

Minimal PI-systems with all points are multiply minimal

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Abstract

We construct a minimal subshift (X^*, σ) that serves as an open proximal extension of its maximal equicontinuous factor. We establish that every point in this subshift is multiply recurrent minimal. This work solves an open problem raised by Huang, Shao and Ye regarding the existence of minimal PI-systems such that each point is multiply minimal.

Keywords. minimal systems, symbolic systems, multiply minimal.

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1 Introduction

Denote by \mathbb{N} the set of all natural numbers including 0, and \mathbb{Z}_+ the set of all positive integers. Let X be a compact metric space with a metric ρ , and $T : X \rightarrow X$ be a continuous surjection. The pair (X, T) is called a *topological dynamical system*. For $d \geq 2$, write $x^{(d)} := (x, x, \dots, x) \in X^d$ and $\tau_d := T \times T^2 \times \dots \times T^d$.

Recurrence and minimality are important in the study of topological dynamical systems. The Birkhoff recurrence theorem shows that each topological dynamical system has at least one recurrent point. For a topological dynamical system (X, T) , a point $x \in X$ is called *recurrent* for the map T if for some strictly increasing sequence $\{n_i\}$ of \mathbb{Z}_+ , $T^{n_i}x \rightarrow x$ as $i \rightarrow \infty$. Recall that the *orbit closure* of a point $x \in X$ under the map T is the closure of its orbit $\mathcal{O}(x, T) := \{T^n x : n \in \mathbb{N}\}$. A point $x \in X$ is called *minimal* or *uniform recurrent* for a map T if its orbit closure under the map T is *minimal*, that is, does not contain any closed T -invariant subset. It is known that each topological dynamical system has at least one minimal point. Furstenberg [3] proves the multiple recurrence theorem, which gives a dynamical proof of Szemerédi

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theorem. The multiple recurrence theorem can shows the multiple Birkhoff recurrence theorem, that is, the existence of multiply recurrent points. A point $x \in X$ is called *multiply recurrent* if the point $x^{(d)} \in X^d$ is recurrent for τ_d , that is, there is a strictly increasing sequence $\{n_i\}$ of \mathbb{Z}_+ such that $T^{jn_i}x \rightarrow x$ as $i \rightarrow \infty$ for $1 \leq j \leq d$. Furstenberg and Weiss [6] show the existence of multiply recurrent points by topologically dynamical tools, where Furstenberg [3] uses ergodic theory. Furstenberg [4] asks whether there is always a *multiple minimal* point, that is, a point $x \in X$ such that $x^{(d)}$ is a minimal point for τ_d . However, Huang, Shao and Ye [8] give a counterexample to this question. They show that there is a minimal weakly mixing system (X, T) such that for all $x \in X$, (x, x) is not minimal for $T \times T^2$.

On the other hand, there are some minimal systems whose points are all multiply minimal. For a *doubly minimal* system (X, T) , that is, the orbit closure of any pair (x, y) under $T \times T$ is either $X \times X$ or the graph of a power of T , Auslander and Markley [1] prove that (X^d, τ_d) is minimal for all $d \in \mathbb{Z}_+$. This show that all the points of a doubly minimal system are multiply minimal. Besides, a *distal* system is a topological system (X, T) such that $\inf_{n \in \mathbb{N}} \rho(T^n x, T^n y) > 0$ for any $x \neq y \in X$. All points in a distal system are multiply minimal. Weiss [12] show that any ergodic system with zero entropy has a uniquely ergodic model that is doubly minimal. Huang and Ye [9] show that all the doubly minimal systems are subshifts.

In [8], the authors ask the following question:

Question 1.1. *Is there a minimal PI-system (X, T) which is a non-trivial open proximal extension of an equicontinuous system (Y, T) such that for each $x \in X$, (x, x) is minimal for $T \times T^2$?*

This question brings our attention to study the multiply minimal points in Proximal-Isometric (PI) systems (See definition in Section 2.1). We give a positive answer to this question by constructing a minimal subshift (X^*, σ) . Write $\sigma_d := \sigma \times \sigma^2 \times \dots \times \sigma^d$.

Theorem 1.2. *There is a minimal subshift (X^*, σ) such that it is an open proximal extension of its maximal equicontinuous factor, and for $d \geq 2$ and $x \in X^*$, $x^{(d)}$ is σ_d -minimal.*

Recall that a map π is *open* if the image under π of each open set is also an open set. Our example is an extension of an odometer, and the corresponding factor map is open and not a bijection. This fact shows that our example is not a Toeplitz subshift, which is an almost one-to-one extension of an odometer [13].

1.1 Main ideas

The idea of our example is to realize the “multiple recurrence” in a finite segment. Roughly speaking, in the setting of symbolic dynamics, let all the words “multiply” appear in each longer word. This can be done by Lemma 3.4 in Section 3. The proof is mainly based on the example constructed by Oprocha [11].

He give an example of doubly minimal subshift (X, σ) which admits any given two words of two symbols. And by [1], (X^d, σ_d) is minimal. By this fact, we can get a minimal subshift (Z, σ) of more symbols such that (Z^d, σ_d) is minimal and all the pair of symbols appear in the subshift. By those property, we get Lemma 3.4. Turning to the setting of symbolic dynamics, this lemma lets all the pair of symbols “multiply” appear in a word.

Now, we can construct a subshift of $\{0, 1, 2\}^{\mathbb{N}}$. Using Lemma 3.4, we construct a sequence of sets of words inductively. We construct the first set of words. Let symbol 2 be the prefix of those words, and the suffix consists of symbols 0 and 1. Temporarily add one more symbol *, and use Lemma 3.4 for symbols 0, 1 and *. Then we can get a word whose first symbol is symbol 2 as prefix and the rest consists of symbols 0, 1 and * as suffix. Replacing * by 0 or 1, we can get a set of words. Choose a word of this set called a marker word. To ensure the openness of the factor map corresponding to the maximal equicontinuous factor, we should make some adjustment, that is, exchanging two symbols in the suffix of words in this set twice. Then we get a larger set of words, which is the first set of words. The words generated by adjusting the marker word are called prefix words, which will be the prefix of the words in the next step of constructing longer words. The rest words are called suffix words, which form the suffix of words in the next step. View the suffix words as symbols, temporarily add one more symbol * and use Lemma 3.4 again. Then we get some words whose prefixes are the prefix words and the rest consist of suffix words and *. Replace * by suffix words, we can get a new set of words. In this set, choose a word whose prefix is the marker word as a new marker word, and make a similar adjustment to words in the new set by exchanging two suffix words twice. Then we get a larger set of words, which is the second set of words. The words generated by adjusting the new marker word are the new prefix words, and the rest are the new suffix words. After finishing this procedure inductively, we get sequences of marker words and sets of words. The sequence of marker words give a point in $\{0, 1, 2\}^{\mathbb{N}}$, and the orbit closure of this point is our constructed subshift.

The prefix words can induce the factor map corresponding the maximal equicontinuous factor of the constructed subshift. For any point in the constructed subshift, by the locations prefix words appear, we can get a sequence of integer, which corresponds to a point in an adding machine. This leads to a factor map from the constructed subshift to an adding machine. Since every two prefix words are almost same, we can show the factor map is proximal, which implies that the adding machine is the maximal equicontinuous factor of the constructed subshift. Through iterative adjustments of word constructions, we ensure that any word in the subshift can be embedded at arbitrary positions within prefix words. This fact leads to the openness of the factor map. Finally, by Lemma 3.4, each words can “multiply” appear in a longer word so many times that the adjustment of the words can preserve the “multiple recurrence”.

2 Preliminaries

In this section, we give some definitions and notations.

2.1 Structure theorem of minimal systems

For two topological dynamical systems $(X, T), (Y, S)$, a map $\pi : (X, T) \rightarrow (Y, S)$ is called a *factor map* if π is a continuous surjection and satisfies $\pi \circ T = S \circ \pi$. In this case, call (X, T) is an *extension* of (Y, S) and (Y, S) is a *factor* of (X, T) .

Let $\pi : (X, T) \rightarrow (Y, S)$ be a factor map of minimal systems. The factor map π is called *proximal* if every two points $x, y \in X$ with same image under π are *proximal*, that is, $\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0$. A topological dynamical system is called *equicontinuous* if for any $\epsilon > 0$, there is $\delta > 0$ such that $\sup_{n \in \mathbb{N}} \rho(T^n x, T^n y) < \epsilon$ whenever $\rho(x, y) < \delta$. Since each topological dynamical system has a maximal equicontinuous factor, we denote by (X_{eq}, T_{eq}) the *maximal equicontinuous factor* of a topological dynamical system (X, T) . If a topological dynamical system is a proximal extension of some equicontinuous system, then this equicontinuous system is a maximal equicontinuous factor of the topological dynamical system.

Before introducing the structure theorem for minimal systems [2], let us give some definitions. For two topological dynamical systems (X, T) and (Y, T) , let $\pi : X \rightarrow Y$ be the factor map from (X, T) to (Y, T) . The factor map π is called *equicontinuous* if for any $\epsilon > 0$, there is $\delta > 0$ such that $\sup_{n \in \mathbb{N}} \rho(T^n x, T^n y) < \epsilon$ whenever $\rho(x, y) < \delta$ and $\pi(x) = \pi(y)$. A topological dynamical system (X, T) is called *transitive* if for any two non-empty open subset U, V , there is $n \in \mathbb{N}$ such that $U \cap T^{-n}V \neq \emptyset$. The factor map π is called *weakly mixing* if the subsystem $R_\pi := \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$ of $(X \times X, T \times T)$ is transitive. The factor map π is called *relatively incontractible* (RIC) if it is open and for every $n \geq 1$, let $R_\pi^n := \{(x_1, \dots, x_n) \in X^n : \pi(x_1) = \dots = \pi(x_n)\}$, and the minimal points of the system $(R_\pi^n, T \times T \times \dots \times T)$ are dense in R_π^n .

A minimal system (X, T) is called a *strictly PI-system* if there is an ordinal α and a family of systems $\{(X_\beta, T_\beta)\}_{\beta \leq \alpha}$ such that the followings hold:

- W_0 is a singleton;
- for any $\beta < \alpha$, there is a factor map from $(X_{\beta+1}, T_{\beta+1})$ to (X_β, T_β) which is either proximal or equicontinuous;
- for any limit ordinal $\beta \leq \alpha$, (X_β, T_β) is the inverse limit of system $\{(X_\gamma, T_\gamma)\}_{\gamma < \beta}$;
- $W_\alpha = X$.

The system (X, T) is called a *PI-system* if there is a proximal factor map from some strictly PI-system to (X, T) .

Now, we introduce the structure theorem for minimal systems [2].

Theorem 2.1 (Structure theorem for minimal systems). *Let (X, T) be a minimal system. Then there are a proximal extension X_∞ of X and a RIC weakly mixing extension π_∞ from X_∞ to a strictly PI-system Y_∞ . The extension π_∞ is a bijection if and only if X is a PI-system.*

2.2 Odometer

In this subsection, we recall some basic notations of odometers [10]. Say a *periodic structure* $\{p_k\}_{k \geq 0}$ is a sequence $\{p_k\}_{k \geq 0}$ such that $p_0 > 1$ and p_k divides p_{k+1} . For a positive integer p , let $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ be a discrete topological group under the addition module p . Then, the product space of \mathbb{Z}_p ,

$$\mathbb{Z}_{\{p_k\}} = \prod_{k=0}^{\infty} \mathbb{Z}_{p_k},$$

can be viewed as a topological space endowed with product topology. The space $\mathbb{Z}_{\{p_k\}}$ is also a topological group. Then, the zero element of $\mathbb{Z}_{\{p_k\}}$ is $\underline{0} = (0, 0, 0, \dots)$. Define $R_{\underline{1}} : \mathbb{Z}_{\{p_k\}} \rightarrow \mathbb{Z}_{\{p_k\}}$ by

$$R_{\underline{1}}((y_0, y_1, y_2, \dots)) = (y_0, y_1, y_2, \dots) + (1, 1, 1, \dots).$$

The orbit closure

$$Y = \overline{\mathcal{O}}(\underline{0}, R_{\underline{1}}) = \{(y_k)_{k \geq 0} \in \mathbb{Z}_{\{p_k\}} : y_{k'} \equiv y_k \pmod{p_k} \text{ for all } k' \geq k\}$$

is called an *odometer* (or *adding machine*) with respect to the periodic structure $\{p_k\}_{k \geq 0}$. It is clear that all the odometers are minimal equicontinuous systems.

2.3 Symbolic dynamics

In this subsection, we give some definitions and notations about symbolic dynamics.

Given a finite alphabet Σ , the set

$$\Sigma^{\mathbb{N}} = \{(x_i)_{i=0}^{\infty} : x_i \in \Sigma\}$$

is a compact metric space with metric ρ defined as follows:

$$\rho(x, y) = 2^{-\min\{i \in \mathbb{N} : x_i \neq y_i\}}$$

and $\rho(x, x) = 0$ for two distinct points $x = (x_i)_{i=0}^{\infty}$, $y = (y_i)_{i=0}^{\infty} \in \Sigma^{\mathbb{N}}$. For a positive integer n , the elements of Σ^n are called **words**, and their **lengths** are n . For a word $w = w_0 w_1 \cdots w_{n-1} \in \Sigma^n$, a **subword** of w is the word

$$w|_{[i,j)} := w_i w_{i+1} \cdots w_{j-1}.$$

for some $0 \leq i < j \leq n$. For convenience, we sometime write $w|_{[i,j)}$ by $w|_{[i,j-1]}$, using the closed interval. Also, for a point $x = x_0 x_1 \cdots \in \Sigma^n$, a subword of x is

the word $x|_{[i,j)} := x_i x_{i+1} \cdots x_{j-1}$ for some $0 \leq i < j$. For a word $w \in \Sigma^n$, the **cylinder set** of w is the set

$$[w] := \{x \in \Sigma^{\mathbb{N}} : x|_{[0,n)} = w\}.$$

For two words $w = w_0 w_1 \cdots w_{n-1}$ and $v = v_0 v_1 \cdots v_{m-1}$, their **concatenation** wv is the word $w_0 w_1 \cdots w_{n-1} v_0 v_1 \cdots v_{m-1}$.

