

SIR and mask

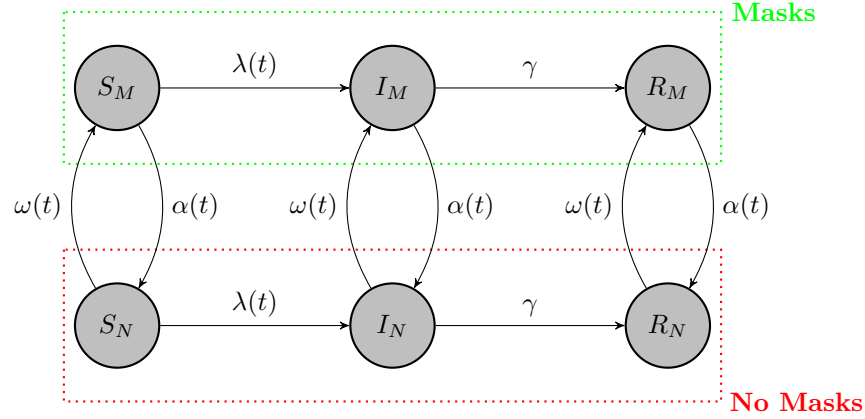
Basic idea

- Simple SIR model disease progression with strata for mask-wearing status, resulting in compartments for S-masked, S-no mask, I-masked, I-no mask, R-masked, R-no mask.
- Supposing wearing a mask reduces both susceptibility and infectiousness (depending on who is wearing it)
- Assume transitions between masked-not masked based on social influence and fear of disease (noting FD type “infection” structure for fear of disease, when could actually be more about absolute number/density dependence – interesting question to explore how structure affects dynamics)
- Work in proportions such that $S + I + R = 1$. Similarly, $M + N = 1$.
- I am thinking of this as two distinct (but coupled) models on the same population.
- no demography
- Han2021 - Effects of masks on the transmission of infectious diseases. Similar approach
- Del Valle 2012 - Modeling the Impact of Behavior Changes on the Spread of Pandemic Influenza: They do essentially the same model as I do, however they suggest a simpler transition for behaviour based off of a simple step function. They also investigate a large scale agent-based model (approx 20mil agents I think).

Where to go

- My model acts as an extension of Perra 2013 and De Valle 2012. Their models are recoverable by changing the α and ω parameters.
- Social exhaustion is also something I might consider.
- customise α and ω to each SIR class
- Can compare current “fear + social” model, TPB model, and HBM. Make for a long, but comparative piece.
- Incorporate behaviour burn out?
- Incorporate sub-classes with “always behaving” in certain way
- include reactance to behaviours, i.e. when it gets too much people blow up and refuse
- Include burn-out of behaviour, first steps towards a constant exit rate from behaviour.

State diagram



Parameter description

- λ is the force of infection, assuming a form of FD
- $\frac{1}{\gamma}$ is average infectious period
- α is transition rate from wearing a mask to not wearing a mask
- ω is transition rate from not wearing a mask to wearing a mask
- ν is the waning immunity term

Further parameters that will pop up

- β is the transmission rate
- c is the reduction in chance of catching disease if S is wearing a mask # this description is inconsistent with code comment definition and 1-c model structure (I think)
- p is the reduction in chance of spreading disease if I is wearing a mask # this description is inconsistent with code comment definition and 1-p model structure (I think)
- α_1 is the rate of transition to no mask based on social influence of no mask
- α_2 is the rate of transition to no mask based on fear of disease
- ω_1 is the rate of transition to mask based on social influence of mask
- ω_2 is the rate of transition to no mask based on fear of disease

Basic system of ODEs

$$\begin{aligned}
 \dot{S}_M &= -\lambda(t)(1-c)S_M - \alpha(t)S_M + \omega(t)S_N + \nu R_M \\
 \dot{S}_N &= -\lambda(t)S_N + \alpha(t)S_M - \omega(t)S_N + \nu R_N \\
 \dot{I}_M &= \lambda(t)(1-c)S_M - \alpha(t)I_M + \omega(t)I_N - \gamma I_M \\
 \dot{I}_N &= \lambda(t)S_N + \alpha(t)I_M - \omega(t)I_N - \gamma I_N \\
 \dot{R}_M &= \gamma I_M - \alpha(t)R_M + \omega(t)R_N - \nu R_M \\
 \dot{R}_N &= \gamma I_N + \alpha(t)R_M - \omega(t)R_N - \nu R_N,
 \end{aligned}$$

where

$$\begin{aligned}
 \lambda(t) &= \beta (I_N + (1-p)I_M) \\
 \omega(t) &= \omega_1(S_M + I_M + R_M) + \omega_2(I_M + I_N) + \omega_3 \\
 \alpha(t) &= \alpha_1(S_N + I_N + R_N) + \alpha_2(1 - (I_M + I_N)) + \alpha_3.
 \end{aligned}$$

Disease-free equilibrium of the behaviour

Case when $\omega_3 + \alpha_2 + \alpha_3 = 0$ and case when those equal 0 and $\omega_2 \neq 0$.

The disease-free equilibrium of the behaviour (M^*, N^*) is

$$(1 - \Delta, \Delta)$$

where

$$\Delta = \begin{cases} \frac{(\omega_1 - \alpha_1 + \omega_3 + \alpha_2 + \alpha_3) - \sqrt{(\omega_1 - \alpha_1 + \omega_3 + \alpha_2 + \alpha_3)^2 - 4(\omega_1 - \alpha_1)(\alpha_2 + \alpha_3)}}{2(\omega_1 - \alpha_1)} & \text{if } \omega_1 - \alpha_1 \neq 0, \\ \frac{\alpha_2 + \alpha_3}{\omega_3 + \alpha_2 + \alpha_3} & \text{else.} \end{cases}$$

Proof:

Considering $N = S_N + I_N + R_N$, we can write

$$\dot{N} = -\omega(t)N + \alpha(t)M.$$

In the disease-free equilibrium, $I_N = I_M = 0$, so this reduces to

$$\dot{N} = -(\omega_1 M + \omega_3)N + (\alpha_1 N + (\alpha_2 + \alpha_3))M.$$

Using $M + N = 1$, we can reduce this to

$$\dot{N} = cN^2 - (c + k)N + (k - \omega_3)$$

where $c = (\omega_1 - \alpha_1)$ and $k = \alpha_2 + \alpha_3 + \omega_3$. Equating $\dot{N} = 0$, we need to consider two situations. First, when $c = 0$ we get

$$N = \frac{k - \omega_3}{k}.$$

Second, when $c \neq 0$ (and using $0 \leq N \leq 1$) we solve the quadratic equation to get

$$N = \frac{(c + k) - \sqrt{(c + k)^2 - 4c(k - \omega_3)}}{2c}.$$

Subbing in c and k gets the result.

NOTE: Due to the constraint that parameters are positive, we find that no combination of c, k, ω_3 can lead to N being undefined.

Next generation matrix

- The disease states are I_M and I_N
- Cannot ignore S_M and S_N !
- THIS NEEDS TO BE REDONE

The flow in is

$$\mathcal{F} = \begin{bmatrix} \beta(1-c)(I_N + (1-p)I_M)S_M \\ \beta(I_N + (1-p)I_M)S_N \end{bmatrix}$$

and flow out is

$$\mathcal{V} = \begin{bmatrix} -\omega I_N + (\alpha + \gamma)I_M \\ (\omega + \gamma)I_N - \alpha I_M \end{bmatrix}.$$

The differential of these with respect to the disease states (remembering that α and ω depend on I_N and I_M) are:

$$F = \begin{bmatrix} \beta(1-c)(1-p)S_M & \beta(1-c)S_M \\ \beta(1-p)S_N & \beta S_N \end{bmatrix},$$

$$V = \begin{bmatrix} \gamma + \alpha - \alpha_2 I_M - (\omega_1 + \omega_2)I_N & (\alpha_1 - \alpha_2)I_M - \omega - \omega_2 I_N \\ (\omega_1 + \omega_2)I_N - \alpha + \alpha_2 I_M & \gamma - (\alpha_1 - \alpha_2)I_M + \omega + \omega_2 I_N \end{bmatrix}.$$

Then $R(t) = \sigma(t)$ where $\sigma(t)$ is the largest (absolute) eigenvalue of $(FV^{-1})(t)$. Defining

