Extended BaD Testing and Isolation with optimisation

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1 Introduction

During COVID-19, multiple testing regimes were implemented using RATs over the course of an infectious period. For example, people were recommended to test on days 3, 5, and 7 after coming into contact with a known positive case.

We want to explore these testing regimes under dynamically evolving behaviour and see if and how recommendations might change.

Research question

Suppose each individual can test n times in k days over their infectious period, $n < k$. Define the optimal testing strategy as the selection of n days on which to test which minimises a particular epidemiological outcome (e.g. final size, long term infections, peaks, etc.).

What is the optimal testing strategy for each of the scenarios described below?

Let $\Delta \in \{0,1\}^n$ be such that $\sum \Delta_i = n$. The optimal testing strategy is found by

$$\min_{\Delta}\{s(T) + \kappa s(I)\}$$

where s is a summary of the compartments T and I that is of interest, for example,

$$s(T) = \begin{cases} T(\infty) & \text{(Final size, steady state)} \\ \max(T) & \text{(Peaks)}\,. \end{cases}$$

The first component $s(T)$ captures the observed cases from individual testing, whereas $\kappa s(I)$ captures testing from severe cases, i.e. hospitalisations.

Probably need to workshop this. Likely to return "optimal" testing to be such that we have hardly any T, which could make epidemic much worse.

Research scenarios

We consider:

1. A constant behaviour scenario - $X\%$ of the population are willing to test if showing symptoms.
2. A Dynamic behaviour scenario - Allow behaviour to adapt to the epidemic using behavioural feedback.
3. Dynamic behaviour with time-varying viral load.
4. All of the above scenarios with a test stockpile. For testing stockpile we might investigate $\sum \Delta_i \leq k$ to see whether it is optimal to test less so we observe more in the long run (by not running out of tests).

2 Model

We extend the behaviour and disease model for testing and isolation by Ryan *et al.* [1] to include multiple testing. This is to capture situations of readily available tests that can be conducted multiple times throughout an infectious period, similar to RATs during COVID-19.

We consider an $SEIR$ infection structure with asymptomatic A and isolated individuals T. We allow multiple testing through considering k compartments for each infectious class (I, A, T). We consider two behavioural strata: B - those who are willing to test if showing symptoms, N - those who are not. We consider the same behavioural dynamics of [1], specifically:

$$\omega(B, T) = \phi_1(B) + \phi_2(T) + \omega_3 = \omega_1 B + \omega_2 T + \omega_3 + \mathcal{O}(B, T)^2 \tag{1}$$

where ϕ_1 represents the influence of social contagion on testing behaviour, ϕ_2 accounts for behaviour uptake driven by the perception of illness threat, and ω_3 captures spontaneous adoption of the behaviour. The rate of behaviour loss is

$$\alpha(N) = \psi(N) + \alpha_2 = \alpha_1 N + \alpha_2 + \mathcal{O}(N)^2 \tag{2}$$

where ψ represents the influence of social contagion on not seeking testing and α_2 captures spontaneous abandonment of the behaviour.

The extend BaD model for symptomatic testing is described by the following differential equations with the parameters detailed in Table 1 and visualised in Figure 1.

$$\frac{\mathrm{d}S_N}{\mathrm{d}t} = -\lambda S_N + \nu R_N - \omega(B,T)S_N + \alpha(N)S_B \tag{3}$$

$$\frac{\mathrm{d}E_N}{\mathrm{d}t} = \lambda S_N - \sigma E_N - \omega(B,T)E_N + \alpha(N)E_B \tag{4}$$

$$\frac{\mathrm{d}A_N^{(1)}}{\mathrm{d}t} = p_A \sigma E_N - k\gamma A_N^{(1)} - \omega(B,T)A_N^{(1)} + \alpha(N)A_B^{(1)} \tag{5}$$

$$\frac{\mathrm{d}A_N^{(l)}}{\mathrm{d}t} = k\gamma A_N^{(l-1)} - k\gamma A_N^{(l)} - \omega(B,T)A_N^{(l)} + \alpha(N)A_B^{(l)}\text{ , for } l = 2,3,\ldots,k \tag{6}$$

$$\frac{\mathrm{d}I_N^{(1)}}{\mathrm{d}t} = (1-p_A)\,\sigma E_N - k\gamma I_N^{(1)} - \omega(B,T)I_N^{(1)} + \alpha(N)I_B^{(1)} \tag{7}$$

$$\frac{\mathrm{d}I_N^{(l)}}{\mathrm{d}t} = k\gamma I_N^{(l-1)} - k\gamma I_N^{(l)} - \omega(B,T)I_N^{(l)} + \alpha(N)I_B^{(l)}\text{ , for } l = 2,3,\ldots,k \tag{8}$$

$$\frac{\mathrm{d}R_N}{\mathrm{d}t} = k\gamma\left(A_N^{(k)} + I_N^{(k)}\right) - \nu R_N - \omega(B,T)R_N + \alpha(N)R_B \tag{9}$$

$$\frac{\mathrm{d}S_B}{\mathrm{d}t} = -q_B\lambda S_B + \nu R_B + \omega(B,T)S_N - \alpha(N)S_B \tag{10}$$