Let (X, σ) be a subshift. The **language** $\mathcal{L}(X)$ of X is the collection of all the subwords of all the points in X , that is,

$$\mathcal{L}(X) := \{w \in \bigcup_{n=1}^{\infty} \Sigma^n : w \text{ is a subword of some } x \in X\} = \{w : [w] \cap X \neq \emptyset\}.$$

2.4 Permutations

For an integer $n \geq 1$, denote by $S(n)$ the set consists of all the permutations $\phi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ permuting i and j where $1 \leq i < j \leq n$, that is,

$$\phi(x) = \begin{cases} j, & x = i, \\ i, & x = j, \\ x, & \text{otherwise.} \end{cases}$$

And set $S^2(n) = \{\phi \circ \phi' : \phi, \phi' \in S(n)\}$. Then the cardinality of $S^2(n)$ is not greater than n^4 , which ensures that our example has positive entropy. It is easy to see that for large enough n and any $\phi \in S^2(n)$, there is some $1 \leq m_{\phi} \leq n$ such that $\phi(m_{\phi}) = m_{\phi}$, which admits the factor map corresponding to the maximal equicontinuous factor is proximal in our example.

3 Combinatorial lemmas

In this section, we give a combinatorial lemma which is used throughout the construction of our example. This lemma is used to ensure the multiple minimality of our example. And it is induced by an example of minimal subshifts with the properties that all points are multiply minimal, or moreover, this example is doubly minimal.

Theorem 3.1. [11, Theorem 4.3] *For any two distinct words $A, B \in \{0, 1\}^s$, where $s \geq 1$, and for any $K > 0$, there exists a minimal subshift $X \subset \{0, 1\}^{\mathbb{Z}}$ and N such that*

1. *(X, σ) is doubly minimal, weakly mixing and has zero topological entropy,*
2. *$A, B \in \mathcal{L}(X)$,*
3. *$\frac{1}{n} |\{j : x_{i+j} = 1, j < n\}| \leq 1/K$ for every $x \in X$, $i \in \mathbb{Z}$ and $n \geq N$,*
4. *each $x \in X$ is a concatenation of words A, B separated by sequences of zeros.*

Now, by this theorem, we give an example as following, which is more convenient for us to prove some combinatorial lemmas.

Lemma 3.2. *Let integer $Q \geq 1$. There exists a minimal subshift $Z \subset \{1, 2, 3, \dots, Q\}^{\mathbb{Z}}$ such that*

- (1) (Z^d, σ_d) is minimal for all $d \geq 1$,
- (2) For any $a, b \in \{1, 2, 3, \dots, Q\}$, $[ab] \cap Z \neq \emptyset$.

Proof. Take an integer $s' \geq 1$ with $2^{s'} > Q$. Then choose $s \geq 1$ large enough and distinct $A, B \in \{0, 1\}^s$ which are concatenations of words in $\{0, 1\}^{s'}$ and include all concatenations of any two words in $\{0, 1\}^{s'}$, that is, for each $w = A, B$,

$$\{w|_{[is', (i+2)s']} : 0 \leq i \leq \frac{s}{s'} - 2\} = \{uv : u, v \in \{0, 1\}^{s'}\}.$$

By Theorem 3.1, there is a minimal subshift (X, σ) which is doubly minimal and $A, B \in \mathcal{L}(X)$. Choose a bijection $\xi : \{0, 1\}^{s'} \rightarrow \{1, 2, 3, \dots, 2^{s'}\}$. Define $\xi^{\mathbb{Z}} : \{0, 1\}^{\mathbb{Z}} \rightarrow \{1, 2, 3, \dots, 2^{s'}\}^{\mathbb{Z}}$ by

$$\xi^{\mathbb{Z}}((x_i)_{i \in \mathbb{Z}}) = (\xi(x|_{[is', is'+s']}))_{i \in \mathbb{Z}}.$$

So $\xi^{\mathbb{Z}}$ is a continuous bijection and $\xi^{\mathbb{Z}} \circ \sigma^{s'} = \sigma \circ \xi^{\mathbb{Z}}$. Choose $x_* \in [A] \cap X$ and let $y_* = \xi^{\mathbb{Z}}(x_*) \in \{1, 2, 3, \dots, 2^{s'}\}^{\mathbb{Z}}$ and $Y = \overline{\mathcal{O}}(y_*, \sigma) \subset \{1, 2, 3, \dots, 2^{s'}\}^{\mathbb{Z}}$. Since (X, σ) is doubly minimal, by [1], (X^d, σ_d) is minimal for all d , which implies that (Y^d, σ_d) is also minimal for all d .

Finally, choose a surjection $\eta : \{1, 2, 3, \dots, 2^{s'}\} \rightarrow \{1, 2, 3, \dots, Q\}$, define $\eta^{\mathbb{Z}} : \{1, 2, 3, \dots, 2^{s'}\}^{\mathbb{Z}} \rightarrow \{1, 2, 3, \dots, Q\}^{\mathbb{Z}}$ by $\eta^{\mathbb{Z}}((x_i)_{i \in \mathbb{Z}}) = (\eta(x_i))_{i \in \mathbb{Z}}$ and let $Z := \eta^{\mathbb{Z}}(Y)$. By multiple minimality of (Y, σ) , (1) is proved. Since A is a concatenations of all words in $\{0, 1\}^{s'}$ and includes all concatenations of any two words in $\{0, 1\}^{s'}$, (2) is proved. So (Z, σ) is as required. \square

Next, we apply Lemma 3.2 to establish the following weaker combinatorial result, which encapsulates the core idea of the proof strategy.

Lemma 3.3. *Let $d \geq 1$ and $Q \geq 1$. Then there are $n > Q$ and a partition*

$$\{0, 1, 2, \dots, n\} = \biguplus_{q=1}^Q R_q$$

such that for any $0 \leq l \leq n$ and $1 \leq q \leq Q$, there is K such that for $1 \leq i \leq d$,

$$iK + l \in R_q \cup (R_q + n + 1).$$

Proof. Fix $d \geq 1$ and $Q \geq 1$. Let (Z, σ) be as in Lemma 3.2 for Q . We will use the multiple minimality of (Z, σ) to prove the lemma. For any $x \in Z$, there is $n_x > 0$ such that

$$x_0 = x_{n_x} = \dots = x_{dn_x}.$$

Then

$$\bigcup_{x \in Z} [x|_{[0, dn_x]}] = Z.$$

So there is a finite subset $I \subset Z$ such that

$$\bigcup_{x \in I} [x|_{[0, dn_x]}] = Z.$$

Since for any $1 \leq a \leq Q$, $[a] \cap Z \neq \emptyset$, we have $\{x_0 : x \in I\} = \{1, 2, 3, \dots, Q\}$.

Fix $z_* \in Z$. For each $x \in I$, there is $l_x > 0$ such that $\sigma^{l_x} z_*, \dots, \sigma^{dl_x} z_* \in [x|_{[0, dn_x]}]$. Let $L = \max\{dl_x + dn_x : x \in I\}$. Since (Z^d, σ_d) is minimal, there is $M > 0$ such that

$$\bigcup_{n=0}^M \sigma_d^{-n} [z_*|_{[0, L]}]^d = Z^d.$$

For integer $H \geq 1$, there exists $m_{t,H} \in [0, M] \cap \mathbb{N}$ such that $(\sigma^{H-t} z_*)^{(d)} \in \sigma_d^{-m_{t,H}} [z_*|_{[0, L]}]^d$ for all $1 \leq t \leq dM+L$. Let $\mathbf{m}_H = (m_{1,H}, m_{2,H}, \dots, m_{dM+l,H})$. Then we can find $L_1 < L_2$ such that $L_2 - L_1 > dM + L$ and $\mathbf{m}_{L_1} = \mathbf{m}_{L_2}$. We denote $(m_1, m_2, \dots, m_{dM+l}) = \mathbf{m}_{L_1} = \mathbf{m}_{L_2}$. Then

$$(\sigma^{L_1-t} z_*)^{(d)}, (\sigma^{L_2-t} z_*)^{(d)} \in \sigma_d^{-m_t} [z_*|_{[0, L]}]^d \text{ for all } 1 \leq t \leq dM + L.$$

Then let $n = L_2 - L_1 - 1$,

$$\{0, 1, 2, \dots, n\} = \biguplus_{a=1}^Q \{k \in \{0, 1, 2, \dots, n\} : \sigma^{L_1+k} z_* \in [a]\} := \biguplus_{a=1}^Q R_a.$$

Fix $1 \leq a \leq Q$ and $0 \leq l \leq n$. Divide l into 2 cases:

Case 1: $0 \leq l \leq n - dM - L$. Since

$$\bigcup_{n=0}^M \sigma_d^{-n} [z_*|_{[0, L]}]^d = Z^d,$$

there is $0 \leq m \leq M$,

$$(\sigma^{L_1+l} z_*)^{(d)} \in \sigma_d^{-m} [z_*|_{[0, L]}]^d,$$

that is, for $1 \leq i \leq d$,

$$\sigma^{L_1+l+im} z_* \in [z_*|_{[0, L]}].$$

Then for $x \in I$ with $x_0 = a$, we have

$$\sigma^{L_1+l+im+i(l_x+n_x)} z_* \in [x_0] = [a].$$

Since $l \leq n - dM - L$, we have $l + im + i(l_x + n_x) \in R_a$.

Case 2: $n - dM - L < l \leq n$. Let $t = n + 1 - l \leq dM + L$. Then we have

$$\sigma^{L_1+l+im_t} z_* \in [z_*|_{[0,L]}]$$

for $1 \leq i \leq d$. Then for $x \in I$ with $x_0 = a$, we have

$$\sigma^{L_1+l+im_t+i(l_x+n_x)} z_* \in [x_0] = [a]$$

for $1 \leq i \leq d$. For $1 \leq i \leq d$ with $l + im_t + i(l_x + n_x) \leq n$, we have already know $l + im_t + i(l_x + n_x) \in R_a$. For $1 \leq i \leq d$ with $l + im_t + i(l_x + n_x) > n$, noticing that

$$(\sigma^{L_1-t} z_*)^{(d)}, (\sigma^{L_2-t} z_*)^{(d)} \in \sigma_d^{-m_t} [z_*|_{[0,L]}]^d$$

and $L_1 - t = L_1 + l - (L_2 - L_1) = L_1 + l - n - 1$, we have

$$\sigma^{L_1+l-n-1} z_* \in \sigma_d^{-m_t} [z_*|_{[0,L]}],$$

that is,

$$\sigma^{L_1+l-n-1+im_t+i(l_x+n_x)} z_* \in [x_0] = [a].$$

So we have $l - n - 1 + im_t + i(l_x + n_x) \in R_a$.

Sum up with above cases, there are $0 \leq m \leq M$ such that for $1 \leq i \leq d$,

$$l + i(m + l_x + n_x) \in R_a \cup (R_a + n + 1)$$

where $x \in I$ with $x_0 = a$. □

Indeed, by choosing more n_x that x hits cylinder set $[x_0]$ and longer $L_2 - L_1$, we can prove the following stronger combinatorial lemma. For $c \geq 1$ and finite subset $R \subset \mathbb{N}$, write

$$\mathcal{R}_c(R) = \{R' \subset \mathbb{N} : \#R - \#(R \cap R') < c\}$$

Lemma 3.4. *Let positive integers $d \geq 1$, $c \geq 1$, $N \geq 1$ and $Q \geq 1$. Then there are $n \geq N$ and a partition*

$$\{0, 1, 2, \dots, n\} = \biguplus_{q=1}^Q R_q$$

satisfying the following property:

(i) $\#R_q \geq \sqrt{n} - 1$ for all $1 \leq q \leq Q$, and

(ii) for any $0 \leq l \leq n$, $1 \leq q, q' \leq Q$, $R_q^{(1)}, R_q^{(2)}, \dots, R_q^{(2d)} \in \mathcal{R}_c(R_q)$ and $R_{q'}^{(1)}, R_{q'}^{(2)}, \dots, R_{q'}^{(2d)} \in \mathcal{R}_c(R_{q'})$, there is an positive integer K such that for $1 \leq i \leq d$,

$$iK + l \in R_q^{(2i-1)} \cup (R_q^{(2i)} + n + 1)$$

and

$$iK + l + 1 \in R_{q'}^{(2i-1)} \cup (R_{q'}^{(2i)} + n + 1).$$

Proof. Fix $d \geq 1$, $c \geq 1$, $N \geq 1$ and $Q \geq 1$. Let (Z, σ) be as in Lemma 3.2 for d and Q .

