

THE OZONE GROUPS OF PI ARTIN–SHELTER REGULAR ALGEBRAS ARE ABELIAN

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ABSTRACT. We prove that the ozone group of any PI Artin–Schelter regular algebra is abelian, which answers a question of Chan–Gaddis–Won–Zhang. For any Calabi–Yau PI Artin–Schelter regular algebra, we prove that the homological determinant of its ozone group acting on it is trivial.

1. INTRODUCTION

The concept of ozone groups was introduced in [CGWZ24, CGWZ25]. For a noncommutative ring A , the ozone group of A , denoted by $\text{Oz}(A)$, is defined as the group of automorphisms of A as a $Z(A)$ -algebra, where $Z(A)$ refers to the center of A . According to [CGWZ25] and [Zhu23], ozone groups of PI Artin–Schelter regular algebras are finite groups consisting of conjugations induced by regular normal elements. Therefore the study of ozone groups helps identifying normal elements of Artin–Schelter regular algebras. Moreover, ozone groups were applied to study the interplay between the PI Artin–Schelter regular algebras and their centers, which was effective for skew polynomial rings.

Skew polynomial rings are special examples of Artin–Schelter regular algebras. Therefore it is natural to ask whether some properties of PI skew polynomial rings are in fact shared by PI AS regular algebras. For instance, ozone groups of skew polynomial rings are not only finite but also abelian. Therefore a natural question was proposed in [CGWZ25, Question 0.3].

Question 1.1. Is the ozone group of a noetherian PI Artin–Schelter regular algebra abelian?

We answer this question affirmatively in this note.

Theorem 1.2. (See Theorem 3.4) Let A be a noetherian PI Artin–Schelter regular algebra. Then $\text{Oz}(A)$ is a finite abelian group.

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It was showed in [CGWZ24] that ozone groups can be applied to characterize Calabi–Yau property of PI skew polynomial rings. To be precise, a skew polynomial ring is Calabi–Yau if and only if its ozone group action has the trivial homological determinant [CGWZ24, Theorem 0.10]. It is a widely concerned question how to characterize Calabi–Yau property of Artin–Schelter regular algebras. Consequently the following question arose.

Question 1.3. [CGWZ25, Question 5.2] What role does Nakayama automorphism play in the study of the ozone group and the center? If $\text{Oz}(A)$ is assumed to be abelian, what is the connection between A being Calabi–Yau and the action of $\text{Oz}(A)$ on A having trivial homological determinant?

We also provide a partial result concerning this question.

Theorem 1.4. (See Theorem 3.8) Let A be a noetherian PI AS regular algebra which is Calabi–Yau. Then the action of $\text{Oz}(A)$ on A has the trivial homological determinant.

2. PRELIMINARIES

Throughout \mathbb{k} is assumed to be an algebraically closed field of characteristic 0. For a ring A , A° refers to the opposite ring of A . By A -modules we mean left A -modules.

A **polynomial identity ring** (or PI ring for short) is a ring satisfying a polynomial identity. A ring which is finitely generated as a module over a central subring is a PI ring. The converse does not hold in general. We refer to [MR01, Chapter 13] for the facts about PI rings. The main fact we need about PI rings is the following structure theorem.

Proposition 2.1. (Posner’s theorem, [MR01, Theorem 6.5]) Let A be a prime PI ring with the center Z . Let K be the fraction field of Z , and let $D := A \otimes_Z K$. Then D is a central simple K -algebra.

A \mathbb{k} -algebra A is **connected graded** if A is an \mathbb{N} -graded algebra with $A_0 = \mathbb{k}$.

Definition 2.2. A noetherian connected graded \mathbb{k} -algebra A is called Artin–Schelter Gorenstein (or AS Gorenstein for short) of dimension d if

- (1) A has finite left and right injective dimensions;
- (2) $\begin{cases} \text{Ext}_A^i(\mathbb{k}, A) \cong \text{Ext}_{A^\circ}^i(\mathbb{k}, A) = 0, \forall i \neq d; \\ \text{Ext}_A^d(\mathbb{k}, A) \cong \text{Ext}_{A^\circ}^d(\mathbb{k}, A) \cong \mathbb{k}(\mathfrak{l}) \text{ for some } \mathfrak{l} \in \mathbb{Z}, \end{cases}$

where \mathfrak{l} is called the Gorenstein parameter of A .

If moreover A has finite global dimension then A is called Artin–Schelter regular (or AS regular for short).

Noetherian PI AS regular algebras were well studied in [SZ94].

Theorem 2.3. [SZ94, Corollary 1.2] Let A be a noetherian PI AS regular algebra. Then A is a domain and a finitely generated module over its center Z , which is a Krull domain. Moreover $\text{Kdim}(A) = \text{GKdim}(A) = \text{gldim}(A)$.