$$\frac{\mathrm{d}E_B}{\mathrm{d}t} = q_B\lambda S_B - \sigma E_B + \omega(B,T)E_N - \alpha(N)E_B \tag{11}$$

$$\frac{\mathrm{d}A_B^{(1)}}{\mathrm{d}t} = p_A \sigma E_B - k\gamma A_B^{(1)} + \omega(B,T)A_N^{(1)} - \alpha(N)A_B^{(1)} \tag{12}$$

$$\frac{\mathrm{d}A_B^{(l)}}{\mathrm{d}t} = k\gamma A_B^{(l-1)} - k\gamma A_B^{(l)} + \omega(B,T)A_N^{(l)} - \alpha(N)A_B^{(l)}\text{ , for } l = 2,3,\ldots,k \tag{13}$$

$$\frac{\mathrm{d}I_B^{(1)}}{\mathrm{d}t} = (1-p_A)\left(1-\delta_0 p_T^{(0)}\right)\sigma E_B - k\gamma I_B^{(1)} + \omega(B,T)I_N^{(1)} - \alpha(N)I_B^{(1)} \tag{14}$$

$$\frac{\mathrm{d}I_B^{(l)}}{\mathrm{d}t} = \left(1-\delta_{l-1}p_T^{(l-1)}\right)k\gamma I_B^{(l-1)} - k\gamma I_B^{(l)} + \omega(B,T)I_N^{(l)} - \alpha(N)I_B^{(l)}\text{ , for } l = 2,3,\ldots,k \tag{15}$$

$$\frac{\mathrm{d}T^{(1)}}{\mathrm{d}t} = (1-p_A)\,\delta_0 p_T^{(0)}\sigma E_B - k\gamma T^{(1)} \tag{16}$$

$$\frac{\mathrm{d}T^{(l)}}{\mathrm{d}t} = \delta_{l-1}p_T^{(l-1)}k\gamma I_B^{(l-1)} + k\gamma T^{(l-1)} - k\gamma T^{(l)}\text{ , for } l = 2,3,\ldots,k \tag{17}$$

$$\frac{\mathrm{d}R_B}{\mathrm{d}t} = \gamma(A_B^{(k)} + I_B^{(k)} + T^{(k)}) - \nu R_B + \omega(B,T)R_N - \alpha(N)R_B. \tag{18}$$

In the above ODEs, we have that

1. δ_j is an indicator which is 1 if testing happens on day j and 0 otherwise.
2. $p_T^{(j)}$ is the testing effectiveness on day j.
3. k is the division of the infectious period on which individuals may test.

The force of infection on non-behavers is given by

$$\lambda = \sum_{l=1}^{k} \beta_l \left(I_N^{(l)} + I_B^{(l)} + q_A \left(A_N^{(l)} + A_B^{(l)} \right) + q_T T^{(l)} \right)$$

whereas for behavers it is $q_B \lambda$. The behavioural dynamics are governed by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha(N)(B - T) - \omega(B, T)N \,, \tag{19}$$

where

$$T = \sum_{l=1}^{k} T^{(l)} \,.$$

3 Theoretical Analyses

Steady States

We determine the steady states in the simplified case when $\beta = \beta_l$ for all $l = 1, 2, \ldots, k$. Note that this simplifies the force of infection to be

$$\lambda = \beta \left(I_N + I_B + q_A \left(A_N + A_B \right) + q_T T \right)$$

