NGM

1. Infected comp: I_M, I_N

2. Infection terms:

$$\mathcal{F} = \begin{array}{c} I_m \\ I_N \end{array}\begin{bmatrix} \beta(1-c) I_N S_M + (1-c)(1-p)\beta I_m S_m \\ \beta I_N S_N + (1-p)\beta I_m S_N \end{bmatrix}$$

$$f = \begin{array}{c|cc} & \frac{\partial F}{\partial I_m} & \frac{\partial F}{\partial I_N} \\ \hline I_m & & \\ I_N & & \end{array} \quad \begin{array}{c} I_m \\ \\ I_N \end{array}\begin{bmatrix} (1-c)(1-p)\beta S_m & \beta(1-c) S_m \\ & \\ (1-p)\beta S_N & \beta S_N \end{bmatrix}$$

3. $$\mathcal{V} = \begin{array}{c} I_m \\ I_N \end{array}\begin{bmatrix} (\alpha+\gamma) I_m - \omega I_N \\ -\alpha I_M + (\omega+\gamma) I_N \end{bmatrix}$$

But $\alpha = \alpha_1 (S_N + I_N + R_N) + \alpha_2 (1 - (I_M + I_N))$

$\omega = \omega_1 (S_M + I_M + R_M) + \omega_2 (I_M + I_N)$

$\therefore$

$$V = \begin{bmatrix} \alpha_1 I_M (S_N + I_N + R_N) + \alpha_2 I_M (1 - I_M - I_N) + \gamma I_M \\ \quad - \omega_1 I_N (S_M + I_M + R_M) - \omega_2 I_N (I_M + I_N) \\ \\ -\alpha_1 I_M (S_N + I_N + R_N) - \alpha_2 I_M (1 - I_M - I_N) + \gamma I_N \\ \quad + \omega_1 I_N (S_M + I_M + R_M) + \omega_2 I_N (I_M + I_N) \end{bmatrix}$$

$P_N = S_N + I_N + R_N$

$P_M = S_M + I_M + R_N$

$\frac{\partial V_1}{\partial I_M} = V_{11} = \cancel{\alpha_1} P_N + \alpha_2 \cancel{(1 - I_N)} - \cancel{2\alpha_2 I_M} + \cancel{\gamma} - \cancel{\omega_1 I_N} - \cancel{\omega_2 I_N}$

$= \cancel{\gamma} + \cancel{\alpha_1 P_N} - \cancel{2\alpha_2} I_M + \cancel{\alpha_2} - I_N(+\cancel{\alpha_2} + \omega_1 + \omega_2)$

$\frac{\partial V_1}{\partial I_N} = V_{12} = \cancel{\alpha_1 I_M} - \cancel{\alpha_2 I_M} - \omega_1 P_M - \cancel{\omega_2 I_M} - 2\omega_2 I_N$

$= I_M(\cancel{\alpha_1} - \cancel{\alpha_2} - \cancel{\omega_2}) - \cancel{\omega_1 P_M} - 2\cancel{\omega_2 I_N}$

$\frac{\partial V_2}{\partial I_M} = V_{21} = -\cancel{\alpha_1 P_N} - \alpha_2 \cancel{(1 - I_N)} + \cancel{2\alpha_2 I_M} + \cancel{\omega_1 I_N} + \omega_2 I_N$

$$= -\cancel{\alpha_1 P_N} + 2\alpha_2 I_M - \cancel{\alpha_2} + I_N\left(\cancel{\alpha_2} + \cancel{\omega_1} + \cancel{\omega_2}\right)$$

$$\frac{H_2}{dI_N} \equiv V_{22} = -\cancel{\alpha_1 I_M} + \cancel{\alpha_2 I_M} + \cancel{\gamma} + \cancel{\omega_1} P_M + \cancel{\omega_2} I_M + 2\omega_2 I_N$$

$$= \cancel{\gamma} + \cancel{\omega_1 P_M} + I_M\left(-\cancel{\alpha_1} + \cancel{\alpha_2} + \cancel{\omega_2}\right) + \cancel{2\omega_2 I_N}$$

$$\alpha = \alpha_1 P_N + \alpha_2 (1 - I_M - I_N)$$

$$\omega = \omega_1 P_M + \omega_2 (I_M + I_N)$$

$$V_{11} = \cancel{\gamma} + \cancel{\alpha_1 P_N} + \cancel{\alpha_2 (1 - I_M - I_N)} - \cancel{\alpha_2 I_M} - \cancel{I_N (\omega_1 + \omega_2)}$$

$$= \gamma + \alpha - a_2 I_M - (\omega_1 + \omega_2) I_N$$

$$V_{12} = \cancel{(\alpha_1 - \alpha_2)} I_M - \cancel{\omega_1 P_M - \omega_2 (I_M + I_N)} - \omega_2 I_N$$

$$= (\alpha_1 - \alpha_2) I_M - \omega - \omega_2 I_N$$

$$V_{21} = (\omega_1 + \omega_2) I_N - \alpha_1 P_N - \alpha_2 (1 - I_N - I_M) + \alpha_2 I_M$$

$$= (\omega_1 + \omega_2) I_N - \alpha + \alpha_2 I_M$$

$$V_{22} = \gamma - (\alpha_1 - \alpha_2) I_M + \omega_1 P_M + \omega_2 (I_M + I_N) + \omega_2 I_N$$

$$= \gamma - (\alpha_1 - \alpha_2) I_M + \omega + \omega_2 I_N$$

$$V = \begin{bmatrix} \gamma + \alpha - \alpha_2 I_M - (\omega_1 + \omega_2) I_N, & (\alpha_1 - \alpha_2) I_M - \omega - \omega_2 I_N \\ (\omega_1 + \omega_2) I_N - \alpha + \alpha_2 I_M & , \gamma - (\alpha_1 - \alpha_2) I_M + \omega + \omega_2 I_N \end{bmatrix}$$

Sanity check

if $\rho = 0, \quad c = 0, \qquad R_0 = \frac{\beta}{\gamma}$

$\alpha_1 = 0 = \alpha_2 \qquad \omega_1 = 0 = \omega_2,$

$$F = \begin{bmatrix} \beta S_M & \beta S_M \\ \beta S_N & \beta S_N \end{bmatrix}$$

$$V = \begin{bmatrix} \gamma & , & 0 \\ 0 & , & \gamma \end{bmatrix}$$

$$V^{-1} = \frac{1}{\gamma^2} \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} = \begin{bmatrix} \frac{1}{\gamma} & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\beta}{\gamma} S_M & , & \frac{\beta S_M}{\gamma} \\ \frac{\beta}{\gamma} S_N & , & \frac{\beta}{\gamma} S_N \end{bmatrix}$$

$$R_0 = \frac{\beta}{\gamma}\left(S_m + S_N\right)$$