# SIR and mask

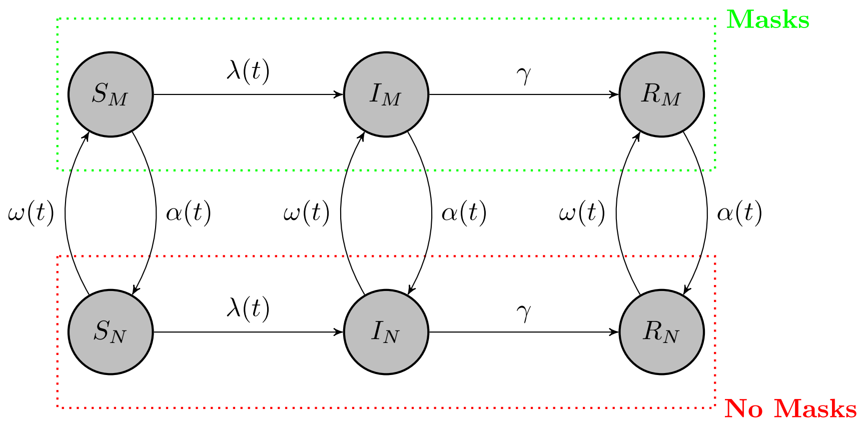
## Basic idea

- Simple SIR model disease progression with strata for mask-wearing status, resulting in compartments for S-masked, S-no mask, I-masked, I-no mask, R-masked, R-no mask.
- Supposing wearing a mask reduces both susceptibility and infectiousness (depending on who is wearing it)
- Assume transitions between masked-not masked based on social influence and fear of disease (noting FD type "infection" structure for fear of disease, when could actually be more about absolute number/density dependence – interesting question to explore how structure affects dynamics)
- Work in proportions such that $S + I + R = 1$. Similarly, $M + N = 1$.
- I am thinking of this as two distinct (but coupled) models on the same population.
- no demography
- Han2021 - Effects of masks on the transmission of infectious diseases. Similar approach
- Del Valle 2012 - Modeling the Impact of Behavior Changes on the Spread of Pandemic Influenza: They do essentially the same model as I do, however they suggest a simpler transition for behaviour based off of a simple step function. They also investigate a large scale agent-based model (approx 20mil agents I think).

## Where to go

- My model acts as an extension of Perra 2013 and De Valle 2012. Their models are recoverable by changing the $\alpha$ and $\omega$ parameters.
- Social exhaustion is also something I might consider.
- customise $\alpha$ and $\omega$ to each SIR class
- Can compare current "fear + social" model, TPB model, and HBM. Make for a long, but comparative piece.
- Incorporate behaviour burn out?
- Incorporate sub-classes with "always behaving" in certain way
- include reactance to behaviours, i.e. when it gets too much people blow up and refuse
- Include burn-out of behaviour, first steps towards a constant exit rate from behaviour.

## State diagram




## Parameter description

- $\lambda$ is the force of infection, assuming a form of FD
- $\frac{1}{\gamma}$ is average infectious period
- $\alpha$ is transition rate from wearing a mask to not wearing a mask
- $\omega$ is transition rate from not wearing a mask to wearing a mask
- $\nu$ is the waning immunity term

Further parameters that will pop up

- $\beta$ is the transmission rate
- $c$ is the reduction in chance of catching disease if S is wearing a mask # this description is inconsistent with code comment definition and `1-c` model structure (I think)
- $p$ is the reduction in chance of spreading disease if I is wearing a mask # this description is inconsistent with code comment definition and `1-p` model structure (I think)
- $\alpha_1$ is the rate of transition to no mask based on social influence of no mask
- $\alpha_2$ is the rate of transition to no mask based on fear of disease
- $\omega_1$ is the rate of transition to mask based on social influence of mask
- $\omega_2$ is the rate of transition to no mask based on fear of disease