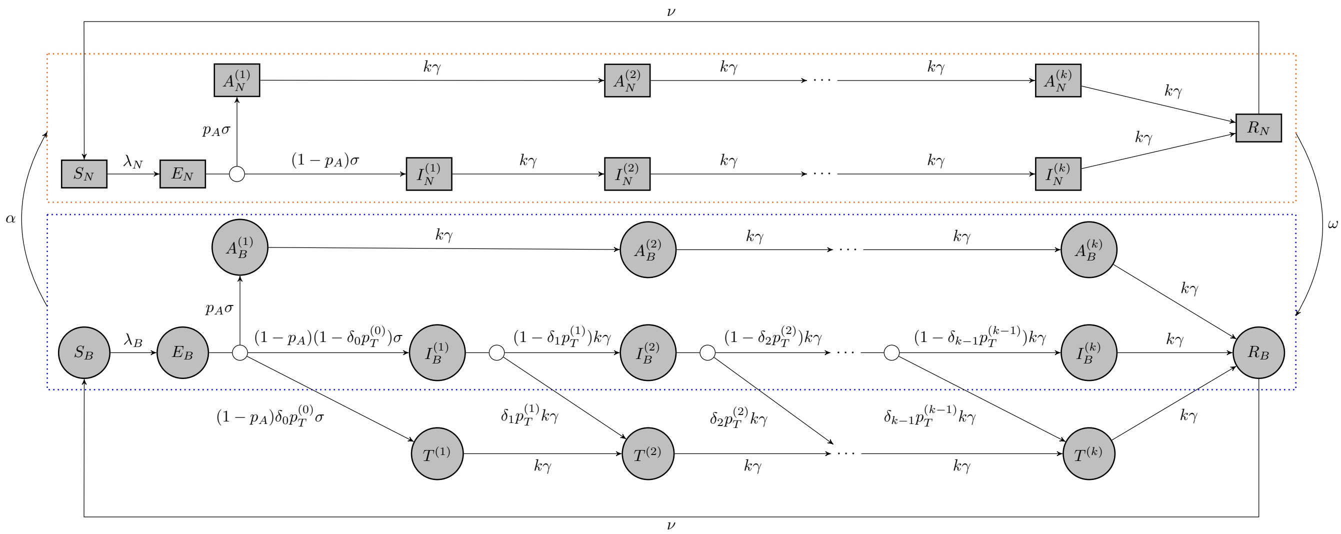

Figure 1: **A compartmental diagram illustrating the extended BaD model for symptomatic testing.** We distinguish between individuals who do not seek testing when symptomatic, referred to as *non-behavers* (labelled N, rectangles) and those who seek testing when symptomatic, referred to as *behavers* (labelled B, circles). Changes in testing behaviour are governed by the rates of behaviour uptake (ω) and abandonment (α), denoted as single arrows between the two dotted bounding boxes - note there is no behaviour transition out of the testing states T. The epidemiological states are susceptible (S), exposed (E), asymptomatic infectious (or pauci-symptomatics) labelled A, symptomatic infectious who are labelled I if they do not seek *or* do not receive a positive test and T if they seek *and* receive a positive test, and those who are recovered (R). The transitions between epidemiological states follow a standard SEIRS framework, with k compartments for each infectious state and waning immunity where recovered individuals can become susceptible again. We define the model parameters in Table 1.

where we have written

$$A_N = \sum_{l=1}^{k} A_N^{(l)}$$

$$A_B = \sum_{l=1}^{k} A_B^{(l)}$$

$$I_N = \sum_{l=1}^{k} I_N^{(l)}$$

$$I_B = \sum_{l=1}^{k} I_B^{(l)}$$

Denote the steady state of compartment X and X^*. First observe that knowing the steady state T^* fixes the steady state of behaviour N^* and $B^* = 1 - N^*$ by Equation (19) (setting the non-linear terms to zero). In turn, knowing T^* fixes $\alpha^* = \alpha(N^*)$ and $\omega^* = \omega(B^*, T^*)$.

Now, adding across the behavioural strata and writing $O^{(l)} = I_N^{(l)} + I_B^{(l)} + T^{(l)}$ reduces Equations

Table 1: **Behavioural and epidemiological parameters for the model.** The parameters and functions defined by Greek letters are in units of time^{-1}, which have been scaled to such that the infectious period is one. All other parameters are dimensionless. Where possible, the last column contains references for the parameter choices. We chose all other parameters to take *a priori* reasonable values with a non-trivial impact on the dynamics.

Parameter	**Description**	**Value**	**Ref.**
$\lambda(t)$	the force of infection		
β_l	the transmission rate on day l	4.151	[?]
q_A	the reduction in infectiousness from being asymptomatic	0.58	[?]
q_T	the reduction in infectiousness from isolation	0.25	
q_B	the reduction in susceptibility from the testing behaviour	0.5	[?]
σ	the transition rate from E to I (average latency period is $1/\sigma$)	2.5	[?]
γ	the recovery rate (average infectious period is $1/\gamma$)	1.0	
ν	the rate of waning immunity (average immune period is $1/\nu$)	0.05	[?]
k	the division of the infectious period to allow for multiple testing	7	
p_A	probability of not developing symptoms	0.18	[?]
$p_T^{(l)}$	test effectiveness for symptomatic individuals on day l	0.9	
δ_j	An indicator of whether testing is done on day j	0.9	
$\omega(t)$	the "force of infection" for testing behaviour uptake		
ω_1	the social transmission rate	0.25	
ω_2	rate of response to perceived illness threat	7.0	
ω_3	spontaneous uptake rate	0.0125	
$\alpha(t)$	the "force of infection" for testing behaviour abandonment		
α_1	the social abandonment rate	0.2	
α_2	spontaneous abandonment rate	0.1	

(3)–(18) to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \lambda S_N + q_A \lambda S_B - \sigma E \tag{20}$$

$$\frac{\mathrm{d}A^{(1)}}{\mathrm{d}t} = p_A \sigma E - k\gamma A^{(1)} \tag{21}$$

$$\frac{\mathrm{d}A^{(l)}}{\mathrm{d}t} = k\gamma A^{(l-1)} - k\gamma A^{(l)} \tag{22}$$

$$\frac{\mathrm{d}O^{(1)}}{\mathrm{d}t} = (1-p_A)\sigma E - k\gamma O^{(1)} \tag{23}$$

$$\frac{\mathrm{d}O^{(l)}}{\mathrm{d}t} = k\gamma O^{(l-1)} - k\gamma O^{(l)} \tag{24}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = k\gamma (A^{(k)} + O^{(k)}) - \nu R \tag{25}$$