- $x = \alpha - \alpha_2 I_M - (\omega_1 + \omega_2)I_N$, (rate of change in I_M due to mask transitions)
- $y = \omega - (\alpha_1 - \alpha_2)I_M + \omega_2 I_N$, (rate of change in I_N due to mask transitions)
- $a = (1-p)(\gamma + y) + x$, (total rate of change in I_N accounting for reduced infections due to mask (1-p)?)
- $b = (1-p)y + \gamma + x$, and (total rate of change in I_M accounting for reduced infections due to mask (1-p)?)
- $\Gamma = \frac{\beta}{\gamma(\gamma+x+y)}$ (R_0 with no mask effects ($p = c = 0$))

we can succinctly write

$$\sigma(t) = \Gamma((1-c)S_M a + S_N b).$$

Misc

- Change α and ω to incorporate theory of planned behaviour? Use logistic regression to get probability of compliance per capita, convert to rate (odds ratio?) and use this instead of alpha and omega. In this setting, probably very much need different probabilities for each class.
- Can look at $I = I_N + I_M$ and $M = I_M + S_M + R_M$ and maybe work out some equilibriums? First (wrong) solution thinks that

$$S_N = \frac{\gamma(1 + \frac{\omega}{\alpha})}{\beta(1 + (1-p)\frac{\omega}{\alpha})(1 + (1-c)\frac{\omega}{\alpha})}$$

Problem is that ω and α depend on the disease states

Improving the behaviour transitions

These are thoughts on how to improve the transition rates α and ω to be (in some sense) more realistic of human behaviour. I found the paper “Incorporating human behaviour in simulation models of screening for breast cancer” (Brailsford et al., 2012) that builds a DES incorporating human behaviour models. They argue that the theory of planned behaviour is the better model to implement because it nicely lends itself to mathematical modelling (through a logistic regression). They also discuss the health belief model (preferred by Emily), but argue it is not clear how to model it because the relationships between the bubbles is not obvious.

My thought is to build a global behaviour logistic regression to replace α and ω with the probability of wearing as mask. This will obviously be very coarse and run under many assumptions. I am also not sure how to integrate the probability as a reasonable rate in the DE model. Thoughts for each model are discussed below.

The theory of planned behaviour

The theory of planned behaviour (Figure 1) is a cognitive-social behavioural model that predicts *intention* to perform a behaviour off of three main inputs:

1. **attitude towards behaviour:** This measures an individual's personal beliefs around how beneficial the behaviour is. Theorised to be constructed of how beneficial the supposed outcomes of the behaviour is and how likely those outcomes are to occur.
2. **Subjective norms:** This is the social influence, basically along the lines of “what do people I respect think of this behaviour?”
3. **Perceived behavioural control:** This is the idea of “How well can I do this? How much say to I have in this matter?” This captures the “me” aspect of behaviour.

We can model the theory of planned behaviour in the following way. Let π_c be the probability that a randomly chosen individual in compartment $c \in \{S, I, R\}$ intends to wear a mask. Let X_1, X_2 and X_3 denote scores reflecting their attitude towards the behaviour, subjective norms, and perceived behavioural control. We model

$$\log \left(\frac{\pi_c}{1 - \pi_c} \right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3.$$

Then

- β_0 measures the natural bias of the full population towards wearing a mask, i.e. if $\beta_0 < 0$ then people are more inclined to **not** wear masks
- β_1 measures the influence of a person's attitude towards a behaviour on them intending to do the behaviour
- β_2 measures the influence of subjective norms on performing the behaviour
- β_3 measures the influence of an individual's perceived control of the behaviour on their intention to perform it

Two issues arise in this model. First, it predicts *intention* to perform the behaviour, which is approx 20% predictive of actually performing the behaviour. Second, this model does not account for past behaviour, as a strong indicator of current behaviour. Possible work arounds for this:

1. Build the uncertainty between intention and performance into the rate derived from π_c .
2. Expand the index set of c to include the masked/non-masked components.

A final deficit of this model is that it does not include risk, which is going to be an important measure to account for in disease modelling.

My thoughts to applying this to my model

I have tossed up some ideas of generalising my fear of disease and social influence into the theory of planned behaviour. Based on the nature of the theory, the probability π_c will be different depending on your compartment because what you deem important will change depending on your condition. In the following, I am making the following assumptions (at least):

- Everyone is altruistic and rational
- Everyone has perfect knowledge of mask efficacy and disease prevalence
- Everyone knows the condition of the pope they mix with in a day
- At each time, individuals will make an independent decision on whether to wear a mask or not, i.e. the model is memoryless in knowledge of mask wearing. Put another way, wearing a mask today has no effect on wearing a mask tomorrow.
- Everyone knows their own disease state

The measures I am currently considering are represented by survey-like questions below for each state. Where possible, I have suggested the model parameters that will input into this. (These concepts need to be discussed with Emily to assess suitability)

Behavioural beliefs

- **S** - wearing a mask will prevent me from catching the disease. This is captured by the reduction in chance of catching the disease (c).
- **I** - wearing a mask will prevent me spreading the disease. This is captured by the reduction in chance of spreading the disease (p)
- **S/I/R** - Wearing a mask will help convince others to do the same thing.

Normative beliefs

- **S/I/R** - Most people around me are wearing masks. I suggest this be captured by the log odds of encountering someone wearing a mask ($\log(M/N)$).

Perceived Control

- **S/I/R** - I expect to have access to a mask (assumed constant)
- **S** - I expect to encounter people with the disease. I am associating this to disease prevalence (I)
- **I** - I expect to encounter people without the disease. I am associating this to the amount of people not with the disease ($1 - I$).

The health belief model

- Durham2012 - Agent based model using HBM, masks and infectious diseases

The health belief model has gone through some renditions (Figure 2), but suggests that actions taken for health benefit are influenced by the following factors (Re. <http://www.sbccimplementationkits.org/demandrnmch/wp-content/uploads/2014/02/Theory-at-a-Glance-A-Guide-For-Health-Promotion-Practice.pdf>):

1. **Self efficacy**: Confidence of one's ability to perform the behaviour

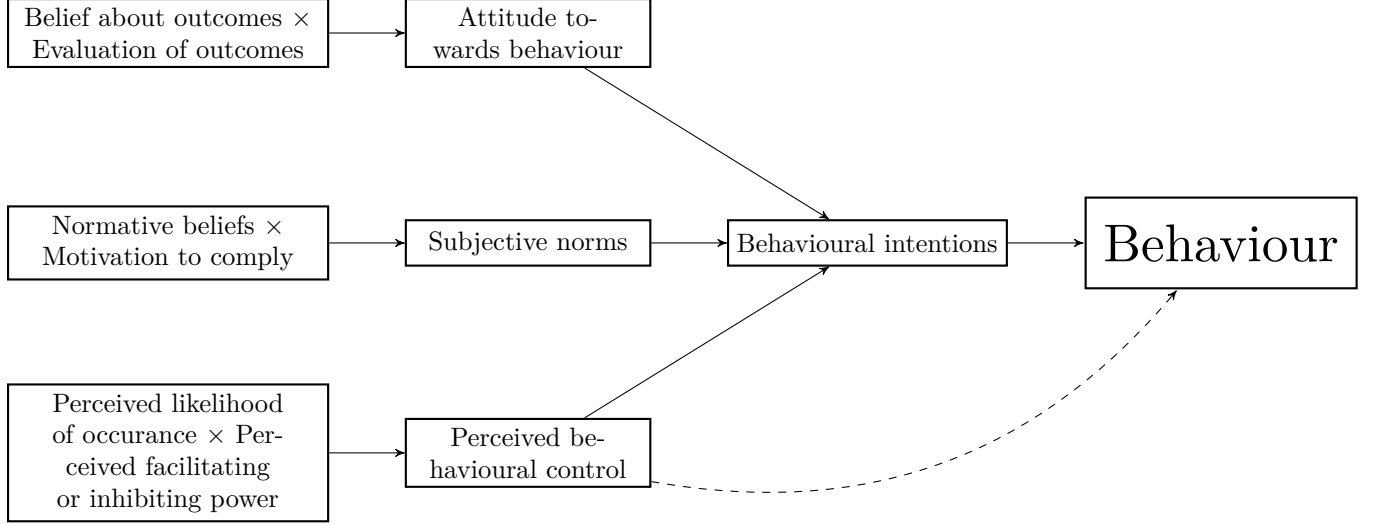


Figure 1: The theory of planned behaviour, from <https://www.sciencedirect.com/science/article/pii/S0377221711009817>

2. **Perception of illness threat:** This captures both perceived susceptibility to the illness and perceived severity of the illness.
 - a. *Perceived susceptibility:* Belief about chance of getting disease
 - b. *Perceived severity:* Beliefs about the seriousness of the condition and the consequences
3. **Evaluation of behaviours to counteract threat:** This captures the perceived benefits of the behaviour as well as the perceived barriers to performing the behaviour
 - a. *Perceived Benefits:* Beliefs about the effectiveness of the behaviour to reduce seriousness
 - b. *Perceived barriers:* Belief about material and psychological costs of taking the action
4. **Cues to action:** External or internal factors that activate the “readiness to change”

Durham and Casman includes the HBM in an agent-based model for disease spread to investigate the impact of behaviour on diseases spread, but decouples this from the epidemiology. Durham and Casman focuses only on the second two inputs, investigating the sub-inputs to those drivers. They suggest a fairly simply approach using a logistic regression in the following way. Let π_i be the probability that individual i chooses to take up the behaviour. Let X_1, X_2, X_3 and X_4 be an individuals binary indicators of low or high risk for susceptibility, severity, benefits, and barriers respectively. They model

$$\log \left(\frac{\pi_i}{1 - \pi_i} \right) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4,$$

where

- β_0 is the baseline attitude towards the behaviour.
- β_i is the effect of having a high risk value for measure i .