## Basic system of ODEs

$$\begin{aligned}
\dot{S}_M &= -\lambda(t)(1-c)S_M - \alpha(t)S_M + \omega(t)S_N + \nu R_M \\
\dot{S}_N &= -\lambda(t)S_N + \alpha(t)S_M - \omega(t)S_N + \nu R_N \\
\dot{I}_M &= \lambda(t)(1-c)S_M - \alpha(t)I_M + \omega(t)I_N - \gamma I_M \\
\dot{I}_N &= \lambda(t)S_N + \alpha(t)I_M - \omega(t)I_N - \gamma I_N \\
\dot{R}_M &= \gamma I_M - \alpha(t)R_M + \omega(t)R_N - \nu R_M \\
\dot{R}_N &= \gamma I_N + \alpha(t)R_M - \omega(t)R_N - \nu R_N \,,
\end{aligned}$$

where

$$\begin{aligned}
\lambda(t) &= \beta\left(I_N + (1-p)I_M\right) \\
\omega(t) &= \omega_1(S_M + I_M + R_M) + \omega_2(I_M + I_N) + \omega_3 \\
\alpha(t) &= \alpha_1(S_N + I_N + R_N) + \alpha_2(1 - (I_M + I_N)) + \alpha_3 \,.
\end{aligned}$$

## Disease-free equilibrium of the behaviour

**Case when $\omega_3 + \alpha_2 + \alpha_3 = 0$ and case when those equal 0 and $\omega_2 \neq 0$.**

The disease-free equilibrium of the behaviour $(M^*, N^*)$ is

$$(1 - \Delta, \Delta)$$

where

$$\Delta = \begin{cases} \frac{(\omega_1-\alpha_1+\omega_3+\alpha_2+\alpha_3)-\sqrt{(\omega_1-\alpha_1+\omega_3+\alpha_2+\alpha_3)^2-4(\omega_1-\alpha_1)(\alpha_2+\alpha_3)}}{2(\omega_1-\alpha_1)} & \text{if } \omega_1 - \alpha_1 \neq 0\,, \\ \frac{\alpha_2+\alpha_3}{\omega_3+\alpha_2+\alpha_3} & \text{else.} \end{cases}$$

**Proof**:

Considering $N = S_N + I_N + R_N$, we can write

$$\dot{N} = -\omega(t)N + \alpha(t)M\,.$$

In the disease-free equilibrium, $I_N = I_M = 0$, so this reduces to

$$\dot{N} = -(\omega_1 M + \omega_3)N + (\alpha_1 N + (\alpha_2 + \alpha_3))M\,.$$

Using $M + N = 1$, we can reduce this to

$$\dot{N} = cN^2 - (c+k)N + (k - \omega_3)$$

where $c = (\omega_1 - \alpha_1)$ and $k = \alpha_2 + \alpha_3 + \omega_3$. Equating $\dot{N} = 0$, we need to consider two situations. First, when $c = 0$ we get

$$N = \frac{k - \omega_3}{k}\,.$$

Second, when $c \neq 0$ (and using $0 \leq N \leq 1$) we solve the quadratic equation to get

$$N = \frac{(c+k) - \sqrt{(c+k)^2 - 4c(k-\omega_3)}}{2c}\,.$$

Subbing in $c$ and $k$ gets the result.

**NOTE**: Due to the constraint that parameters are positive, we find that no combination of $c, k, \omega_3$ can lead to $N$ being undefined.

## Next generation matrix

- The disease states are $I_M$ and $I_N$
- Cannot ignore $S_M$ and $S_N$!
- THIS NEEDS TO BE REDONE

The flow in is

$$\mathscr{F} = \begin{bmatrix} \beta(1-c)\left(I_N + (1-p)I_M\right)S_M \\ \beta\left(I_N + (1-p)I_M\right)S_N \end{bmatrix}$$

and flow out is

$$\mathscr{V} = \begin{bmatrix} -\omega I_N + (\alpha+\gamma)I_M \\ (\omega+\gamma)I_N - \alpha I_M \end{bmatrix}.$$

The differential of these with respect to the disease states (remembering that $\alpha$ and $\omega$ depend on $I_N$ and $I_M$) are:

$$F = \begin{bmatrix} \beta(1-c)(1-p)S_M & \beta(1-c)S_M \\ \beta(1-p)S_N & \beta S_N \end{bmatrix},$$

$$V = \begin{bmatrix} \gamma + \alpha - \alpha_2 I_M - (\omega_1+\omega_2)I_N & (\alpha_1-\alpha_2)I_M - \omega - \omega_2 I_N \\ (\omega_1+\omega_2)I_N - \alpha + \alpha_2 I_M & \gamma - (\alpha_1-\alpha_2)I_M + \omega + \omega_2 I_N \end{bmatrix}.$$

Then $R(t) = \sigma(t)$ where $\sigma(t)$ is the largest (absolute) eigenvalue of $(FV^{-1})(t)$. Defining

