

SHORTCUT LAAKSO SPACES, PURE PI UNRECTIFIABILITY AND DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS

DAVID BATE AND PIETRO WALD

ABSTRACT. We construct a family of purely PI unrectifiable Lipschitz differentiability spaces and investigate the possible of Banach spaces targets for which Lipschitz differentiability holds. We provide a general investigation into the geometry of *shortcut* metric spaces and characterise when such spaces are PI rectifiable, and when they are Y -LDS, for a given Y . The family of spaces arises as an example of our characterisations. Indeed, we show that Laakso spaces satisfy the required hypotheses.

CONTENTS

1. Introduction	2
1.1. Metric spaces with shortcuts	3
1.2. Quantitative differentiation on Laakso spaces	4
1.3. Shortcut Laakso spaces	4
1.4. Acknowledgements	4
2. Preliminaries	4
2.1. General notation	4
2.2. Lipschitz differentiability spaces	5
2.3. Three useful lemmas	10
2.4. PI spaces	11
2.5. PI rectifiability	13
2.6. Ahlfors-David regular spaces and Borel-Cantelli-type lemmas	16
3. Shortcut metric spaces	18
4. Shortcut metric spaces and PI (un)rectifiability	25
4.1. Pure PI unrectifiability	25
4.2. PI rectifiability	26
5. Shortcut metric spaces and Lipschitz differentiability	28
6. Laakso spaces	32
6.1. Laakso spaces as metric graphs	34
6.2. Cubical covers of Laakso spaces	34
6.3. Shortcuts in Laakso spaces	36
7. Harmonic approximation on LDS of analytic dimension 1	38
8. Quantitative differentiation on Laakso spaces	40
9. Shortcut Laakso spaces	42
10. Almost nowhere differentiable Lipschitz maps	42
11. A characterisation of non-superreflexive spaces	46
11.1. Diamond graphs	47
11.2. Projection of Laakso graphs onto diamond graphs	51
12. Lipschitz dimension and biLipschitz embeddability	51
Appendix A. Metric spaces with shortcuts uniformly and Carnot groups	57
A.1. Comparison with [LDLR17]	57
A.2. Carnot groups have shortcuts	58
References	62

Date: October 30, 2025.

1. INTRODUCTION

Recent years have seen much activity in the study of first order calculus on metric measure spaces (X, d, μ) satisfying a Poincaré inequality with a doubling measure (a *PI space*, see Definition 2.26). First introduced by [HK98], these spaces allow a general theory of first order Sobolev spaces to be developed, see the monographs [HKST15, BB11] for more information.

A cornerstone of the theory of PI spaces is the seminal work of Cheeger [Che99] that provides a generalisation of Rademacher's differentiation theorem to PI spaces. That is, up to a countable decomposition of X , there exists a Lipschitz function $\varphi: X \rightarrow \mathbb{R}^n$ such that the following holds. For any Lipschitz $f: X \rightarrow \mathbb{R}$ and μ -a.e. $x \in X$, there exists a unique linear $D: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$f(y) - f(x) = D(\varphi(y) - \varphi(x)) + o(d(y, x)).$$

Cheeger's theorem was later strengthened by Cheeger and Kleiner [CK09], who proved that this differentiability theory holds for any Lipschitz $f: X \rightarrow Y$, for any Banach space Y with the Radon-Nikodym property. These results have found numerous applications in biLipschitz non-embeddability problems [Che99, Section 14], [CK09, Corollary 1.7 and 1.8], [LN06, CKN11].

Cheeger's work stimulated many new lines of research. Notably, Keith [Kei04] shows that the conclusion of Cheeger's theorem continues to hold if one replaces the local Poincaré inequality with an infinitesimal condition regarding pointwise Lipschitz constants; this fits more closely with the infinitesimal conclusion of the theorem. Bate [Bat15] gives a study of *Lipschitz differentiability spaces* (LDS), metric measure spaces that satisfy the conclusion of Cheeger's theorem, without any further assumption such as a Poincaré inequality. In particular, several characterisations of LDS are given in terms of a rich structure of rectifiable curves known as an Alberti representation of the space. Since these results, it has been an important open question to determine to what extent the PI hypothesis is necessary for the conclusion of Cheeger's theorem to hold, see [Hei07, Introduction] and [CKS16, Question 1.17].

The work of Bate-Li [BL18] and Eriksson-Bique [EB19b] give a partial answer to this. They show that the *RNP* differentiability theory of Cheeger-Kleiner characterises PI spaces. More precisely, it characterises *PI rectifiable* spaces, metric measure spaces that can be covered by a countable union of measurable subsets of PI spaces (see Definition 2.32), for which porous sets have measure zero (see Definition 2.8). The question for LDS was answered, negatively, with a remarkable, and unfortunately unpublished, counterexample of Schioppa [Sch16b]: There is an LDS (\mathcal{S}, d, μ) which is *purely PI unrectifiable*. That is, any biLipschitz image of a PI space into \mathcal{S} has μ -measure zero. (see Definition 2.32). The space \mathcal{S} has analytic dimension 3 and Nagata and Lipschitz dimension at least 3. Moreover, any Lipschitz function $f: \mathcal{S} \rightarrow \ell_2$ is differentiable almost everywhere.

Schioppa's example, and its construction, immediately open many new questions: Can the dimensions of such an example be reduced below 3? Is ℓ_2 the only target for which Lipschitz functions are differentiable? More generally, for Banach spaces Y, Z , under what conditions does Y valued differentiability imply Z valued differentiability? For what Y there are purely PI unrectifiable spaces for which Y valued differentiability holds? Schioppa's construction produces a *single* space with the desired properties, from which it is not possible to extract underlying reasons why a purely PI unrectifiable space is an LDS. Indeed, Schioppa's example remains the *only* example. Is there a systematic approach to constructing such a space?

In this article we answer these questions and more. To state our main results we introduce the following terminology. For a Banach space Y , a metric measure space (X, d, μ) is a *Y -LDS* if every Lipschitz function $f: X \rightarrow Y$ is differentiable μ -almost everywhere. See Definition 2.2 for a precise definition.

Theorem 1.1. *Let $2 < s < \infty$. There is a compact s -Ahlfors-David regular metric measure space $\mathcal{L} = (\mathcal{L}, d, \mu)$ with the following properties:*

- (i) \mathcal{L} is purely PI unrectifiable;
- (ii) if $2 \leq q < s$ and Y is a q -uniformly convex Banach space, then \mathcal{L} is a Y -LDS;
- (iii) if $s < q < \infty$, no positive-measure μ -measurable subset of \mathcal{L} is an ℓ_q -LDS;

- (iv) if Y is a non-superreflexive Banach space, then no positive-measure μ -measurable subset of \mathcal{L} is a Y -LDS;
- (v) \mathcal{L} has analytic, Nagata, and Lipschitz dimension 1; this is minimal for purely PI unrectifiable LDS and answers [Sch16b, (Q1)];
- (vi) \mathcal{L} biLipschitz embeds in $L^1([0, 1])$, but no positive-measure μ -measurable subset admits a David-Semmes regular (in particular, biLipschitz) embedding in a Banach space with RNP.

Recall that a Banach space Y is non-superreflexive if and only if it does not have a uniformly convex renorming ([Pis16, Chapter 10 and 11]). Moreover, every uniformly convex space has q -uniformly convex renorming for some $2 \leq q < \infty$ ([Pis16, Theorem 10.2]). Hence, the spaces \mathcal{L} as in Theorem 1.1 are Y -LDS provided Y has a renorming which is ‘sufficiently’ uniformly convex. Furthermore, the existence of a uniformly convexity renorming is a necessary condition for \mathcal{L} to be Y -LDS. The precise threshold for the convexity modulus is given by the Hausdorff dimension of \mathcal{L} . Since ℓ_1 is non-superreflexive, Theorem 1.1 provides partial information towards [Sch16b, (Q2)].

The proof of Theorem 1.1 demonstrates, in a systematic way, *how* a purely PI unrectifiable LDS can arise. Indeed, our first contribution consists of the following, alternative point of view on the construction of Schioppa. Being purely PI unrectifiable, one could believe that (\mathcal{S}, d, μ) looks ‘far’ from a PI space. However, this is not exactly the case. Indeed, our work began with the observation that, when equipped with the length distance \hat{d} generated by d , $(\mathcal{S}, \hat{d}, \mu)$ is in fact a PI space. Moreover, the geometry of $(\mathcal{S}, \hat{d}, \mu)$ allows one to find, for every RNP Banach space Y , every Lipschitz $f: (X, \hat{d}) \rightarrow Y$, and μ -a.e. $x \in X$, a sequence $y_j \rightarrow x$, such that

$$f(y_j) - f(x) = o(\hat{d}(y_j, x)). \quad (1.2)$$

Further, using the ideas behind [Sch16b, Theorems 4.30 and 5.8], one may obtain a *quantitative* version of Eq. (1.2), for ℓ_2 -valued Lipschitz maps. One may then distort \hat{d} , contracting the distance between all of the (y_j, x) , in order to make the space purely PI unrectifiable. The fact that $(\mathcal{S}, \hat{d}, \mu)$ is an ℓ_2 -LDS, together with the quantitative Eq. (1.2), yield a direct proof that \mathcal{S} , equipped with the contracted distance, is an ℓ_2 -LDS. This contraction recovers the distance and result of Schioppa.

The proof of Theorem 1.1 builds upon our observation in two ways: We develop a general theory of contracting a metric space first introduced by Le Donne, Rajala and Li [LDLR17]; We show that it is possible to perform the construction on a more flexible and familiar PI space, the Laakso space [Laa00].

These results illustrate how Lipschitz differentiability can arise in purely PI unrectifiable spaces. Indeed, in the latter, the available curves cannot by themselves control Lipschitz oscillations and a new mechanism is necessary. Theorem 1.1 shows that, for pairs of points where curves lack, the geometry of the underlying metric space and target Banach space can interact in such a way to force wild oscillation to concentrate on a null set.

1.1. Metric spaces with shortcuts. Le Donne, Rajala and Li [LDLR17] introduce a method to construct Lipschitz images of an Ahlfors-David regular metric space by permitting *shortcuts* to be taken between carefully chosen points. This method answers a conjecture of Semmes by showing that the Heisenberg group is not *minimal in looking down*. Roughly speaking, by identifying pairs of points throughout the space that witness an excess of the triangle inequality, one can shorten distances between these and near by points, without affecting distances far away. Slightly more precisely, for some $0 < \delta < 1$ and each $i \in \mathbb{N}$, one can find δ^i separated nets formed of shortcuts, and we contract the distances between shortcuts by a factor of $\eta_i \rightarrow 0$. It is shown that the new metric space is not biLipschitz equivalent to the original space on any set of positive measure. Moreover, it is shown that the Heisenberg group and snowflake metric spaces possess shortcuts.

In order to construct the metric space in Theorem 1.1, we first generalise this construction by allowing the shortcuts to appear at an arbitrarily rate $\delta_i \rightarrow 0$ (for a precise comparison, see Appendix A.1). This generalisation is necessary so that Laakso spaces possess shortcuts, see Definition 6.17. We then proceed to give a general study of the resulting *shortcut metric space*.

In particular, in Section 4 we give conditions on η_i that characterise when a shortcut metric space is PI rectifiable and purely PI unrectifiable. By ensuring that the latter condition is satisfied, the first item of Theorem 1.1 is automatically satisfied.

In Section 5 we characterise at which points the derivative of a Lipschitz function on (X, d) is preserved under the shortcut construction. As a result, when the non-contracted space is a Y -LDS, we find a condition characterising when the contracted space is also a Y -LDS. The latter condition is not true in general. For instance, in Proposition 10.15, we give an example of an LDS, in fact, any s -ADR Laakso space for $1 < s < 2$ works, for which any (non-trivial) shortcut metric space is not an LDS. More results of this type are found in the same section, Section 10. Carnot groups (of step ≥ 2) have shortcuts (see Appendix A.2), but we do not know if they satisfy this differentiability condition. This question is related to the ‘Vertical vs Horizontal’ Poincaré inequalities in Carnot groups, see [ANT13, Ryo25], but does not seem to be directly implied by such results. However, it is satisfied when a suitable quantitative differentiation theory holds in the contracted or non-contracted space, see Corollary 5.18. We are able to prove such an estimate in s -ADR Laakso spaces with $s > 2$, leading to the second point of Theorem 1.1.

1.2. Quantitative differentiation on Laakso spaces. Laakso spaces were introduced in [Laa00] as examples of Ahlfors-David regular 1-PI spaces whose Hausdorff dimension varies continuously in $(1, \infty)$. Consequently, Laakso spaces are also LDS. It is natural to ask whether a quantitative differentiation theory holds in Laakso spaces. One conjecture is that a generalisation of the celebrated Dorronsoro theorem [Dor85] holds in Laakso spaces, analogously to the case in Heisenberg groups [FO20]. We do not know the answer to this question. Instead we prove a weaker form of quantitative differentiation holds in s -ADR Laakso spaces with $s > 2$, see Theorem 8.3. This suffices to apply Corollary 5.18 discussed above.

The first step of the proof of Theorem 8.3 is to prove a harmonic approximation of Lipschitz functions on LDS with analytic dimension 1 (that is, the map φ takes values in \mathbb{R}), see Proposition 7.8. When applied to Laakso spaces, the symmetry of the space is inherited by the harmonic approximation, see Lemma 8.2. This additional symmetry allows us to control the oscillation of a Lipschitz function by the oscillation of its harmonic approximation, and hence obtain Theorem 8.3.

1.3. Shortcut Laakso spaces. By combining the work discussed above, we show that a suitable shortcut space formed from an s -ADR Laakso space, $s > 2$, satisfies the first two points of Theorem 1.1. We call the resulting metric space a *shortcut Laakso space*, see Section 9. The remaining points of Theorem 1.1 are proven via a direct analysis of shortcut Laakso spaces in Sections 10 and 12. Point (iv) of Theorem 1.1 requires a new characterisation of non-superreflexive Banach spaces in terms of Laakso spaces, which may be of independent interest, see Theorem 10.17. The proof of this characterisation is given in Section 11.

1.4. Acknowledgements. We would like to thank Sylvester Eriksson-Bique for conversations on [Sch16b] and on obstructions to the construction of a similar space out of lower-dimensional cube complexes. In particular, the idea of using orthogonality in \mathbb{R}^2 to control the Lipschitz constant in Lemma 10.5 is due to Eriksson-Bique.

D.B. was supported by the European Union’s Horizon 2020 research and innovation programme (Grant agreement No. 948021). P.W. was supported by the Warwick Mathematics Institute Centre for Doctoral Training, and gratefully acknowledges funding from the University of Warwick and the UK Engineering and Physical Sciences Research Council (Grant number: EP/W524645/1).

2. PRELIMINARIES

2.1. General notation. We set $\inf \emptyset := \infty$ and $\sup \emptyset := 0$.

The set positive integers is denoted with $\mathbb{N} = \{1, 2, \dots\}$, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, we set $[n] := \{1, \dots, n\}$.

For two non-negative functions f, g on a set, we write $f \lesssim g$ or $g \gtrsim f$ if there is $C > 0$ such that $f \leq Cg$, and $f \sim g$ if $f \lesssim g \lesssim f$.

For a metric space X , we denote its distance as either d or d_X . Similarly, when a quantity depends on the distance of X and we wish to emphasise such dependence, we add the subscript X (or d). For instance, we may write $\text{diam}_X A$ (or $\text{diam}_d A$) for the diameter of a set $A \subseteq X$, $\text{diam } A := \sup\{d(x, y) : x, y \in A\}$.

For sets $A, B \subseteq X$ and $x \in X$, we set

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$$

and $d(A, x) := d(A, \{x\})$. For $A \subseteq X$ and $r > 0$, we define

$$B(A, r) := \{x \in X : d(A, x) \leq r\},$$

$B(x, r) := B(\{x\}, r)$, and

$$U(A, r) := \{x \in X : d(A, x) < r\},$$

$U(x, r) := U(\{x\}, r)$.

We call *measure* on a set Ω any map $\mu : \mathcal{P}(\Omega) := \{A : A \subseteq \Omega\} \rightarrow [0, \infty]$ which is monotone, countably subadditive, and vanishes at \emptyset . The restriction of μ to a set $E \subseteq \Omega$ is the measure defined as $\mu \llcorner E(A) := \mu(E \cap A)$ for $A \subseteq \Omega$. If Z is a set and $f : E \subseteq \Omega \rightarrow Z$ a map, we set $f_\# \mu(A) := \mu(f^{-1}(A))$ for $A \subseteq Z$. A set $E \subseteq \Omega$ is μ -measurable if

$$\mu(A \cap E) + \mu(A \setminus E) = \mu(A)$$

for all $A \subseteq \Omega$. The collection of μ -measurable sets forms a σ -algebra on which μ is countably additive; see [Fed69, Theorem 2.1.3].

We call *metric measure space* a triplet (X, d, μ) , where (X, d) is a separable metric space and μ a locally finite Borel regular measure satisfying

$$\mu(V) = \sup\{\mu(K) : K \text{ compact and } K \subseteq V\} \quad (2.1)$$

for all open sets $V \subseteq X$. From [Fed69, Theorem 2.2.2, 2.2.3] and (the argument of) [Fed69, 2.2.5], we deduce that μ is a Radon measure, i.e. Eq. (2.1) holds for any μ -measurable set V and, for any set $A \subseteq X$, we have

$$\mu(A) = \inf\{\mu(V) : V \text{ open and } A \subseteq V\}.$$

Recall that if $E \subseteq X$ is a non-empty set and $B \subseteq X$ Borel, then $B \cap E$ is a Borel set in $(E, d|_{E \times E})$; see e.g. [HKST15, Lemma 3.3.4]. From this (and the above), it is not difficult to see that, for any μ -measurable $E \subseteq X$, the triplet $(E, d|_{E \times E}, \mu|_{\mathcal{P}(E)})$ is also a metric measure space. In such cases, we will write (E, d, μ) in place of $(E, d|_{E \times E}, \mu|_{\mathcal{P}(E)})$.

If μ is a Borel measure on a separable metric space X , we set

$$\text{spt } \mu := X \setminus \bigcup \{V : V \text{ is open and } \mu(V) = 0\}.$$

2.2. Lipschitz differentiability spaces. Let X, Y be metric spaces. A function $f : X \rightarrow Y$ is *Lipschitz* if there is $L \geq 0$ such that $d_Y(f(x), f(y)) \leq Ld(x, y)$ for $x, y \in X$. The least such L is denoted by $\text{LIP}(f)$; if f is not Lipschitz, we set $\text{LIP}(f) := \infty$. We define

$$\text{LIP}(X; Y) := \{f : X \rightarrow Y : \text{LIP}(f) < \infty\}$$

and $\text{LIP}(X) := \text{LIP}(X; \mathbb{R})$.

For metric space X, Y and a function $f : X \rightarrow Y$, we define its *pointwise Lipschitz constant* at x as

$$\text{Lip}(f; x) := \limsup_{y \rightarrow x} \frac{d_Y(f(x), f(y))}{d(x, y)} = \limsup_{r \rightarrow 0} \sup_{y \in B(x, r)} \frac{d_Y(f(x), f(y))}{r}$$

if $x \in X$ is a limit point and $\text{Lip}(f; x) := 0$ otherwise. If f is continuous, $x \mapsto \text{Lip}(f; x)$ is Borel measurable.

Definition 2.2. Let Y be a Banach space. A *Y -Lipschitz differentiability space* (Y -LDS) (or Lipschitz differentiability space, LDS, if $Y = \mathbb{R}$) is a metric measure space (X, d, μ) satisfying the following condition. There is a countable collection of (*Cheeger*) *charts* (U_i, φ_i) , where $U_i \subseteq X$

is μ -measurable and $\varphi_i: X \rightarrow \mathbb{R}^{n_i}$ Lipschitz ($n_i \in \mathbb{N}_0$), such that $\mu(X \setminus \bigcup_i U_i) = 0$, and for every Lipschitz $f: X \rightarrow Y$, i , and μ -a.e. $x \in U_i$, there is a unique linear map $D: \mathbb{R}^{n_i} \rightarrow Y$ such that

$$\text{Lip}(f - D \circ \varphi_i; x) = 0. \quad (2.3)$$

We call $d_x f := d_x^{\varphi_i} f := D$ (*Cheeger*) *differential* (or φ_i -*differential*) of f at x and any collection $\{(U_i, \varphi_i)\}_i$ as above (*Cheeger*) *atlas*.

If Y is not the zero-dimensional Banach space, then any Y -LDS is an LDS.

Remark 2.4. Note that we allow $n_i = 0$, i.e. charts $(U_i, \varphi_i: X \rightarrow \mathbb{R}^0 = \{0\}) = (U_i, 0)$. In this case, a function $f: X \rightarrow Y$ has a φ_i -differential at $x \in U_i$ if and only if $\text{Lip}(f; x) = 0$. By definition of Lip , the latter condition is satisfied for every isolated point $x \in X$ and so $(\{x \in X: x \text{ is isolated}\}, 0)$ is a chart in any LDS (X, d, μ) . The converse is also true, see Proposition 2.17 (and [Bat15, Remark 4.10]).

We mostly need elementary properties of LDS and Y -LDS.

Lemma 2.5. *Let X be a metric space, $x \in X$, $n \in \mathbb{N}$, and let $\varphi: X \rightarrow \mathbb{R}^n$ be Lipschitz. The following are equivalent:*

- $\text{Lip}(\langle v, \varphi \rangle; x) > 0$ for $v \in \mathbb{R}^n \setminus \{0\}$;
- there is a constant $c > 0$ such that for every Banach space Y and linear $T: \mathbb{R}^n \rightarrow Y$ it holds

$$\text{Lip}(T \circ \varphi; x) \geq c \|T\|;$$

- for every Lipschitz function $f: X \rightarrow \mathbb{R}$ there is at most one $D \in (\mathbb{R}^n)^*$ such that $\text{Lip}(f - D \circ \varphi; x) = 0$;
- for every Banach space Y and function $f: X \rightarrow Y$ there is at most one linear map $D: \mathbb{R}^n \rightarrow Y$ such that $\text{Lip}(f - D \circ \varphi; x) = 0$.

Proof. The main implication is that the first point implies the second; we begin by establishing the others. The second point implies the first considering $Y = (\mathbb{R}^n)^*$. It also yields the fourth by taking $T = D_1 - D_2$ for differentials D_1, D_2 of f at x and using triangle inequality (for Lip). The fourth point implies the third, which implies the first by taking $f = \langle v, \varphi \rangle$ for $v \in \mathbb{R}^n \setminus \{0\}$ and observing that, by uniqueness, $\text{Lip}(f; x) = \text{Lip}(f - \langle 0, \varphi \rangle; x) > 0$.

It remains to prove that the first point implies the second. Arguing as in [BS13, Lemma 2.1], we find, for $1 \leq i \leq n$, sequences $(x_j^i)_j \subseteq X \setminus \{x\}$, $x_j^i \rightarrow x$, such that

$$\lim_{j \rightarrow \infty} \frac{\varphi(x_j^i) - \varphi(x)}{d(x_j^i, x)} =: b_i$$

exists for $1 \leq i \leq n$ and b_1, \dots, b_n form a basis of \mathbb{R}^n . Let $b_1^*, \dots, b_n^* \in (\mathbb{R}^n)^*$ denote the dual basis of b_1, \dots, b_n and set $p(v) := \sum_{i=1}^n |b_i^*(v)|$ for $v \in \mathbb{R}^n$. It is clear that $p(\cdot)$ is a norm on \mathbb{R}^n , and so there is $C \geq 1$ such that $C^{-1} \|\cdot\| \leq p(\cdot) \leq C \|\cdot\|$. Let Y be a Banach space, $T: \mathbb{R}^n \rightarrow Y$ linear, and let $v \in \mathbb{R}^n$, $\|v\| = 1$, be such that $\|T(v)\|_Y = \|T\|$. Then,

$$\|T\| \leq \sum_{i=1}^n |b_i^*(v)| \|T(b_i)\|_Y \leq p(v) \max_{1 \leq i \leq n} \|T(b_i)\|_Y \leq C \max_{1 \leq i \leq n} \|T(b_i)\|_Y.$$

Let $1 \leq i \leq n$ be such that $\|T\| \leq C \|T(b_i)\|_Y$. Then

$$\text{Lip}(T \circ \varphi; x) \geq \limsup_{j \rightarrow \infty} \frac{\|T \circ \varphi(x_j^i) - T \circ \varphi(x)\|_Y}{d(x_j^i, x)} = \|T(b_i)\|_Y \geq C^{-1} \|T\|.$$

□

Lemma 2.6. *Let Y be a non-zero Banach space, (X, d, μ) a Y -LDS, and $\{(U_i, \varphi_i: X \rightarrow \mathbb{R}^{n_i})\}_i$ and $\{(V_j, \psi_j: X \rightarrow \mathbb{R}^{m_j})\}_j$ be Cheeger atlases of (X, d, μ) when viewed as an \mathbb{R} -LDS and Y -LDS, respectively. Then, for i, j such that $\mu(U_i \cap V_j) > 0$, we have $n_i = m_j$ and the differentials $d_x^{\varphi_i} \psi_j$, $d_x^{\psi_j} \varphi_i$ are invertible at μ -a.e. $x \in U_i \cap V_j$.*

Proof. Drop i, j from the notation and set $W := U \cap V$. On W , we may differentiate one chart w.r.t the other obtaining, for μ -a.e. $x \in W$, linear maps $d_x^\varphi \psi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $d_x^\psi \varphi: \mathbb{R}^m \rightarrow \mathbb{R}^n$ which satisfy

$$\text{Lip}(\langle v, d_x^\psi \varphi \circ \psi \rangle; x) = \text{Lip}(\langle v, \varphi \rangle; x) > 0$$

and $\text{Lip}(\langle w, d_x^\varphi \psi \rangle; x) > 0$ for $v \in \mathbb{R}^n \setminus \{0\}$ and $w \in \mathbb{R}^m \setminus \{0\}$. In particular, $d_x^\psi \varphi$ and $d_x^\varphi \psi$ are surjective linear maps, proving $n = m$ and hence the thesis. \square

Lemma 2.6 implies that Cheeger atlases do not depend on the target and, moreover, that differentiability depends on the choice of Cheeger atlas only up to a μ -null set. Given a chart $(U, \varphi: X \rightarrow \mathbb{R}^n)$ with $\mu(U) > 0$, we call n the *analytic dimension* of U . More generally, if E is μ -measurable with $\mu(E) > 0$ and there are charts $(U_i, \varphi_i: X \rightarrow \mathbb{R}^n)$ with $\mu(E \setminus \bigcup_i U_i) = 0$, we say that E has analytic dimension n .

Lemma 2.7. *Let X be a metric space, $x \in X$, $n \in \mathbb{N}$, and let $\varphi: X \rightarrow \mathbb{R}^n$ be Lipschitz such that $\text{Lip}(\langle v, \varphi \rangle; x) > 0$ for $v \in \mathbb{R}^n \setminus \{0\}$. Then, for any Banach space Y and function $f: X \rightarrow Y$, the following are equivalent:*

- f is φ -differentiable at x , i.e. there is a unique linear map $T: \mathbb{R}^n \rightarrow Y$ such that

$$\text{Lip}(f - T \circ \varphi; x) = 0;$$

- for every $\epsilon > 0$ there is a linear map $T_\epsilon: \mathbb{R}^n \rightarrow Y$ such that

$$\text{Lip}(f - T_\epsilon \circ \varphi; x) \leq \epsilon.$$

A similar argument appears at the end of [BL18, Proof of backward implication of Theorem 8.2].

Proof. We need only to prove that the second item implies the first. Triangle inequality and Lemma 2.5 show that $(T_\epsilon)_\epsilon$ is Cauchy as $\epsilon \rightarrow 0$. Its limit $T: \mathbb{R}^n \rightarrow Y$ then satisfies

$$\text{Lip}(f - T \circ \varphi; x) \leq \text{Lip}(f - T_\epsilon \circ \varphi; x) + \|T - T_\epsilon\| \text{Lip}(\varphi; x) \rightarrow 0$$

as $\epsilon \rightarrow 0$. Uniqueness follows from Lemma 2.5. \square

Definition 2.8. Let (X, d) be a metric space and $E \subseteq X$ a set. We say that E is *porous at* $x \in E$ (w.r.t. d) if there is $\epsilon > 0$ and $(y_i) \subseteq X$, $y_i \rightarrow x$, such that $E \cap B(y_i, \epsilon d(x, y_i)) = \emptyset$ for all $i \in \mathbb{N}$. We call *porous* those sets E which are porous at every $x \in E$.

Observe that, for a non-empty subset E of a metric space X , the map $x \in X \mapsto d(E, x)$ is 1-Lipschitz.

Lemma 2.9. *Let X be a metric space, $E \subseteq X$ a set, and $x \in E$. Then the following are equivalent:*

- E is not porous at x ;
- for every metric space Y and $f: X \rightarrow Y$ Lipschitz, it holds $\text{Lip}(f|_E; x) = \text{Lip}(f; x)$;
- $\text{Lip}(d(E, \cdot); x) = 0$.

Proof. The fact that the first item implies the second is in [BL18, Lemma 2.6]. Then the second clearly implies the third, which implies the first by definition of porosity. \square

Remark 2.10. From Lemma 2.9, it is easy to see that for any non-empty set $E \subseteq X$ there is a porous Borel set E' such that

$$\{x \in E: E \text{ is porous at } x\} \subseteq E'.$$

Indeed, since $d(E, \cdot) = d(\overline{E}, \cdot)$, we may take $E' = \{x \in \overline{E}: \text{Lip}(d(E, \cdot); x) > 0\}$.

A Borel measure μ on a metric space X is called *doubling* if there is $C \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$$

for $x \in X$ and $0 < r < \text{diam } X$. More generally, a locally finite Borel measure μ on a separable metric space X is *asymptotically doubling* if $\limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty$ at μ -a.e. $x \in \text{spt } \mu$. Lebesgue differentiation theorem holds w.r.t. any asymptotically doubling measure (see e.g. [HKST15, Section 3.4]) and, in particular, almost every point of a measurable set is a Lebesgue density point.

Lemma 2.11. *Let X be a metric space and μ a doubling measure on X . Let $E \subseteq X$ be a μ -measurable set and $x \in E$ a Lebesgue density point of E w.r.t. μ . Then E is not porous at x .*

In particular, for any set $E \subseteq X$, μ -a.e. $x \in E$ is not a porosity point of E .

Proof. We prove the contrapositive. Suppose E is porous at $x \in E$ and let $\epsilon > 0$, $(y_i) \subseteq X \setminus \{x\}$ as in Definition 2.8. Set $r_i := d(x, y_i)$ and observe that

$$\mu(E \cap B(x, (1 + \epsilon)r_i)) \leq \mu(B(x, (1 + \epsilon)r_i)) - \mu(B(y_i, \epsilon r_i)).$$

By [HKST15, Lemma 8.1.13], there are constants $C \geq 1$ and $0 < q < \infty$ such that

$$\frac{\mu(B(y_i, \epsilon r_i))}{\mu(B(x, (1 + \epsilon)r_i))} \geq C^{-1} \left(\frac{\epsilon r_i}{(1 + \epsilon)r_i} \right)^q = C^{-1} (\epsilon / (1 + \epsilon))^q,$$

proving

$$\liminf_{r \rightarrow 0} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} < 1.$$

The ‘In particular’ part of the statement follows taking E' as in Remark 2.10 and observing that, by Lebesgue differentiation theorem and the above, it must be $\mu(E') = 0$. \square

Lemma 2.12 ([BS13, Theorem 2.4]). *Let (X, d, μ) be an LDS and $E \subseteq X$ a porous set. Then $\mu(E) = 0$.*

In particular, for any set $E \subseteq X$, μ -a.e. $x \in E$ is not a porosity point of E .

Proof. Bate and Speight [BS13] consider complete and separable metric measure spaces with a locally finite Borel measure, but, in fact, the same proof applies to any metric space with a σ -finite Borel measure. \square

The proof of [BL18, Lemma 2.10] gives the following.

Lemma 2.13 ([BL18, Lemma 2.10]). *Let Y be a Banach space, (X, d, μ) a Y -LDS, $\{(U_i, \varphi_i: X \rightarrow \mathbb{R}^{n_i})\}_i$ a Cheeger atlas, and let $E \subseteq X$ be a μ -measurable set. Then (E, d, μ) is a Y -LDS and $\{(U_i \cap E, \varphi_i)\}_i$ a Cheeger atlas.*

We also need the following.

Lemma 2.14. *Let Y be a Banach space, (X, d, μ) a Y -LDS, and $f, f_j: X \rightarrow Y$ Lipschitz maps. Suppose $f_j \rightarrow f$ pointwise on X and $\sup_j \text{Lip}(f_j) < \infty$. Then, for non-negative $g \in L^1(\mu)$, it holds*

$$\int_X g \text{Lip}(f; \cdot) d\mu \leq \liminf_{j \rightarrow \infty} \int_X g \text{Lip}(f_j; \cdot) d\mu. \quad (2.15)$$

In particular, for μ -measurable $U \subseteq X$ and $p \in [1, \infty]$, we have

$$\|\chi_U \text{Lip}(f; \cdot)\|_{L^p(\mu)} \leq \liminf_{j \rightarrow \infty} \|\chi_U \text{Lip}(f_j; \cdot)\|_{L^p(\mu)}. \quad (2.16)$$

Proof. Possibly replacing Y with the closed span of $\bigcup_j f_j(X)$, we may assume Y to be separable. Hence, there is a countable 1-norming set $\{y_n^*: n \in \mathbb{N}\} \subseteq Y^*$, i.e. $\sup_n |\langle y_n^*, y \rangle| = \|y\|_Y$ for $y \in Y$. It is then not difficult to see that $\text{Lip}(f; x) = \sup_n \text{Lip}(y_n^* \circ f; x)$ whenever $x \in X$ is a differentiability point f w.r.t. some chart. Let $g \in L^1(\mu)$ be non-negative and fix $\epsilon \in (0, 1)$. Set $\tilde{E}_n := \{\text{Lip}(y_n^* \circ f; \cdot) \geq (1 - \epsilon) \text{Lip}(f; \cdot)\}$, $E_n := \tilde{E}_n \setminus \bigcup_{m < n} \tilde{E}_m$, and observe that (E_n) is a disjoint collection of Borel sets covering μ -almost all of X . Since μ is Radon, there are compact sets $K_n \subseteq E_n$ such that $\int_{E_n \setminus K_n} g d\mu \leq \epsilon 2^{-n}$ for $n \in \mathbb{N}$. By Lemmas 2.9 and 2.12, we have $\text{Lip}((y_n^* \circ f)|_{K_n}; x) = \text{Lip}(y_n^* \circ f; x)$ at μ -a.e. $x \in K_n$. Hence, [BEBS24, Lemma 7.3] applied to the compact LDS (K_n, d, μ) and Lipschitz maps $(y_n^* \circ f)|_{K_n}$, $(y_n^* \circ f_j)|_{K_n}$ yields

$$\begin{aligned} (1 - \epsilon) \int_{K_n} g \text{Lip}(f; \cdot) d\mu &\leq \int_{K_n} g \text{Lip}(y_n^* \circ f; \cdot) d\mu \leq \liminf_{j \rightarrow \infty} \int_{K_n} g \text{Lip}((y_n^* \circ f_j)|_{K_n}; \cdot) d\mu \\ &\leq \liminf_{j \rightarrow \infty} \int_{K_n} g \text{Lip}(f_j; \cdot) d\mu, \end{aligned}$$

where we have also used $\|y_n^*\|_{Y^*} \leq 1$. Then, summing over $n \in \mathbb{N}$ and using $\int_{X \setminus \bigcup_n K_n} g \, d\mu \leq \epsilon$, we have

$$(1 - \epsilon) \int_X g \operatorname{Lip}(f; \cdot) \, d\mu \leq \liminf_{j \rightarrow \infty} \int_X g \operatorname{Lip}(f_j; \cdot) \, d\mu + \operatorname{LIP}(f)(1 - \epsilon)\epsilon,$$

which implies Eq. (2.15) because $\epsilon \in (0, 1)$ was arbitrary.

Finally, Eq. (2.16) follows from Eq. (2.15) by Hölder inequality and a standard dual characterisation of the $L^p(\mu)$ -norm. \square

Recall that a Banach space Y has the *Radon-Nikodym property (RNP)* if every Lipschitz function $f: \mathbb{R} \rightarrow Y$ is differentiable Lebesgue a.e., see [BL00, Pis16] for more on Banach spaces with RNP.

In the definition of Y -LDS, we do not require the Banach space Y to have the Radon-Nikodym property, because such assumption is a priori unnecessary. Nonetheless, it is not difficult to see that RNP is required in all but trivial cases. The equivalence between the first two points is due to Bate [Bat15, Remark 4.10].

Proposition 2.17. *Let $X = (X, d, \mu)$ be a metric measure space. Then the following are equivalent:*

- μ -a.e. $x \in X$ is isolated;
- X is an LDS of analytic dimension 0;
- X is a Y -LDS for every Banach space Y ;
- X is a Y -LDS for some Banach space Y without the Radon-Nikodym property.

For the proof of Proposition 2.17, we need the following lemma.

Lemma 2.18. *Let $E \subseteq \mathbb{R}$ be a set, Y a Banach space, $\varphi: E \rightarrow \mathbb{R}$ a function, and $f: \mathbb{R} \rightarrow Y$ a Lipschitz function. Suppose E is not porous at $t_0 \in E$, φ and $f \circ \varphi$ are differentiable at t_0 , and $\varphi'(t_0) \neq 0$. Then f is differentiable at $\varphi(t_0)$.*

Proof. We may assume w.l.o.g. $t_0 = 0 = \varphi(t_0)$ and $\varphi'(0) = 1$, so that $D := (f \circ \varphi)'(0) = (f \circ \varphi)'(0)/\varphi'(0) \in Y$. Let $(x_j) \subseteq \mathbb{R} \setminus \{0\}$ be a sequence converging to 0. Since E is not porous at $0 \in E$, there is $(t_j) \subseteq E \setminus \{0\}$ such that

$$\frac{|t_j - x_j|}{|x_j|} \rightarrow 0 \quad \text{and} \quad \frac{|t_j|}{|x_j|} \rightarrow 1, \quad (2.19)$$

where the second condition follows from the first. Then, if we divide by $|x_j|$ all terms of the inequality

$$\begin{aligned} \|f(x_j) - f(0) - Dx_j\|_Y &\leq \operatorname{LIP}(f)(|x_j - t_j| + |\varphi(t_j) - t_j|) + \|f \circ \varphi(t_j) - f(0) - Dt_j\|_Y \\ &\quad + \|D(t_j - x_j)\|_Y, \end{aligned}$$

we see from Eq. (2.19) that $\|f(x_j) - f(0) - Dx_j\|_Y/|x_j| \rightarrow 0$ as $j \rightarrow \infty$. Since (x_j) was arbitrary, this shows $f'(0) = D$, as claimed. \square

Proof of Proposition 2.17. It is clear that the first point implies all others (see also Remark 2.4) and that the third implies the fourth. We show that the fourth implies the second, which implies the first by [Bat15, Remark 4.10].

We begin by proving the following claim: If (X, d, μ) is a Y -LDS and there is a chart $(K, \varphi: X \rightarrow \mathbb{R}^n)$ with $n \geq 1$, K compact, and $\mu(K) > 0$, then Y has RNP.

By Lemma 2.13, φ is a chart also in the Y -LDS (K, d, μ) , so we may assume $K = X$. Since necessarily $Y \neq 0$, (X, d, μ) is also an LDS, and we are then in the setting of [Bat15]¹. Let $f: \mathbb{R} \rightarrow Y$ be Lipschitz; by [BL00, Theorem 5.21], we only need to show that f has a point of differentiability.

Write $\varphi = (\varphi^1, \dots, \varphi^n)$, set $F := f \circ \varphi^1$, and observe that, since $F: X \rightarrow Y$ is Lipschitz, we have

$$F(y) - F(x) = d_x^\varphi F(\varphi(y) - \varphi(x)) + o(d(x, y)), \quad (2.20)$$

¹In [Bat15], it is assumed that the (countable) set of isolated points is μ -null. This condition is satisfied in our case because $X = K$ has analytic dimension 1; see (the first part of) Remark 2.4.

for μ -a.e. $x \in X$. By [Bat15, Corollary 6.7], for μ -a.e. $x \in X$ and $1 \leq k \leq n$, there are biLipschitz functions $\gamma^{x,k}: K^{x,k} \rightarrow X$ such that: $K^{x,k} \subseteq \mathbb{R}$ is compact and $0 \in K^{x,k}$ is a Lebesgue density point w.r.t. Lebesgue measure, $\gamma^{x,k}(0) = x$, $(\varphi \circ \gamma^{x,1})'(0), \dots, (\varphi \circ \gamma^{x,n})'(0)$ exist and form a basis of \mathbb{R}^n .

Fix $x \in X$ such that the above and Eq. (2.19) hold, let $1 \leq k \leq n$ be such that $(\varphi^1 \circ \gamma^{x,k})'(0) \neq 0$, and observe that, by Eq. (2.19), $(F \circ \gamma^{x,k})'(0)$ exists. Then $f \circ (\varphi^1 \circ \gamma^{x,k}) = F \circ \gamma^{x,k}$ and $(\varphi^1 \circ \gamma^{x,k})$ are differentiable at 0, $(\varphi^1 \circ \gamma^{x,k})'(0) \neq 0$, and $0 \in K^{x,k}$ is not a porosity point of $K^{x,k}$ (by Lemma 2.11). Since f is Lipschitz, by Lemma 2.18 we conclude that f is differentiable at $(\varphi^1 \circ \gamma^{x,k})(0)$, proving the claim.

We now conclude the proof of the statement. Since μ is Radon, there is a Cheeger atlas $\{(K_i, \varphi_i: X \rightarrow \mathbb{R}^{n_i})\}_i$, where K_i is compact and $\mu(K_i) > 0$. (We may assume w.l.o.g. $\mu \neq 0$.) Since, by assumption, Y does not have RNP, the claim implies $n_i = 0$ for every i . In particular, X is an LDS of analytic dimension 0. Then, by Lemma 2.13, $(K_i, 0)$ is a chart in the complete LDS (K_i, d, μ) and so, by [Bat15, Remark 4.10], μ -a.e. $x \in K_i$ is isolated in K_i . Finally, by Lemma 2.12, μ -a.e. $x \in K_i$ is isolated in X , concluding the proof. \square

2.3. Three useful lemmas. In this subsection we collect three lemmas which will be used repeatedly in the rest of the paper.

Although we state the following for sequences, in reality it is a fact about measures.

