e-07
	(1.1357877)	(0.144242)	(0.0010034)	(1e-07)	(0.0001523)	(1.2681683)	(0.0002294)	(2e-07)
VOD	34.5696753	0.0076454	-7.8e-06	-5e-07	-0.0002337	-4.2250149	1.14e-05	1.2e-06
	(1.2324129)	(0.1045205)	(0.0017905)	(2e-07)	(0.0002649)	(1.2164438)	(0.0001424)	(4e-07)
GOOG	542.8302421	0.328999	-0.0009426	-4.72e-05	-0.0145369	-3.1616662	0.0006758	9.34e-05
	(6.1604382)	(3.1658284)	(0.235805)	(2.7e-05)	(0.017287)	(0.9419513)	(0.0055439)	(2.64e-05)
MSFT	48.3980857	-0.0037844	-1.15e-05	-4e-07	-0.0001448	-3.7722162	-4.43e-05	8e-07
	(0.7234343)	(0.171465)	(0.0008954)	(1e-07)	(0.0001574)	(1.0471774)	(0.0002614)	(2e-07)

Appendix B. Proof of Proposition 3.4

Proof. We take the Gâteaux derivative,

$$
\begin{aligned}
\left\langle D_i J^i(\nu^i; \boldsymbol{\nu}^{-i}), w^i \right\rangle &= \lim_{\epsilon \to 0} \frac{J_i(\nu^i + \epsilon w^i, \boldsymbol{\nu}^{-i}) - J_i(\nu^i, \boldsymbol{\nu}^{-i})}{\epsilon} \\
&= \mathbb{E}\Bigg[\int_0^T w_t^i \left\{ p_T(\nu^i, \boldsymbol{\nu}^{-i}) - p_t(\nu^i, \boldsymbol{\nu}^{-i}) - 2A q_T^i - 2\kappa \nu_t^i - 2\phi \int_t^T q_u^i \, du \right\} dt \\
&\quad + q_T^i \left\langle D_i p_T(\nu^i, \boldsymbol{\nu}^{-i}), w^i \right\rangle - \int_0^T \nu_t^i \left\langle D_i p_t(\nu^i, \boldsymbol{\nu}^{-i}), w^i \right\rangle dt \Bigg].
\end{aligned}
$$

By our assumptions (**A1**) and (**A2**) this leads to

$$
\begin{aligned}
\left\langle D_i J^i(\nu^i, \boldsymbol{\nu}^{-i}), w^i \right\rangle &= \mathbb{E}\Bigg[\int_0^T w_t^i \Bigg\{ p_T(\nu^i, \boldsymbol{\nu}^{-i}) - p_t(\nu^i, \boldsymbol{\nu}^{-i}) - (2A - \xi_{T,t}^{i,N}(\nu^i, \boldsymbol{\nu}^{-i})) q_T^i \\
&\qquad\qquad - 2\kappa \nu_t^i - 2\phi \int_t^T q_u^i \, du - \int_t^T \nu_u^i \xi_{u,t}^i(\nu^i, \boldsymbol{\nu}^{-i}) \, du \Bigg\} dt \Bigg] \\
&= \mathbb{E}\Bigg[\int_0^T w_t^i \Bigg\{ -2\kappa \nu_t^i - p_t\left(\nu^i, \boldsymbol{\nu}^{-i}\right) + 2\phi \int_0^t q_u^i \, du \\
&\qquad + \mathbb{E}\left[\xi_{T,t}^i(\nu^i, \boldsymbol{\nu}^{-i}) q_T^i - \int_t^T \nu_u^i \xi_{u,t}^i(\nu^i, \boldsymbol{\nu}^{-i}) \, du \Bigg| \mathcal{F}_t^{i0} \right] + M_t^i \Bigg\} dt \Bigg],
\end{aligned}
\tag{B.1}
$$

where we have written

$$
M_t^i := \mathbb{E}\left[-2A q_T^i + p_T(\nu^i, \boldsymbol{\nu}^{-i}) - 2\phi \int_0^T q_u^i \, du \Bigg| \mathcal{F}_t^{i0} \right] \in \mathbb{L}^2.
$$

A Nash equilibrium $\boldsymbol{\nu}^*$ is characterized by

$$
\nu^{*i} = \operatorname{argmax}_{\nu^i} J_i(\nu^i, \boldsymbol{\nu}^{*-i}), \text{ for each } i \in \mathcal{N}.
$$

Under the assumptions presently made, this is equivalent to

$$
\left\langle D_i J_i(\nu^{*i}, \boldsymbol{\nu}^{*-i}), w^i \right\rangle = 0,
\tag{B.2}
$$

for all $i \in \mathcal{N}$ and w^i. From (B.1) and (B.2), the characterization of $\boldsymbol{\nu}^*$ as the solution of (3.8) follows. $\quad\square$

Appendix C. Further discussion on the admissible prices class

We begin by remarking that we can generalize Example 3.1 as follows.