- $x = \alpha - \alpha_2 I_M - (\omega_1+\omega_2)I_N$, (rate of change in $I_M$ due to mask transitions)
- $y = \omega - (\alpha_1-\alpha_2)I_M + \omega_2 I_N$, (rate of change in $I_N$ due to mask transitions)
- $a = (1-p)(\gamma+y) + x$, (total rate of change in $I_N$ accounting for reduced infections due to mask $(1-p)$?)
- $b = (1-p)y + \gamma + x$, and (total rate of change in $I_M$ accounting for reduced infections due to mask $(1-p)$?)
- $\Gamma = \frac{\beta}{\gamma(\gamma+x+y)}$ ($R_0$ with no mask effects ($p=c=0$))

we can succinctly write

$$\sigma(t) = \Gamma\left((1-c)S_M a + S_N b\right).$$

**Misc**

- Change $\alpha$ and $\omega$ to incorporate theory of planned behaviour? Use logistic regression to get probability of compliance per capita, convert to rate (odds ratio?) and use this instead of alpha and omega. In this setting, probably very much need different probabilities for each class.

- Can look at $I = I_N + I_M$ and $M = I_M + S_M + R_M$ and maybe work out some equilibriums? First (wrong) solution thinks that

$$S_N = \frac{\gamma(1+\frac{\omega}{\alpha})}{\beta(1+(1-p)\frac{\omega}{\alpha})(1+(1-c)\frac{\omega}{\alpha})}$$

Problem is that $\omega$ and $\alpha$ depend on the disease states

# Improving the behaviour transitions

These are thoughts on how to improve the transition rates $\alpha$ and $\omega$ to be (in some sense) more realistic of human behaviour. I found the paper "Incorporating human behaviour in simulation models of screening for breast cancer" (Brailsford et al., 2012) that builds a DES incorporating human behaviour models. They argue that the theory of planned behaviour is the better model to implement because it nicely lends itself to mathematical modelling (through a logistic regression). The also discuss the health belief model (preferred by Emily), but argue it is not clear how to model it because the relationships between the bubbles is not obvious.

My thought is to build a global behaviour logistic regression to replace $\alpha$ and $\omega$ with the probability of wearing as mask. This will obviously be very coarse and run under many assumptions. I am also not sure how to integrate the probability as a reasonable rate in the DE model. Thoughts for each model are discussed below.

## The theory of planned behaviour

The theory of planned behaviour (Figure 1) is a cognitive-social behavioural model that predicts *intention* to perform a behaviour off of three main inputs:

1. **attitude towards behaviour**: This measures an individuals personal beliefs around how beneficial the behaviour is. Theorised to be constructed of how beneficial the supposed outcomes of the behaviour is and how likely those outcomes are to occur.
2. **Subjective norms**: This is the social influence, basically along the lines of "what do people I respect think of this behaviour?"
3. **Perceived behavioural control**: This is the idea of "How well can I do this? How much say to I have in this matter?" This capture the "me" aspect of behaviour.

We can model the theory of planned behaviour in the following way. Let $\pi_c$ be the probability that a randomly chosen individual in compartment $c \in \{S, I, R\}$ intends to wear a mask. Let $X_1, X_2$ and $X_3$ denote scores reflecting their attitude towards the behaviour, subjective norms, and perceived behavioural control. We model

$$\log\left(\frac{\pi_c}{1-\pi_c}\right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3\,.$$

Then

- $\beta_0$ measures the natural bias of the full population towards wearing a mask, i.e. if $\beta_0 < 0$ then people are more inclined to **not** wear masks
- $\beta_1$ measures the influence of a persons attitude towards a behaviour on them intending to do the behaviour
- $\beta_2$ measures the influence of subjective norms on performing the behaviour
- $\beta_3$ measures the influence of an individuals perceived control of the behaviour on their intention to perform it

Two issues arise in this model. First, it predicts *intention* to perform the behaviour, which is approx 20% predictive of actually performing the behaviour. Second, this model does not account for past behaviour, as strong indicator of current behaviour. Possible work arounds for this:

1. Build the uncertainty between intention and performance into the rate derived from $\pi_c$.
2. Expand the index set of $c$ to include the masked/non-masked components.

A final deficit of this model is that it does not include risk, which is going to be an important measure to account for in disease modelling.