Lemma 2.21. *Let $(\alpha_i) \subseteq (0, \infty)$ be a sequence. Then the following are equivalent:*

- $\inf\{\alpha_i: i \notin I\} = 0$ for any $I \subseteq \mathbb{N}$ with $\sum_{i \in I} \alpha_i < \infty$;
- for $I \subseteq \mathbb{N}$ and $\sum_{i \in I} \alpha_i < t < \infty$, there is $J \supseteq I$ such that

$$\sum_{i \in J} \alpha_i = t.$$

Proof. If the first item fails, then it is clear that the second is false as well. Conversely, if the first condition holds, then we may argue as in the proof of [FL07, Proposition 1.20] (from ‘Next we claim [...]’). \square

Lemma 2.22. *Let $(\alpha_i) \subseteq (0, \infty)$ be a sequence. Suppose $\inf\{\alpha_i: i \notin I\} = 0$ whenever $\sum_{i \in I} \alpha_i < \infty$, $I \subseteq \mathbb{N}$. Then, there is a disjoint collection $(I_n)_n$ of finite non-empty subsets of \mathbb{N} such that*

$$\sum_{n \in \mathbb{N}} \sum_{i \in I_n} \alpha_i = \infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} \left(\sum_{i \in I_n} \alpha_i^p \right)^\sigma < \infty$$

for all $0 < \sigma \leq 1 < p < \infty$. In particular, there is a subsequence $(\alpha_{i_j})_j$ satisfying

$$\sum_{j \in \mathbb{N}} \alpha_{i_j} = \infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} \alpha_{i_j}^p < \infty,$$

for $1 < p < \infty$.

Proof. We first need to construct an auxiliary sequence. Recall that for all $\epsilon > 0$ and $1 < p < \infty$ there is $(t_k) \subseteq (0, \infty)$ such that $\sum_k t_k = \infty$ and $\sum_k t_k^p \leq \epsilon$. It follows that for all $0 < \sigma \leq 1 < p < \infty$ and $\epsilon > 0$ there are $N = N(\sigma, p, \epsilon) \in \mathbb{N}$ and $t_1, \dots, t_N \in (0, 1)$ satisfying $\sum_{k=1}^N t_k = 1$ and $\left(\sum_{k=1}^N t_k^p \right)^\sigma \leq \epsilon$.

Fix $(\sigma_n)_n \subseteq (0, 1]$ and $(p_n)_n \subseteq (1, \infty)$ with $\sigma_n \rightarrow 0$ and $p_n \rightarrow 1$. From the above, we find a strictly increasing sequence $(k_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{N}_0$ with $k_0 = 0$ and $(t_k)_k \subseteq (0, 1)$ such that

$$\sum_{k=k_{n-1}+1}^{k_n} t_k = 1, \quad \left(\sum_{k=k_{n-1}+1}^{k_n} t_k^{p_n} \right)^{\sigma_n} \leq 2^{-n},$$

for $n \in \mathbb{N}$. It is then not difficult to see that

$$\sum_{k \in \mathbb{N}} t_k = \infty, \quad \sum_{n \in \mathbb{N}} \left(\sum_{k=k_{n-1}+1}^{k_n} t_k^p \right)^\sigma < \infty \quad (2.23)$$

for all $0 < \sigma \leq 1 < p < \infty$.

Applying Lemma 2.21 inductively, we find non-empty disjoint $(\tilde{J}_k)_k$ with $\sum_{i \in \tilde{J}_k} \alpha_i = t_k$ for $k \in \mathbb{N}$. Then there is a collection $(J_k)_k$ of finite non-empty sets with $J_k \subseteq \tilde{J}_k$ and $\sum_{i \in J_k} \alpha_i \sim t_k$ for $k \in \mathbb{N}$. Define $I_n := \bigcup_{k=k_{n-1}+1}^{k_n} J_k$ for $n \in \mathbb{N}$. By Eq. (2.23), we have for $0 < \sigma \leq 1 < p < \infty$

$$\sum_{n \in \mathbb{N}} \sum_{i \in I_n} \alpha_i = \sum_{n \in \mathbb{N}} \sum_{k=k_{n-1}+1}^{k_n} \sum_{i \in J_k} \alpha_i \sim \sum_{n \in \mathbb{N}} \sum_{k=k_{n-1}+1}^{k_n} t_k = \infty$$

and

$$\sum_{i \in I_n} \alpha_i^p = \sum_{k=k_{n-1}+1}^{k_n} \sum_{i \in J_k} \alpha_i^p \leq \sum_{k=k_{n-1}+1}^{k_n} \left(\sum_{i \in J_k} \alpha_i \right)^p \sim \sum_{k=k_{n-1}+1}^{k_n} t_k^p,$$

which implies $\sum_{n \in \mathbb{N}} \left(\sum_{i \in I_n} \alpha_i^p \right)^\sigma < \infty$ and concludes the proof. \square

Let $X = (X, d)$ be a metric space, $E \subseteq \mathbb{R}$ closed and non-empty, and $\gamma: E \rightarrow X$ a continuous function. We define the *variation* of γ as

$$\text{var } \gamma := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : n \in \mathbb{N}, t_0 \leq \dots \leq t_n \in E \right\}. \quad (2.24)$$

We say that X is C -*quasiconvex*, $C \geq 1$, if for every $x, y \in X$ there is a continuous $\gamma: [0, 1] \rightarrow X$ with $\gamma(0) = x$, $\gamma(1) = y$, and $\text{var } \gamma \leq Cd(x, y)$. A metric space is *quasiconvex* if it is C -quasiconvex for some $C \geq 1$ and a *length space* if it is C -quasiconvex for every $C > 1$.

Lemma 2.25 ([IP00, Lemma 2]). *Let X be a quasiconvex metric space and set*

$$C_0 := \inf\{C \in [1, \infty) : X \text{ is } C\text{-quasiconvex}\}.$$

Suppose Y a metric space, \mathcal{Q} a locally finite closed cover of X , and $f: X \rightarrow Y$ a function such that $L := \sup_{Q \in \mathcal{Q}} \text{LIP}(f|_Q) < \infty$. Then f is $C_0 L$ -Lipschitz on X . In particular, if X is a length space, then f is L -Lipschitz.

Proof. Let $\epsilon > 0$, $x, y \in X$, and $\gamma: [0, 1] \rightarrow X$ be continuous with $\gamma(0) = x$, $\gamma(1) = y$, and $\text{var } \gamma \leq (C_0 + \epsilon)d(x, y) =: L_0$. By [HKST15, Proposition 5.1.8], we may assume γ to be L_0 -Lipschitz. Since $\gamma([0, 1])$ is compact and \mathcal{Q} locally finite, there are finitely many $Q_1, \dots, Q_n \in \mathcal{Q}$ covering $\gamma([0, 1])$. Set $Z_i := \gamma^{-1}(Q_i)$ and observe that $f \circ \gamma$ is LL_0 -Lipschitz on Z_i for each i . In particular, $f \circ \gamma$ is continuous on the closed sets Z_1, \dots, Z_n and the latter cover $[0, 1]$; this implies that $f \circ \gamma$ is continuous. Then, the proof of [IP00, Lemma 2] shows that $d_Y(f(x), f(y)) = d_Y(f \circ \gamma(0), f \circ \gamma(1)) \leq L_0 L = (C_0 + \epsilon)Ld(x, y)$. \square

2.4. PI spaces.

Definition 2.26. We say that a metric measure space (X, d, μ) supports a p -*Poincaré inequality*, $p \in [1, \infty)$, if μ is positive and finite on balls and there are $C > 0$ and $\lambda \geq 1$ such that for every Lipschitz function $f: X \rightarrow \mathbb{R}$ and ball $B = B(x, r)$

$$\int_B |f - f_B| d\mu \leq C \text{diam}(B) \left(\int_{\lambda B} \text{Lip}(f; \cdot)^p d\mu \right)^{\frac{1}{p}},$$

where $\lambda B = B(x, \lambda r)$ and $f_B := \int_B f d\mu / \mu(B)$. We refer to (C, λ) as the constants in the p -Poincaré inequality of (X, d, μ) .

We call (X, d, μ) a p -*PI space* if, in addition, μ is doubling and (X, d) complete, and a *PI space* if it is a p -PI space for some $p \in [1, \infty)$.

Although not immediately apparent from the definition, any two points $x, y \in X$ in a PI space X may be connected by a rich family of curves (in fact, this is a characterisation; see [Kei03, EBG21, CC25a]). In particular, PI spaces are quasiconvex; see [Che99, Appendix].

We refer the reader to [HKST15, BB11] for more information on PI spaces, including several characterisations. More recent equivalent conditions may be found in [EB19b, EBG21, CC25b, CC25a] and the survey [Cap25].

For $s \in (0, \infty)$, we say that a Borel measure μ on a metric space X is *Ahlfors-David regular of dimension s* (or s -ADR) if there is $C \geq 1$ such that

$$C^{-1}r^s \leq \mu(B(x, r)) \leq Cr^s, \quad (2.27)$$

for all $x \in X$ and $0 < r < \text{diam } X$; see also Section 2.6. We say that a metric measure space (X, d, μ) is s -ADR if μ is. We refer to the least $C \geq 1$ as in Eq. (2.27) as the ADR constant of μ .

The proof of the following lemma is inspired by (and very similar to) [BBL17, Lemma 4.1, Proposition 6.1(a)].

Lemma 2.28. *Let (X, d, μ) be an s -ADR metric measure space and suppose it supports a p_0 -Poincaré inequality with constants (C_P, λ_P) , for some $1 \leq p_0 < s$. Then, for $1 \leq p < s$, there is a constant $C > 0$ such that for every metric space Y , L -Lipschitz function $f: X \rightarrow Y$, and $x, y \in X$, we have*

$$d_Y(f(x), f(y))^s \leq CL^{s-p} \int_{B(x, 2\lambda_P d(x, y))} \text{Lip}(f; \cdot)^p d\mu. \quad (2.29)$$

More precisely, C depends only on C_P, λ_P, s, p , and the ADR constant of μ .

Remark 2.30. By [KZ08] (see also [EB19a]), a complete metric measure space satisfies the assumptions of Lemma 2.28 if and only if it is s -ADR and s -PI.

Remark 2.31. Let $X = (X, d, \mu)$ be a complete and quasiconvex s -ADR metric measure space, for some $s \in (1, \infty)$. If Eq. (2.29) holds on X (even just for $p = 1$), then for every $q > s$ and $\delta \in (0, 1)$ there are $C \geq 1$ and $\tau_0 \in (0, 1)$ such that X is (C, δ, τ_0, q) -connected; see [EBG21, Definition 2.16]. To prove this, for a given closed ‘obstacle’ $E \subseteq X$ with ‘density’ at most $\tau_0 \in (0, 1)$ and $x, y \in X$, consider the Lipschitz function

$$f(z) := \inf_{\gamma} \int_{\gamma} \chi_E ds + \tau_0 \text{var } \gamma,$$

where the infimum is taken over all rectifiable curves $\gamma: [0, 1] \rightarrow X$ from x to z . See Eq. (2.24) for the definition of $\text{var } \gamma$. For measurable obstacles, one can argue as in [EBG21, Remark 2.15]. Then, by [EBG21, Theorem 2.19] we deduce that X supports a q -Poincaré inequality for every $q > s$. This cannot be improved to $q = s$, as Eq. (2.29) is stable under gluing at a point, while s -Poincaré inequalities are not, because points have zero s -capacity in s -ADR spaces; see e.g. [HKST15, Corollary 5.3.11].

Eq. (2.29) is (at least formally) stronger for larger values of p and fails for $p = s$ on every metric space with at least two points. By Remark 2.30 and Remark 2.31, Eq. (2.29) may be interpreted as a condition in-between an s -Poincaré inequality and q -Poincaré inequality for every $q > s$. We also point out that Eq. (2.29) (even just for $p = 1$) already implies $\mu(B(x, 2\lambda_P d(x, y))) \gtrsim d(x, y)^s$ for $x, y \in X$, as can be seen taking $f = d(x, \cdot)$.

Proof. If we prove Eq. (2.29) for $p = p_0$, then we obtain the inequality also for $1 \leq p < p_0$, because Eq. (2.29) is weaker for smaller values of p . Also, if X has p_0 -Poincaré inequality, it also has a q_0 -Poincaré inequality for all $q_0 \geq p_0$ (with the same constants). Hence, it is enough to prove Eq. (2.29) with $p = p_0$.

Post-composing f with the 1-Lipschitz function $d_Y(f(x), \cdot)$, we see that we may assume w.l.o.g. $Y = \mathbb{R}$. Suppose $r := |f(x) - f(y)| > 0$ and set $R := d(x, y) > 0$, $k := \lceil \log_2(8LR/r) \rceil \geq 3$. Define $B_i := B(x, 2^{1-i}R)$ for $0 \leq i \leq k$ and $B_i := B(y, 2^{i+1}R)$ for $-k \leq i < 0$. Since $B_k \subseteq B(x, r/4L)$ and $B_{-k} \subseteq B(y, r/4L)$, we have $|f_{B_k} - f_{B_{-k}}| \geq r/2$. Hence,

$$r \lesssim \sum_{i=1-k}^k |f_{B_{i-1}} - f_{B_i}| \sim \sum_{i=1-k}^0 \int_{B_i} |f - f_{B_i}| d\mu + \sum_{i=1}^k \int_{B_{i-1}} |f - f_{B_{i-1}}| d\mu.$$

By the p -Poincaré inequality and s -AD regularity, we have for $-k \leq i \leq k$

$$\int_{B_i} |f - f_{B_i}| d\mu \lesssim \text{diam } B_i \left(\int_{\lambda_P B_i} \text{Lip}(f; \cdot)^p d\mu \right)^{1/p} \lesssim (2^{-|i|} R)^{1-s/p} \left(\int_{\lambda_P B_0} \text{Lip}(f; \cdot)^p d\mu \right)^{1/p},$$

thus

$$r \lesssim \sum_{i=-\infty}^k (2^{-i} R)^{1-s/p} \left(\int_{\lambda_P B_0} \text{Lip}(f; \cdot)^p d\mu \right)^{1/p} \sim (2^{-k} R)^{1-s/p} \left(\int_{B(x, 2\lambda_P R)} \text{Lip}(f; \cdot)^p d\mu \right)^{1/p}.$$

Since $2^{-k} R \sim r/L$, rearranging and raising to the power of p concludes the proof. \square

2.5. PI rectifiability. We adopt the following (equivalent) variant of [EB19b, Definition 5.1].

Definition 2.32 ([EB19b, Definition 5.1]). Let (X, d, μ) be a metric measure space. It is *PI rectifiable* if it can be μ -almost all covered by countably many μ -measurable sets (E_i) for which there are PI spaces (Y_i, d_{Y_i}, ν_i) and biLipschitz embeddings $f_i: E_i \rightarrow Y_i$ satisfying $f_{i\#}\mu \ll \nu_i$.

It is *purely PI unrectifiable* if there is no positive-measure μ -measurable set $E \subseteq X$ such that (E, d, μ) is PI rectifiable.

Remark 2.33. The PI condition (Definition 2.26) is stable under biLipschitz change of the distance and multiplication of the measure by a positive measurable function which is both bounded and bounded away from zero. From this, it is not difficult to see that in the definition of PI rectifiability we may equivalently require (E_i) to be disjoint compact sets and (f_i) isometric embeddings² with $f_{i\#}\mu(A) = \nu_i \lfloor f(E_i)(A)$ for all sets $A \subseteq Y_i$ ³.

Let X be a metric space, $C \subseteq \mathbb{R}$ closed and non-empty, and $\gamma: C \rightarrow X$ continuous. Let $J \subseteq \mathbb{R}$ be the least interval containing C and $\{(a_i, b_i)\}_{i \in I}$ the unique at most countable collection of disjoint open intervals satisfying $C = J \setminus \bigcup_{i \in I} (a_i, b_i)$. We then define

$$\text{gap } \gamma := \sum_{i \in I} d(\gamma(a_i), \gamma(b_i)).$$

We call *curve fragment* any continuous map $\gamma: K \rightarrow X$ with $\text{var } \gamma < \infty$, where $K \subseteq \mathbb{R}$ is a non-empty compact set; see Eq. (2.24) for the definition of $\text{var } \gamma$.

Definition 2.34 ([EB19b, Definition 3.1]). Let $X = (X, d, \mu)$ be a metric measure space, $x \neq y \in X$, $C \geq 1$, $\delta > 0$, and $\epsilon > 0$. The pair (x, y) is (C, δ, ϵ) -*connected* (in X) if for every μ -measurable set $E \subseteq X$ with

$$\mu(E \cap B(x, Cd(x, y))) < \epsilon \mu(B(x, Cd(x, y)))$$

there is a curve fragment $\gamma: K \rightarrow X$ with $\gamma(\min K) = x$, $\gamma(\max K) = y$, $\gamma^{-1}(E) \subseteq \{\min K, \max K\}$, and

$$\begin{aligned} \text{var } \gamma &\leq Cd(x, y), \\ \text{gap } \gamma &< \delta d(x, y). \end{aligned}$$

For $E \subseteq X$ and $r > 0$, we say that X is (C, δ, ϵ, r) -*connected along* E if for every $x \in E$ and $y \in B(x, r) \setminus \{x\}$ the pair (x, y) is (C, δ, ϵ) -connected in X .

²To prove this, one may use the following fact. Let (X, d) be a metric space, $A \subseteq X$ a non-empty subset, d_A a distance on A , and $C \geq 1$. Suppose $C^{-1}d_A \leq d|_{A \times A} \leq Cd_A$ and set, for $x, y \in X$,

$$\begin{aligned} \rho(x, y) &:= \begin{cases} d_A(x, y), & x, y \in A \\ Cd(x, y), & \text{otherwise} \end{cases}, \\ \hat{d}(x, y) &:= \inf \left\{ \sum_{i=1}^n \rho(x_{i-1}, x_i) : n \in \mathbb{N}, x = x_0, \dots, x_n = y \right\}. \end{aligned}$$

Then \hat{d} is a distance on X , $C^{-1}d \leq \hat{d} \leq Cd$, and $\hat{d}|_{A \times A} = d_A$.

³To establish equality for all sets, and not just Borel sets, observe that $f_{i\#}\mu$ and $\nu_i \lfloor f(E_i)$ are both Borel regular. To prove this, for the former one may use the Lusin-Souslin Theorem (see [Kec95, Theorem 15.1]), while for the latter it follows by Borel regularity of ν_i (and ν_i -measurability of $f_i(E_i)$).

Definition 2.34 is closely related to Definition 2.26. Indeed, let $X = (X, d, \mu)$ be a complete metric measure space with $\mu \neq 0$. Then X is a PI space if and only if there are $C \geq 2$ and $\delta, \epsilon \in (0, 1)$ such that every pair $(x, y) \in X \times X$ with $x \neq y$ is (C, δ, ϵ) -connected; see [EB19b, Theorem 1.2]⁴. We are going to use the ‘qualitative’ analog of this statement.

Definition 2.35 ([EB19b, Definition 5.5]). Let $X = (X, d, \mu)$ be a metric measure space. We say that X is *asymptotically well-connected* if for $\delta \in (0, 1)$ and μ -a.e. $x \in X$ there are $C_x \geq 1$, $\epsilon_x \in (0, 1)$, and $r_x > 0$, such that, for $y \in B(x, r_x) \setminus \{x\}$, the pair (x, y) is $(C_x, \delta, \epsilon_x)$ -connected in X .

Remark 2.36. Definition 2.34 differs slightly from [EB19b, Definition 3.1], but is equivalent if the measure μ is doubling (which is the only case we need and, moreover, is implied by (C, δ, ϵ) -connectivity for $C \geq 2$ and $\delta, \epsilon \in (0, 1)$; see [EB19b, Lemma 3.4]). Similarly, also Definition 2.35 differs from [EB19b, Definition 5.5], but is equivalent if μ is asymptotically doubling.

In [EB19b], it is proven that if $X = (X, d, \mu)$ is an asymptotically well-connected metric measure space with μ asymptotically doubling, then it is PI rectifiable [EB19b, Theorem 5.6]⁵. Contrary to what claimed in [EB19b, Theorem 5.6], the reverse implication is however not true. Indeed, one might verify that if (X, d, μ) is asymptotically well-connected and μ is asymptotically doubling, then μ vanishes on porous sets, a property which need not hold on PI rectifiable spaces (as trivial examples in the plane show). The following is an amended statement.

Theorem 2.37 ([EB19b, Theorem 5.6]). *Let $X = (X, d, \mu)$ be a metric measure space. Then X is PI rectifiable if and only if μ is asymptotically doubling and there are countably many μ -measurable sets $(E_i)_i$ such that $\mu(X \setminus \bigcup_i E_i) = 0$ and (E_i, d, μ) is asymptotically well-connected.*

The precise connection between PI rectifiability and differentiability is given by the following theorem.

Theorem 2.38 ([CK09, Theorem 1.5], [BL18, Lemma 3.4], [EB19b, Theorem 1.1]). *Let $X = (X, d, \mu)$ be a metric measure space. Then the following are equivalent:*

- X is asymptotically well-connected and μ asymptotically doubling;
- X is PI rectifiable and μ vanishes on porous sets;
- X is a Y -LDS for every Banach space Y with RNP.

We are going to need the following.

Lemma 2.39. *Let $X = (X, d, \mu)$ be a metric measure space. Suppose μ is asymptotically doubling and X asymptotically well-connected. Then, for $\delta \in (0, 1)$, the parameters $C_x \geq 1, \epsilon_x \in (0, 1), r_x > 0$ (appearing in the definition of asymptotically well-connected) may be chosen such that $x \mapsto (C_x, \epsilon_x, r_x)$ is Borel measurable.*

Proof. Assume first $\mu(X) < \infty$. For μ -measurable $E \subseteq X$, $C \geq 1$, $r > 0$, $\delta \in (0, 1)$, and $x, y \in X$, define

$$\begin{aligned} \rho_{E, C, \delta}(x, y) &:= \inf \{ \max(\text{var } \gamma / C, \text{gap } \gamma / \delta) : \gamma \text{ is curve fragment from } x \text{ to } y \text{ avoiding } E \}, \\ \kappa_{E, C, r, \delta}(x) &:= \sup_{y \in U(x, r)} \frac{\rho_{E, C, \delta}(x, y)}{r}, \\ \theta_{E, C, r}(x) &:= \begin{cases} \frac{\mu(B(x, Cr) \cap E)}{\mu(U(x, 2Cr))}, & x \in \text{spt } \mu, \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

⁴Arguing essentially as in [EB19b, Lemma 3.4], it is not difficult to see that, under the present assumption, $0 < \mu(B) < \infty$ for every ball B .

⁵Our definition of metric measure space is slightly more general than the one adopted in [EB19b]. However, to apply [EB19b, Theorem 5.6], it is enough to cover μ -almost all of X with countably many compact sets $(K_i)_i$, observe that the assumptions on X are also satisfied by (K_i, d, μ) and that the latter is a metric measure space in the sense of [EB19b].

Concatenating curve fragments, it follows that $\rho_{E,C,r}$ satisfies triangle inequality. Also, considering the trivial curve fragment $\gamma: \{0, 1\} \rightarrow X$, $\gamma(0) := x$, $\gamma(1) := y$, we see that $\rho_{E,C,\delta}(x, y) \leq d(x, y)/\delta$. In particular, $\rho_{E,C,\delta}(\cdot, y): X \rightarrow (0, \infty)$ is continuous for each $y \in X$, and so $\kappa_{E,C,r,\delta}$ is lower semi-continuous. Since $\theta_{E,C,r}: X \rightarrow [0, \infty)$ is upper semi-continuous, the map $\tau_{C,R,\epsilon,\delta}: X \rightarrow (0, \infty]$

$$\tau_{C,R,\epsilon,\delta}(x) := \sup_E \sup_{0 < r < R} \min(\epsilon/\theta_{E,C,r}(x), \kappa_{E,C,r,\delta}(x)), \quad x \in X,$$

is lower semi-continuous and hence Borel measurable. The supremum is taken over μ -measurable sets $E \subseteq X$ and we interpret $1/0$ as ∞ .

Let $C \geq 1$, $R > 0$, $\epsilon, \delta \in (0, 1)$, $x \in \{\tau_{C,R,\epsilon,\delta} < 1\}$ and $y \in U(x, R)$, $y \neq x$. We claim that (x, y) is $(2C, \delta', \epsilon)$ -connected in (X, d, μ) for all $\delta < \delta' < 1$. Let $E \subseteq X$ be μ -measurable, and let $\alpha \in (1, 2)$ with $r := \alpha d(x, y) < R$ and $\alpha\delta \leq \delta'$. Recall that $\min(\epsilon/\theta_{E,C,r}(x), \kappa_{E,C,r,\delta}(x)) < 1$. If $\theta_{E,C,r}(x) > \epsilon$, then

$$\mu(B(x, 2Cd(x, y)) \cap E) \geq \mu(B(x, Cr) \cap E) > \epsilon\mu(U(x, 2Cr)) \geq \epsilon\mu(B(x, 2Cd(x, y)))$$

and there is nothing prove. If $\kappa_{E,C,r,\delta}(x) < 1$, then there is a curve fragment γ from x to y , avoiding E , satisfying

$$\text{var } \gamma < Cr \leq 2Cd(x, y) \quad \text{and} \quad \text{gap } \gamma < \delta r \leq \delta' d(x, y).$$

This proves the claim.

To conclude the proof of the lemma (under the additional assumption $\mu(X) < \infty$), it remains to show that, for $\delta \in (0, 1)$, the set

$$A := \bigcap_{\substack{C \in [1, \infty) \cap \mathbb{Q} \\ R \in (0, \infty) \cap \mathbb{Q} \\ \epsilon \in (0, 1) \cap \mathbb{Q}}} \{\tau_{C,R,\epsilon,\delta} \geq 1\}$$

is μ -null. Let $x \in A \cap \text{spt } \mu$ such that $\limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty$ and let $M_x, R_x > 0$ be such that $\mu(B(x, 2r)) \leq M_x \mu(B(x, r))$ for $0 < r < R_x$. By definition of $\tau_{C,R,\epsilon,\delta}(x)$ and A , we see that for every rational $C \geq 1$, $R > 0$, $\epsilon \in (0, 1)$, there are $0 < r < R$ and a μ -measurable set $E \subseteq X$ such that $\min(\epsilon/\theta_{E,C,r}(x), \kappa_{E,C,r,\delta}(x)) > 1/2$. That is,

$$\mu(E \cap B(x, Cr)) < 2\epsilon\mu(U(x, 2Cr))$$

and there is $y \in U(x, r)$ such that $\rho_{E,C,r}(x, y) > r/2$, i.e.

$$\text{gap } \gamma > \delta r/2 > (\delta/2)d(x, y) \quad \text{or} \quad \text{var } \gamma > Cr/2 > (C/2)d(x, y) \quad (2.40)$$

for all curve fragments γ from x to y which avoid E . Since $\rho_{E,C,r}(x, y) \leq d(x, y)/\delta$, we have $\delta r/2 < d(x, y) < r$ and therefore

$$\begin{aligned} \mu(E \cap B(x, (C/2)d(x, y))) &\leq \mu(E \cap B(x, Cr)) < 2\epsilon\mu(B(x, 2Cr)) \leq 2\epsilon\mu(B(x, 4Cd(x, y)/\delta)) \\ &\leq C_1(M_x, \delta)\epsilon\mu(B(x, (C/2)d(x, y))), \end{aligned} \quad (2.41)$$

provided $R < C_2(R_x, \delta, C)$. Here $C_1(M_x, \delta)$ and $C_2(R_x, \delta, C)$ are constants depending only on M_x, δ and R_x, δ, C , respectively. Since we may choose C, R, ϵ , Eqs. (2.40) and (2.41) show that there are no (rational or irrational) $R > 0$, $C \geq 1$, $\epsilon \in (0, 1)$ such that (x, y) is $(C, \delta/2, \epsilon)$ -connected for all $y \in B(x, R)$. Since (X, d, μ) is asymptotically well-connected and μ asymptotically doubling, we conclude that $\mu(A) = 0$.

If $\mu(X) = \infty$, let $(V_i)_i$ be an open cover of X of sets of finite μ -measure and observe that (V_i, d, μ) satisfies the assumption of the previous step. The thesis follows by modifying the measurable parameters on V_i in the obvious way. \square

2.6. Ahlfors-David regular spaces and Borel-Cantelli-type lemmas. For a metric space X , $s \in [0, \infty)$, and a set $E \subseteq X$, the s -dimensional Hausdorff measure of E is defined as

$$\mathcal{H}^s(E) := \sup_{\delta > 0} \inf \left\{ \sum_{i \in \mathbb{N}} \text{diam}(E_i)^s : \text{diam}(E_i) < \delta \text{ and } E \subseteq \bigcup_{i \in \mathbb{N}} E_i \subseteq X \right\};$$

the map $E \mapsto \mathcal{H}^s(E)$ is a Borel regular measure on X ; see [Fed69, Section 2.10].

Remark 2.42. An application of the 5r-Vitali covering lemma (see e.g. [HKST15, 5B-covering lemma]) shows that if μ is an s -ADR measure on X (see Eq. (2.27)), then $\mathcal{H}^s \sim \mu$ on Borel sets. In particular, \mathcal{H}^s is also s -ADR.

Let X be a metric space. We say that X is s -ADR if \mathcal{H}^s is s -ADR or, equivalently, if there is an s -ADR measure on X . For $\delta > 0$ and $\mathcal{N} \subseteq X$, we say that \mathcal{N} is a δ -net if $B(\mathcal{N}, \delta) = X$, while it is δ -separated if $d(x, y) \geq \delta$ for $x \neq y \in \mathcal{N}$ and *strictly* δ -separated if $d(x, y) > \delta$ for $x \neq y \in \mathcal{N}$.

The main results of this subsection are Lemmas 2.43 and 2.44.

Lemma 2.43 (First Borel-Cantelli lemma in ADR spaces). *Let X be an s -ADR metric space, $C \geq 1$, and let $(\delta_i) \subseteq (0, \infty)$ be a sequence with $\delta_i \rightarrow 0$. For $i \in \mathbb{N}$, let $\mathcal{N}_i \subseteq X$ be a set satisfying*

$$\#(B(x, \delta_i/C) \cap \mathcal{N}_i) \leq C, \quad x \in \mathcal{N}_i.$$

Then, for $(\alpha_i) \subseteq [0, \infty)$, $\mathcal{H}^s(\limsup_i B(\mathcal{N}_i, \alpha_i \delta_i)) = 0$ whenever $\sum_i \alpha_i^s < \infty$.

Lemma 2.44 (Second Borel-Cantelli lemma in ADR spaces). *Let X be an s -ADR metric space, $C \geq 1$, and let $(\delta_i) \subseteq (0, \infty)$ be a sequence with $\delta_i \rightarrow 0$. For $i \in \mathbb{N}$, let $\mathcal{N}_i \subseteq X$ be a $C\delta_i$ -net in X . Let $(\alpha_i) \subseteq [0, \infty)$ and suppose that for each $i, j \in \mathbb{N}$ with $i < j$ (at least) one of the following holds:*

- $\alpha_i \delta_i \geq C^{-1} \delta_j$;
- $d(\mathcal{N}_i, \mathcal{N}_j) \geq C^{-1} \min(\delta_i, \delta_j)$.

Then $\mathcal{H}^s(X \setminus \limsup_i B(\mathcal{N}_i, \alpha_i \delta_i)) = 0$ whenever $\sum_i \alpha_i^s = \infty$.

The proofs of Lemmas 2.43 and 2.44 consist of simple estimates based on Ahlfors-David regularity and an application of the corresponding Borel-Cantelli lemma. For both, the basic intuition is that a separated net approximates the underlying s -ADR measure, therefore its α -enlargement in an R -ball should have volume $\sim (R\alpha)^s$. For Lemma 2.44, the separation condition ensures that one net is diffused enough w.r.t. the other, providing enough independence for a Borel-Cantelli-type result.

We will need the following refinement of the classical second Borel-Cantelli lemma. A short proof can be found in [Yan06].

Theorem 2.45 ([Pet02, Theorem 2.1]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(A_i) \subseteq \mathcal{F}$ a sequence of events. Suppose there are $C \geq 1$ and $n \in \mathbb{N}$ such that $\mathbb{P}(A_i \cap A_j) \leq C\mathbb{P}(A_i)\mathbb{P}(A_j)$ for $j > i \geq n$. Then $\mathbb{P}(\limsup_i A_i) \geq 1/C$ whenever $\sum_i \mathbb{P}(A_i) = \infty$.*

We now require the aforementioned estimates, which we collect in a few lemmas. Throughout this subsection, the implicit constants in \lesssim, \gtrsim , and \sim , depend on s , the ADR constant of \mathcal{H}^s , and any additional parameters indicated as subscript.

Lemma 2.46. *Let X be an s -ADR metric space, $x_0 \in X$, $R, \delta > 0$, and let $\mathcal{N} \subseteq X$ be a δ -separated set. Then $\#(\mathcal{N} \cap B(x_0, R)) \lesssim \max\{(R/\delta)^s, 1\}$.*

Proof. If $\delta > 2R$, then $\#(\mathcal{N} \cap B(x_0, R)) \leq 1$. Suppose now $\delta \leq 2R$. Since $\{B(x, \delta/3) : x \in \mathcal{N}\}$ is pairwise disjoint, we have

$$\#(\mathcal{N} \cap B(x_0, R)) \delta^s \lesssim \mathcal{H}^s \left(\bigcup \{B(x, \delta/3) : x \in \mathcal{N} \cap B(x_0, R)\} \right) \leq \mathcal{H}^s(B(x_0, R + 2R/3)) \lesssim R^s.$$

□

Proof of Lemma 2.43. Possibly enlarging C , we can assume $0 < \alpha_i \leq C$ for all i . Set $E := \limsup_i B(\mathcal{N}_i, \alpha_i \delta_i)$. Let \mathcal{N}'_i be a maximal $C^{-1}\delta_i$ -separated subset of \mathcal{N}_i and observe that $\mathcal{N}'_i \subseteq U(\mathcal{N}'_i, \delta_i/C)$. Fix $x_0 \in X$, $0 < R < \text{diam } X$, and let $n \in \mathbb{N}$ be such that $\delta_i \leq R$ for $i \geq n$. Then

$$B(x_0, R) \cap B(\mathcal{N}_i, \alpha_i \delta_i) \subseteq \bigcup_{x \in \mathcal{N}_i \cap B(x_0, 3CR)} B(x, 2\alpha_i \delta_i) \subseteq \bigcup_{x' \in \mathcal{N}'_i \cap B(x_0, 4CR)} \bigcup_{x \in B(x', \delta_i/C) \cap \mathcal{N}_i} B(x, 2\alpha_i \delta_i), \quad (2.47)$$

which implies $\mathcal{H}^s(B(x_0, R) \cap B(\mathcal{N}_i, \alpha_i \delta_i)) \lesssim_C \#(\mathcal{N}'_i \cap B(x_0, 4CR))(\alpha_i \delta_i)^s$. Since $\mathcal{N}'_i \cap B(x_0, 4CR)$ is $C^{-1}\delta_i$ -separated and $\delta_i \leq R$, Lemma 2.46 and Eq. (2.47) imply

$$\mathcal{H}^s(B(x_0, R) \cap B(\mathcal{N}_i, \alpha_i \delta_i)) \lesssim_C (R\alpha_i)^s, \quad i \geq n.$$

From the above equation, arguing as in the proof of the classical first Borel-Cantelli lemma (or applying it to the probability measure $\mathcal{H}^s \llcorner B(x_0, R)/\mathcal{H}^s(B(x_0, R))$) we conclude that $\mathcal{H}^s(E \cap B(x_0, R)) = 0$. Since x_0 and R were arbitrary, this concludes the proof. \square

Lemma 2.48. *Let X be an s -ADR metric space, $x_0 \in X$, $R, \delta > 0$, and $C \geq 1$. Suppose $8C\delta \leq R < \text{diam } X$ and let $\mathcal{N} \subseteq X$ be a $C\delta$ -net. Then*

$$\mathcal{H}^s(B(x_0, R) \cap B(\mathcal{N}, \alpha\delta)) \gtrsim_C (R\alpha)^s,$$

for $\alpha \in [0, C]$.

Proof. Let \mathcal{N}' be a maximal $C\delta$ -separated subset of \mathcal{N} and set $E := B(x_0, R) \cap B(\mathcal{N}', \alpha\delta)$, $\mathcal{N}_0 := \mathcal{N}' \cap B(x_0, R - \alpha\delta)$. Since $\{B(x, \alpha\delta/3) : x \in \mathcal{N}_0\}$ is a pairwise disjoint collection of balls contained in E , we have

$$\mathcal{H}^s(E) \gtrsim \#\mathcal{N}_0(\alpha\delta)^s. \quad (2.49)$$

Since \mathcal{N}' is maximal, we have $B(\mathcal{N}', 2C\delta) = X$. Then, for $y \in B(x_0, R/2)$, there is $x \in \mathcal{N}'$ with $d(x, y) \leq 3C\delta$ and hence $d(x_0, x) \leq R/2 + 3C\delta \leq R - \alpha\delta$. That is, $\{B(x, 3C\delta) : x \in \mathcal{N}_0\}$ covers $B(x_0, R/2)$, proving

$$\#\mathcal{N}_0\delta^s \gtrsim_C \mathcal{H}^s\left(\bigcup_{x \in \mathcal{N}_0} B(x, 3C\delta)\right) \gtrsim R^s. \quad (2.50)$$

Equations (2.49) and (2.50) conclude the proof. \square

Lemma 2.51. *Let X be an s -ADR metric space, $C \geq 1$, $\delta, \delta' > 0$, $x_0 \in X$, and suppose $C^{-1}\delta \leq R < \text{diam } X$. Let $\mathcal{N}, \mathcal{N}' \subseteq X$ be $C^{-1}\delta$ and $C^{-1}\delta'$ -separated, respectively. Then, for $\alpha, \alpha' \in (0, C]$ with $\alpha\delta \geq C^{-1}\delta'$ it holds*

$$\mathcal{H}^s(B(x_0, R) \cap B(\mathcal{N}, \alpha\delta) \cap B(\mathcal{N}', \alpha'\delta')) \lesssim_C (R\alpha\alpha')^s.$$

Proof. Set

$$\begin{aligned} E &:= B(x_0, R) \cap B(\mathcal{N}, \alpha\delta) \cap B(\mathcal{N}', \alpha'\delta'), \\ \mathcal{N}_0 &:= \mathcal{N} \cap B(x_0, R + 2\alpha\delta), \\ \mathcal{N}'_x &:= \mathcal{N}' \cap B(x, 2\alpha\delta + 2\alpha'\delta'), \quad x \in \mathcal{N}_0. \end{aligned}$$

From the definitions, we have

$$\begin{aligned} B(x_0, R) \cap B(\mathcal{N}, \alpha\delta) &\subseteq \bigcup \{B(x, 2\alpha\delta) : x \in \mathcal{N}_0\}, \\ B(x, 2\alpha\delta) \cap B(\mathcal{N}', \alpha'\delta') &\subseteq \bigcup \{B(x', 2\alpha'\delta') : x' \in \mathcal{N}'_x\}, \quad x \in \mathcal{N}_0, \end{aligned}$$

and therefore

$$E \subseteq \bigcup_{x \in \mathcal{N}_0} \bigcup_{x' \in \mathcal{N}'_x} B(x', 2\alpha'\delta'). \quad (2.52)$$

Since, for $x \in \mathcal{N}_0$, the set $\mathcal{N}'_x \subseteq B(x, 2\alpha\delta + 2\alpha'\delta')$ is $C^{-1}\delta'$ -separated, Lemma 2.46 implies that its cardinality is at most $\lesssim_C \max\{(\alpha\delta + \alpha'\delta')^s(\delta')^{-s}, 1\}$. By assumption, $\alpha\delta + \alpha'\delta' \sim_C \alpha\delta$, $(\alpha\delta)/\delta' \gtrsim_C 1$, and therefore

$$\#\mathcal{N}'_x \lesssim_C \left(\frac{\alpha\delta}{\delta'}\right)^s, \quad x \in \mathcal{N}_0. \quad (2.53)$$

Similarly, \mathcal{N}_0 is a $C^{-1}\delta$ -separated set in $B(x_0, R + 2\alpha\delta)$ and $R + 2\alpha\delta \sim_C R \gtrsim_C \delta$, yielding $\#\mathcal{N}_0 \lesssim_C (R/\delta)^s$. Finally, Eqs. (2.52) and (2.53) give

$$\mathcal{H}^s(E) \lesssim \#\mathcal{N}_0 \left(\max_{x \in \mathcal{N}_0} \#\mathcal{N}'_x \right) (\alpha'\delta')^s \lesssim_C (R\alpha\alpha')^s.$$

□

Lemma 2.54. *Let X be an s -ADR metric space, $C \geq 1$, $\delta, \delta' > 0$, $x_0 \in X$, and suppose $C^{-1}\max(\delta, \delta') \leq R < \text{diam } X$. Let $\mathcal{N}, \mathcal{N}' \subseteq X$ be $C^{-1}\delta$ and $C^{-1}\delta'$ -separated, respectively, and assume $d(\mathcal{N}, \mathcal{N}') \geq C^{-1}\min(\delta, \delta')$. Then, for $\alpha, \alpha' \in [0, C]$ it holds*

$$\mathcal{H}^s(B(x_0, R) \cap B(\mathcal{N}, \alpha\delta) \cap B(\mathcal{N}', \alpha'\delta')) \lesssim_C (R\alpha\alpha')^s.$$

Proof. Set $E := B(x_0, R) \cap B(\mathcal{N}, \alpha\delta) \cap B(\mathcal{N}', \alpha'\delta')$ and assume w.l.o.g. $\delta' \leq \delta$ and $\alpha, \alpha' > 0$.

Suppose first $\max\{\alpha', (\alpha\delta)/\delta'\} < 1/4C$. Then $B(x, 2\alpha\delta) \cap B(x', 2\alpha'\delta') = \emptyset$ for $x \in \mathcal{N}$ and $x' \in \mathcal{N}'$, for otherwise we would find x, x' with $d(x, x') \leq 2\alpha\delta + 2\alpha'\delta' < (4/4C)\delta' = C^{-1}\delta'$, contradicting $d(\mathcal{N}, \mathcal{N}') \geq C^{-1}\delta'$. Since

$$E \subseteq B(x_0, R) \cap \bigcup_{\substack{x \in \mathcal{N} \\ x' \in \mathcal{N}'}} B(x, 2\alpha\delta) \cap B(x', 2\alpha'\delta'),$$

we have $E = \emptyset$ and the thesis is trivially satisfied.