For any $x \in Z$, since (Z^d, σ_d) is minimal, choose $4cd + 1$ positive integers $n_x^{(1)}, n_x^{(2)}, \dots, n_x^{(4cd+1)} \in N_{\sigma_d}(x^{(d)}, [x_0 x_1]^d)$ such that

$$dn_x^{(i)} < n_x^{(i+1)}, \quad 1 \leq i \leq 4cd.$$

Then we have

$$\bigcup_{x \in Z} [x]_{[0, dn_x^{(4cd+1)}]} = Z.$$

So there is a finite $I \subset Z$ such that

$$\bigcup_{x \in I} [x]_{[0, dn_x^{(4cd+1)}]} = Z.$$

Since for any word $w \in \{1, 2, 3, \dots, Q\}^2$, $[w] \cap Z \neq \emptyset$, we have $\{[x_0 x_1] : x \in I\} = \{[w] : w \in \{1, 2, 3, \dots, Q\}^2\}$.

Fix $z_* \in Z$. For $x \in I$, by minimality of (Z^d, σ_d) , there is $l_x > dn_x^{(4cd+1)}$ such that

$$\sigma_d^{l_x} ((z_*)^{(d)}) \in [x]_{[0, dn_x^{(4cd+1)}]}^d.$$

Then for any $n_x^{(i)}$, $1 \leq i \leq 4cd + 1$, we have $\sigma_d^{l_x + n_x^{(i)}} ((z_*)^{(d)}) \in [x_0 x_1]^d$.

Let $L = 2 + \max\{d(l_x + n_x^{(4cd+1)}) : x \in I\}$. By minimality of (Z^d, σ_d) , there is $M > 0$ such that

$$\bigcup_{m=0}^M \sigma_d^{-m} [z_*]_{[0, L]} = Z. \quad (3.1)$$

Then there are $0 \leq m_1, m_2, \dots, m_{dM+L} \leq M$ and $L_1 < L_2$ such that $L_2 - L_1 > \max\{(dM + L)^2, N\}$ and

$$(\sigma^{L_1-t} z_*)^{(d)}, (\sigma^{L_2-t} z_*)^{(d)} \in \sigma_d^{-m_t} [z_*]_{[0, L]}^d \text{ for all } 1 \leq t \leq dM + L. \quad (3.2)$$

Then let $n = L_2 - L_1 - 1$ and for $1 \leq q \leq Q$,

$$R_q = \{k \in \{0, 1, 2, \dots, n\} : \sigma^{L_1+k} z_* \in [q]\}.$$

So we have $\{0, 1, 2, \dots, n\} = \biguplus_{q=1}^Q R_q$.

We will show this partition is as required.

To prove (i), fix $1 \leq q \leq Q$. Since $n \geq (dM + L)^2$, we have

$$D := \left\lfloor \frac{n}{dM + L} \right\rfloor \geq \lfloor \sqrt{n} \rfloor \geq \sqrt{n} - 1$$

where $\lfloor x \rfloor$ denote the integer part of real number x . And then for $0 \leq D' < D$, $[D'(dM + L), (D' + 1)(dM + L)) \cap \mathbb{N} \subset \{0, 1, 2, \dots, n\}$. For each $0 \leq D' < D$, there is $0 \leq m \leq M$ such that $(\sigma^{L_1+D'(dM+L)} z_*)^{(d)} \in \sigma_d^{-m} [z_*]_{[0, L]}^d$. So we have $R_q \cap [D'(dM + L), (D' + 1)(dM + L)) \neq \emptyset$ for all $0 \leq D' < D$. So $\#R_q \geq D \geq \sqrt{n} - 1$. Therefore, (i) is proved.

To prove (ii), fix $0 \leq l \leq n$, $1 \leq q, q' \leq Q$, $R_q^{(1)}, R_q^{(2)}, \dots, R_q^{(2d)} \in \mathcal{R}_c(R_q)$ and $R_{q'}^{(1)}, R_{q'}^{(2)}, \dots, R_{q'}^{(2d)} \in \mathcal{R}_c(R_{q'})$. Divide l into 2 cases:

Case 1: $0 \leq l \leq n - dM - L$. Since

$$\bigcup_{m=0}^M \sigma_d^{-m}[z_*|_{[0,L]}]^d = Z^d,$$

there is $0 \leq m \leq M$,

$$(\sigma^{L_1+l} z_*)^{(d)} \in \sigma_d^{-m}[z_*|_{[0,L]}]^d,$$

that is, for $1 \leq i \leq d$,

$$\sigma^{L_1+l+im} z_* \in [z_*|_{[0,L]}].$$

Then for $x \in I$ with $x_0 = q$ and $x_1 = q'$, we have

$$\sigma_d^{m+l_x+n_x^{(j)}} (\sigma^{L_1+l} z_*)^{(d)} \in [x_0 x_1]^d = [qq']^d$$

for $1 \leq j \leq 4cd + 1$. Since $d(m + l_x + n_x^{(4cd+1)}) < dM + L$, we have

$$\begin{aligned} & \{m + l_x + n_x^{(j)} : 1 \leq j \leq 4cd + 1\} \\ & \subset \{m' \in \{0, 1, 2, \dots, n\} : l + im' \in R_q, l + im' + 1 \in R_{q'} \text{ for all } 1 \leq i \leq d\}. \end{aligned}$$

Since $l_x > dn_x^{(4cd+1)}$ and $dn_x^{(j)} < n_x^{(j+1)}$ for $1 \leq j < 4cd + 1$, then

$$l + im + i(l_x + n_x^{(j)}), 1 \leq i \leq d, 1 \leq j \leq 4cd + 1 \text{ are distinct.}$$

For $1 \leq j \leq 4cd + 1$ satisfying that there is some $1 \leq i \leq d$ such that $l + im + i(l_x + n_x^{(j)}) \notin R_q^{(2i-1)}$ or $l + im + i(l_x + n_x^{(j)}) + 1 \notin R_{q'}^{(2i-1)}$, since $R_q^{(2i-1)} \in \mathcal{R}_c(R_q)$ and $R_{q'}^{(2i-1)} \in \mathcal{R}_c(R_{q'})$, the amount of those j is not greater than $2cd$.

More strictly, for $1 \leq i \leq d$, set

$$J_i = \{1 \leq j \leq 4cd + 1 : l + im + i(l_x + n_x^{(j)}) \notin R_q^{(2i-1)}\}$$

and

$$J'_i = \{1 \leq j \leq 4cd + 1 : l + im + i(l_x + n_x^{(j)}) + 1 \notin R_{q'}^{(2i-1)}\}.$$

Since $R_q^{(2i-1)} \in \mathcal{R}_c(R_q)$, we have $\#J_i < c$. By a similar argument, we have $\#J'_i < c$. So

$$\# \left(\bigcup_{i=1}^d (J_i \cup J'_i) \right) \leq \sum_{i=1}^d (\#J_i + \#J'_i) < 2cd$$

Therefore, there is $1 \leq j \leq 4cd + 1$ such that for $1 \leq i \leq d$,

$$l + im + i(l_x + n_x^{(j)}) \in R_q^{(2i)} \text{ and } l + im + i(l_x + n_x^{(j)}) + 1 \in R_{q'}^{(2i)}.$$

Case 2: $n - dM - L < l \leq n$. Let $t = n + 1 - l \in [1, dM + L]$. So

$$(\sigma^{L_1-t} z_*)^{(d)}, (\sigma^{L_2-t} z_*)^{(d)} \in \sigma_d^{-m_t} [z_*|_{[0,L]}]^d.$$

Fix $x \in I$ with $x_0 = q$ and $x_1 = q'$. For $1 \leq j \leq 4cd + 1$ and $1 \leq i \leq d$, we have

$$\sigma^{L_1-t+im_t+i(l_x+n_x^{(j)})} z_*, \sigma^{L_2-t+im_t+i(l_x+n_x^{(j)})} z_* \in [x_0 x_1] = [qq'].$$

Noting that $L_2 - t = L_1 + l$ and $L_1 - t = L_1 + l - (n+1)$, for $1 \leq j \leq 4cd + 1$ and $1 \leq i \leq d$, we have

$$l + im_t + i(l_x + n_x^{(j)}) \in \begin{cases} R_q, & l + im_t + i(l_x + n_x^{(j)}) \leq n, \\ R_q + n + 1, & l + im_t + i(l_x + n_x^{(j)}) > n, \end{cases}$$

and

$$l + im_t + i(l_x + n_x^{(j)}) + 1 \in \begin{cases} R_{q'}, & l + im_t + i(l_x + n_x^{(j)}) < n, \\ R_{q'} + n + 1, & l + im_t + i(l_x + n_x^{(j)}) \geq n. \end{cases}$$

For $1 \leq i \leq d$, set

$$J_{2i-1} = \{1 \leq j \leq 4cd + 1 : n \geq l + im_t + i(l_x + n_x^{(j)}) \notin R_q^{(2i-1)}\},$$

$$J_{2i} = \{1 \leq j \leq 4cd + 1 : n < l + im_t + i(l_x + n_x^{(j)}) \notin R_q^{(2i)} + n + 1\},$$

$$J'_{2i-1} = \{1 \leq j \leq 4cd + 1 : n \geq l + im_t + i(l_x + n_x^{(j)}) + 1 \notin R_{q'}^{(2i-1)}\},$$

and

$$J'_{2i} = \{1 \leq j \leq 4cd + 1 : n < l + im_t + i(l_x + n_x^{(j)}) + 1 \notin R_{q'}^{(2i)} + n + 1\}.$$

Since $R_q^{(2i-1)} \in \mathcal{R}_c(R_q)$, we have $\#J_{2i-1} < c$. By similar arguments, we have $\#J_{2i}, \#J'_{2i-1}, \#J'_{2i} < c$. So

$$\# \left(\bigcup_{i=1}^{2d} J_i \cup J'_i \right) \leq \sum_{i=1}^{2d} (\#J_i + \#J'_i) < 4cd.$$

So there is $1 \leq j \leq 4cd + 1$ such that for $1 \leq i \leq d$,

$$l + im_t + i(l_x + n_x^{(j)}) \in R_q^{(2i-1)} \cup (R_q^{(2i)} + n + 1)$$

and

$$l + im_t + i(l_x + n_x^{(j)}) + 1 \in R_{q'}^{(2i-1)} \cup (R_{q'}^{(2i)} + n + 1).$$

Sum up with above 2 cases, we prove the partition is as required, which ends the proof. \square

4 Proof of Theorem 1.2

In this section, we will construct a minimal subshift which is an open proximal extension of its maximal equicontinuous factor. We will also show that all points in this subshift are multiply minimal.

4.1 Inductive construction of words

In this subsection, we construct a minimal subshift inductively. Given a finite alphabet $\Sigma = \{0, 1, 2\}$, our example is a subshift of $\{0, 1, 2\}^{\mathbb{N}}$.

Let word $A_0 = 2$. Choose $N_* > 100$ such that $2^{\sqrt{n}-1} > n^4 + 2$ for all $n \geq N_*$.

Step 0: By Lemma 3.4 for $d = 1$, $c = 6$, $N = N_*$ and $Q = 3$, there is $n_0 > N_*$ and a partition

$$\{0, 1, 2, \dots, n_0\} = R_{*,0} \cup \bigcup_{a \in \{0, 1\}} R_{a,0}$$

satisfying the properties in Lemma 3.4. Let

$$W_0 = \{A_0 w_1^{(0)} \cdots w_{n_0}^{(0)} : w_l^{(0)} \in \{0, 1\}, 1 \leq l \leq n_0\}.$$

For $\phi \in S^2(n_0)$, define $P_\phi^{(0)} : W_0 \rightarrow W_0$ by

$$A_0 w_1^{(0)} \cdots w_{n_0}^{(0)} \mapsto A_0 w_{\phi(1)}^{(0)} w_{\phi(2)}^{(0)} \cdots w_{\phi(n_0)}^{(0)}.$$

Let

$$\mathcal{W}_0 = \left\{ P_\phi^{(0)}(A_0 w_1^{(0)} \cdots w_{n_0}^{(0)}) : \begin{array}{l} w_l^{(0)} = a \text{ for } l \in R_{a,0}, a \in \{0, 1\}, \\ w_l^{(0)} \in \{0, 1\} \text{ for } l \in R_{*,0}, \phi \in S^2(n_0) \end{array} \right\} \subset W_0.$$

So for any word $w \in \mathcal{W}_0$, we have

$$\{n \in \{0, 1, 2, \dots, n_0\} : w|_{[n,n+1]} = a\} \in \mathcal{R}_6(R_{a,0})$$

for all $a \in \{0, 1\}$. Since all words in \mathcal{W}_0 have same length, denote by $p_0 = |\mathcal{W}_0|$ be the length of words in \mathcal{W}_0 .

Fix $A_1 = A_0 w_1^{(*,0)} w_2^{(*,0)} \cdots w_{n_0}^{(*,0)} \in \mathcal{W}_0$ such that

$$\text{for } 1 \leq l \leq n_0, w_l^{(*,0)} \begin{cases} = a, & l \in R_{a,0}, a \in \{0, 1\}, \\ \in \{0, 1\}, & l \in R_{*,0}. \end{cases}$$

Then the set $\mathcal{W}_0^{\text{pre}} := \{P_\phi^{(0)}(A_1) : \phi \in S^2(n_0)\} \subset \mathcal{W}_0$. Fix an enumeration $A_1^{(0)} = A_1, A_1^{(1)}, A_1^{(2)}, \dots, A_1^{(\hat{n}_0)}$ of $\mathcal{W}_0^{\text{pre}}$. Since

$$\#\mathcal{W}_0 \geq 2^{\#R_{*,0}} \geq 2^{\sqrt{n_0}-1} > n_0^4 + 2 > \hat{n}_0 + 1,$$

the set $\mathcal{W}_0^{\text{suf}} := \mathcal{W}_0 \setminus \mathcal{W}_0^{\text{pre}}$ contains at least 2 elements.