**My thoughts to applying this to my model**

I have tossed up some ideas of generalising my fear of disease and social influence into the theory of planned behaviour. Based on the nature of the theory, the probability $\pi_c$ will be different depending on your compartment because what you deem important will change depending on your condition. In the following, I am making the following assumptions (at least):

- Everyone is altruistic and rational
- Everyone has perfect knowledge of mask efficacy and disease prevalence
- Everyone knows the condition of the pope they mix with in a day
- At each time, individuals will make an independent decision on whether to wear a mask or not, i.e. the model is memoryless in knowledge of mask wearing. Put another way, wearing a mask today has no effect on wearing a mask tomorrow.
- Everyone knows their own disease state

The measures I am currently considering are represented by survey-like questions below for each state. Where possible, I have suggested the model parameters that will input into this. (These concepts need to be discussed with Emily to assess suitability)

*Behavioural beliefs*

- **S** - wearing a mask will prevent me from catching the disease. This is captured by the reduction in chance of catching the disease ($c$).
- **I** - wearing a mask will prevent me spreading the disease. This is captured by the reduction in chance of spreading the disease ($p$)
- **S/I/R** - Wearing a mask will help convince others to do the same thing.

*Normative beliefs*

- **S/I/R** - Most people around me are wearing masks. I suggest this be captured by the log odds of encountering someone wearing a mask ($\log(M/N)$).

*Perceived Control*

- **S/I/R** - I expect to have access to a mask (assumed constant)
- **S** - I expect to encounter people with the disease. I am associating this to disease prevalence ($I$)
- **I** - I expect to encounter people without the disease. I am associating this to the amount of people not with the disease ($1 - I$).

## The health belief model

- Durham2012 - Agent based model using HBM, masks and infectious diseases

The health belief model has gone through some renditions (Figure 2), but suggests that actions taken for health benefit are influenced by the following factors:

1. **Health motivation**: TBD
2. **Perception of illness threat**: This captures both perceived susceptibility to the illness and perceived severity of the illness.

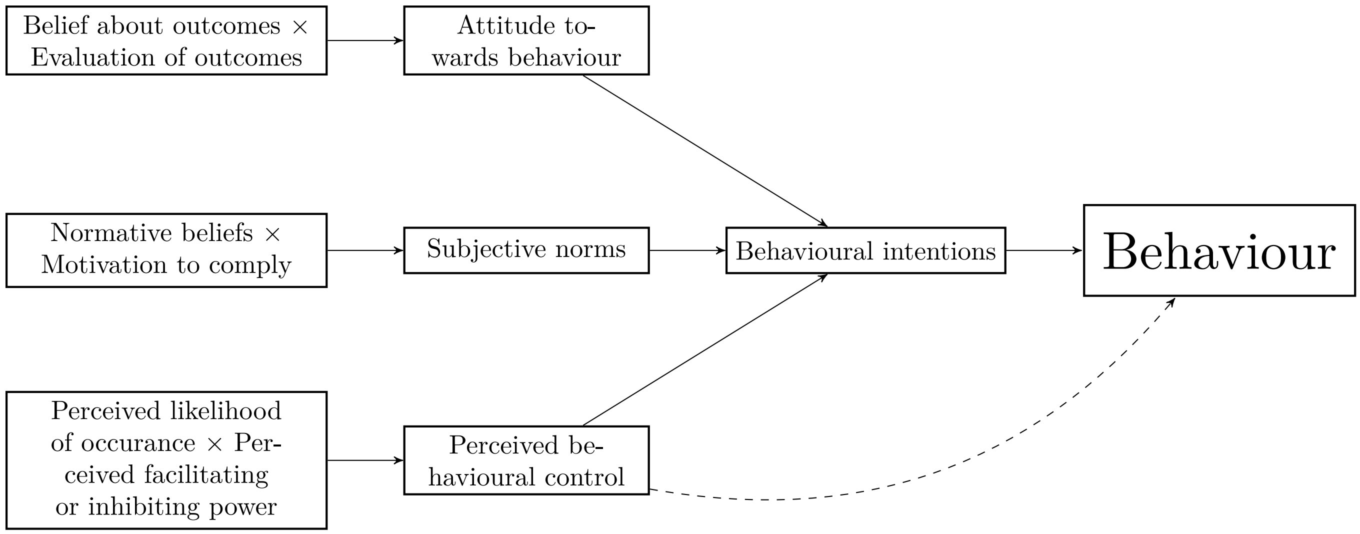



Figure 1: The theory of planned behaviour, from https://www.sciencedirect.com/science/article/pii/S0377221711009817