$$\tag{26}$$

and $S = 1-(E+A+O+R)$, where $\mathcal{X} = \mathcal{X}_N + \mathcal{X}_B$ for a compartment $\mathcal{X} \in \{S, E, A, O, R\}$. Suppose now, in addition to knowing T^*, we know $O^{(k)*}$. Then adding Equations (23) and (24) and setting it to zero gives

$$E^* = \frac{k\gamma}{1-p_A}\sigma O^{(k)*} \, ,$$

and setting Equation (24) to zero gives

$$O^{(l)*} = O^{(k)^*}$$

for each $l = 1, 2, 3, \ldots, k-1$. Now, adding Equations (21) and (22) and setting to zero gives

$$A^{(k)*} = \frac{p_A \sigma}{k\gamma} E^* \, ,$$

and setting Equation (22) to zero gives

$$A^{(l)*} = A^{(k)^*}$$

for each $l = 1, 2, 3, \ldots, k-1$. Then setting Equation (25) to zero gives

$$R^* = \frac{k\gamma}{\nu}\left(A^{(k)*} + O^{(k)*}\right)$$

and $S^* = 1 - (E^* + A^* + O^* + R^*)$. Finally, we fix

$$\lambda^* = \beta \left(O^* - T^* + q_A \left(A^*\right) + q_T T^*\right) \, .$$

In summary, knowing T^* and $O^{(k)*}$ fixes values for N^*, B^*, α^*, ω^*, S^*, E^*, $A^{(l)*}$, $O^{(j)*}$, R^*, and λ^*, where $l = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k-1$.

We now turn our attention to calculating the steady states of the behavioural strata in each epidemiological compartment. First, setting Equations (3) and (10) to zero gives

$$\lambda^* S_N^* = (\omega^* + \sigma)E^* - (\alpha^* + \omega^* + \sigma)E_B^* \tag{27}$$

$$\lambda^* S_B^* = \frac{-\omega^* E^* - (\alpha^* + \omega^* + \sigma)E_B^*}{q_B} \, . \tag{28}$$

Adding Equations (27) and (28) and solving for E_B^* gives

$$E_B^* = \frac{q_B \lambda^* S^* + (\omega^* - q_B(\omega^* + \sigma^*) E^*}{(1-q_B)(\alpha^* + \omega^* + \sigma)} \,. \tag{29}$$

Setting Equation (12) to zero gives

$$A_B^{(1)*} = \frac{p_A \sigma E_B^* + \omega^* A^{(1)*}}{k\gamma + \alpha^* + \omega^*} \,, \tag{30}$$

and setting Equation (13) to zero gives

$$A_B^{(l)*} = \frac{k\gamma A_B^{(l-1)} + \omega^* A^{(l)*}}{k\gamma + \alpha^* + \omega^*} \,. \tag{31}$$

Setting Equations (14) and (15) to zero gives

$$I_B^{(1)*} = \frac{(1-p_A)\left(1 - \delta_0 p_T^{(0)}\right)\sigma E_B^* + \omega^* I^{(1)*}}{k\gamma + \alpha^* + \omega^*} \tag{32}$$

$$I_B^{(l)*} = \frac{\left(1 - \delta_{l-1} p_T^{(l-1)}\right) k\gamma I_B^{(l-1)*} + \omega^* I^{(l)*}}{k\gamma + \alpha^* + \omega^*} \,. \tag{33}$$

Setting Equation (18) to zero gives

$$R_B^* = \frac{k\gamma\left(I_B^{(k)*} + A_B^{(k)*} + T^{(k)*}\right) + \omega^* R^*}{\nu + \alpha^* + \omega^*} \,. \tag{34}$$

Setting Equations (16) and (17) to zero gives

$$T^{(1)*} = \frac{(1-p_A)\delta_0 p_T^{(0)} \sigma E_B^*}{k\gamma} \tag{35}$$

$$T^{(l)*} = T^{(l-1)*} + \delta_{l-1} p_T^{(l-1)} I_B^{(l-1)*} \,. \tag{36}$$

Finally, we obtain $I^{(l)*} = O^{(l)*} - T^{(l)*}$ for $l = 1, 2, \ldots, k$. All other compartments are determined by subtraction.

What remains is to fix the values of T^* and $O^{(k)^*}$ to determine the system. Adding Equations

(35) and (36) gives

$$T^* = \frac{(1-p_A)\delta_0 p_T^{(0)} \sigma E_B^*}{k\gamma} + \sum_{l=2}^{k} \left\{ T^{(l-1)*} + \delta_{l-1} p_T^{(l-1)} I_B^{(l-1)*} \right\} ,$$

which fixes T^*. Setting Equation (3) to zero gives

$$(\lambda^* + \omega^*) S_N^* = \alpha^* S_B^* + \nu R_N^* ,$$

which fixes I^*, which fixes $O^* = I^* + T^*$ and hence $O^{(k)*}$.

Reproduction number

To simplify presentation of equations, we assume that $\delta_l = \delta$ and $p_T^{(l)} = p_T$ for all $l = 1, 2, \ldots, k$; please note this is not necessary to derive the reproduction number.