Now we assume that A is AS Gorenstein of dimension d with Gorenstein parameter \mathfrak{l} . Let G be a finite subgroup of $\text{Aut}_{gr}(A)$, and $\sigma \in G$. Then $\sigma : A \rightarrow A$, $a \mapsto \sigma(a)$ is a σ -linear morphism between graded A^o -modules. It induces a σ -linear morphism $H_{\mathfrak{m}_A}^d(\sigma) : A'(\mathfrak{l}) \rightarrow A'(\mathfrak{l})$. On the other hand, it is direct to verify that the only σ -linear morphisms between $A'(\mathfrak{l})$ are of form $c \cdot (\sigma^{-1})'$ for some $c \in \mathbb{k}$. Therefore there exists unique $c_\sigma \in \mathbb{k}$ such that $H_{\mathfrak{m}_A}^d(\sigma) = c_\sigma(\sigma^{-1})'$.

Definition 2.4. [JZ00, Definition 2.3] The homological determinant of G acting on A is defined as $\text{hdet}_A : G \rightarrow \mathbb{k}$, $\sigma \mapsto c_\sigma^{-1}$.

It is easy to check that $\text{hdet}_A(\sigma\tau) = \text{hdet}_A(\sigma)\text{hdet}_A(\tau)$ for all $\sigma, \tau \in G$. Therefore $\text{hdet}_A(\sigma)\text{hdet}_A(\sigma^{-1}) = 1$. Since G is a finite group, it follows that $\text{hdet}_A(\sigma)$ is a root of unity. If $\text{hdet}_A(\sigma) = 1$ for all $\sigma \in G$, then we say that the G -action on A has the **trivial homological determinant**.

3. OZONE GROUPS OF PI AS REGULAR ALGEBRAS

Definition 3.1. [CGWZ25, Definition 1.1] Let A be a ring with the center Z . The ozone group of A is defined as $\text{Oz}(A) := \{\sigma \in \text{Aut}_{\mathbb{k}}(A) \mid \sigma(z) = z, \forall z \in Z\}$.

An element a of a ring A is called **normal** if $Aa = aA$. If $ab \neq 0$ for any nonzero element b of A , then a is called a **regular** element. Note that normal elements of a prime ring are always regular. Suppose that a is a normal regular element of A . Then a induces an automorphism η_a of A , given by $ab = \eta_a(b)a$, for all $b \in A$. Obviously, $\eta_a \in \text{Oz}(A)$. In fact, under some mild assumptions, the elements in $\text{Oz}(A)$ are of the above form.

Lemma 3.2. [CGWZ25, Lemmas 1.9] Let A be a prime ring which is finitely generated as a module over its center Z . For any $\sigma \in \text{Oz}(A)$, there exists a nonzero normal element $a \in A$ such that $ab = \sigma(b)a$ for all $b \in A$. In other words, $\sigma = \eta_a$.

Lemma 3.2 can be applied to noetherian PI AS regular algebras by Theorem 2.3. For any $\sigma \in \text{Oz}(A)$, let $A_\sigma := \{a \in A \mid \sigma(b)a = ab, \forall b \in A\}$.

Lemma 3.3. Let A be a domain such that A is a finitely generated module over its center Z , which contains all the roots of unity. Suppose x and y are two nonzero normal elements of A . If the order of the automorphism $\eta_x \eta_y \eta_x^{-1} \eta_y^{-1}$ is finite, then $xy = \zeta yx$ for some root of unity ζ .

Proof. Let K be the fraction field of Z , and D be the division ring $A \otimes_Z K$. Let $u := xyx^{-1}y^{-1} \in D$. Then

$$ub = xyx^{-1}y^{-1}b = (\eta_x \eta_y \eta_x^{-1} \eta_y^{-1}(b))u, \forall b \in A.$$

If $(\eta_x \eta_y \eta_x^{-1} \eta_y^{-1})^m = \text{id}$, then $u^m b = bu^m$ for all $b \in A$. Hence $u^m \in Z(D) = K$.

Consider the following norm map

$$N : D \longrightarrow \text{End}_K(D) = M_n(K) \xrightarrow{\det} K$$

where $D \rightarrow \text{End}_K(D)$ is given by the left multiplications. Since the norm N is multiplicative, it follows that

$$N(u) = N(x)N(y)N(x^{-1})N(y^{-1}) = 1, \text{ and } 1 = N(u)^m = N(u^m) = (u^m)^n$$

as $u^m \in K$.

By assumption, Z contains all mn -th roots of unity $\zeta_1, \dots, \zeta_{mn}$. Then

$$t^{mn} - 1 = \prod_{i=1}^{mn} (t - \zeta_i) \in Z[t] \subseteq K[t].$$

It follows from $u^{mn} = 1$ that $\prod_{i=1}^{mn} (u - \zeta_i) = 0 \in D$. Since D is a domain, $u = \zeta_i \in Z \subseteq K$ for some i . Hence $xy = \zeta_i yx$. \square

Theorem 3.4. Let A be a noetherian PI AS regular algebra. Then $\text{Oz}(A)$ is a finite abelian group.