3. **Evaluation of behaviours to counteract threat**: This captures the perceived benefits of the behaviour as well as the perceived barriers to performing the behaviour
4. **Cues to action**: TBD

Durham and Casman includes the HBM in an gent-based model for disease spread to investigate the impact of behaviour on diseases spread, but decouples this from the epidemiology. Durham and Casman focuses only on the second two inputs, investigating the sub-inputs to those drivers. They suggest a fairly simply approach using a logistic regression in the following way. Let $\pi_i$ be the probability that individual $i$ chooses to take up the behaviour. Let $X_1, X_2, X_3$ and $X_4$ be an individuals binary indicators of low or high risk for susceptibility, severity, benefits, and barriers respectively. They model

$$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 \,,$$

where

- $\beta_0$ is the baseline attitude towards the behaviour.
- $\beta_i$ is the effect of having a high risk value for measure $i$.

They actually present model with respect to odds ratios, but I find this a lot more opaque than the standard logistic regression format I am used to.

In their work, they suggest a cut off such that if $\hat{\pi}_i > 0.5$ at a given time for an agent, they will adopt the behaviour. They also argue that $\beta_3 = 0$, that is, perceived benefit has no effect in their model, based off little correlation between the benefit and act of wearing a mask. Further, they suggest models for $X_1, X_2$ and $X_4$ roughly based around the following:

- $X_1$ captures disease prevalence. Measures the time dependant knowledge of disease today and in the past, with waning weight towards past cases. Once this measure exceeds a threshold, they consider $X_1$ to be high risk.
- $X_2$ captures the fear of severity of the disease. They suggest two possible models for this. The first provides a ratio of deaths to infections as a measure of how sever it is. The second measures the media coverage of the disease. In both situations, once this fear exceeds a threshold, the $X_2$ variable is considered high.

- $X_4$ captures the social influence of the behaviour. They argue that a barrier to mask wearing can be social stigma for wearing a mask when no one else is wearing one. Thus, they give a weighted sum to observing those around you wearing a mask as an indicator for if you should wear one. Again, they give this a threshold for it to tick over. Due to the contradictory nature of a barrier, the barrier is considered low when many people are wearing a mask and high otherwise.