Step 1: By Lemma 3.4 for $d = 2$, $c = 6$, $N = N_*$ and $Q = 1 + \#\mathcal{W}_0^{\text{suf}}$, there is $n_1 > N_*$ and a partition

$$\{0, 1, 2, \dots, n_1\} = R_{*,1} \cup \bigcup_{a \in \mathcal{W}_0^{\text{suf}}} R_{a,1}$$

satisfying the properties in Lemma 3.4. Let

$$W_1 = \{A_1^{(i)} w_1^{(1)} \cdots w_{n_1}^{(1)} : 0 \leq i \leq \hat{n}_0, w_l^{(0)} \in \mathcal{W}_0^{\text{suf}}, 1 \leq l \leq n_1\}.$$

For $\phi \in S^2(n_1)$, define $P_\phi^{(1)} : W_1 \rightarrow W_1$ by

$$A_1^{(i)} w_1^{(1)} \cdots w_{n_1}^{(1)} \mapsto A_1^{(i)} w_{\phi(1)}^{(1)} w_{\phi(2)}^{(1)} \cdots w_{\phi(n_1)}^{(1)}.$$

Let

$$\mathcal{W}_1 = \left\{ \begin{array}{l} P_\phi^{(1)}(A_1^{(i)} w_1^{(1)} \cdots w_{n_1}^{(1)}) : 0 \leq i \leq \hat{n}_0, \phi \in S^2(n_1), \\ w_l^{(1)} = a \text{ for } l \in R_{a,1}, a \in \mathcal{W}_0^{\text{suf}}, \\ w_l^{(1)} \in \mathcal{W}_0^{\text{suf}} \text{ for } l \in R_{*,1} \end{array} \right\} \subset W_1.$$

So for any word $w \in \mathcal{W}_1$, we have

$$\{n \in \{0, 1, 2, \dots, n_1\} : w|_{[np_0, (n+1)p_0]} = a\} \in \mathcal{R}_6(R_{a,1})$$

for all $a \in \mathcal{W}_0^{\text{suf}}$. Since all words in \mathcal{W}_1 have same length, denote by $p_1 = |\mathcal{W}_1|$ be the length of words in \mathcal{W}_1 . Fix $A_2 = A_1 w_1^{(*,1)} w_2^{(*,1)} \cdots w_{n_1}^{(*,1)} \in \mathcal{W}_1$ such that

$$\text{for } 1 \leq l \leq n_1, w_l^{(*,1)} \left\{ \begin{array}{ll} = a, & l \in R_{a,1}, a \in \mathcal{W}_0^{\text{suf}}, \\ \in \mathcal{W}_0^{\text{suf}}, & l \in R_{*,1}. \end{array} \right.$$

Then the set

$$\mathcal{W}_1^{\text{pre}} := \{P_\phi^{(1)}(A_1^{(i)} w_1^{(*,1)} \cdots w_{n_1}^{(*,1)}) : 0 \leq i \leq \hat{n}_0, \phi \in S^2(n_1)\} \subset \mathcal{W}_1.$$

Fix an enumeration $A_2^{(0)} = A_2, A_2^{(1)}, A_2^{(2)}, \dots, A_2^{(\hat{n}_1)}$ of $\mathcal{W}_1^{\text{pre}}$. Since

$$\#\mathcal{W}_1 \geq 2^{\#R_{*,1}} \geq 2^{\sqrt{n_1}-1} > n_1^4 + 2 > \hat{n}_1 + 1,$$

the set $\mathcal{W}_1^{\text{suf}} := \mathcal{W}_1 \setminus \mathcal{W}_1^{\text{pre}}$ contains at least 2 elements.

Step $k+1$: Assume that, for some $k \geq 1$, we have defined integers $n_k > N_*$ and \hat{n}_k , sets $\mathcal{W}_k^{\text{pre}}, \mathcal{W}_k^{\text{suf}} \subset \mathcal{W}_k \subset W_k$ of words, the length $p_k = |\mathcal{W}_k|$ of words in \mathcal{W}_k , words $A_{k+1}^{(i)}$, $0 \leq i \leq \hat{n}_k$, and a partition

$$\{0, 1, 2, \dots, n_k\} = R_{*,k} \cup \bigcup_{a \in \mathcal{W}_{k-1}^{\text{suf}}} R_{a,k}$$

satisfying Lemma 3.4.

By Lemma 3.4 for $d = k + 2$, $c = 6$, $N = N_*$ and $Q = 1 + \#\mathcal{W}_k^{\text{suf}}$, there are $n_{k+1} > N_*$ and a partition

$$\{0, 1, 2, \dots, n_{k+1}\} = R_{*,k+1} \cup \bigcup_{a \in \mathcal{W}_k^{\text{suf}}} R_{a,k+1}$$

satisfying the properties in Lemma 3.4. Let

$$W_{k+1} = \{A_{k+1}^{(i)} w_1^{(k+1)} \cdots w_{n_{k+1}}^{(k+1)} : 0 \leq i \leq \hat{n}_k, w_l^{(k+1)} \in \mathcal{W}_k^{\text{suf}}, 1 \leq l \leq n_{k+1}\}.$$

For $\phi \in S^2(n_{k+1})$, define $P_\phi^{(k+1)} : W_{k+1} \rightarrow W_{k+1}$ by

$$A_{k+1}^{(i)} w_1^{(k+1)} \cdots w_{n_{k+1}}^{(k+1)} \mapsto A_{k+1}^{(i)} w_{\phi(1)}^{(k+1)} w_{\phi(2)}^{(k+1)} \cdots w_{\phi(n_{k+1})}^{(k+1)}.$$

Let

$$\mathcal{W}_{k+1} = \left\{ \begin{array}{l} P_\phi^{(k+1)}(A_{k+1}^{(i)} w_1^{(k+1)} \cdots w_{n_{k+1}}^{(k+1)}) : 0 \leq i \leq \hat{n}_k, \phi \in S^2(n_{k+1}), \\ w_l^{(k+1)} = a \text{ for } l \in R_{a,k+1}, a \in \mathcal{W}_k^{\text{suf}}, \\ w_l^{(k+1)} \in \mathcal{W}_k^{\text{suf}} \text{ for } l \in R_{*,k+1} \end{array} \right\} \subset W_{k+1}.$$

So for any word $w \in \mathcal{W}_{k+1}$, we have

$$\{n \in \{0, 1, 2, \dots, n_{k+1}\} : w|_{[np_k, (n+1)p_k]} = a\} \in \mathcal{R}_6(R_{a,k+1})$$

for all $a \in \mathcal{W}_k^{\text{suf}}$. Since all words in \mathcal{W}_{k+1} have same length, denote by $p_{k+1} = |\mathcal{W}_{k+1}|$ be the length of words in \mathcal{W}_{k+1} .

Fix $A_{k+2} = A_{k+1} w_1^{(*,k+1)} w_2^{(*,k+1)} \cdots w_{n_{k+1}}^{(*,k+1)} \in \mathcal{W}_{k+1}$ such that

$$\text{for } 1 \leq l \leq n_{k+1}, w_l^{(*,k+1)} \begin{cases} = a, & l \in R_{a,k+1}, a \in \mathcal{W}_k^{\text{suf}}, \\ \in \mathcal{W}_k^{\text{suf}}, & l \in R_{*,k+1}. \end{cases}$$

Then the set

$$\mathcal{W}_{k+1}^{\text{pre}} := \{P_\phi^{(k+1)}(A_{k+1}^{(i)} w_1^{(*,k+1)} \cdots w_{n_{k+1}}^{(*,k+1)}) : 0 \leq i \leq \hat{n}_k, \phi \in S^2(n_{k+1})\} \subset \mathcal{W}_{k+1}.$$

Fix an enumeration $A_{k+2}^{(0)} = A_{k+2}, A_{k+2}^{(1)}, A_{k+2}^{(2)}, \dots, A_{k+2}^{(\hat{n}_{k+1})}$ of the set $\mathcal{W}_{k+1}^{\text{pre}}$. Since, by $n_{k+1} > N_*$,

$$\#\mathcal{W}_{k+1} \geq (\hat{n}_k + 1)2^{\#R_{*,k+1}} \geq (\hat{n}_k + 1)2^{\sqrt{n_{k+1}} - 1} > (\hat{n}_k + 1)(n_{k+1}^4 + 2) > \hat{n}_{k+1} + 1,$$

the set $\mathcal{W}_{k+1}^{\text{suf}} = \mathcal{W}_{k+1} \setminus \mathcal{W}_{k+1}^{\text{pre}}$ contains at least 2 elements.

By the above process, we construct the followings: for $k \geq 0$,

- integers $n_k > N_*$ and \hat{n}_k ;

- maps $P_\phi^{(k)} : W_k \rightarrow W_k$, $\phi \in S^2(n_k)$, where

$$P_\phi^{(k)} : A_k^{(i)} w_1^{(k)} \cdots w_{n_k}^{(k)} \mapsto A_k^{(i)} w_{\phi(1)}^{(k)} w_{\phi(2)}^{(k)} \cdots w_{\phi(n_k)}^{(k)};$$

- a partition

$$\{0, 1, 2, \dots, n_k\} = R_{*,k} \cup \bigcup_{a \in \mathcal{W}_{k-1}^{\text{suf}}} R_{a,k}$$

satisfying Lemma 3.4 for $d = k + 1$, $c = 6$, $N = N_*$ and $Q = 1 + \#\mathcal{W}_{k-1}^{\text{suf}}$;

- word $A_{k+1} = A_k w_1^{(*,k)} \cdots w_{n_k}^{(*,k)} \in \mathcal{W}_k$ such that

$$\text{for } 1 \leq l \leq n_k, w_l^{(*,k)} \begin{cases} = a, & l \in R_{a,k}, a \in \mathcal{W}_{k-1}^{\text{suf}}, \\ \in \mathcal{W}_{k-1}^{\text{suf}}, & l \in R_{*,k}; \end{cases}$$

- sets of words

$$W_k = \{A_k^{(i)} w_1^{(k)} \cdots w_{n_k}^{(k)} : 0 \leq i \leq \hat{n}_{k-1}, w_l^{(k)} \in \mathcal{W}_{k-1}^{\text{suf}}, 1 \leq l \leq n_k\},$$

$$\mathcal{W}_k = \left\{ \begin{array}{l} P_\phi^{(k)}(A_k^{(i)} w_1^{(k)} \cdots w_{n_k}^{(k)}) : 0 \leq i \leq \hat{n}_{k-1}, \phi \in S^2(n_k), \\ w_l^{(k)} = a \text{ for } l \in R_{a,k}, a \in \mathcal{W}_{k-1}^{\text{suf}}, \\ w_l^{(k)} \in \mathcal{W}_{k-1}^{\text{suf}} \text{ for } l \in R_{*,k} \end{array} \right\},$$

$$\begin{aligned} \mathcal{W}_k^{\text{pre}} &= \{P_\phi^{(k)}(A_k^{(i)} w_1^{(*,k)} \cdots w_{n_k}^{(*,k)}) : 0 \leq i \leq \hat{n}_{k-1}, \phi \in S^2(n_k)\} \\ &= \{A_{k+1} = A_{k+1}^{(0)}, A_{k+1}^{(1)}, \dots, A_{k+1}^{(\hat{n}_k)}\}, \end{aligned}$$

and $\mathcal{W}_k^{\text{suf}} = \mathcal{W}_k \setminus \mathcal{W}_k^{\text{pre}}$, where we set $\mathcal{W}_{-1}^{\text{suf}} := \{0, 1\}$, $\hat{n}_{-1} := 0$, $A_0^{(0)} := A_0$, $p_{-1} = 1$ and $p_k = |W_k|$ the length of words in W_k .