Example Appendix C.1. *Both conditions (**C1**) and (**C2**) hold if $p_t(\boldsymbol{\nu}) = P_t\left(\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N q_t^i \Big| \mathcal{F}_t^0 \right] \right)$, where $(t, Q) \in [0, T] \times \mathbb{R} \mapsto P_t(Q) \in \mathbb{R}$ satisfies:*

(i) For each $\{Q_t\}_{0 \leqslant t \leqslant T} \in \mathcal{A}_0$, the process $\{P_t(Q_t)\}_{0 \leqslant t \leqslant T}$ belongs to $\mathcal{A}_0$.

(ii) For each $t \in [0, T]$, the mapping $Q \mapsto P_t(Q)$ is of class C^1, $\mathbb{P}-$almost surely.

In effect, in this setting we have

$$
\left\langle D_i p_t(\boldsymbol{\nu}), w^i \right\rangle = \mathbb{E}\left[\int_0^t \frac{1}{N} \partial_Q P_t\left(\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N q_t^i \Big| \mathcal{F}_t^0 \right] \right) w_u^i \, du \Bigg| \mathcal{F}_t^0 \right],
$$

*whence (**C1**) is valid with $\xi_{t,u}^{i,N}(\boldsymbol{\nu}) = \frac{1}{N} \partial_Q P_t\left(\mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N q_t^i \Big| \mathcal{F}_t^0 \right] \right)$, $0 \leqslant u \leqslant t \leqslant T$, from where it follows that (**C2**) holds too.*

The works Féron et al. (2020b,a) focus on the analysis of the price formation problem in the case information is symmetric among players and p is has the form we described in (3.1). In this case, understanding the formed price amounts to understanding the dynamics of the optimal trading rates. In the present framework, in which we allow for a broader class of admissible prices, we gain more flexibility for explaining the price in terms of the order flow, as we have already seen in Section 2.

We remark that (**C2**) is an integrability condition we stipulate to allow us to apply variational techniques. Regarding the condition (**C1**), it is of a rather general nature. The intuition behind it is that a mapping $p \in \mathfrak{P}$ ought to have the following properties:

(i) Prices are *non-anticipative* functionals: for each $t \in [0, T]$, $\boldsymbol{\nu} \mapsto p_t(\boldsymbol{\nu})$ depends only on the restriction $\boldsymbol{\nu}|_{[0,t]}$ of the path of $\boldsymbol{\nu}$;

(ii) For each $t \in [0, T]$, the mapping

$$\boldsymbol{\nu}(\omega)|_{[0,t]} = \{\boldsymbol{\nu}_u(\omega)\}_{0 \leqslant u \leqslant t} \in L^2(0,t)^N \mapsto p_t(\boldsymbol{\nu})(\omega) \in \mathbb{R}$$

is Gâteaux differentiable $\mathbb{P}-$almost surely;

(iii) For every $\boldsymbol{\nu} \in \mathcal{A}$, the process $\{p_t(\boldsymbol{\nu})\}_{0 \leqslant t \leqslant T}$ belongs to $\mathcal{A}_0$.

Technically, it is worthwhile to remark an implication of the relation $p \in \mathfrak{P}$. We expose it in the sequel.

Lemma Appendix C.2. *Let us assume $p \in \mathfrak{P}$. Then,*

$$p_t(\boldsymbol{\nu}) = p_t(\mathbf{0}) + \mathbb{E}\left[\sum_{i=1}^{N} \int_0^t \zeta_{t,u}^i(\boldsymbol{\nu}) \nu_u^i \, du \middle| \mathcal{F}_t^0\right],$$

where

$$\zeta_{t,u}^i(\boldsymbol{\nu}) = \int_0^1 \xi_{t,u}^i\left(0, \ldots, 0, \lambda \nu^i, \nu^{i+1}, \ldots, \nu^N\right) d\lambda.$$

Proof. It suffices to observe that

$$p_t(\boldsymbol{\nu}) - p_t(\mathbf{0}) = \sum_{i=1}^{N} \left\{p_t(0, \ldots, 0, \nu^i, \nu^{i+1}, \ldots, \nu^N) - p_t(0, \ldots, 0, 0, \nu^{i+1}, \ldots, \nu^N)\right\}$$

$$= \int_0^1 \frac{d}{d\lambda} p_t(0, \ldots, 0, \lambda \nu^i, \nu^{i+1}, \ldots, \nu^N) \, d\lambda$$

$$= \int_0^1 \sum_{i=1}^{N} \mathbb{E}\left[\int_0^t \xi_{t,u}^{i,N}(0, \ldots, 0, \lambda \nu^i, \nu^{i+1}, \ldots, \nu^N) \nu_u^i \, du \middle| \mathcal{F}_t^0\right] d\lambda$$

$$= \mathbb{E}\left[\sum_{i=1}^{N} \int_0^t \left(\int_0^1 \xi_{t,u}^{i,N}(0, \ldots, 0, \lambda \nu^i, \nu^{i+1}, \ldots, \nu^N) \, d\lambda\right) \nu_u^i \, du \middle| \mathcal{F}_t^0\right].$$

$\square$

Lemma Appendix C.2 indicates that our class $\mathfrak{P}$ generalizes in a natural way the concept of transient price impact Gatheral et al. (2012).