**My thoughts of applying this approach to my model**

TBD

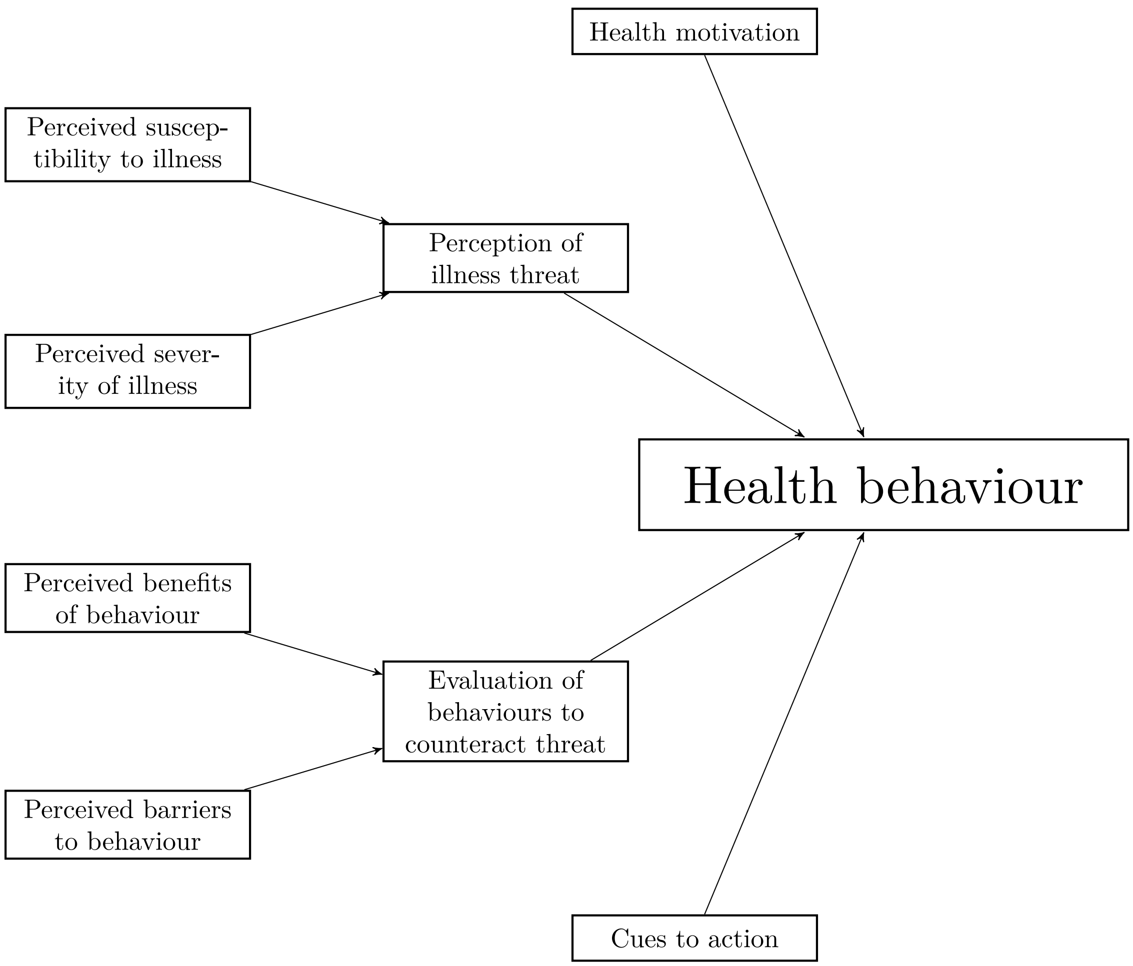



Figure 2: The health belief model, from https://www.sciencedirect.com/science/article/pii/S0377221711009817

## Protection motivation theory

I have not had many thoughts on this one, but Emily has suggested it. She also believes that they have untapped data relating to COVID and this theory.

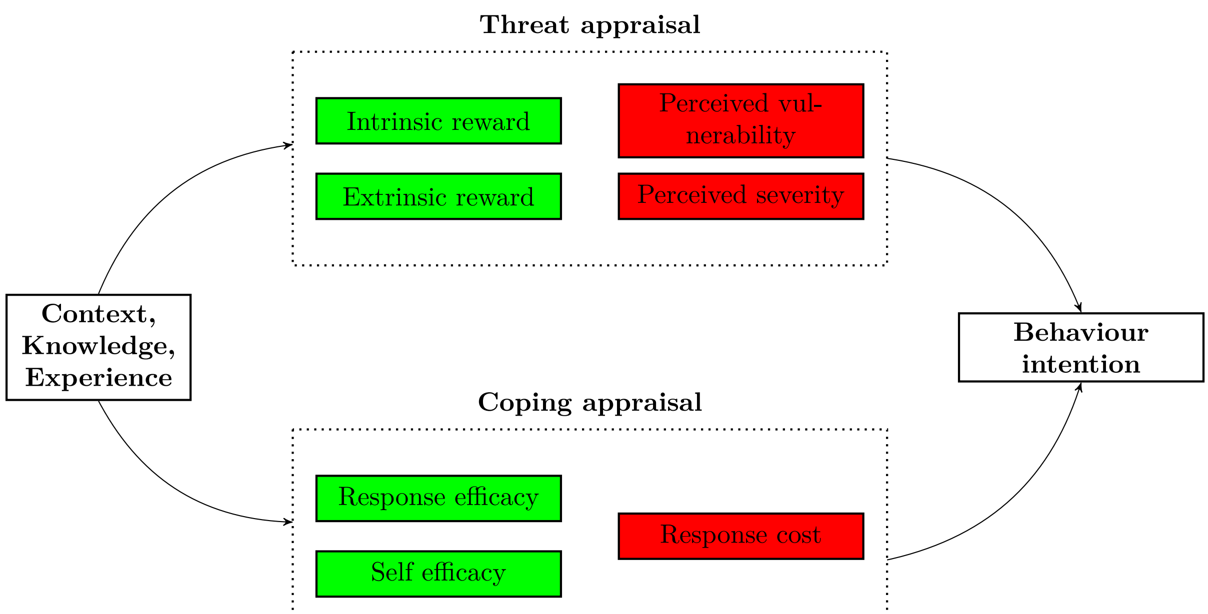



Figure 3: The protection motivation theory, red is negative effect, green is positive effect. Version from https://www.researchgate.net/figure/Schematic-presentation-of-the-Protection-Motivation-Theory-PMT-and-its-seven_fig1_267102412