Let $t_{E_N}(\mathcal{X})$ be the average time spent in $\mathcal{X}$ with state-at-infection E_N.

For $l \leq k$, we have that

$$\begin{aligned} t_{E_N}(A_N^{(l)}) = \frac{p_A}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l} & \Bigg\{ (\sigma+\alpha_0) \Bigg[(k\gamma+\alpha_0)^l + \\ & \sum_{j=1}^{\lceil \frac{l}{2} \rceil} (\alpha_0\omega_0)^j Z(l-2j) \sum_{h=0}^{l-2j} C_h^j(A_N^{(l)}) (k\gamma+\alpha_0)^h (k\gamma+\omega_0)^{l-2j-h} \Bigg] + \\ & \sum_{j=1}^{\lceil \frac{l}{2} \rceil} (\alpha_0\omega_0)^j \left[\sum_{i=0}^{l+1-2j} D_i^j(A_N^{(l)}) (k\gamma+\alpha_0)^i (k\gamma+\omega_0)^{l+1-2j-i} \right] \Bigg\} \end{aligned}$$

where $C_i^j(A_N^{(l)})$ and $D_h^j(A_N^{(l)})$ are coefficient terms in the sum dependent on the loop number j, the indices i and h, and the target compartment $A_N^{(l)}$, and $Z\,(l-2j)$ is an indicator that is one when $l - 2j \geq 0$ and zero otherwise.

For $l \leq k$, we have that

$$t_{E_N}(I_N^{(l)}) = \frac{(1-p_A)}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l} \left\{(\sigma+\alpha_0)\left[(k\gamma+\alpha_0)^l + \sum_{j=1}^{\lceil \frac{l}{2} \rceil}(\alpha_0\omega_0(1-\delta p_T))^j Z\left(l-2j\right)\sum_{h=0}^{l-2j} C_h^j(I_N^{(l)})(k\gamma+\alpha_0)^h \left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-2j-h}\right] + \sum_{j=1}^{\lceil \frac{l}{2} \rceil}(\alpha_0\omega_0(1-\delta p_T))^j \sum_{i=0}^{l+1-2j} D_i^j(I_N^{(l)})(k\gamma+\alpha_0)^i \left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l+1-2j-i}\right\}$$

For $l \leq k$, we have that

$$t_{E_N}(A_B^{(l)}) = \frac{p_A}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l} \left\{\omega_0\left[(k\gamma+\omega_0)^l + Z(l-2)\sum_{j=1}^{\lfloor \frac{l}{2} \rfloor}(\alpha_0\omega_0)^j \sum_{i=0}^{l-2j} C_i^j(A_B^{(l)})(k\gamma+\alpha_0)^i(k\gamma+\omega_0)^{l-2j-i}\right] + (\sigma+\alpha_0)\omega_0\left[\sum_{h=0}^{l-1}(k\gamma+\alpha_0)^h(k\gamma+\omega_0)^{l-1-h} + \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor}(\alpha_0\omega_0)^j Z(l-1-2j)\sum_{h=0}^{l-1-2j} D_h^j(A_B^{(l)})(k\gamma+\alpha_0)^h(k\gamma+\omega_0)^{l-1-2j-h}\right]\right)$$

For $l \leq k$, we have that

$$t_{E_N}(I_B^{(l)}) = \frac{(1-p_A)}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l} \left\{\omega_0\left[(k\gamma+\omega_0)^l(1-\delta p_T)^l + Z(l-2)\sum_{j=1}^{\lfloor \frac{l}{2} \rfloor}(\alpha_0\omega_0(1-\delta p_T))^j \sum_{i=0}^{l-2j} C_i^j(A_B^{(l)})(k\gamma+\alpha_0)^i \left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-2j-i}\right] + \omega_0(\sigma+\alpha_0)\left[\sum_{h=0}^{l-1}(k\gamma+\alpha_0)^h \left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-1-h} + Z(l-2)\times \sum_{j=1}^{\lfloor \frac{l}{2} \rfloor}(\alpha_0\omega_0(1-\delta p_T))^j Z(l-1-2j)\sum_{h=0}^{l-1-2j} D_h^j(A_B^{(l)})(k\gamma+\alpha_0)^h \left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-1-2j-h}\right]\right\}$$

For $l \le k$ we have

$$t_{E_N}\left(T^{(l)}\right) =$$

$$\frac{(1-p_A)}{k\gamma(\sigma+\alpha_0+\omega_0)}\left\{Z(l-2)\frac{\delta p_T\omega_0(\sigma+\alpha_0)}{(k\gamma+\alpha_0+\omega_0)}\left[\sum_{j=0}^{l-2}\left(\frac{(k\gamma+\alpha_0)}{(k\gamma+\alpha_0+\omega_0)}\right)^j\times\right.\right.$$

$$\left.\left[\sum_{i=0}^{l-2-j}\left(\frac{(k\gamma+\omega)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T)\right)^i\right]\right] + Z(l-4)\sum_{h=1}^{\lfloor\frac{l-2}{2}\rfloor}\left(\frac{\alpha_0\omega_0(1-\delta p_T)}{(k\gamma+\alpha_0+\omega_0)^2}\right)^h\times$$