The construction ensures the following properties: for $k \geq 0$,

- (i) $A_{k+1}|_{[0, p_{k-1})} = A_k$;
- (ii) $\mathcal{W}_k^{\text{pre}} \uplus \mathcal{W}_k^{\text{suf}} = \mathcal{W}_k \subset W_k$;
- (iii) for any word $w \in \mathcal{W}_k$ and $a \in \mathcal{W}_{k-1}^{\text{suf}}$, we have

$$\{n \in \{0, 1, 2, \dots, n_k\} : w|_{[np_{k-1}, (n+1)p_{k-1})} = a\} \in \mathcal{R}_6(R_{a,k}). \quad (4.1)$$

By (i), let $x^* = \lim_{k \rightarrow \infty} A_k$. Then x^* has forms like:

$$\begin{aligned}
x^* &= \underbrace{A_0 w_1^{(*,0)} w_2^{(*,0)} \cdots w_{n_0}^{(*,0)}}_{A_1} \underbrace{A_0 w_1^{(1,0)} w_2^{(1,0)} \cdots w_{n_0}^{(1,0)}}_{w_1^{(*,1)}} \underbrace{A_0 w_1^{(2,0)} w_2^{(2,0)} \cdots w_{n_0}^{(2,0)}}_{w_2^{(*,1)}} \cdots \\
&= \underbrace{A_1 w_1^{(*,1)} w_2^{(*,1)} \cdots w_{n_1}^{(*,1)}}_{A_2} \underbrace{A_1^{(i_1,1)} w_1^{(1,1)} w_2^{(1,1)} \cdots w_{n_1}^{(1,1)}}_{w_1^{(*,2)}} \underbrace{A_1^{(i_2,1)} w_1^{(2,1)} w_2^{(2,1)} \cdots w_{n_1}^{(2,1)}}_{w_2^{(*,2)}} \cdots \\
&= \underbrace{A_2 w_1^{(*,2)} w_2^{(*,2)} \cdots w_{n_2}^{(*,2)}}_{A_3} \underbrace{A_2^{(i_1,2)} w_1^{(1,2)} w_2^{(1,2)} \cdots w_{n_2}^{(1,2)}}_{w_1^{(*,3)}} \underbrace{A_2^{(i_2,2)} w_1^{(2,2)} w_2^{(2,2)} \cdots w_{n_2}^{(2,2)}}_{w_2^{(*,3)}} \cdots \\
&\cdots \\
&= \underbrace{A_k w_1^{(*,k)} w_2^{(*,k)} \cdots w_{n_k}^{(*,k)}}_{A_{k+1}} \underbrace{A_k^{(i_1,k)} w_1^{(1,k)} w_2^{(1,k)} \cdots w_{n_k}^{(1,k)}}_{w_1^{(*,k+1)}} \underbrace{A_k^{(i_2,k)} w_1^{(2,k)} w_2^{(2,k)} \cdots w_{n_k}^{(2,k)}}_{w_2^{(*,k+1)}} \cdots
\end{aligned}$$

Write $X^* = \overline{\mathcal{O}}(x^*, \sigma) \subset \Sigma^{\mathbb{N}}$. We will show that all the points in (X^*, σ) are multiply minimal in the next subsection.

First, we show some basic properties of (X^*, σ) .

Lemma 4.1. *Let $K \geq 1$. Then for any $0 \leq k < K$ and $w \in \mathcal{W}_K$,*

$$w|_{[ip_k, (i+1)p_k)} \in \mathcal{W}_k, \quad 0 \leq i < \frac{p_K}{p_k}.$$

In particular, for any $k \geq 0$, $x^*|_{[ip_k, (i+1)p_k)} \in \mathcal{W}_k$ for any $i \in \mathbb{N}$.

Proof. We prove it by induction. By the definition of \mathcal{W}_1 , it holds for $K = 1$.

Suppose that it holds for some $K \geq 1$. By the definition of \mathcal{W}_{K+1} , for any word $w \in \mathcal{W}_{K+1}$,

$$w|_{[ip_K, (i+1)p_K)} \in \mathcal{W}_K$$

holds for any $0 \leq i < \frac{p_{K+1}}{p_K}$. So by the induction hypothesis, for any $0 \leq k < K$, we have

$$w|_{[ip_k, (i+1)p_k)} \in \mathcal{W}_k$$

holds for any $0 \leq i < \frac{p_{K+1}}{p_k}$. So we prove the case of $K + 1$, which ends the proof.

For any $k, i \in \mathbb{N}$, choose some K such that $p_K > (i+1)p_k$ which implies $K > k$. By the definition of x^* , $x^*|_{[0, p_K)} = A_{K+1} \in \mathcal{W}_K$. Then we have

$$x^*|_{[ip_k, (i+1)p_k)} = A_{K+1}|_{[ip_k, (i+1)p_k)} \in \mathcal{W}_k.$$

□

Lemma 4.2. *Let x^* and X^* be defined as above. For any $k \geq 0$,*

$$(1) \quad N_\sigma(x^*, \bigcup_{i=0}^{\hat{n}_k-1} [A_k^{(i)}]) = p_k \mathbb{N};$$

(2) for any $x \in X^*$, there is a unique integer $0 \leq r_k(x) < p_k$ such that

$$N_\sigma(x, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]) = r_k(x) + p_k \mathbb{N}.$$

Proof. To prove (1), for any $K \geq 1$, by Lemma 4.1 and $A_{K+1} \in \mathcal{W}_K$, we have

$$x^*|_{[ip_k, (i+1)p_k)} = A_{K+1}|_{[ip_k, (i+1)p_k)} \in \mathcal{W}_k$$

for all $0 \leq i < \frac{p_K}{p_k}$ and $0 \leq k < K$. So by the definition of \mathcal{W}_k , we have $p_k \mathbb{N} \subset N_\sigma(x^*, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}])$ for all $k \geq 0$.

For $k \geq 0$, we prove the converse by induction. Since

$$x^*|_{[ip_0, (i+1)p_0)} \in \mathcal{W}_0$$

for all $i \in \mathbb{N}$, by the definition of \mathcal{W}_0 , we have $N_\sigma(x^*, [A_0]) \subset p_0 \mathbb{N}$. Suppose that $N_\sigma(x^*, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]) \subset p_k \mathbb{N}$ holds for some $k \geq 0$. For any $np_{k+1} + r \in N_\sigma(x^*, \bigcup_{i=0}^{\hat{n}_k} [A_{k+1}^{(i)}])$ where $0 \leq r < p_{k+1}$, since

$$\sigma^{np_{k+1}+r} x^* \in \bigcup_{i=0}^{\hat{n}_k} [A_{k+1}^{(i)}] \subset \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}],$$

by the induction hypothesis, we have $np_{k+1} + r \in p_k \mathbb{N}$ and $p_{k+1} = p_k(n_{k+1} + 1)$, which implies that $r \in p_k \mathbb{N}$. Let $l = \frac{r}{p_k} \in [0, n_{k+1}] \cap \mathbb{N}$. We have

$$x^*|_{[np_{k+1} + lp_k, np_{k+1} + (l+1)p_k)} \in \mathcal{W}_k^{\text{pre}},$$

and by $x^*|_{[np_{k+1}, (n+1)p_{k+1}]} \in \mathcal{W}_{k+1}$ and the definition of \mathcal{W}_{k+1} , we can conclude that $l = 0$. So we have $N_\sigma(x^*, \bigcup_{i=0}^{\hat{n}_k} [A_{k+1}^{(i)}]) \subset p_{k+1} \mathbb{N}$, which ends the proof of (1).

To prove (2), fix any $x \in X^*$ and $k \geq 0$. There is a sequence $\{m_i\}$ such that $\sigma^{m_i} x^* \rightarrow x$ as $i \rightarrow \infty$. So there are an integer $0 \leq r < p_k$ and a subsequence $\{m'_i\} \subset \{m_i\}$ such that $m'_i + r \equiv 0 \pmod{p_k}$ for all i . We will show that $N_\sigma(x, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]) = r + p_k \mathbb{N}$, which also shows the uniqueness of r . For any $n \in \mathbb{N}$, by (1) and $m'_i + r \equiv 0 \pmod{p_k}$, we have $\sigma^{np_k+r+m'_i} x^* \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$, which implies that $\sigma^{np_k+r} x \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$. So $r + p_k \mathbb{N} \subset N_\sigma(x, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}])$.

For the converse, for any $n \in N_\sigma(x, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}])$, that is, $\sigma^n x \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$, there is i large enough such that $\sigma^{n+m'_i} x^* \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$. By (1), we have $n + m'_i \in p_k \mathbb{N}$. Since $m'_i + r \equiv 0 \pmod{p_k}$, we have $n \in r + p_k \mathbb{N}$. So it ends the proof of (2). \square

By (2) of above lemma, we can define $r_k : X^* \rightarrow \{0, 1, 2, \dots, p_k - 1\}$ for $k \geq 0$, which not only induces the factor map corresponding the maximal equicontinuous factor, but also characterizes positions which words in \mathcal{W}_k appear in x .

The following lemma shows that each the concatenation of words in $\mathcal{W}_k^{\text{suf}}$ can appear in a given position of words in both $\mathcal{W}_{k+1}^{\text{pre}}$ and $\mathcal{W}_{k+1}^{\text{suf}}$. This property is a basic property of words in X^* , which is useful for the proof of multiple minimality and the openness of factor map.

Lemma 4.3. *Let $k \geq 0$. Then for any $A_{k+1}^{(i)} \in \mathcal{W}_k^{\text{pre}}$, $1 \leq l_1 < l_2 \leq n_{k+1}$ and $w_1^{(k+1)}, w_2^{(k+1)} \in \mathcal{W}_k^{\text{suf}}$, there exist $A_{k+2}^{(i')} \in \mathcal{W}_{k+1}^{\text{pre}}$ and $w^{(k+2)} \in \mathcal{W}_{k+1}^{\text{suf}}$ such that for each $w \in \{A_{k+2}^{(i')}, w^{(k+2)}\}$, we have*

$$w|_{[0,p_k]} = A_{k+1}^{(i)}, w|_{[l_1 p_k, (l_1+1)p_k]} = w_1^{(k+1)} \text{ and } w|_{[l_2 p_k, (l_2+1)p_k]} = w_2^{(k+1)}.$$

Proof. Fix $k \geq 0$, $A_{k+1}^{(i)} \in \mathcal{W}_k^{\text{pre}}$, $1 \leq l_1 < l_2 \leq n_{k+1}$ and $w_1^{(k+1)}, w_2^{(k+1)} \in \mathcal{W}_k^{\text{suf}}$. By the definition of $A_{k+2} = A_{k+1} w_1^{(*,k+1)} \dots w_{n_{k+1}}^{(*,k+1)} \in \mathcal{W}_{k+1}$, there is $1 \leq l'_1, l'_2 \leq n_{k+1}$ such that $w_{l'_1}^{(*,k+1)} = w_1^{(k+1)}$ and $w_{l'_2}^{(*,k+1)} = w_2^{(k+1)}$. Choose some $\phi_1 \in S^2(n_{k+1})$ such that $\phi_1(l'_1) = l_1$ and $\phi_1(l'_2) = l_2$. Then for some i' , word $A_{k+2}^{(i')} = P_{\phi_1}^{(k+1)}(A_{k+1}^{(i)} w_1^{(*,k+1)} \dots w_{n_{k+1}}^{(*,k+1)}) \in \mathcal{W}_{k+1}^{\text{pre}}$ satisfies that

$$A_{k+2}^{(i')}|_{[0,p_k]} = A_{k+1}^{(i)}, A_{k+2}^{(i')}|_{[l_1 p_k, (l_1+1)p_k]} = w_1^{(k+1)} \text{ and } A_{k+2}^{(i')}|_{[l_2 p_k, (l_2+1)p_k]} = w_2^{(k+1)}.$$

Since $\#R_{*,k+2} > 2\sqrt{n_{k+1}} - 1 \geq 9$, we can choose some $w \in \mathcal{W}_{k+1}^{\text{suf}}$ such that $w|_{[0,p_k]} = A_{k+1}^{(i)}$ and

$$\#\{1 \leq n \in R_{*,k+1} : w|_{[np_k, (n+1)p_k]} \neq w_n^{(*,k+1)}\} \geq 9.$$

So for any $\phi \in S^2(n_{k+1})$, we have $P_\phi^{(k+1)}(w) \notin \mathcal{W}_{k+1}^{\text{pre}}$. Since $w \in \mathcal{W}_{k+1}^{\text{suf}}$, there are $1 \leq l''_1, l''_2 \leq n_{k+1}$ such that $w|_{[l''_1 p_k, (l''_1+1)p_k]} = w_1^{(k+1)}$ and $w|_{[l''_2 p_k, (l''_2+1)p_k]} = w_2^{(k+1)}$. Choose some $\phi_2 \in S^2(n_{k+1})$ such that $\phi_2(l''_1) = l_1$ and $\phi_2(l''_2) = l_2$. Then let $w^{(k+2)} = P_{\phi_2}^{(k+1)}(w)$, and we have

$$w^{(k+2)}|_{[0,p_k]} = A_{k+1}^{(i)}, w^{(k+2)}|_{[l_1 p_k, (l_1+1)p_k]} = w_1^{(k+1)} \text{ and } w^{(k+2)}|_{[l_2 p_k, (l_2+1)p_k]} = w_2^{(k+1)}.$$

□

By the induction, we have the following corollary.

Corollary 4.4. *Let $k' \geq k$. For any $A_{k+1}^{(i)} \in \mathcal{W}_k^{\text{pre}}$, $w^{(k+1)} \in \mathcal{W}_k^{\text{suf}}$, $l_0 \in [0, p_{k'+1}] \cap p_{k+1}\mathbb{Z}$ and $l_1 \in [0, p_{k'+1}) \cap p_k\mathbb{Z} \setminus p_{k+1}\mathbb{Z}$, there exist $A_{k'+2}^{(i')} \in \mathcal{W}_{k'+1}^{\text{pre}}$ and $w^{(k'+2)} \in \mathcal{W}_{k'+1}^{\text{suf}}$ such that for each $w \in \{A_{k'+2}^{(i')}, w^{(k'+2)}\}$, we have*

$$w|_{[l_0, l_0+p_k]} = A_{k+1}^{(i)} \text{ and } w|_{[l_1, l_1+p_k]} = w^{(k+1)}.$$

Proof. Fix $k \geq 0$, $A_{k+1}^{(i)} \in \mathcal{W}_k^{\text{pre}}$, $w^{(k+1)} \in \mathcal{W}_k^{\text{suf}}$. We will prove by induction.

When $k' = k$, it is directly deduced from Lemma 4.3.