We can now assume $\max\{\alpha', (\alpha\delta)/\delta'\} \geq 1/4C$. Since Lemma 2.51 covers the case $\alpha\delta \geq (1/4C)\delta'$, it only remains to consider the case $\alpha\delta < (1/4C)\delta'$ and $\alpha' \geq (1/4C)$. Set $\mathcal{N}_0 := \mathcal{N} \cap B(x_0, R + 2\alpha\delta)$ and observe that it is a $C^{-1}\delta$ -separated subset of $B(x_0, R + 2C^2R)$. Since $\delta \lesssim_C R$, by Lemma 2.46 we conclude $\#\mathcal{N}_0 \lesssim_C (R/\delta)^s$. The inclusion $E \subseteq \bigcup_{x \in \mathcal{N}_0} B(x, 2\alpha\delta)$ then implies

$$\mathcal{H}^s(E) \lesssim_C (R/\delta)^s (\alpha\delta)^s \sim_C (R\alpha\alpha')^s,$$

where in the last equality we have used $\alpha' \sim_C 1$. □

Proof of Lemma 2.44. Replacing \mathcal{N}_i with a maximal $C^{-1}\delta_i$ -separated subset and increasing $C \geq 1$ we can assume \mathcal{N}_i to be a $C^{-1}\delta_i$ -separated $C\delta_i$ -net. We can also assume $\alpha_i < C$ for $i \in \mathbb{N}$, for otherwise the thesis holds trivially.

Set $E := \limsup_i B(\mathcal{N}_i, \alpha_i\delta_i)$, fix $x_0 \in X$, $0 < R < \text{diam } X$, and let $n \in \mathbb{N}$ be such that $8C\delta_i \leq R$ for $i \geq n$. Observe that for $n \leq i < j$, depending on the separation condition holding for i, j , we can apply either Lemma 2.51 or Lemma 2.54 to deduce

$$\mathcal{H}^s(B(x_0, R) \cap B(\mathcal{N}_i, \alpha_i\delta_i) \cap B(\mathcal{N}_j, \alpha_j\delta_j)) \lesssim_C (R\alpha_i\alpha_j)^s.$$

By Lemma 2.48, we also have $\mathcal{H}^s(B(x_0, R) \cap B(\mathcal{N}_i, \alpha_i\delta_i)) \gtrsim_C (R\alpha_i)^s$ for $i \geq n$. Hence, setting $\mathbb{P} := \mathcal{H}^s \llcorner B(x_0, R) / \mathcal{H}^s(B(x_0, R))$, the above shows that there is a constant $C_1 \geq 1$ independent of x_0 and R such that

$$\mathbb{P}(B(\mathcal{N}_i, \alpha_i\delta_i) \cap B(\mathcal{N}_j, \alpha_j\delta_j)) \leq C_1 \mathbb{P}(B(\mathcal{N}_i, \alpha_i\delta_i)) \mathbb{P}(B(\mathcal{N}_j, \alpha_j\delta_j))$$

for $j > i \geq n$. Since $\sum_i \mathbb{P}(B(\mathcal{N}_i, \alpha_i\delta_i)) \gtrsim_C \sum_{i \geq n} \alpha_i^s = \infty$, Theorem 2.45 gives $\mathbb{P}(E) \geq 1/C_1$. Recalling the definition of \mathbb{P} and the fact that x_0 and $0 < R < \text{diam } X$ were arbitrary, we have

$$\mathcal{H}^s(B(x_0, R) \setminus E) \leq (1 - 1/C_1) \mathcal{H}^s(B(x_0, R)),$$

for all x_0 and R . Since $(1 - 1/C_1) < 1$, the above inequality implies that $X \setminus E$ does not have any Lebesgue density point w.r.t. the doubling measure \mathcal{H}^s . □

3. SHORTCUT METRIC SPACES

Given a metric space (X, d) and a sequence $\eta_i \in (0, 1]$, we now construct a metric space (X, d_η) by allowing shortcuts in the metric at scales $\delta_i \rightarrow 0$; the cost of each shortcut is $\eta_i d$. The notion of shortcuts is inspired by [LDLR17]; see Appendix A.1 for a comparison of the definitions.

Definition 3.1 (Metric space with shortcuts). Let (X, d) be a metric space. Suppose there are a sequence $(\delta_i) \subseteq (0, \infty)$ with $\delta_i \rightarrow 0$, non-empty families (\mathcal{J}_i) of disjoint subsets of X , and constants $a, a_0, b, M > 0$, such that:

- $a_0\delta_i \leq d(x, y) \leq a\delta_i$ for $x, y \in S \in \mathcal{J}_i$, $x \neq y$;
- $d(S, S') \geq b\delta_j$ for distinct $S \in \mathcal{J}_i$, $S' \in \mathcal{J}_j$ with $i \leq j$;
- for $S \in \mathcal{J}_i$, $x, y \notin U(S, b\delta_i)$ and $z, w \in S$ it holds

$$d(x, y) \leq d(x, z) + d(w, y); \quad (3.2)$$

- $2 \leq \#S \leq M$ for $S \in \mathcal{J}_i$;
- $B(\cup_i \mathcal{J}_i, M\delta_i) = X$.

We then say that (X, d) is a *metric space with shortcuts* $\{(\mathcal{J}_i, \delta_i)\}_i$ and call *shortcut* any $S \in \mathcal{J} := \cup_i \mathcal{J}_i$.

If a metric space X has shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$, then \mathcal{J} is a collection of disjoint sets.

Remark 3.3. Let X be a metric space with shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$, $(i_j) \subseteq \mathbb{N}$ strictly increasing and, for $j \in \mathbb{N}$ and $S \in \mathcal{J}_{i_j}$, let $\hat{S} \subseteq S$ be a subset with at least two elements. Then

$$(\{\hat{S} : S \in \mathcal{J}_{i_j}\}, \delta_{i_j})$$

are shortcuts in X . Hence, it is more interesting to find large families of shortcuts, although for our results this does not really matter.

Equation (3.2) is the most important condition in Definition 3.1. Informally, points in $S \in \mathcal{J}$ are, in some sense, indistinguishable when looked at from afar. For an illustrative example, consider a connected graph $G = (V, E)$ containing a complete bipartite graph $K_{2,M}$, let S be the part of $K_{2,M}$ with M elements, and endow G with the length distance d_G . Then, for $x \notin U_{d_G}(S, 1) = S$ and $z, w \in S$, it holds

$$d_G(x, z) = d_G(x, w),$$

and Eq. (3.2) follows. With this example in mind, many of the results of this section should become more transparent.

Notation 3.4. Throughout the remainder of this subsection, we consider a fixed a metric space $X = (X, d)$ having shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$ with constants a_0, a, b , and M . We also fix a sequence $\eta = (\eta_i)_i \subseteq (0, 1]$.

We can distort the distance d in the natural way.

Definition 3.5. For $x, y \in X$, define

$$\rho_\eta(x, y) := \begin{cases} \eta_i d(x, y), & x, y \in S \in \mathcal{J}_i \\ d(x, y), & \text{otherwise} \end{cases}.$$

The *shortcut distance* between x and y is

$$d_\eta(x, y) := \inf \left\{ \sum_{i=0}^n \rho_\eta(x_{i-1}, x_i) : n \in \mathbb{N}, x = x_0, \dots, x_n = y \right\}. \quad (3.6)$$

The map $d_\eta : X \times X \rightarrow [0, \infty)$ is a pseudo-distance. We will shortly show that it is actually a distance (Proposition 3.11). The space (X, d_η) is the *shortcut metric space* generated by (X, d) and η .

Notation 3.7. To avoid ambiguity, we always use the subscript η when a quantity depends on the distance d_η . For instance, the Lipschitz constant w.r.t. d_η of a function $f : X \rightarrow Y$ will be denoted as $\text{LIP}_\eta(f)$.

The following lemma is analogous to [LDLR17, Lemma 3.2].

Lemma 3.8. Let $x, y \in X$ with $d_\eta(x, y) < d(x, y)$. Then, for $\epsilon > 0$, there are $n \in \mathbb{N}$, $i_1, \dots, i_n \in \mathbb{N}$, $p_j^-, p_j^+ \in S_j \in \mathcal{J}_{i_j}$, and $1 \leq k \leq n$ such that

$$d_\eta(x, y) + \epsilon \geq d(x, p_1^-) + \rho_\eta(p_1^-, p_1^+) + d(p_1^+, p_2^-) + \dots + \rho_\eta(p_n^-, p_n^+) + d(p_n^+, y), \quad (3.9)$$

$$i_1 \geq \dots \geq i_k, \quad i_k \leq \dots \leq i_n, \quad (3.10)$$

with p_1^\pm, \dots, p_n^\pm distinct and $S_j \neq S_{j+1}$ for $1 \leq j < n$.

Proof. Assume w.l.o.g. $d_\eta(x, y) + \epsilon < d(x, y)$ and let $m \in \mathbb{N}$, $x = x_0, \dots, x_m = y \in X$ be such that $\sum_{i=1}^m \rho_\eta(x_{i-1}, x_i) \leq d_\eta(x, y) + \epsilon$ and $x_i \neq x_j$ for $i \neq j$. Observe that, since the total ‘cost’ of $(x_i)_{i=0}^m$ w.r.t ρ_η is less than $d(x, y)$, there are consecutive points which belong to the same shortcut.

By triangle inequality of d , we may discard terms from $(x_i)_{i=0}^m$ to ensure that, for $0 < i < m$ and $S \in \mathcal{J}$, if $x_{i-1} \notin S$ and $x_i \in S$, then $x_{i+1} \in S$. When restricted to a shortcut ρ_η satisfies triangle inequality, so we may further assume that no shortcut contains more than two consecutive points of $(x_i)_{i=0}^m$. Then, possibly adjoining x and y at the beginning and end of $(x_i)_{i=0}^m$, we find $n \in \mathbb{N}$, $i_1, \dots, i_n, p_j^-, p_j^+ \in S_j \in \mathcal{J}_{i_j}$ such that $S_j \neq S_{j+1}$ for $1 \leq j < n$, p_1^\pm, \dots, p_n^\pm are distinct, and $(x_i)_{i=0}^m = (x, p_1^-, \dots, p_n^+, y)$.

We now show by induction on $n \in \mathbb{N}$ that, for any discrete path $(x, p_1^-, \dots, p_n^+, y)$ as above, we may discard some terms and obtain one as in the thesis.

For $1 \leq n \leq 2$ Eq. (3.10) clearly holds, so we may assume $n \geq 3$. If Eq. (3.10) holds for no k , it is then not difficult to see that there is $1 < l < n$ such that $i_{l-1} \leq i_l$ and $i_{l+1} < i_l$. The second point of Definition 3.1 implies $d(p_{l-1}^+, S_l) \geq b\delta_{i_l}$, $d(p_{l+1}^-, S_l) \geq b\delta_{i_l}$ and, from the third, we obtain

$$d(p_{l-1}^+, p_l^-) + \rho_\eta(p_l^-, p_l^+) + d(p_l^+, p_{l+1}^-) \geq d(p_{l-1}^+, p_{l+1}^-).$$

Hence, discarding (p_l^-, p_l^+) yields a discrete path with no larger ‘cost’ w.r.t. ρ_η . Then, the first part of the proof shows that from $(x, p_1^-, \dots, p_{l-1}^-, p_{l+1}^-, \dots, p_n^+, y)$ we may obtain a new discrete path $(x, q_1^-, \dots, q_t^+, y)$ with $t \in \mathbb{N}$, $t < n$, satisfying all conditions of the thesis except possibly for Eq. (3.10). Since $t < n$, the induction hypothesis concludes the proof. \square

The following marks our point of departure from [LDLR17].

Proposition 3.11 (Single jump). *Let $x, y \in X$ with $d_\eta(x, y) < d(x, y)$. Then, for $\epsilon > 0$, there are $p_- \neq p_+ \in S \in \mathcal{J}$ such that*

$$d_\eta(x, y)(1 + a/b + \epsilon) \geq d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y). \quad (3.12)$$

In particular, d_η is a distance on X .

Proof. We first prove that, for $\epsilon > 0$, if i_j, S_j, p_j^\pm , and k are as in Lemma 3.8, then

$$d_\eta(x, y) + \epsilon \geq (1 + a/b)^{-1} (d(x, p_k^-) + \rho_\eta(p_k^-, p_k^+) + d(p_k^+, y)).$$

Later, we will show that $d_\eta(x, y) > 0$ whenever $x \neq y$, which, combined with the above, immediately gives the thesis.

From $S_j \neq S_{j+1}$ and $i_j \geq i_{j+1}$ for $1 \leq j < k$, we have

$$\sum_{j=1}^{k-1} d(p_j^+, p_{j+1}^-) \geq b \sum_{j=1}^{k-1} \delta_{i_j} \geq (b/a) \sum_{j=1}^{k-1} d(p_j^-, p_j^+)$$

and similarly $\sum_{j=k}^{n-1} d(p_j^+, p_{j+1}^-) \geq (b/a) \sum_{j=k}^{n-1} d(p_{j+1}^-, p_{j+1}^+)$. Set $t := a/(a+b)$ and observe that

$$\begin{aligned} \sum_{j=1}^{k-1} d(p_j^+, p_{j+1}^-) &\geq t(b/a) \sum_{j=1}^{k-1} d(p_j^-, p_j^+) + (1-t) \sum_{j=1}^{k-1} d(p_j^+, p_{j+1}^-) \\ &= \frac{b}{a+b} \sum_{j=1}^{k-1} d(p_j^-, p_j^+) + d(p_j^+, p_{j+1}^-) \geq (1+a/b)^{-1} d(p_1^-, p_k^-) \end{aligned}$$

and $\sum_{j=k}^{n-1} d(p_j^+, p_{j+1}^-) \geq (1+a/b)^{-1} d(p_k^+, p_n^+)$. Inequality (3.9) and the above give

$$\begin{aligned} d_\eta(x, y) + \epsilon &\geq d(x, p_1^-) + \sum_{j=1}^{n-1} (\rho_\eta(p_j^-, p_j^+) + d(p_j^+, p_{j+1}^-)) + \rho_\eta(p_n^-, p_n^+) + d(p_n^+, y) \\ &\geq d(x, p_1^-) + (1+a/b)^{-1} (d(p_1^-, p_k^-) + d(p_k^+, p_n^+)) + \rho_\eta(p_k^-, p_k^+) + d(p_n^+, y) \\ &\geq (1+a/b)^{-1} (d(x, p_k^-) + \rho_\eta(p_k^-, p_k^+) + d(p_k^+, y)), \end{aligned}$$

as claimed.

Suppose now there are $x \neq y \in X$ with $d_\eta(x, y) = 0$ and let $\epsilon_j \downarrow 0$. By the first part of the proof, we find $i_j \in \mathbb{N}$ and $p_j^- \neq p_j^+ \in S_j \in \mathcal{J}_{i_j}$ such that $d(x, p_j^-) + \rho_\eta(p_j^-, p_j^+) + d(p_j^+, y) \leq \epsilon_j$ for $j \in \mathbb{N}$. Since $\epsilon_j \rightarrow 0$, (i_j) is unbounded, and so it has a (not relabelled) divergent subsequence. But then

$$d(x, y) \leq d(x, p_j^-) + d(p_j^-, p_j^+) + d(p_j^+, y) \leq \epsilon_j + a\delta_{i_j} \rightarrow 0,$$

a contradiction. \square

Lemma 3.13. *For $r > 0$ there is $\rho(r) > 0$ such that $B_\eta(x, \rho(r)) \subseteq U_X(x, r)$ for $x \in X$. Hence, d and d_η induce the same topology.*

Proof. Set $C := 1 + 2a/b$. Since $\delta_i \rightarrow 0$ and $\eta_i > 0$ for all i , for $r > 0$ there is $\rho = \rho(r) \in (0, r/2C)$ such that $\rho_\eta(x, y) \leq C\rho$ implies $d(x, y) \leq r/2$, for all $x, y \in X$. Let $x \in X$, $y \in B_\eta(x, \rho)$, and $p_- \neq p_+ \in S \in \mathcal{J}_i$ such that $d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq Cd_\eta(x, y)$. Since $\rho_\eta(p_-, p_+) \leq C\rho$, it holds $d(p_-, p_+) \leq r/2$ and so $d(x, y) \leq C\rho + r/2 < r$. \square

Recall that a metric space is *proper* if closed and bounded subsets are compact.

Lemma 3.14. *The following hold:*

- X is complete if and only if (X, d_η) is complete;
- X is proper if and only if (X, d_η) is proper.

Proof. From Lemma 3.13, X and (X, d_η) have the same Cauchy sequences and topology; this proves the first point. For the second point, it is enough to show that sets which are bounded w.r.t. d_η are also bounded w.r.t. d . This follows easily from Proposition 3.11 and $\sup_i \delta_i < \infty$. \square

Definition 3.15. For metric space Z, Y , we say that a map $f: Z \rightarrow Y$ is *David-Semmes regular* (or DS-regular) if there is $C \geq 1$ such that $\text{LIP}(f) \leq C$ and for every ball $B_Y(y, r) \subseteq Y$ there are $z_1, \dots, z_n \in Z$, $n \leq C$, satisfying $f^{-1}(B_Y(y, r)) \subseteq \bigcup_{i=1}^n B_Z(z_i, Cr)$.

Let $f: Z \rightarrow Y$ be DS-regular. It follows from the definition that

$$f_\# \mathcal{H}_Z^s(E) \underset{C, s}{\sim} \mathcal{H}_Y^s \lfloor f(Z)(E) \quad (3.16)$$

for $E \subseteq Y$ and $s \in [0, \infty)$. Note also that if $A \subseteq Z$ is non-empty, then $f|_A$ is also DS-regular, with comparable constant. Moreover, if μ is an s -ADR measure on Z and in addition f is surjective, then $f_\# \mu$ is s -ADR on Y .

Definition 3.17. A metric space Z is (*metric*) *doubling* if there is $N \in \mathbb{N}$ such that every ball can be covered by at most N balls of half the radius. This is equivalent to the following: For every $\epsilon \in (0, 1)$, there is $N(\epsilon) \in \mathbb{N}$ such that if $W \subseteq B_Z(z, r)$ is ϵr -separated, then $\#W \leq N(\epsilon)$.

For instance, a metric space Z supporting a doubling measure is metric doubling and, if Z is complete, the converse is also true [LS98].

One may verify that a DS-regular image of a doubling metric space is also metric doubling.

Lemma 3.18. *Suppose X is metric doubling. Then the identity map $(X, d) \rightarrow (X, d_\eta)$ is David-Semmes regular.*

Proof. Set $C := 1 + 2a/b$. Since $d_\eta \leq d$, the identity map $(X, d) \rightarrow (X, d_\eta)$ is 1-Lipschitz. Let $x \in X$ and $r > 0$. Set $J := \{S \in \mathcal{J}_i : a\delta_i \geq r, S \cap B(x, Cr) \neq \emptyset\}$. Then

$$B_\eta(x, r) \subseteq B(x, (C+1)r) \cup \bigcup_{z \in S \in J} B(z, Cr).$$

Indeed, let $y \in B_\eta(x, r)$. If $d_\eta(x, y) = d(x, y)$, then $y \in B(x, (C+1)r)$. Otherwise, let $p_-, p_+ \in S \in \mathcal{J}$ be such that $d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq Cr$. Observe that $p_- \in B(x, Cr)$, so $S \cap B(x, Cr) \neq \emptyset$. If $S \in J$, then $y \in B(p_+, Cr)$. Otherwise, $d(x, y) \leq Cr + r$ and $y \in B(x, (C+1)r)$, as required.

Since $\#S \leq M$ for each $S \in \mathcal{J}$ (see Definition 3.1), to conclude the proof of David-Semmes regularity it only remains to show that the cardinality of J can be bounded independently of x and r . For each $S \in J$, pick $z_S \in B(x, Cr) \cap S$ and set $Z := \{z_S : S \in J\}$. Observe that $\#Z = \#J$.

From the separation condition in Definition 3.1 and the definition of J , we see that $d(z, z') \geq (b/a)r$ for $z, z' \in Z$, $z \neq z'$. That is, Z is a $(b/a)r$ -separated set contained in $B(x, Cr)$. Since X is metric doubling, there is a constant $N > 0$, depending only on the doubling constant of X and Ca/b , such that $\#J \leq N$. \square

The following variant of Lemma 3.18 will be used in Section 5.

Lemma 3.19. *Suppose there is a measure μ on X such that (X, d, μ) is a metric measure space and μ vanishes on sets porous w.r.t. d .*

Then there are countably many μ -measurable sets $(E_i)_i$ with $\mu(X \setminus \bigcup_i E_i) = 0$ such that the identity map $(E_i, d) \rightarrow (E_i, d_\eta)$ is David-Semmes regular for each i (and η).

Proof. Set $C := 1 + 2a/b$ and $\epsilon := b/4Ca$. Since μ vanishes on porous sets, by [MMPZ03, Theorem 3.6(ii),(iv)], μ is asymptotically doubling. Then, by [Bat15, Lemma 8.3], we find countably many metric doubling μ -measurable sets E_j with $\mu(X \setminus \bigcup_j E_j) = 0$. For $R > 0$ and j , define $g_{j,R}: E_j \rightarrow [0, 1]$ as

$$g_{j,R}(x) := \sup_{y \in U(x, R) \setminus \{x\}} \frac{d(E_j, y)}{d(x, y)}, \quad x \in E_j,$$

and observe that it is lower semi-continuous on E_j . Also, by Lemma 2.9, we see that $g_{j,R}(x) \rightarrow 0$ as $R \rightarrow 0$ for every j and μ -a.e. $x \in E_j$. Hence, $E_{j,k} := \{x \in E_j: g_{j,1/k}(x) \leq \epsilon/2\}$ defines a countable collection of μ -measurable sets which cover μ -almost all of X . Since $E_{j,k}$ is doubling in X , it is in particular separable, and we may find countably many μ -measurable sets $(E_{j,k,l})_l$ such that $E_{j,k} = \bigcup_l E_{j,k,l}$ and $\text{diam}_\eta E_{j,k,l} \leq \text{diam}_X E_{j,k,l} \leq 1/Ck$ for all l .

We claim that the identity $(E_{j,k,l}, d) \rightarrow (E_{j,k,l}, d_\eta)$ is DS-regular. We argue as in Lemma 3.18, with minor adaptation. Let $x \in E_{j,k,l}$ and $r > 0$. If $r \geq 1/Ck$, then $B_\eta(x, r) \cap E_{j,k,l} = E_{j,k,l} = B_X(x, r) \cap E_{j,k,l}$ and there is nothing to prove. Suppose $0 < r < 1/Ck$, define $J \subseteq \mathcal{J}$ as in the proof of Lemma 3.18, and recall that

$$B_\eta(x, r) \subseteq B_X(x, (C+1)r) \cup \bigcup_{z \in S \in J} B_X(z, Cr).$$

Observe that to conclude the proof of the claim (and the lemma), we need only to show that the cardinality of J is bounded by a constant independent of x and r .

Let Z as in the proof of Lemma 3.18, recall that $\#Z = \#J$, $Z \subseteq B_X(x, Cr)$, and that Z is $(b/a)r$ -separated in X . Recall also that $Cr < 1/k$ and $g_{j,1/k}(x) \leq \epsilon/2$. Since $Z \subseteq B_X(x, Cr) \subseteq U_X(x, 1/k)$, for $z \in Z$ there is $w_z \in E_j$ such that $d(w_z, z) \leq \epsilon d(x, z) \leq (b/4a)r$. Since Z is $(b/a)r$ -separated in X , we deduce that $W := \{w_z: z \in Z\}$ is $(b/2a)r$ -separated set in X and contained in $B_X(x, C(1+\epsilon)r) \cap E_j$. Then the cardinality of W is bounded by a constant depending only on the doubling constant of E_j and $2aC(1+\epsilon)/b$. Finally, the separation condition of Z implies also $\#W = \#Z = \#J$, concluding the proof of the claim and hence the lemma. \square

It turns out that, regardless of the contraction η , the separation condition in Definition 3.1 is preserved, albeit with worse constants.

Lemma 3.20. *There is a constant $c \in (0, 1)$ such that, for $i \leq j$, $S \in \mathcal{J}_i$ and $S' \in \mathcal{J}_j$ distinct, it holds*

$$d_\eta(S, S') \geq c\delta_j. \quad (3.21)$$

We may take $c = b/(1 + a/b)$.

Proof. Let $x \in S$ and $y \in S'$ be such that $d_\eta(x, y) = d_\eta(S, S')$. If $d_\eta(x, y) = d(x, y)$, then Eq. (3.21) follows from Definition 3.1. Suppose $d_\eta(x, y) < d(x, y)$, fix $C > 1 + a/b$, and let $p_- \neq p_+ \in S'' \in \mathcal{J}_k$ be such that

$$d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq Cd_\eta(x, y). \quad (3.22)$$

Suppose first $1 \leq k \leq j$. If $S' \neq S''$, then $d(p_+, y) \geq d(S'', S') \geq b\delta_j$. Otherwise, $d(x, p_-) \geq d(S, S') \geq b\delta_j$. In either case, from Eq. (3.22) we deduce $d_\eta(x, y) \geq C^{-1}b\delta_j$. Suppose now $k > j$. Then $d(x, y) \leq d(x, p_-) + d(p_+, y)$ and $d(x, y) \geq b\delta_j$, from which $d_\eta(x, y) \geq C^{-1}b\delta_j$ follows. \square

Given $x, y \in S \in \mathcal{J}$, it is immediate that $d_\eta(x, y) \leq \rho_\eta(x, y)$, but it is not apriori clear how much smaller $d_\eta(x, y)$ can be. We show that it cannot be too much smaller.

Lemma 3.23. *There is a constant $c > 0$ such that, for $x, y \in S \in \mathcal{J}$, it holds $d_\eta(x, y) \geq c\rho_\eta(x, y)$. We may take $c = \min(2b, a_0)/a(1 + a/b)$.*

Proof. We can assume $d_\eta(x, y) < d(x, y)$. Let $i \in \mathbb{N}$ be such that $S \in \mathcal{J}_i$. Fix $C > 1 + a/b$ and let $p_- \neq p_+ \in S' \in \mathcal{J}_j$ be such that $d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq Cd_\eta(x, y)$. If $1 \leq j \leq i$ and $S \neq S'$, then $d(S, S') \geq b\delta_i$ and so $Cd_\eta(x, y) \geq 2b\delta_i \geq 2(b/a)\rho_\eta(x, y)$. If $S = S'$, then $\rho_\eta(p_-, p_+) \geq (a_0/a)\rho_\eta(x, y)$. Finally, if $j > i$, then $d(x, p_-) + d(p_+, y) \geq d(x, y) \geq \rho_\eta(x, y)$. \square

Corollary 3.24. *Let (X, d, μ) be an s -ADR metric measure space with shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$. Let (α_i) be a non-negative sequence and set*

$$E := \limsup_{i \rightarrow \infty} B_X(\cup \mathcal{J}_i, \alpha_i \delta_i), \quad E_\eta := \limsup_{i \rightarrow \infty} B_\eta(\cup \mathcal{J}_i, \alpha_i \delta_i).$$

Then E and E_η have full μ -measure if $\sum_i \alpha_i^s = \infty$, while are μ -null if $\sum_i \alpha_i^s < \infty$.

Proof. We need only to consider E_η , since $E = E_\eta$ for $\eta = (1, 1, \dots)$. Recall that, by Lemma 3.18, (X, d_η, μ) is s -ADR. Let $c > 0$ be the constant of Lemma 3.20, set $C := \max(M, 2/c)$, and define $\mathcal{N}_i := \cup \mathcal{J}_i$ for $i \in \mathbb{N}$. Then Lemma 3.20 and the last two conditions of Definition 3.1 imply that $(\mathcal{N}_i)_i$ satisfies the assumptions of Lemmas 2.43 and 2.44 with our choice of C , concluding the proof. \square

Lemma 3.25. *There are $R_0 > 0$ and $C \geq 1$ such that for any $S \in \mathcal{J}_i$ and $0 < R < R_0$, it holds*

$$B_\eta(S, R\delta_i) \subseteq B_X(S, CR\delta_i).$$

We may take $R_0 = b/(1 + a/b)$ and $C = (1 + a/b)^2$.

Proof. Let $z \in S$ and $x \in B_\eta(z, R\delta_i)$. Set $C := 1 + a/b$, let $\epsilon > 0$ be such that $R(C + \epsilon) < b$, and pick $p_- \neq p_+ \in S' \in \mathcal{J}_j$ such that $d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, z) \leq (C + \epsilon)d_\eta(x, z)$. If $1 \leq j \leq i$ and $S \neq S'$, then $b\delta_i \leq d(p_+, z) \leq (C + \epsilon)R\delta_i$, a contradiction. If $S = S'$, then $p_- \in S$ and so $d(x, S) \leq (C + \epsilon)d_\eta(x, z)$. Lastly, if $j > i$, we have $d(p_+, z) \geq b\delta_j \geq (b/a)d(p_-, p_+)$, which yields $d(x, z) \leq (C + \epsilon)(1 + a/b)d_\eta(x, z)$. Hence, $d(x, S) \leq C(C + \epsilon)R\delta_i$ for all $\epsilon > 0$. \square

Lemma 3.26. *There is a constant $c > 0$ such that the following holds for any $i \in \mathbb{N}$, $R > 0$, and $C \geq 1$ with $C(R + 1)\eta_i < c$. Let $p_- \neq p_+ \in S \in \mathcal{J}_i$ and $x \in B_X(p_-, R\eta_i\delta_i\eta_i)$, $y \in B_X(p_+, R\delta_i\eta_i)$. Then $d_\eta(x, y) < d(x, y)$ and for any $q_-, q_+ \in S'$ satisfying*

$$d(x, q_-) + \rho_\eta(q_-, q_+) + d(q_+, y) \leq Cd_\eta(x, y), \quad (3.27)$$

we have $q_\pm = p_\pm$. We may take $c = \min(a_0, b)/\max(4, a)(1 + a/2b)$.

Remark 3.28. We stress that Lemma 3.26 does not guarantee the validity of Eq. (3.27) with $q_\pm = p_\pm$, but only that if Eq. (3.27) holds for some q_\pm , then it must be $q_\pm = p_\pm$. However, combined with Proposition 3.11, it does imply Eq. (3.27) with $q_\pm = p_\pm$ and $C := 1 + a/b$. Indeed, if $C(R + 1)\eta_i < c$, then $(C + \epsilon)(R + 1)\eta_i < c$ for all sufficiently small $\epsilon > 0$, so from Proposition 3.11 and Lemma 3.26 we deduce

$$d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq (C + \epsilon)d_\eta(x, y),$$

for all $x \in B_X(p_-, R\delta_i\eta_i)$, $y \in B_X(p_+, R\delta_i\eta_i)$, and $\epsilon > 0$ small.

Proof. Let $i \in \mathbb{N}$, $R > 0$, $C \geq 1$, $p_- \neq p_+ \in S \in \mathcal{J}_i$, and $x \in B_X(p_-, R\delta_i\eta_i)$, $y \in B_X(p_+, R\delta_i\eta_i)$. We show that there is $c = c(a, a_0, b) > 0$ such that, if the conclusion fails, then $C(R + 1)\eta_i \geq c$.

Since $d_\eta \leq d$ and $\rho_\eta(p_-, p_+) \leq a\delta_i\eta_i$, we have

$$d_\eta(x, y) \leq (2R + a)\delta_i\eta_i, \quad (3.29)$$

and so, if $d_\eta(x, y) = d(x, y)$, then

$$(2R + a)\delta_i\eta_i \geq d(x, y) \geq d(p_-, p_+) - d(x, p_-) - d(p_+, y) \geq a_0\delta_i - 2R\delta_i\eta_i,$$

which implies $(4R + a)\eta_i \geq a_0$.

Assume $d_\eta(x, y) < d(x, y)$ and that $q_-, q_+ \in S' \in \mathcal{J}_j$ satisfy Eq. (3.27). Suppose $1 \leq j \leq i$ and either $q_- \neq p_-$ or $q_+ \neq p_+$; w.l.o.g. $q_- \neq p_-$. Then

$$d(x, q_-) \geq d(q_-, p_-) - d(x, p_-) \geq \min(a_0, b)\delta_i - R\delta_i\eta_i,$$

Eqs. (3.27) and (3.29) imply $((2C + 1)R + Ca)\eta_i \geq \min(a_0, b)$.

It remains to consider the case $j > i$. We note that

$$\begin{aligned} d(q_-, q_+) &\leq a\delta_j \leq \frac{a}{2b}(d(q_-, p_-) + d(q_+, p_+)) \\ &\leq \frac{a}{2b}(d(q_-, x) + d(x, p_-) + d(q_+, y) + d(y, p_+)) \\ &\leq \frac{a}{2b}(d(x, q_-) + d(q_+, y) + 2R\delta_i\eta_i), \end{aligned}$$

and by triangle inequality

$$d(x, y) \leq (1 + a/2b)(d(x, q_-) + d(q_+, y)) + \frac{a}{2b}2R\delta_i\eta_i. \quad (3.30)$$

Since $p_- \neq p_+$, it holds $d(x, y) \geq a_0\delta_i - 2R\delta_i\eta_i$, and so Eqs. (3.27), (3.29) and (3.30) imply $(1 + a/2b)(2(C + 1)R + Ca)\eta_i \geq a_0$.

If $c > 0$ is defined as in the statement, then $C(R + 1)\eta_i \geq c$ in any of the above cases. \square

Lemma 3.31. *There is a constant $C \geq 1$ such that the following holds for $i \in \mathbb{N}$ and $R > 0$ with $(R + 1)\eta_i \leq C^{-1}$. For $S \in \mathcal{J}_i$ and $0 < r \leq R$, there are closed sets $\{B_z(r) : z \in S\}$ satisfying:*

- $B_\eta(S, r\delta_i\eta_i) = \bigcup_{z \in S} B_z(r)$;
- $B_X(z, r\delta_i\eta_i) \subseteq B_z(r) \subseteq B_X(z, Cr\delta_i\eta_i)$;
- $d_\eta(B_z(r), B_{z'}(r)) \geq C^{-1}\delta_i\eta_i$ and $d_X(B_z(r), B_{z'}(r)) \geq C^{-1}\delta_i$ for $z \neq z' \in S$.

Moreover, for $0 < r < r' \leq R$ and $z \in S$, it holds $B_z(r) \subseteq B_z(r')$. We may take

$$C = \max(4, a)(1 + a/b)^4 / \min(a_0, b).$$

Proof. Set $C := \max(4, a)(1 + a/b)^4 / \min(a_0, b) \geq 1$ and let $R > 0$, $i \in \mathbb{N}$, and $S \in \mathcal{J}_i$ be as in the statement. Define $C_0 := (1 + a/b)^2$, $B_z(r) := B_\eta(S, r\delta_i\eta_i) \cap B_X(z, C_0r\delta_i\eta_i)$, $z \in S$, and observe that $\{B_z(r) : z \in S\}$ is a collection of closed sets (Lemma 3.13) satisfying the second point of the statement, because $C \geq C_0$. Since $r\eta_i < C^{-1} < b/(1 + a/b)$, Lemma 3.25 shows that $\{B_z(r) : z \in S\}$ covers $B_\eta(S, r\delta_i\eta_i)$. Let $z \neq z' \in S$. Let $c_0 \in (0, 1)$ be defined as in Lemma 3.26 and note that $(C_0r + 1)\eta_i < C_0C^{-1} < c_0/(1 + a/b)$. Then, by Remark 3.28, for $x \in B_z(r)$ and $y \in B_{z'}(r)$ we have

$$(1 + a/b)d_\eta(x, y) \geq d(x, z) + \rho_\eta(z, z') + d(z', y) \geq a_0\delta_i\eta_i,$$

which implies $d_\eta(B_z(r), B_{z'}(r)) \geq C^{-1}\delta_i\eta_i$ because $C^{-1} \leq a_0/(1 + a/b)$. If x, y are as above, then

$$d(x, y) \geq d(z, z') - d(x, z) - d(z', y) \geq a_0\delta_i - 2C_0R\delta_i\eta_i \geq (a_0 - 2C_0C^{-1})\delta_i \geq C^{-1}\delta_i$$

concludes the proof. \square

The following was already proven in [DLR17, Theorem 1.3] under slightly more restrictive assumption (see Appendix A.1). We include it here for comparison with, and as warm-up to, Theorem 4.3 and Theorem 4.1. Our proof relies on the Borel-Cantelli-type lemmas of Section 2.6 and the properties of d_η developed so far.

Proposition 3.32. *Let (X, d, μ) be an s -ADR metric measure space with shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$ and $\eta = (\eta_i) \subseteq (0, 1]$. Then the following are equivalent:*

- *there is a positive-measure μ -measurable subset of X on which d and d_η are biLipschitz equivalent;*
- $\inf\{\eta_i : i \in \mathbb{N}\} > 0$;
- *d and d_η are biLipschitz equivalent on X .*

Proof. We need only to prove that the first point implies the second. We prove the contrapositive. Suppose $\inf\{\eta_i : i \in \mathbb{N}\} = 0$, let $(\eta_{i_j})_j$ and $(\alpha_j) \subseteq [1, \infty)$ be such that $\alpha_j \eta_{i_j} \rightarrow 0$ and $\sum_j (\alpha_j \eta_{i_j})^s = \infty$. Set $E_0 := \limsup_j B_X(\cup \mathcal{J}_{i_j}, \alpha_j \eta_{i_j} \delta_{i_j})$ and fix a positive-measure μ -measurable set $E \subseteq X$. By Lemma 3.18, the measure μ is s -ADR (in particular doubling) also on (X, d_η) . Since, by Corollary 3.24, E_0 has full measure, we may then find a Lebesgue density point x of E w.r.t. d_η with $x \in E \cap E_0$.

Since $x \in E_0$, there are a subsequence $(i_{j_k})_k$ and $p_k^- \in S_k \in \mathcal{J}_{i_{j_k}}$ such that $d(x, p_k^-) \leq 2\alpha_{j_k} \eta_{i_{j_k}} \delta_{i_{j_k}}$ for $k \in \mathbb{N}$. Let $C_0 \geq 1$ denote the constant of Lemma 3.31 and observe that, since $(\alpha_{j_k} + 1)\eta_{i_{j_k}} \rightarrow 0$, we may assume w.l.o.g. $(2\alpha_{j_k} + 1)\eta_{i_{j_k}} \leq C_0^{-1}$ for all $k \in \mathbb{N}$. Let $\{B_{z,k} : z \in S_k\}$ be as in Lemma 3.31 with $r = R = 2\alpha_{j_k}$ (and $i = i_{j_k}$, $S = S_k$), let $p_k^+ \in S_k \setminus \{p_k^-\}$, set $B_k^- := B_{p_k^-, k}$ and $B_k^+ := B_{p_k^+, k}$, and observe that $x \in B_k^-$. From $d_\eta(p_k^+, x) \leq (2\alpha_{j_k} + a)\eta_{i_{j_k}} \delta_{i_{j_k}} \leq (2 + a)\alpha_{j_k} \eta_{i_{j_k}} \delta_{i_{j_k}}$, we have

$$B_k^+ \subseteq B_\eta(S_k, 2\alpha_{j_k} \eta_{i_{j_k}} \delta_{i_{j_k}}) \subseteq B_\eta(x, (4 + a)\alpha_{j_k} \eta_{i_{j_k}} \delta_{i_{j_k}}). \quad (3.33)$$

By Lemma 3.31, we have $\mu(B_k^+) \sim (\alpha_{j_k} \eta_{i_{j_k}} \delta_{i_{j_k}})^s$. Then, since x is a Lebesgue density point of E w.r.t. d_η , from the above and Eq. (3.33) we see that $E \cap B_k^+ \neq \emptyset$ for all sufficiently large k . Assume w.l.o.g. $E \cap B_k^+ \neq \emptyset$ for all $k \in \mathbb{N}$ and let $y_k \in E \cap B_k^+$. Then Eq. (3.33) implies $d_\eta(x, y_k) \leq (4 + a)\alpha_{j_k} \eta_{i_{j_k}} \delta_{i_{j_k}}$, while Lemma 3.31 and $x \in B_k^-$ give $d(x, y_k) \geq C_0^{-1} \delta_{i_{j_k}}$. We finally have

$$\limsup_{E \ni y \rightarrow x} \frac{d(x, y)}{d_\eta(x, y)} \geq \limsup_{k \rightarrow \infty} \frac{d(x, y_k)}{d_\eta(x, y_k)} \geq \lim_{k \rightarrow \infty} \frac{C_0^{-1}}{(4 + a)\alpha_{j_k} \eta_{i_{j_k}}} = \infty.$$

□

4. SHORTCUT METRIC SPACES AND PI (UN)RECTIFIABILITY

In this section we determine the sequences $\eta = (\eta_i)_i \subseteq (0, 1]$ for which a shortcut space (X, d_η, μ) is PI rectifiable or purely PI unrectifiable, and show that there is no other possibility.

4.1. Pure PI unrectifiability. The main theorem regarding pure PI unrectifiability of shortcut spaces is the following.

Theorem 4.1. *Let (X, d, μ) be an s -ADR metric measure space with shortcuts and $\eta = (\eta_i)_i \subseteq (0, 1]$. Suppose $\inf\{\eta_i : i \notin I\} = 0$ whenever $\sum_{i \in I} \eta_i^s < \infty$, $I \subseteq \mathbb{N}$. Then (X, d_η, μ) is purely PI unrectifiable.*

In particular, if $\eta_i \rightarrow 0$, then (X, d_η, μ) is purely PI unrectifiable whenever $\sum_{i \in \mathbb{N}} \eta_i^s = \infty$.

For the proof of Theorem 4.1, we need the following lemma.

Lemma 4.2. *Let X be a metric space, $E, F \subseteq X$ sets, and let $\gamma : C \rightarrow E \cup F$ be a continuous function from a non-empty closed set $C \subseteq \mathbb{R}$. Suppose $\gamma(C) \cap E \neq \emptyset$ and $\gamma(C) \cap F \neq \emptyset$. Then $\text{gap } \gamma \geq d(E, F)$.*

Proof. We may assume w.l.o.g. $d(E, F) > 0$, $0 \in C \subseteq [0, \infty)$, and $\gamma(0) \in E$. Set

$$\alpha := \max\{t \in C : \gamma([0, t] \cap C) \subseteq E\}.$$

It is not difficult to see that $C \cap (\alpha, \alpha + \epsilon) = \emptyset$ for some $\epsilon > 0$. Also, by assumption $\gamma^{-1}(F) \cap C \neq \emptyset$, and therefore $C \setminus [0, \alpha]$ is a non-empty closed set which is bounded from below. We can therefore define $\beta := \min(C \setminus [0, \alpha])$. By definition of α , it must be $\gamma(\beta) \in F$, and thus $\text{gap } \gamma \geq d(\gamma(\alpha), \gamma(\beta)) \geq d(E, F)$. □

Proof of Theorem 4.1. By Lemma 2.22, there is an subsequence $(\eta_{i_j})_j$ with $\eta_{i_j} \rightarrow 0$ and $\sum_j \eta_{i_j}^s = \infty$. Define

$$E_0 := \limsup_{j \rightarrow \infty} B_\eta(\cup \mathcal{J}_{i_j}, \delta_{i_j} \eta_{i_j}),$$

and note that by Corollary 3.24 it has full measure. Let $E \subseteq X$ be a positive-measure μ -measurable set, $x \in E_0 \cap E$ a Lebesgue density point of E w.r.t. d_η , $C_0 \geq 1$ the constant in Lemma 3.31, and $0 < \delta \leq 1/C_0(4 + a)$. Let $C \geq 1$ and $\epsilon, r > 0$. We claim that there is $y \in B_\eta(x, r) \cap E$, $y \neq x$, such

that (x, y) is not (C, δ, ϵ) -connected in (E, d_η, μ) ; see Definition 2.34. By Theorem 2.37, this will conclude the proof.