They actually present model with respect to odds ratios, but I find this a lot more opaque than the standard logistic regression format I am used to.

In their work, they suggest a cut off such that if $\hat{\pi}_i > 0.5$ at a given time for an agent, they will adopt the behaviour. They also argue that $\beta_3 = 0$, that is, perceived benefit has no effect in their model, based off little correlation between the benefit and act of wearing a mask. Further, they suggest models for X_1, X_2 and X_4 roughly based around the following:

- X_1 captures disease prevalence. Measures the time dependant knowledge of disease today and in the past, with waning weight towards past cases. Once this measure exceeds a threshold, they consider X_1 to be high risk.
- X_2 captures the fear of severity of the disease. They suggest two possible models for this. The first provides a ratio of deaths to infections as a measure of how sever it is. The second measures the media coverage of the disease. In both situations, once this fear exceeds a threshold, the X_2 variable is considered high.
- X_4 captures the social influence of the behaviour. They argue that a barrier to mask wearing can be social stigma for wearing a mask when no one else is wearing one. Thus, they give a weighted sum to observing those around you wearing a mask as an indicator for if you should wear one. Again, they give this a threshold for it to tick over. Due to the contradictory nature of a barrier, the barrier is considered low when many people are wearing a mask and high otherwise.

My thoughts of applying this approach to my model

TBD

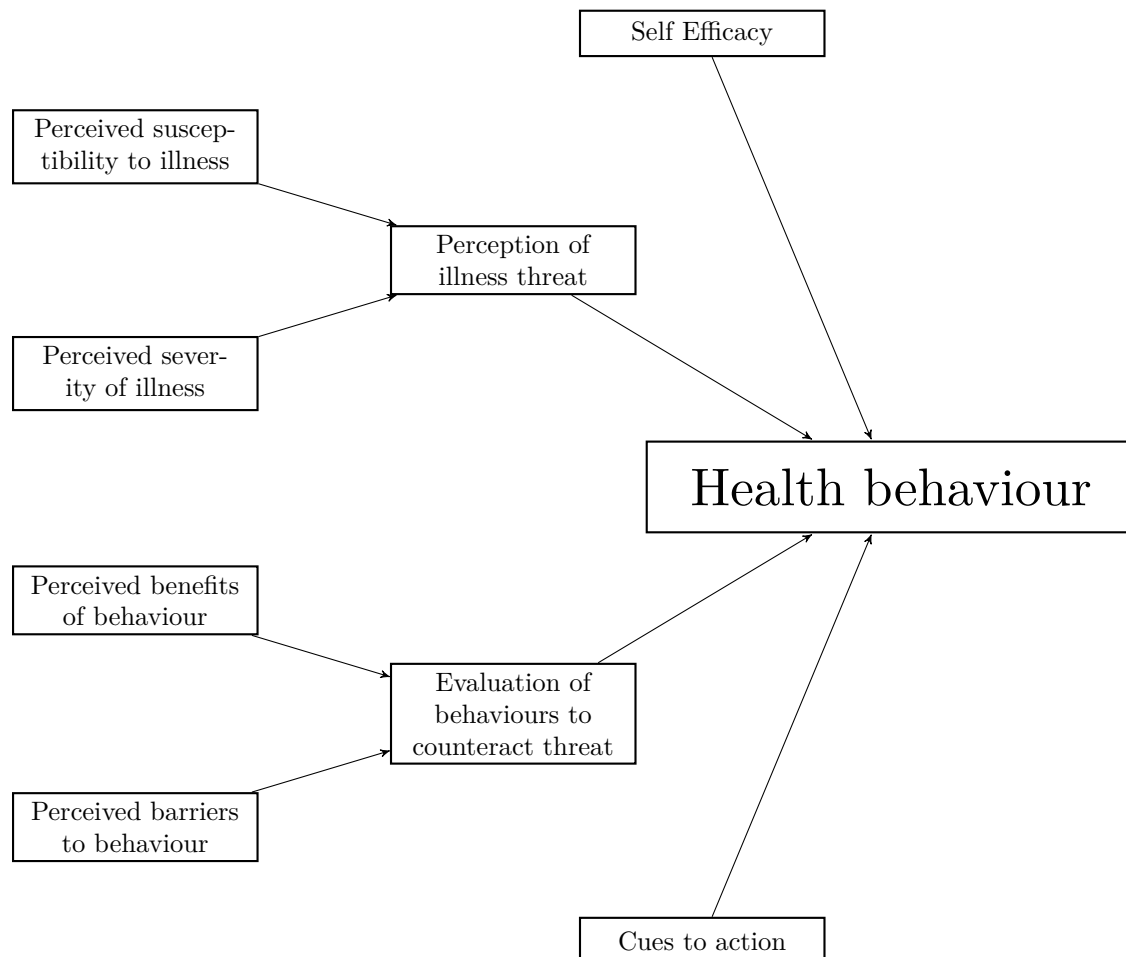


Figure 2: The health belief model, from <https://www.sciencedirect.com/science/article/pii/S0377221711009817>

Protection motivation theory

I have not had many thoughts on this one, but Emily has suggested it. She also believes that they have untapped data relating to COVID and this theory.

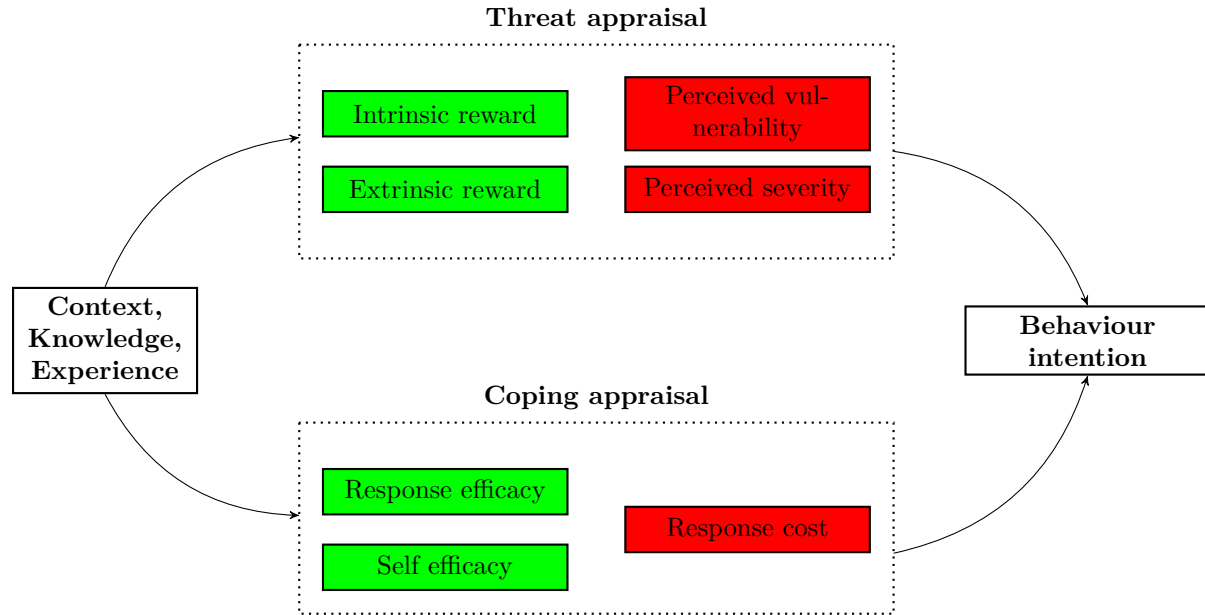
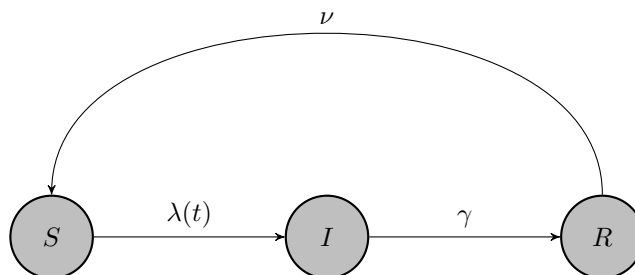