Suppose that the corollary holds for some $k' \geq k$. Fix $l_0 \in [0, p_{k'+2}) \cap p_{k+1}\mathbb{Z}$ and $l_1 \in [0, p_{k'+2}) \cap p_k\mathbb{Z} \setminus p_{k+1}\mathbb{Z}$. Then for each $i = 0, 1$, set

$$l_i = l'_i p_{k'+1} + l''_i$$

for some $0 \leq l'_i \leq n_{k'+2}$ and $0 \leq l''_i < p_{k'+1}$. So $l''_0 \in [0, p_{k'+1}) \cap p_{k+1}\mathbb{Z}$ and $l''_1 \in [0, p_{k'+1}) \cap p_k\mathbb{Z} \setminus p_{k+1}\mathbb{Z}$. By the assumption, there exist $A_{k'+2}^{(i')} \in \mathcal{W}_{k'+1}^{\text{pre}}$ and $w^{(k'+2)} \in \mathcal{W}_{k'+1}^{\text{suf}}$ such that for each $w \in \{A_{k'+2}^{(i')}, w^{(k'+2)}\}$, we have

$$w|_{[l''_0, l''_0 + p_k)} = A_{k+1}^{(i)} \text{ and } w|_{[l''_1, l''_1 + p_k)} = w^{(k+1)}.$$

By Lemma 4.3 for $k' + 1$, there exist $A_{k'+3}^{(i')} \in \mathcal{W}_{k'+2}^{\text{pre}}$ and $w^{(k'+3)} \in \mathcal{W}_{k'+2}^{\text{suf}}$ such that for each $w \in \{A_{k'+3}^{(i')}, w^{(k'+3)}\}$ and $i = 0, 1$,

$$w|_{[l'_i p_{k'+1}, (l'_i + 1)p_{k'+1})} = \begin{cases} A_{k'+2}^{(i')}, & l'_i = 0, \\ w^{(k'+2)}, & l'_i \neq 0. \end{cases}$$

Thus for each $w \in \{A_{k'+3}^{(i')}, w^{(k'+3)}\}$,

$$w|_{[l_0, l_0 + p_k)} = A_{k+1}^{(i)} \text{ and } w|_{[l_1, l_1 + p_k)} = w^{(k+1)},$$

which shows that the corollary holds for $k' + 1$. \square

4.2 Multiple minimality

In this subsection, we will show that all points in X^* are multiply minimal. Recall that for $d \geq 2$, $\sigma_d := \sigma \times \sigma^2 \times \cdots \times \sigma^d : (X^*)^d \rightarrow (X^*)^d$, and $x^{(d)} := (x, x, \dots, x) \in (X^*)^d$ for $x \in X^*$. And then we will prove that for all $x \in X$ and any neighbourhood $[w]$ of x , $N_{\sigma_d}(x^{(d)}, [w]^d)$ is syndetic.

Recall that for all $k \geq 0$, the partition

$$\{0, 1, 2, \dots, n_k\} = R_{*,k} \cup \bigcup_{a \in \mathcal{W}_{k-1}^{\text{suf}}} R_{a,k}$$

satisfying Lemma 3.4 for $d = k + 1$, $c = 6$, $N = N_*$ and $Q = 1 + \#\mathcal{W}_{k-1}^{\text{suf}}$, and for any word $w \in \mathcal{W}_k$ and $a \in \mathcal{W}_{k-1}^{\text{suf}}$, we have

$$\{n \in \{0, 1, 2, \dots, n_k\} : w|_{[n p_{k-1}, (n+1)p_{k-1})} = a\} \in \mathcal{R}_6(R_{a,k}). \quad (4.2)$$

The two properties mentioned above can ensure that all points in X^* are multiply minimal.

Proposition 4.5. *Let $d \geq 2$. For all $x \in X^*$, $(x, x, \dots, x) \in (X^*)^d$ is σ_d -minimal.*

Proof. Fix $d \geq 2$, $x \in X^*$ and $k \geq 0$. Let $w = x|_{[0, p_k]}$. Fix $K > d + k + 2$ and $n \geq 1$. We will show that there is $0 \leq m < 2(n_{k+2} + 1)$ such that $np_K + mp_{k+1} \in N_{\sigma_d}(x^{(d)}, [w]^d)$, which implies that $N_{\sigma_d}(x^{(d)}, [w]^d)$ is syndetic.

First, we need some preparations. Choose M large enough such that

$$\sigma^M x^*|_{[0, (d+2)np_K]} = x|_{[0, (d+2)np_K]}.$$

By Lemma 4.2, $\sigma^{r_k(x)} x \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$. So $\sigma^{M+r_k(x)} x^* \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$, by Lemma 4.2 again, $M + r_k(x) \in p_k \mathbb{N}$, where $r_k(x)$ is as in Lemma 4.2. Similarly, we have $M + r_{k+1}(x) \in p_{k+1} \mathbb{N}$ and $M + r_{k+2}(x) \in p_{k+2} \mathbb{N}$, where $r_{k+1}(x), r_{k+2}(x)$ are as in Lemma 4.2. Let $y_{k+i} = (p_{k+i} - r_{k+i}(x)) \bmod p_{k+i}$, $i = 0, 1, 2$. Then for $i = 0, 1, 2$, we have

$$M - y_{k+i} \in p_{k+i} \mathbb{N}.$$

By Lemma 4.1, suppose that

$$x^*|_{[M-y_{k+1}, M-y_{k+1}+p_{k+1}+p_k]} = A_{k+1}^{(j_0)} w_1^{(0,k+1)} \cdots w_{n_{k+1}}^{(0,k+1)} A_{k+1}^{(j'_0)}.$$

Then we have

$$\left(A_{k+1}^{(j_0)} w_1^{(0,k+1)} \cdots w_{n_{k+1}}^{(0,k+1)} A_{k+1}^{(j'_0)} \right) \Big|_{[y_{k+1}, y_{k+1}+p_k]} = x^*|_{[M, M+p_k]} = w.$$

Set $q := \frac{y_{k+1}-y_k}{p_k} \in \mathbb{N}$ and

$$l := \frac{y_{k+2}-y_{k+1}}{p_{k+1}} \in \mathbb{N}. \quad (4.3)$$

So we have $0 \leq q \leq n_{k+1}$ and $0 \leq l \leq n_{k+2}$.

Recall that $M - y_{k+2} \in p_{k+2} \mathbb{N}$. For $1 \leq i \leq d$, let

$$M_i := M - y_{k+2} + inp_K \in p_{k+2} \mathbb{N} \quad (4.4)$$

since $p_K \equiv 0 \pmod{p_{k+2}}$. By Lemma 4.1, for $1 \leq i \leq d$,

$$w_{2i-1} := x^*|_{[M_i, M_i+p_{k+2}]}, \quad w_{2i} := x^*|_{[M_i+p_{k+2}, M_i+2p_{k+2}]} \in \mathcal{W}_{k+2}. \quad (4.5)$$

We aim to prove that w appears in a suitable position of w_i . For the rest of the proof, we divide q into 2 cases:

Case 1: $0 \leq q < n_{k+1}$. In this case,

$$w = \begin{cases} \left(A_{k+1}^{(j_0)} w_1^{(0,k+1)} \right) \Big|_{[y_k, y_k+p_k]}, & q = 0, \\ \left(w_q^{(0,k+1)} w_{q+1}^{(0,k+1)} \right) \Big|_{[y_k, y_k+p_k]}, & 1 \leq q < n_{k+1}. \end{cases}$$

Next, we will find some $w_1^{(k+2)} \in \mathcal{W}_{k+1}^{\text{suf}}$ such that

$$\left(w_1^{(k+2)} \right) \Big|_{[y_{k+1}, y_{k+1}+p_k]} = w.$$

By Lemma 4.3, there is $w_1^{(k+2)} \in \mathcal{W}_{k+1}^{\text{suf}}$ such that

$$w_1^{(k+2)}|_{[qp_k, (q+2)p_k)} = \begin{cases} A_{k+1}^{(j_0)} w_1^{(0,k+1)}, & q = 0, \\ w_q^{(0,k+1)} w_{q+1}^{(0,k+1)}, & 1 \leq q < n_{k+1}. \end{cases}$$

Then we have

$$w_1^{(k+2)}|_{[y_{k+1}, y_{k+1} + p_k)} = w. \quad (4.6)$$

By (4.1) and (4.5), for $1 \leq i \leq 2d$, we have

$$R^{(i)} := \{n \in \{0, 1, 2, \dots, n_{k+2}\} : w_i|_{[np_{k+1}, (n+1)p_{k+1})} = w_1^{(k+2)}\} \in \mathcal{R}_6(R_{w_1^{(k+2)}, k+2}).$$

For those $R^{(i)}$, $1 \leq i \leq 2d$ and $0 \leq l = \frac{y_{k+2} - y_{k+1}}{p_{k+1}} \leq n_{k+2}$, by Lemma 3.4, there is a positive integer m such that for $1 \leq i \leq d$,

$$im + l \in R^{(2i-1)} \cup (R^{(2i)} + n_{k+2} + 1).$$

Noting that $m + l < 2(n_{k+2} + 1)$, then $m < 2(n_{k+2} + 1)$.

For each $1 \leq i \leq d$, we show that

$$\sigma^{M_i + (im+l)p_{k+1}} x^* \in [w_1^{(k+2)}].$$

If $im + l \in R^{(2i-1)}$, we have

$$w_{2i-1}|_{[(im+l)p_{k+1}, (im+l+1)p_{k+1})} = w_1^{(k+2)},$$

that is,

$$\sigma^{M_i + (im+l)p_{k+1}} x^* \in [w_1^{(k+2)}].$$

If $im + l \in R^{(2i)} + n_{k+2} + 1$, then

$$w_{2i}|_{[(im+l-(n_{k+2}+1))p_{k+1}, (im+l+1-(n_{k+2}+1))p_{k+1})} = w_1^{(k+2)}.$$

Noting that $(n_{k+2} + 1)p_{k+1} = p_{k+2}$, we have

$$\sigma^{M_i + p_{k+2} + (im+l-(n_{k+2}+1))p_{k+1}} x^* \in [w_1^{(k+2)}],$$

that is, $\sigma^{M_i + (im+l)p_{k+1}} x^* \in [w_1^{(k+2)}]$.

So we have proved $\sigma^{M_i + (im+l)p_{k+1}} x^* \in [w_1^{(k+2)}]$ for all $1 \leq i \leq d$. By (4.3) and (4.4), we have

$$M_i + (im + l)p_{k+1} = M + i(np_K + mp_{k+1}) - y_{k+1}.$$

So

$$\sigma^{M+i(np_K + mp_{k+1}) - y_{k+1}} x^* \in [w_1^{(k+2)}],$$

which implies that

$$\sigma^{M+i(np_K + mp_{k+1})} x^* \in \sigma^{y_{k+1}} [w_1^{(k+2)}] \subset [w],$$

where the last inclusion is implied by (4.6). Since $i(np_K + mp_{k+1}) + p_k < (d+2)np_K$, we have

$$x|_{[i(np_K + mp_{k+1}), i(np_K + mp_{k+1}) + p_k]} = x^*|_{[M+i(np_K + mp_{k+1}), M+i(np_K + mp_{k+1}) + p_k]} = w,$$

that is, $\sigma^{i(np_K + mp_{k+1})}x \in [w]$ for $1 \leq i \leq d$. Therefore, we have proved that $np_K + mp_{k+1} \in N_{\sigma_d}(x^{(d)}, [w]^d)$ for Case 1.

Case 2: $q = n_{k+1}$, that is, $y_{k+1} = y_k + n_{k+1}p_k$. In this case,

$$\left(w_{n_{k+1}}^{(0,k+1)} A_{k+1}^{(j'_0)}\right)|_{[y_k, y_k + p_k]} = w.$$

By Lemma 4.3, there are $w_1^{(k+2)}, w_2^{(k+2)} \in \mathcal{W}_{k+1}^{\text{suf}}$ such that

$$w_1^{(k+2)}|_{[n_{k+1}p_k, (n_{k+1}+1)p_k]} = w_{n_{k+1}}^{(0,k+1)} \text{ and } w_2^{(k+2)}|_{[0, p_k]} = A_{k+1}^{(j'_0)}.$$

So we have

$$\left(w_1^{(k+2)} w_2^{(k+2)}\right)|_{[y_{k+1}, y_{k+1} + p_k]} = \left(w_{n_{k+1}}^{(0,k+1)} A_{k+1}^{(j'_0)}\right)|_{[y_k, y_k + p_k]} = w. \quad (4.7)$$

In this case, it is different from Case 1 that we should ensure words $w_1^{(k+2)}$ and $w_2^{(k+2)}$ appear in w_i consecutively, which is implied by the full conclusion of Lemma 3.4, while Case 1 uses half of them. By (4.1) and (4.5), for $1 \leq i \leq 2d$, we have

$$R_1^{(i)} := \{n \in \{0, 1, 2, \dots, n_{k+2}\} : w_i|_{[np_{k+1}, (n+1)p_{k+1}]} = w_1^{(k+2)}\} \in \mathcal{R}_6(R_{w_1^{(k+2)}, k+2}),$$

and

$$R_2^{(i)} := \{n \in \{0, 1, 2, \dots, n_{k+2}\} : w_i|_{[np_{k+1}, (n+1)p_{k+1}]} = w_2^{(k+2)}\} \in \mathcal{R}_6(R_{w_2^{(k+2)}, k+2}).$$

For those $R_1^{(i)}, R_2^{(i)}$, $1 \leq i \leq 2d$ and $0 \leq l = \frac{y_{k+2}-y_{k+1}}{p_{k+1}} \leq n_{k+2}$, by Lemma 3.4, there is a positive integer m such that for $1 \leq i \leq d$,

$$im + l \in R_1^{(2i-1)} \cup (R_1^{(2i)} + n_{k+2} + 1),$$

and

$$im + l + 1 \in R_2^{(2i-1)} \cup (R_2^{(2i)} + n_{k+2} + 1).$$

Noting that $m + l < 2(n_{k+2} + 1)$, then $m < 2(n_{k+2} + 1)$.