Let $R \geq C(4 + a)$ and $j \in \mathbb{N}$ be such that $d_\eta(x, \cup \mathcal{J}_{i_j}) \leq \delta_{i_j} \eta_{i_j}$, $(R + 1)\eta_{i_j} \leq C_0^{-1}$, and $(4 + a)\delta_{i_j} \eta_{i_j} \leq r$. Let $S \in \mathcal{J}_{i_j}$ be such that $d_\eta(x, S) \leq 2\delta_{i_j} \eta_{i_j}$ and let $\{B_z(t) : z \in S\}$ be as in Lemma 3.31 for $0 < t \leq R$. Since x is a Lebesgue density point of E w.r.t. d_η , we may assume that $j \in \mathbb{N}$ was taken so large that $B_z(2) \cap E \neq \emptyset$ for all $z \in S$. Indeed, if this failed for infinitely many j ,

$$B_X(z, 2\delta_{i_j} \eta_{i_j}) \subseteq B_z(2) \subseteq B_\eta(x, 4\delta_{i_j} \eta_{i_j})$$

and s -AD regularity of μ on (X, d) and (X, d_η) would contradict the fact that x is a Lebesgue density point of E w.r.t. d_η .

Let $z_0 \in S$ be such that $x \in B_{z_0}(2)$, pick $z \in S \setminus \{z_0\}$, $y \in B_z(2) \cap E$, and observe that $d_\eta(x, y) \leq (4 + a)\delta_{i_j} \eta_{i_j} \leq r$, i.e. $y \in B_\eta(x, r) \cap E$.

Let $K \subseteq \mathbb{R}$ be a non-empty compact set and $\gamma : K \rightarrow X$ a continuous map with $\gamma(\min K) = x$ and $\gamma(\max K) = y$. If $\text{var}_\eta \gamma > R\delta_{i_j} \eta_{i_j}$, the choices of R and y imply

$$\text{var}_\eta \gamma > C(4 + a)\delta_{i_j} \eta_{i_j} \geq C d_\eta(x, y).$$

Suppose $\text{var}_\eta \gamma \leq R\delta_{i_j} \eta_{i_j}$. Then $\gamma(K) \subseteq B_\eta(S, R\delta_{i_j} \eta_{i_j})$, $\gamma(K) \cap B_{z_0}(R) \neq \emptyset$, and $\gamma(K) \cap \bigcup \{B_{z'}(R) : z' \in S \setminus \{z_0\}\} \neq \emptyset$. From Lemmas 3.31 and 4.2, we deduce

$$\text{gap}_\eta \gamma \geq d_\eta \left(B_{z_0}(R), \bigcup \{B_{z'}(R) : z' \in S \setminus \{z_0\}\} \right) \geq C_0^{-1} \delta_{i_j} \eta_{i_j} \geq \delta d_\eta(x, y).$$

This proves the claim and hence the thesis. \square

4.2. PI rectifiability. In this subsection we prove the following.

Theorem 4.3. *Let (X, d, μ) be a PI rectifiable s -ADR metric measure space with shortcuts and $\eta = (\eta_i)_i \subseteq (0, 1]$. Suppose there is $I \subseteq \mathbb{N}$ with $\sum_{i \in I} \eta_i^s < \infty$ and $\inf \{\eta_i : i \notin I\} > 0$. Then (X, d_η, μ) is PI rectifiable.*

In particular, if $\eta_i \rightarrow 0$, then (X, d_η, μ) is PI rectifiable whenever $\sum_{i \in \mathbb{N}} \eta_i^s < \infty$.

Remark 4.4. The hypotheses on the sequence η in Theorem 4.1 and Theorem 4.3 exactly complement each other. Therefore, for a given PI rectifiable s -ADR metric measure space (X, d, μ) with shortcuts, these results characterise when the shortcut metric space is PI rectifiable or purely PI unrectifiable.

Definition 4.5. Given a metric space X with shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$ and $\eta = (\eta_i)_i \subseteq (0, 1]$, we set

$$\text{Bad} := \bigcup_{\alpha \in \mathbb{N}} \limsup_{i \rightarrow \infty} B_\eta(\cup \mathcal{J}_i, \alpha \delta_i \eta_i). \quad (4.6)$$

One may verify that Bad may be equivalently defined replacing the set $B_\eta(\cup \mathcal{J}_i, \alpha \delta_i \eta_i)$ with $B_X(\cup \mathcal{J}_i, \alpha \delta_i \eta_i)$.

For Theorem 4.3, we need only to focus on points *not* in Bad .

Lemma 4.7. *Let X be a metric space with shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$, let a, b be as in Definition 3.1, and $\eta = (\eta_i)_i \subseteq (0, 1]$. For any constant $C > 1 + a/b$, $x \notin \text{Bad}$, and $\epsilon > 0$, there is $R > 0$ with the following property. If $y \in B_\eta(x, R)$ and $p_- \neq p_+ \in S \in \mathcal{J}$ satisfy*

$$d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq C d_\eta(x, y),$$

then

$$\rho_\eta(p_-, p_+) \leq \epsilon d_\eta(x, y).$$

Proof. Define $\alpha := aC/\epsilon$. Since $x \in X \setminus \text{Bad}$, there is $i_0 \in \mathbb{N}$ such that

$$d_\eta(x, \cup \mathcal{J}_i) \geq \alpha \delta_i \eta_i, \quad i \geq i_0.$$

Let $R > 0$ be such that $CR < a_0 \delta_i \eta_i$ for $1 \leq i < i_0$, where a_0 is as in Definition 3.1. Let y and $p_- \neq p_+ \in S \in \mathcal{J}_i$ be as in the statement. Then $a_0 \delta_i \eta_i \leq CR$ and, from the choice of R , it follows that $i \geq i_0$ and thus $\alpha \delta_i \eta_i \leq d_\eta(x, p_-) \leq C d_\eta(x, y)$. Then

$$\rho_\eta(p_-, p_+) \leq a \delta_i \eta_i \leq (a/\alpha) C d_\eta(x, y) = \epsilon d_\eta(x, y).$$

□

Lemma 4.8. *Let (X, d, μ) be an s -ADR metric measure space with shortcuts. There are constants $C_0 \geq 1$ and $c_0 \in (0, 1)$ such that the following holds. Let $\eta = (\eta_i)_i \subseteq (0, 1]$ be a sequence, $E \subseteq X \setminus \text{Bad}$ a μ -measurable set, and suppose (X, d, μ) is (C, δ, ϵ, r) -connected along E for some $C \geq 1$, $\delta, \epsilon \in (0, 1)$, and $r > 0$. Then, for every Lebesgue density point x of E w.r.t. d_η , there is $r_x > 0$ such that (x, y) is $(\bar{C}, \bar{\delta}, \bar{\epsilon})$ -connected in (X, d_η, μ) for all $y \in B_\eta(x, r_x) \setminus \{x\}$, where $\bar{C} = C_0 C$, $\bar{\delta} = C_0 \delta$, and $\bar{\epsilon} = c_0 \delta^s \epsilon$.*

Proof. Let a, b be as in Definition 3.1, fix $\tilde{C}_0 > 1 + a/b$, set $C_0 := 7\tilde{C}_0$, and let $c_0 \in (0, 1)$ to be determined later. Let $R_x > 0$ be the radius given by Lemma 4.7 applied to x with $\epsilon \equiv \delta$ and $C \equiv \tilde{C}_0$. Fix $0 < \alpha < \min(\delta, 1/2)$. Since x is a Lebesgue point of E w.r.t. d_η , there is $0 < r_x < \min(r, R_x)/\tilde{C}_0(1 + \alpha)$ such that for every $y \in B_\eta(x, r_x) \setminus \{x\}$ there is $z \in B_\eta(y, \alpha d_\eta(x, y)) \cap E$. We claim that the thesis holds with this value of r_x , C_0 , and c_0 sufficiently small.

Let y and z be as above, set $\rho_y := d_\eta(x, y)$, $\rho_z := d_\eta(x, z)$, and observe that $(1 - \alpha)\rho_y \leq \rho_z \leq (1 + \alpha)\rho_y$. Let $A \subseteq X$ be a μ -measurable set with

$$\mu(A \cap B_\eta(x, C_0 C \rho_y)) < c_0 \delta^s \epsilon \mu(B_\eta(x, C_0 C \rho_y)).$$

Suppose first $\rho_z = d(x, z)$. Then $B_X(x, C \rho_z) \subseteq B_\eta(x, C(1 + \alpha)\rho_y) \subseteq B_\eta(x, C_0 C \rho_y)$ and $\rho_z \geq \rho_y/2$ imply for $c_0 \in (0, 1)$ small enough

$$\mu(A \cap B_X(x, C \rho_z)) < c_0 \delta^s \epsilon \mu(B_\eta(x, C_0 C \rho_y)) \lesssim (c_0 \epsilon)(C \rho_z)^s \leq \epsilon \mu(B_X(x, C \rho_z)),$$

where we have also used the fact that μ is s -ADR also on (X, d_η) , see Lemma 3.18. Since $x \in E$ and $d(x, z) = \rho_z \leq (1 + \alpha)\rho_y < r$, the connectivity assumption on E yields a curve fragment γ_z from x to z , satisfying $\text{var}_X \gamma_z \leq C \rho_z$, $\text{gap}_X \gamma_z < \delta \rho_z$, and which may meet A only at the endpoints of its domain. The curve fragment γ , obtained following γ_z and then jumping from z to y , satisfies

$$\begin{aligned} \text{var}_\eta \gamma &\leq C \rho_z + \alpha \rho_y \leq (C(1 + \alpha) + \alpha) \rho_y < C_0 C \rho_y, \\ \text{gap}_\eta \gamma &< \delta \rho_z + \alpha \rho_y \leq (2 + \alpha) \delta \rho_y < C_0 \delta \rho_y, \end{aligned}$$

and may meet A at its endpoints or in z . It is not difficult to see that restricting the domain of γ we can find a curve fragment $\tilde{\gamma}$ from x to y having slightly larger gap and variation (in d_η) which does not intersect z .

Suppose now $\rho_z < d(x, z)$, let $p_- \neq p_+ \in S \in \mathcal{J}$ be such that

$$d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, z) \leq \tilde{C}_0 \rho_z,$$

and recall that, by the choice of r_x , we have $\rho_\eta(p_-, p_+) \leq \delta \rho_z$. We first construct curve fragments γ_x, γ_z from x to p_- and z to p_+ , respectively, with controlled variation and gap. Let (w, p) be one of the pairs $(x, p_-), (z, p_+)$. Suppose $t := d(w, p) \geq \delta \rho_z$. Since $t \leq \tilde{C}_0 \rho_z$ and $d_\eta(x, w) \leq \rho_z$, we have $B_X(w, Ct) \subseteq B_\eta(x, (\tilde{C}_0 C + 1)\rho_z) \subseteq B_\eta(x, C_0 C \rho_y)$ and therefore

$$\begin{aligned} \mu(A \cap B_X(w, Ct)) &< c_0 \delta^s \epsilon \mu(B_\eta(x, C_0 C \rho_y)) \lesssim (c_0 \epsilon)(C \delta \rho_z)^s \\ &\lesssim (c_0 \epsilon)(Ct)^s \leq \epsilon \mu(B_X(w, Ct)), \end{aligned} \tag{4.9}$$

provided $c_0 \in (0, 1)$ is taken small enough. Then, $t \leq \tilde{C}_0 \rho_z \leq \tilde{C}_0(1 + \alpha)\rho_y < r$, $w \in E$, the connectivity assumption on E , and Eq. (4.9) ensure the existence of a curve fragment γ_w from w to p satisfying

$$\begin{aligned} \text{var}_\eta \gamma_w &\leq \text{var}_X \gamma_w \leq Ct \leq \tilde{C}_0 C \rho_z, \\ \text{gap}_\eta \gamma_w &\leq \text{gap}_X \gamma_w < \delta t \leq \tilde{C}_0 \delta \rho_z, \end{aligned} \tag{4.10}$$

and, moreover, γ_w may meet A only at the endpoints of its domain. If, instead, $t = d(w, p) < \delta \rho_z$, let $\gamma_w: \{0, 1\} \rightarrow X$ be given by $\gamma_w(0) := w$, $\gamma_w(1) := p$, and observe that it satisfies Eq. (4.10) and avoids A in the same way. Then, the curve fragment $\gamma: K \rightarrow X$ which follows γ_x , then γ_z with reverse orientation, and, if $z \neq y$, finally jumps from z to y , satisfies

$$\begin{aligned} \text{var}_\eta \gamma &\leq (2\tilde{C}_0 C + \delta) \rho_z + \alpha \rho_y < C_0 C \rho_y, \\ \text{gap}_\eta \gamma &\leq (2\tilde{C}_0 + 1) \delta \rho_z + \alpha \rho_y < C_0 \delta \rho_y, \end{aligned} \tag{4.11}$$

where we have used Eq. (4.10), $\rho_\eta(p_-, p_+) \leq \delta \rho_z$, and $d_\eta(z, y) \leq \delta \rho_y$. Note that γ may meet A only at its endpoints or at p_-, p_+ , or z . By construction, there are $t_- < t_+ < t_z \in K$ such that $\gamma^{-1}\{p_-, p_+, z\} = \{t_-, t_+, t_z\}$. Hence, there is a curve fragment $\tilde{\gamma}$, obtained restricting slightly the domain of γ , which may meet A only at its endpoints, and moreover still satisfies Eq. (4.11). \square

Proof of Theorem 4.3. Let $I \subseteq \mathbb{N}$ be such that $\sum_{i \in I} \eta_i^s < \infty$ and $\inf\{\eta_i : i \notin I\} > 0$. If I is empty or finite, then d and d_η are biLipschitz and (X, d_η, μ) is trivially PI rectifiable. Assume I is infinite and suppose we have established the thesis for $I = \mathbb{N}$. Let $(i_j)_j \subseteq \mathbb{N}$ be the strictly increasing sequence with $I = \{i_j : j \in \mathbb{N}\}$ and observe that d_η is biLipschitz to the distance $d_{\tilde{\eta}}$ obtained from the shortcuts $\{(\mathcal{J}_{i_j}, \delta_{i_j})\}_j$ and the sequence $\tilde{\eta}_j := \eta_{i_j}$. That is, (X, d_η, μ) is biLipschitz to the PI rectifiable space $(X, d_{\tilde{\eta}}, \mu)$. Hence, we need only to consider the case $I = \mathbb{N}$.

By Corollary 3.24 and $I = \mathbb{N}$, we have $\mu(\text{Bad}) = 0$. Let C_0, c_0 be as in Lemma 4.8 and let $\delta \in (0, 1)$. Since (X, d, μ) is PI rectifiable, by Theorem 2.38 and Lemma 2.11, for μ -a.e. $x \in X$ there are (C_x, ϵ_x, r_x) as in Definition 2.35 with $\delta \equiv \delta/C_0$. We may also assume $x \mapsto (C_x, \epsilon_x, r_x)$ to be Borel measurable; see Lemma 2.39. Then, there are countably many disjoint Borel sets $E_n \subseteq X$ and constants (C_n, ϵ_n, r_n) such that (X, d, μ) is $(C_n, \epsilon_n, \delta/C_0, r_n)$ -connected along E_n and $\mu(X \setminus \bigcup_n E_n) = 0$. Then, by Lemma 4.8, for every n and μ -a.e. $x \in E_n$ there are constants $(\tilde{C}_x, \tilde{\epsilon}_x, \tilde{r}_x)$ such that (x, y) is $(\tilde{C}_x, \delta, \tilde{\epsilon}_x)$ -connected in (X, d_η, μ) for $y \in B_\eta(x, \tilde{r}_x) \setminus \{x\}$. By Theorem 2.38 we conclude that (X, d_η, μ) is PI rectifiable. \square

5. SHORTCUT METRIC SPACES AND LIPSCHITZ DIFFERENTIABILITY

Let Y be a Banach space, (X, d, μ) a Y -LDS with shortcuts, and $f : (X, d_\eta) \rightarrow Y$ Lipschitz. In this section we give a precise description of the set of points in X where the derivative of f on (X, d) defines a derivative of f on (X, d_η) . We also show that this is the only way in which a differentiable structure on (X, d_η, μ) can arise, see Lemma 5.3 and Proposition 5.7. We require some definitions in order to state the main result in Theorem 5.6.

Definition 5.1 (Compatible differentiable structure). Let Y be a Banach space, (X, d, μ) a Y -LDS, and suppose (X, d) has shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$. We say that a Cheeger atlas

$$\{(U_j, \varphi_j : X \rightarrow \mathbb{R}^{n_j})\}_j$$

of (X, d, μ) is *compatible* with the shortcuts if

$$\text{diam } \varphi_j(S) = 0 \tag{5.2}$$

for each j and $S \in \mathcal{J}$. If (X, d, μ) has a compatible atlas, we then say that it is a Y -LDS with compatible shortcuts, or that the shortcuts and the differentiable structure are compatible.

Under the compatibility condition of Definition 5.1, the only possible differentiable structure on (X, d_η, μ) is the one of (X, d, μ) .

Lemma 5.3. *Let (X, d, μ) be an LDS with compatible shortcuts. Let Y be a non-zero Banach space, $E \subseteq X$ a positive-measure μ -measurable set, and suppose (E, d_η, μ) is a Y -LDS. Then, the restriction to E of any compatible atlas of (X, d, μ) is an atlas of (E, d_η, μ) .*

Proof. The proof consists of standard reductions and the combination of Lemma 3.19 with [CEB23, Theorem 3.26].

Since the differentiable structure of an LDS does not depend on the target (see Lemma 2.6), it is enough to consider the case $Y = \mathbb{R}$. Let $\{(U_j, \varphi_j : X \rightarrow \mathbb{R}^{n_j})\}_j$ be a compatible atlas, observe that φ_j is Lipschitz also w.r.t. d_η , and let j be such that $\mu(E \cap U_j) > 0$. If $n_j = 0$, then by Proposition 2.17 and Lemma 3.13 μ -a.e. $x \in U_j$ is isolated and therefore $(E \cap U_j, \varphi_j) = (E \cap U_j, 0)$ is trivially a chart. Suppose now $n_j \geq 1$. By Lemmas 2.12 and 3.19 and inner regularity of μ , there are countably many compact sets $K_{j,k} \subseteq E \cap U_j$ with $\mu(E \cap U_j \setminus \bigcup_k K_{j,k}) = 0$ and such that the identity map $\iota|_{K_{j,k}} : (K_{j,k}, d) \rightarrow (K_{j,k}, d_\eta)$ is DS-regular. Then, for each k , $(K_{j,k}, d, \mu)$ and $(K_{j,k}, d_\eta, \mu)$ are compact (hence proper) LDS, μ is Radon, $\iota|_{K_{j,k}} : (K_{j,k}, d) \rightarrow (K_{j,k}, d_\eta)$ is a surjective DS-regular map preserving μ , $\varphi_j|_{K_{j,k}} : (K_{j,k}, d_\eta) \rightarrow \mathbb{R}^{n_j}$ Lipschitz, and $\varphi_j|_{K_{j,k}} \circ \iota|_{K_{j,k}} : (K_{j,k}, d) \rightarrow \mathbb{R}^{n_j}$ is a global chart on $(K_{j,k}, d, \mu)$. Thus, by [CEB23, Theorem 3.26], $\varphi_j|_{K_{j,k}}$ is a chart on $(K_{j,k}, d_\eta, \mu)$.

Then, for any Lipschitz $f: (E, d_\eta) \rightarrow \mathbb{R}$, any j, k , and μ -a.e. $x \in K_{j,k}$, there is a unique linear map $T_x: \mathbb{R}^{n_j} \rightarrow \mathbb{R}$ such that $\text{Lip}_\eta((f - T_x \circ \varphi_j)|_{K_{j,k}}; x) = 0$. Since (E, d_η, μ) is an LDS, by Lemmas 2.9 and 2.12 we conclude $\text{Lip}_\eta((f - T_x \circ \varphi_j)|_E; x) = 0$ at μ -a.e. $x \in E \cap U_j$, and any j .

Uniqueness of the φ_j -differentials follows from

$$\text{Lip}_\eta(\langle v, \varphi_j \rangle|_E; x) \geq \text{Lip}(\langle v, \varphi_j \rangle|_E; x) = \text{Lip}(\langle v, \varphi_j \rangle; x) > 0$$

for $v \in \mathbb{R}^{n_j} \setminus \{0\}$ and μ -a.e. $x \in E \cap U_j$; see Lemmas 2.5, 2.9 and 2.12. The first inequality holds because $d_\eta \leq d$. \square

We now introduce a definition related to the set Bad given in Definition 4.5.

Definition 5.4. Let Y be a Banach space, (X, d, μ) be a Y -LDS with compatible shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$, and $f: (X, d_\eta) \rightarrow Y$ a map. For $\epsilon > 0$ and $i \in \mathbb{N}$, gather in $\mathcal{J}_i^\epsilon(f)$ the shortcuts $S \in \mathcal{J}_i$ for which $\text{diam } f(S) \geq \epsilon \eta_i \delta_i$. We define

$$\begin{aligned} \text{Bad}_\epsilon(f) &:= \limsup_{i \rightarrow \infty} B_\eta(\cup \mathcal{J}_i^\epsilon(f), \eta_i \delta_i / \epsilon), \\ \text{Bad}(f) &:= \bigcup_{\epsilon > 0} \text{Bad}_\epsilon(f). \end{aligned} \tag{5.5}$$

If $0 < \epsilon' < \epsilon$, then $\text{Bad}_\epsilon(f) \subseteq \text{Bad}_{\epsilon'}(f)$. Hence, $\text{Bad}(f) = \bigcup_j \text{Bad}_{1/j}(f)$ shows that $\text{Bad}(f)$ is Borel. We show that, if f is Lipschitz on (X, d_η) , $\text{Bad}(f)$ coincides with the set of non-differentiability points of f , up to a null set.

The main result of this section is the following.

Theorem 5.6. Let Y be a Banach space, (X, d, μ) a Y -LDS with compatible shortcuts, and suppose μ vanishes on sets porous w.r.t. d_η . Then, for any Lipschitz map $f: (X, d_\eta) \rightarrow Y$, we have

- f is μ -a.e. differentiable on $X \setminus \text{Bad}(f)$ w.r.t. d_η and any compatible atlas;
- f is μ -a. nowhere differentiable on any subset of $\text{Bad}(f)$ w.r.t. d_η and any compatible atlas.

Moreover, the Cheeger differentials of f w.r.t. d and d_η agree μ -a.e. on $X \setminus \text{Bad}(f)$.

Before proving Theorem 5.6, let us single out two immediate applications.

Proposition 5.7. Let Y be a non-zero Banach space and (X, d, μ) a Y -LDS with compatible shortcuts. Then the following are equivalent:

- (X, d_η, μ) is a Y -LDS;
- $\mu(\text{Bad}(f)) = 0$ for every Lipschitz map $f: (X, d_\eta) \rightarrow Y$ and μ vanishes on sets porous w.r.t. d_η .

Proof. Suppose (X, d_η, μ) is a Y -LDS. Then, by Lemma 2.12, sets porous w.r.t. d_η are μ -null. By Lemma 5.3, there is a compatible atlas of (X, d, μ) which restricts to an atlas of (X, d_η, μ) and, by Theorem 5.6, we then have $\mu(\text{Bad}(f)) = 0$ for any Lipschitz $f: (X, d_\eta) \rightarrow Y$. Conversely, if the assumptions of the second point are satisfied, then (X, d_η, μ) is a Y -LDS by Theorem 5.6. \square

Proposition 5.8. Let Y be a non-zero Banach space, (X, d, μ) a Y -LDS with compatible shortcuts, and suppose μ vanishes on sets porous w.r.t. d_η . Let $f: (X, d_\eta) \rightarrow Y$ be a Lipschitz map and $E \subseteq \text{Bad}(f)$ a positive-measure μ -measurable set. Then (E, d_η, μ) is not a Y -LDS.

Proof. Suppose by contradiction that (E, d_η, μ) is a Y -LDS. By Lemma 5.3, there is a compatible atlas of (X, d, μ) which restricts to an atlas of (E, d_η, μ) . But then, by Theorem 5.6, $f|_E: (E, d_\eta) \rightarrow Y$ is μ -almost nowhere differentiable on E , from which we conclude $\mu(E) = 0$. \square

We now turn to the proof of Theorem 5.6. Unsurprisingly (in view of Theorem 2.38), the proof of Theorem 5.6 is reminiscent of the ones of Theorem 4.3 and Theorem 4.1. The crucial point is that $\text{Bad}(f)$ may be much smaller than Bad or, in other words, Lipschitz maps $f: (X, d_\eta) \rightarrow Y$ may be unable to fully capture the lack of ‘connectivity’ (in the sense of Definition 2.35) of the underlying space. This is the fundamental phenomenon behind Theorem 1.1. Theorem 8.3 and the constructions of non-differentiable maps in Section 10 also serve to illustrative this principle.

The following lemma should be compared to Lemma 4.7. It will be used for the proof of the first claim in Theorem 5.6.

Lemma 5.9. *Let (X, d) be a metric space with shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$ and $C > 1 + a/b$, where a, b are as in Definition 3.1. Then, for every $\eta = (\eta_i)_i \subseteq (0, 1]$, Banach space Y , $f: (X, d_\eta) \rightarrow Y$ Lipschitz, $x \notin \text{Bad}(f)$, and $\epsilon > 0$, there is $R > 0$ with the following property. If $y \in B_\eta(x, R)$ and $p_- \neq p_+ \in S \in \mathcal{J}$ satisfy*

$$d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq C d_\eta(x, y),$$

then

$$\|f(p_-) - f(p_+)\|_Y \leq \epsilon d_\eta(x, y). \quad (5.10)$$

Proof. Set $L := \text{LIP}_\eta(f)$ and $\epsilon_0 := \epsilon \min(a_0, 1/aL)/C$, where a_0 is as in Definition 3.1. Since $x \notin \text{Bad}(f)$, in particular $x \notin \text{Bad}_{\epsilon_0}(f)$, we find $i_0 \in \mathbb{N}$ such that

$$d_\eta(x, \cup \mathcal{J}_i^{\epsilon_0}(f)) \geq \eta_i \delta_i / \epsilon_0, \quad (5.11)$$

for $i \geq i_0$. Let $R > 0$ be such that $a_0 \eta_i \delta_i \leq CR$ implies $i \geq i_0$. Let y and $p_- \neq p_+ \in S \in \mathcal{J}_i$ be as in the statement and observe that

$$a_0 \eta_i \delta_i \leq \rho_\eta(p_-, p_+) \leq C d_\eta(x, y) \leq CR,$$

and so, by the choice of R , we have $i \geq i_0$ and hence Eq. (5.11) holds. If $S \notin \mathcal{J}_i^{\epsilon_0}(f)$, then

$$\|f(p_-) - f(p_+)\|_Y \leq \epsilon_0 \eta_i \delta_i \leq \epsilon_0 (C/a_0) d_\eta(x, y) \leq \epsilon d_\eta(x, y).$$

Otherwise, Eq. (5.11) gives

$$\eta_i \delta_i \leq \epsilon_0 d_\eta(x, \cup \mathcal{J}_i^{\epsilon_0}(f)) \leq \epsilon_0 d(x, S) \leq \epsilon_0 C d_\eta(x, y)$$

and, since f is L -Lipschitz on (X, d_η) , we finally have

$$\|f(p_-) - f(p_+)\|_Y \leq L a \eta_i \delta_i \leq \epsilon_0 a L C d_\eta(x, y) \leq \epsilon d_\eta(x, y).$$

□

We require the following lemma for the proof of the non-differentiability claim in Theorem 5.6.

Lemma 5.12. *Let (X, d) be a metric space with shortcuts $\{(\mathcal{J}_i, \delta_i)\}_i$ and let $\eta = (\eta_i)_i \subseteq (0, 1]$. Let $n \in \mathbb{N}$ and suppose $\varphi: X \rightarrow \mathbb{R}^n$ is a function satisfying $\text{diam } \varphi(S) = 0$ for $S \in \mathcal{J}$. Then, for any Banach space Y , $f: (X, d_\eta) \rightarrow Y$, and $x \in \text{Bad}(f)$, there are a constant $c > 0$ and a sequence $(x_j) \subseteq X \setminus \{x\}$, $x_j \rightarrow x$, such that*

$$\limsup_{j \rightarrow \infty} \frac{\|f(x_j) - f(x) - T(\varphi(x_j) - \varphi(x))\|_Y}{d_\eta(x, x_j)} \geq c,$$

for every linear $T: \mathbb{R}^n \rightarrow Y$.

Proof. Since $x \in \text{Bad}(f)$ there are $\epsilon > 0$, a strictly increasing sequence $(i_k) \subseteq \mathbb{N}$, and $S_k \in \mathcal{J}_{i_k}^\epsilon(f)$ such that $d_\eta(x, S_k) \leq 2\eta_{i_k} \delta_{i_k} / \epsilon$. Since $d_\eta(z, w) \leq a\eta_{i_k} \delta_{i_k}$ for $z, w \in S_k$, possibly shrinking $\epsilon > 0$, we may assume

$$d_\eta(x, p) \leq \eta_{i_k} \delta_{i_k} / \epsilon \quad (5.13)$$

for $p \in S_k$ and $k \in \mathbb{N}$. Also, since (S_k) are pairwise disjoint, we may assume $x \notin S_k$ for $k \in \mathbb{N}$. By definition of $\mathcal{J}_{i_k}^\epsilon(f)$ and $S_k \in \mathcal{J}_{i_k}^\epsilon(f)$, there are $p_k^- \neq p_k^+ \in S_k$ such that $\|f(p_k^-) - f(p_k^+)\|_Y \geq \epsilon \eta_{i_k} \delta_{i_k}$. For $k \in \mathbb{N}$, define $x_{2k-1} := p_k^-$, $x_{2k} := p_k^+$, and observe that $(x_j) \subseteq X \setminus \{x\}$ and $x_j \rightarrow x$. Let $T: \mathbb{R}^n \rightarrow Y$ be linear and note that, by triangle inequality and $\varphi(p_k^-) = \varphi(p_k^+)$, there are $\sigma_k \in \{-1, +1\}$ such that

$$\|f(p_k^{\sigma_k}) - f(x) - T(\varphi(p_k^{\sigma_k}) - \varphi(x))\|_Y \geq \epsilon \eta_{i_k} \delta_{i_k} / 2 \quad (5.14)$$

for $k \in \mathbb{N}$. Since $(p_k^{\sigma_k})_k$ is a subsequence of (x_j) , we finally have from Eq. (5.13), Eq. (5.14)

$$\begin{aligned} \limsup_{j \rightarrow \infty} \frac{\|f(x_j) - f(x) - T(\varphi(x_j) - \varphi(x))\|_Y}{d_\eta(x, x_j)} &\geq \limsup_{k \rightarrow \infty} \frac{\|f(p_k^{\sigma_k}) - f(x) - T(\varphi(p_k^{\sigma_k}) - \varphi(x))\|_Y}{\eta_{i_k} \delta_{i_k} / \epsilon} \\ &\geq \frac{\epsilon^2}{2}. \end{aligned}$$

□

Proof of Theorem 5.6. We begin with the statement regarding points where f is differentiable.

Possibly replacing Y with the closed span of $f(X)$, we may assume Y to be separable. Then, for $n \in \mathbb{N}$, the Banach space of linear maps $\mathbb{R}^n \rightarrow Y$ is also separable. Let $\{(U_j, \varphi_j: X \rightarrow \mathbb{R}^{n_j})\}_j$ be a Cheeger atlas compatible with the shortcuts and fix $\epsilon > 0$. Since f is in particular Lipschitz on (X, d) and (X, d, μ) a Y -LDS, we may cover μ -almost all of $X \setminus \text{Bad}(f)$ with countably many Borel sets $E \subseteq X \setminus \text{Bad}(f)$ having the following properties. There is j with $E \subseteq U_j$, $T: \mathbb{R}^{n_j} \rightarrow Y$ linear, and $R_1 > 0$ such that:

(i) for $x \in E$ and $y \in B_X(x, R_1)$, it holds

$$\|f(y) - f(x) - T(\varphi_j(y) - \varphi_j(x))\|_Y \leq \epsilon d(x, y);$$

(ii) E is not porous w.r.t. d_η at any $x \in E$.

Set $C := 1 + 2a/b$, let $R > 0$ be as in Lemma 5.9 with the current choices of parameters, and assume further $R \leq R_1/C$. We are now ready to prove differentiability. Let $y \in B_\eta(x, R) \cap E$. If $d_\eta(x, y) = d(x, y)$, then $R \leq R_1$ and (i) give

$$\|f(y) - f(x) - T(\varphi_j(y) - \varphi_j(x))\|_Y \leq \epsilon d_\eta(x, y). \quad (5.15)$$

Suppose now $d_\eta(x, y) < d(x, y)$. By Proposition 3.11, we find $p_- \neq p_+ \in S \in \mathcal{J}$ such that

$$d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_+, y) \leq C d_\eta(x, y); \quad (5.16)$$

in particular, $d(x, p_-), d(y, p_+) \leq CR \leq R_1$ and y, p_-, p_+ are as in Lemma 5.9. Thus, item (i) and Eq. (5.10) give

$$\begin{aligned} \|f(y) - f(x) - T(\varphi_j(y) - \varphi_j(x))\|_Y &\leq \|f(y) - f(p_+) - T(\varphi_j(y) - \varphi_j(p_+))\|_Y + \|f(p_+) - f(p_-)\|_Y \\ &\quad + \|f(p_-) - f(x) - T(\varphi_j(p_-) - \varphi_j(x))\|_Y \\ &\leq \epsilon d(y, p_+) + \epsilon d_\eta(x, y) + \epsilon d(x, p_-) \\ &\leq \epsilon(C + 1)d_\eta(x, y), \end{aligned} \quad (5.17)$$

where we have also used $\varphi_j(p_-) = \varphi_j(p_+)$ and Eq. (5.16). Hence, Eq. (5.15) and Eq. (5.17) imply $\text{Lip}_\eta((f - T \circ \varphi_j)|_E; x) \leq (C + 1)\epsilon$ for $x \in E$. But then, by item (ii) and Lemma 2.9, we have $\text{Lip}_\eta(f - T \circ \varphi_j; x) \leq (C + 1)\epsilon$ for every $x \in E$. Since, for every $\epsilon > 0$, $X \setminus \text{Bad}(f)$ is μ -almost all covered by sets such as E , we have proven that for every $\epsilon > 0$, j , and μ -a.e. $x \in U_j \setminus \text{Bad}(f)$, there is a linear map $T_{x, \epsilon}: \mathbb{R}^{n_j} \rightarrow Y$ such that $\text{Lip}_\eta(f - T_{x, \epsilon} \circ \varphi_j; x) \leq \epsilon$. Lemma 2.7 and

$$\text{Lip}_\eta(\langle v, \varphi_j \rangle; x) \geq \text{Lip}(\langle v, \varphi_j \rangle; x) > 0$$

for $v \in \mathbb{R}^{n_j} \setminus \{0\}$ and μ -a.e. $x \in U_j$ conclude the proof of differentiability; see Lemma 2.5. Lastly, the differentials w.r.t. the two distances agree because they are unique μ -a.e. and $d_\eta \leq d$.

We now consider points where f is not differentiable.

Let $\{(U_j, \varphi_j: X \rightarrow \mathbb{R}^{n_j})\}_j$ be a compatible atlas and $A \subseteq \text{Bad}(f)$ a set. Let $N \subseteq X$ be the set of porosity points of A and recall that $\mu(N) = 0$ by assumption. By Lemma 5.12, for every j and $x \in A \cap U_j$, we have $\text{Lip}_\eta(f - T \circ \varphi_j; x) > 0$ for every linear $T: \mathbb{R}^{n_j} \rightarrow Y$. But then, by Lemma 2.9, we have $\text{Lip}_\eta((f - T \circ \varphi_j)|_A; x) > 0$ for every linear map $T: \mathbb{R}^{n_j} \rightarrow Y$ and $x \in A \cap U_j \setminus N$. □

As mentioned in the introduction, in general the set $\text{Bad}(f)$ may have positive μ measure, as is demonstrated by the example in Proposition 10.15 and, for different targets, by Proposition 10.13 and Proposition 10.16. However, this is not possible if the space satisfies the following quantitative

differentiation hypothesis. Note that if it holds on the non-contracted space, then it holds also on the contracted one.

Corollary 5.18. *Let Y be a Banach space and let (X, d, μ) be an s -ADR Y -LDS with compatible shortcuts. Suppose that, for any Lipschitz map $f: (X, d_\eta) \rightarrow Y$ with bounded support, it holds*

$$\sum_{S \in \mathcal{J}} (\text{diam } f(S))^s < \infty.$$

Then $\mu(\text{Bad}(f)) = 0$ for all Lipschitz $f: (X, d_\eta) \rightarrow Y$. In particular, (X, d_η, μ) is a Y -LDS.

Proof. Let $f: (X, d_\eta) \rightarrow Y$ be Lipschitz. Fix $x_0 \in X$ and, for $R > 0$, let $f_R: (X, d_\eta) \rightarrow Y$ be a Lipschitz function with support in $B_\eta(x_0, 2R)$ and which agrees with f on $B_\eta(x_0, R)$. Let $\epsilon > 0$ and observe that for $S \in \mathcal{J}_i^\epsilon(f_R)$, we have $\mu(B_\eta(S, \eta_i \delta_i / \epsilon)) \lesssim \epsilon^{-2s} (\text{diam } f_R(S))^s$. Hence, the hypothesis gives

$$\sum_{i \in \mathbb{N}} \sum_{S \in \mathcal{J}_i^\epsilon(f_R)} \mu(B_\eta(S, \eta_i \delta_i / \epsilon)) \lesssim \epsilon^{-2s} \sum_{S \in \mathcal{J}} (\text{diam } f_R(S))^s < \infty,$$

which implies $\mu(\text{Bad}_\epsilon(f_R)) = 0$ by the first Borel-Cantelli lemma. By the definition of $\text{Bad}_\epsilon(f)$, it follows that $\text{Bad}_\epsilon(f) \cap B_\eta(x_0, R/2) \subseteq \text{Bad}_\epsilon(f_R)$, and therefore $\mu(\text{Bad}_\epsilon(f) \cap B_\eta(x_0, R/2)) = 0$ for all $\epsilon, R > 0$. Thus, we have $\mu(\text{Bad}(f)) = 0$ for Lipschitz $f: (X, d_\eta) \rightarrow Y$, and rest of the thesis follows from Proposition 5.7, Lemma 2.11, and Lemma 3.18. \square

6. LAAKSO SPACES

Laakso spaces were introduced in [Laa00] as examples of Ahlfors-David regular 1-PI spaces whose Hausdorff dimension varies continuously in $(1, \infty)$. We now describe the construction.

Let $M \in \mathbb{N}$, $M \geq 2$, $\theta \in (0, 1/4]$, and $(N_n) \subseteq 2\mathbb{N}$, $N_n \geq 4$, such that

$$\prod_{i=1}^n \frac{1}{N_i} \sim \theta^n, \quad n \in \mathbb{N}. \quad (6.1)$$

It is convenient to metrize $[M]^\mathbb{N}$ as follows. For $x \neq y \in [M]^\mathbb{N}$, let $n \in \mathbb{N}_0$ be the greatest integer such that $x(i) = y(i)$ for $1 \leq i < n$, and set $d(x, y) = \theta^n$.

Set $\tilde{X} := [0, 1] \times [M]^\mathbb{N}$ and endow it with the product topology (and metric). Let $h: \tilde{X} \rightarrow [0, 1]$ denote the projection onto the first coordinate, which we will call *height*. For $i \in \mathbb{N}$, set

$$W_i^h := \left\{ \sum_{j=1}^i \frac{t_j}{N_1 \cdots N_j} : t_j \in \mathbb{N}_0, 0 \leq t_j < N_j \text{ for } 1 \leq j < i \text{ and } 1 \leq t_i < N_i \right\},$$

whose elements will be called *wormhole heights of level i* . We also set

$$W_{\leq i}^h := \bigcup_{j=1}^i W_j^h = \left\{ \frac{m}{N_1 \cdots N_i} : m \in \mathbb{N}, 0 < m < N_1 \cdots N_i \right\}. \quad (6.2)$$

We now define an equivalence relation on \tilde{X} . Given $x, y \in \tilde{X}$, we write $x \sim y$ if either $x = y$, or there is $i \in \mathbb{N}$ such that $h(x) = h(y) \in W_i^h$ and x, y differ at most in the i -th digit, i.e. $x(j) = y(j)$ for all $j \neq i$. Let X denote the quotient space and $q: \tilde{X} \rightarrow X$ the quotient map.

We call *wormholes of level i* points $x \in X$ for which $h(x) \in W_i^h$. Note that for $x \in X$, $q^{-1}\{x\}$ is a singleton unless x is a wormhole of some level $i \in \mathbb{N}$, in which case $\#q^{-1}\{x\} = M$ and for $a \in [M]$ there is $y \in q^{-1}\{x\}$ with $y(i) = a$. Hence, if $x \in X$ is not a wormhole, we can talk about its digits, denoted by $(x(j))_j$. Instead, if x is a wormhole of some level $i \in \mathbb{N}$, its digit of level i is not defined, but the ones of level $j \neq i$ are, which we still denote as $x(j)$. We then say that $x, y \in X$ *differ at the i -th digit* or *differ at the digit of level i* if $\tilde{x}(i) \neq \tilde{y}(i)$ for all representatives $\tilde{x}, \tilde{y} \in \tilde{X}$ of x, y respectively. In this case, we also write $x(i) \neq y(i)$.

We metrize X by setting

$$d(x, y) := \inf\{\mathcal{H}^1(\gamma) : q \circ \gamma \text{ is a continuous curve from } x \text{ to } y\},$$

for $x, y \in X$.

Remark 6.3. The above distance can be informally described as follows. Let $x, y \in X$ and let $\Delta \subseteq \mathbb{N}$ be the set of levels j for which $x(j) \neq y(j)$. Any path from x to y has to change digits at each level in Δ , which can occur only by traveling through wormholes of the corresponding level. Paths from x to y can be explicitly constructed as follows. Let $I \subseteq [0, 1]$ be an interval containing $h(x), h(y)$ and intersecting W_i^h for each $i \in \Delta$. We can then travel from x to the closest boundary point of I , then along I , and lastly to y , changing all necessary digits along the way; the resulting path has length $2|I| - |h(x) - h(y)|$. It turns out there is always an optimal path of this form.

Proposition 6.4 ([Laa00, Proposition 1.1]). *Let $x, y \in X$ and ℓ the minimum among lengths of intervals containing $h(x), h(y)$, and all wormhole heights necessary to travel from x to y . Then*

$$d(x, y) = 2\ell - |h(x) - h(y)|.$$

It is not difficult to verify that d induces the quotient topology on X , $q: \tilde{X} \rightarrow X$ is David-Semmes regular, and $h: X \rightarrow [0, 1]$ is open [Laa00].