Appendix D. Examples from the Brazilian and Crypto markets

Here we illustrate two examples of our price formation approach applied to the Brazilian market (PETR3, Petrobras ordinary stock) and crypto market (BTCBRL, Bitcoin traded on Binance exchange in Brazil), quoted in BRL. For both of them, we illustrate the formed prices from the four methods we developed (MFG with TI, MFG with OFI, Finite population with TI, and finite population with OFI), along with the corresponding benchmarks. Figures D.6 and D.7 correspond to PETR3, and in Table D.5 we compute the average of adjusted R^2 for this stock. Moreover, Figures D.8 and D.9 relate to BTCBRL, and in Table D.6 we compute the average of its adjusted R^2.

Table D.5: Average adjusted R^2 results

Stock	R^2_{TI}	R^2_{OFI}	$R^2_{TI_{micro}}$	$R^2_{OFI_{micro}}$	$R^2_{TI_{bench}}$	$R^2_{OFI_{bench}}$
PETR3	0.486544	0.51164	0.59331	0.848529	0.004201	0.460331
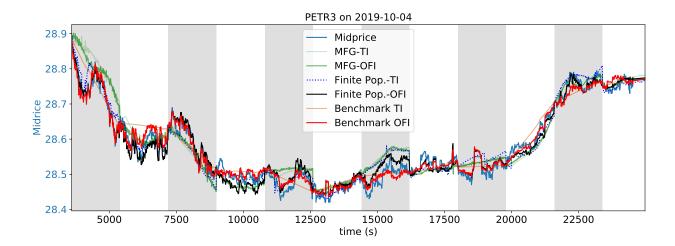
Figure D.6: Comparison between the midprice and formed price models: MFG with TI, MFG with OFI, Finite population with TI, and Finite population with OFI.
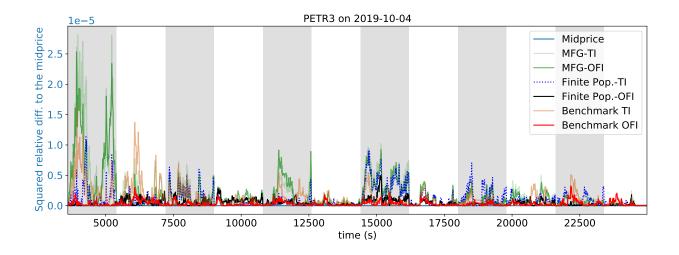

Figure D.7: Squared relative difference with respect to the midprice of each formed price models: MFG with TI, MFG with OFI, Finite population with TI, and Finite population with OFI.

Table D.6: Average adjusted R^2 results

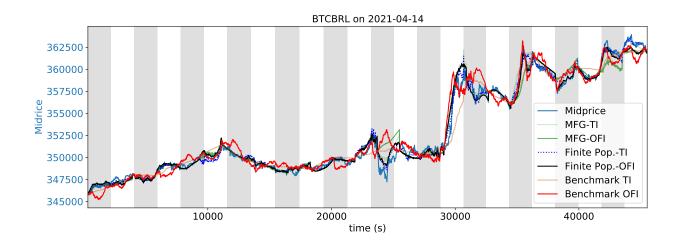| Stock	R^2_{TI}	R^2_{OFI}	$R^2_{TI_{micro}}$	$R^2_{OFI_{micro}}$	$R^2_{TI_{bench}}$	$R^2_{OFI_{bench}}$
BTCBRL	0.62019	0.609347	0.687367	0.743341	0.02188	0.409336

Figure D.8: Comparison between the midprice and formed price models: MFG with TI, MFG with OFI, Finite population with TI, and Finite population with OFI.
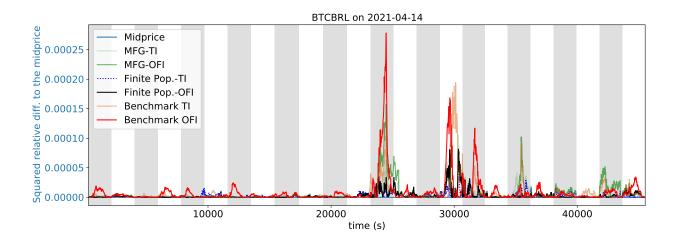

Figure D.9: Squared relative difference with respect to the midprice of each formed price models: MFG with TI, MFG with OFI, Finite population with TI, and Finite population with OFI.