For each $1 \leq i \leq d$, we show that

$$\sigma^{M_i + (im+l)p_{k+1}} x^* \in [w_1^{(k+2)}] \quad (4.8)$$

and

$$\sigma^{M_i + (im+l+1)p_{k+1}} x^* \in [w_2^{(k+2)}], \quad (4.9)$$

which imply that

$$\sigma^{M_i + (im+l)p_{k+1}} x^* \in [w_1^{(k+2)} w_2^{(k+2)}]. \quad (4.10)$$

We only prove (4.9) since the proof of (4.8) is similar. If $im + l + 1 \in R_2^{(2i-1)}$, then

$$w_{2i-1}|_{[(im+l+1)p_{k+1}, (im+l+2)p_{k+1})} = w_2^{(k+2)},$$

that is,

$$\sigma^{M_i+(im+l+1)p_{k+1}}x^* \in [w_2^{(k+2)}].$$

If $im + l + 1 \in R^{(2i)} + n_{k+2} + 1$, then

$$w_{2i}|_{[(im+l+1-(n_{k+2}+1))p_{k+1}, (im+l+2-(n_{k+2}+1))p_{k+1})} = w_2^{(k+2)}.$$

Noting that $(n_{k+2} + 1)p_{k+1} = p_{k+2}$, we have

$$\sigma^{M_i+p_{k+2}+(im+l+1-(n_{k+2}+1))p_{k+1}}x^* \in [w_2^{(k+2)}],$$

that is, $\sigma^{M_i+(im+l+1)p_{k+1}}x^* \in [w_2^{(k+2)}]$.

Now we have proved (4.10). By (4.3) and (4.4), we have

$$M_i + (im + l)p_{k+1} = M + i(np_K + mp_{k+1}) - y_{k+1}.$$

And by (4.7), we have $\sigma^{M+i(np_K+mp_{k+1})}x^* \in [w]$. Since $i(np_K + mp_{k+1}) < (d+2)np_K$,

$$x|_{[i(np_K+mp_{k+1}), i(np_K+mp_{k+1})+p_k]} = x^*|_{[M+i(np_K+mp_{k+1}), M+i(np_K+mp_{k+1})+p_k]} = w,$$

that is, $\sigma^{i(np_K+mp_{k+1})}x \in [w]$ for all $1 \leq i \leq d$. Therefore, we have proved $np_K + mp_{k+1} \in N_{\sigma_d}(x^{(d)}, [w]^d)$. \square

4.3 Open and proximal extensions

In this subsection, we will prove that (X^*, σ) is an open proximal extension of its maximal equicontinuous factor.

First, we characterize its maximal equicontinuous factor. For a positive integer n , we define $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$. Recall that p_k is the length of words in \mathcal{W}_k . Then the sequence $\{p_k\}_{k=0}^\infty$ is a periodic structure.

By Lemma 4.2, for $k \geq 0$, define $\pi_k : X^* \rightarrow \mathbb{Z}_{p_k}$ by

$$\pi_k(x) = (p_k - r_k(x)) \bmod p_k. \quad (4.11)$$

And $\pi : X^* \rightarrow \mathbb{Z}_{\{p_k\}}$ by

$$\pi(x) = (\pi_k(x))_{k=0}^\infty. \quad (4.12)$$

Let Y be the odometer with respect to the periodic structure $\{p_k\}_{k=0}^\infty$.

We will show that Y is a factor X by the factor map π .

Lemma 4.6. *Let π be defined as (4.12). For any $x \in X^*$, there is a sequence $\{m_k\}_{k=0}^\infty$ such that*

- (i) $\sigma^{m_k}x^* \rightarrow x$ as $k \rightarrow \infty$;

(ii) $R_{\underline{1}}^{m_k}(\underline{0})|_{[0, K+1]} = (\pi_0(x), \pi_1(x), \dots, \pi_K(x))$ for all $k \geq K \geq 0$.

In particular, $\pi(X^*) \subset Y$.

Proof. For any $x \in X^*$, there is a sequence $\{m'_j\}_{j=0}^\infty$ such that $\sigma^{m'_j}x^* \rightarrow x$ as $j \rightarrow \infty$. For $k \geq 0$, since $\sigma^{r_k(x)}x \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$ by Lemma 4.2, there is J_k such that for any $j \geq J_k$, $\sigma^{m'_j+r_k(x)}x^* \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$. Then by Lemma 4.2, $m'_j + r_k(x) \equiv 0 \pmod{p_k}$, that is, $m'_j \equiv \pi_k(x) \pmod{p_k}$ for $j > J_k$. Without loss of generality, assume that $J_{k+1} > J_k$ for all $k \geq 0$. So for any $k \geq 0$ and $j \geq J_k$,

$$\left(R_{\underline{1}}^{m'_j}(\underline{0})\right)_k = \pi_k(x).$$

Let $m_k = m'_{J_k}$. Then (ii) holds. Since $\{m_k\}$ is a subsequence of $\{m'_j\}$, (i) holds.

By (ii), it implies that $\pi(X^*) \subset Y$. \square

Lemma 4.7. *Let π be defined as (4.12). Then $\pi(X^*) = Y$.*

Proof. By Lemma 4.6, we prove that $\pi(X^*) \subset Y$. To prove $Y \subset \pi(X^*)$, fix $y = (y_k)_{k=0}^\infty \in Y$. Choose a sequence $\{m_j\}$ such that $R_{\underline{1}}^{m_j}(\underline{0}) \rightarrow y$ as $j \rightarrow \infty$, and

$$\left(R_{\underline{1}}^{m_j}(\underline{0})\right)_k = y_k \text{ for any } j > k \geq 0. \quad (4.13)$$

By choosing a subsequence, we can assume that $\sigma^{m_j}x^* \rightarrow x$ as $j \rightarrow \infty$. Fix $k \geq 0$. By (4.13), $m_j + p_k - y_k \in p_k \mathbb{N}$ for $j > k$. So

$$\sigma^{p_k - y_k + m_j}x^* \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}] \text{ for } j > k.$$

Then we have $\sigma^{p_k - y_k}x \in \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]$. By Lemma 4.2, $p_k - y_k \in r_k(x) + p_k \mathbb{N}$, that is, $\pi_k(x) = y_k$. Therefore, $\pi(x) = y$, which implies that $Y \subset \pi(X^*)$. \square

Now, we restrict π on Y . Indeed, $\pi : X^* \rightarrow Y$ is a factor map.

Lemma 4.8. *Let π be defined as (4.12). Then $\pi : X^* \rightarrow Y$ is a factor map.*

Proof. By Lemma 4.7, π is a surjection. By Lemma 4.2, for any $k \geq 0$ and $r \in \mathbb{Z}_{p_k}$,

$$\pi_k^{-1}(\{r\}) = X^* \cap \sigma^{-(p_k - r)} \left(\bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}] \right),$$

which is an open subset of X^* . So π_k , $k \geq 0$ are continuous, which implies that π is continuous.

Fix $k \geq 0$. To prove $\pi_k(\sigma x) \equiv (\pi_k(x) + 1) \pmod{p_k}$, by (4.11), it is sufficient to show that $r_k(x) \equiv (r_k(\sigma x) + 1) \pmod{p_k}$. Note that

$$N_\sigma(\sigma x, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]) = \left(N_\sigma(x, \bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}]) - 1 \right) \cap \mathbb{N},$$

which ends the proof. \square

First, we will show that π is not a bijection, and moreover, the pre-image of each $y \in Y$ is an infinite set. By the proof of Lemma 4.8, it is sufficient to show that $X^* \cap [A_k^{(i)}] \neq \emptyset$ for all $0 \leq i \leq \hat{n}_{k-1}$ and $k \geq 1$, which is the following lemma.

Lemma 4.9. *Let $k \geq 0$. Then for any $A_{k+1}^{(i)} \in \mathcal{W}_k^{\text{pre}}$, there is m such that $\sigma^{mp_{k+1}}x^* \in [A_{k+1}^{(i)}]$.*

Proof. Fix $k \geq 0$ and $A_{k+1}^{(i)} \in \mathcal{W}_k^{\text{pre}}$. By Lemma 4.3, there is $w \in \mathcal{W}_{k+1}^{\text{suf}}$ such that $w|_{[0, p_k]} = A_{k+1}^{(i)}$. By the definition of A_{k+2} , there is m such that $A_{k+2}|_{[mp_{k+1}, (m+1)p_{k+1}]} = w$. By the definition of x^* , $\sigma^{mp_{k+1}}x^* \in [w] \subset [A_{k+1}^{(i)}]$. \square

By the above lemma, we can show that π is a non-trivial factor map.

Lemma 4.10. *For each $y \in Y$, $\pi^{-1}(y)$ is an infinite set.*

Proof. Fix $y = (y_k)_{k \geq 0} \in Y$. By Lemma 4.2, for any $k \geq 0$,

$$\pi^{-1}(y) \subset \pi_k^{-1}(y_k) = X^* \cap \sigma^{-(p_k - y_k)} \left(\bigcup_{i=0}^{\hat{n}_{k-1}} [A_k^{(i)}] \right).$$

Fix some $k_0 \geq 0$ and $A_{k_0+1}^{(i_0)}$ for some $0 \leq i_0 \leq \hat{n}_{k_0}$. For any $k \geq k_0$, let

$$l_k = p_{k+1} - y_{k+1} - (p_{k_0+1} - y_{k_0+1}) \in [0, p_{k+1}) \cap p_{k_0+1}\mathbb{Z}.$$

Then by Corollary 4.4, there is some $A_{k+2}^{(i)} \in \mathcal{W}_{k+1}^{\text{pre}}$ such that

$$A_{k+2}^{(i)}|_{[l_k, l_k + p_{k_0}]} = A_{k_0+1}^{(i_0)}.$$

By Lemma 4.9, we have

$$X^* \cap \left(\bigcup_{i=0}^{\hat{n}_k} [A_{k+1}^{(i)}] \right) \cap \sigma^{-l_k} [A_{k_0+1}^{(i_0)}] \neq \emptyset.$$

So for any $k \geq k_0$,

$$X^* \cap \sigma^{-(p_{k+1} - y_{k+1})} \left(\bigcup_{i=0}^{\hat{n}_k} [A_{k+1}^{(i)}] \right) \cap \sigma^{-(p_{k_0+1} - y_{k_0+1})} [A_{k_0+1}^{(i_0)}] \neq \emptyset.$$

Thus there is some $\delta_{k_0} > 0$ and $E_{k,k_0} \subset \pi_{k+1}^{-1}(y_{k+1})$ with $\#E_{k,k_0} = \hat{n}_{k_0} + 1$ such that for any distinct $x, x' \in E_{k,k_0}$, $\rho(x, x') > \delta_{k_0}$. Therefore, there is $E_{k_0} \subset \pi^{-1}(y)$ with $\#E_{k_0} = \hat{n}_{k_0} + 1$. By the arbitrariness of k_0 , we have $\#\pi^{-1}(y) = \infty$. \square

Next, we will prove the openness of π . Before the proof, we need the following lemma, which plays a key role in the proof of openness.

Lemma 4.11. Let an integer $K \geq 0$ and a word $w \in \mathcal{L}_{p_K}(X^*)$. Then there is K' such that for any $k > K'$, the following holds:

if $m \in \mathbb{N}$ and $0 \leq y_k < p_k$ satisfy

$$\sigma^{mp_k+y_k}x^* \in [w],$$

then for any $0 \leq y_{k+1} < p_{k+1}$ with $y_{k+1} - y_k \equiv 0 \pmod{p_{k+1}}$, there is $m' \in \mathbb{N}$ such that

$$\sigma^{m'p_{k+1}+y_{k+1}}x^* \in [w].$$

Proof. Fix $K \geq 0$ and a word $w \in \mathcal{L}_{p_K}(X^*)$. Let $K' > K + 2$. Fix $k > K$, $m \in \mathbb{N}$ and $0 \leq y_k < p_k$ with $\sigma^{mp_k+y_k}x^* \in [w]$. By Lemma 4.1,

$$\sigma^{mp_k}x^*|_{[0,p_k+p_{k-1})} = A_k^{(i)}w_1^{(k)} \cdots w_{n_k}^{(k)}A_k^{(i')}$$

for some $A_k^{(i)}, A_k^{(i')} \in \mathcal{W}_{k-1}^{\text{pre}}$ and $w_l^{(k)} \in \mathcal{W}_{k-1}^{\text{suf}}$, $l = 1, 2, \dots, n_k$.