For $M, (N_n)$, and d as above, we call *Laakso space* the metric space (X, d) and, when we want to highlight the dependence on M and (N_n) , we write $X(M; (N_n))$.

Remark 6.5. In [Laa00], Laakso allows for a slightly larger family of sequences (N_n) than us. Hence, what we call Laakso spaces form a strict subset of the ones considered in [Laa00]. Nonetheless, as will be clarified shortly, the spaces we consider can still achieve any Hausdorff dimension in $(1, \infty)$.

Laakso proves the following.

Theorem 6.6 ([Laa00]). *Let $X = X(M; (N_n))$ be a Laakso space and $s \in (1, \infty)$ such that $M\theta^{s-1} = 1$, i.e. $s = 1 + \log M / \log(1/\theta)$. Then (X, d, \mathcal{H}^s) is a compact geodesic s -ADR 1-PI space.*

In particular, by [CK09], (X, d, \mathcal{H}^s) is a Y -LDS for any Banach space Y with RNP.

The following lemma gives more precise information on the charts.

Lemma 6.7. *Let (X, d, \mathcal{H}^s) be an s -ADR Laakso space. Then the height function $h: X \rightarrow \mathbb{R}$ is a global Cheeger chart, satisfying $\text{Lip}(h; \cdot) = 1$ everywhere.*

Proof. The equality $\text{Lip}(h; \cdot) = 1$ is clear.

One can prove directly that h is a chart following an argument analogous to [CPS25]. For brevity, we explain how this follows from results in the literature.

From [Wea00, 5.A], the module of Weaver derivations on (X, d, \mathcal{H}^s) has exactly one independent derivation. Then, by [Sch16a, Theorem 3.24], (X, d, \mathcal{H}^s) has exactly one independent Alpert representation and hence, by [Bat15, Theorem 6.6], the analytic dimension of (X, d, \mathcal{H}^s) is 1. \square

To construct non-differentiable maps, it will be useful to consider the following sequence of approximations of $X = X(M; (N_n))$.

Definition 6.8. Set $\tilde{X}_n := [0, 1] \times [M]^n$, $n \in \mathbb{N}_0$, and let $\pi_n^\infty: \tilde{X} \rightarrow \tilde{X}_n$ denote the coordinate projection map. Similarly to \sim on \tilde{X} , we can define an equivalence relation on \tilde{X}_n , which then satisfies

$$x \sim y \text{ if and only if } \pi_n^\infty(x) \sim \pi_n^\infty(y) \text{ for all } n \in \mathbb{N}_0.$$

We let X_n denote the quotient of \tilde{X}_n under the above equivalence relation and still denote with q the quotient map; this should cause no confusion. The maps π_n^∞ induce maps $X \rightarrow X_n$, which we still denote with π_n^∞ . We similarly have maps $\pi_n^m: \tilde{X}_m \rightarrow \tilde{X}_n$, $\pi_n^m: X_m \rightarrow X_n$ for $0 \leq n \leq m$. We endow the spaces X_n with the quotient topology and define a distance similarly to X .

One can then recover X from $(X_n)_n$ in several ways. Indeed, it is then not difficult to verify that X corresponds to the inverse limit of the inverse system given by $(X_n)_n$ and $(\pi_n^{n+1})_n$, and that $(X_n)_n$ Gromov-Hausdorff converges to X ; see e.g. [CK15].

6.1. Laakso spaces as metric graphs. Let $G = (V, E)$ be a connected (abstract) graph, $\ell > 0$, and endow it with its length distance d_G , scaled such that $d_G(u, v) = \ell$ for each edge $\{u, v\} \in E$. We denote with X_G the metric space obtained by attaching an isometric copy of $[0, \ell]$ to each pair of neighbouring vertices. We can then identify V with an isometric copy in X_G and each $e \in E$ with the copy of $[0, \ell]$ in X_G joining ∂e . With these identifications, we call (X_G, V, E) the *metric graph* associated to G .

The following will be useful in the construction of non-differentiable functions. We omit the proof.

Lemma 6.9. *Let (X_G, V, E) be a metric graph associated to a connected graph $G = (V, E)$, Y a Banach space, and $f: V \rightarrow Y$ a function. Then there is a unique extension $F: X_G \rightarrow Y$ of f which is affine linear on each edge of X_G . Moreover, $\text{LIP}(F) = \text{LIP}(f)$.*

Let $n \in \mathbb{N}$. We now define a graph $G_n = (V, E)$ for which $X_n = X_{G_n}$ isometrically. The most natural choice for G_n would be a multigraph, but since we prefer working with graphs, our choice of G_n will be a topological minor of the natural multigraph. For $1 \leq l \leq n$, set

$$V_l := W_l^h \times ([M]^{l-1} \times \{0\} \times [M]^{n-l})$$

and

$$V := \bigcup_{l=1}^n V_l \cup \left(\left\{ \frac{m}{N_1 \cdots N_{n+1}} : m \in \mathbb{N}_0, 0 \leq m \leq N_1 \cdots N_{n+1} \right\} \setminus W_{\leq n}^h \right) \times [M]^n.$$

In the definition of V_l , the digit 0 should be considered simply as a ‘symbol’ outside the ‘alphabet’ $[M]$. For $v, v' \in V$, we say that they are adjacent if $|h(v) - h(v')| = \frac{1}{N_1 \cdots N_{n+1}}$ and $v(j) = v'(j)$ for all $1 \leq j \leq n$ for which $v(j), v'(j) \neq 0$. This adjacency relation defines an edge set E and we set $G_n = (V, E)$. It is then not difficult to verify that, taking $\ell = \frac{1}{N_1 \cdots N_{n+1}}$, we have $X_n = X_{G_n}$ isometrically.

6.2. Cubical covers of Laakso spaces. We now define a family of covers of a Laakso space.

Definition 6.10. Let $X = X(M; (N_n))$ be a Laakso space, fixed for the rest of the subsection. For $i \in \mathbb{N}$, define

$$\begin{aligned} \mathcal{I}_0 &:= \{[0, 1]\}, \\ \mathcal{I}_i &:= \left\{ \left[\frac{m}{N_1 \cdots N_i}, \frac{m+1}{N_1 \cdots N_i} \right] : m \in \mathbb{N}_0, 0 \leq m < N_1 \cdots N_i \right\}. \end{aligned}$$

Further, for $i, j \in \mathbb{N}_0$ and $I \in \mathcal{I}_i$, $a \in [M]^j$, define

$$\begin{aligned} \tilde{Q}_{I,a} &:= I \times \{a\} \times [M]^{\mathbb{N}}, \\ Q_{I,a} &:= q(I \times \{a\} \times [M]^{\mathbb{N}}), \\ \mathcal{Q}_i &:= \{Q_{I,a} : I \in \mathcal{I}_i, a \in [M]^i\}, \\ \mathcal{Q} &:= \bigcup_{i \in \mathbb{N}_0} \mathcal{Q}_i, \end{aligned}$$

where we interpret $I \times \{a\} \times [M]^{\mathbb{N}} = I \times [M]^{\mathbb{N}}$ if $j = 0$.

Recall that given two sets Ω, Z and a function $f: \Omega \rightarrow Z$, a set $E \subseteq \Omega$ is *f-saturated* if $f^{-1}(f(E)) = E$.

Lemma 6.11. *Let $i, j \in \mathbb{N}_0$ with $j \leq i$ and $I \in \mathcal{I}_i$, $a \in [M]^j$. Then $\text{Int } \tilde{Q}_{I,a}$ is q -saturated and $\text{Int } Q_{I,a} = q(\text{Int } \tilde{Q}_{I,a})$.*

Proof. We first prove that $\text{Int } \tilde{Q}_{I,a}$ is q -saturated. We may suppose $i \geq 1$, for otherwise $\text{Int } \tilde{Q}_{I,a} = \text{Int } \tilde{X} = \tilde{X}$, which is q -saturated. Let J denote the interior of I as a subset of $[0, 1]$. Then $\text{Int } \tilde{Q}_{I,a} = J \times \{a\} \times [M]^{\mathbb{N}}$ and J does not intersect $W_{\leq i}^h$ (see Eq. (6.2)). Thus, if $x \in \text{Int } \tilde{Q}_{I,a}$ and $y \in q^{-1}\{q(x)\}$, then $h(y) = h(x) \in J$ and $y(k) = x(k)$ for $k > i \geq j$, proving that $\text{Int } \tilde{Q}_{I,a}$ is q -saturated.

Being q a quotient map (in the topological sense), $q(\text{Int } \tilde{Q}_{I,a})$ is open in X and so $q(\text{Int } \tilde{Q}_{I,a}) \subseteq \text{Int } Q_{I,a}$. For the reverse inclusion, let $V \subseteq Q_{I,a}$ be open. Then $h(V)$ is open in $[0, 1]$ and contained in I . Hence, $h(V) \subseteq J$ and V does not contain wormholes of level $> j$, implying $q^{-1}(V) \subseteq \text{Int } \tilde{Q}_{I,a}$. \square

Saturated sets have the following useful properties. We include proofs for completeness.

Lemma 6.12. *Let Ω, Z be sets, $f: \Omega \rightarrow Z$ a function, and suppose $E \subseteq \Omega$ is f -saturated. Then $f(E \cap A) = f(E) \cap f(A)$ and $f(A \setminus E) = f(A) \setminus f(E)$ for all $A \subseteq \Omega$.*

Proof. The inclusion $f(E \cap A) \subseteq f(E) \cap f(A)$ is clear. Let $z \in f(E) \cap f(A)$ and $x \in A$ with $f(x) = z$. Then $x \in f^{-1}\{z\} \subseteq f^{-1}(f(E)) = E$, proving $z \in f(E \cap A)$ and hence $f(E \cap A) = f(E) \cap f(A)$.

The inclusion $f(A) \setminus f(E) \subseteq f(A \setminus E)$ is clear. Let $z \in f(A \setminus E)$ and pick $x \in A \setminus E$ such that $f(x) = z$. If $z \in f(E)$, then $x \in f^{-1}\{z\} \subseteq f^{-1}(f(E)) = E$, a contradiction. Thus, we have $f(A \setminus E) \subseteq f(A) \setminus f(E)$, concluding the proof. \square

Lemma 6.13. *Let $i, j \in \mathbb{N}_0$ with $j \leq i$ and $I, I' \in \mathcal{I}_i$, $a, a' \in [M]^j$. Then $\partial Q_{I,a} = q(\partial \tilde{Q}_{I,a})$ and $Q_{I,a} \cap Q_{I',a'} = \partial Q_{I,a} \cap \partial Q_{I',a'}$, whenever $(I, a) \neq (I', a')$.*

Proof. We only need to apply Lemmas 6.11 and 6.12. We have

$$\begin{aligned} \partial Q_{I,a} &= q(\tilde{Q}_{I,a}) \setminus q(\text{Int } \tilde{Q}_{I,a}) = q(\tilde{Q}_{I,a} \setminus \text{Int } \tilde{Q}_{I,a}) = q(\partial \tilde{Q}_{I,a}), \\ Q_{I,a} \cap \text{Int } Q_{I',a'} &= q(\tilde{Q}_{I,a}) \cap q(\text{Int } \tilde{Q}_{I',a'}) = q(\tilde{Q}_{I,a} \cap \text{Int } \tilde{Q}_{I',a'}) = \emptyset, \end{aligned}$$

where in the second line we have used $\tilde{Q}_{I,a} \cap \text{Int } \tilde{Q}_{I',a'} = \emptyset$ for $(I, a) \neq (I', a')$. \square

To conclude, we gather the most relevant properties of $(Q_i)_{i \in \mathbb{N}_0}$.

Lemma 6.14. *Let $X = X(M; (N_n))$ be an s -ADR Laakso space and let $(Q_i)_{i \in \mathbb{N}_0}$ be as in Definition 6.10. For $i \in \mathbb{N}_0$, Q_i is a finite compact cover of X satisfying:*

- (i) $\mathcal{H}^s(\partial Q) = 0$ for $Q \in Q_i$;
- (ii) $Q \cap Q' = \partial Q \cap \partial Q'$ for $Q \neq Q' \in Q_i$;
- (iii) for $Q_0 \in Q_i$ and $j > i$ it holds

$$Q_0 = \bigcup \{Q \in Q_j : Q \subseteq Q_0\}.$$

Moreover, there is $C = C(X) \geq 1$ such that for $i \in \mathbb{N}_0$ and $Q \in Q_i$ it holds

$$\begin{aligned} C^{-1}\theta^i &\leq \text{diam } Q \leq C\theta^i \\ C^{-1}\theta^{is} &\leq \mathcal{H}^s(Q) \leq C\theta^{is} \end{aligned} \tag{6.15}$$

Here θ is as in Eq. (6.1).

Proof. To see that Q_i is compact, recall that q is continuous. Item (ii) is the content of Lemma 6.13, while (iii) follows from the definition of Q_i . Let $i \in \mathbb{N}_0$ and $Q = Q_{I,a} \in Q_i$. Since q is David-Semmes regular, by Lemma 6.13 we have (see Eq. (3.16))

$$\mathcal{H}_X^s(\partial Q) \sim q_{\#} \mathcal{H}_{\tilde{X}}^s(q(\partial \tilde{Q}_{I,a})) = \mathcal{H}_{\tilde{X}}^s(q^{-1}(q(\partial \tilde{Q}_{I,a}))). \tag{6.16}$$

Let J denote the interior of I as a subset of $[0, 1]$. Then $\partial \tilde{Q}_{I,a} = (I \setminus J) \times \{a\} \times [M]^{\mathbb{N}}$, $\#(I \setminus J) \leq 2$, and $q^{-1}(q(\partial \tilde{Q}_{I,a})) \subseteq (I \setminus J) \times [M]^{\mathbb{N}}$. Since $\mathcal{H}_{\tilde{X}}^s((I \setminus J) \times [M]^{\mathbb{N}}) = 0$, Eq. (6.16) concludes the proof of (i).

Let $i \in \mathbb{N}_0$ and $Q \in Q_i$. Since $Q_0 = \{X\}$, we may assume $i \in \mathbb{N}$ and take $I \in \mathcal{I}_i$, $a \in [M]^i$ such that $Q = Q_{I,a}$. From Proposition 6.4, it is clear that $\text{diam } Q_{I,a} \sim \theta^i$. Since \mathcal{H}^s is s -ADR (see Theorem 6.6), it follows $\mathcal{H}^s(Q) \lesssim \theta^{is}$. From the choice of distance on $[M]^{\mathbb{N}}$ and Eq. (6.1), it is not difficult to see that $\tilde{Q}_{I,a}$ contains a ball of \tilde{X} of radius $\sim \theta^i$. Since $q: \tilde{X} \rightarrow X$ is DS-regular, we finally have

$$\mathcal{H}_X^s(Q) \sim q_{\#} \mathcal{H}_{\tilde{X}}^s(Q) \geq \mathcal{H}_{\tilde{X}}^s(\tilde{Q}_{I,a}) \gtrsim \theta^{is}.$$

\square

6.3. Shortcuts in Laakso spaces. In this subsection we show that Laakso spaces have shortcuts, see Proposition 6.25.

Let $X = X(M; (N_n))$ be a Laakso space, fixed for the subsection.

Definition 6.17. For $i \in \mathbb{N}$, set

$$\begin{aligned} \mathcal{J}_i^h &:= \left\{ \sum_{j=1}^i \frac{t_j}{N_1 \cdots N_j} + \frac{1}{2N_1 \cdots N_i} : t_j \in \mathbb{N}_0, 0 \leq t_j < N_j \text{ for } 1 \leq j < i \text{ and } 1 \leq t_i < N_i - 1 \right\} \\ &= \left\{ \frac{a+b}{2} : [a, b] \in \mathcal{I}_i \text{ and } a, b \in W_i^h \right\}. \end{aligned}$$

When we wish to emphasise the dependence on N_1, \dots, N_i in \mathcal{J}_i^h , we then write

$$\mathcal{J}_i^h = \mathcal{J}_i^h(N_1, \dots, N_i). \quad (6.18)$$

Since N_{i+1} is even, $\mathcal{J}_i^h \subseteq W_{i+1}^h$. We define the following set of *shortcuts of level i*

$$\mathcal{J}_i := \{q(\{t\} \times \{a\} \times [M] \times \{1\}^{\mathbb{N}}) : t \in \mathcal{J}_i^h \text{ and } a \in [M]^{i-1}\},$$

where, for $i = 1$, we interpret

$$\{t\} \times \{a\} \times [M] \times \{1\}^{\mathbb{N}} = \{t\} \times [M] \times \{1\}^{\mathbb{N}}.$$

Also define

$$\delta_i := \frac{1}{N_1 \cdots N_i}.$$

Finally, for $n \in \mathbb{N}$, consider X_n defined as in Section 6.1. We also define $(\mathcal{J}_i)_{i=1}^n$ similarly as above also in X_n .

We now proceed to prove that the sets \mathcal{J}_i defined above are indeed shortcuts in the Laakso space X .

We will need the following fact, often used without mention. Let $N_1, \dots, N_i \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \leq m < N_1 \cdots N_i$, and set $t := \frac{m}{N_1 \cdots N_i}$. Then, there are unique $t_1, \dots, t_i \in \mathbb{N}_0$, $0 \leq t_j < N_j$, such that

$$t = \sum_{j=1}^i \frac{t_j}{N_1 \cdots N_j}.$$

Lemma 6.19. *Let $i \in \mathbb{N}$, $S \in \mathcal{J}_i$, and let $t \in (0, 1)$, $a \in [M]^{i-1}$ be such that $S = q(\{t\} \times \{a\} \times [M] \times \{1\}^{\mathbb{N}})$. Let $I \in \mathcal{I}_i$ be the interval having t as midpoint and suppose $x \notin \text{Int } Q_{I,a}$.*

Then $d(x, z) = d(x, S) \geq \frac{1}{2N_1 \cdots N_i}$ for $z \in S$. In particular,

$$d(x, y) \leq d(x, z) + d(w, y)$$

for $x, y \notin \text{Int } Q_{I,a}$ and $z, w \in S$.

Proof. If $h(x) \notin \text{Int } I$, then any geodesic from x to $z \in S$ has to cross $\partial I \subseteq W_i^h$ and therefore meet a wormhole of level i . At such intersection point we can modify the i -th digit and obtain a geodesic from x to w , for any $w \in S$. Hence, $d(x, z) = d(x, S)$ for $z \in S$. Since h is 1-Lipschitz, $d(x, S) \geq |h(x) - t| = \frac{1}{2N_1 \cdots N_i}$.

Suppose now $x \in Q_{I,b}$ for some $b \in [M]^{i-1}$, $b \neq a$, and note that I does not contain wormholes heights of level $\leq i-1$. Since $b \neq a$, there is $1 \leq j \leq i-1$ such that $x(j) = b(j) \neq a(j)$. Again, since $W_j^h \cap I = \emptyset$, any geodesic from x to $z \in S$ has to cross ∂I and therefore meet a wormhole of level i . It follows that $d(x, S) = d(x, z)$ and $d(x, S) \geq \frac{3}{2N_1 \cdots N_i}$. \square

Lemma 6.20. *Let $i \in \mathbb{N}$, $S \in \mathcal{J}_i$, and let $t \in (0, 1)$, $a \in [M]^{i-1}$ be such that $S = q(\{t\} \times \{a\} \times [M] \times \{1\}^{\mathbb{N}})$. Let $I \in \mathcal{I}_i$ be the interval having t as midpoint. Then $\text{Int } Q_{I,a} = U_X(S, \frac{1}{2N_1 \cdots N_i})$.*

Proof. Lemma 6.19 gives $U_X(S, \frac{1}{2N_1 \dots N_i}) \subseteq \text{Int } Q_{I,a}$. Fix $x \in \text{Int } Q_{I,a}$ and let $p \in S$ be such that $x(j) = p(j)$ for $1 \leq j \leq i$. Since $\mathcal{J}_i^h \subseteq W_{i+1}^h$, we need to change only digits of level $j \geq i+2$. If the interval with endpoints $\{h(x), t\}$ intersects W_{i+2}^h , then $d(x, p) = |h(x) - t|$, because $t \in W_{i+1}^h$ and between wormhole heights of levels l_1, l_2 there are wormholes heights of level l for each $l \geq \max(l_1, l_2)$. Suppose now there are no wormholes heights of level $i+2$ between t and $h(x)$. Then $|h(x) - t| < \frac{1}{N_1 \dots N_{i+2}}$. Let $y \in X$ be such that $h(y) \in W_{i+2}^h$, $|h(y) - h(x)| = d(h(x), W_{i+2}^h)$, and $y(j) = x(j)$ for all j . Then $d(y, p) = |h(y) - t| = \frac{1}{N_1 \dots N_{i+2}}$, $d(x, p) = d(x, y) + d(y, p)$, and $d(x, y) = |h(x) - h(y)| = |h(y) - t| - |t - h(x)|$. Hence,

$$d(x, p) \leq 2|h(y) - t| = \frac{2}{N_1 \dots N_{i+2}} < \frac{1}{2N_1 \dots N_i},$$

because $N_{i+1}N_{i+2} > 4$. □

Lemma 6.21. *Let $i \leq j$ and $S \in \mathcal{J}_i$, $S' \in \mathcal{J}_j$ be distinct. Then $d(S, S') \geq \frac{1}{2N_1 \dots N_j}$.*

Proof. Let $z \in S$, $z' \in S'$, and $t = h(z)$, $t' = h(z')$. If $t \neq t'$, then

$$d(z, z') \geq |t - t'| \geq \frac{1}{2N_1 \dots N_j}.$$

Suppose now $t = t'$. Since $(\mathcal{J}_k^h)_k$ is pairwise disjoint, in particular we have $i = j \geq 2$. Let $a \neq a' \in [M]^{j-1}$ such that S, S' are determined by (t, a) and (t, a') respectively, and let $I \in \mathcal{I}_j$ be the interval having t as midpoint. Let $1 \leq k \leq j-1$ be such that $a(k) \neq a'(k)$ and hence $z(k) \neq z'(k)$. Since there are no wormholes heights of level k in I , we have

$$d(z, z') \geq 2d(t, W_k^h) = \frac{3}{N_1 \dots N_j}.$$

□

Lemma 6.22. *For $i \in \mathbb{N}$ and $z \neq w \in S \in \mathcal{J}_i$ it holds $d(z, w) = \frac{1}{N_1 \dots N_i}$.*

Proof. The points z, w differ exactly in the i -th digit and $d(h(z), W_i^h) = d(h(w), W_i^h) = \frac{1}{2N_1 \dots N_i}$. □

Lemma 6.23. *Let $i \in \mathbb{N}$ and $t \in W_i^h$. Then there is $I \in \mathcal{I}_i$ such that $t \in \partial I \subseteq W_i^h$.*

Proof. We can write $t = \sum_{j=1}^i \frac{t_j}{N_1 \dots N_j}$ with $t_j \in \mathbb{N}_0$, $0 \leq t_j < N_j$ for $1 \leq j < i$ and $1 \leq t_i \leq N_i - 1$. If $t_i \leq N_i - 2$, then $t' := t + \frac{1}{N_1 \dots N_i} \in W_i^h$ and $[t, t'] \in \mathcal{I}_i$. If $t_i = N_i - 1 > 1$, then $t' := t - \frac{1}{N_1 \dots N_i} \in W_i^h$ and $[t', t] \in \mathcal{I}_i$. □

Lemma 6.24. *For $i \in \mathbb{N}$ and $x \in X$, it holds $d(x, \cup \mathcal{J}_i) \leq \frac{3}{2N_1 \dots N_i}$.*

Proof. Let $I \in \mathcal{I}_i$ be such that $h(x) \in I$. Since $\partial I \cap W_i^h \neq \emptyset$, by Lemma 6.23 there is $I' \in \mathcal{I}_i$ with $\partial I' \subseteq W_i^h$ and $\partial I \cap \partial I' \neq \emptyset$. Let $z \in X$ be such that $h(z) \in \partial I \cap \partial I'$ and $z(j) = x(j)$ for all j . Then

$$d(x, \cup \mathcal{J}_i) \leq d(x, z) + d(z, \cup \mathcal{J}_i) \leq \frac{3}{2N_1 \dots N_i}.$$

□

Proposition 6.25. *Let $X = X(M; (N_n))$ be an s -ADR Laakso space. Then $\{(\mathcal{J}_i, \delta_i)\}_{i \geq 1}$, defined as in Definition 6.17, are shortcuts compatible with the differentiable structure of (X, d, \mathcal{H}^s) .*

More precisely, it satisfies the conditions of Definition 3.1 with parameters $a = a_0 = 1$, $b = 1/2$, and $M = M$, and $\text{diam } h(S) = 0$ for $S \in \mathcal{J}$.

Proof. This follows combining the previous lemmas. The first condition of Definition 3.1 is given by Lemma 6.22, the second by Lemma 6.21, the third by Lemma 6.19 and Lemma 6.20, the fourth by the definition of $(\mathcal{J}_i)_i$, and the last by Lemma 6.24 (and $M \geq 2$).

It is a PI space [Laa00] and hence a Y -LDS for any Banach space Y with RNP by [CK09]. Compatibility of the shortcuts with the differentiable structure follows by definition of $(\mathcal{J}_i)_i$ and Lemma 6.7. \square

7. HARMONIC APPROXIMATION ON LDS OF ANALYTIC DIMENSION 1

The main result of this subsection is Proposition 7.8. Under suitable assumptions, it decomposes a Lipschitz map $f: X \rightarrow Y$ into a sum of Lipschitz maps $(F_{i+1} - F_i)$ with control of the total energy of the sequence (see Definition 7.2), where F_i agrees with f on a ‘grid’ of the space. This will be used later, in Theorem 8.3, to prove that shortcut Laakso spaces are Y -LDS for some Banach spaces Y .

The proof of Proposition 7.8 is inspired by Pisier’s martingale inequality [Pis16, Theorem 10.6], where we replace Jensen’s inequality with a local minimality condition.

Piecewise harmonic approximations of ℓ_2 valued maps appear in [Sch16b]. Our construction differs by building the harmonic approximations in an LDS, rather than a cube complex, and so we only have access to the Cheeger derivative. On the other hand, our construction does not rely on piecewise Euclidean approximations of a metric space, nor Euler-Lagrange equations, and holds for general (superreflexive) Banach space targets. We also do not need to study the regularity of minimisers, nor find covers with regular boundaries (in the potential-theoretic sense).

Recall that a Banach space Y is q -uniformly convex, $2 \leq q < \infty$, if its modulus of uniform convexity $\delta_Y(\cdot)$ satisfies $\delta_Y(\epsilon) \geq c\epsilon^q$ for some $c > 0$ and all $\epsilon \in (0, 2]$. Equivalently, there is $K > 0$ such that the inequality

$$\left\| \frac{x+y}{2} \right\|_Y^q + \frac{1}{K^q} \left\| \frac{x-y}{2} \right\|_Y^q \leq \frac{\|x\|_Y^q}{2} + \frac{\|y\|_Y^q}{2} \quad (7.1)$$

holds for $x, y \in Y$; see e.g. [Pis16, Proposition 10.31]. We let $K_q(Y)$ denote the least K as in Eq. (7.1).

Definition 7.2. Let (X, d, μ) is a metric measure space, $1 \leq q < \infty$, and Y a Banach space. For $f: X \rightarrow Y$ Lipschitz and $Q \subseteq X$ μ -measurable, we define the q -energy of f on Q as

$$E_q(f, Q) := \int_Q \text{Lip}(f; x)^q d\mu(x).$$

Lemma 7.3. Let Y be a q -uniformly convex Banach space, (X, d, μ) a Y -LDS, and $Q \subseteq X$ a μ -measurable set of analytic dimension 1. Then, for any $u, v: X \rightarrow Y$ Lipschitz, we have

$$2E_q\left(\frac{u+v}{2}, Q\right) + 2K_q(Y)^{-q}E_q\left(\frac{u-v}{2}, Q\right) \leq E_q(u, Q) + E_q(v, Q).$$

Proof. By definition of analytic dimension, there are charts $(U_i, \varphi_i: X \rightarrow \mathbb{R})$ such that $U_i \subseteq Q$ and $\mu(Q \setminus \bigcup_i U_i) = 0$. Since (U_i, φ_i) is 1-dimensional, we may identify φ_i -differentials with elements of Y , so that $\text{Lip}(f; x) = \|d_x f\|_Y \text{Lip}(\varphi_i; x)$ for any $f: X \rightarrow Y$ which is φ_i -differentiable at $x \in U_i$. Hence, Eq. (7.1) implies

$$2 \text{Lip}\left(\frac{u+v}{2}; x\right)^q + 2K_q(Y)^{-q} \text{Lip}\left(\frac{u-v}{2}; x\right)^q \leq \text{Lip}(u; x)^q + \text{Lip}(v; x)^q$$

at μ -a.e. $x \in Q$. Integrating over Q concludes the proof. \square

Definition 7.4. Let (X, d, μ) be a metric measure space, $1 \leq q < \infty$, and Y a Banach space. For $f: X \rightarrow Y$ Lipschitz and $Q \subseteq X$ μ -measurable, we define the following sets of admissible functions and minimisers, respectively,

$$\begin{aligned} \mathcal{A}(f, Q) &:= \{u: \overline{Q} \rightarrow Y: \text{LIP}(u) \leq \text{LIP}(f), u|_{\partial Q} = f|_{\partial Q}\}, \\ \mathcal{E}_q(f, Q) &:= \{u \in \mathcal{A}(f, Q): E_q(u, Q) = \inf_{v \in \mathcal{A}(f, Q)} E_q(v, Q)\}. \end{aligned}$$

Both $\mathcal{A}(f, Q)$ and $\mathcal{E}_q(f, Q)$ are convex subsets of $\text{LIP}(X; Y)$ and $\mathcal{A}(f, Q)$ is never empty, since $f|_{\overline{Q}} \in \mathcal{A}(f, Q)$.

Lemma 7.5. *Let Y be a q -uniformly convex Banach space, (X, d, μ) a Y -LDS, $Q \subseteq X$ a μ -measurable set, and $f: X \rightarrow Y$ a Lipschitz map.*

Then $\mathcal{E}_q(f, Q)$ is non-empty. Moreover, if Q has analytic dimension 1, the elements $u \in \mathcal{E}_q(f, Q)$ are characterised as those $u \in \mathcal{A}(f, Q)$ satisfying

$$E_q(u, Q) + 2(2K_q(V))^{-q} E_q(u - v, Q) \leq E_q(v, Q), \quad v \in \mathcal{A}(f, Q). \quad (7.6)$$

Proof. Proving existence requires a simple application of the direct method of calculus of variations. If $\partial Q = \emptyset$, any constant function is admissible and a minimiser, hence $\mathcal{E}_q(f, Q) \neq \emptyset$. Suppose $\partial Q \neq \emptyset$ and let (u_j) be a minimising sequence. Since (u_j) is equiLipschitz and bounded at a point, it is pointwise bounded on \overline{Q} . Using reflexivity of Y (see e.g. [Pis16, Theorem 10.3]), Mazur's lemma, and arguing as in Arzelà-Ascoli's theorem, we may suppose (u_j) converges pointwise to a function $u: \overline{Q} \rightarrow Y$, which then belongs to $\mathcal{A}(f, Q)$. Moreover, by Lemma 2.14, we have $E_q(u, Q) \leq \lim_j E_q(u_j, Q) = \inf_{v \in \mathcal{A}(f, Q)} E_q(v, Q)$, proving $u \in \mathcal{E}_q(f, Q)$.

It remains to prove the variational characterisation of minimisers. Let $u \in \mathcal{E}_q(f, Q)$, $v \in \mathcal{A}(f, Q)$ and observe that $(u + v)/2 \in \mathcal{A}(f, Q)$. We may assume $E_q(v, Q) < \infty$, so that $E_q(u, Q) \leq E_q(\frac{u+v}{2}, Q) < \infty$. Then Lemma 7.3 and minimality of u give

$$\begin{aligned} 2K_q(Y)^{-q} E_q\left(\frac{u-v}{2}, Q\right) &\leq E_q(u, Q) + E_q(v, Q) - 2E_q\left(\frac{u+v}{2}, Q\right) \\ &\leq E_q(v, Q) - E_q(u, Q). \end{aligned}$$

Lastly, if $u \in \mathcal{A}(f, Q)$ satisfies Eq. (7.6), non-negativity of $E_q(\cdot, Q)$ implies $u \in \mathcal{E}_q(f, Q)$. \square

The following is an immediate application of Lemma 2.25.

Lemma 7.7. *Let Y be a Banach space, $1 \leq q < \infty$, and let (X, d, μ) be a metric measure space. Suppose (X, d) is a length space and let \mathcal{Q} be a locally finite closed cover of X satisfying $Q \cap Q' = \partial Q \cap \partial Q'$ for $Q \neq Q' \in \mathcal{Q}$.*

Then, for any Lipschitz $f: X \rightarrow Y$ and $(u_Q)_{Q \in \mathcal{Q}}$ with $u_Q \in \mathcal{E}_q(f, Q)$, $Q \in \mathcal{Q}$, there is a unique function $F: X \rightarrow Y$ satisfying $F|_Q = u_Q$. Moreover, $\text{LIP}(F) \leq \text{LIP}(f)$.

Note that a locally finite cover \mathcal{Q} of X is necessarily countable. Indeed, there is an open cover \mathcal{U} of X of sets intersecting only finitely many elements of \mathcal{Q} . Since X is separable, it has the Lindelöf property, and so there is a countable subcover $\mathcal{U}' \subseteq \mathcal{U}$. Then $\mathcal{Q} \setminus \{\emptyset\} = \bigcup_{U \in \mathcal{U}'} \{Q \in \mathcal{Q} : Q \cap U \neq \emptyset\}$ is a countable union of finite sets.

Proposition 7.8. *Let Y be a q -uniformly convex Banach space, (X, d, μ) a Y -LDS of analytic dimension 1, and suppose (X, d) is a length space.*

For $i \in \mathbb{N}$, let \mathcal{Q}_i be a locally finite closed cover of X satisfying:

- (i) $\mu(\partial Q) = 0$ for $Q \in \mathcal{Q}_i$;
- (ii) $Q \cap Q' = \partial Q \cap \partial Q'$ for $Q \neq Q' \in \mathcal{Q}_i$;
- (iii) for $Q_0 \in \mathcal{Q}_i$ and $j > i$ it holds

$$Q_0 = \bigcup \{Q \in \mathcal{Q}_j : Q \subseteq Q_0\}.$$

Let $f: X \rightarrow Y$ be L -Lipschitz and, for $i \in \mathbb{N}$, let $F_i: X \rightarrow Y$ be such that $F_i|_Q \in \mathcal{E}_q(f, Q)$ for $Q \in \mathcal{Q}_i$. Then, for $i_0 \in \mathbb{N}$ and $Q_0 \in \mathcal{Q}_{i_0}$, we have

$$\sum_{i \geq i_0} E_q(F_i - F_{i+1}, Q_0) \leq \frac{1}{2} (2K_q(Y)L)^q \mu(Q_0).$$

Proof. Set $c := 2(2K_q(Y))^{-q}$ and assume w.l.o.g. $\mu(Q_0) < \infty$. Observe that $F_{i+1}|_Q \in \mathcal{A}(f, Q)$ for $Q \in \mathcal{Q}_i$. Indeed, $\partial Q \subseteq \bigcup \{\partial Q' \in \mathcal{Q}_{i+1} : Q' \subseteq Q\}$ implies $F_{i+1}|_{\partial Q} = f|_{\partial Q}$, while Lemma 7.7 gives $\text{LIP}(F_{i+1}) \leq \text{LIP}(f)$. Then, for $i \geq i_0$ and $Q \in \mathcal{Q}_i$ with $Q \subseteq Q_0$, we have (by Lemma 7.5)

$$cE_q(F_i - F_{i+1}, Q) \leq E_q(F_{i+1}, Q) - E_q(F_i, Q),$$

where we have also used $\mu(Q) \leq \mu(Q_0) < \infty$. Hence, for $i \geq i_0$,

$$\begin{aligned} cE_q(F_i - F_{i+1}, Q_0) &= \sum_{\substack{Q \in \mathcal{Q}_i: \\ Q \subseteq Q_0}} cE_q(F_i - F_{i+1}, Q) \leq \sum_{\substack{Q \in \mathcal{Q}_i: \\ Q \subseteq Q_0}} E_q(F_{i+1}, Q) - E_q(F_i, Q) \\ &= E_q(F_{i+1}, Q_0) - E_q(F_i, Q_0), \end{aligned} \quad (7.9)$$

because the collection $\{Q \in \mathcal{Q}_i: Q \subseteq Q_0\}$ is an essentially disjoint countable cover of Q_0 . Finally, the thesis follows from

$$\sum_{i \geq i_0} cE_q(F_i - F_{i+1}, Q_0) \leq \sum_{i \geq i_0} E_q(F_{i+1}, Q_0) - E_q(F_i, Q_0) \leq \lim_i E_q(F_i, Q_0) \leq L^q \mu(Q_0).$$

□

Remark 7.10. Note that the proof of Proposition 7.8 relies on the exact cancellation of terms in the telescopic sum Eq. (7.9). This is possible due to the precise constants appearing in Lemmas 7.3 and 7.5, which is where we use the assumption that the space has analytic dimension 1.

8. QUANTITATIVE DIFFERENTIATION ON LAAKSO SPACES

In this section we first apply our harmonic decomposition given in Proposition 7.8 to the cubical cover of Laakso spaces defined in Definition 6.10. Because of the symmetry of Laakso spaces, the approximation satisfies an additional property (see Lemma 8.2, item (ii)), which is crucial to examine the behaviour of Lipschitz functions on Laakso spaces. In particular, Theorem 8.3 establishes the quantitative differentiation hypothesis of Corollary 5.18 for Laakso spaces. This will give the second item of Theorem 1.1 in Theorem 9.2.

Let $X = X(M; (N_n))$ be a Laakso space. For $i \in \mathbb{N}$, let σ_i be a permutation of $[M]$. Then the map $\tilde{X} \rightarrow \tilde{X}$ which applies σ_i to the i -th digit, for every $i \in \mathbb{N}$, descends to an isometry $X \rightarrow X$. We now expand upon this fact.

Lemma 8.1. *Let $X = X(M; (N_n))$ be an s -ADR Laakso space. Let $i \in \mathbb{N}$, $I \in \mathcal{I}_i$ with $\partial I \subseteq W_i^h$, and $a, b \in [M]^i$ with $a(j) = b(j)$ for $1 \leq j < i$ (any $a, b \in [M]$ if $i = 1$). Let $T: X \rightarrow X$ denote the isometry induced by the transposition $(a(i), b(i))$ acting on the i -th digit. Then, for every Banach space Y , $1 \leq q < \infty$, $f: X \rightarrow Y$ Lipschitz, and $u: Q_{I,a} \rightarrow Y$ there hold:*

- $u \in \mathcal{A}(f, Q_{I,a})$ if and only if $u \circ T \in \mathcal{A}(f, Q_{I,b})$;
- $E_q(u, Q_{I,a}) = E_q(u \circ T, Q_{I,b})$, provided u (equivalently, $u \circ T$) is Lipschitz.

Proof. We first focus on the first point. The condition on the Lipschitz constant in the definition of $\mathcal{A}(f, Q_{I,a})$ is preserved by T because it is an isometry. For the boundary conditions, by Lemma 6.13, we know that $\partial Q_{I,b} = q(\partial I \times \{b\} \times [M])$ and for $t \in \partial I$, $w \in [M]^\mathbb{N}$, we have

$$u \circ T \circ q(t, b, w) = u \circ q(t, a, w) = f \circ q(t, a, w) = f \circ q(t, b, w),$$

where we have used the definition of T , $u \in \mathcal{A}(f, Q_{I,a})$, and $q(t, a, w) = q(t, b, w)$. This shows that $u \circ T \in \mathcal{A}(f, Q_{I,b})$ whenever $u \in \mathcal{A}(f, Q_{I,a})$; the reverse statement holds by symmetry of the claim. For the last point, using the fact that T is an isometry, we have $\text{Lip}(u \circ T; x) = \text{Lip}(u; T(x))$ for \mathcal{H}^s -a.e. $x \in Q_{I,a}$ and so

$$E_q(u \circ T, Q_{I,b}) = \int_{T^{-1}(Q_{I,b})} \text{Lip}(u; T(x))^q d\mathcal{H}^s(x) = \int_{Q_{I,a}} \text{Lip}(u; x)^q d(T_\# \mathcal{H}^s)(x),$$

concluding the proof (because $T_\# \mathcal{H}^s = \mathcal{H}^s$). □

Lemma 8.2. *Let $X = X(M; (N_n))$ be an s -ADR Laakso space, Y a q -uniformly convex Banach space, and $f: X \rightarrow Y$ an L -Lipschitz map. Then, for $i \in \mathbb{N}_0$, there are L -Lipschitz maps $F_i: X \rightarrow Y$ satisfying:*

- (i) $F_i|_Q \in \mathcal{E}_q(f, Q)$ for $Q \in \mathcal{Q}_i$;

(ii) if $i \geq 1$, for $I \in \mathcal{I}_i$ with $\partial I \subseteq W_i^h$ and $a, b \in [M]^i$ with $a(j) = b(j)$, $1 \leq j < i$, (any $a, b \in [M]$ if $i = 1$) we have

$$F_i \circ q(t, a, w) = F_i \circ q(t, b, w),$$

for $t \in I$ and $w \in [M]^{\mathbb{N}}$;

(iii) $\sum_{i \geq i_0} E_q(F_{i+1} - F_i, Q_0) \leq \frac{1}{2}(2K_q(Y)L)^q \mathcal{H}^s(Q_0)$ for $i_0 \in \mathbb{N}_0$ and $Q_0 \in \mathcal{Q}_{i_0}$.

Proof. By Lemma 6.14 and Proposition 7.8, item (iii) is implied by (i) and it is then enough to construct $(F_i)_{i \in \mathbb{N}_0}$ satisfying (i) and (ii). By Lemma 7.5, for $i \in \mathbb{N}_0$ and $Q \in \mathcal{Q}_i$ the set $\mathcal{E}_q(f, Q)$ is non-empty, while, by Lemma 7.7, for any collection $(u_Q)_{Q \in \mathcal{Q}_i}$ with $u_Q \in \mathcal{E}_q(f, Q)$ there is a unique L -Lipschitz map $F_i: X \rightarrow Y$ with $F_i|_Q = u_Q$ for all $Q \in \mathcal{Q}_i$.