$$\left.\left[\sum_{g=0}^{l-2-2h}\left(C_g^h\left(T^{(l)}\right)\frac{(k\gamma+\omega_0)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T)+D_g^h\left(T^{(l)}\right)\frac{(k\gamma+\alpha_0)}{(k\gamma+\alpha_0+\omega_0)}\right)^g\right]\right]+$$

$$\delta p_T\omega_0\left[\sum_{j=0}^{l-1}\left[\frac{(k\gamma+\omega_0)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T)\right]^j + Z(l-3)\sum_{h=1}^{\lfloor\frac{l-1}{2}\rfloor}\left(\frac{\alpha_0\omega_0(1-\delta p_T)}{(k\gamma+\alpha_0+\omega_0)^2}\right)^h\times\right.$$

$$\left.\left.\left[\sum_{g=0}^{l-1-2h}\left(E_g^h\left(T^{(l)}\right)\frac{(k\gamma+\omega_0)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T)+F_g^h\left(T^{(l)}\right)\frac{(k\gamma+\alpha_0)}{(k\gamma+\alpha_0+\omega_0)}\right)^g\right]\right]\right\}$$

From symmetry, we get for $l \le k$, we have that

$$t_{E_B}(A_N^{(l)}) = \frac{p_A}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l}\left\{\alpha_0\left[(k\gamma+\alpha_0)^l+\right.\right.$$

$$\left.\sum_{j=1}^{\lceil\frac{l}{2}\rceil}(\alpha_0\omega_0)^j Z\left(l-2j\right)\sum_{h=0}^{l-2j}C_h^j(A_N^{(l)})(k\gamma+\alpha_0)^h(k\gamma+\omega_0)^{l-2j-h}\right]+$$

$$\left.(\sigma+\omega_0)\alpha_0\sum_{j=1}^{\lceil\frac{l}{2}\rceil}(\alpha_0\omega_0)^{j-1}\left[\sum_{i=0}^{l+1-2j}D_i^j(A_N^{(l)})(k\gamma+\alpha_0)^i(k\gamma+\omega_0)^{l+1-2j-i}\right]\right\}$$

$$t_{E_B}(I_N^{(l)}) = \frac{(1-p_A)}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l}\left\{\alpha_0\left[(k\gamma+\alpha_0)^l+\right.\right.$$

$$\left.\sum_{j=1}^{\lceil\frac{l}{2}\rceil}(\alpha_0\omega_0(1-\delta p_T))^j Z\left(l-2j\right)\sum_{h=0}^{l-2j}C_h^j(I_N^{(l)})(k\gamma+\alpha_0)^h\left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-2j-h}\right]+$$

$$\left.\alpha_0(\sigma+\omega_0)(1-\delta p_T)\sum_{j=1}^{\lceil\frac{l}{2}\rceil}(\alpha_0\omega_0(1-\delta p_T))^{j-1}\sum_{i=0}^{l+1-2j}D_i^j(I_N^{(l)})(k\gamma+\alpha_0)^i\left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l+1-2j-i}\right\}$$

$$t_{E_B}(A_B^{(l)}) = \frac{p_A}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l}\Big\{(\sigma+\omega_0)\Big[(k\gamma+\omega_0)^l + Z(l-2)\sum_{j=1}^{\lfloor\frac{l}{2}\rfloor}(\alpha_0\omega_0)^j\sum_{i=0}^{l-2j}C_i^j(A_B^{(l)})(k\gamma+\alpha_0)^i(k\gamma+\omega_0)^{l-2j-i}\Big] + \alpha_0\omega_0\Big[\sum_{h=0}^{l-1}(k\gamma+\alpha_0)^h(k\gamma+\omega_0)^{l-1-h} + Z(l-2)\times \sum_{j=1}^{\lfloor\frac{l}{2}\rfloor}(\alpha_0\omega_0)^j Z(l-1-2j)\sum_{h=0}^{l-1-2j}D_h^j(A_B^{(l)})(k\gamma+\alpha_0)^h(k\gamma+\omega_0)^{l-1-2j-h}\Big]\Big\}$$

$$t_{E_B}(I_B^{(l)}) = \frac{(1-p_A)}{k\gamma(\sigma+\omega_0+\alpha_0)(k\gamma+\omega_0+\alpha_0)^l}\Big\{(\sigma+\omega_0)\Big[(k\gamma+\omega_0)^l(1-\delta p_T)^l + Z(l-2)\sum_{j=1}^{\lfloor\frac{l}{2}\rfloor}(\alpha_0\omega_0(1-\delta p_T))^j\sum_{i=0}^{l-2j}C_i^j(A_B^{(l)})(k\gamma+\alpha_0)^i\left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-2j-i}\Big] + \omega_0\alpha_0\Big[\sum_{h=0}^{l-1}(k\gamma+\alpha_0)^h\left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-1-h} + Z(l-2)\times \sum_{j=1}^{\lfloor\frac{l}{2}\rfloor}(\alpha_0\omega_0(1-\delta p_T))^j Z(l-1-2j)\sum_{h=0}^{l-1-2j}D_h^j(A_B^{(l)})(k\gamma+\alpha_0)^h\left\{(k\gamma+\omega_0)(1-\delta p_T)\right\}^{l-1-2j-h}\Big]\Big\}$$