Proof. The finiteness of $\text{Oz}(A)$ is proved in [Zhu23, Theorem 2.5] and [CGWZ25, Theorem E]. It suffices to show that $\sigma\tau = \tau\sigma$ for any $\sigma, \tau \in \text{Oz}(A)$. By Lemma 3.2, there exist $0 \neq x \in A_\sigma$ and $0 \neq y \in A_\tau$. Lemma 3.3 then guarantees the existence of a root of unity ζ such that $xy = \zeta yx$. Since $x \in A_\sigma$ and $y \in A_\tau$, the product xy lies in $A_{\sigma\tau}$. Similarly, $yx \in A_{\sigma\tau}$. Then $0 \neq xy = \zeta yx$ implies $A_{\sigma\tau} \cap A_{\tau\sigma} \neq 0$, which forces $\sigma\tau = \tau\sigma$. \square

As a result of Theorem 3.4, some results of [CGWZ25] can be restated as follows.

Corollary 3.5. Let A be a noetherian PI AS regular algebra with the center Z .

- (1) [CGWZ25, Theorem B] A finite group G is isomorphic to the ozone group of some noetherian PI AS regular algebra if and only if G is abelian.
- (2) [CGWZ25, Lemma 4.10] $\sum_{\sigma \in \text{Oz}(A)} A_\sigma$ is a direct sum of Z -modules.
- (3) [CGWZ25, Theorem F] Suppose moreover that A is generated in degree one. Then the following are equivalent.
 - (a) A is a skew polynomial ring.
 - (b) $|\text{Oz}(A)| = \text{rank}_Z A$.
 - (c) A is generated by normal elements.
 - (d) $A^{\text{Oz}(A)} = Z$.

Although the ozone group action of A may be complicated, the action of $\text{Oz}(A)$ on the normal elements of A is quite simple.

Corollary 3.6. Let x be a normal element of a noetherian PI AS regular algebra A . For any $\tau \in \text{Oz}(A)$, $\tau(x) = \zeta x$ for some root of unity ζ .

Proof. We may assume that $x \neq 0$. By Lemma 3.2 $x \in A_\sigma$ for some $\sigma \in \text{Oz}(A)$. Take $0 \neq y \in A_\tau$. It follows from Lemma 3.3 that there exists some root of unity ζ such that $xy = \zeta yx$. By the definition of A_τ , $yx = \tau(x)y$. The conclusion follows as A is a domain. \square

An AS regular algebra A is called **Calabi–Yau** if the Nakayama automorphism μ_A of A is the identity map. Note that μ_A belongs to $\text{Oz}(A)$ [BZ08, Proposition 4.4]. With the aid of Corollary 3.6, the homological determinants of the $\text{Oz}(A)$ action on A can be calculated through μ_A .

Lemma 3.7. [RRZ17, Corollary 5.4] Let A be a connected graded noetherian AS Gorenstein algebra which is a surjective image of a noetherian AS regular algebra. Then $\text{hdet}_A(\mu_A) = 1$.

Theorem 3.8. Let A be a noetherian PI AS regular algebra. For any $\sigma \in \text{Oz}(A)$ and $a \in A_\sigma$, $\mu_A(a) = \text{hdet}_A(\sigma)a$. Consequently, if A is moreover Calabi–Yau, then the $\text{Oz}(A)$ -action on A has the trivial homological determinant.

Proof. Obviously, $\sigma(a) = a$ as A is a domain. By Corollary 3.6, $\mu_A(a) = \zeta a$ for some root of unity ζ . We may assume $a \neq 0$. By Rees' lemma $A/(a)$ is AS Gorenstein. It follows from [RRZ14, Lemma 1.5] that

$$\mu_{A/(a)} = (\mu_A \circ \sigma)|_{A/(a)}.$$

According to [JZ99, Proposition 2.4],

$$\begin{aligned} \text{hdet}_A(\sigma) &= \text{hdet}_{A/(a)}(\sigma|_{A/(a)}); \\ \text{hdet}_A(\mu_A) &= \zeta \text{hdet}_{A/(a)}(\mu_A|_{A/(a)}). \end{aligned}$$

The rest follows from Lemma 3.7. \square

From Theorem 3.8, we conclude that $\text{Oz}(A)$ acts on A with the trivial homological determinant if and only if μ_A fixes all normal elements. Since skew polynomial rings are generated by normal elements, $\text{Oz}(A)$ acting on A with the trivial homological determinant indicates that μ_A fixes A , or equivalently, that A is Calabi–Yau. See [CGWZ24, Theorem 0.10] for further connections. At the end, we show by an example that this does not necessarily hold for PI AS regular algebras.

Example 3.9. Consider the down-up algebra

$$A := A(0, 1) = \mathbb{k}\langle x, y \rangle / (x^2y - yx^2, xy^2 - y^2x).$$

Then $\text{Oz}(A) = \{1, \mu_A\}$, where $\mu_A : x \mapsto -x, y \mapsto -y$ [CGWZ25, Example 5.4]. Therefore A is not Calabi–Yau. By Lemma 3.7 $\text{hdet}_A(\mu_A) = 1$, which implies that $\text{hdet}_A(\text{Oz}(A)) = 1$.

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