Let F_0 be defined as above, for an arbitrary choice of $(u_Q)_{Q \in \mathcal{Q}_0} = (u_X)$ (e.g. $F_0 := 0$) and let $i \geq 1$. By Lemma 8.1, there is a collection $(u_Q)_{Q \in \mathcal{Q}_i}$ with $u_Q \in \mathcal{E}_q(f, Q)$ and, if $I \in \mathcal{I}_i$ and $a, b \in [M]^i$ are as in (ii), we have for $Q = Q_{I,a}, Q' = Q_{I,b} \in \mathcal{Q}_i$

$$u_Q \circ q(t, a, w) = u_{Q'} \circ q(t, b, w),$$

for $t \in I$ and $w \in [M]^{\mathbb{N}}$. Let $F_i: X \rightarrow Y$ be the L -Lipschitz map given by Lemma 7.7 with this collection $(u_Q)_{Q \in \mathcal{Q}_i}$. It is clear that it satisfies (i) and (ii). \square

Theorem 8.3. Suppose $2 \leq q < s$. Let (X, d, \mathcal{H}^s) be an s -ADR Laakso space and Y a q -uniformly convex Banach space. Then, there is a constant $C = C(X, q) > 0$ such that for any L -Lipschitz function $f: (X, d) \rightarrow Y$ and $Q \in \mathcal{Q}$ we have

$$\sum_{\substack{S \in \mathcal{J}: \\ S \subseteq Q}} (\text{diam } f(S))^s \leq CK_q(Y)^q L^s \mathcal{H}^s(Q). \quad (8.4)$$

Here \mathcal{J} is the collection of shortcuts in X defined in Definition 6.17.

Proof. Let $(F_i)_{i \in \mathbb{N}_0}$ be as in Lemma 8.2 and observe that, for $i \in \mathbb{N}$, we have

$$\text{diam}(F_{i+1} - F_i)(S) = \text{diam } f(S), \quad S \in \mathcal{J}_i. \quad (8.5)$$

Indeed, for $S \in \mathcal{J}_i$ and $x \in S$ there is $Q \in \mathcal{Q}_{i+1}$ with $x \in \partial Q$, thus $F_{i+1}|_S = f|_S$, while $\text{diam } F_i(S) = 0$ by Lemma 8.2 (item (ii)).

We now prove the statement. Fix $i_0 \in \mathbb{N}_0$ and $Q \in \mathcal{Q}_{i_0}$. Let $i_1 \in \mathbb{N}$ be the least integer $i \in \mathbb{N}$ for which there is $S \in \mathcal{J}_i$ with $S \subseteq Q$. Since $\text{diam } S = \delta_i \sim \theta^i$ for $S \in \mathcal{J}_i$ (see Proposition 6.25 and Eq. (6.1)), it follows that

$$\theta^{i_1} \lesssim \text{diam}_X Q \sim \theta^{i_0}, \quad (8.6)$$

where we have used Eq. (6.15) of Lemma 6.14. Recall that, by Lemma 2.28, there is a constant $C_1 = C_1(X, q) \geq 1$ such that

$$(\text{diam}(F_{i+1} - F_i)(S))^s \leq C_1 L^{s-q} \int_{B(S, C_1 \delta_i)} \text{Lip}(F_{i+1} - F_i; x)^q d\mathcal{H}^s(x) \quad (8.7)$$

for $i \in \mathbb{N}$ and $S \in \mathcal{J}_i$. Since X is metric doubling and $d(S, S') \geq \delta_i/2$ for $S \neq S' \in \mathcal{J}_i$ (see Lemma 6.21), the collection $\{B(S, C_1 \delta_i): S \in \mathcal{J}_i\}$ has uniformly bounded overlap,

$$\sum_{S \in \mathcal{J}_i} \chi_{B(S, C_1 \delta_i)} \lesssim_{C_1, X} 1.$$

Hence, for $i \geq i_1$, summing Eq. (8.7) over $S \in \mathcal{J}_i$ with $S \subseteq Q$ and recalling Eq. (8.5), we have

$$\begin{aligned} \sum_{\substack{S \in \mathcal{J}_i: \\ S \subseteq Q}} (\text{diam } f(S))^s &\lesssim_{X, q} L^{s-q} \int_{B(Q, C_1 \delta_i)} \text{Lip}(F_{i+1} - F_i; x)^q d\mathcal{H}^s(x) \\ &\leq L^{s-q} \int_{B(Q, C_2 \theta^{i_0})} \text{Lip}(F_{i+1} - F_i; x)^q d\mathcal{H}^s(x), \end{aligned} \quad (8.8)$$

where we have used Eq. (8.6) in the last line. Define

$$B := \bigcup \{Q' \in \mathcal{Q}_{i_1}: Q' \cap B(Q, C_2 \theta^{i_0}) \neq \emptyset\}$$

and observe that, by Eq. (8.6) and Eq. (6.15), $B(Q, C_2\theta^{i_0}) \subseteq B \subseteq B(Q, C_3\theta^{i_0})$. In particular, $\mathcal{H}^s(B) \sim \mathcal{H}^s(Q)$, again by Eq. (6.15). Summing Eq. (8.8) over $i \geq i_1$, we have

$$\sum_{\substack{S \in \mathcal{J}: \\ S \subseteq Q}} (\text{diam } f(S))^s \lesssim L^{s-q} \sum_{i \geq i_1} E_q(F_{i+1} - F_i, B) \lesssim L^s K_q(Y)^q \mathcal{H}^s(B),$$

where the last inequality follows summing both sides of item (iii) of Lemma 8.2 over $Q' \in \mathcal{Q}_{i_1}$ with $Q' \subseteq B$. Finally, $\mathcal{H}^s(B) \sim \mathcal{H}^s(Q)$ concludes the proof. \square

Remark 8.9. The conclusion of Theorem 8.3 fails for $s < q$ or if the Banach space Y does not have a uniformly convex renorming. Indeed, if it were to hold, the proof of Theorem 9.2 would show that $(X, d_\eta, \mathcal{H}^s)$ is a Y -LDS for any η , contradicting the non-differentiability results of Section 10. Alternatively, the maps built in Section 10 fail Eq. (8.4).

9. SHORTCUT LAAKSO SPACES

In this short section we combine the results of the previous section to prove the first two points of Theorem 1.1.

Definition 9.1. Let (X, d, \mathcal{H}^s) be an s -ADR Laakso space, $\eta = (\eta_i)_i \subseteq (0, 1]$, and define d_η as in Definition 3.5. We call $(X, d_\eta, \mathcal{H}^s)$ a *shortcut Laakso space*.

Recall that Theorem 4.1 and Theorem 4.3 characterise sequences $\eta = (\eta_i)_i \subseteq (0, 1]$ for which $(X, d_\eta, \mathcal{H}^s)$ is PI rectifiable or purely PI unrectifiable. To prove Theorem 1.1, we will choose η so that the latter holds. For instance, any $\eta_i \rightarrow 0$ with $\sum_i \eta_i^s = \infty$ works.

Theorem 9.2. Suppose $2 \leq q < s$. Let $(X, d_\eta, \mathcal{H}^s)$ be a s -ADR shortcut Laakso space and Y a q -uniformly convex Banach space. Then $(X, d_\eta, \mathcal{H}^s)$ is a Y -LDS with global chart $h: (X, d_\eta) \rightarrow \mathbb{R}$.

Proof. By Theorem 6.6 and Proposition 6.25, (X, d, \mathcal{H}^s) is an s -ADR Y -LDS with shortcuts compatible with the global chart $h: X \rightarrow \mathbb{R}$. The thesis then follows from Corollary 5.18 and Theorem 8.3 (and $d_\eta \leq d$). \square

10. ALMOST NOWHERE DIFFERENTIABLE LIPSCHITZ MAPS

In this section, we construct almost nowhere differentiable Lipschitz functions on those shortcut Laakso spaces which are purely PI unrectifiable. These functions take values in a variety of Banach spaces Y , depending on the precise setting. In particular, we will prove the third and fourth items of Theorem 1.1.

Theorem 5.6 shows that a function $f: (X, d_\eta) \rightarrow Y$ is almost nowhere differentiable if it separate sufficiently many shortcuts. Corollary 3.24 makes ‘sufficiently many’ precise. In the first lemma, we reduce the construction of a single map $f: (X, d_\eta) \rightarrow Y$ to a sequence of building blocks $f_n: X \rightarrow Y$.

Lemma 10.1. Let (X, d, \mathcal{H}^s) be an s -ADR Laakso space, $\eta = (\eta_i)_i \subseteq (0, 1]$, Y a Banach space, $C \geq 1$, and $(I_n)_{n \in \mathbb{N}}$ a disjoint family of subsets of \mathbb{N} . Suppose there are maps $f_n: X \rightarrow Y$, $n \in \mathbb{N}$, such that:

(i) for $S \in \bigcup_{i \in I_n} \mathcal{J}_i$,

$$C^{-1} \text{diam}_\eta S \leq \text{diam } f_n(S) \leq C \text{diam}_\eta S;$$

(ii) for $S \in \mathcal{J}$, $S \notin \bigcup_{i \in I_n} \mathcal{J}_i$,

$$\text{diam } f_n(S) = 0;$$

(iii) $\sum_{n \in \mathbb{N}} \text{LIP}_X(f_n) < \infty$ and $\sum_{n \in \mathbb{N}} \sum_{i \in I_n} \eta_i^s = \infty$.

Then no positive-measure measurable subset of $(X, d_\eta, \mathcal{H}^s)$ is a Y -LDS.

Proof. By Proposition 2.17, we may assume Y has RNP. Possibly adding a constant to f_n , we can assume there is $x_0 \in X$ such that $f_n(x_0) = 0$ for all $n \in \mathcal{N}$. Hence, from (iii), $f := \sum_{n \in \mathcal{N}} f_n$ is well-defined and a Lipschitz map from (X, d) to Y . Set $I := \bigcup_{n \in \mathcal{N}} I_n$. We are going to use the following properties of f , implied by items (i) and (ii)

$$\begin{aligned} \text{diam } f(S) &\leq C \text{diam}_\eta S, & S \in \mathcal{J}, \\ \text{diam } f(S) &\geq C^{-1} \text{diam}_\eta S, & S \in \bigcup_{i \in I} \mathcal{J}_i. \end{aligned} \quad (10.2)$$

We first show that f is Lipschitz also as a map from (X, d_η) . Set $L := \text{LIP}_X(f) < \infty$ and let $x, y \in X$. If $d_\eta(x, y) = d(x, y)$, then $\|f(x) - f(y)\|_Y \leq L d_\eta(x, y)$. Suppose $d_\eta(x, y) < d(x, y)$ and let $p_- \neq p_+ \in S \in \mathcal{J}_i$ be such that

$$4d_\eta(x, y) \geq d(x, p_-) + \rho_\eta(p_-, p_+) + d(p_-, y), \quad (10.3)$$

where we used Proposition 3.11 and Proposition 6.25. By triangle inequality, Eq. (10.2), and Eq. (10.3), we have

$$\begin{aligned} \|f(x) - f(y)\|_Y &\leq Ld(x, p_-) + \|f(p_-) - f(p_+)\|_Y + Ld(p_-, y) \\ &\leq Ld(x, p_-) + \rho_\eta(p_-, p_+) + Ld(p_-, y) \lesssim (L+1)d_\eta(x, y), \end{aligned}$$

showing that $f: (X, d_\eta) \rightarrow Y$ is Lipschitz. Observe that Eq. (10.2) implies, for all sufficiently small $\epsilon > 0$,

$$\text{Bad}_\epsilon(f) \supseteq \bigcup_{i \in I} B_\eta(\cup \mathcal{J}_i, \delta_i \eta_i / \epsilon).$$

From (iii) and Corollary 3.24, it follows that $\text{Bad}_\epsilon(f)$ has full \mathcal{H}^s -measure for all sufficiently small $\epsilon > 0$ and therefore $\mathcal{H}^s(X \setminus \text{Bad}(f)) = 0$. The thesis then follows from Proposition 5.8 (and Lemma 2.11). \square

We may explicitly construct real valued building blocks as follows.

Lemma 10.4. *Let $i \in \mathbb{N}$ and $X_i = X_i(M; (N_n))$. There is a 1-Lipschitz function $f: X_i \rightarrow \mathbb{R}$ satisfying:*

- (i) $\text{diam } f(S) = \text{diam } S$ for $S \in \mathcal{J}_i$;
- (ii) $\text{diam } f(S) = 0$ for $S \in \bigcup_{j < i} \mathcal{J}_j$;
- (iii) $f \circ q|_{I \times \{a\}}: I \times \{a\} \rightarrow \mathbb{R}$ is affine linear for $I \in \mathcal{I}_{i+1}$ and $a \in [M]^i$;
- (iv) $f \circ q|_{\partial I \times [M]^i} = 0$ for $I \in \mathcal{I}_i$.

Here $(\mathcal{J}_j)_{j=1}^i$ are the families of subsets of X_i defined in Definition 6.17.

Proof. We first construct $\tilde{f}: \tilde{X}_i \rightarrow \mathbb{R}$. Let $I \in \mathcal{I}_i$ and $a \in [M]^{i-1}$. If $\partial I \not\subseteq W_i^h$, set $\tilde{f}|_{I \times \{a\} \times [M]} := 0$. Suppose now $\partial I \subseteq W_i^h$ and let t_0 denote the midpoint of I . We define

$$\tilde{f}(t, a', M) := -\tilde{f}(t, a', 1) := \left(\frac{|I|}{2} - |t - t_0| \right)^+, \quad t \in I,$$

and $\tilde{f}|_{I \times \{(a', a(i))\}} := 0$ for $a(i) \neq 1, M$, where $t^+ := \max(0, t)$ denotes the positive part of $t \in \mathbb{R}$. Since $\tilde{f}|_{\partial I \times \{a\}} = 0$ for $I \in \mathcal{I}_i$ and $a \in [M]^i$, $\tilde{f}: \tilde{X}_i \rightarrow \mathbb{R}$ is well-defined and induces a unique function $f: X_i \rightarrow \mathbb{R}$ satisfying $f \circ q = \tilde{f}$. It is easy to verify that f has the desired properties. \square

We also construct building blocks into \mathbb{R}^2 .

Lemma 10.5. *Let $i \in \mathbb{N}$ and $X_i = X_i(M; (N_n))$. Let $f: X_i \rightarrow \mathbb{R}^2$ a function such that, for $J \in \mathcal{I}_i$ and $a \in [M]^i$,*

$$f \circ q|_{J \times \{a\}}: J \times \{a\} \rightarrow \mathbb{R}^2 \text{ is affine linear.}$$

Set $L := \text{LIP}(f)$. Then, for $A > 0$, there is a function $F: X_i \rightarrow \mathbb{R}^2$ satisfying:

- (i) $\text{diam } F(S) = \text{diam } f(S)$ for $S \in \bigcup_{j < i} \mathcal{J}_j$;
- (ii) $\text{diam } F(S) = A \text{diam } S$ for $S \in \mathcal{J}_i$;
- (iii) $\text{LIP}(F)^2 \leq L^2 + A^2$;

- (iv) $F \circ q|_{J \times \{a\}}: J \times \{a\} \rightarrow \mathbb{R}^2$ is affine linear for $J \in \mathcal{I}_{i+1}$ and $a \in [M]^i$;
- (v) $F \circ q|_{\partial J \times [M]^i} = f \circ q|_{\partial J \times [M]^i}$ for $J \in \mathcal{I}_i$.

Here $(\mathcal{J}_j)_{j=1}^i$ are the families of subsets of X_i defined in Definition 6.17.

Proof. We first establish a few properties of f . We can apply Lemma 2.25 to see that f is in fact Lipschitz, i.e. $L < \infty$. Next, let $J \in \mathcal{I}_i$ with $\partial J \subseteq W_i^h$ and pick $a \in [M]^{i-1}$. Then, for $a_1(i), a_2(i) \in [M]$, $f \circ q|_{J \times \{(a, a_1(i))\}}, f \circ q|_{J \times \{(a, a_2(i))\}}$ are two affine functions agreeing at the endpoints of the interval J . Hence, there is an affine function $\alpha_{J,a}: J \rightarrow \mathbb{R}^2$ such that

$$f \circ q(t, a, a_1(i)) = f \circ q(t, a, a_2(i)) = \alpha_{J,a}(t), \quad t \in J \quad (10.6)$$

and, in particular, $\text{diam } f(S) = 0$ for $S \in \mathcal{J}_i$.

We now describe the construction of F . For each $J \in \mathcal{I}_i$ and $a \in [M]^{i-1}$, we will define a function $F_{J,a}: J \times \{a\} \times [M] \rightarrow \mathbb{R}^2$ with the property

$$F_{J,a}|_{\partial J \times \{a\} \times [M]} = f \circ q|_{\partial J \times \{a\} \times [M]}. \quad (10.7)$$

The function in the thesis will then be the unique $F: X_i \rightarrow \mathbb{R}^2$ satisfying $F \circ q|_{J \times \{a\} \times [M]} = F_{J,a}$ for all $J \in \mathcal{I}_i$ and $a \in [M]^{i-1}$; its existence easily follows from Eq. (10.7). Also, (10.7) implies (v), which in turn gives (i), because for $S \in \mathcal{J}_j$, $j < i$, there is $J \in \mathcal{I}_i$ with $S \subseteq q(\partial J \times [M]^i)$. By Lemma 2.25, we know that (iii) holds as soon as

$$\text{LIP}(F_{J,a})^2 \leq L^2 + A^2 \quad (10.8)$$

for all J, a . Hence, during the construction of $\{F_{J,a}\}$, we need only to ensure the validity of Eq. (10.7), Eq. (10.8) and items (ii), (iv).

We are ready to proceed with the construction. Let $g: X_i \rightarrow \mathbb{R}$ be the 1-Lipschitz function given by Lemma 10.4. Let $J \in \mathcal{I}_i$. If $\partial J \not\subseteq W_i^h$, then $J \cap \mathcal{J}_i^h = \emptyset$ and so $q(J \times [M]^i) \cap S = \emptyset$ for $S \in \mathcal{J}_i$. We set $F_{J,a} := f \circ q|_{J \times \{a\} \times [M]}$ for each $a \in [M]^{i-1}$. With this choice, Eq. (10.7), Eq. (10.8) and items (ii), (iv) are clearly satisfied.

Suppose now $\partial J \subseteq W_i^h$ and let $a \in [M]^{i-1}$. Let $\alpha_{J,a}: J \rightarrow \mathbb{R}^2$ be as in Eq. (10.6) and choose $v_{J,a} \in \mathbb{S}^1$ such that

$$\langle v_{J,a}, \alpha_{J,a}(t_1) - \alpha_{J,a}(t_2) \rangle = 0, \quad t_1, t_2 \in J. \quad (10.9)$$

(Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{R}^2 .) We set $F_{J,a} := (f \circ q + A g \circ q v_{J,a})|_{J \times \{a\} \times [M]}$. By Lemma 10.4 we know that g satisfies the condition of item (iv) and $g \circ q|_{\partial J \times [M]^i} = 0$. It follows that $F_{J,a}$ satisfies (iv) and Eq. (10.7). By Lemma 10.4 and Eq. (10.6) it is clear that (ii) is also satisfied. It remains to show Eq. (10.8). Let $t_1, t_2 \in J$, $a_1(i), a_2(i) \in [M]$, and set $x_k := (t_k, a, a_k(i))$, $1 \leq k \leq 2$. Then Eq. (10.6) and Eq. (10.9) imply

$$\begin{aligned} \|F_{J,a}(t_1, a_1(i)) - F_{J,a}(t_2, a_2(i))\|^2 &= \|f \circ q(x_1) - f \circ q(x_2)\|^2 + A^2 \|g \circ q(x_1) - g \circ q(x_2)\|^2 \\ &\leq (L^2 + A^2) d(q(x_1), q(x_2))^2. \end{aligned}$$

As discussed, this concludes the proof. \square

Lemma 10.10. *There is a constant $C \geq 1$ such that the following holds. For any Laakso space X , $\eta = (\eta_i)_i \subseteq (0, 1]$, and non-empty $I \subseteq \mathbb{N}$, there is a function $f: X \rightarrow \mathbb{R}^2$ satisfying:*

- (i) $C^{-1} \leq \text{diam } f(S) / \text{diam}_\eta S \leq C$ for $S \in \bigcup_{i \in I} \mathcal{J}_i$;
- (ii) $\text{diam } f(S) = 0$ for $S \in \mathcal{J}$, $S \notin \bigcup_{i \in I} \mathcal{J}_i$;
- (iii) $\text{LIP}_X(f)^2 \leq \sum_{i \in I} \eta_i^2$.

Proof. We first establish the following claim. For any (possibly finite) strictly increasing sequence $(i_k)_k \subseteq \mathbb{N}$ there are functions $f_k: X_{i_k} \rightarrow \mathbb{R}^2$ satisfying:

- (1) $\text{diam } f_k(S) = \eta_{i_l}$ $\text{diam } S$ for $S \in \mathcal{J}_{i_l}$, $l \leq k$;
- (2) $\text{diam } f_k(S) = 0$ for $S \in \bigcup_{j \leq i_k} \mathcal{J}_j$, $S \notin \bigcup_{l \leq k} \mathcal{J}_{i_l}$;
- (3) $\text{LIP}(f_k)^2 \leq \sum_{l \leq k} \eta_{i_l}^2$;
- (4) $f_k \circ q|_{J \times \{a\}}$ is affine for $J \in \mathcal{I}_{i_{k+1}}$ and $a \in [M]^{i_k}$;
- (5) $f_k \circ q|_{\partial J \times [M]^{i_k}} = f_{k-1} \circ \pi_{i_{k-1}}^{i_k} \circ q|_{\partial J \times [M]^{i_k}}$ for $J \in \mathcal{I}_{i_k}$.

If $g: X_{i_1} \rightarrow \mathbb{R}$ is the function given by Lemma 10.4, then $f_1 := (\eta_{i_1}g, 0): X_{i_1} \rightarrow \mathbb{R}^2$ has all the desired properties. Hence, if $(i_k)_k \subseteq \mathbb{N}$ is a (possibly finite) strictly increasing sequence of N terms, $N \in \mathbb{N} \cup \{\infty\}$, we can at least find $f_1: X_{i_1} \rightarrow \mathbb{R}^2$. Suppose now $1 < n \leq N$, $n \in \mathbb{N}$, and that we can find f_1, \dots, f_{n-1} as in the claim. We show that we can then construct $f_n: X_{i_n} \rightarrow \mathbb{R}^2$ satisfying (1-5). Since $i_{n-1} < i_n$, it is not difficult to see that $f_{n-1} \circ \pi_{i_{n-1}}^{i_n}: X_{i_n} \rightarrow \mathbb{R}^2$ satisfies the assumption of Lemma 10.5 with $i \equiv i_n$. Taking $A \equiv \eta_{i_n}$, the lemma gives a function $f_n: X_{i_n} \rightarrow \mathbb{R}^2$ satisfying (1-5), hence proving the claim.

We can now prove the lemma. If I is finite, $I = \{i_1 < \dots < i_n\}$, we can take $f := f_n \circ \pi_{i_n}^\infty$, where f_n is as in the claim. Suppose now I is infinite and let $i_k \uparrow \infty$ be such that $I = \{i_k: k \in \mathbb{N}\}$. We can assume

$$L := \sum_{k \in \mathbb{N}} \eta_{i_k}^2 < \infty, \quad (10.11)$$

for otherwise the statement is trivial. Let $\tilde{f}_k: X_{i_k} \rightarrow \mathbb{R}^2$ be the function provided by the claim and set $f_k := \tilde{f}_k \circ \pi_{i_k}^\infty: X \rightarrow \mathbb{R}^2$. By item (5), (f_k) converges pointwise on the set

$$E := q \left(\left\{ \frac{m}{N_1 \dots N_n} : n \in \mathbb{N}, m \in \mathbb{N}_0, 0 \leq m \leq N_1 \dots N_n \right\} \times [M]^\mathbb{N} \right).$$

By item (3) and Eq. (10.11), (f_k) is equicontinuous. Since E is dense in X , it follows that f_k converges pointwise on X to an L -Lipschitz function $f: X \rightarrow \mathbb{R}^2$. By pointwise convergence and (1-2) (or by (1-2) and (5)), we conclude that f satisfies also (i),(ii). \square

We are now in a position to construct non-differentiable Banach space valued functions on shortcut Laakso spaces.

For $q \in [1, \infty]$ and $n \in \mathbb{N}$, we denote $\ell_q^n = (\mathbb{R}^n, \|\cdot\|_{\ell_q})$. We say that a Banach space Y contains ℓ_q^n uniformly if there are $C \geq 1$ and, for $n \in \mathbb{N}$, linear operators $T_n: \ell_q^n \rightarrow Y$ satisfying

$$C^{-1} \|x\|_{\ell_q^n} \leq \|T_n x\|_Y \leq C \|x\|_{\ell_q^n} \quad (10.12)$$

for all $x \in \ell_q^n$ and $n \in \mathbb{N}$. By a theorem of Krivine [Kri75] (see [MS86, 10.5] for the precise statement we use), we may equivalently require Eq. (10.12) for all $C > 1$.

Proposition 10.13. *Suppose $2 \leq s < q < \infty$. Let $(X, d_\eta, \mathcal{H}^s)$ be an s -ADR shortcut Laakso space, assume it is purely PI unrectifiable, and let Y be a Banach space containing ℓ_q^n uniformly (e.g. $Y = \ell_q$). Then, no positive-measure measurable subset of $(X, d_\eta, \mathcal{H}^s)$ is a Y -LDS.*

Proof. By Lemma 2.22, we find countably many finite non-empty disjoint sets $I_n \subseteq \mathbb{N}$ such that

$$\sum_{n \in \mathbb{N}} \left(\sum_{i \in I_n} \eta_i^q \right)^{1/q} < \infty, \quad \sum_{n \in \mathbb{N}} \sum_{i \in I_n} \eta_i^s = \infty. \quad (10.14)$$

For $n \in \mathbb{N}$, denote with N_n the cardinality of I_n . We will first construct a sequence of functions $\tilde{f}_n: X \rightarrow \ell_q^{N_n}$ satisfying items (i), (ii), (iii) of Lemma 10.1. It will then be easy to produce a sequence $f_n: X \rightarrow Y$ with the same property.

For $n \in \mathbb{N}$ and $i \in I_n$, let $g_i: X_i \rightarrow \mathbb{R}$ be given by Lemma 10.4 and set $\tilde{f}_n := (\eta_i g_i \circ \pi_i^\infty)_{i \in I_n}: X \rightarrow (\mathbb{R}^{I_n}, \|\cdot\|_{\ell_q}) \equiv \ell_q^{N_n}$. Items (ii) and (iii) follow easily from Lemma 10.4, the definition of \tilde{f}_n , and Eq. (10.14). Recall that by Lemma 3.23 and Proposition 6.25, it holds

$$\text{diam}_\eta S \sim \eta_i \delta_i \sim \eta_i \text{diam } S, \quad S \in \mathcal{J}_i.$$

Hence, also item (i) follows from Lemma 10.4 and the definition of \tilde{f}_n .

Since N_n is finite and Y contains ℓ_q^n uniformly, there are a constant $C \geq 1$ and linear operators $T_n: \ell_q^{N_n} \rightarrow Y$ satisfying Eq. (10.12). It is then clear that the maps $f_n := T_n \circ \tilde{f}_n: X \rightarrow Y$ satisfy the assumptions of Lemma 10.1, from which the thesis follows. \square

Proposition 10.13 also holds if $1 < s < 2$. However, for this range we can establish a much stronger conclusion, namely that no positive-measure measurable subset of $(X, d_\eta, \mathcal{H}^s)$ is an LDS.

Proposition 10.15. *Suppose $1 < s < 2$. Let $(X, d_\eta, \mathcal{H}^s)$ be an s -ADR shortcut Laakso space and assume it is purely PI unrectifiable. Then, no positive-measure measurable subset of $(X, d_\eta, \mathcal{H}^s)$ is an LDS.*

Proof. By Lemma 2.22, $\eta = (\eta_i)$ has a subsequence $(\eta_{i_j})_j$ satisfying

$$\sum_{j \in \mathbb{N}} \eta_{i_j}^2 < \infty, \quad \sum_{j \in \mathbb{N}} \eta_{i_j}^s = \infty.$$

Let $I := \{i_j : j \in \mathbb{N}\}$ and $f : X \rightarrow \mathbb{R}^2$ be given by Lemma 10.10. The single map f and set I satisfy the assumptions of Lemma 10.1, concluding the proof. \square

The following is our final non-differentiability result.

Proposition 10.16. *Let $(X, d_\eta, \mathcal{H}^s)$ be an s -ADR shortcut Laakso space, suppose it is purely PI unrectifiable, and let Y be a non-superreflexive Banach space. Then, no positive-measure measurable subset of $(X, d_\eta, \mathcal{H}^s)$ is a Y -LDS.*

The proof relies on the following theorem, whose proof is postponed to the next section.

Theorem 10.17. *Let Y be a Banach space and X a Laakso space. Then Y is non-superreflexive if and only if there is $C \geq 1$ such that for any finite non-empty set $I \subseteq \mathbb{N}$ there is a 1-Lipschitz function $f : X \rightarrow Y$ satisfying:*

- (i) $\text{diam } f(S) \geq C^{-1} \text{diam } S$ for $S \in \bigcup_{i \in I} \mathcal{J}_i$;
- (ii) $\text{diam } f(S) = 0$ for $S \in \mathcal{J}$, $S \notin \bigcup_{i \in I} \mathcal{J}_i$.

Moreover, if Y is non-superreflexive, the above holds for any $C > 8$.

Proof of Proposition 10.16. By Lemma 2.22, there is a subsequence $(\eta_{i_j})_j$ such that $\eta_{i_j} \rightarrow 0$ and $\sum_j \eta_{i_j}^s = \infty$. For $n \in \mathbb{N}$, set

$$I_n := \{i_j : 2^{-n} < \eta_{i_j} \leq 2^{1-n}\}$$

and note that they are all finite sets. Let \tilde{f}_n be the map given by Theorem 10.17 with $I \equiv I_n$ and define $f_n := 2^{-n} \tilde{f}_n$ (or $f_n := 0$ if $I_n = \emptyset$). To conclude, it is enough to verify that f_n and I_n satisfy the assumptions of Lemma 10.1. Let $n \in \mathbb{N}$. If $i \in I_n$ and $S \in \mathcal{J}_i$, we have

$$\text{diam } f_n(S) \sim \eta_i \text{diam } S \sim \text{diam}_\eta S,$$

while if $i \notin I_n$, $S \in \mathcal{J}_i$, Lemma 10.1 gives $\text{diam } f_n(S) = 0$. Lastly, $\sum_n \text{LIP}_X(f_n) \leq 1$ and

$$\sum_{n \in \mathbb{N}} \sum_{i \in I_n} \eta_i^s = \sum_{j \in \mathbb{N}} \eta_{i_j}^s = \infty$$

conclude the proof. \square

11. A CHARACTERISATION OF NON-SUPERREFLEXIVE SPACES

The goal of this section is to prove Theorem 10.17. We exploit well-known characterisations of non-superreflexivity in terms of biLipschitz embeddability of families of graphs, first introduced by Bourgain [Bou86]; see also [Bau07]. More specifically, we (indirectly) use a technique introduced in [OR17] and later applied in [Swi18] to a larger family of graphs.

The above works consider ‘Laakso graphs’. However, in spite of the shared name, none of the above works deal with the Laakso spaces defined in Section 6 nor their finite approximations G_n in Section 6.1. Indeed, [OR17, Swi18] consider the family of graphs introduced in [LP01, Section 2] which is inspired by [Laa02] and are often called ‘Laakso graphs’; their Gromov-Hausdorff limit is also sometimes called ‘Laakso space’. However, the spaces we are considering, those of [Laa00], are not directly related to those of [Laa02]. For instance, all of the latter graphs are planar, while G_n is non-planar for $n \geq 2$ and $M \geq 3$ (since it contains $K_{3,3}$). Nonetheless, the techniques of [OR17] can be used to construct the map required in Theorem 10.17. Instead of doing so directly, we shall reduce the construction of our map to the embedding of a graph simpler than G_n , for which [Swi18] can be applied.

We refer the reader to [Swi18, Definitions 2.1 and 2.4] for the definition of M -branching bundle graph and to [Die25] for graph theory background. We stress that, compared to [Swi18], we adopt a slightly different vertex set, with the only difference being a different scaling of the ‘height’ coordinate. We also scale the distance accordingly.

11.1. Diamond graphs. We define a collection of graphs which we call *diamond graphs*. Let $n, M \in \mathbb{N}$, $M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$ and set

$$\begin{aligned}\widehat{W}_1^h &:= \widehat{W}_1^h(N_1, \dots, N_n) := W_1^h(N_1, \dots, N_n) \cup \{0, 1\}, \\ \widehat{W}_j^h &:= \widehat{W}_j^h(N_1, \dots, N_n) := W_j^h(N_1, \dots, N_n), \\ \widehat{W}_{\leq l}^h &:= \widehat{W}_{\leq l}^h(N_1, \dots, N_n) := \bigcup_{i=1}^l \widehat{W}_i^h \\ &= \left\{ \frac{m}{N_1 \cdots N_l} : m \in \mathbb{N}_0, 0 \leq m \leq N_1 \cdots N_l \right\}\end{aligned}\tag{11.1}$$

for $1 < j \leq n$ and $1 \leq l \leq n$. For $1 \leq j \leq n$ and $t \in \widehat{W}_j^h$, set $w_t := j - 1$. We then denote with

$$\widehat{G}(M; N_1, \dots, N_n)\tag{11.2}$$

the M -branching bundle graph associated with (w_t) ; see [Swi18, Definition 2.9].

Our first main goal is to show that [Swi18] implies the existence of biLipschitz embeddings of our diamond graphs with uniformly bounded distortion. Swift proves that suitable graphs, including our diamond graphs \widehat{G} , admit biLipschitz embeddings in non-superreflexive Banach space. The distortion of such embeddings depends on a geometric parameter $p_{\widehat{G}}$ of the graph, which we now define.

Let $\widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$ and define for $t \in \widehat{W}_{\leq n}^h$, $t = \sum_{i=1}^n \frac{t_i}{N_1 \cdots N_i}$, and $1 \leq l \leq n$

$$\begin{aligned}x(t, l) &:= t - d([0, t] \cap \widehat{W}_{\leq l}^h, t) = \sum_{1 \leq i \leq l} \frac{t_i}{N_1 \cdots N_i}, \\ y(t, l) &:= t + d([t, 1] \cap \widehat{W}_{\leq l}^h, t) = \frac{1}{N_1 \cdots N_l} + \sum_{1 \leq i \leq l} \frac{t_i}{N_1 \cdots N_i},\end{aligned}\tag{11.3}$$

and $x(t, 0) := 0$, $y(t, 0) := 1$. We also set $x(t, l) := y(t, l) := t$ for $l > n$ ⁶. Observe that equality between the leftmost and rightmost sides of Eq. (11.3) holds even for $l = 0$, provided we interpret empty sums as 0 and empty products as 1. Let $p_{\widehat{G}}$ be the least $p \in \mathbb{N}$ such that for all $0 \leq j < n$ and $t \in \widehat{W}_{\leq n}^h$

- $t \geq (x(t, j) + y(t, j))/2$ implies $x(t, j + p) \geq (x(t, j) + y(t, j))/2$;
- $t \leq (x(t, j) + y(t, j))/2$ implies $y(t, j + p) \leq (x(t, j) + y(t, j))/2$.

We shall verify that $p_{\widehat{G}} = 1$.

Remark 11.4. For $t \in \widehat{W}_j^h$ it holds $x(t, l) = y(t, l) = t$, $j \leq l \leq n$. We may then restrict our attention to $1 \leq l + 1 < j \leq n$. Also, since N_{l+1} is even, $(x(t, l) + y(t, l))/2 \in \widehat{W}_{l+1}^h$ for $0 \leq l < n$ and in particular $t \neq (x(t, l) + y(t, l))/2$ whenever $t \notin \widehat{W}_{l+1}^h$.

Lemma 11.5. *Let $n, M \in \mathbb{N}$, $M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$, and recall the definitions Eqs. (11.1) and (11.3). Let $1 \leq l + 1 < j \leq n$ and $t \in \widehat{W}_j^h$, $t = \sum_{i=1}^j \frac{t_i}{N_1 \cdots N_i}$. Then $t > (x(t, l) + y(t, l))/2$ if and only if $t_{l+1} \geq N_{l+1}/2$.*

⁶The definitions of $x(t, l), y(t, l)$ we adopt differs slightly from the one of [Swi18], but it is not difficult to see that they are equivalent.

Proof. Under the present assumptions $t \neq (x(t, l) + y(t, l))/2$; see Remark 11.4. To treat the cases $l = 0$ and $l > 0$ in a unified way, we adopt the conventions on empty sums and product mentioned after Eq. (11.3). Observe that $t \geq (x(t, l) + y(t, l))/2$ if and only if

$$\sum_{l < i \leq j} \frac{t_i}{N_1 \cdots N_i} \geq \frac{1}{2N_1 \cdots N_l}.$$

Now, if $t_{l+1} < N_{l+1}/2$, then

$$\sum_{l < i \leq j} \frac{t_i}{N_1 \cdots N_i} \leq \frac{t_{l+1}}{N_1 \cdots N_{l+1}} + \sum_{i=l+2}^j \frac{N_i - 1}{N_1 \cdots N_i} < \frac{t_{l+1} + 1}{N_1 \cdots N_{l+1}} \leq \frac{1}{2N_1 \cdots N_l},$$

proving $t < (x(t, l) + y(t, l))/2$. If instead $t_{l+1} \geq N_{l+1}/2$, then

$$\sum_{l < i \leq j} \frac{t_i}{N_1 \cdots N_i} \geq \frac{1}{2N_1 \cdots N_l},$$

proving $t > (x(t, l) + y(t, l))/2$. □

Lemma 11.6. *Let $n, M \in \mathbb{N}$, $M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$, consider $\widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$, and recall the definitions Eqs. (11.1) and (11.3). Let $0 \leq l < n$ and $t \in \widehat{W}_{\leq n}^h$. Then*

- $t \geq (x(t, l) + y(t, l))/2$ implies $x(t, l+1) \geq (x(t, l) + y(t, l))/2$;
- $t \leq (x(t, l) + y(t, l))/2$ implies $y(t, l+1) \leq (x(t, l) + y(t, l))/2$.

Equivalently, $p_{\widehat{G}} = 1$.

Proof. Let $1 \leq j \leq n$ be such that $t \in \widehat{W}_j^h$ and write $t = \sum_{i=1}^j \frac{t_i}{N_1 \cdots N_i}$. Assume w.l.o.g. $j > l+1$ and hence $t \neq (x(t, l) + y(t, l))/2$; see Remark 11.4. The thesis will follow from Eq. (11.3), Lemma 11.5 and some simple computations. We adopt the conventions on empty sums and product mentioned after Eq. (11.3). If $t > (x(t, l) + y(t, l))/2$, then $t_{l+1} \geq N_{l+1}/2$ and therefore

$$x(t, l+1) = \sum_{1 \leq i \leq l+1} \frac{t_i}{N_1 \cdots N_i} \geq \frac{1}{2N_1 \cdots N_l} + \sum_{1 \leq i \leq l} \frac{t_i}{N_1 \cdots N_i} = (x(t, l) + y(t, l))/2.$$

If instead $t < (x(t, l) + y(t, l))/2$, then $t_{l+1} < N_{l+1}/2$ and therefore

$$y(t, l+1) = \frac{1}{N_1 \cdots N_{l+1}} + \sum_{1 \leq i \leq l+1} \frac{t_i}{N_1 \cdots N_i} \leq \frac{1}{2N_1 \cdots N_l} + \sum_{1 \leq i \leq l} \frac{t_i}{N_1 \cdots N_i} = (x(t, l) + y(t, l))/2.$$

Recalling the definition of $p_{\widehat{G}}$ (see the paragraph after Eq. (11.3)), we immediately have $p_{\widehat{G}} = 1$. □

Theorem 11.7 ([Swi18, Theorem 5.4]). *Let $n, M \in \mathbb{N}$, $M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$, and let $\widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$. Then, for any non-superreflexive Banach space Y and $\epsilon > 0$, there is a map $f: \widehat{G} \rightarrow Y$ such that*

$$(8 + \epsilon)^{-1} d_{\widehat{G}}(u, v) \leq \|f(u) - f(v)\|_Y \leq d_{\widehat{G}}(u, v), \quad (11.8)$$

for u, v in \widehat{G} .

Proof. The thesis follows from [Swi18, Theorem 5.4] and [BS75], as we now describe. We refer to [Swi18, Definition 5.1] or [OR17, Definition 2.1] for the definition of equal-sign-additive (ESA) basis and [Pis16, Chapter 11] for finite representability.

Since Y is non-superreflexive, there is a non-reflexive Banach space Z_1 which is finitely representable in Y . By [BS75] (see the paragraph after [OR17, Theorem 2.3] for details), there is a Banach space Z with an equal-sign-additive (ESA) basis which is finitely representable in Z_1 and hence in Y . Thus, [Swi18, Theorem 5.4] and Lemma 11.6 ensure the existence of a map $g: \widehat{G} \rightarrow Z$ satisfying

$$\frac{1}{8} d_{\widehat{G}}(u, v) \leq \|g(u) - g(v)\|_Z \leq d_{\widehat{G}}(u, v),$$

for u, v in \widehat{G} . Since \widehat{G} is finite, the thesis follows by definition of finite representability. □

To prove Theorem 10.17, we need to define suitable ‘projections’ between diamond graphs and establish some basic properties.