Figure 3: The protection motivation theory, red is negative effect, green is positive effect. Version from https://www.researchgate.net/figure/Schematic-presentation-of-the-Protection-Motivation-Theory-PMT-and-its-seven_fig1_267102412

A behaviour-infection model

Consider an influenza like illness governed by an *SIRS* structure:

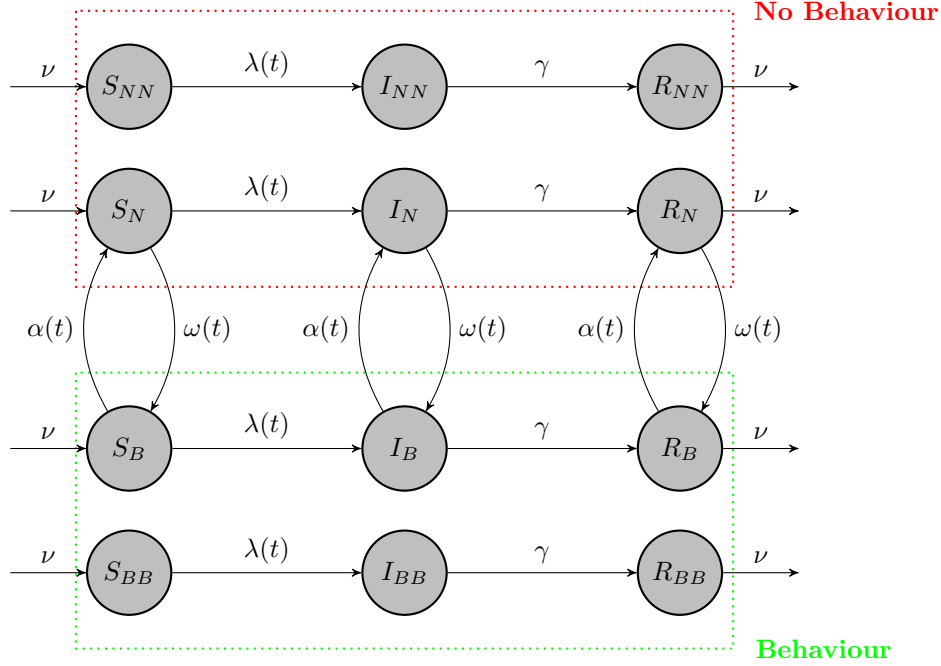


where

- $\lambda(t)$ is the force of infection;
- $1/\gamma$ is the average infection period;
- $1/\nu$ is the average immunity period, and;

- $S + I + R = P = 1$.

Suppose we introduce a preventative health behaviour B into the population that individuals can choose to take up. We model this by stratifying the S , I , and R components into X_N and X_B , $X \in \{S, I, R\}$. The strata X_N represents those in the populations who are not performing the behaviour, and the strata X_B represents those in the population who *are* performing the behaviour. In fact, to capture radicals for each strata, we add further strata X_{NN} and X_{BB} of individuals who will never/always do the behaviour respectively. Stratifying the population like this gives the following stratified SIRS model:



where:

- $\lambda(t)$, γ , and ν are as above for the *SIRS* model;
- $\omega(t)$ is the rate someone in strata N takes up the behaviour B ;
- $\alpha(t)$ is the rate someone in strata B gives up the behaviour B ;
- To simplify the model, we have assumed $\omega(t)$ and $\alpha(t)$ are the same across disease states S, I, R , and;
- $1 = P = \sum_{X \in \{S, I, R\}} (X_N + X_{NN} + X_B + X_{BB})$.

Notice how the radical groups X_{NN} and X_{BB} are “immune” to the behavioural changes $\omega(t)$ and $\alpha(t)$. However, this model does preclude the idea that an individual can become radicalised to a behaviour, or a radical may become less radical.

Assuming homogeneous mixing of the population, we propose the frequency dependent (noting everything is in proportions) force of infection give by

$$\lambda(t) = \beta ((I_N + I_{NN}) + (1 - p)(I_B + I_{BB})) ,$$

where

- β contains information on the average contact rate and probability of infection, and;

- $p \times 100\%$ is the decrease in infectiousness due to the behaviour B . In other words, p describes how effective the behaviour B is at reducing the infectiousness of an individual.

Assuming there is a parameter c that describes how effective the behaviour B is at reducing an individuals *susceptibility*, we may write the following set of differential equations to describe our model:

- NN

$$\begin{cases} \dot{S}_{NN} &= -\lambda(t)S_{NN} + \nu R_{NN} \\ \dot{I}_{NN} &= \lambda(t)S_{NN} - \gamma I_{NN} \\ \dot{R}_{NN} &= \gamma I_{NN} - \nu R_{NN} \end{cases}$$

- N

$$\begin{cases} \dot{S}_N &= -\lambda(t)S_N + \alpha(t)S_B - \omega(t)S_N + \nu R_N \\ \dot{I}_N &= \lambda(t)S_N + \alpha(t)I_B - \omega(t)I_N - \gamma I_N \\ \dot{R}_N &= \gamma I_N + \alpha(t)R_B - \omega(t)R_N - \nu R_N, \end{cases}$$

- B

$$\begin{cases} \dot{S}_B &= -\lambda(t)(1-c)S_B - \alpha(t)S_B + \omega(t)S_N + \nu R_B \\ \dot{I}_B &= \lambda(t)(1-c)S_B - \alpha(t)I_B + \omega(t)I_N - \gamma I_B \\ \dot{R}_B &= \gamma I_B - \alpha(t)R_B + \omega(t)R_N - \nu R_B \end{cases}$$

- BB

$$\begin{cases} \dot{S}_{BB} &= -\lambda(t)(1-c)S_{BB} + \nu R_{BB} \\ \dot{I}_{BB} &= \lambda(t)(1-c)S_{BB} - \gamma I_{BB} \\ \dot{R}_{BB} &= \gamma I_{BB} - \nu R_{BB} \end{cases}$$

where the dot represents the derivative with respect to time. When we wish to discuss the *behavioural states* (red/green rectangles in state diagram) of the model, we will commonly write

- $N = S_N + S_{NN} + I_N + I_{NN} + R_N + R_{NN}$, and;
- $B = S_B + S_{BB} + I_B + I_{BB} + R_B + R_{BB}$.

Note then that $N + B = 1$ and we find that

- $\dot{N} = -\omega(t)(S_N + I_N + R_N) + \alpha(t)(S_B + I_B + R_B)$, and;
- $\dot{B} = \omega(t)(S_N + I_N + R_N) - \alpha(t)(S_B + I_B + R_B)$.

This demonstrates that we have an *SIS*-like structure on our behaviour in the sense that “susceptible” individuals (N) can become “infected” with the behaviour (B) and eventually “recover” back to “susceptible”.

Using behavioural models to describe ω and α

To determine the uptake (ω) and the upkeep (α) of the behaviour B , we propose using the health belief model (Figure 2). The health belief model is a popular behavioural science model that aims to explain why an individual chooses to follow a behaviour that can be directly beneficial for their health (such as wearing a mask or getting vaccinated). The health belief model is made up of six inputs, so it is reasonable to write

$$\omega(t) = f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t) + f_6(t),$$

where

- $f_1(t)$ represents the influence of perceived susceptibility to the disease on performing the behaviour

- $f_2(t)$ represents the influence of perceived severity of the disease on performing the behaviour
- $f_3(t)$ represents the influence of perceived benefits of the behaviour on performing the behaviour
- $f_4(t)$ represents the influence of perceived barriers to the behaviour on performing the behaviour
- $f_5(t)$ represents the influence of cues to action on performing the behaviour
- $f_6(t)$ represents the influence of self efficacy on performing the behaviour

The functional forms of $f_i(t)$, $i = 1, 2, \dots, 6$, can be highly complex and depend both on the behaviour and the specific pathogen. For example, if the behaviour is getting a vaccine, we may expect $f_4(t)$ to be (negatively) large due to general fear and possible side effects of the vaccine. However, if the pathogen has a high mortality rate we may expect the effect of $f_2(t)$ to be (positively) large. As a general rule to avoid parameter blow out, we suggest to parameterise each f_i around the disease states (S, I, R), the behaviour states (N, B), the model parameters (β, γ, ν, p, c), and time (t).