Let $0 \leq y_{k-1} < p_{k-1}$ with $y_k \equiv y_{k-1} \pmod{p_{k-1}}$. Depending on which two consecutive words w appear, we divide y_k into 3 Cases:

Case 1: $0 \leq y_k < p_{k-1}$. In this case, we have $y_{k-1} = y_k$ and

$$w = \left(A_k^{(i)}w_1^{(k)} \right) \Big|_{[y_{k-1}, y_{k-1}+p_K)}. \quad (4.14)$$

For any $0 \leq y_{k+1} < p_{k+1}$ with $y_{k+1} \equiv y_k \pmod{p_k}$, by Lemma 4.3, there is a word

$$u = \begin{cases} A_{k+1}^{(i'')} \in \mathcal{W}_k^{\text{pre}}, & y_{k+1} = y_k, \\ w^{(k+1)} \in \mathcal{W}_k^{\text{suf}}, & y_{k+1} > y_k, \end{cases}$$

such that

$$u|_{[0, 2p_{k-1})} = A_k^{(i)}w_1^{(k)}. \quad (4.15)$$

Let $l = \frac{y_{k+1}-y_k}{p_k}$. By Lemma 4.3 again, there is $A_{k+2}^{(i''')} \in \mathcal{W}_{k+1}^{\text{pre}}$ such that

$$A_{k+2}^{(i''')}|_{[lp_{k+1}, (l+1)p_{k+1})} = u.$$

So combining this with (4.14) and (4.15), we have

$$A_{k+2}^{(i''')}|_{[y_{k+1}, y_{k+1}+p_K)} = w,$$

noting that $y_{k-1} = y_k$ in this case. By Lemma 4.9, there is $m'' \in \mathbb{N}$ such that $\sigma^{m''p_{k+2}}x^* \in [A_{k+2}^{(i''')}]$, which implies that $\sigma^{m''p_{k+2}+y_{k+1}}x^* \in [w]$.

Case 2: $p_{k-1} \leq y_k < p_k - p_{k-1}$. In this case, $1 \leq q := \frac{y_k-y_{k-1}}{p_{k-1}} < n_k$, and

$$w = \left(w_q^{(k)}w_{q+1}^{(k)} \right) \Big|_{[y_{k-1}, y_{k-1}+p_K)}. \quad (4.16)$$

For any $0 \leq y_{k+1} < p_{k+1}$ with $y_{k+1} \equiv y_k \pmod{p_k}$, by Lemma 4.3, there is a word

$$u = \begin{cases} A_{k+1}^{(i'')} \in \mathcal{W}_k^{\text{pre}}, & y_{k+1} = y_k, \\ w^{(k+1)} \in \mathcal{W}_k^{\text{suf}}, & y_{k+1} > y_k, \end{cases}$$

such that

$$u|_{[qp_{k-1},(q+2)p_{k-1}]} = w_q^{(k)} w_{q+1}^{(k)}. \quad (4.17)$$

Let $l = \frac{y_{k+1}-y_k}{p_k}$. By Lemma 4.3 again, there is $A_{k+2}^{(i''')} \in \mathcal{W}_{k+1}^{\text{pre}}$ such that

$$A_{k+2}^{(i''')}|_{[lp_{k+1},(l+1)p_{k+1}]} = u.$$

So combining this with (4.16) and (4.17), we have

$$A_{k+2}^{(i''')}|_{[y_{k+1},y_{k+1}+p_K]} = w,$$

noting that $y_k = y_{k-1} + qp_{k-1}$ in this case. By Lemma 4.9, there is $m'' \in \mathbb{N}$ such that $\sigma^{m''p_{k+2}}x^* \in [A_{k+2}^{(i''')}]$, which implies that $\sigma^{m''p_{k+2}+y_{k+1}}x^* \in [w]$.

Case 3: $p_k - p_{k-1} \leq y_k < p_k$. In this case, $y_k = n_k p_{k-1} + y_{k-1}$ and

$$w = \left(w_{n_k}^{(k)} A_k^{(i')} \right)|_{[y_{k-1},y_{k-1}+p_K]}. \quad (4.18)$$

For any $0 \leq y_{k+1} < p_{k+1}$ with $y_{k+1} \equiv y_k \pmod{p_k}$, let $l := \frac{y_{k+1}-y_k}{p_k}$ and we have $0 \leq l \leq n_{k+1}$.

We divide l into 2 cases:

Case 3.1: $0 \leq l < n_{k+1}$. By Lemma 4.3, there are

$$u_1 = \begin{cases} A_{k+1}^{(i'')} \in \mathcal{W}_k^{\text{pre}}, & l = 0, \\ w_1^{(k+1)} \in \mathcal{W}_k^{\text{suf}}, & 0 < l < n_{k+1}, \end{cases}$$

and $u_2 \in \mathcal{W}_k^{\text{suf}}$ such that

$$u_1|_{[n_k p_{k-1},(n_k+1)p_{k-1}]} = w_{n_k}^{(k)} \text{ and } u_2|_{[0,p_{k-1}]} = A_k^{(i')},$$

which implies that

$$(u_1 u_2)|_{[n_k p_{k-1},(n_k+2)p_{k-1}]} = w_{n_k}^{(k)} A_k^{(i')}. \quad (4.19)$$

Noting that the choice of u_1, u_2 depends on l , by Lemma 4.3 again, there is $A_{k+2}^{(i''')} \in \mathcal{W}_{k+1}^{\text{pre}}$ such that

$$A_{k+2}^{(i''')}|_{[lp_k,(l+2)p_k]} = u_1 u_2.$$

So combining this with (4.18) and (4.19), we have

$$A_{k+2}^{(i''')}|_{[y_{k+1},y_{k+1}+p_K]} = w$$

noting that $y_k = n_k p_{k-1} + y_{k-1}$ in this case. By Lemma 4.9, there is $m'' \in \mathbb{N}$ such that $\sigma^{m''p_{k+2}}x^* \in [A_{k+2}^{(i''')}]$, which implies that $\sigma^{m''p_{k+2}+y_{k+1}}x^* \in [w]$.

Case 3.2: $l = n_{k+1}$. By Lemma 4.3, there are $w^{(k+1)} \in \mathcal{W}_k^{\text{suf}}$ and $A_{k+1}^{(i'')} \in \mathcal{W}_k^{\text{pre}}$ such that

$$w^{(k+1)}|_{[n_k p_{k-1}, (n_k + 1)p_{k-1})} = w_{n_k}^{(k)} \text{ and } A_{k+1}^{(i'')}|_{[0, p_{k-1})} = A_k^{(i')},$$

which implies that

$$\left(w^{(k+1)} A_{k+1}^{(i'')}\right)|_{[n_k p_{k-1}, (n_k + 2)p_{k-1})} = w_{n_k}^{(k)} A_k^{(i')}.$$
 (4.20)

By Lemma 4.3 again, there are $A_{k+2}^{(i''')} \in \mathcal{W}_{k+1}^{\text{pre}}$ and $w^{(k+2)} \in \mathcal{W}_{k+1}^{\text{suf}}$ such that

$$A_{k+2}^{(i''')}|_{[n_k p_{k-1}, (n_k + 1)p_{k-1})} = w^{(k+1)} \text{ and } w^{(k+2)}|_{[0, p_{k-1})} = A_{k+1}^{(i'')},$$

which implies that

$$\left(A_{k+2}^{(i''')} w^{(k+2)}\right)|_{[l p_k, (l+2)p_k)} = w^{(k+1)} A_{k+1}^{(i'')}.$$
 (4.21)

Then by Lemma 4.3, there is $A_{k+3}^{(i^{(4)})} \in \mathcal{W}_{k+2}^{\text{pre}}$ such that

$$A_{k+3}^{(i^{(4)})}|_{[0, 2p_{k+1})} = A_{k+2}^{(i''')} w^{(k+2)}.$$

So combining this with (4.18), (4.20) and (4.21), we have

$$A_{k+3}^{(i^{(4)})}|_{[y_{k+1}, y_{k+1} + p_K)} = w,$$

noting that $y_k = n_k p_{k-1} + y_{k-1}$ in this case. By Lemma 4.9, there is $m'' \in \mathbb{N}$ such that $\sigma^{m'' p_{k+3}} x^* \in [A_{k+3}^{(i^{(4)})}]$, which implies that $\sigma^{m'' p_{k+3} + y_{k+1}} x^* \in [w]$.

Sum up with above cases, since $p_{k+3}, p_{k+2} \equiv 0 \pmod{p_{k+1}}$, then there is $m' \in \mathbb{N}$ such that $\sigma^{m' p_{k+1} + y_{k+1}} x^* \in [w]$, which ends the proof. \square

By the above lemma, we can prove that π is open by showing that the image under π of each open set contains some open neighborhood of each point in the image.

Proposition 4.12. *Let π be defined as (4.12). Then $\pi : X^* \rightarrow Y$ is open.*

Proof. Fix $K \geq 0$ and $x \in X^*$. Let $w = x|_{[0, p_K)}$ and $y = (y_k)_{k=0}^\infty = \pi(x) \in Y$. By Lemma 4.6, there is a sequence $\{m_j\}$ such that $\sigma^{m_j} x^* \rightarrow x$ as $j \rightarrow \infty$ and $m_j \equiv y_J \pmod{p_J}$ for all $j \geq J \geq 0$.

Let $M > K' > K$ where K' is as in Lemma 4.11. We will prove that

$$\{y' = (y'_k)_{k=0}^\infty \in Y : y'_j = y_j, j = 0, 1, 2, \dots, M\} \subset \pi([w] \cap X^*).$$
 (4.22)

To prove (4.22), fix a point $y' = (y'_k)_{k=0}^\infty$ that belongs to the set of the left-hand side of (4.22).

We first choose a sequence in $[w] \cap X^*$ by induction. Since $\sigma^{m_j} x^* \rightarrow x \in [w]$ as $j \rightarrow \infty$, there is $j_0 > M$ such that $\sigma^{m_{j_0}} x^* \in [w]$. Noting that

$m_{j_0} \equiv y_M \pmod{p_M}$, there is $m'_0 \in \mathbb{N}$ such that $m_{j_0} = m'_0 p_M + y_M$ and $\sigma^{m'_0 p_M + y_M} x^* \in [w]$. Since $y' \in Y$, $y'_{M+1} \equiv y'_M \pmod{p_M}$. And by $y'_M = y_M$, $y'_{M+1} \equiv y_M \pmod{p_M}$. Then by Lemma 4.11, there is $m'_1 \in \mathbb{N}$ such that $\sigma^{m'_1 p_{M+1} + y'_{M+1}} x^* \in [w]$. Suppose that for some $k \geq 1$ we have defined m'_k with $\sigma^{m'_k p_{M+k} + y'_{M+k}} x^* \in [w]$. Since $y' \in Y$, $y'_{M+k+1} \equiv y'_{M+k} \pmod{p_{M+k}}$. Then by Lemma 4.11, there is $m'_{k+1} \in \mathbb{N}$ such that $\sigma^{m'_{k+1} p_{M+k+1} + y'_{M+k+1}} x^* \in [w]$.

We have define a sequence $\{\sigma^{m'_k p_{M+k} + y'_{M+k}} x^*\}_{k=0}^\infty$ in $[w] \cap X^*$. Taking a subsequence, assume that $\sigma^{m'_{k_i} p_{M+k_i} + y'_{M+k_i}} x^* \rightarrow x'$ as $i \rightarrow \infty$. So $x' \in [w] \cap X^*$. Noting that for all $i \geq I \geq 0$,

$$\left(R_1^{m'_{k_I} p_{M+k_I} + y'_{M+k_I}}(\underline{0}) \right)_{M+k_I} = y'_{M+k_I},$$

since $\pi(x^*) = \underline{0}$ and Lemma 4.8, we have $\pi(x') = y'$.

Since the left-hand side of (4.22) is an open neighbourhood of y in Y , we prove that π is open. \square

Finally, we show the proximality of π .

Proposition 4.13. *Let π be defined as (4.12). Then $\pi : X^* \rightarrow Y$ is a proximal extension.*

Proof. Fix $x, y \in X^*$ with $\pi(x) = \pi(y)$. By Lemma 4.2 and the definition of π , $r_k := r_k(x) = r_k(y)$ for all $k \geq 0$.

Fix any $K \geq 0$. For any $k > K + 2$, by Lemma 4.2,

$$\sigma^{p_k + r_{K+2}} x, \sigma^{p_k + r_{K+2}} y \in \bigcup_{i=0}^{\hat{n}_{K+1}} [A_{K+2}^{(i)}].$$

Since all $A_{K+2}^{(i)} \in \mathcal{W}_{K+1}^{\text{pre}}$ is the image of A_{K+2} under $P_\phi^{(K+2)}$ for some $\phi \in S^2(n_{K+1})$, then for any $0 \leq i, i' \leq \hat{n}_{K+1}$, there is some $m_{i,i'}$ such that

$$A_{K+2}^{(i)}|_{[m_{i,i'} p_K, (m_{i,i'} + 1)p_K]} = A_{K+2}^{(i')}|_{[m_{i,i'} p_K, (m_{i,i'} + 1)p_K]}.$$

So let $\sigma^{p_k + r_{K+2}} x \in [A_{K+2}^{(i_k)}]$ and $\sigma^{p_k + r_{K+2}} y \in [A_{K+2}^{(i'_k)}]$, we have

$$(\sigma^{p_k + r_{K+2} + m_{i_k, i'_k} p_K} x)|_{[0, p_K]} = (\sigma^{p_k + r_{K+2} + m_{i_k, i'_k} p_K} y)|_{[0, p_K]}.$$

By the arbitrariness of K , it ends the proof. \square

Since $\pi : X^* \rightarrow Y$ is a proximal extension and Y is equicontinuous, Y is the maximal equicontinuous factor of X^* .

Proof of Theorem 1.2. Sum up with Proposition 4.12 and Proposition 4.13, we show that (X^*, σ) is an open proximal extension of its maximal equicontinuous factor. And by Proposition 4.5, for any $d \geq 2$ and $x \in X^*$, $x^{(d)}$ is σ_d -minimal. \square

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