Let $n, M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$ and let $(\widehat{V}, \widehat{E}) = \widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$ denote the corresponding diamond graph. Let $k \in \mathbb{N}$ and $I = \{i_1 < \dots < i_k\}$ be a subset of $[n-1]$. For $t \in \widehat{W}_{\leq n}^h$, set $w_{I,t} := \#\{j \in [k]: i_j \leq w_t\}$,

$$\widehat{V}_I := \bigcup \left\{ \{t\} \times [M]^{w_{I,t}} : t \in \widehat{W}_{\leq n}^h \right\}$$

and

$$\pi_I: \widehat{V} \rightarrow \widehat{V}_I, \quad \pi_I(t, a) := (t, a|_I). \quad (11.9)$$

Set $i_0 := 0$, $i_{k+1} := n$, and for $1 \leq j \leq k+1$

$$\widehat{W}_{I,j}^h := \bigcup_{l=i_{j-1}+1}^{i_j} \widehat{W}_l^h, \quad N_{I,j} := \prod_{i=i_{j-1}+1}^{i_j} N_i. \quad (11.10)$$

Observe that $N_{I,1}, \dots, N_{I,k+1} \in 2\mathbb{N}$ and

$$\widehat{W}_{I,j}^h = \widehat{W}_j^h(N_{I,1}, \dots, N_{I,k+1}).$$

It is then clear that $w_{I,t} = j-1$ for $t \in \widehat{W}_{I,j}^h$, $1 \leq j \leq k+1$. Hence, \widehat{V}_I coincides with the vertex set of $\widehat{G}(M; N_{I,1}, \dots, N_{I,k+1})$. Let \widehat{E}_I denote the smallest edge set for which $\pi_I: \widehat{G} \rightarrow (\widehat{V}_I, \widehat{E}_I)$ is a graph homomorphism and define

$$\widehat{G}_I := \widehat{G}(M; N_1, \dots, N_n)_I := (\widehat{V}_I, \widehat{E}_I). \quad (11.11)$$

Lemma 11.12. *Let $n, M \in \mathbb{N}$, $n, M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$, and let $\widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$. Let $I = \{i_1 < \dots < i_k\} \subseteq [n-1]$ be a non-empty set and define $N_{I,1}, \dots, N_{I,k+1} \in 2\mathbb{N}$ and \widehat{G}_I as in Eq. (11.10) and Eq. (11.11) respectively. Then*

$$\widehat{G}_I = \widehat{G}(M; N_1, \dots, N_n)_I = \widehat{G}(M; N_{I,1}, \dots, N_{I,k+1}).$$

Proof. We use the notation \leq from [Swi18, Section 2]. Let $u = (t, a), v = (s, b) \in \widehat{V}_I$. We may assume $t \in \widehat{W}_{l(t)}^h(N_1, \dots, N_n)$, $s \in \widehat{W}_{l(s)}^h(N_1, \dots, N_n)$ with $1 \leq l(t) \leq l(s) \leq n$. Suppose u, v are adjacent in $\widehat{G}(M; N_{I,1}, \dots, N_{I,k+1})$, i.e. $|t-s| = \frac{1}{N_{I,1} \dots N_{I,k+1}} = \frac{1}{N_1 \dots N_n}$ and $a \leq b$ or $b \leq a$. Since $l(t) \leq l(s)$, we have $w_{I,l(t)} \leq w_{I,l(s)}$, and therefore it must be $a \leq b$. The conditions $l(t) \leq l(s)$ and $a \leq b$ guarantee the existence of a_0, b_0 with $a_0 \leq b_0$ or $b_0 \leq a_0$ and $\pi_I(t, a) = (t, a_0)$, $\pi_I(s, b) = (s, b_0)$, proving that u, v are adjacent also in \widehat{G}_I . Suppose now u, v are adjacent in \widehat{G}_I , i.e. there are $(t, a_0), (s, b_0)$ adjacent in \widehat{G} such that $u = \pi_I(t, a_0)$ and $v = \pi_I(s, b_0)$. Then $|t-s| = \frac{1}{N_{I,1} \dots N_{I,k+1}} = \frac{1}{N_1 \dots N_n}$, $a_0 \leq b_0$, hence $a \leq b$ and therefore u, v are adjacent in $\widehat{G}(M; N_{I,1}, \dots, N_{I,k+1})$. \square

Recall the notation introduced in Eq. (6.18).

Lemma 11.13. *Let $n, M \in \mathbb{N}$, $n, M \geq 2$, and $N_1, \dots, N_n \in 2\mathbb{N}$. Let $I = \{i_1 < \dots < i_k\} \subseteq [n-1]$ be a non-empty set and define $N_{I,1}, \dots, N_{I,k+1}$ as in Eq. (11.10). Then, for $1 \leq j \leq k$, we have*

$$\mathcal{J}_{i_j}^h(N_1, \dots, N_{i_j}) \subseteq \mathcal{J}_j^h(N_{I,1}, \dots, N_{I,j}).$$

Proof. Let $t \in \mathcal{J}_{i_j}^h(N_1, \dots, N_{i_j})$ and $t_i \in \mathbb{N}_0$, $0 \leq t_i < N_i$ for $1 \leq i < i_j$ and $0 < t_{i_j} < N_{i_j} - 1$ such that

$$t = \sum_{i=1}^{i_j} \frac{t_i}{N_1 \dots N_i} + \frac{1}{2N_1 \dots N_{i_j}}.$$

There are unique $\tau_l \in \mathbb{N}_0$, $0 \leq \tau_l < N_{I,l}$, satisfying

$$\frac{\tau_l}{N_{I,l}} = \sum_{i=i_{l-1}+1}^{i_l} \frac{t_i}{N_1 \dots N_i}$$

for $1 \leq l \leq j$. Since $0 < t_{i_j} < N_{i_j} - 1$, we have $\tau_j > 0$ and

$$\tau_j = \left\lfloor \sum_{i=i_{j-1}+1}^{i_j} \frac{t_i}{N_1 \cdots N_i} \right\rfloor (N_{i_{j-1}+1} \cdots N_{i_j}) < N_{i_{j-1}+1} \cdots N_{i_j} - 1,$$

i.e. $\tau_j < N_{I,j} - 1$. Hence, t can be written as

$$t = \sum_{l=1}^j \frac{\tau_l}{N_{I,1} \cdots N_{I,l}} + \frac{1}{2N_{I,1} \cdots N_{I,j}}$$

with $0 \leq \tau_l < N_{I,l}$ and $0 < \tau_j < N_{I,j} - 1$, proving $t \in \mathcal{J}_j^h(N_{I,1}, \dots, N_{I,j})$. \square

Let $n, M \in \mathbb{N}$, $n, M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$, and $\widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$. We then set, for $1 \leq i \leq n-1$,

$$\widehat{\mathcal{J}}_i := \widehat{\mathcal{J}}_i(M; N_1, \dots, N_i) := \{\{t\} \times \{a\} \times [M] : t \in \mathcal{J}_i^h(N_1, \dots, N_i) \text{ and } a \in [M]^{i-1}\}. \quad (11.14)$$

Lemma 11.15. *Let $n, M \in \mathbb{N}$, $n, M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$, and $\widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$. For $1 \leq i \leq n-1$, $u \neq v \in S \in \widehat{\mathcal{J}}_i$, we have*

$$d_{\widehat{G}}(u, v) = \frac{1}{N_1 \cdots N_i}.$$

Proof. Write $u = (t, a, u(i))$, $v = (t, a, v(i))$, for some $a \in [M]^{i-1}$ and $t \in \mathcal{J}_i^h \subseteq \widehat{W}_{i+1}^h$. Recall that in the definition of \mathcal{J}_i^h we require $t \pm \frac{1}{2N_1 \cdots N_i} \in W_i^h$ and so, for $1 \leq j \leq i$,

$$d_{\mathbb{R}}(t, \widehat{W}_j^h) \geq \frac{1}{2N_1 \cdots N_i} = d_{\mathbb{R}}(t, \widehat{W}_i^h) = d_{\mathbb{R}}(t, W_i^h).$$

Since u, v differ exactly in the i -th digit, from [Swi18, Proposition 2.9] we then have

$$d_{\widehat{G}}(u, v) = \frac{2}{2N_1 \cdots N_i} = \frac{1}{N_1 \cdots N_i}.$$

\square

Lemma 11.16. *Let $n, M \in \mathbb{N}$, $n, M \geq 2$, $N_1, \dots, N_n \in 2\mathbb{N}$, and $\widehat{G} = \widehat{G}(M; N_1, \dots, N_n)$. Let $I \subseteq [n-1]$ be a non-empty set and let \widehat{G}_I and $\pi_I: \widehat{G} \rightarrow \widehat{G}_I$ be as in Eq. (11.11) and Eq. (11.9) respectively. Then the map π_I is 1-Lipschitz and for $1 \leq i \leq n-1$, $u, v \in S \in \widehat{\mathcal{J}}_i$, it holds*

$$d_{\widehat{G}_I}(\pi_I(u), \pi_I(v)) = \begin{cases} d_{\widehat{G}}(u, v), & i \in I \\ 0, & i \notin I \end{cases}.$$

Proof. Since π_I is a graph homomorphism, it follows that it is 1-Lipschitz. Let $I = \{i_1 < \dots < i_k\}$ and $N_{I,1}, \dots, N_{I,k+1}$ be as in Eq. (11.10). We can assume $u \neq v$ and write $u = (t, a, u(i))$, $v = (t, a, v(i))$, for some $a \in [M]^{i-1}$ and $t \in \mathcal{J}_i^h(N_1, \dots, N_i)$. Suppose $i \in I$ and let $1 \leq j \leq k$ be such that $i = i_j$. Then $\pi_I(S) = \{(t, a|_I)\} \times [M]$ with $a|_I \in [M]^{j-1}$ and $t \in \mathcal{J}_j^h(N_{I,1}, \dots, N_{I,j})$ (by Lemma 11.13). Hence, $\pi_I(u) \neq \pi_I(v) \in \pi_I(S) \in \widehat{\mathcal{J}}_j(M; N_{I,1}, \dots, N_{I,j})$ and so Lemma 11.15 and Eq. (11.10) yield

$$d_{\widehat{G}_I}(\pi_I(u), \pi_I(v)) = \frac{1}{N_{I,1} \cdots N_{I,j}} = \frac{1}{N_1 \cdots N_{i_j}} = d_{\widehat{G}}(u, v).$$

If $i \notin I$, from $1 \leq i \leq n-1$ we find $1 \leq j \leq k+1$ such that $i_{j-1} < i < i_j$ and therefore $\pi_I(u) = (t, a|_I) = \pi_I(v)$. \square

11.2. Projection of Laakso graphs onto diamond graphs. We use the following notation for the rest of the section. Fix a Laakso space $X = X(M; (N_n))$ and, for $n \in \mathbb{N}$, let G_n be the graph having $X_n = X_n(M; N_1, \dots, N_{n+1})$ as metric graph (see Section 6.1), and set $\hat{G}_n = \hat{G}(M; N_1, \dots, N_{n+1})$ (Eq. (11.2)). We let $\hat{\pi}_n$ denote the obvious projection from the vertex set of G_n onto the one of \hat{G}_n . For a non-empty $I \subseteq [n]$, set $\hat{G}_{n,I} := (\hat{G}_n)_I$ (Eq. (11.11)) and $\hat{\pi}_{n,I} := \pi_I \circ \hat{\pi}_n$, where π_I is as in Eq. (11.9).

Lemma 11.17. *Let $n \in \mathbb{N}$ and $I \subseteq [n]$ be a non-empty set. Let G_n , $\hat{G}_{n,I}$, and $\hat{\pi}_{n,I}: G_n \rightarrow \hat{G}_{n,I}$ be as above. Then the map $\hat{\pi}_{n,I}$ is 1-Lipschitz and for $1 \leq i \leq n$ and $u, v \in S \in \mathcal{J}_i$, we have*

$$d_{\hat{G}_{n,I}}(\hat{\pi}_{n,I}(u), \hat{\pi}_{n,I}(v)) = \begin{cases} d_{G_n}(u, v), & i \in I \\ 0, & i \notin I \end{cases}.$$

Proof. It is not difficult to see that $\hat{\pi}_n: G_n \rightarrow \hat{G}_n$ is a graph homomorphism and therefore 1-Lipschitz. Thus, Lemma 11.16 shows that $\hat{\pi}_{n,I}$ is 1-Lipschitz. Let S be as in the statement and assume w.l.o.g. $u \neq v \in S$. From the definitions of G_n , \hat{G}_n , and $\hat{\pi}_n: G_n \rightarrow \hat{G}_n$, we have $\hat{\pi}_n(u) \neq \hat{\pi}_n(v) \in \hat{\pi}_n(S) \in \hat{\mathcal{J}}_i(M; N_1, \dots, N_i)$ (Eq. (11.14)). Hence, Lemma 11.15 and (the argument of) Lemma 6.22 give

$$d_{\hat{G}_n}(\hat{\pi}_n(u), \hat{\pi}_n(v)) = \frac{1}{N_1 \dots N_i} = d_{G_n}(u, v).$$

Finally, Lemma 11.16 concludes the proof. \square

Proof of Theorem 10.17. Suppose first Y is non-superreflexive, set $n := \max(I)$, and let $G_n, \hat{G}_{n,I}$, and $\hat{\pi}_{n,I}: G_n \rightarrow \hat{G}_{n,I}$ be as above. By Lemma 11.12 and Theorem 11.7, we deduce the existence of a map $\hat{f}: \hat{G}_{n,I} \rightarrow Y$ for which Eq. (11.8) holds. Set $\tilde{f} := \hat{f} \circ \hat{\pi}_{n,I}: G_n \rightarrow Y$ and observe that, by Lemma 11.17, it is 1-Lipschitz and satisfies both items of Theorem 10.17 (on the graph) for $1 \leq i \leq n$. With small abuse of notation, let $\tilde{f}: X_n \rightarrow Y$ denote the extension of $\tilde{f}|_{G_n}$ to X_n given by Lemma 6.9, and finally set $f := \tilde{f} \circ \pi_n^\infty: X \rightarrow Y$. Since $\text{diam } \pi_n^\infty(S) = 0$ for $S \in \bigcup_{i > n} \mathcal{J}_i$, the map f is as claimed.

For the converse direction, we use the differentiability theory of Cheeger-Kleiner [CK09] and a characterisation of superreflexivity via ultrapowers. We refer the reader to [Hei80] for background on ultrafilters, ultrapowers, and their applications to Banach space theory.

Suppose Y is a Banach space, $C \geq 1$, and that, for $n \in \mathbb{N}$, there is a 1-Lipschitz map $f_n: X \rightarrow Y$ satisfying (i) with $I = [n]$. We may assume w.l.o.g. $\sup_n \|f_n\|_\infty < \infty$. Let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} , let $Y^\mathcal{U}$ denote the corresponding ultrapower of Y , and define $f: X \rightarrow Y^\mathcal{U}$ as $f(x) := [(f_n(x))]$, where, for bounded $(y_n) \subseteq Y$, $[(y_n)] \in Y^\mathcal{U}$ denotes its equivalence class. Since \mathcal{U} is non-principal and $\|[(y_n)]\|_{Y^\mathcal{U}} = \lim_{\mathcal{U}} \|y_n\|_Y$ for $(y_n) \subseteq Y$ bounded, it is not difficult to verify that f is 1-Lipschitz and $\text{diam } f(S) \geq C^{-1} \text{diam } S$ for $S \in \mathcal{J}$. Then, from Lemma 10.1 with the single map f and $\eta = (1, 1, \dots)$, we deduce that no positive-measure measurable subset of (X, d, \mathcal{H}^s) is a $Y^\mathcal{U}$ -LDS, where s denotes the Hausdorff dimension of X . Since (X, d, \mathcal{H}^s) is a PI space, by [CK09, Theorem 1.5] we see that $Y^\mathcal{U}$ does not have RNP and therefore it is not reflexive [BL00, Corollary 5.12(ii)]. Finally, by [Hei80, Proposition 6.4], we conclude that Y is non-superreflexive. \square

12. LIPSCHITZ DIMENSION AND BI-LIPSCHITZ EMBEDDABILITY

The notion of Lipschitz dimension was introduced in [CK13], where Cheeger and Kleiner prove a structure theorem for metric spaces of Lipschitz dimension at most 1 (and, more generally, of Lipschitz light maps) and their biLipschitz embeddability into L^1 spaces. It was further studied in [Dav21, Dav24].

In this section we prove that any shortcut Laakso space (X, d_η) has topological, Nagata, and Lipschitz dimension 1 (Proposition 12.26). As a corollary, all such spaces admit a biLipschitz embedding in $L^1([0, 1])$, and a short argument yields the non-embeddability claimed in Theorem 1.1.

We begin focusing on Lipschitz dimension.

Definition 12.1. Given $r \in (0, \infty)$, a finite sequence (x_0, \dots, x_n) of points in X is an r -chain or r -path (from x_0 to x_n) if $d(x_{i-1}, x_i) \leq r$ for $1 \leq i \leq n$. Two points $x, y \in X$ are r -connected if there is an r -path from x to y . This defines an equivalence relation on X , whose equivalence classes are called r -components.

Definition 12.2. Let X, Y be metric spaces, $f: X \rightarrow Y$, and $C \in (0, \infty)$. We say that f is C -Lipschitz light if

- f is C -Lipschitz, and
- for every $E \subseteq Y$ and $r \geq \text{diam } E$, the r -components of $f^{-1}(E)$ have diameter at most Cr .

We say that f is Lipschitz light if it is C -Lipschitz light for some $C \in (0, \infty)$.

Remark 12.3. Definition 12.2 differs from [CK13, Definition 1.14], but is equivalent for maps into \mathbb{R}^n , $n \in \mathbb{N}$; see [Dav21, Section 1.3].

Remark 12.4. Let $f: X \rightarrow Y$ be Lipschitz. To see if f is Lipschitz light, it suffices to check if the second condition of Definition 12.2 is satisfied for closed sets $E \subseteq Y$ or, if Y is a normed space, for E closed and convex. In particular, if $Y = \mathbb{R}$, it is enough to consider compact intervals.

Definition 12.5 ([CK13, Definition 1.15]). The *Lipschitz dimension* of a metric space X is defined as

$$\dim_L X := \inf\{n \in \mathbb{N}_0 : \text{there exists a Lipschitz light map } f: X \rightarrow \mathbb{R}^n\}.$$

The following lemma will soon be useful.

Lemma 12.6. For $n \in \mathbb{N}$ and $C_1, \dots, C_n > 0$ there is a constant $C = C(C_1, \dots, C_n) > 0$ with the following property. Let X be a metric space, $A_1, \dots, A_n \subseteq X$ non-empty, and $r_0 > 0$. Suppose that for $1 \leq i \leq n$ and $r \geq r_0$, every r -component of A_i has diameter at most $C_i r$. Then, for $r \geq r_0$, every r -component of $\bigcup_{i=1}^n A_i$ has diameter at most Cr .

Proof. By induction on $n \in \mathbb{N}$, it is enough to consider the case $n = 2$. Let $r \geq r_0$ and $(x_t)_{t=0}^m = (x(t))_{t=0}^m$ be an r -path in $A_1 \cup A_2$; we need to show that $d(x_0, x_m) \leq C(C_1, C_2)r$. We may suppose w.l.o.g. $x_0 \in A_1$ and $\{x_t : 0 \leq t \leq m\} \cap A_2 \neq \emptyset$. Then, there is $k \in \mathbb{N}$ and integers $0 = t_0 < \dots < t_{k-1} < m =: t_k - 1$ such that, for $1 \leq i \leq k$, $\{x_t : t_{i-1} \leq t < t_i\} \subseteq A_1$ if i is odd and $\{x_t : t_{i-1} \leq t < t_i\} \subseteq A_2$ otherwise. That is, t_i is the time at which $(x_t)_t$ changes set. In particular, for $1 \leq i \leq k$, $(x_t : t_{i-1} \leq t < t_i)$ is an r -path fully contained in either A_1 or A_2 , thus

$$d(x(t_{i-1}), x(t_i - 1)) \leq C_0 r, \quad (12.7)$$

where $C_0 := \max(C_1, C_2)$. Therefore, for $0 \leq i < k/2 - 1$,

$$\begin{aligned} d(x(t_{2i}), x(t_{2i+2})) &\leq d(x(t_{2i}), x(t_{2i+1} - 1)) + d(x(t_{2i+1} - 1), x(t_{2i+1})) + d(x(t_{2i+1}), x(t_{2i+2} - 1)) \\ &\quad + d(x(t_{2i+2} - 1), x(t_{2i+2})) \leq 2(C_0 + 1)r; \end{aligned}$$

that is, $(x(t_{2i}) : 0 \leq i < k/2 - 1)$ is a $2(C_0 + 1)r$ -path in A_1 . Hence, setting $T := \max\{0 \leq i \leq k - 1 : i \in 2\mathbb{N}_0\}$, we deduce

$$d(x(t_0), x(t_T)) \leq 2C_0(C_0 + 1)r. \quad (12.8)$$

If k is odd, then $T = k - 1$, and by Eqs. (12.7) and (12.8), we have

$$d(x_0, x_m) = d(x(t_0), x(t_k - 1)) \leq d(x(t_0), x(t_{k-1})) + d(x(t_{k-1}), x(t_k - 1)) \leq C_0(2C_0 + 3)r.$$

If k is even, then $T = k - 2$, and we similarly obtain from Eqs. (12.7) and (12.8)

$$\begin{aligned} d(x_0, x_m) &\leq d(x(t_0), x(t_{k-2})) + d(x(t_{k-2}), x(t_{k-1} - 1)) + d(x(t_{k-1} - 1), x(t_{k-1})) \\ &\quad + d(x(t_{k-1}), x(t_k - 1)) \\ &\leq 2C_0(C_0 + 1)r + C_0 r + r + C_0 r = (2C_0^2 + 4C_0 + 1)r. \end{aligned}$$

□

Lemma 12.9. Let (X, d_η) be a shortcut Laakso space. For $j \in \mathbb{N}$ and $x, y \in X$ with $x(j) \neq y(j)$, we have

$$d_\eta(x, y) \geq \frac{1}{3} \min \{2d(h\{x, y\}, W_j^h), 2d(\{x, y\}, \cup \mathcal{J}_j) + \eta_j \delta_j\}.$$

Proof. If $d_\eta(x, y) = d(x, y)$, by Proposition 6.4 we have

$$d_\eta(x, y) \geq 2d(h\{x, y\}, W_j^h). \quad (12.10)$$

Suppose now $d_\eta(x, y) < d(x, y)$, $\epsilon > 0$, and let $i \in \mathbb{N}$, $p_- \neq p_+ \in S \in \mathcal{J}_i$ be such that $(3+\epsilon)d_\eta(x, y) \geq d(x, p_-) + \eta_i \delta_i + d(p_+, y)$; they exist by Proposition 3.11 and Proposition 6.25. If $i = j$, then

$$(3+\epsilon)d_\eta(x, y) \geq 2d(\{x, y\}, \cup \mathcal{J}_j) + \eta_j \delta_j. \quad (12.11)$$

If $i = j - 1$, then $h(p_\pm) \in \mathcal{J}_i^h \subseteq W_j^h$ and so

$$(3+\epsilon)d_\eta(x, y) \geq 2d(h\{x, y\}, W_j^h). \quad (12.12)$$

If $i \neq j - 1, j$, then $p_-(j) = p_+(j)$ and therefore either $x(j) \neq p_\pm(j)$ or $y(j) \neq p_\pm(j)$. Hence, by Proposition 6.4, we have

$$(3+\epsilon)d_\eta(x, y) \geq \max\{d(x, p_-), d(p_+, y)\} \geq 2d(h\{x, y\}, W_j^h) \quad (12.13)$$

Finally, combining Eqs. (12.10) to (12.13), we have the thesis. \square

Remark 12.14. Let $n \in \mathbb{N}$ and $I \in \mathcal{I}_n$. Then, for $1 \leq j \leq n$, $W_j^h \cap I = W_j^h \cap \partial I$, and moreover $W_n^h \cap I \neq \emptyset$.

Lemma 12.15. *Let X be a Laakso space. For $n \geq 2$ and $I \in \mathcal{I}_n$, it holds*

$$\#\{l \in [n-1]: \mathcal{J}_l^h \cap I \neq \emptyset\} \leq 1.$$

Proof. Let $\mathcal{L} \subseteq [n-1]$ be the set of the thesis, suppose $\mathcal{L} \neq \emptyset$, and set $l := \min \mathcal{L}$. If $l = n-1$, then $\#\mathcal{L} = 1$, so we may assume $n \geq 3$ and $1 \leq l \leq n-2$. Since $\mathcal{J}_l^h \subseteq W_{l+1}^h$ and $l+1 < n$, from Remark 12.14 we deduce that either $\mathcal{J}_{n-1}^h \cap \partial I = \emptyset$ or $\mathcal{L} = \{l\}$. Let $t \in I$ be such that $\mathcal{J}_l^h \cap I = \mathcal{J}_l^h \cap \partial I = \{t\}$. From the inclusion $\partial I \subseteq \{t - \frac{1}{N_1 \cdots N_n}, t, t + \frac{1}{N_1 \cdots N_n}\}$, it is then enough to show $t_\pm := t \pm \frac{1}{N_1 \cdots N_n} \notin \mathcal{J}_{n-1}^h$. Write $t = \sum_{i=1}^n \frac{t_i}{N_1 \cdots N_i}$ with $t_i \in \mathbb{N}_0$, $0 \leq t_i < N_i$, and recall that $t \in \mathcal{J}_l^h$ is equivalent to $1 \leq t_l < N_l - 1$, $t_{l+1} = N_{l+1}/2$, and $t_i = 0$ for $l+2 \leq i \leq n$. Define $t_{\pm, i}$ in the analogous way. Then $t_n = 0$ implies $t_{+, n} = 1 < N_n/2$, proving $t_+ \notin \mathcal{J}_{n-1}^h$. For t_- , we have

$$\begin{aligned} t_- &= \sum_{i=1}^l \frac{t_i}{N_1 \cdots N_i} + \frac{N_{l+1}/2 - 1}{N_1 \cdots N_{l+1}} + \frac{1}{N_1 \cdots N_{l+1}} - \frac{1}{N_1 \cdots N_n} \\ &= \sum_{i=1}^l \frac{t_i}{N_1 \cdots N_i} + \frac{N_{l+1}/2 - 1}{N_1 \cdots N_{l+1}} + \sum_{i=l+2}^n \frac{N_i - 1}{N_1 \cdots N_i} \end{aligned}$$

and so $t_{-, n} = N_n - 1 > N_n/2$, proving $t_- \notin \mathcal{J}_{n-1}^h$. \square

Lemma 12.16. *Let X be a Laakso space. Let $n \geq 2$, $I \in \mathcal{I}_n$, $l \in [n-1]$ and suppose $\mathcal{J}_l^h \cap I \neq \emptyset$. Then $W_l^h \cap I = \emptyset$.*

Proof. From Remark 12.14 we see that, if $n \geq 3$ and $l+1 < n$, then $\emptyset \neq \mathcal{J}_l^h \cap I \subseteq W_{l+1}^h \cap \partial I$ and so $\partial I \setminus \mathcal{J}_l^h \subseteq W_n^h$. Suppose $n \geq 2$, $l = n-1$, and let $t \in I$ be such that $\mathcal{J}_l^h \cap \partial I = \{t\}$. Arguing as in Lemma 12.15, it is not difficult to see that $t \pm \frac{1}{N_1 \cdots N_n} \notin W_{n-1}^h$. Since $\partial I \subseteq \{t - \frac{1}{N_1 \cdots N_n}, t, t + \frac{1}{N_1 \cdots N_n}\}$, this implies $\emptyset = W_{n-1}^h \cap \partial I = W_{n-1}^h \cap I$ and concludes the proof. \square

Notation 12.17. Let $X = X(M; (N_n))$ be a Laakso space and $\eta = (\eta_i)_i \subseteq (0, 1]$. For $n \in \mathbb{N}$, $n \geq 2$, and $I \in \mathcal{I}_n$, set

$$\begin{aligned} \mathcal{L}_W(I) &:= \{l \in [n-1]: W_l^h \cap I \neq \emptyset\}, \\ \mathcal{L}_{\mathcal{J}}(I, \eta) &:= \{l \in [n-1]: \mathcal{J}_l^h \cap I \neq \emptyset \text{ and } \eta_l \delta_l \leq |I|\}, \\ \mathcal{L}(I, \eta) &:= \mathcal{L}_W(I) \cup \mathcal{L}_{\mathcal{J}}(I, \eta), \end{aligned}$$

and recall that $\mathcal{L}_W(I), \mathcal{L}_{\mathcal{J}}(I, \eta)$ are disjoint and contain at most one element each; see Lemmas 12.15 and 12.16. For $a \in [M]^{n-1}$, let $A(I, \eta; a)$ denote the set of all $b \in [M]^{n-1}$ for which, setting

$$\Delta := \{1 \leq j \leq n-1: a(j) \neq b(j)\},$$

we have $\Delta \subseteq \mathcal{L}(I, \eta)$ and, if $\Delta \cap \mathcal{L}_{\mathcal{J}}(I, \eta) \neq \emptyset$, then

$$1 = a(j)(= b(j))$$

for $\min(\Delta \cap \mathcal{L}_{\mathcal{J}}(I, \eta)) < j \leq n-1$ with $j \notin \mathcal{L}_W(I)$. The set $A(I, \eta; a)$ is well-defined even if $\mathcal{L}(I, \eta) = \emptyset$, and it always contains a . Moreover, $\{(a, b) \in [M]^{n-1} \times [M]^{n-1} : a \in A(I, \eta; b)\}$ is an equivalence relation on $[M]^{n-1}$, therefore

$$\mathcal{A}_{\eta}(I) := \{A(I, \eta; a) : a \in [M]^{n-1}\}$$

is a partition of $[M]^{n-1}$. For $A \in \mathcal{A}_{\eta}(I)$, we define

$$E_A := \bigcup_{a \in A} Q_{I, a}.$$

Lemma 12.18. *Let (X, d_{η}) be a shortcut Laakso space. For $n \geq 2$, $I \in \mathcal{I}_n$, and $A_1 \neq A_2 \in \mathcal{A}_{\eta}(I)$, we have*

$$d_{\eta}(E_{A_1}, E_{A_2}) \geq \frac{1}{3}|I|.$$

Proof. Let $a_i \in A_i$, and define Δ as in Notation 12.17, but with a_1, a_2 in place of a, b . Since $A_1 \neq A_2$, we have $\Delta \neq \emptyset$ and $a_1 \notin A(I, \eta; a_2)$. Let $x_i \in Q_{I, a_i}$, $1 \leq i \leq 2$, and observe that by definition of Q_{I, a_i} , we have $x_i(j) = a_i(j)$ for $1 \leq j \leq n-1$. Suppose first $\Delta \not\subseteq \mathcal{L}(I, \eta)$, so that there is $1 \leq j \leq n-1$, $j \notin \mathcal{L}(I, \eta)$, such that $x_1(j) \neq x_2(j)$. Since $j \notin \mathcal{L}(I, \eta)$, $W_j^h \cap I = \emptyset$ and either $\eta_j \delta_j > |I|$ or $\mathcal{J}_j^h \cap I = \emptyset$. Then

$$\begin{aligned} d(h\{x, y\}, W_j^h) &\geq d(I, W_{\leq n}^h \setminus I) \geq |I|, \\ 2d(\{x, y\}, \cup \mathcal{J}_j) + \eta_j \delta_j &\geq 2d(I, \mathcal{J}_j^h) + \eta_j \delta_j \geq |I|, \end{aligned}$$

and, by Lemma 12.9, we have

$$d_{\eta}(x_1, x_2) \geq \frac{1}{3}|I|. \quad (12.19)$$

Suppose now $\Delta \subseteq \mathcal{L}(I, \eta)$, i.e. $x_1(j) = x_2(j)$ for $1 \leq j \leq n-1$ with $j \notin \mathcal{L}(I, \eta)$. Then $\Delta \cap \mathcal{L}_{\mathcal{J}}(I, \eta) \neq \emptyset$, $\Delta \cap \mathcal{L}_{\mathcal{J}}(I, \eta) = \{l\}$ for some $l \in [n-1]$ (by Lemma 12.15), and there is $l < j \leq n-1$ with $j \notin \mathcal{L}_W(I)$ and $x_1(j) = x_2(j) \neq 1$. Since $l \in \mathcal{L}_{\mathcal{J}}(I, \eta)$, by Lemma 12.16 we have $W_l^h \cap I = \emptyset$ and therefore $d(W_l^h, I) \geq |I|$. Since $l \in \Delta$ and $l \leq n-1$, we have $x_1(l) = a_1(l) \neq a_2(l) = x_2(l)$ and Lemma 12.9 gives

$$\begin{aligned} d_{\eta}(x_1, x_2) &\geq \frac{1}{3} \min\{2d(h\{x_1, x_2\}, W_l^h), 2d(\{x_1, x_2\}, \cup \mathcal{J}_l) + \eta_l \delta_l\} \\ &\geq \frac{2}{3} \min\{|I|, d(\{x_1, x_2\}, \cup \mathcal{J}_l)\} \end{aligned}$$

Let $x \in \{x_1, x_2\}$ and $y \in \cup \mathcal{J}_l$. The inequality $l < j$ implies $x(j) \neq 1 = y(j)$, and Proposition 6.4 gives

$$d(x, y) \geq d(h(x), W_j^h) \geq |I|,$$

because $I \cap W_j^h = \emptyset$. Hence

$$d_{\eta}(x_1, x_2) \geq \frac{2}{3}|I|. \quad (12.20)$$

Since $a_i \in A_i$ and $x_i \in Q_{I, a_i}$ were arbitrary, Eq. (12.19) and Eq. (12.20) conclude the proof. \square

Lemma 12.21. *Let (X, d_{η}) be a shortcut Laakso space. For $n \geq 2$, $I \in \mathcal{I}_n$, and $A \in \mathcal{A}_{\eta}(I)$, we have*

$$\text{diam}_{\eta}(E_A) \leq 5|I|.$$

Proof. Let $a_1, a_2 \in A$ and $x_i \in Q_{I, a_i}$ for $1 \leq i \leq 2$ and define Δ as in Notation 12.17, with a_1, a_2 in place of a, b . Recall that $W_l^h \cap I \neq \emptyset$ for $l \geq n$. Thus, if $\Delta \cap \mathcal{L}_{\mathcal{J}}(I, \eta) = \emptyset$, then $\Delta \subseteq \mathcal{L}_W(I)$ and so I contains all wormhole heights necessary to travel from x_1 to x_2 , which, by Proposition 6.4, gives

$$d_{\eta}(x_1, x_2) \leq d(x_1, x_2) \leq 2|I|.$$

Suppose now $\Delta \cap \mathcal{L}_{\mathcal{J}}(I, \eta) \neq \emptyset$; in particular $a_1 \neq a_2$. By Lemma 12.15, there is $1 \leq l \leq n-1$ such that $\Delta \cap \mathcal{L}_{\mathcal{J}}(I, \eta) = \{l\}$. Let $t \in \mathcal{J}_l^h \cap I$ and let $p_1, p_2 \in X$ be given by $h(p_1) = h(p_2) = t$,

$$p_i(j) = \begin{cases} x_i(j), & 1 \leq j \leq l \text{ and } j \notin \mathcal{L}_W(I) \\ 1, & l+2 \leq j < \infty \text{ or } j \in \mathcal{L}_W(I) \end{cases},$$

where we have not prescribed the $(l+1)$ -th digit because $h(p_i) \in \mathcal{J}_l^h \subseteq W_{l+1}^h$. Since $\Delta \subseteq \mathcal{L}(I, \eta)$ by assumption, it follows that p_1, p_2 agree at all digits except possibly the l -th one. Hence, there is $S \in \mathcal{J}_l$ for which $p_1, p_2 \in S$, and so $d_{\eta}(p_1, p_2) \leq \eta_l \delta_l \leq |I|$ by definition of $\mathcal{L}_{\mathcal{J}}(I, \eta)$. By definition of $\mathcal{A}_{\eta}(I)$, it follows that x_i, p_i may differ only at digits $j \geq n$ or $j \in \mathcal{L}_W(I)$. Thus, all wormhole heights necessary to travel from x_i to p_i are contained in I , and Proposition 6.4 gives

$$d_{\eta}(x_i, p_i) \leq d(x_i, p_i) \leq 2|I|.$$

The thesis then follows by triangle inequality. \square

Lemma 12.22. *Let (X, d_{η}) be a shortcut Laakso space. Then there is $C \geq 1$ such that for $n \in \mathbb{N}$, $I \in \mathcal{I}_n$, and $r \geq |I|$, the r -components of $h^{-1}(I)$ in (X, d_{η}) have diameter at most Cr w.r.t. d_{η} .*

The constant C depends only on $\max_n N_n$, where (N_n) is such that $X = X(M; (N_n))$ for some M .

Proof. By Lemma 12.18 and Lemma 12.21, there is a universal constant $C_1 \geq 1$ such that for $n \in \mathbb{N}$, $n \geq 2$, and $I \in \mathcal{I}_n$, the sets $\{E_A : A \in \mathcal{A}_{\eta}(I)\}$ are $C_1|I|$ -bounded and strictly $C_1^{-1}|I|$ -separated in (X, d_{η}) . Since

$$h^{-1}(I) = \bigcup_{A \in \mathcal{A}_{\eta}(I)} E_A,$$

the strict separation shows that any $C_1^{-1}|I|$ -path must be contained in a single set E_A , for some $A \in \mathcal{A}_{\eta}(I)$. Hence, the $C_1^{-1}|I|$ -components of $h^{-1}(I)$ in (X, d_{η}) have diameter at most $C_1|I|$ w.r.t. d_{η} .

Let $0 < r \leq C_1^{-1}\delta_2$, where $\delta_i = \frac{1}{N_1 \dots N_i}$ is as in Definition 6.17, and let $k \in \mathbb{N}$ be the greatest integer for which $r \leq C_1^{-1}\delta_k$. By maximality, we have

$$C_1^{-1}\delta_{k+1} < r \leq C_1^{-1}\delta_k \quad (12.23)$$

and $k \geq 2$. Let $I \in \mathcal{I}_n$, with $r \geq |I|$. Then $n \geq 2$ and there is $J \in \mathcal{I}_k$ with $I \subseteq J$. Since $h^{-1}(I) \subseteq h^{-1}(J)$, each r -component of $h^{-1}(I)$ is contained in an r -component of $h^{-1}(J)$, which in turn is contained in a $C_1^{-1}|J|$ -component of $h^{-1}(J)$, because $r \leq C_1^{-1}|J|$. This shows that every r -component of $h^{-1}(I)$ has diameter w.r.t. d_{η} at most

$$C_1|J| \leq C_1^2 N_{k+1} r \leq C_1^2 C_2 r,$$

where we have used Eq. (12.23) and set $C_2 := \max_n N_n$, which (exists and) is finite by Eq. (6.1). If $r > C_1^{-1}\delta_2$ and $I \in \mathcal{I}_n$, the r -components of $h^{-1}(I)$ have diameter at most $\text{diam}_{\eta} X \leq 2 \leq 2C_1\delta_2^{-1}r \leq 2C_1C_2^2r$. \square

Lemma 12.24. *Let (X, d_{η}) be a shortcut Laakso space. Then there is $C \geq 1$ such that the map $h : (X, d_{\eta}) \rightarrow \mathbb{R}$ is C -Lipschitz light.*

The constant C depends only on $\max_n N_n$, where (N_n) is such that $X = X(M; (N_n))$ for some M .

Proof. We already know that $\text{LIP}_{\eta}(h) = \text{LIP}(h) = 1$. Let $C_1 \geq 1$ be the constant of Lemma 12.22. It is not difficult to see that there is an integer $N \geq 2$, depending only on $\max_n N_n$, with the following property. For every interval $J \subseteq [0, 1]$ there are $n \in \mathbb{N}$ and $J_1, \dots, J_N \in \mathcal{I}_n$ satisfying

$$J \subseteq \bigcup_{i=1}^N J_i, \quad |J_i| \leq |J|$$

for $1 \leq i \leq N$. (We can take $N = \max_n N_n + 2$.) It is now easy to conclude. Let $J \subseteq [0, 1]$ be an interval, $r \geq |J|$, and J_1, \dots, J_N as above. Since $J_i \in \mathcal{I}_n$ and $r \geq |J| \geq |J_i|$, Lemma 12.22 shows that the r -components of $h^{-1}(J_i)$ have diameter at most $C_1 r$. By Lemma 12.6, there is a constant

$C \geq 1$, depending only on C_1 and N , such that the r -components of $h^{-1}(J)$ have diameter at most Cr . Hence, from Remark 12.4, h is C -Lipschitz light. Since C_1 and N depend only on $\max_n N_n$, so does C . \square

The notion of Nagata dimension was introduced in [Ass82], see also [LS05]. For our purposes, it is enough to know that

$$\dim_T Z \leq \dim_N Z \leq \dim_L Z, \quad (12.25)$$

where Z is a compact metric space and $\dim_T Z$, $\dim_N Z$ denote respectively its topological and Nagata dimension. See [LS05, Theorem 2.2] for the first inequality and [Dav21, Corollary 3.5] for the second.

Proposition 12.26. *Let (X, d_η) be a shortcut Laakso space. Then*

$$\dim_T(X, d_\eta) = \dim_N(X, d_\eta) = \dim_L(X, d_\eta) = 1.$$

Proof. By Lemma 12.24, we have $\dim_L(X, d_\eta) \leq 1$. Since (X, d_η) contains a homeomorphic copy of $[0, 1]$, we also have $\dim_T(X, d_\eta) \geq 1$. Finally, Eq. (12.25) concludes the proof. \square

As a corollary, we have the following.

Proposition 12.27. *Let (X, d_η) be a shortcut Laakso space. There are $C \geq 1$ and a map $f: X \rightarrow L^1([0, 1])$ satisfying*

$$d_\eta(x, y) \leq \|f(x) - f(y)\|_{L^1([0, 1])} \leq C d_\eta(x, y),$$

for $x, y \in X$.

The constant C depends only on $\max_n N_n$, where (N_n) is such that $X = X(M; (N_n))$ for some M .

Proof. Let $C_1 \geq 1$ be the constant of Lemma 12.24. By [CK13, Theorem 1.11] (with e.g. constant $m = 2$) and Lemma 12.24, (X, d_η) admits a biLipschitz embedding into an admissible inverse system [CK13, Definition 1.8], with biLipschitz constant depending only on C_1 . But then, by [CK13, Theorem 1.16], there are a measure space $(\Omega, \mathcal{F}, \nu)$ and a biLipschitz embedding $\tilde{f}: (X, d_\eta) \rightarrow L^1(\nu)$, with biLipschitz constant depending only on C_1 . Since the closed span Y of $\tilde{f}(X)$ is a separable subspace of $L^1(\nu)$, by [Ost13, Fact 1.20], there is a linear isometry $\iota: Y \rightarrow L^1([0, 1])$. Finally, setting $f := \iota \circ \tilde{f}$ concludes the proof. \square

We now turn to non-embeddability in Banach spaces with RNP. Note that, if η is chosen such that $(X, d_\eta, \mathcal{H}^s)$ is purely PI unrectifiable, we cannot appeal to the differentiability theory of Cheeger and Kleiner [CK09], at least not directly. Nonetheless, the controlled behaviour of the contraction η (Lemma 3.18) paired with a weak Sard theorem for Lipschitz functions on LDS ([Dav15, Theorem 1.4]) is sufficient to rule out DS-regular embeddability. Indeed, this is implied by the following more general result.

Proposition 12.28. *Let Y be a Banach space, (X, d, μ) an s -ADR Y -LDS with shortcuts, and $\eta = (\eta_i)_i \subseteq (0, 1]$. Then*

$$\mathcal{H}_Y^s(f(E)) = 0$$

for every μ -measurable set $E \subseteq X$ and Lipschitz $f: (E, d_\eta) \rightarrow Y$.

In particular, no positive-measure μ -measurable subset of (X, d_η) admits a David-Semmes regular embedding in Y .

Proof. First, let us verify that if $(U, \varphi: X \rightarrow \mathbb{R}^n)$ is a chart in (X, d, μ) with $\mu(U) > 0$, then $n < s$. To see this, suppose not. Then $n \in \mathbb{N}$, $n = s$ (by [Sch16a, Theorem 5.99] or [BKO23, Theorem 5.3]) and (U, d) is s -rectifiable by [BL17], but by Proposition 3.32 it has a Lipschitz image with positive measure and no biLipschitz piece, which is not possible by Kirchheim's differentiation theorem [Kir94] and $\mu(U) > 0$.