In a similar manner, we may write

$$\alpha(t) = g_1(t) + g_2(t) + g_3(t) + g_4(t) + g_5(t) + g_6(t),$$

where each g_i now describes the influence of the components of the HBM in *maintaining* the health behaviour B . For example, $g_1(t)$ describes the influence of the perceived susceptibility of the disease on an individuals ability to continue performing a behaviour. Once again these are very behaviour and disease dependant. Taking the example of vaccination as the behaviour, we note this is not a behaviour that someone can simply “stop doing”. In such a situation, many components of $\alpha(t)$ would be identically zero.

Behavioural model for mask wearing

Suppose our behaviour B is the act of wearing a mask. We consider the following parameterisation for uptake of mask wearing ($\omega(t)$):

1. $f_1(t) \propto I(t) = I_N(t) + I_{NN}(t) + I_B(t) + I_{BB}(t)$. This assumes that each individual in the N population is instantaneously aware of the exact prevalence of the disease at time t .
2. $f_2(t) \propto \sigma I(t)$, where σ is the (assumed constant) proportion of infected people who get severe cases of the disease (hospitalisation or death). Assuming a mild pathogen, we take $f_2(t) \approx 0$.
3. $f_3(t) = b_\omega(p, c)$, where f is a function describing the public perception of mask efficacy for both susceptible and infected individuals. Assuming (on average) little knowledge of the efficacy of masks for each individual, we take $b_\omega(p, c) = \text{constant}$.
4. $f_4(t) \approx 0$. We assume everyone has access to affordable mask options.
5. $f_5(t) \propto B + \text{constant}$. We consider the cues to action of wearing a mask to be described by the amount of people you see around you wearing a mask, as well as a constant source due to government advertising and signage and internal cues such as empathy for others.
6. $f_6(t) \approx \text{constant}$. We take self efficacy to be (on average) constant across the population as wearing a mask is a relatively low-effort behaviour.

Combining these assumptions, we write

$$\omega(t) = \omega_1 B(t) + \omega_2 I(t) + \omega_3$$

where $\omega_1, \omega_2, \omega_3$ are all positive constants.

In a similar manner we can build up the functional form for *dropping* the behaviour of mask wearing. We suggest

1. $g_1(t) \propto 1 - I(t)$. This assumes someone doing the behaviour assess their susceptibility by how many people in the population are *not* infected. This should capture the idea that many people seems to ignore that the behaviour is directly related to the reduction of infected peoples in the community. Note that since $1 - I \gg I$ (in general), the constant of proportionality should be appropriately small to avoid a “flooding effect”.
2. $g_2(t) \propto \sigma I(t)$. However, as before we assume a mild pathogen such that $g_2(t) \approx 0$.
3. $g_3(t) = b_\alpha(p, c)$, where g is a function describing the public perception of mask efficacy for both susceptible and infected individuals. Assuming (on average) little knowledge of the efficacy of masks for each individual, we take $b_\alpha(p, c) = \text{constant}$.
4. $g_4(t) \approx 0$. We assume there are little to no barriers for continued mask wearing.
5. $g_5(t) \propto N + \text{constant}$. We need to consider the cues to action for an individual to **stop** wearing a mask. We take this to be social cues of other individuals not wearing masks (N) as well as an “exhaustion” effect, which is an internal cue to action we capture through a constant.
6. $g_6(t) \approx \text{constant}$. We take the self efficacy of maintaining the behaviour of mask wearing as constant.

As with $\omega(t)$, we may write

$$\alpha(t) = \alpha_1 N(t) + \alpha_2 (1 - I(t)) + \alpha_3 ,$$

where $\alpha_1, \alpha_2, \alpha_3$ are all positive constants.

Notes

1. Assume that radicals are negligible for mask wearing, such that $BB \approx NN \approx 0$. This simplifies the model.
2. Look over short time span (more reasonable for masks?) so take $\nu = 0$.
3. Calculate R-effective using NGM
4. Calculate equilibrium of behaviour
5. If possible, final size of disease (is the behaviour useful?)
 - a. Very likely no analytic or semi analytic form

Behavioural model for vaccination

Suppose the behaviour B is the behaviour of getting vaccinated. We build the functional form $\omega(t)$ as follows:

1. $f_1(t) \propto I(t)$. This assumes that each individual in the N population is instantaneously aware of the exact prevalence of the disease at time t .
2. $f_2(t) \propto \sigma I(t)$, where σ is the (assumed constant) proportion of infected people who get sever cases of the disease (hospitalisation or death). Assuming a moderately high mortality rate, we consider $f_2(t) \neq 0$.
3. $f_3(t) = b_\omega(p, c) + \sigma_N (I_N(t) + I_{NN}(t)) / \sigma_B I_B(t)$, where f is a function describing the public perception of vaccination efficacy for both susceptible and infected individuals and σ_N and σ_B describe the mortality rates of non-vaccinated and vaccinated peoples respectively. Assuming (on average) little knowledge of the true efficacy of vaccination for each individual, we take $b_\omega(p, c) = \text{constant}$.
4. $f_4(t) \propto NN + \text{constant}$. This assumes the vocal “anti-vax” group may affect the uptake of a vaccination program, and the constant represents individual hesitancy and fear of possible side-effects of the vaccine.
5. $f_5(t) \propto \sigma_N (I_N(t) + I_{NN}(t)) + \text{constant}$. We consider the cues to action of wearing a mask to be described by the amount of unvaccinated people getting sever illness, as well as a constant source due to government advertising and signage. Note that vaccination is not a visible behaviour, so the proportion of people vaccinated will not affect the number of people to take up vaccination.
6. $f_6(t) \approx \text{constant}$. We take self efficacy to be (on average) constant across the population.

This creates the functional form

$$\omega(t) = \omega_1(1 + \sigma)I(t) + \omega_2\sigma_N(I_N(t) + I_{NN}(t))(1 + 1/(\sigma_B I_B(t) + \varepsilon)) - \omega_3 NN(t) + \omega_4.$$

Note that we have included the $\varepsilon \ll 1$ term to prevent dividing by zero. Note that although $\omega_1, \omega_2, \omega_3, \omega_4$ are all positive constants, they are constrained such that $\omega(t) \geq 0$ for all time t . ***As a note, we could extend this such that if $\omega(t) < 0$ people transition from N to NN .***

Now consider the “dropping of behaviour” $\alpha(t)$. Vaccination is not a behaviour that can be stopped as once you are vaccinated there is no going back. We could consider a two-step behaviour that captures “partial vaccination” and “full vaccination,” but for simplicity we will assume one vaccination does is enough for full coverage. However, although we take $g_1(t) = g_2(t) = g_3(t) = g_4(t) = g_5(t)$, we do consider $g_6(t) = \alpha$, a positive constant. We take the “self efficacy of maintaining vaccination status” to be described by the longevity of the vaccine coverage. As such,

$$\alpha(t) = \alpha$$

and we can interpret $1/\alpha$ as the average time that an individual remains effectively vaccinated before re-turning to “unvaccinated” status.

Notes

1. Assume that radicals are negligible for getting vaxxed, but vocal for not getting vaxxed. i.e. take $BB \approx 0$ and $NN > 0$.
2. Look over long time span (so we can consider vaccine wear off) so take $\nu > 0$.
3. Calculate R-effective using NGM?
4. Calculate equilibrium of behaviour?
5. If possible, steady state of disease (is the behaviour useful?)
 - a. Very likely no analytic or semi analytic form

The “behavioural R0”

I want to think about how “infectious” our behaviour is in the mask setting. Consider the behavioural transition equation:

$$\dot{B} = (\omega_1 B + \omega_2 I + \omega_3)N - (\alpha_1 N + \alpha_2(1 - I) + \alpha_3)B.$$

Suppose t is such that $I \approx \varepsilon$ where $\varepsilon \ll 1$ is a constant. Using $N = 1 - B$ we may write

$$\begin{aligned} \dot{B} &= (\omega_1 B + \omega_2 \varepsilon + \omega_3)N - (\alpha_1 N + \alpha_2(1 - \varepsilon) + \alpha_3)B \\ &= (\omega_1 B + \omega_2 \varepsilon + \omega_3)(1 - B) - (\alpha_1(1 - B) + \alpha_2(1 - \varepsilon) + \alpha_3)B \\ &= \omega_1 B + \omega_2 \varepsilon + \omega_3 - \omega_1 B^2 - \omega_2 \varepsilon B - \omega_3 B - \alpha_1 B + \alpha_1 B^2 - \alpha_2(1 - \varepsilon)B - \alpha_3 B \\ &= (\omega_2 \varepsilon + \omega_3) + (\omega_1 - \omega_2 \varepsilon - \omega_3 - \alpha_1 - \alpha_2(1 - \varepsilon) - \alpha_3)B + (\alpha_1 - \omega_1)B^2 \\ &= a_0 + a_1 B + a_2 B^2, \quad (*) \end{aligned}$$

where

$$\begin{aligned} a_0 &= \omega_2 \varepsilon + \omega_3 \\ a_1 &= \omega_1 - \omega_2 \varepsilon - \omega_3 - \alpha_1 - \alpha_2(1 - \varepsilon) - \alpha_3 \\ a_2 &= \alpha_1 - \omega_1. \end{aligned}$$