$$t_{E_B}\left(T^{(l)}\right) = \frac{(1-p_A)}{k\gamma(\sigma+\alpha_0+\omega_0)}\left\{Z(l-2)\frac{\delta p_T\omega_0\alpha_0}{(k\gamma+\alpha_0+\omega_0)}\left[\sum_{j=0}^{l-2}\left(\frac{(k\gamma+\alpha_0)}{(k\gamma+\alpha_0+\omega_0)}\right)^j \times \left[\sum_{i=0}^{l-2-j}\left(\frac{(k\gamma+\omega)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T)\right)^i\right] + Z(l-4)\sum_{h=1}^{\lfloor\frac{l-2}{2}\rfloor}\left(\frac{\alpha_0\omega_0(1-\delta p_T)}{(k\gamma+\alpha_0+\omega_0)^2}\right)^h \times \left[\sum_{g=0}^{l-2-2h}\left(C_g^h\left(T^{(l)}\right)\frac{(k\gamma+\omega_0)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T) + D_g^h\left(T^{(l)}\right)\frac{(k\gamma+\alpha_0)}{(k\gamma+\alpha_0+\omega_0)}\right)^g\right]\right] + \delta p_T(\sigma+\omega_0)\left[\sum_{j=0}^{l-1}\left[\frac{(k\gamma+\omega_0)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T)\right]^j + Z(l-3)\sum_{h=1}^{\lfloor\frac{l-1}{2}\rfloor}\left(\frac{\alpha_0\omega_0(1-\delta p_T)}{(k\gamma+\alpha_0+\omega_0)^2}\right)^h \times \left[\sum_{g=0}^{l-1-2h}\left(E_g^h\left(T^{(l)}\right)\frac{(k\gamma+\omega_0)}{(k\gamma+\alpha_0+\omega_0)}(1-\delta p_T) + F_g^h\left(T^{(l)}\right)\frac{(k\gamma+\alpha_0)}{(k\gamma+\alpha_0+\omega_0)}\right)^g\right]\right]\right\}$$

The coefficient terms follow no obvious pattern, but can be calculated using a symbolic manipulator such as the SYMPY package in PYTHON.

For $X \in \{N, B\}$ we define

$$\begin{aligned}\Lambda_X &= \lambda \circ t_{E_X} \\ &= \sum_{l=1}^{k} \beta_k \left(t_{E_X}(I_N^{(l)}) + t_{E_X}(I_B^{(l)}) + q_A \left(t_{E_X}(I_N^{(l)}) + t_{E_X}(I_B^{(l)}) \right) + q_T t_{E_X}(T^{(l)}) \right) .\end{aligned}$$

The next generation matrix is then

$$\mathbf{K} = \begin{bmatrix} \Lambda_N N_0 & \lambda_B N_0 \\ q_B \lambda_N B_0 & q_B \lambda_B B_0 \end{bmatrix} .$$

Since $\det \mathbf{K} = 0$, we get that

$$\mathcal{R}_0 = \Lambda_N N_0 + q_B \Lambda_B B_0 .$$

4 Numerics

References

[1] Ryan M, Hickson RI, Hill EM, House T, Isham V, Zhang D, et al. A Behaviour and Disease Model of Testing and Isolation. arXiv preprint arXiv:250402488. 2025.