We now turn to the proof of the statement. It is enough to consider the case where $\mu(E) > 0$ and there is $n \in \mathbb{N}_0$ and $\varphi: (X, d) \rightarrow \mathbb{R}^n$ so that (E, φ) is a chart of (X, d, μ) . If $f: (E, d_\eta) \rightarrow Y$

is Lipschitz, then so is $f: (E, d) \rightarrow Y$, and [Dav15, Theorem 1.4] gives $\mathcal{H}^s(f(E)) = 0$, as claimed. Note that the proof of [Dav15, Theorem 1.4] generalises verbatim to our setting.

Finally, suppose $E \subseteq X$ is μ -measurable and $f: (E, d_\eta) \rightarrow Y$ is David-Semmes regular. Then, by Lemma 3.18, $f: (E, d) \rightarrow Y$ is also DS-regular and so (by Eq. (3.16)) we have

$$\mu(E) \sim \mathcal{H}_Y^s(f(E)) = 0,$$

concluding the proof. \square

APPENDIX A. METRIC SPACES WITH SHORTCUTS UNIFORMLY AND CARNOT GROUPS

A.1. Comparison with [LDLR17]. Our definition of metric spaces with shortcuts (Definition 3.1) is inspired by [LDLR17], where the following condition is introduced (without giving it a name).

Definition A.1. We say that a metric space X has *shortcuts δ -uniformly*, $\delta \in (0, 1)$, if for $x_0 \in X$ and $0 < r < \text{diam } X$ there is a δr -separated set $S \subseteq U(x_0, r)$ with $\#S \geq 2$ satisfying

$$d(x, y) \leq d(x, z) + d(w, y)$$

for $x, y \notin U(x_0, r)$ and $z, w \in S$.

Definition A.1 differs slightly from [LDLR17, Equation (1.1)] in that we allow for sets S with $\#S > 2$. As pointed out in Remark 3.3, this is only a minor change. Metric spaces as in Definition A.1 appeared also in [JR17], where they are called spaces with δ -invisible pieces.

Under mild assumptions, Definition A.1 implies the existence of shortcuts as in Definition 3.1 (see Lemma A.2), but Definition 3.1 allows for more flexibility, since the scales (δ_i) are allowed to decay arbitrarily quickly.

Recall that a metric space X is *C -uniformly perfect*, $C > 1$, if $U(x, r) \setminus U(x, r/C) \neq \emptyset$ for $x \in X$ and $0 < r < \text{diam } X$, and uniformly perfect if there is $C > 1$ as above. Examples of uniformly perfect metric spaces include connected and ADR metric spaces.

Lemma A.2. *Let X be a C -uniformly perfect metric space with shortcuts δ_0 -uniformly. Then X has shortcuts (Definition 3.1).*

More precisely, for $0 < \delta \leq \delta_0/16C$, $0 < A < \text{diam } X/4\delta$, and $M \geq 4C + 5$, X has shortcuts with parameters $\delta_i = A\delta^i$, $a = b = 2$, $a_0 = \delta_0$, and M as above.

Lemma A.2 follows arguing as in [LDLR17, Section 3.1]. We provide a complete proof since our setting differs slightly and we need to keep track of the several constants $(a_0, a, b, \text{ and } M)$.

Lemma A.3. *Let X be a C -uniformly perfect metric space, $0 < R < \text{diam } X$, and $\mathcal{N}_0 \subseteq X$ an R -separated set. Then, for $0 < r \leq R/4C$, any r -net \mathcal{N} of $X \setminus U(\mathcal{N}_0, r)$ is a $(C+1)r$ -net of X , i.e. $B(\mathcal{N}, (C+1)r) = X$.*

Proof. Since X is C -uniformly perfect, we have $U(x, \rho) \setminus U(y, t) \neq \emptyset$ for $x, y \in X$, $0 < \rho < \text{diam } X$, and $0 < t < \rho/C$. Let $Cr < \rho \leq R/2 - r$ and pick $x \in X$. Since \mathcal{N}_0 is R -separated, the choice of ρ implies that there is at most one $y \in \mathcal{N}_0$ such that $U(x, \rho) \cap U(y, r) \neq \emptyset$. Hence, there is $y_0 \in \mathcal{N}_0$ such that $U(x, \rho) \setminus U(\mathcal{N}_0, r) = U(x, \rho) \setminus U(y_0, r)$, which is then non-empty because $r < \rho/C$. Since $X \setminus U(\mathcal{N}_0, r) \subseteq B(\mathcal{N}, r)$, we have $d(x, \mathcal{N}) < \rho + r$ for every $x \in X$ and ρ as above, proving $B(\mathcal{N}, (C+1)r) = X$. \square

Remark A.4. The conclusion of Lemma A.3 characterises uniformly perfect metric spaces. Indeed, taking $x \in X$, $\mathcal{N}_0 = \{x\}$, $0 < R < \text{diam } X$, and $r = R/4C$, we find $y \in X$ with $r \leq d(x, y) < 4Cr$, i.e. $y \in U(x, R) \setminus U(x, R/4C)$.

Proof of Lemma A.2. We first construct \mathcal{J}_1 . Let $\mathcal{N}_1 \subseteq X$ be a maximal $4A\delta$ -separated set and, for $x \in \mathcal{N}_1$, let $S_x \subseteq U(x, A\delta)$ be as in Definition A.1 with $2 \leq \#S_x \leq M$. Set $\mathcal{J}_1 := \{S_x : x \in \mathcal{N}_1\}$. It is then not difficult to see that \mathcal{J}_1 satisfies the conditions of Definition 3.1 with the current values of a_0, a, b, M , and $\delta_1 = A\delta$.

Let $n \in \mathbb{N}$, $n \geq 2$, and suppose we have found $\mathcal{J}_1, \dots, \mathcal{J}_{n-1}$ satisfying the conditions of Definition 3.1 with a_0, a, b, M as above and $\delta_i = A\delta^i$ for $1 \leq i \leq n-1$. Set $\tilde{\mathcal{N}} := \bigcup_{i=1}^{n-1} \mathcal{J}_i$, let

$\mathcal{N}_n \subseteq X \setminus U(\tilde{\mathcal{N}}, 4A\delta^n)$ be a maximal $4A\delta^n$ -separated set, and, for $x \in \mathcal{N}_n$, let $S_x \subseteq U(x, A\delta^n)$ be as in Definition A.1 with $2 \leq \#S_x \leq M$. Set $\mathcal{J}_n := \{S_x : x \in \mathcal{N}_n\}$. Observe that $\tilde{\mathcal{N}}$ is $A\delta_0\delta^{n-1}$ -separated, $A\delta_0\delta^{n-1} < \text{diam } X$, $4A\delta^n \leq (A\delta_0\delta^{n-1})/4C$, and hence, by Lemma A.3, $\mathcal{N}_n \neq \emptyset$ and $B(\mathcal{N}_n, (C+1)4A\delta^n) = X$. Since $\mathcal{N}_n \subseteq U(\cup \mathcal{J}_n, A\delta^n)$, \mathcal{J}_n satisfies also the last condition of Definition 3.1 (with M as above). Lastly, it is not difficult to verify that $\{(\mathcal{J}_i, A\delta^i)\}_{i=1}^n$ satisfies the first four conditions of Definition 3.1 with $\delta_n = A\delta^n$. \square

Remark A.5. Let X be a metric space as Lemma A.2 and suppose there is $N \geq 2$ such that the sets S as in Definition A.1 may be taken with $\#S = N$. It follows from the proof of Lemma A.2 that we can then construct $\{(\mathcal{J}_i, A\delta^i)\}_{i \geq 1}$ as in the thesis (with $M \geq \max(N, 4C+5)$), which additionally satisfies $\#S = N$ for $S \in \mathcal{J} = \bigcup_i \mathcal{J}_i$.

A.2. Carnot groups have shortcuts. Carnot groups are a distinguished class of Lie groups, whose underlying smooth manifold may be identified with a finite dimensional Euclidean space. We refer the reader to [LD17, SC16, BLU07, Don24] for more information on the topic. From our point of view, Carnot groups are a class of ADR 1-PI spaces which can be equipped with a distance with shortcuts compatible with the Cheeger differentiable structure. We will recall the few basic facts we need.

We identify Carnot groups with their Lie algebra via exponential coordinates (see [BLU07, Proposition 2.2.22]) and write $\mathbb{G} = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}$, where \mathbb{R}^{m_i} corresponds to the i -th layer of the stratification. We denote points of $\mathbb{G} = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}$ as $x = (x^{(1)}, \dots, x^{(k)})$ and define $\Delta_\lambda : \mathbb{G} \rightarrow \mathbb{G}$, $\Delta_\lambda x := (\lambda x^{(1)}, \dots, \lambda^k x^{(k)})$, for $\lambda > 0$ and $x \in \mathbb{G}$. The dilations Δ_λ are group automorphisms; in particular, $\Delta_\lambda(x \cdot y) = \Delta_\lambda x \cdot \Delta_\lambda y$ and $\Delta_\lambda(x^{-1}) = (\Delta_\lambda x)^{-1}$, where \cdot denotes the group law. Also, the identity element $e_{\mathbb{G}}$ of \mathbb{G} satisfies $\Delta_\lambda e_{\mathbb{G}} = e_{\mathbb{G}}$ for $\lambda > 0$, proving $e_{\mathbb{G}} = 0 = (0, \dots, 0)$. A distance $d : \mathbb{G} \times \mathbb{G} \rightarrow [0, \infty)$ is *invariant* if it satisfies

$$d(z \cdot x, z \cdot y) = d(x, y), \quad d(\Delta_\lambda x, \Delta_\lambda y) = \lambda d(x, y),$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$. Note that we do not require d to induce the (Euclidean) topology of \mathbb{G} , since this follows from the definition, see [LDG21, Theorem 1.1]. Invariant distances are closely related to *homogeneous norms*, namely functions $N : \mathbb{G} \rightarrow [0, \infty)$ such that

$$\begin{aligned} N(x) = 0 &\text{ implies } x = 0, \\ N(x^{-1}) &= N(x), \\ N(\Delta_\lambda x) &= \lambda N(x), \\ N(x \cdot y) &\leq N(x) + N(y), \end{aligned} \tag{A.6}$$

for $x, y \in \mathbb{G}$ and $\lambda > 0$. Indeed, if d is an invariant distance, then $d(\cdot, 0)$ is a homogeneous norm and, conversely, if N is a homogeneous norm, then $d(x, y) := N(y^{-1} \cdot x)$ defines an invariant distance. In particular, homogeneous norms are continuous and one may verify that they are all equivalent, i.e. for any two homogeneous norms N_1, N_2 there is a constant $C \geq 1$ such that $C^{-1}N_1 \leq N_2 \leq CN_1$; see [BLU07, Proposition 5.1.4]⁷. It follows that all invariant distances are biLipschitz equivalent.

Proposition A.7. *Let $\mathbb{G} = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}$ be a Carnot group of step $k \geq 2$ and $\delta \in (0, 1/2)$. Then there is an invariant distance d on \mathbb{G} such that*

$$d(x, y) \leq d(x, z) + d(w, y), \tag{A.8}$$

for $x, y \notin U_d(0, 1)$ and $z, w \in B_d(0, \delta)$ with $z^{(1)} = w^{(1)}$.

By left-translation and dilation, Eq. (A.8) implies

$$d(x, y) \leq d(x, z) + d(w, y),$$

whenever there are $x_0 \in \mathbb{G}$, $r > 0$ such that $x, y \notin U_d(x_0, r)$ and $z, w \in B_d(x_0, \delta r)$ with $z^{(1)} = w^{(1)}$.

⁷Note that, following [SC16, Definition 2.6], our definition of homogeneous norm is more restrictive than [BLU07, Definition 5.1.1]. Also, we do not explicitly require continuity, since it follows from the other conditions; see [LDG21, Theorem 1.1].

The proof of Proposition A.7 relies on elementary estimates which already appeared in [FSSC03, Theorem 5.1], which we follow closely. They are based on a few properties of the group laws of Carnot groups, which we now recall; see [BLU07, Remark 1.4.4]. For any Carnot group $\mathbb{G} = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k}$, there are polynomial functions $\mathcal{Q}^{(i)}: \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}^{m_i}$ such that the group law may be written as

$$(x \cdot y)^{(i)} = x^{(i)} + y^{(i)} + \mathcal{Q}^{(i)}(x, y) \quad (\text{A.9})$$

and, moreover, $\mathcal{Q}^{(i)}(x, y)$ depends only on $x^{(1)}, \dots, x^{(i-1)}, y^{(1)}, \dots, y^{(i-1)}$; in particular, $\mathcal{Q}^{(1)} = 0$. From this, one may verify by induction that $(x^{-1})^{(i)} = -x^{(i)}$ for $1 \leq i \leq k$, i.e. $x^{-1} = -x = (-x^{(1)}, \dots, -x^{(k)})$.

Lemma A.10. *Let $\mathbb{G} = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k}$ be a Carnot group of step $k \geq 2$ and, for $1 \leq i \leq k$, let $\|\cdot\|_i$ be a norm on \mathbb{R}^{m_i} . Let $\mathcal{Q}^{(2)}, \dots, \mathcal{Q}^{(k)}$ be as in Eq. (A.9) and set $\|x\|_{\mathbb{G}, i} := \max_{1 \leq j \leq i} \|x^{(j)}\|_j^{\frac{1}{j}}$ for $x \in \mathbb{G}$ and $1 \leq i \leq k$. Then, there is a constant $C > 0$, such that for $2 \leq i \leq k$ and $x, y \in \mathbb{G}$, we have*

$$\|\mathcal{Q}^{(i)}(x, y)\|_i \leq C \max_{1 \leq j \leq i-1} \|x\|_{\mathbb{G}, i-1}^j \|y\|_{\mathbb{G}, i-1}^{i-j}.$$

Proof. This lemma is essentially extracted from the proof of [FSSC03, Theorem 5.1]. Set $h_0 := 0$, $h_i := \sum_{j=1}^i m_j$ for $1 \leq i \leq k$, and define $|\sigma|_{\mathbb{G}} := \sum_{i=1}^k \sum_{l=h_{i-1}+1}^{h_i} i \sigma_l$, for $\sigma = (\sigma_1, \dots, \sigma_{h_k}) \in \mathbb{N}_0^{h_k}$. Fix $2 \leq i \leq k$, $h_{i-1} + 1 \leq l \leq h_i$, and let \mathcal{Q}_l denote the $(l - h_{i-1})$ -th entry of $\mathcal{Q}^{(i)}$. Since $\Delta_\lambda: \mathbb{G} \rightarrow \mathbb{G}$ is a group homomorphism, Eq. (A.9) implies $\mathcal{Q}_l(\Delta_\lambda x, \Delta_\lambda y) = \lambda^i \mathcal{Q}_l(x, y)$, and therefore (see [BLU07, Proposition 1.3.4])

$$\mathcal{Q}_l(x, y) = \sum_{\substack{\sigma, \theta \in \mathbb{N}_0^{h_k}, \\ |\sigma|_{\mathbb{G}} + |\theta|_{\mathbb{G}} = i}} q_l(\sigma, \theta) x^\sigma y^\theta,$$

for some $q_l(\sigma, \theta) \in \mathbb{R}$. By definition of $|\cdot|_{\mathbb{G}}$ and $\mathcal{Q}_l(x, 0) = \mathcal{Q}_l(0, x) = 0$ (which follows from 0 being the identity element of \mathbb{G}), we deduce $q_l(\sigma, \theta) = 0$ if $\sigma = 0$ or $\theta = 0$ or $\sigma_t + \theta_t > 0$ for some $t > h_{i-1}$. Thus, $q_l(\sigma, \theta) \neq 0$ implies

$$\begin{aligned} |x^\sigma| &= |x_1^{\sigma_1} \dots x_{h_{i-1}}^{\sigma_{h_{i-1}}}| \leq \prod_{j=1}^{i-1} \prod_{t=h_{j-1}+1}^{h_j} |x_t|^{\frac{j\sigma_t}{j}} \lesssim \prod_{j=1}^{i-1} \prod_{t=h_{j-1}+1}^{h_j} \|x^{(j)}\|_j^{\frac{j\sigma_t}{j}} \\ &\leq \|x\|_{\mathbb{G}, i-1}^{\sum_{j=1}^{i-1} \sum_{t=h_{j-1}+1}^{h_j} j \sigma_t} = \|x\|_{\mathbb{G}, i-1}^{|\sigma|_{\mathbb{G}}} \end{aligned}$$

and similarly $|y^\theta| \lesssim \|y\|_{\mathbb{G}, i-1}^{|\theta|_{\mathbb{G}}}$. The thesis then follows from

$$|\mathcal{Q}_l(x, y)| \lesssim \sum_{\substack{\sigma, \theta \in \mathbb{N}_0^{h_k} \setminus \{0\}, \\ |\sigma|_{\mathbb{G}} + |\theta|_{\mathbb{G}} = i}} \|x\|_{\mathbb{G}, i-1}^{|\sigma|_{\mathbb{G}}} \|y\|_{\mathbb{G}, i-1}^{|\theta|_{\mathbb{G}}} \lesssim \sum_{j=1}^{i-1} \|x\|_{\mathbb{G}, i-1}^j \|y\|_{\mathbb{G}, i-1}^{i-j} \lesssim \max_{1 \leq j \leq i-1} \|x\|_{\mathbb{G}, i-1}^j \|y\|_{\mathbb{G}, i-1}^{i-j}.$$

□

Proof of Proposition A.7. The thesis is equivalent to the existence of a homogeneous norm N on \mathbb{G} satisfying

$$N(x \cdot y) \leq N(x \cdot z) + N(w \cdot y)$$

for $x, y, z, w \in \mathbb{G}$ with $z^{(1)} = -w^{(1)}$ and $N(z), N(w) \leq \delta < 1 \leq N(x), N(y)$. Fix norms $\|\cdot\|_1, \dots, \|\cdot\|_k$ on $\mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_k}$ respectively. For $\omega = (\omega_1, \dots, \omega_k) \in (0, \infty)^k$ and $1 \leq i \leq k$, set

$$N_{\omega, i}(x) := \max_{1 \leq j \leq i} \omega_j \|x^{(j)}\|_j^{\frac{1}{j}}, \quad N_\omega(x) := N_{\omega, k}(x), \quad (\text{A.11})$$

for $x = (x^{(1)}, \dots, x^{(k)}) \in \mathbb{G}$. It is clear that N_ω satisfies all conditions in Eq. (A.6), except possibly for triangle inequality. The proof of [FSSC03, Theorem 5.1] shows that for any $1 < i \leq k$ if $N_{\omega, i-1}$ satisfies triangle inequality, then so does $N_{\omega, i}$, provided ω_i is chosen small enough w.r.t.

$\omega_1, \dots, \omega_{i-1} > 0$. We claim that for $1 \leq i \leq k$ there are $\omega_1, \dots, \omega_i > 0$ such that $N_{\omega,i}$ satisfies triangle inequality and, if so does N_ω , then

$$N_{\omega,i}(x \cdot y) \leq N_\omega(x \cdot z) + N_\omega(w \cdot y), \quad (\text{A.12})$$

for x, y, z, w as in Eq. (A.11) with $N = N_\omega$. The thesis will then follow taking $i = k$ and setting $N := N_\omega$.

We proceed by induction on $1 \leq i \leq k$. If $i = 1$, then triangle inequality and Eq. (A.12) hold for any $\omega_1 > 0$. Suppose now $i > 1$ and let $\omega_1, \dots, \omega_{i-1} > 0$ be such that the claim holds with $i \equiv i-1$. Let $\omega_i > 0$ be small (to be determined depending on $\omega_1, \dots, \omega_{i-1}$), suppose N_ω satisfies triangle inequality, and let $x, y, z, w \in \mathbb{G}$ be as in Eq. (A.11) with $N = N_\omega$. Since $\|\cdot\|_i$ is a norm on \mathbb{R}^{m_i} , we have

$$\begin{aligned} \|x^{(i)} + y^{(i)} + \mathcal{Q}^{(i)}(x, y)\|_i &\leq \|x^{(i)} + z^{(i)} + \mathcal{Q}^{(i)}(x, z)\|_i + \|w^{(i)} + y^{(i)} + \mathcal{Q}^{(i)}(w, y)\|_i \\ &\quad + \|\mathcal{Q}^{(i)}(x, y)\|_i + \|\mathcal{Q}^{(i)}(x, z)\|_i + \|\mathcal{Q}^{(i)}(w, y)\|_i \\ &\quad + \|z^{(i)}\|_i + \|w^{(i)}\|_i. \end{aligned} \quad (\text{A.13})$$

The assumptions on x, y, z, w and N_ω imply

$$1 - \delta \leq (1 - \delta)N_\omega(x) \leq N_\omega(x \cdot z) \leq (1 + \delta)N_\omega(x)$$

and similarly for $N_\omega(w \cdot y)$; in particular,

$$\max(N_\omega(z), N_\omega(w)) \leq \delta \leq \frac{\delta}{1 - \delta} \min(N_\omega(x \cdot z), N_\omega(w \cdot y)). \quad (\text{A.14})$$

We then deduce from Lemma A.10

$$\begin{aligned} \|\mathcal{Q}^{(i)}(x, y)\|_i &\lesssim \max_{1 \leq j \leq i-1} \|x\|_{\mathbb{G}, i-1}^j \|y\|_{\mathbb{G}, i-1}^{i-j} \\ &\leq \min_{1 \leq j \leq i-1} \omega_j^{-i} \max_{1 \leq j \leq i-1} N_{\omega, i-1}(x)^j N_{\omega, i-1}(y)^{i-j} \\ &\lesssim \min_{1 \leq j \leq i-1} \omega_j^{-i} \max_{1 \leq j \leq i-1} N_\omega(x \cdot z)^j N_\omega(w \cdot y)^{i-j}, \\ \|\mathcal{Q}^{(i)}(x, z)\|_i &\lesssim \min_{1 \leq j \leq i-1} \omega_j^{-i} \max_{1 \leq j \leq i-1} N_\omega(x \cdot z)^j N_\omega(w \cdot y)^{i-j}, \\ \|\mathcal{Q}^{(i)}(w, y)\|_i &\lesssim \min_{1 \leq j \leq i-1} \omega_j^{-i} \max_{1 \leq j \leq i-1} N_\omega(x \cdot z)^j N_\omega(w \cdot y)^{i-j}. \end{aligned} \quad (\text{A.15})$$

Eq. (A.14) also implies

$$\begin{aligned} \|z^{(i)}\|_i + \|w^{(i)}\|_i &\leq \omega_i^{-i} (N_\omega(z)^i + N_\omega(w)^i) \leq \frac{2\delta^i}{\omega_i^i (1 - \delta)^i} \min(N_\omega(x \cdot z), N_\omega(w \cdot y))^i \\ &\leq \frac{2\delta^i}{\omega_i^i (1 - \delta)^i} \max_{1 \leq j \leq i-1} N_\omega(x \cdot z)^j N_\omega(w \cdot y)^{i-j} \end{aligned}$$

which, together with Eqs. (A.9), (A.13) and (A.15), gives

$$\begin{aligned} \omega_i^i \|(x \cdot y)^{(i)}\|_i &\leq \omega_i^i \|(x \cdot z)^{(i)}\|_i + \omega_i^i \|(w \cdot y)^{(i)}\|_i \\ &\quad + \left[\frac{C\omega_i^i}{\min_{1 \leq j \leq i-1} \omega_j^i} + \frac{2\delta^i}{(1 - \delta)^i} \right] \max_{1 \leq j \leq i-1} N_\omega(x \cdot z)^j N_\omega(w \cdot y)^{i-j}, \end{aligned} \quad (\text{A.16})$$

for some constant $C > 0$ independent of ω . Since $\delta \in (0, 1/2)$, it holds $\delta(1 - \delta)^{-1} < 1$ and, if $\omega_i > 0$ was chosen small enough w.r.t. $\omega_1, \dots, \omega_{i-1}$, we have

$$\frac{C\omega_i^i}{\min_{1 \leq j \leq i-1} \omega_j^i} + \frac{2\delta^i}{(1 - \delta)^i} \leq 2;$$

by (the proof of) [FSSC03, Theorem 5.1]⁸, we may take $\omega_i > 0$ so small that, in addition, $N_{\omega,i}$ satisfies triangle inequality. Since $\min_{1 \leq j \leq i-1} \binom{i}{j} = \binom{i}{\lfloor i/2 \rfloor} \geq 2^{\lfloor i/2 \rfloor} \geq 2$, Eq. (A.16) yields

$$\omega_i^j \|(x \cdot y)^{(i)}\|_i \leq \sum_{j=0}^i \binom{i}{j} N_{\omega}(x \cdot z)^j N_{\omega}(w \cdot y)^{i-j} = (N_{\omega}(x \cdot z) + N_{\omega}(w \cdot y))^i.$$

Hence, the induction hypothesis and the definition of $N_{\omega,i}$ imply that Eq. (A.12) holds, concluding the proof of the claim. \square

Remark A.17. We do not know what is the optimal value of δ in Proposition A.7; nonetheless, one cannot do better than $\frac{1}{\sqrt{2}}$. To see this, let $\mathbb{G} = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k}$ be a Carnot group of step $k \geq 2$, d an invariant distance on \mathbb{G} , $\delta > 0$, and suppose they satisfy the conclusion of Proposition A.7. Let $m_1 + 1 \leq l \leq m_1 + m_2$, $u = (0, \dots, 0, u_l, 0, \dots, 0) \in \mathbb{G}$ with $d(u, 0) = 1$, and set $x := y^{-1} := u$, $z := w^{-1} := \Delta_{\delta} u$, so that $d(x, 0) = d(y, 0) = 1$ and $d(z, 0) = d(w, 0) = \delta$. Observe that $\mathcal{Q}^{(i)}(u, u) = \mathcal{Q}^{(i)}(-\Delta_{\delta} u, u) = \mathcal{Q}^{(i)}(u, -\Delta_{\delta} u) = 0$ for $1 \leq i \leq k$; see [BLU07, Proposition 2.22(4)]. It follows that $y^{-1} \cdot x = \Delta_{\sqrt{2}} u$, $z^{-1} \cdot x = y^{-1} \cdot w = \Delta_{\sqrt{1-\delta^2}} u$ and therefore

$$\sqrt{2} = d(x, y) \leq d(x, z) + d(w, y) = 2\sqrt{1-\delta^2},$$

which implies $\delta \leq \frac{1}{\sqrt{2}}$.

Consider a Carnot group $\mathbb{G} = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k}$ of step k , let $s = \sum_{i=1}^k i m_i$ denote its *homogeneous dimension*, and let d be an invariant distance on \mathbb{G} . Then there is a constant $c(d) > 0$ such that

$$\mathcal{H}_d^s(B_d(x, r)) = c(d)r^s$$

$x \in \mathbb{G}$, and $0 < r < \infty$; see e.g. [BLU07, Remark 1.4.5] (and Remark 2.42). In particular, if d is chosen as in Proposition A.7, then $(\mathbb{G}, d, \mathcal{H}_d^s)$ is an s -ADR metric measure space with shortcuts (by Lemma A.2). Since it is also a 1-PI space (see e.g. [SC16, Theorem 3.25]), by [CK09] we deduce that it is a Y -LDS for any Banach space Y with RNP. From Pansu's differentiation theorem with \mathbb{R} as target (see [Pan89] or [Don24, Theorem 10.3.2]), we see that $(\mathbb{G}, d, \mathcal{H}_d^s)$ is an LDS with single chart $(\mathbb{G}, \varphi: \mathbb{G} \rightarrow \mathbb{R}^{m_1})$, $\varphi(x) = x^{(1)}$, for $x = (x^{(1)}, \dots, x^{(k)}) \in \mathbb{G}$. In particular, the Cheeger differentiable structure is compatible with the shortcuts.

Hence, we have the following.

Corollary A.18. *Let $\mathbb{G} = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_k}$ be a Carnot group of step $k \geq 2$, homogeneous dimension s , and let d be an invariant distance as in Proposition A.7.*

Then $(\mathbb{G}, d, \mathcal{H}_d^s)$ is an s -ADR 1-PI space with compatible shortcuts. Hence, for $\eta = (\eta_i) \subseteq (0, 1]$, the following hold:

- $(\mathbb{G}, d_{\eta}, \mathcal{H}_d^s)$ is PI rectifiable if there is $I \subseteq \mathbb{N}$ such that $\sum_{i \in I} \eta_i^s < \infty$ and $\inf\{\eta_i: i \notin I\} > 0$, and purely PI unrectifiable otherwise;
- for any non-zero Banach space Y with RNP, $(\mathbb{G}, d_{\eta}, \mathcal{H}_d^s)$ is a Y -LDS if and only if $\mathcal{H}_d^s(\text{Bad}(f)) = 0$ for every Lipschitz map $f: (\mathbb{G}, d_{\eta}) \rightarrow Y$;
- for any Banach space Y with RNP and Lipschitz map $f: (\mathbb{G}, d_{\eta}) \rightarrow Y$, no positive-measure \mathcal{H}_d^s -measurable subset of $\text{Bad}(f)$ is a Y -LDS.

Proof. The first item follows from Theorem 4.3 and Theorem 4.1, while the second and third from (Lemma 2.11 and) Proposition 5.7 and Proposition 5.8, respectively. \square

We do not know if $\mathcal{H}_d^s(\text{Bad}(f)) = 0$ for Lipschitz $f: (\mathbb{G}, d_{\eta}) \rightarrow \mathbb{R}$ (let alone for $f: (\mathbb{G}, d_{\eta}) \rightarrow Y$ and $Y \neq 0$ Banach space with RNP). This question is related to the ‘Vertical vs Horizontal’ Poincaré inequalities in Carnot groups, see [ANT13, Ryo25], but does not seem to be directly implied by such results.

Nonetheless, we have the following application.

⁸In fact, the proof of [FSSC03, Theorem 5.1] consists of the the same argument exhibited here, but with $z = w = 0$ and triangle inequality of $N_{\omega,i}$ as induction hypothesis.

Corollary A.19. *Let $\mathbb{G} = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k}$ be a Carnot group of step $k \geq 2$, $d_{\mathbb{G}}$ an invariant distance on \mathbb{G} , and let s denote the homogeneous dimension of \mathbb{G} .*

Then there is a complete s -ADR purely PI unrectifiable metric measure space $(X, d_X, \mathcal{H}_X^s)$ and a David-Semmes regular homeomorphism $f: (\mathbb{G}, d_{\mathbb{G}}) \rightarrow (X, d_X)$. In particular, f is biLipschitz on no positive-measure $\mathcal{H}_{d_{\mathbb{G}}}^s$ -measurable set.

Proof. Follows from Corollary A.18, Lemmas 3.13, 3.14 and 3.18, and the fact that \mathbb{G} is a PI space (or by Proposition 3.32). \square

Corollary A.19 provides a complete negative answer to [Sem01, Conjecture 5.2] and one of its variants [Sem01, Section 5.2]. When \mathbb{G} is the first Heisenberg group \mathbb{H}^1 , this is due to [LDLR17]. In the recent [LS25], the authors provide a close-to-optimal answer to the question of when Lipschitz (or DS-regular) maps have biLipschitz pieces. In particular, [LS25] also resolves the above questions of Semmes.

REFERENCES

- [ANT13] Tim Austin, Assaf Naor, and Romain Tessera. Sharp quantitative nonembeddability of the Heisenberg group into superreflexive Banach spaces. *Groups Geom. Dyn.*, 7(3):497–522, 2013.
- [Ass82] Patrice Assouad. Sur la distance de Nagata. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(1):31–34, 1982.
- [Bat15] David Bate. Structure of measures in Lipschitz differentiability spaces. *Journal of the American Mathematical Society*, 28(2):421–482, 2015.
- [Bau07] Florent Baudier. Metrical characterization of super-reflexivity and linear type of Banach spaces. *Arch. Math. (Basel)*, 89(5):419–429, 2007.
- [BB11] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.
- [BBL17] Anders Björn, Jana Björn, and Juha Lehrbäck. Sharp capacity estimates for annuli in weighted \mathbb{R}^n and in metric spaces. *Mathematische Zeitschrift*, page 1173–1215, 2017.
- [BEBS24] David Bate, Sylvester Eriksson-Bique, and Eleftherios Soultanis. Fragment-wise differentiable structures, 2024.
- [BKO23] David Bate, Ilmari Kangasniemi, and Tuomas Orponen. Cheeger’s differentiation theorem via the multilinear Kakeya inequality. *Pure Appl. Funct. Anal.*, 8(6):1587–1602, 2023.
- [BL00] Yoav Benyamini and Joram Lindenstrauss. *Geometric nonlinear functional analysis. Vol. 1*, volume 48 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2000.
- [BL17] David Bate and Sean Li. Characterizations of rectifiable metric measure spaces. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(1):1–37, 2017.
- [BL18] David Bate and Sean Li. Differentiability and Poincaré-type inequalities in metric measure spaces. *Advances in Mathematics*, 333:868–930, 2018.
- [BLU07] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [Bou86] J. Bourgain. The metrical interpretation of superreflexivity in Banach spaces. *Israel J. Math.*, 56(2):222–230, 1986.
- [BS75] Antoine Brunel and Louis Sucheston. On J -convexity and some ergodic super-properties of Banach spaces. *Trans. Amer. Math. Soc.*, 204:79–90, 1975.
- [BS13] David Bate and Gareth Speight. Differentiability, porosity and doubling in metric measure spaces. *Proceedings of the American Mathematical Society*, 141(3):971–985, 2013.
- [Cap25] Emanuele Caputo. Geometric characterizations of PI spaces: An overview of some modern techniques. *Indagationes Mathematicae*, 2025.
- [CC25a] Emanuele Caputo and Nicola Cavallucci. A geometric approach to Poincaré inequality and Minkowski content of separating sets. *Int. Math. Res. Not. IMRN*, (1):Paper No. rnae276, 30, 2025.
- [CC25b] Emanuele Caputo and Nicola Cavallucci. Poincaré inequality and energy of separating sets. *Adv. Calc. Var.*, 18(3):915–942, 2025.
- [CEB23] Jeff Cheeger and Sylvester Eriksson-Bique. Thin Loewner carpets and their quasisymmetric embeddings in S^2 . *Comm. Pure Appl. Math.*, 76(2):225–304, 2023.
- [Che99] Jeff Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geometric and Functional Analysis*, 9:428–517, 1999.
- [CK09] Jeff Cheeger and Bruce Kleiner. Differentiability of Lipschitz maps from metric measure spaces to Banach spaces with the Radon-Nikodým property. *Geom. Funct. Anal.*, 19(4):1017–1028, 2009.
- [CK13] Jeff Cheeger and Bruce Kleiner. Realization of metric spaces as inverse limits, and bilipschitz embedding in L_1 . *Geom. Funct. Anal.*, 23(1):96–133, 2013.

- [CK15] Jeff Cheeger and Bruce Kleiner. Inverse limit spaces satisfying a Poincaré inequality. *Anal. Geom. Metr. Spaces*, 3(1):15–39, 2015.
- [CKN11] Jeff Cheeger, Bruce Kleiner, and Assaf Naor. Compression bounds for Lipschitz maps from the Heisenberg group to L_1 . *Acta Math.*, 207(2):291–373, 2011.
- [CKS16] Jeff Cheeger, Bruce Kleiner, and Andrea Schioppa. Infinitesimal structure of differentiability spaces, and metric differentiation. *Anal. Geom. Metr. Spaces*, 4(1):104–159, 2016.
- [CPS25] Marco Capolli, Andrea Pinamonti, and Gareth Speight. Maximal directional derivatives in Laakso space. *Communications in Contemporary Mathematics*, 27(04):2450017, 2025.
- [Dav15] Guy C. David. Tangents and rectifiability of Ahlfors regular Lipschitz differentiability spaces. *Geom. Funct. Anal.*, 25(2):553–579, 2015.
- [Dav21] Guy C. David. On the Lipschitz dimension of Cheeger-Kleiner. *Fund. Math.*, 253(3):317–358, 2021.
- [Dav24] Guy C. David. A non-injective Assouad-type theorem with sharp dimension. *J. Geom. Anal.*, 34(2):Paper No. 45, 19, 2024.
- [Die25] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, sixth edition, [2025] ©2025.
- [Don24] Enrico Le Donne. Metric Lie groups. Carnot-Carathéodory spaces from the homogeneous viewpoint, 2024.
- [Dor85] José R. Dorronsoro. A characterization of potential spaces. *Proc. Amer. Math. Soc.*, 95(1):21–31, 1985.
- [EB19a] Sylvester Eriksson-Bique. Alternative proof of Keith-Zhong self-improvement and connectivity. *Ann. Acad. Sci. Fenn. Math.*, 44(1):407–425, 2019.
- [EB19b] Sylvester Eriksson-Bique. Characterizing spaces satisfying Poincaré inequalities and applications to differentiability. *Geometric and Functional Analysis*, 29:119–189, 2019.
- [EBG21] Sylvester Eriksson-Bique and Jasun Gong. Almost uniform domains and Poincaré inequalities. *Trans. London Math. Soc.*, 8(1):243–298, 2021.
- [Fed69] Herbert Federer. *Geometric measure theory*, volume Band 153 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag New York, Inc., New York, 1969.
- [FL07] Irene Fonseca and Giovanni Leoni. *Modern methods in the calculus of variations: L^p spaces*. Springer Monographs in Mathematics. Springer, New York, 2007.
- [FO20] Katrin Fässler and Tuomas Orponen. Dorronsoro’s theorem in Heisenberg groups. *Bull. Lond. Math. Soc.*, 52(3):472–488, 2020.
- [FSSC03] Bruno Franchi, Raul Serapioni, and Francesco Serra Cassano. On the structure of finite perimeter sets in step 2 Carnot groups. *J. Geom. Anal.*, 13(3):421–466, 2003.
- [Hei80] Stefan Heinrich. Ultraproducts in Banach space theory. *J. Reine Angew. Math.*, 313:72–104, 1980.
- [Hei07] Juha Heinonen. Nonsmooth calculus. *Bull. Amer. Math. Soc. (N.S.)*, 44(2):163–232, 2007.
- [HK98] Juha Heinonen and Pekka Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998.
- [HKST15] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy T. Tyson. *Sobolev spaces on metric measure spaces*, volume 27 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015. An approach based on upper gradients.
- [IP00] D. J. Ives and D. Preiss. Not too well differentiable Lipschitz isomorphisms. *Israel J. Math.*, 115:343–353, 2000.
- [JR17] Matthieu Joseph and Tapio Rajala. Products of snowflaked Euclidean lines are not minimal for looking down. *Anal. Geom. Metr. Spaces*, 5(1):78–97, 2017.
- [Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [Kei03] Stephen Keith. Modulus and the Poincaré inequality on metric measure spaces. *Math. Z.*, 245(2):255–292, 2003.
- [Kei04] Stephen Keith. A differentiable structure for metric measure spaces. *Adv. Math.*, 183(2):271–315, 2004.
- [Kir94] Bernd Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.*, 121(1):113–123, 1994.
- [Kri75] Jean-Louis Krivine. Représentation finie des l^p dans les espaces de Banach réticulés. *C. R. Acad. Sci. Paris Sér. A-B*, 280:A1061–A1062, 1975.
- [KZ08] Stephen Keith and Xiao Zhong. The Poincaré inequality is an open ended condition. *Ann. of Math. (2)*, 167(2):575–599, 2008.
- [Laa00] T.J. Laakso. Ahlfors Q -regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geometric & Functional Analysis GAFA*, 10:111–123, 2000.
- [Laa02] Tomi J. Laakso. Plane with A_∞ -weighted metric not bi-Lipschitz embeddable to \mathbb{R}^N . *Bull. London Math. Soc.*, 34(6):667–676, 2002.
- [LD17] Enrico Le Donne. A primer on Carnot groups: homogenous groups, Carnot-Carathéodory spaces, and regularity of their isometries. *Anal. Geom. Metr. Spaces*, 5(1):116–137, 2017.
- [LDG21] Enrico Le Donne and Sebastiano Nicolussi Golo. Metric Lie groups admitting dilations. *Ark. Mat.*, 59(1):125–163, 2021.

- [LDLR17] Enrico Le Donne, Sean Li, and Tapio Rajala. Ahlfors-regular distances on the Heisenberg group without biLipschitz pieces. *Proceedings of the London Mathematical Society*, 115(2):348–380, 2017.
- [LN06] James R Lee and Assaf Naor. l^p metrics on the heisenberg group and the goemans-linial conjecture. In *FOCS*, pages 99–108, 2006.
- [LP01] Urs Lang and Conrad Plaut. Bilipschitz embeddings of metric spaces into space forms. *Analysis and Geometry in Metric Spaces*, 87:285–307, 2001.
- [LS98] Jouni Luukkainen and Eero Saksman. Every complete doubling metric space carries a doubling measure. *Proc. Amer. Math. Soc.*, 126(2):531–534, 1998.
- [LS05] Urs Lang and Thilo Schlichenmaier. Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions. *Int. Math. Res. Not.*, (58):3625–3655, 2005.
- [LS25] Sean Li and Raanan Schul. Characterizing rectifiability via bilipschitz pieces of lipschitz mappings on the space, 2025.
- [MMPZ03] M. E. Mera, M. Morán, D. Preiss, and L. Zajíček. Porosity, σ -porosity and measures. *Nonlinearity*, 16(1):247–255, 2003.
- [MS86] Vitali D. Milman and Gideon Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.
- [OR17] Mikhail I. Ostrovskii and Beata Randrianantoanina. A new approach to low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces. *J. Funct. Anal.*, 273(2):598–651, 2017.
- [Ost13] Mikhail I. Ostrovskii. *Metric embeddings*, volume 49 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2013. Bilipschitz and coarse embeddings into Banach spaces.
- [Pan89] Pierre Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.
- [Pet02] Valentin V. Petrov. A note on the Borel–Cantelli lemma. *Statistics & Probability Letters*, 58(3):283–286, 2002.
- [Pis16] Gilles Pisier. *Martingales in Banach Spaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.
- [Ryo25] Seung-Yeon Ryoo. Quantitative nonembeddability of groups of polynomial growth into uniformly convex spaces, 2025.
- [SC16] Francesco Serra Cassano. Some topics of geometric measure theory in Carnot groups. In *Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. 1*, EMS Ser. Lect. Math., pages 1–121. Eur. Math. Soc., Zürich, 2016.
- [Sch16a] Andrea Schioppa. Derivations and Alberti representations. *Advances in Mathematics*, 293:436–528, 2016.
- [Sch16b] Andrea Schioppa. An example of a differentiability space which is PI-unrectifiable, 2016.
- [Sem01] Stephen Semmes. *Some novel types of fractal geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.
- [Swi18] Andrew Swift. A coding of bundle graphs and their embeddings into Banach spaces. *Mathematika*, 64(3):847–874, 2018.
- [Wea00] Nik Weaver. Lipschitz algebras and derivations II. exterior differentiation. *Journal of Functional Analysis*, 178(1):64–112, 2000.
- [Yan06] Jia-An Yan. *A Simple Proof of Two Generalized Borel-Cantelli Lemmas*, pages 77–79. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.