From Wolfram Alpha, this has the solution

$$B(t) = \frac{\sqrt{4a_0a_2 - a_1^2} \tan\left(\frac{1}{2}(c_1 + t)\sqrt{4a_0a_2 - a_1^2}\right) - a_1}{2a_2}, \quad (**)$$

where c_1 is a constant determined by the initial condition $B(0) = B_0$. Noting that $t \geq 0$, this solution exists under the dual conditions

$$\begin{aligned} a_1^2 &\leq 4a_0a_2 \quad \text{and,} \\ t &\leq \frac{\pi}{\sqrt{4a_0a_2 - a_1^2}} - c_1. \end{aligned}$$

Consider the first condition. For simplicity, take $\varepsilon \approx 0$ (alternatively, absorb it into appropriate constants). For this condition to hold, we need

$$(\omega_1 - \omega_3 - \alpha_1 - \alpha_2 - \alpha_3)^2 \leq 4(\alpha_1 - \omega_1)\omega_3.$$

Since every parameter here is positive, if $\omega_1 > \alpha_1$ this condition cannot hold. So consider $\omega_1 < \alpha_1$ (we ignore equality as this requires all parameters to be 0). We may then write

$$(\alpha_1 - \omega_1)^2 + 2(\alpha_1 - \omega_1)(\omega_3 + \alpha_2 + \alpha_3) + (\omega_3 + \alpha_2 + \alpha_3)^2 \leq 4(\alpha_1 - \omega_1)\omega_3,$$

or

$$(\alpha_1 - \omega_1)^2 + 2(\alpha_1 - \omega_1)(\alpha_2 + \alpha_3 - \omega_3) + (\omega_3 + \alpha_2 + \alpha_3)^2 \leq 0.$$

Since this is a convex quadratic in $(\alpha_1 - \omega_1)$, this inequality only holds if the determinant of the quadratic is greater than or equal to 0, i.e this quadratic needs to have real solutions. The determinant is

$$\begin{aligned} 4(\alpha_2 + \alpha_3 - \omega_3)^2 - 4(\omega_3 + \alpha_2 + \alpha_3)^2 &= 4(\alpha_2 + \alpha_3 - \omega_3 + \alpha_2 + \alpha_3 + \omega_3)(\alpha_2 + \alpha_3 - \omega_3 - (\alpha_2 + \alpha_3 + \omega_3)) \\ &= 4(2(\alpha_2 + \alpha_3))(-2\omega_3) \\ &= -16(\alpha_2 + \alpha_3)\omega_3 \\ &\leq 0 \end{aligned}$$

for all valid choices of $\alpha_2, \alpha_3, \omega_3$. This means that

$$(\alpha_1 - \omega_1)^2 + 2(\alpha_1 - \omega_1)(\alpha_2 + \alpha_3 - \omega_3) + (\omega_3 + \alpha_2 + \alpha_3)^2 \geq 0.$$

for all choices of $\alpha_1 - \omega_1$ and $(**)$ is never a valid solution to DE $(*)$.

Extensions

- Idea sparked by Emily: Consider the health belief model with distributions on the parameters, i.e. self efficacy is given by $b_6 \sim N(\mu_6, \sigma_6^2)$. Use this to capture individuality in behaviour.

Linearising the behavioural equations

Consider the transition of behaviour for mask wearing, where

$$\dot{B} = \omega(t)N(t) - \alpha(t)B(t).$$

Suppose we are considering an interval $[t_0, t_1]$ such that $B(t) \ll 1$ for $t \in [t_0, t_1]$. Then $N \approx 1$, so

$$\begin{aligned}\dot{B} &= (\omega_1 B(t) + \omega_2 I(t) + \omega_3) - (\alpha_1 + \alpha_2(1 - I(t)) + \alpha_3)B(t) \\ &= (\omega_1 - \alpha_1 - \alpha_2 - \alpha_3)B(t) + \alpha_2 I(t)B(t) + \omega_2 I(t) + \omega_3 \\ &= k_3 B(t) + k_2 I(t)B(t) + k_1 I(t) + k_0.\end{aligned}$$

For simplicity, assume $t_0 = 0$. Then the solution to this system is given by

$$B(t) = C_1 e^{\int_0^t k_3 + k_2 I(\xi) d\xi} + e^{\int_0^t k_3 + k_2 I(\xi) d\xi} \int_0^t e^{-\int_0^\zeta k_3 + k_2 I(\xi) d\xi} (k_0 + k_1 I(\zeta)) d\zeta.$$

1. This solution was originally derived by wolfram
2. Wolfram uses “1” at the bottom of the integrand rather than “0”

Derivation:

Consider the DE

$$\begin{aligned}\dot{B}(t) &= k_3 B(t) + k_2 I(t)B(t) + k_1 I(t) + k_0 \\ &= (k_3 + k_2 I(t))B(t) + (k_0 + k_1 I(t)).\end{aligned}\quad (1)$$

This is of the form

$$\dot{B}(t) - g(t)B(t) = r(t)$$

where

$$g(t) = k_3 + k_2 I(t) \quad \text{and} \quad r(t) = k_0 + k_1 I(t).$$

First consider the homogeneous equation

$$\dot{B}(t) - g(t)B(t) = 0. \quad (2)$$

Substituting $B(t) = e^{\sigma(t)}$ provides the characteristic equation

$$\dot{\sigma}(t) = g(t)$$

and so

$$\sigma(t) = \int_0^t g(\xi) d\xi = \int_0^t k_3 + k_2 I(\xi) d\xi.$$

Then

$$B(t) = C_1 e^{\sigma(t)} = C_1 e^{\int_0^t k_3 + k_2 I(\xi) d\xi}$$

is the solution to the homogeneous equation (2). Now consider the particular solution to (1). Using variation of parameters, we consider

$$B(t) = e^{\sigma(t)} u(t)$$

as a solution to (2). Differentiating and substituting, we find

$$\dot{\sigma}(t) e^{\sigma(t)} u(t) + e^{\sigma(t)} \dot{u}(t) - g(t) e^{\sigma(t)} u(t) = r(t).$$

Since $e^{\sigma(t)}$ is a solution to (2), this reduces to

$$e^{\sigma(t)} \dot{u}(t) = r(t),$$

or

$$\dot{u}(t) = e^{-\sigma(t)} r(t).$$

Integrating gives

$$u(t) = \int_0^t e^{-\sigma(\zeta)} r(\zeta) d\zeta = \int_0^t e^{-\int_0^\zeta k_3 + k_2 I(\xi) d\xi} (k_0 + k_1 I(\zeta)) d\zeta.$$

Combining the homogeneous and particular solutions gives the general solution to (1) of the form

$$B(t) = C_1 e^{\int_0^t k_3 + k_2 I(\xi) d\xi} + e^{\int_0^t k_3 + k_2 I(\xi) d\xi} \int_0^t e^{-\int_0^\zeta k_3 + k_2 I(\xi) d\xi} (k_0 + k_1 I(\zeta)) d\zeta.$$

END DERIVATION

Using the assumption $B(0) \approx 0$, this reduces to

$$B(t) = e^{\int_0^t k_3 + k_2 I(\xi) d\xi} \int_0^t e^{-\int_0^\zeta k_3 + k_2 I(\xi) d\xi} (k_0 + k_1 I(\zeta)) d\zeta. \quad (\star)$$

Simplifying $B(t)$ through estimating $I(t)$

We consider the following approximations of $I(t)$:

1. Constant
2. Linear
3. Quadratic
4. Exponential
5. Keeling and Rohani

With the exception of 1., no approximations yield a (useful) functional form for $B(t)$, but can be used in numerical approximations.

Constant

Suppose t is such that $I(t) \approx \varepsilon$, a constant. Then Equation (\star) reduces to

$$B(t) = \frac{k_0 + k_1\varepsilon}{k_3 + k_2\varepsilon} \left(e^{(k_3 + k_2\varepsilon)t} - 1 \right).$$

Linear

Suppose $I(t) \approx a_0 + a_1 t$. Recalling that $B(t)$ is small (approximately 0), and the population is normalised to 1, using a Taylor approximation about $t = 0$ we get

- $a_0 = I(0)$
- $a_1 = \dot{I}(0) \approx \beta I(0)(1 - I(0)) - \gamma I(0)$.