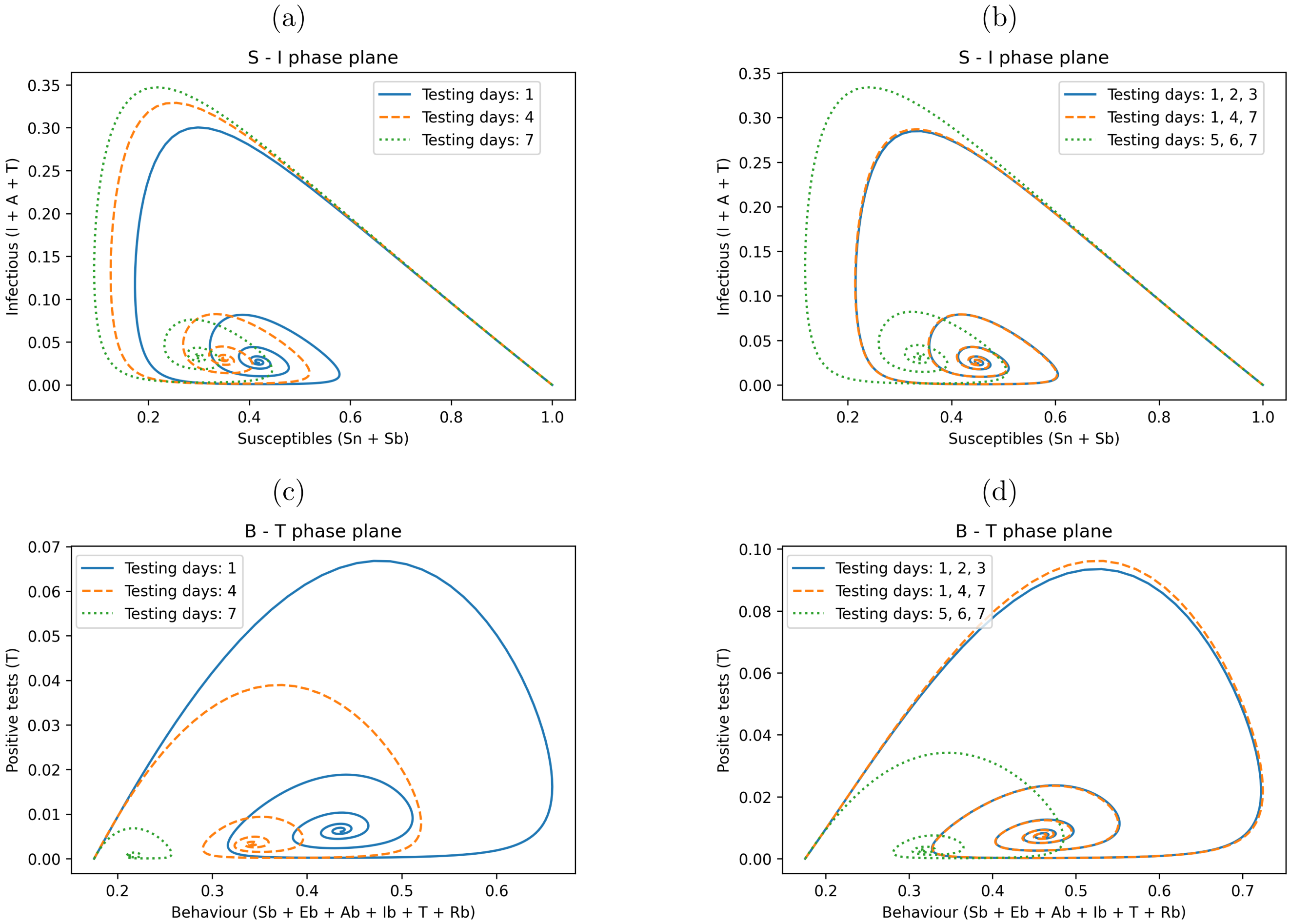

Figure 2: **Phase planes for testing on different days of the infectious period.** Top row: The susceptible - infectious phase plane for allowing (a) one test over the infectious period and (b) three tests over the infectious period. Bottom row: The behaviour - testing phase plane for allowing (c) one test over the infectious period and (d) three tests over the infectious period. Testing days are determined by switching the value of the appropriate δ_i from 0 to 1. Parameter values are from Table 1.

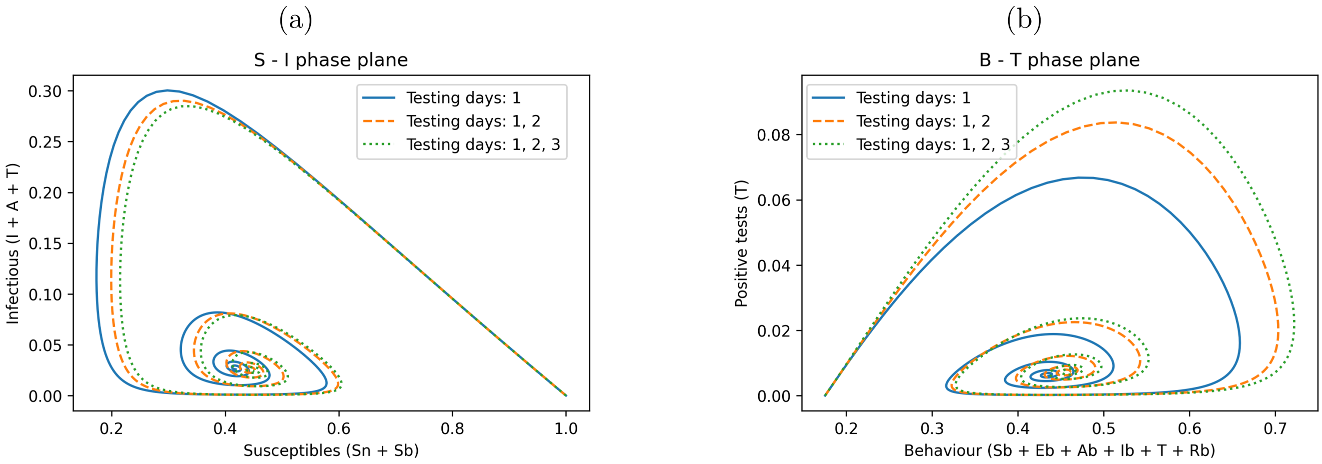

Figure 3: **Phase planes for testing on different days of the infectious period.** (a) The susceptible - infectious phase plane for allowing different numbers of tests over the period and (b) The behaviour - testing phase plane for allowing different number of tests over the period. Testing days are determined by switching the value of the appropriate δ_i from 0 to 1. Parameter values are from Table 1.