Quadratic

Suppose $I(t) \approx a_0 + a_1 t + \frac{a_2}{2} t^2$. Using a Taylor approximation about $t = 0$, we have a_0 and a_1 as above, and a_2 given by

$$a_2 = \frac{d^2 I}{dt^2}(0).$$

Now,

$$\dot{I}(t) = \beta(S_N(t) + (1 - c)S_B(t))(I_N(t) + (1 - p)I_B(t)) - \gamma I(t),$$

where $I(t) = I_N(t) + I_B(t)$. Thus the second derivative is

$$\begin{aligned} \frac{d}{dt} \dot{I}(t) &= \beta [(\dot{S}_N(t) + (1 - c)\dot{S}_B(t))(I_N(t) + (1 - p)I_B(t)) + (S_N(t) + (1 - c)S_B(t))(\dot{I}_N(t) + (1 - p)\dot{I}_B(t))] \\ &\quad - \gamma \dot{I}(t). \end{aligned}$$

Evaluating this at $t = 0$ and using $B(t) \approx 0$ gives

$$\frac{d}{dt} \dot{I}(0) \approx \beta [(\dot{S}_N(0) + (1 - c)\dot{S}_B(0))I(0) + (1 - I(0))(\dot{I}_N(0) + (1 - p)\dot{I}_B(0))] - \gamma a_1,$$

so we need to calculate $\dot{S}_N(0)$, $\dot{S}_B(0)$, $\dot{I}_N(0)$, and $\dot{I}_B(0)$. We see that

$$\begin{aligned} \dot{S}_N(0) &= -\beta(I_N(0) + (1 - p)I_B(0))S_N(0) - \omega(0)S_N(0) + \alpha(0)S_B(0) + \nu R_N(0) \\ &\approx -\beta I(0)(1 - I(0)) - (\omega_1 B(0) + \omega_2 I(0) + \omega_3)(1 - I(0)) \\ &\approx [-\beta I(0) - (\omega_2 I(0) + \omega_3)](1 - I(0)), \end{aligned}$$

and

$$\begin{aligned} \dot{S}_B(0) &= -\beta(1 - c)(I_N(0) + (1 - p)I_B(0))S_B(0) + \omega(0)S_N(0) - \alpha(0)S_B(0) + \nu R_B(0) \\ &\approx (\omega_2 I(0) + \omega_3)(1 - I(0)). \end{aligned}$$

Similarly, we find

$$\begin{aligned}\dot{I}_N(0) &= \beta(I_N(0) + (1-p)I_B(0))S_N(0) - \omega(0)I_N(0) + \alpha(0)I_B(0) - \gamma I_N(0) \\ &\approx \beta I(0)(1 - I(0)) - (\omega_1 B(0) + \omega_2 I(0) + \omega_3)I(0) - \gamma I(0) \\ &\approx [\beta(1 - I(0)) - (\omega_2 I(0) + \omega_3) - \gamma] I(0),\end{aligned}$$

and

$$\begin{aligned}\dot{I}_B(0) &= \beta(1 - c)(I_N(0) + (1-p)I_B(0))S_B(0) + \omega(0)I_N(0) - \alpha(0)I_B(0) - \gamma I_B(0) \\ &\approx (\omega_2 I(0) + \omega_3)I(0).\end{aligned}$$

Putting these together gives

$$a_2 = \beta(1 - I(0))I(0) (\beta(1 - 2I(0)) - (c + p)(\omega_2 I(0) + \omega_3)) - \gamma a_1.$$

Exponential

For small B and I , we can approximate

$$\dot{I}(t) \approx \beta I(t) - \gamma(t),$$

i.e. we return the basic SIR model for small time and behaviour. Solving this gives

$$I(t) \approx I(0) \exp((\beta - \gamma)t).$$

Keeling and Rohani

From Box 2.2 in K& R we get

$$I(t) = \frac{\sigma^2}{2S(0)\mathcal{R}_0^2} \operatorname{sech}^2 \left(\frac{1}{2} \sigma \gamma t - \phi \right),$$

where

$$\sigma = [(S(0)\mathcal{R}_0 - 1)^2 + 2S(0)I(0)\mathcal{R}_0^2]^{1/2},$$

and

$$\phi = \tanh^{-1} \left[\frac{1}{\sigma} (S(0)\mathcal{R}_0 - 1) \right].$$

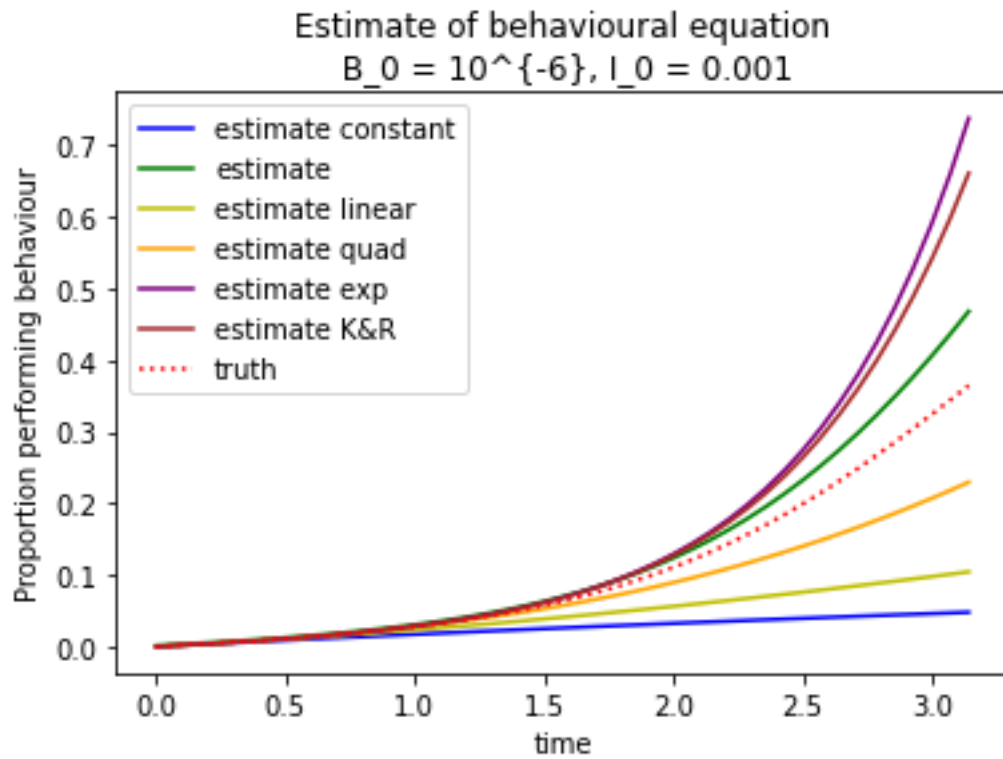


Figure 4: Estimates of the early term behaviour for the time period they are valid. The green 'estimate' line shows the estimate of B using the observed values of $I(t)$ from the solve ivp function.

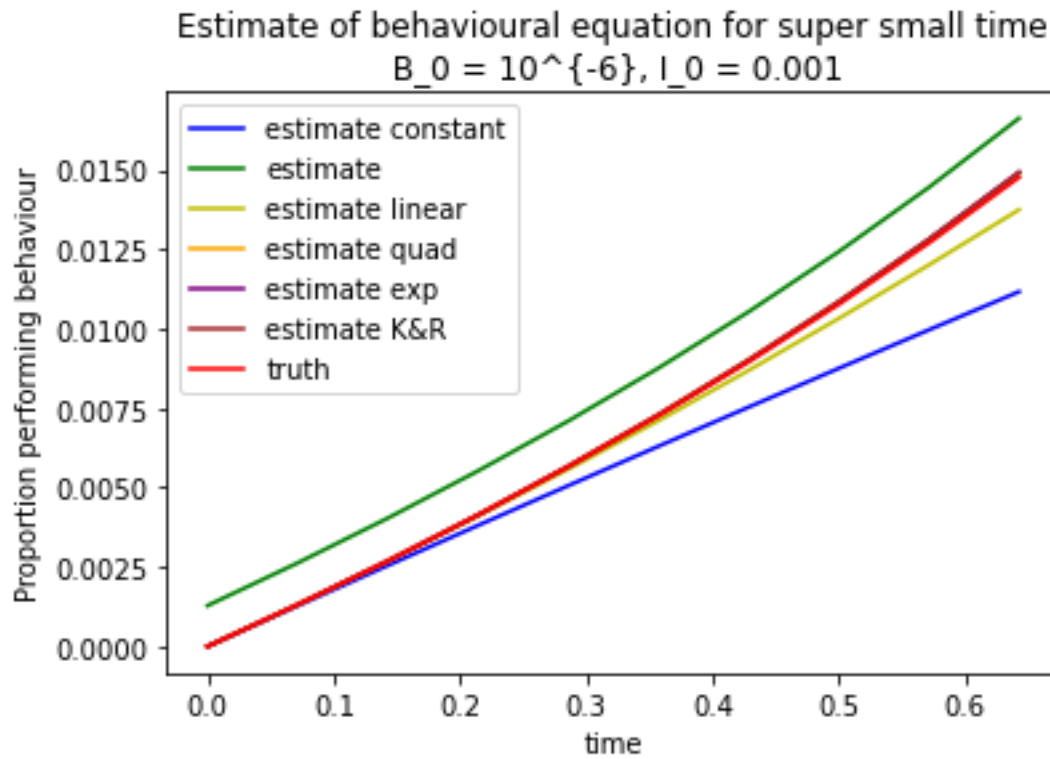


Figure 5: Estimates of the early term behaviour for early time. The green 'estimate' line shows the estimate of B using the observed values of $I(t)$ from the solve ivp function.

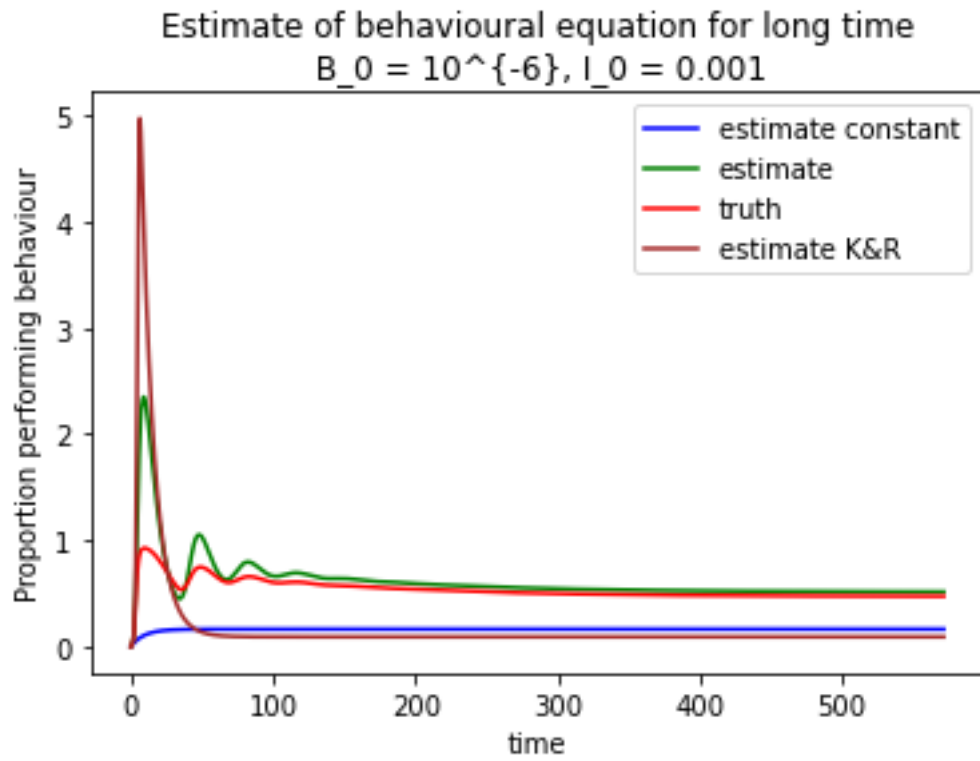


Figure 6: Estimates of the early term behaviour for all time simulated. The green 'estimate' line shows the estimate of B using the observed values of $I(t)$ from the solve ivp function.

Simplifying the behaviour model estimate

Although (\star) does not have an analytic solution in general, we can derive many analytic solutions when $k_2 = \alpha_2 = 0$. This assumption implies there is **no** effect of “fear of disease” for stopping people doing the behaviour. That is, people will not stop doing the behaviour because there is “little” disease in the community. Under this assumption, we can rewrite (\star) as

$$\begin{aligned} B(t) &= e^{k_3 t} \int_0^t e^{-k_3 \zeta} (k_0 + k_1 I(\zeta)) d\zeta \\ &= \frac{k_0}{k_3} (e^{k_3 t} - 1) + k_1 e^{k_3 t} \int_0^t e^{-k_3 \zeta} I(\zeta) d\zeta. \quad (\star\star) \end{aligned}$$

We can see then that $B(t)$ has a “behaviour affecting behaviour” term ($b(t) = \frac{k_0}{k_3} (e^{k_3 t} - 1)$) and an “infection affecting behaviour” term ($i(t) = k_1 e^{k_3 t} \int_0^t e^{-k_3 \zeta} I(\zeta) d\zeta$). We now solve $i(t)$ under some approximation of $I(t)$. The approximations considered are

1. Polynomial
2. Exponential
3. Keeling and Rohani

Polynomial

Suppose we write

$$I(t) \approx \sum_{m=0}^M a_m t^m$$

for some constants a_0, a_1, \dots, a_M (think Taylor expansion). This scenario captures the constant, linear, and quadratic approximations considered above. Fix M , and consider

$$\begin{aligned} i(t) &= k_1 e^{k_3 t} \int_0^t e^{-k_3 \zeta} I(\zeta) d\zeta \\ &\approx k_1 e^{k_3 t} \sum_{m=0}^M a_m \int_0^t \zeta^m e^{-k_3 \zeta} d\zeta \\ &= k_1 e^{k_3 t} \sum_{m=0}^M a_m E_m(t), \end{aligned}$$

where we have written

$$E_m(t) = \int_0^t \zeta^m e^{-k_3 \zeta} d\zeta.$$

Using integration by parts on $E_m(t)$ we find

$$\begin{aligned} E_m(t) &= \int_0^t \zeta^m e^{-k_3 \zeta} d\zeta \\ &= -\frac{\zeta^m e^{-k_3 \zeta}}{k_3} \Big|_0^t + \frac{m}{k_3} \int_0^t \zeta^{m-1} e^{-k_3 \zeta} d\zeta \\ &= \frac{m}{k_3} E_{m-1}(t) - \frac{t^m e^{-k_3 t}}{k_3}, \end{aligned}$$

which defines a recurrence relation noting that

$$\begin{aligned} E_0(t) &= \int_0^t e^{-k_3 \zeta} d\zeta \\ &= \frac{1}{k_3} (1 - e^{-k_3 t}) . \end{aligned}$$

Defining

$$F_m(t) = e^{k_3 t} E_m(t)$$

gives the relation

$$F_m(t) = \frac{m}{k_3} F_{m-1}(t) - \frac{t^m}{k_3}$$

with starting point

$$F_0(t) = \frac{1}{k_3} (e^{k_3 t} - 1) .$$

Then

$$\begin{aligned} i(t) &\approx k_1 e^{k_3 t} \sum_{m=0}^M a_m E_m(t) \\ &= k_1 \sum_{m=0}^M a_m F_m(t) \end{aligned}$$

gives a solution for the affect of infection on behaviour under a polynomial approximation. Note that this agrees with the previous solution under a constant approximation for $I(t)$ when $k_2 = 0$.

Exponential

Suppose

$$I(t) \approx I(0) \exp((\beta - \gamma)t) .$$

Then we may write

$$\begin{aligned} i(t) &= k_1 e^{k_3 t} \int_0^t e^{-k_3 \zeta} I(\zeta) d\zeta \\ &\approx I(0) k_1 e^{k_3 t} \int_0^t e^{(\beta - \gamma - k_3)\zeta} d\zeta \\ &= \frac{I(0) k_1}{\beta - \gamma - k_3} (e^{(\beta - \gamma)t} - e^{k_3 t}) \end{aligned}$$

Keeling and Rohani

The approximation of the SIR solution from K& R is

$$I(t) = \frac{\sigma^2}{2S(0)\mathcal{R}_0^2} \operatorname{sech}^2 \left(\frac{1}{2} \sigma \gamma t - \phi \right) ,$$

where

$$\sigma = \left[(S(0)\mathcal{R}_0 - 1)^2 + 2S(0)I(0)\mathcal{R}_0^2 \right]^{1/2} ,$$

and

$$\phi = \tanh^{-1} \left[\frac{1}{\sigma} (S(0)\mathcal{R}_0 - 1) \right] .$$

Inserting this into $i(t)$ gives a solution using hypergeometric functions. Not convinced this is a tree I want to bark up.